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Wheeled Pro(p)file of Batalin-Vilkovisky Formalism

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Abstract: Using a technique of wheeled props we establish a correspondence between the homotopy theory of unimodular Lie 1-bialgebras and the famous Batalin-Vilkovisky formalism. Solutions of the so-called quantum master equation satisfying certain boundary conditions are proven to be in 1-1 correspondence with representations of a wheeled dg prop which, on the one hand, is isomorphic to the cobar construction of the prop of unimodular Lie 1-bialgebras and, on the other hand, is quasi-isomorphic to the dg wheeled prop of unimodular Poisson structures. These results allow us to apply properadic methods for computing formulae for a homotopy transfer of a unimodular Lie 1-bialgebra structure on an arbitrary complex to the associated quantum master function on its cohomology. It is proven that in the category of quantum BV manifolds associated with the homotopy theory of unimodular Lie 1-bialgebras quasi-isomorphisms are equivalence relations.

It is shown that Losev-Mnev's BF theory for unimodular Lie algebras can be naturally extended to the case of unimodular Lie 1-bialgebras (and, eventually, to the case of unimodular Poisson structures). Using a finite-dimensional version of the Batalin-Vilkovisky quantization formalism it is rigorously proven that the Feynman integrals computing the effective action of this new BF theory describe precisely homotopy transfer formulae obtained within the wheeled properadic approach to the quantum master equation. Quantum corrections (which are present in our BF model to all orders of the Planck constant) correspond precisely to what are often called “higher Massey products” in the homological algebra.

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1. Introduction

The theory of operads and props has grown nowadays from a useful technical tool into a kind of universal mathematical language with the help of which topologists, algebraists, homotopy theorists and geometers can fruitfully communicate with each other. For example, one and the same operad of little 2-disks (i) solves the recognition problem for based 2-loop spaces in algebraic topology, (ii) describes homotopy Gerstenhaber structure on the Hochschild deformation complex in homological algebra, and (iii) controls diffeomorphism invariant Hertling-Manin integrability equations [HeMa] in differential geometry. It is yet to see whether or not basic concepts and constructions of theoretical physics can be understood and developed in the framework of operads and props, but the fact that space-time, "the background of everything", can be turned into an ordinary observable — a certain function (representation) on a prop — is rather intriguing.

This paper attempts to tell a story of the famous theoretical physics *quantum master equation*,

$$\hbar\Delta\Gamma + \frac{1}{2}\{\Gamma, \Gamma\} = 0, \tag{1}$$

in the language of wheeled prop(erad)s. It is shown that an important class of its solutions (specified by certain boundary conditions in the quasi-classical limit) is controlled by a surprisingly simple wheeled prop of unimodular Lie 1-bialgebras and hence can be understood as a class of strongly homotopy algebras. It is proven that the homotopy classification of this class of quantum master functions is as simple as, for example, the homotopy classification of strongly homotopy Lie algebras given in [Ko]. These results allow us to compare the standard Feynman technique of producing new quantum master functions (called often in physics literature "effective actions") by integrating the original ones along certain Lagrangian submanifolds with the purely properadic homotopy transfer method which uses Koszul duality theory, and conclude (in a mathematically rigorous way) that they are identical to each other.

Here is a detailed description of paper's content. Section 2 gives a self-contained introduction into the theory of wheeled props, their bar and cobar constructions [Me3, MMS]. We introduce and study *Koszul duality* theory for quadratic wheeled properads[1] having in mind applications (in § 4 and 5) of the Koszul duality technique to two example important for us, the first of which controls the local finite-dimensional Poisson geometry, and the other one the local geometry of master equation (1). The content of this theory is standard (cf. [GeJo]):

- For any quadratic wheeled properad $\mathcal{P}$ there is a naturally associated Koszul dual wheeled coproperad $\mathcal{P}^{\perp}$ which comes together with a canonical monomorphism of dg coproperads, $\imath : \mathcal{P}^{\perp} \to B(\mathcal{P})$, into the bar construction on $\mathcal{P}$.
- The cobar construction, $B^c(\mathcal{P}^{\perp})$, is a dg free wheeled properad denoted in this paper by $\mathcal{P}_\infty$.
- There exists an epimorphism, $\mathcal{P}_\infty \to \mathcal{P}$, which is a quasi-isomorphism if $\mathcal{P}$ is Koszul.

The main result in § 2 is Theorem 2.7.1 which, if reformulated shortly, says that *given an arbitrary (not necessarily Koszul) quadratic wheeled properad and an arbitrary dg $\mathcal{P}$-algebra V, then every cohomological splitting of V makes canonically its cohomology, $H(V)$, into a $\mathcal{P}_\infty$-algebra; moreover, this induced $\mathcal{P}_\infty$ structure is given precisely by that sum of decorated graphs which describe the image of the canonical monomorphism* $\imath : \mathcal{P}^{\perp} \to B(\mathcal{P})$. This result gives a conceptual explanation of the well-known "experimental" fact that the homotopy transfer formulae of infinity structures can be given in terms of graphs. A closely related result (for ordinary operads) has been obtained recently in [ChLa]. The first explicit graphic formulae have been obtained by Kontsevich and Soibelman [KoSo] who have rewritten in terms of graphs the homotopy transfer formulae of [Me1] for the case when $\mathcal{P}$ is an operad of associative algebras. Another example can be found in the work of Mnev [Mn] who treated the case when $\mathcal{P}$ is a wheeled operad of unimodular Lie algebras. One more example of explicit transfer formulae (related to the master equation (1)) is given below in § 6.

