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Localization of Analytic Regularity Criteria on the Vorticity and Balance Between the Vorticity Magnitude and Coherence of the Vorticity Direction in the 3D NSE

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Abstract: The first part of the paper provides spatio-temporal localization of a family of analytic regularity classes for the 3D NSE obtained by Beirao Da Veiga (space-time integrability of the gradient of the velocity on $\mathbb{R}^3 \times (0, T)$ which is out of the range of the Sobolev embedding theorem reduction to the classical Foias-Ladyzhenskaya-Prodi-Serrin space-time integrability conditions on the velocity) as well as the localization of the Beale-Kato-Majda regularity criterion (time integrability of the L^∞-norm of the vorticity). The second part introduces a family of local, scaling invariant, hybrid geometric-analytic classes in which coherence of the vorticity direction serves as a weight in the local spatio-temporal integrability of the vorticity magnitude.

1. Introduction

The classical Foias-Ladyzhenskaya-Prodi-Serrin regularity criterion (FLPS) for weak solutions to the 3D NSE on an open space-time domain $\Omega \times (0, T)$,

$$u_t - \triangle u + (u \cdot \nabla)u + \nabla p = 0, \tag{1}$$

supplemented with the incompressibility condition $\operatorname{div} u = 0$, where u is the velocity of the fluid and p is the pressure, reads

$$\|u\|_{L^q(\Omega)}^{\frac{2q}{q-3}} \in L^1(0, T) \quad \text{for some } 3 < q \le \infty.$$

Local versions as well as generalizations to the weak Lebesgue spaces can be found, e.g., in [Se,St,T,KK] and the references therein. In the endpoint case $q = 3$, i.e., $\|u\|_{L^3_x} \in L^\infty_t$,"standard methods" require a smallness condition; this smallness condition was removed in [ESS] via a different method utilizing unique continuation across a spatial boundary, backward uniqueness for the heat equation and Carleman-type estimates.

A method for the study of a possible singular set in the class of "suitable weak solutions" was introduced in [CKN]. The method is based on the localized energy inequality, scaling arguments and local estimates on the pressure (this is also for the velocity-pressure formulation), and it provides blow-up criteria on the family of shrinking parabolic cylinders below a spatio-temporal point. One consequence of the blow-up estimates is that the 1D Hausdorff measure of the singular set in $\Omega \times (0, T)$ is zero (see also recent works [ZS and GKT] where a local FLPS criterion is given as a special case of a regularity criterion obtained by the CKN method and a CKN-type condition is presented for a boundary point, respectively).

The following family of regularity classes was introduced in [daVeiga],

$$\|Du\|_{L^q(\mathbb{R}^3)}^{\frac{2q}{2q-3}} \in L^1(0, T) \quad \text{for some } 3 \leq q < \infty. \tag{2}$$

Note that in this range of the parameter q, $3 \leq q < \infty$, in contrast to the case $\frac{3}{2} \leq q < 3$, the regularity criterion (2) can not be reduced to the FLSP criterion via the Sobolev embedding theorem; hence, (2) represents a natural extension of the FLSP regularity classes. (In the case of the whole space, the proper L^p-norms of ω and ∇u are equivalent due to the Calderon-Zygmund theorem; hence, one can view (2) as a regularity criterion on the vorticity.) A generalization to the case of Dirichlet boundary conditions on a C^2-domain was given in [B].

The case $q = \infty$, more precisely, the time integrability of $\|\omega\|_{L^\infty(\mathbb{R}^3)}$, where $\omega = \operatorname{curl} u$ is the vorticity is the Beale-Kato-Majda (BKM) criterion [BKM]; actually, less is required – the time integrability of the BMO-norm of ω [KT] (see also [KOT1,KOT2] for generalizations to the time integrability of the homogeneous Besov norm $\dot{B}^0_{\infty,\infty}$).

Note that all the aforementioned analytic regularity classes are critical in the sense they are invariant with respect to the NSE scaling $u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t)$, $p_\lambda(x, t) = \lambda^2 p(\lambda x, \lambda^2 t)$, $\omega_\lambda(x, t) = \lambda^2 \omega(\lambda x, \lambda^2 t)$ $(\lambda > 0)$.

In the first part of the paper, we present a spatio-temporal localization of the regularity class (2) as well as the BKM regularity criterion to an arbitrarily small space-time cylinder $Q_r(x_0, t_0) = B_r(x_0) \times (t_0 - r^2, t_0)$ utilizing a spatio-temporal localization of vortex-stretching recently obtained in [Gr2]. Localization of the BMO-criterion is addressed in the forthcoming work [GG].

The study of geometric criteria for the regularity was pioneered by Constantin in [Co] where an explicit representation formula for the stretching factor in the evolution of the vorticity magnitude was derived. The representation is in the form of a singular integral whose kernel is depleted by coherence of the vorticity direction. This geometric depletion of the nonlinearity was utilized in [CoFe] where it was shown that Lipschitz coherence in the region of high vorticity on an interval $(0, T)$ prevents a possible formation of singularities at $t = T$. The result was sharpened in [daVeigaBe1] where the Lipschitz coherence was replaced with the $\frac{1}{2}$-Hölder coherence.

A different approach to discovering geometric conditions on the 3D NSE flow preventing the singularity formation was introduced in [Gr] utilizing local-in-time spatial analyticity of the solutions via a plurisubharmonic measure (p-measure) maximum principle in $\mathbb{C}^3$ – an extension of the classical harmonic measure maximum principle in one complex variable (for computing the p-measure using foliations see [GaKa,Ga]). The result states that local existence of a thin direction in the region of intense vorticity on the length scale that is essentially an L^∞-version of the Kolmogorov dissipation scale suffices to control the evolution of the vorticity magnitude preventing finite time blow-up. It is interesting that the same length scale appeared in [RuGr] utilizing a completely different technique; namely, restricting a representation formula for the vortex-stretching term to small (comparable to the aforementioned dissipation scale), medium and large scales. The result reads that sparseness (in the sense of volume) of the region of high vorticity depletes the nonlinearity. Detecting additional cancelation properties in the nonlinearity, it was then shown (cf. [RuGr]) that a certain isotropy condition on the velocity in the region of intense fluid activity prevents the possible formation of singularities.

A mixed geometric-analytic regularity class was presented in [daVeiga2]. For $0 < s \le \frac{1}{2}$, let p be such that $\frac{3}{p} = s + 1$. Then the class in view is defined by requiring s-Hölder coherence of the vorticity direction and the following space-time integrability condition on the vorticity magnitude,

$$\omega \in L^2\left(0, T; L^p(\mathbb{R}^3)\right). \tag{3}$$

Another mixed geometric-analytic regularity class was independently introduced in [GrRu]; namely, $\frac{1}{q}$-Hölder coherence of the vorticity direction supplemented with the following integrability condition on the vorticity magnitude:

$$\|\omega\|_{L^q(\mathbb{R}^3)}^{\frac{q}{q-1}} \in L^1(0, T) \tag{4}$$

for some $2 \le g < \infty$.

Note that the above two conditions are in fact complementary; they meet at the purely geometric case ($\frac{1}{2}-$Hölder coherence of the vorticity direction), and then they run in two separate directions – the first to the endpoint regularity class in (2), and the second to the BKM criterion.

A more general family of mixed geometric-analytic regularity classes was recently obtained in [Ch], where an assumption on the space-time integrability of the vorticity magnitude is paired with an assumption of the membership of the vorticity direction in a homogeneous Tribel-Lizorkin space $\dot{\mathcal{F}}^s_{p,q}$.

In all the aforementioned purely geometric and mixed geometric-analytic regularity criteria the spatial domain is the whole space $\mathbb{R}^3$, the main reason being that the techniques used rely on the Biot-Savart law, a *non-local* representation of the velocity by the vorticity over the whole space. The case of the non-slip boundary conditions on smooth bounded domains was treated in [daVeiga2], where $\frac{1}{2}$-Hölder coherence regularity criterion was established under an additional assumption on the control of the normal derivative of the vorticity magnitude along the boundary. In the case of the slip boundary conditions on the half-space, no additional assumptions are needed (cf. [daVeiga1]). This result was recently generalized to the case of the free boundary-type boundary conditions on smooth bounded domains in [daVeigaBe2]. In each case, the proof is based on a version of the Biot-Savart law corresponding to the particular type of the boundary conditions.

Since the possible formation of singularities is a local phenomenon, a physically relevant question is whether it is possible to localize geometric and mixed geometric-analytic regularity conditions. In the case of the whole space, a localization of the mixed geometric-analytic conditions (4) (including the purely geometric $\frac{1}{2}$-Hölder coherence) to an arbitrarily small space-time cylinder was obtained in [GrZh]. The localization of the transport of the vorticity by the velocity was independent of the choice of the spatial domain/boundary conditions; however, the localization of the vortex-stretching term was based on the restriction of Constantin's representation formula for the stretching factor in the evolution of the vorticity magnitude (valid on the whole space) to small scales. A localized representation formula for the vortex-stretching term based on a localization of the Biot-Savart law was recently derived in [Gr2]. This led to complete localization of the evolution of the enstrophy independently of the type of the boundary conditions, and in particular, to complete localization of the $\frac{1}{2}$-Hölder coherence condition (cf. [Gr2]).