In § 3 we introduce and study a category, $\mathcal{C}at(BV)$, of *(quasi-classically split) quantum BV manifolds* whose

- *objects*, $\mathcal{M}$, are, roughly speaking, formal solutions of all possible quantum master equations (1) with non-degenerate odd Poisson brackets { , } which satisfy in the quasiclassical ($\lim_{\hbar\to 0} + \lim_{\hbar\to 0} \frac{d}{d\hbar}$) limit certain boundary conditions (see § 3.9 for a precise definition); these boundary conditions imply that the tangent space, $\mathcal{T}_*\mathcal{M}$, to the formal manifold $\mathcal{M}$ at the distinguished point comes equipped with an induced differential d; if this induced differential vanishes, then $\mathcal{M}$ is called *minimal*; if, on the other hand, d encodes the full information about the corresponding solution to

[1] Koszul duality for wheeled *operads* has been studied earlier in [MMS].

(1) and the complex $(\mathcal{T}_*\mathcal{M}, d)$ is acyclic, then such a quantum BV manifold $\mathcal{M}$ is called *contractible*;

- *morphisms* are generated by symplectomorphisms, natural projections $\mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{M}_1$, and quantum embeddings, $\mathcal{M}_1 \to \mathcal{M}_1 \times \mathcal{M}_2$, depending on a choice of a Lagrangian submanifold in $\mathcal{M}_2$.

One has the following two results in the category $\mathcal{C}at(BV)$:

(i) *Every quantum BV manifold is isomorphic to the product of a minimal quantum BV manifold and a contractible one.*
(ii) *Quasi-isomorphisms are equivalence relations.*

In § 4 the material of § 2 and § 3 is tied together. We introduce and study a wheeled prop, $\mathcal{ULie}^1\mathcal{B}$, of unimodular Lie 1-bialgebras and prove that *there is a one-to-one correspondence between quantum BV manifolds and representations of the associated dg free wheeled prop* $\mathcal{ULie}^1\mathcal{B}_\infty$. We do not know at present whether or not the wheeled prop(erad) $\mathcal{ULie}^1\mathcal{B}$ is Koszul, i.e. whether or not the natural epimorphism,

$$(\mathcal{ULie}^1\mathcal{B}_\infty, \delta) \longrightarrow (\mathcal{ULie}^1\mathcal{B}, 0),$$

is a quasi-isomorphism. If it is, then the wheeled prop quantization machine of [Me4] would apply to deformation quantization of *unimodular* Poisson structures.

Formal unimodular Poisson structures can be identified with a subclass of solutions, Γ, of the master equation (1) which are independent of $\hbar$. Hence there is a canonical epimorphism of dg wheeled props,

$$F : \mathcal{ULie}^1\mathcal{B}_\infty \longrightarrow \mathcal{UP}oisson,$$

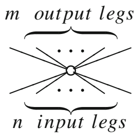
where $\mathcal{UP}oisson$ is a dg prop whose representations in a vector space V are formal unimodular Poisson structures on V vanishing at 0. It is proven in § 5 that F is a quasi-isomorphism.

Section 6 is inspired by the work of Mnev [Mn] on a remarkable approach to the homotopy transfer formulae of unimodular L_∞-algebras which is based on the BV quantization of an extended BF theory and the associated Feynman integrals. We apply in § 6 Losev-Mnev's ideas to unimodular Lie 1-bialgebras and show that the Feynman integrals technique provides us with exactly the same formulae for the homotopy transfer of $\mathcal{ULie}^1\mathcal{B}_\infty$-structures as the ones which follow from the Koszul duality theory for quadratic wheeled properads developed in § 2. These results imply essentially that the Ward identities in a certain class of quantum field theories can be interpreted as equations for a *morphism* of certain dg wheeled (co)props.

A few words about notations. The symbol $\mathbb{S}_n$ stands for the permutation group, that is the group of all bijections, $[n] \to [n]$, where $[n]$ denotes (here and everywhere) the set $\{1, 2, \ldots, n\}$. If $V = \oplus_{i\in\mathbb{Z}} V^i$ is a graded vector space, then $V[k]$ is a graded vector space with $V[k]^i := V^{i+k}$. We work throughout over a field $\mathbb{K}$ of characteristic 0 so that, for an action of finite group G on a vector space V, the subspace of invariants, $\{v \in V | \sigma(v) = v\ \forall \sigma \in G\}$, is canonically isomorphic to the quotient space of coinvariants, $V/span\{v - \sigma(v)\}_{v\in V, \sigma\in G}$, so that we denote them by one and the same symbol V_G.

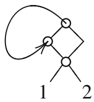
2. Quadratic Wheeled Properads and Homotopy Transfer Formulae

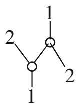
2.1. Wheeled operads, properads and props [Me3, MMS, Me5]. Let $\mathfrak{G}^{\circlearrowright}$ be the family of all possible (not necessarily connected) graphs constructed from the so called directed (m, n)-*corollas*,

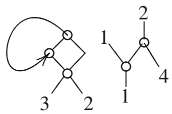


$$, \quad m, n \geq 0, \tag{2}$$

by taking their disjoint unions, and gluing some output legs in such a union with the same number of input legs. The glued legs are called the *internal edges of the graph* and all the rest retain their name *legs of the graph*. Note that every internal edge as well as every leg of a graph is naturally directed; unless otherwise explicitly shown, we tacitly assume in all our pictures that the direction flow runs from the bottom to the top. We have $\mathfrak{G}^{\circlearrowright} = \coprod_{m,n\geq 0} \mathfrak{G}^{\circlearrowright}(m, n)$, where $\mathfrak{G}^{\circlearrowright}(m, n) \subset \mathfrak{G}^{\circlearrowright}$ is the subset of graphs having m output legs and n input legs. We assume from now on that the input legs of each graph $G \in \mathfrak{G}^{\circlearrowright}(m, n)$ are labeled by the natural numbers $\{1, \ldots, n\}$ and the output legs are labeled by $\{1, \ldots, m\}$ so that each set $\mathfrak{G}^{\circlearrowright}(m, n)$ comes equipped with a natural action of the group $\mathbb{S}_m \times \mathbb{S}_n$. For example,



$\in \mathfrak{G}^{\circlearrowright}(0, 2)$,


$\in \mathfrak{G}^{\circlearrowright}(2, 2)$,


$\in \mathfrak{G}^{\circlearrowright}(2, 4)$.

Let E be an $\mathbb{S}$-*bimodule*, that is a family, $\{E(p, q)\}_{p,q\geq 0}$, of vector spaces on which the group $\mathbb{S}_p$ act on the left and the group $\mathbb{S}_q$ act on the right and both actions commute with each other. Fix an arbitrary graph $G \in \mathfrak{G}^{\circlearrowright}(m, n)$ and denote by $V(G)$ the set of its *vertices*, that is the set of its generating corollas (2). For each $v \in V(G)$, denote by In_v (resp. Out_v) the set of the input (resp. output) legs of the vertex v. Assume the cardinality of In_v (resp. Out_v) equals q (resp. p) and note that vector spaces,

$$\langle In_v \rangle := \begin{cases} \text{the } q!\text{-dimensional vector space spanned by all bijections } In_v \to [q] & \text{if } q \geq 1 \\ \mathbb{K} & \text{if } q = 0, \end{cases}$$

and

$$\langle Out_v \rangle := \begin{cases} \text{the } p!\text{-dimensional vector space spanned by all bijections } [p] \to Out_v & \text{if } p \geq 1 \\ \mathbb{K} & \text{if } p = 0 \end{cases}$$

have, respectively, a natural left $\mathbb{S}_q$-module structure and a right $\mathbb{S}_p$-module structure. Hence one can form a quotient,

$$E(Out_v, In_v) := \langle Out_v \rangle \otimes_{\mathbb{S}_p} E(p, q) \otimes_{\mathbb{S}_q} \langle In_v \rangle,$$

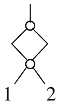
which is (non-canonically) isomorphic to $E(p, q)$ as a vector space and which carries natural actions of the automorphism groups of the sets Out_v and In_v. These actions make a so called *unordered tensor product* over the set $V(G)$ (of cardinality, say, N),

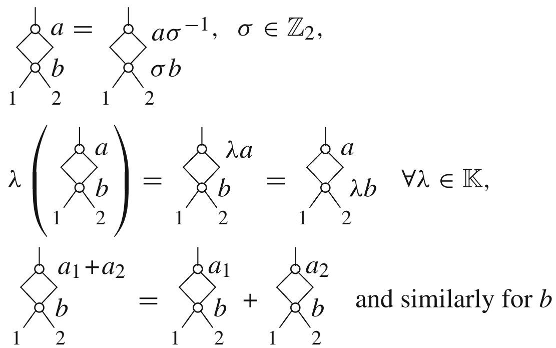
$$\bigotimes_{v\in V(G)} E(Out_v, In_v)$$

$$:= \left(\bigoplus_{i:\{1,\ldots,N\}\to V(G)} E(Out_{i(1)}, In_{i(1)}) \otimes \ldots \otimes E(Out_{i(N)}, In_{i(N)})\right)_{\mathbb{S}_N},$$

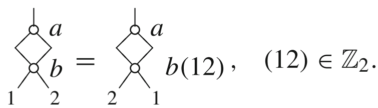
into a representation space of the automorphism group, $Aut(G)$, of the graph G which is, by definition, the subgroup of the symmetry group of the 1-dimensional CW-complex underlying the graph G which fixes its legs. Hence with an arbitrary graph $G \in \mathfrak{G}^{\circlearrowright}$ and an arbitrary $\mathbb{S}$-bimodule E one can associate a vector space,

$$G\langle E\rangle := \left(\otimes_{v\in V(G)} E(Out_v, In_v)\right)_{AutG},$$

whose elements are called *decorated (by E) graphs*. For example, the automorphism group of the graph $G_0 =$  is $\mathbb{Z}_2$ so that $G_0\langle E\rangle = E(1,2) \otimes_{\mathbb{Z}_2} E(2,2)$. It is useful to think of an element in $G_0\langle E\rangle$ as the graph G_0 whose vertices are literarily decorated by some elements $a \in E(1,2)$ and $b \in E(2,1)$; this pictorial representation of $G_0\langle E\rangle$ is correct provided the relations,



are imposed. It also follows from the definition that



Thus one can define alternatively $G_0\langle E\rangle$ as a quotient space, $\prod_{v\in V(G)} E(Out_v, In_v)/\sim$, with respect to the equivalence relation generated by the above pictures.

Note that if E is a *differential graded* (dg, for short) $\mathbb{S}$-bimodule, then, for any graph $G \in \mathfrak{G}^{\circlearrowright}(m,n)$, the associated graded vector space $G\langle E\rangle$ comes equipped with an induced $\mathbb{S}_m \times \mathbb{S}_n$-equivariant differential so that the collection, $\{\bigoplus_{G\in\mathfrak{G}^{\circlearrowright}(m,n)} G\langle E\rangle\}_{m,n\geq 0}$, is again a *dg* $\mathbb{S}$-bimodule. The differential in $G\langle E\rangle$ induced from a differential δ on E is denoted by δ_G or, when no confusion may arise, simply by δ.