In the second part of the paper, we introduce a family of local, critical, hybrid geometric-analytic classes in which coherence of the vorticity direction serves as a weight in the space-time integrability of the vorticity magnitude. (Instead of integrating the vorticity magnitude against the homogeneous Lebesgue measure, we integrate it against a nonhomogeneous measure that depends only on the vorticity direction.)

Denote by η the vorticity direction. Let (x, t) be a spatio-temporal point, $r > 0$ and $0 < \gamma < 1$. A γ-Hölder measure of coherence of the vorticity direction at (x, t) is then given by

$$\rho_{\gamma,r}(x,t) = \sup_{y\in B(x,r),\, y\neq x} \frac{|\sin\varphi\,(\eta(x,t),\eta(y,t))\,|}{|x-y|^{\gamma}}.$$

For a given point (x_0, t_0) and $\alpha, \delta > 0$, the class under consideration is defined by requiring finiteness of

$$\int_{t_0-(2r)^2}^{t_0} \left[\int_{B(x_0,2r)} |\omega(x,t)|^{\alpha}\, \rho^{\alpha}_{\gamma,2r}(x,t)\,dx\right]^{\delta} dt. \tag{5}$$

It will transpire in the proof that

$$\delta = \frac{2}{(2+\gamma)\alpha - 3} \tag{6}$$

with the suitable restrictions on the parameters α and γ. It is worth observing that the condition (6) makes the class (5) scaling invariant (critical).

The case $\gamma = \frac{1}{2}$, $\alpha = 2$ and $\delta = 1$, namely

$$\int_{t_0-(2R)^2}^{t_0} \int_{B(x_0,2R)} |\omega(x,t)|^2\, \rho^2_{\frac{1}{2},2R}(x,t)\,dx\,dt, \tag{7}$$

is included (and is an improvement over the $\frac{1}{2}$-Hölder coherence–the sup in x and t bound on $\rho_{\frac{1}{2}}$). For a comparison with what can be bounded, a local spatio-temporal average of $|\omega||\nabla\eta|^2$ averaged over a ball around a spatial point moving with the fluid over a suitable interval of time is *a priori* bounded (cf. [CPS]).

As in [GrZh, Gr2], for simplicity of the exposition, we present the relevant calculations on smooth solutions. More precisely, we consider a Leray solution on a space-time domain $\Omega \times (0, T)$ and suppose that u is smooth in an open parabolic cylinder $Q_{2R}(x_0, t_0) = B(x_0, 2R) \times \big(t_0 - (2R)^2, t_0\big)$ contained in $\Omega \times (0, T)$. The goal is to show that, under a suitable local condition (analytic or hybrid geometric-analytic) on $Q_{2R}(x_0, t_0)$, the localized enstrophy remains uniformly bounded up to $t = t_0$, i.e.,

$$\sup_{t\in(t_0-R^2,t_0)} \int_{B(x_0,R)} |\omega(x,t)|^2\,dx < \infty.$$

Alternatively, we can consider, e.g., a class of suitable weak solutions constructed in [CKN] as a limit of a family of delayed mollifications (see also [C1]), and perform the calculations on the smooth approximations.

Remark 1. Since all the estimates are local, instead of considering (finite energy) Leray solutions, we can consider a class of local Leray solutions in the sense of [LR] (finite locally uniform energy).

The paper is organized as follows. In Sect. 2 we recall the relevant results from [GrZh, Gr2]. Section 3 is devoted to the localization of the family of analytic regularity classes (2) as well as the BKM regularity criterion, and in Sect. 4 we show that the critical hybrid geometric-analytic condition (5) prevents a possible formation of the singularities in the flow.

Remark 2. After the manuscript was submitted, the authors learned that the paper [ChKaLe] contains a local version of the analytic regularity criteria on the vorticity presented in Theorem 1 excluding the endpoint case $q = \infty$, i.e., the BKM criterion. More precisely, using a localization of the velocity-pressure formulation of the equations, the authors in [ChKaLe] obtained a local version of the regularity criteria on ∇u [ChKaLe, Prop. 1] which, via a localized Biot-Savart law, lead to the local conditions on the vorticity [ChKaLe, Prop. 2]. (Since the L^∞-norms of ω and ∇u are not equivalent, this localization method does not lead to a local BKM criterion.) This was then utilized to obtain a local version of the analytic regularity criteria on the two components of the vorticity derived in [ChCh]. In addition, [ChKaLe] contains a localization of the mixed geometric-analytic regularity criteria obtained in [Ch].