Definition 2.1.1. *A* ***wheeled prop*** *is an $\mathbb{S}$-bimodule $\mathcal{P} = \{\mathcal{P}(m,n)\}$ together with a family of linear $\mathbb{S}_m \times \mathbb{S}_n$-equivariant maps,*

$$\{\mu_G : G\langle\mathcal{P}\rangle \to \mathcal{P}(m,n)\}_{G\in\mathfrak{G}^{\circlearrowright}(m,n),m,n\geq 0},$$

parameterized by elements $G \in \mathfrak{G}^{\circlearrowright}$, which satisfy the condition

$$\mu_G = \mu_{G/H} \circ \mu'_H \tag{3}$$

for any subgraph $H \subset G$. Here G/H is the graph obtained from G by shrinking the whole subgraph H into a single internal vertex, and $\mu'_H : G\langle E\rangle \to (G/H)\langle E\rangle$ stands for the map which equals μ_H on the decorated vertices lying in H and which is identity on all other vertices of G.

If the $\mathbb{S}$-bimodule $\mathcal{P}$ underlying a wheeled prop has a differential δ satisfying, for any $G \in \mathfrak{G}^{\circlearrowright}$, the condition $\delta \circ \mu_G = \mu_G \circ \delta_G$, then the wheeled prop $\mathcal{P}$ is called differential.

Remarks 2.1.2. (i) If $\mathfrak{C}_{m,n}$ denotes (m,n)-corolla (2), then the $\mathbb{S}_m \times \mathbb{S}_n$-module $\mathfrak{C}_{m,n}\langle\mathcal{P}\rangle$ is canonically isomorphic to $\mathcal{P}(m,n)$. Thus the defining linear map $\mu_G : G\langle\mathcal{P}\rangle \to \mathcal{P}(m,n)$ associated to an arbitrary graph $G \in \mathfrak{G}^{\circlearrowright}(m,n)$ can be interpreted as a *contraction* map, $\mu_G : G\langle\mathcal{P}\rangle \to \mathfrak{C}_{m,n}\langle\mathcal{P}\rangle$, contracting all the internal edges and all the internal vertices of G into a single vertex.

(ii) Equation (3) implies $\mu_G = \mu_{G/G} \circ \mu_G$ for any graph $G \in \mathfrak{G}^{\circlearrowright}$, which in turn implies that $\mu_{\mathfrak{C}_{m,n}} : \mathcal{P}(m,n) \to \mathcal{P}(m,n)$ is the identity map.

(iii) Condition (3) can be equivalently rewritten as the equality, $\mu_{G/H_1} \circ \mu'_{H_1} = \mu_{G/H_2} \circ \mu'_{H_2}$, for any subgraphs $H_1, H_2 \subset G$, i.e. it is a kind of associativity condition for the family of contraction operations $\{\mu_G\}$.

(iv) Strictly speaking, the notion introduced in §2.1.1 should be called a wheeled prop *without unit*. A wheeled prop *with unit* can be defined as in §2.1.1 provided one enlarges the family of graphs $\mathfrak{G}^{\circlearrowright}$ by adding the following graphs without vertices,

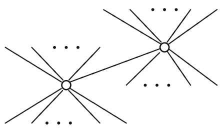
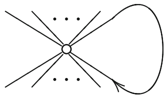
$$\mathfrak{t}_{p,q} := \underbrace{\uparrow\uparrow\uparrow\ \cdots\uparrow}_{p}\underbrace{\circlearrowright\circlearrowright\cdots\circlearrowright}_{q},\ \ p,q \geq 0,\ p+q \geq 1,$$

to the family $\mathfrak{G}^{\circlearrowright}(p,p)$ (see [MMS]). The $\mathbb{S}$-bimodule spanned by such graphs without vertices has an obvious structure of wheeled prop with unit called the *trivial* wheeled prop $\mathfrak{t}$. Similar to the case of an associative algebra, any wheeled prop, $\mathcal{P}$, *without unit* can be made into a wheeled prop, $\mathcal{P}^+ := \mathcal{P} * \mathfrak{t}$, *with unit* by taking the free product of $\mathcal{P}$ and $\mathfrak{t}$. All the unital wheeled props we study in this paper are obtained in this trivial way from non-unital ones prompting us to work in this paper with non-unital props only. A small bonus of this choice is that one can avoid bothering about (co)augmentation (co)ideals when dealing with bar-cobar constructions of wheeled (co)props (see § 2.4 below)

Definitions 2.1.3. *A* ***wheeled properad****, $\mathcal{P} = \{\mathcal{P}(m,n)\}$, is defined exactly as in §2.1.1 except that the graphs G and H are required now to belong to the subfamily, $\mathfrak{G}^{\circlearrowright}_c$, of $\mathfrak{G}^{\circlearrowright}$ consisting of* ***connected*** *graphs.*

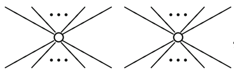
A wheeled operad *is a wheeled properad $\mathcal{P} = \{\mathcal{P}(m,n)\}$ with $\mathcal{P}(m,n) = 0$ for $m \geq 2$.*

2.1.4. Generating compositions Associativity equations (3) imply that for an arbitrary wheeled properad $\mathcal{P}$ the defining family of contraction maps, $\{\mu_G : G\langle\mathcal{P}\rangle \to \mathcal{P}\}_{G\in\mathfrak{G}_c^{\circlearrowright}}$, is uniquely determined (via iteration) by its subfamily, $\{\mu_G : G\langle\mathcal{P}\rangle \to \mathcal{P}\}_{G\in\mathfrak{G}_{gen}^{\circlearrowright}}$, where $\mathfrak{G}_{gen}^{\circlearrowright} \subset \mathfrak{G}_c^{\circlearrowright}$ consists of graphs of the form,

(i)  and (ii)  (4)

i.e. of one-vertex graphs with precisely one internal edge (forming a loop) and of connected two vertex graphs with precisely one internal edge. The set of graphs $\mathfrak{G}_{gen}^{\circlearrowright}$ lies behind the notion of a *quadratic* wheeled properad introduced below in §2.6.1.

Generating compositions of a wheeled prop are given by graphs shown above and the extra ones,

 , (5)

having two vertices and no internal edges.

2.1.5. An endomorphism wheeled prop(erad) For any finite-dimensional vector space V the $\mathbb{S}$-bimodule $\mathcal{E}nd_V := \{\mathrm{Hom}(V^{\otimes n}, V^{\otimes m})\}$ is naturally a wheeled prop(erad) with compositions defined as follows:

- for graphs G of the form (4)(i) the associated composition $\mu_G : G\langle\mathcal{E}nd_V\rangle \to \mathcal{E}nd_V$ is the ordinary composition of two linear maps;
- for graphs G of the form (4)(ii) the associated composition μ_G is the ordinary trace of a linear map;
- for graphs G of the form (5) the associated composition μ_G is the ordinary tensor product of linear maps.

For an arbitrary graph $G \in \mathfrak{G}^{\circlearrowright}$ the associated composition $\mu_G : G\langle\mathcal{E}nd_V\rangle \to \mathcal{E}nd_V$ is defined as an iteration of the above "elementary" compositions, and it is easy to see that such a μ_G is independent of a particular choice of an iteration; this independence means, in fact, that associativity conditions (3) are fulfilled. The prop(erad) $\mathcal{E}nd_V$ is called the *endomorphism wheeled prop(erad)* of V. Note that if V is a complex, then $\mathcal{E}nd_V$ is naturally a *dg* prop(erad).

2.1.6. A free wheeled prop(erad) Given an arbitrary $\mathbb{S}$-bimodule, $E = \{E(m,n)\}$, there is an associated $\mathbb{S}$-bimodule, $\mathcal{F}^{\circlearrowright}\langle E\rangle = \{\mathcal{F}^{\circlearrowright}\langle E\rangle(m,n) := \bigoplus_{G\in\mathfrak{G}^{\circlearrowright}(m,n)} G\langle E\rangle\}$, which has a natural prop structure with the contraction maps $\mu_G : G\langle\mathcal{F}^{\circlearrowright}\langle E\rangle\rangle \to \mathcal{F}^{\circlearrowright}\langle E\rangle$ being tautological. The wheeled prop $\mathcal{F}^{\circlearrowright}\langle E\rangle$ is called the *free wheeled prop generated by an $\mathbb{S}$-bimodule E*.

A *free wheeled properad, $\mathcal{F}_c^{\circlearrowright}\langle E\rangle$, generated by an $\mathbb{S}$-bimodule E* is defined as in the previous paragraph but with the symbol $\mathfrak{G}^{\circlearrowright}$ replaced by $\mathfrak{G}_c^{\circlearrowright}$.

2.1.7. Prop(erad)s, dioperads and operads Consider the follows subsets of the set $\mathfrak{G}^{\circlearrowright}$:

(a) $\mathfrak{G}^{\uparrow}$ is a subset of $\mathfrak{G}^{\circlearrowright}$ consisting of directed graphs with no *wheels*, i.e. directed paths of internal edges which begin and end at the same vertex;
(b) $\mathfrak{G}^{\uparrow}_c := \mathfrak{G}^{\uparrow} \cap \mathfrak{G}^{\circlearrowright}_c$;
(c) $\mathfrak{G}^{\uparrow}_{c,0}$ is a subset of $\mathfrak{G}^{\uparrow}_c$ consisting of graphs of *genus zero*;
(d) $\mathfrak{G}^{\uparrow}_{oper}$ is a subset of $\mathfrak{G}^{\uparrow}_{c,0}$ built from corollas (2) of type $(1, n)$ only, $n \geq 1$.

Let $\mathfrak{G}^{\checkmark}$ be any one of these families of graphs. Then one can define an $\mathfrak{G}^{\checkmark}$-algebra as in §2.1.1 by requiring that all the graphs G, H and G/H involved in that definition belong to the subset $\mathfrak{G}^{\checkmark}$ (cf. [Me5]). Then:

(a) an $\mathfrak{G}^{\uparrow}$- algebra is called a *prop* [Mc];
(b) an $\mathfrak{G}^{\uparrow}_c$-algebra is called a *properad* [Va];
(c) an $\mathfrak{G}^{\uparrow}_{c,0}$-algebra is called a *dioperad* [Ga];
(d) an $\mathfrak{G}^{\uparrow}_{oper}$-algebra is called an *operad* [May].

A *quadratic* $\mathfrak{G}^{\checkmark}$-algebra is defined (in all the above cases) as a quotient of a free $\mathfrak{G}^{\checkmark}$-algebra, $\mathcal{F}^{\checkmark}\langle E\rangle$, by the ideal generated by a subspace $R \subset \mathfrak{G}^{\checkmark}_{gen}\langle E\rangle$, where $\mathfrak{G}^{\checkmark}_{gen}$ is the *minimal* subset of $\mathfrak{G}^{\checkmark}$ whose elements generate all possible compositions, μ_G, via iteration (cf. §2.1.4). We apply the same minimality principle for the definition of a *quadratic* wheeled properad in § 2.6 below.