2. Bounds on the Localized Enstrophy

Taking the curl of the velocity-pressure formulation of the 3D NSE (1) on a space-time domain $\Omega \times (0, T)$ leads to the vorticity-velocity formulation,

$$\omega_t - \triangle\omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u. \tag{8}$$

The left-hand side of the nonlinearity is the transport term and the right-hand side is the vortex-stretching term.

Fix a point (x_0, t_0) in $\Omega \times (0, T)$. Let $0 < R < 1$ be such that $Q_{2R}(x_0, t_0) \subset \Omega \times (0, T)$, $r \le R$, and let $\psi(x, t) = \phi(x)\eta(t)$ be a smooth cut-off function on $Q_{2r}(x_0, t_0)$ satisfying

$$\operatorname{supp}\phi \subset B(x_0, 2r),\ \ \phi = 1 \text{ on } B(x_0, r),\ \ \frac{|\nabla\phi|}{\phi^\rho} \le \frac{c}{r} \text{ for some } \rho \in (0, 1), \quad 0 \le \phi \le 1$$

and

$$\operatorname{supp}\eta \subset (t_0 - (2r)^2, t_0],\ \ \eta = 1 \text{ on } [t_0 - r^2, t_0],\ \ |\eta'| \le \frac{c}{r^2}, \quad 0 \le \eta \le 1.$$

(It was shown in [GrZh and Gr2] that, choosing the parameter ρ sufficiently close to 1, it is possible to control the lower order terms in the localized transport term $\int_{Q_{2r}} (u \cdot \nabla)\omega \cdot \psi^2\omega\, dx dt$ and in the localized vortex-stretching term $\int_{Q_{2r}} (\omega \cdot \nabla)u \cdot \psi^2\omega\, dx dt$, respectively.)

The following localization formula for the vortex-stretching term was obtained in [Gr2]:

$$
\begin{aligned}
&\phi^2(x)(\omega\cdot\nabla)u\cdot\omega(x)\\
&\quad=\phi(x)\,\frac{\partial}{\partial x_i}u_j(x)\,\phi(x)\,\omega_i(x)\,\omega_j(x)\\
&\quad=-c\,P.V.\int_{B(x_0,2r)}\epsilon_{jkl}\,\frac{\partial^2}{\partial x_i\partial y_k}\frac{1}{|x-y|}\,\phi\,\omega_l\,dy\,\phi(x)\,\omega_i(x)\,\omega_j(x)+\text{LOT}\\
&\quad=-c\,P.V.\int_{B(x_0,2r)}(\omega(x)\times\omega(y))\cdot G_\omega(x,y)\,\phi(y)\,\phi(x)\,dy+\text{ LOT}\\
&\quad=\text{VST}_{loc}+\text{LOT}
\end{aligned}
\tag{9}
$$

(ϵ_{jkl} is the Levi-Civita symbol) for x in $B(x_0, 2r)$, uniformly in time, where

$$
(G_\omega(x,y))_k=\frac{\partial^2}{\partial x_i\partial y_k}\frac{1}{|x-y|}\,\omega_i(x)
$$

and LOT denotes terms that are either lower order for at least one order of the differentiation or/and less singular for at least one power of $|x-y|$ then the leading order term VST_{loc}.

Let s in $(t_0-(2r)^2, t_0)$. Then (cf. [GrZh]),

$$
\begin{aligned}
&\frac{1}{2}\int_{B(x_0,2r)}\phi^2(x)|\omega|^2(x,s)\,dx+\int_{Q_{2r}^s}|\nabla(\psi\omega)|^2\,dxdt\\
&\quad\le\frac{1}{2}\int_{Q_{2r}}|\nabla(\psi\omega)|^2\,dxdt+c(r)\int_{Q_{2r}}|\omega|^2\,dxdt+\left|\int_{Q_{2r}^s}(\omega\cdot\nabla)u\cdot\psi^2\omega\,dxdt\right|\\
&\quad=\frac{1}{2}\int_{Q_{2r}}|\nabla(\psi\omega)|^2\,dxdt+c(r)\int_{Q_{2r}}|\omega|^2\,dxdt+I,
\end{aligned}
\tag{10}
$$

where $Q_{2r}^s=B(x_0,2r)\times(t_0-(2r)^2,s)$ (this holds for any $\frac{1}{2}\le\rho<1$).

Notice that

$$
I\le\left|\int_{Q_{2r}^s}\eta^2\,\text{VST}_{loc}\,dxdt\right|+\left|\int_{Q_{2r}^s}\eta^2\,\text{LOT}\,dxdt\right|.
$$

Choosing ρ sufficiently close to 1, the second term (the sum of all the lower order vortex-stretching terms) can be bounded (cf. [Gr2]) by

$$
\begin{aligned}
&\max\left\{c\,\|\nabla u\|_{L^2(Q_{2r})},\frac{1}{4}\right\}\left(\frac{1}{2}\sup_{t\in(t_0-(2r)^2,t_0)}\|\phi\omega(t)\|^2_{L^2(B(x_0,2r))}+\|\nabla(\psi\omega)\|^2_{L^2(Q_{2r})}\right)\\
&\quad+M\left(r,\|\nabla u\|_{L^2(Q_{2r})}\right).
\end{aligned}
$$

Since u is a weak solution, the first term in the above expression can, for a sufficiently small r, eventually (after sending s to t_0) be absorbed by the left-hand side of (10) and the second term is bounded.

Summarizing, in order to control the evolution of the localized enstrophy, we are left to estimate the leading order vortex-stretching term $\int_{Q_{2r}^s} \eta^2 \, \mathrm{VST}_{loc} \, dxdt$, i.e.,

$$\int_{Q_{2r}^s} P.V.\int_{B(x_0,2r)} \epsilon_{jkl} \frac{\partial^2}{\partial x_i \partial y_k} \frac{1}{|x-y|} \, \psi(y,t)\omega_l(y,t)\, dy \; \psi(x,t)\omega_i(x,t)\, \omega_j(x,t)\, dxdt. \tag{11}$$

3. Localization of Analytic Regularity Criteria on the Vorticity

In this section we obtain spatio-temporal localization of the regularity criteria in (2) and the BKM regularity criterion to an arbitrarily small parabolic cylinder.

Theorem 1. *Let u be a Leray solution on a space-time domain $\Omega \times (0,T)$, (x_0,t_0) in $\Omega \times (0,T)$ and $0 < R < 1$ such that the parabolic cylinder $Q_{2R}(x_0,t_0) = B(x_0,2R) \times \big(t_0-(2R)^2, t_0\big)$ is contained in $\Omega \times (0,T)$.*

Suppose that u is smooth in $Q_{2R}(x_0,t_0)$ and

$$\|\omega\|_{L^q(B(x_0,2R))}^{\frac{2q}{2q-3}} \text{ is in } L^1(t_0-(2R)^2, t_0)$$

for some $3 \le q \le \infty$. Then the localized enstrophy remains uniformly bounded up to $t = t_0$, i.e.,

$$\sup_{t\in(t_0-R^2,t_0)} \int_{B(x_0,R)} |\omega(x,t)|^2 \, dx < \infty.$$

Proof. Let $r \le R$. Following a discussion in the previous section, it remains to estimate the leading order vortex-stretching term

$$J = \int_{Q_{2r}^s} P.V.\int_{B(x_0,2r)} \epsilon_{jkl} \frac{\partial^2}{\partial x_i \partial y_k} \frac{1}{|x-y|} \, \psi(y,t)\omega_l(y,t)\, dy \; \psi(x,t)\omega_i(x,t) \\ \times \omega_j(x,t)\, dxdt.$$

Notice that

$$\frac{\partial^2}{\partial x_i \partial y_k} \frac{1}{|x-y|}$$

is a Calderon-Zygmund kernel.

Consider first the case $3 \le q < \infty$. Hölder inequality in x with the exponents $2q'$, $2q'$ and q, where q' is the conjugate exponent of q, followed by an application of the Calderon-Zygmund theorem, yield

$$J \le c \int_{t_0-(2r)^2}^{t_0} \|\omega(t)\|_{L^q(B(x_0,2r))} \|(\psi\omega)(t)\|_{L^{2q'}(B(x_0,2r))}^2 \, dt.$$

Interpolating the second factor in the integrand,

$$J \le c \int_{t_0-(2r)^2}^{t_0} \|\omega(t)\|_{L^q(B(x_0,2r))} \|(\psi\omega)(t)\|_{L^2(B(x_0,2r))}^{\frac{2q-3}{q}} \|\nabla(\psi\omega)(t)\|_{L^2(B(x_0,2r))}^{\frac{3}{q}} \, dt.$$

Hölder inequality in t with the exponents $\frac{2q}{2q-3}$, ∞ and $\frac{2q}{3}$, and the polarization inequality imply our final bound on J,

$$J \leq c \left(\int_{t_0-(2r)^2}^{t_0} \|\omega(t)\|_{L^q(B(x_0,2r))}^{\frac{2q}{2q-3}} \, dt \right)^{\frac{2q-3}{2q}} \times \left(\frac{1}{2} \sup_{t \in (t_0-(2r)^2, t_0)} \|\phi\omega(t)\|_{L^2(B(x_0,2r))}^2 + \|\nabla(\psi\omega)\|_{L^2(Q_{2r})}^2 \right). \tag{12}$$

Collecting all the estimates, (10) yields

$$\begin{aligned}
&\frac{1}{2} \sup_{t \in (t_0-(2r)^2, t_0)} \|\phi\omega(t)\|_{L^2(B(x_0,2r))}^2 + \|\nabla(\psi\omega)\|_{L^2(Q_{2r})}^2 \\
&\leq \frac{1}{2} \|\nabla(\psi\omega)\|_{L^2(Q_{2r})}^2 + c(r)\|\omega\|_{L^2(Q_{2r})}^2 \\
&\quad + \max\left\{ c\,\|\nabla u\|_{L^2(Q_{2r})}, \frac{1}{4} \right\} \left(\frac{1}{2} \sup_{t \in (t_0-(2r)^2, t_0)} \|\phi\omega(t)\|_{L^2(B(x_0,2r))}^2 + \|\nabla(\psi\omega)\|_{L^2(Q_{2r})}^2 \right) \\
&\quad + M\left(r, \|\nabla u\|_{L^2(Q_{2r})}\right) + c \left(\int_{t_0-(2r)^2}^{t_0} \|\omega(t)\|_{L^q(B(x_0,2r))}^{\frac{2q}{2q-3}} \, dt \right)^{\frac{2q-3}{2q}} \\
&\quad \times \left(\frac{1}{2} \sup_{t \in (t_0-(2r)^2, t_0)} \|\phi\omega(t)\|_{L^2(B(x_0,2r))}^2 + \|\nabla(\psi\omega)\|_{L^2(Q_{2r})}^2 \right).
\end{aligned}$$

Since u is a weak solution, choosing r small enough, the third term can be absorbed. Similarly, our assumption on the local integrability of ω implies that the last term can be absorbed too.

This proof actually works for any $\frac{3}{2} < q < \infty$ (and for $q = \frac{3}{2}$ under an additional smallness assumption); however, as already mentioned in the Introduction, in the range of the parameter q, $\frac{3}{2} \leq q < 3$, the regularity criterion (2) can be reduced to the FLSP criterion via the Sobolev embedding theorem.

The case $q = \infty$ requires only minor modifications. Hölder inequality in x with the exponents 2, 2 and ∞ and the Calderon-Zygmund theorem yield

$$J \leq c \int_{t_0-(2r)^2}^{t_0} \|\omega(t)\|_{L^\infty(B(x_0,2r))} \|(\psi\omega)(t)\|_{L^2(B(x_0,2r))}^2 \, dt,$$

which is in turn bounded by

$$c \left(\int_{t_0-(2r)^2}^{t_0} \|\omega(t)\|_{L^\infty(B(x_0,2r))} \, dt \right) \sup_{t \in (t_0-(2r)^2, t_0} \|\phi\omega(t)\|_{L^2(B(x_0,2r))}^2;$$

this can be absorbed for r small enough.

The result for $r = R$ follows by covering $B(x_0, 2R)$ with finitely many balls. $\square$

4. A Local Hybrid Geometric-Analytic Regularity Criterion

This section introduces a two-parameter family of local, scaling invariant (critical), hybrid geometric-analytic classes in which coherence of the vorticity direction serves as a weight in the space-time integrability of the vorticity magnitude. The classes in view represent an improvement over the corresponding mixed geometric-analytic criteria in which coherence of the vorticity direction and space-time integrability of the vorticity magnitude are considered separately (e.g., compared to the classes presented in [GrRu]), but more importantly, provide a local, more refined measure of the balance between the vorticity magnitude and coherence of the vorticity direction preventing the formation of singularities.

As in the Introduction, denote by $\rho_{\gamma,r}$ the following γ-Hölder measure of coherence of the vorticity direction η at a point (x,t),

$$\rho_{\gamma,r}(x,t) = \sup_{y\in B(x,r), y\neq x} \frac{|\sin\varphi\,(\eta(x,t),\eta(y,t))\,|}{|x-y|^\gamma}.$$

Theorem 2. *Let u be a Leray solution on a space-time domain $\Omega\times(0,T)$, (x_0,t_0) in $\Omega\times(0,T)$ and $0<R<1$ such that the parabolic cylinder $Q_{2R}(x_0,t_0) = B(x_0,2R)\times\big(t_0-(2R)^2,t_0\big)$ is contained in $\Omega\times(0,T)$.*

Let $0<\gamma<1, \alpha<\frac{3}{\gamma}, \alpha>\frac{3}{\gamma+2}$ and

$$\delta = \frac{2}{(2+\gamma)\alpha-3}$$

(the scaling invariance). Suppose that u is smooth in $Q_{2R}(x_0,t_0)$ and

$$\int_{t_0-(2r)^2}^{t_0}\left[\int_{B(x_0,2r)} |\omega(x,t)|^\alpha\ \rho_{\gamma,2r}^\alpha(x,t)dx\right]^\delta dt$$

is finite. Then the localized enstrophy remains uniformly bounded up to $t=t_0$, i.e.,

$$\sup_{t\in(t_0-R^2,t_0)}\int_{B(x_0,R)}|\omega(x,t)|^2\,dx<\infty.$$

Proof. Let $r\le R$. As in [Gr2], we utilize the geometric structure of the leading order vortex-stretching term J, and express it as (see (9))

$$J=\int_{Q_{2r}^s} P.V.\int_{B(x_0,2r)}(\omega(x,t)\times\omega(y,t))\cdot G_\omega(x,y,t)\,\psi(y,t)\,\psi(x,t)\,dy\,dxdt, \tag{13}$$

where

$$(G_\omega(x,y,t))_k = \frac{\partial^2}{\partial x_i\partial y_k}\frac{1}{|x-y|}\,\omega_i(x,t).$$

This leads to the following bound:

$$\begin{aligned} J \le c\int_{t_0-(2r)^2}^{t_0}\int_{B(x_0,2r)} &\big(\rho_{\gamma,2r}(x,t)|\omega(x,t)|\big)\\ &\times\left(\int_{B(x_0,2r)}\frac{1}{|x-y|^{3-\gamma}}|(\psi\omega)(y,t)|\,dy\right)(|(\psi\omega)(x,t)|)\ dx\,dt. \end{aligned}\tag{14}$$

As a preview, we present the proof in a particular case when $\gamma = \frac{1}{2}$, $\alpha = 2$ and $\delta = 1$ first.

Hölder inequality in x with the exponents 2, 4 and 4 yields

$$J \le c \int_{t_0-(2r)^2}^{t_0} \|(\rho_{\frac{1}{2},2r}|\omega|)(t)\|_{L^2(B(x_0,2r))} \times \left\| \int_{B(x_0,2r)} \frac{1}{|\cdot - y|^{\frac{5}{2}}} |(\psi\omega)(y,t)|\, dy \right\|_{L^4(B(x_0,2r))} \|(\psi\omega)(t)\|_{L^4(B(x_0,2r))}\, dt.$$

This is by the Hardy-Littlewood-Sobolev inequality bounded by

$$c \int_{t_0-(2r)^2}^{t_0} \|(\rho_{\frac{1}{2},2r}|\omega|)(t)\|_{L^2(B(x_0,2r))} \|(\psi\omega)(t)\|_{L^{\frac{12}{5}}(B(x_0,2r))} \|(\psi\omega)(t)\|_{L^4(B(x_0,2r))}\, dt.$$

Interpolating the last two factors in the integrand implies

$$\begin{aligned} J &\le c \int_{t_0-(2r)^2}^{t_0} \|(\rho_{\frac{1}{2},2r}|\omega|)(t)\|_{L^2(B(x_0,2r))} \|(\psi\omega)(t)\|_{L^2(B(x_0,2r))} \|\nabla(\psi\omega)(t)\|_{L^2(B(x_0,2r))}\, dt \\ &\le c \left(\int_{t_0-(2r)^2}^{t_0} \left[\int_{B(x_0,2r)} \left(\rho_{\frac{1}{2},2r}(x,t)|\omega(x,t)|\right)^2 dx \right] dt \right)^{\frac{1}{2}} \\ &\quad \times \left(\frac{1}{2} \sup_{t\in(t_0-(2r)^2,t_0)} \|\phi\omega(t)\|^2_{L^2(B(x_0,2r))} + \|\nabla(\psi\omega)\|^2_{L^2(Q_{2r})} \right). \end{aligned}$$

As in the proof of Theorem 1, collecting all the estimates, (10) yields

$$\begin{aligned} &\frac{1}{2} \sup_{t\in(t_0-(2r)^2,t_0)} \|\phi\omega(t)\|^2_{L^2(B(x_0,2r))} + \|\nabla(\psi\omega)\|^2_{L^2(Q_{2r})} \\ &\le \frac{1}{2} \|\nabla(\psi\omega)\|^2_{L^2(Q_{2r})} + c(r)\|\omega\|^2_{L^2(Q_{2r})} \\ &\quad + \max\left\{ c\, \|\nabla u\|_{L^2(Q_{2r})}, \frac{1}{4} \right\} \left(\frac{1}{2} \sup_{t\in(t_0-(2r)^2,t_0)} \|\phi\omega(t)\|^2_{L^2(B(x_0,2r))} + \|\nabla(\psi\omega)\|^2_{L^2(Q_{2r})} \right) \\ &\quad + M\left(r, \|\nabla u\|_{L^2(Q_{2r})}\right) \\ &\quad + c \left(\int_{t_0-(2r)^2}^{t_0} \left[\int_{B(x_0,2r)} \left(\rho_{\frac{1}{2},2r}(x,t)|\omega(x,t)|\right)^2 dx \right] dt \right)^{\frac{1}{2}} \\ &\quad \times \left(\frac{1}{2} \sup_{t\in(t_0-(2r)^2,t_0)} \|\phi\omega(t)\|^2_{L^2(B(x_0,2r))} + \|\nabla(\psi\omega)\|^2_{L^2(Q_{2r})} \right), \end{aligned}$$

finishing the argument.

The proof in the general case follows exactly the same steps. Let $0 < \gamma < 1$ and $\alpha > 1$. Hölder inequality in x with the exponents α, p_1 and p_2 applied to (14) yields

$$J \le c \int_{t_0-(2r)^2}^{t_0} \|(\rho_{\gamma,2r}|\omega|)(t)\|_{L^\alpha(B(x_0,2r))} \times \left\| \int_{B(x_0,2r)} \frac{1}{|\cdot - y|^{3-\gamma}} |(\psi\omega)(y,t)|\, dy \right\|_{L^{p_1}(B(x_0,2r))} \|(\psi\omega)(t)\|_{L^{p_2}(B(x_0,2r))}\, dt,$$

which is by the Hardy-Littlewood-Sobolev inequality bounded by

$$c\int_{t_0-(2r)^2}^{t_0} \|(\rho_{\gamma,2r}|\omega|)(t)\|_{L^\alpha(B(x_0,2r))}\|(\psi\omega)(t)\|_{L^{\widetilde{p}}(B(x_0,2r))}\|(\psi\omega)(t)\|_{L^{p_2}(B(x_0,2r))}\,dt,$$

where $\widetilde{p} = \dfrac{3p_1}{3+\gamma p_1}$. Interpolating the last two factors in the integrand implies

$$J \le c\int_{t_0-(2r)^2}^{t_0} \|(\rho_{\gamma,2r}|\omega|)(t)\|_{L^\alpha(B(x_0,2r))} \times\|(\psi\omega)(t)\|_{L^2(B(x_0,2r))}^{\gamma+3(1-\frac{1}{\alpha})-1}\|\nabla(\psi\omega)(t)\|_{L^2(B(x_0,2r))}^{3-\gamma-3(1-\frac{1}{\alpha})}\,dt$$

$\left(\dfrac{6-p_2}{2p_2}+\dfrac{6-\widetilde{p}}{2\widetilde{p}} = \gamma+3(1-\dfrac{1}{\alpha})-1\right)$. An application of the Hölder inequality in t with the exponents $\dfrac{2}{\gamma+3(1-\frac{1}{\alpha})-1}$, ∞ and $\dfrac{2}{3-\gamma-3(1-\frac{1}{\alpha})}$ yields our final bound on J,

$$J \le c\left(\int_{t_0-(2r)^2}^{t_0}\left[\int_{B(x_0,2r)}\left(\rho_{\gamma,2r}(x,t)|\omega(x,t)|\right)^\alpha dx\right]^\delta dt\right)^{\frac{1}{\alpha\delta}} \times\left(\frac{1}{2}\sup_{t\in(t_0-(2r)^2,t_0)}\|\phi\omega(t)\|_{L^2(B(x_0,2r))}^2+\|\nabla(\psi\omega)\|_{L^2(Q_{2r})}^2\right),$$

provided $\alpha < \frac{3}{\gamma}$ and $\alpha > \frac{3}{\gamma+2}$. □

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