2.2. Morphisms of wheeled props. One can make dg wheeled prop(erad)s into a category by defining a morphism, $f : \mathcal{P}_1 \to \mathcal{P}_2$, as a morphism of the underlying dg $\mathbb{S}$-bimodules, $\{f : \mathcal{P}_1(m,n) \to \mathcal{P}_2(m,n)\}_{m,n\geq 0}$, such that, for any graph $G \in \mathfrak{G}^{\circlearrowright}$, one has $f \circ \mu_G = \mu_G \circ (f^{\otimes G})$, where $f^{\otimes G}$ means a map, $G\langle \mathcal{P}_1\rangle \to G\langle \mathcal{P}_2\rangle$, which changes decorations of each vertex in G in accordance with f.

Definition 2.2.1. *A morphism of wheeled prop(erad)s, $\mathcal{P} \to \mathcal{E}nd_V$, is called a* ***representation*** *of the wheeled prop(erad) $\mathcal{P}$ in a graded vector space V.*

Definition 2.2.2. *A morphism of dg wheeled prop(erad)s, $\mathcal{P}_1 \to \mathcal{P}_2$, is called a* ***quasi-isomorphism****, if the induced morphism of cohomology prop(erad)s, $H(\mathcal{P}_1) \to H(\mathcal{P}_2)$, is an isomorphism.*

2.2.3. A useful fact If $\mathcal{P}_2$ is an arbitrary wheeled prop(erad) and $\mathcal{P}_1$ is a free wheeled prop(erad), $\mathcal{F}^{\circlearrowright}\langle E\rangle$, generated by some $\mathbb{S}$-bimodule E, then the set of morphisms of wheeled prop(erad)s, $\{f : \mathcal{P}_1 \to \mathcal{P}_2\}$, is in one-to-one correspondence with the vector space of degree zero morphisms of $\mathbb{S}$-bimodules, $\{f|_E : E \to \mathcal{P}_2\}$, i.e. f is uniquely determined by its values on the generators. In particular, the set of morphisms, $\mathcal{F}^{\circlearrowright}\langle E\rangle \to \mathcal{P}_2$, has a graded vector space structure for any $\mathcal{P}_2$.

Definition 2.2.4. *A* ***free resolution*** *of a dg wheeled prop(erad) $\mathcal{P}$ is, by definition, a dg free wheeled prop(erad), $(\mathcal{F}^{\circlearrowright}\langle E\rangle, \delta)$, generated by some $\mathbb{S}$-bimodule E together with an epimorphism, $\pi : (\mathcal{F}^{\circlearrowright}\langle E\rangle, \delta) \to \mathcal{P}$, which is a quasi-isomorphism. If the differential δ in $\mathcal{F}^{\circlearrowright}\langle \mathcal{E}\rangle$ is decomposable with respect to the compositions μ_G, then $\pi : (\mathcal{F}^{\circlearrowright}\langle E\rangle, \delta) \to \mathcal{P}$ is called a* ***minimal model*** *of $\mathcal{P}$.*

2.3. Coprop(erad)s. A *wheeled coproperad* is an $\mathbb{S}$-bimodule $\mathcal{P} = \{\mathcal{P}(m,n)\}$ together with a family of linear $\mathbb{S}_m \times \mathbb{S}_n$-equivariant maps,

$$\{\Delta_G : \mathcal{P}(m,n) \to G\langle\mathcal{P}\rangle\}_{G\in\mathfrak{G}_c^{\circlearrowright}(m,n),m,n\geq 0},$$

parameterized by elements $G \in \mathfrak{G}_c^{\circlearrowright}$, which satisfy the condition

$$\Delta_G = \Delta'_H \circ \Delta_{G/H} \tag{6}$$

for any connected subgraph $H \subset G$. Here $\Delta'_H : (G/H)\langle E\rangle \to G\langle E\rangle$ is the map which equals Δ_H on the distinguished vertex of G/H and which is identity on all other vertices of G. *Wheeled coprops* are defined analogously.

If the $\mathbb{S}$-bimodule $\mathcal{P}$ underlying a wheeled coprop(erad) has a differential δ satisfying, for any $G \in \mathfrak{G}^{\circlearrowright}$, the condition $\Delta_G \circ \delta = \delta_G \circ \Delta_G$, then the wheeled coprop(erad) $\mathcal{P}$ is called *differential*.
For any $\mathbb{S}$-bimodule, $E = \{E(m,n)\}$, the associated $\mathbb{S}$-bimodule, $\mathcal{F}^{\circlearrowright}\langle E\rangle$, has a natural coproperad structure with the co-contraction map

$$\Delta := \sum_{G\in\mathfrak{G}^{\circlearrowright}(m,n)} \Delta_G : \mathcal{F}^{\circlearrowright}\langle E\rangle \longrightarrow \sum_{G\in\mathfrak{G}^{\circlearrowright}(m,n)} G\langle\mathcal{F}^{\circlearrowright}\langle E\rangle\rangle = \mathcal{F}^{\circlearrowright}\langle\mathcal{F}^{\circlearrowright}\langle E\rangle\rangle$$

given, on an arbitrary element $g \in G\langle E\rangle \subset \mathcal{F}^{\circlearrowright}\langle E\rangle$, by [MMS],

$$\Delta g = \sum_{f:Edg(G)\to\{0,1\}} g_f,$$

where the sums run over markings, $f : Edg(G) \to \{0,1\}$, of the set, $Edg(G)$, of internal edges of G by numbers 0 and 1, and g_f is an element of $\mathcal{F}^{\circlearrowright}\langle\mathcal{F}^{\circlearrowright}\langle E\rangle\rangle$ obtained from g by the following recipe:

(i) cut every internal edge of the graph G marked by 0 in the middle; let $G_1, \ldots, G_k$, for some $k \geq 1$, be the resulting connected components of G; the vertices of the latter graphs inherit E-decorations, and hence the marking f defines elements $g_1 \in G_1\langle E\rangle, \ldots, g_k \in G_k\langle E\rangle$;
(ii) let G' be the graph with k-vertices obtained from G by shrinking each subgraph $G_1, \ldots, G_k$ into a single vertex; then g_f is, by definition, the decorated graph g viewed as an element of $G'\langle\mathcal{F}^{\circlearrowright}\langle E\rangle\rangle$, i.e. it equals G' with vertices decorated by elements $g_1, \ldots, g_k \in \mathcal{F}^{\circlearrowright}\langle E\rangle$.

The wheeled coprop $(\mathcal{F}^{\circlearrowright}\langle E\rangle, \Delta)$ is called the *free* coprop generated by the $\mathbb{S}$-module E.

One can show analogously that $\mathcal{F}_c^{\circlearrowright}\langle E\rangle$ has a natural coproperad structure Δ; the data $(\mathcal{F}_c^{\circlearrowright}\langle E\rangle, \Delta)$ is called the *free coproperad* generated by the $\mathbb{S}$-module E. We denote it by $\mathcal{F}_{co}^{\circlearrowright}\langle E\rangle$ (to avoid confusion with the natural properad structure in $\mathcal{F}_c^{\circlearrowright}\langle E\rangle$).

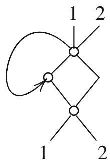
2.4. Bar construction. With an $\mathbb{S}$-module $E = \{E(m,n)\}$ one can associate two other $\mathbb{S}$-bimodules,

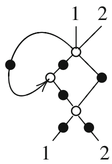
$$\mathsf{w}E =: \{E(m,n) \otimes sgn_n[-n]\}, \quad \mathsf{w}^{-1}E := \{E(m,n) \otimes sgn_n[n]\},$$

where sgn_m stands for the 1-dimensional sign representation of $\mathbb{S}_m$. We shall show in this subsection that for any properad $\mathcal{P}$ the associated free coproperad,

$$B(\mathcal{P}) := \mathcal{F}_{co}^{\circlearrowright}\langle \mathsf{w}^{-1}\mathcal{P}\rangle,$$

comes canonically equipped with a differential, $\delta_{\mathcal{P}}$, encoding all the generating properadic compositions $\{\mu_G : G\langle\mathcal{P}\rangle \to \mathcal{P}\}_{G\in\mathfrak{G}_{gen}^{\circlearrowright}}$. For this purpose let us consider a family of graphs, $\mathfrak{G}_{\bullet}^{\circlearrowright}$, obtained from the family of directed connected graphs $\mathfrak{G}_c^{\circlearrowright}$ by inserting into each input leg and each internal edge of a graph $G \in \mathfrak{G}_c^{\circlearrowright}$ a black (1, 1)-corolla, $\blacklozenge$, and denoting the resulting graph by $G_{\bullet}$. For example,

if $G =$ 

then $G_{\bullet} =$ 

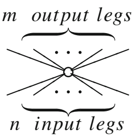
The automorphism group of such a graph $G_{\bullet}$ is defined as in § 2.1 with an extra assumption that the colour is preserved. Then, obviously, $Aut(G) = Aut(G_{\bullet})$.

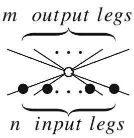
Let 1 stand for the unit in the field $\mathbb{K}$, and $\bar{1}$ for its image under the isomorphism $\mathbb{K} \to \mathbb{K}[1]$. The vector $\bar{1}$ has degree -1. For an arbitrary $\mathbb{S}$-bimodule E and an arbitrary graph $G \in \mathfrak{G}_{\bullet}^{\circlearrowright}$ we denote by $G_{\bullet}\langle E\rangle$ the vector space spanned by the graph $G_{\bullet}$ whose white vertices are decorated by elements of E and the special black (1, 1)-vertices are decorated by $\bar{1}$.

Lemma 2.4.1. *For any $\mathbb{S}$-module E there is a canonical isomorphism of $\mathbb{S}$-modules,*

$$\mathcal{F}_c^{\circlearrowright}\langle \mathsf{w}^{-1}E\rangle = \bigoplus_{G_{\bullet}\in\mathfrak{G}_{\bullet}^{\circlearrowright}(m,n)} G_{\bullet}\langle E\rangle.$$

Proof. It is enough to show a canonical isomorphism $\mathbb{S}_m \times \mathbb{S}_n$-modules, $G\langle \mathsf{w}^{-1}E\rangle = G_{\bullet}\langle E\rangle$ for an arbitrary graph $G \in \mathfrak{G}_c^{\circlearrowright}(m, n)$. The graph $G_{\bullet}$ is obtained from G by replacing each constituting (m, n)-corolla of G as follows:

$\mathfrak{C}_{m,n} =$ 

$\longrightarrow \mathfrak{C}_{m,n\bullet} =$ 

It is obvious that $\mathfrak{C}_{m,n\bullet}\langle E\rangle = \mathfrak{C}_{m,n}\langle \mathsf{w}^{-1}E\rangle$ as $\mathbb{S}_m \times \mathbb{S}_n$-bimodules. If we set $E(Out_v, In_v) := \bar{1}$ for every black vertex v in $G_{\bullet}$, then $\bigotimes_{v\in V(G_{\bullet})} E(Out_v, In_v) = \bigotimes_{v\in V(G)} \mathsf{w}^{-1}E(Out_v, In_v)$ and the claim follows finally from the isomorphism $Aut(G_{\bullet}) = Aut(G)$. □

Corollary 2.4.2. *For any wheeled properad $\mathcal{P}$ there is a canonical isomorphism of $\mathbb{S}$-modules,*

$$B(\mathcal{P}) = \bigoplus_{G_{\bullet}\in\mathfrak{G}_{\bullet}^{\circlearrowright}} G_{\bullet}\langle\mathcal{P}\rangle \tag{7}$$

The r.h.s of (7) is denoted sometimes by $B_{\bullet}(\mathcal{P})$.

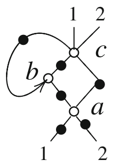
Fact 2.4.3. Let $\mathcal{P}$ be an arbitrary wheeled properad. The $\mathbb{S}$-module $\bigoplus_{G_\bullet \in \mathfrak{G}_\bullet^{\circlearrowright}} G_\bullet\langle\mathcal{P}\rangle$ can be made naturally into a complex with the differential,

$$\delta_{\mathcal{P}} = \text{“}\frac{\partial}{\partial \bullet_{edge}}\text{"},$$

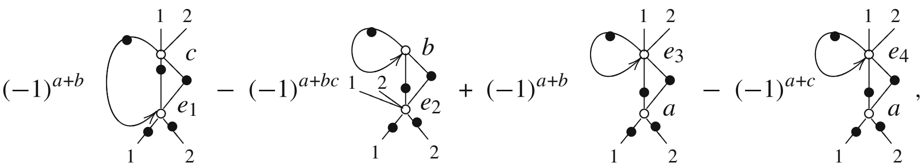
which is equal to zero on all white vertices and all black vertices attached to *legs*, and which deletes a black vertex lying on every *internal edge* and contracts the associated internal edge with the help of the corresponding composition in $\mathcal{P}$; equation $\delta_{\mathcal{P}}^2 = 0$ follows then from associativity conditions (3). More precisely, one defines $\delta_{\mathcal{P}} g$ for some $g \in G_\bullet\langle E\rangle = (\bigotimes_{v\in V(G_\bullet)} E(Out_v, In_v))_{Aut(G_\bullet)}$ as follows: choose first a representative, $\tilde{g} \in E(Out_{v_1}, In_{v_2}) \otimes \ldots E(Out_{v_p}, In_{v_p})$, of the equivalence class g associated with some ordering of all vertices in G, apply then $\delta_{\mathcal{P}}$ to the vertices of $\tilde{g}$ in the chosen order, and finally set $\delta_{\mathcal{P}} g = \pi(\delta_{\mathcal{P}}\tilde{g})$, where π is the natural surjection

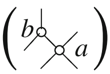
$$\pi : E(Out_{v_1}, In_{v_2}) \otimes \ldots \otimes E(Out_{v_p}, In_{v_p}) \longrightarrow (\bigotimes_{v\in V(G_\bullet)} E(Out_v, In_v))_{Aut(G_\bullet)}.$$

The result does not depend on the choice of a section, $g \to \tilde{g}$, of π used in the definition. For example, if

$g =$  for some $a \in E(2,2), b \in E(1,2), c \in E(3,2)$,

then, ordering the vertices from the bottom to the top, we obtain that $\delta_{\mathcal{P}} g$ is the equivalence class (in the unordered tensor product) of the following graph:



where $e_1 := \mu$  $\in E(2,3)$, etc. Applying $\delta_{\mathcal{P}}$ again and using associativity relations (3), one easily concludes that $\delta_{\mathcal{P}}^2 = 0$.

Isomorphism (7) induces a differential in the free coproperad $B(\mathcal{P})$ which we denoted by the same symbol $\delta_{\mathcal{P}}$. It obviously respects the coproperad structure in $B(\mathcal{P})$. If $\mathcal{P}$ is a *differential* operad with differential d, then the sum $d + \delta_{\mathcal{P}}$ is a differential in $B(\mathcal{P})$.

Definition 2.4.4. *The dg coproperad* $(B(\mathcal{P}), d + \delta_{\mathcal{P}})$ *is called the* ***bar construction*** *of a dg properad* $(\mathcal{P}, d)$.

This notion was first introduced in [MMS] but with a different $\mathbb{S}$-module structure and $\mathbb{Z}$-grading on $B(\mathcal{P})$. We shall be most interested below in the situations when $d = 0$.

2.5. Cobar construction. If $(\mathcal{C}, d)$ is a dg coproperad, then its *cobar construction* is, by definition, a free wheeled properad, $B^c(\mathcal{C}) := \mathcal{F}_c^{\circlearrowright}\langle \mathsf{w}\mathcal{C}\rangle$, equipped with a differential, $d + \partial_{\mathcal{C}}$, where $\partial_{\mathcal{C}}$ is the differential encoding the co-composition maps $\Delta_G : \mathcal{C} \to G\langle\mathcal{C}\rangle$ in a way dual to the definition of $\partial_{\mathcal{P}}$ in § 2.4 (see [MMS]). Let $\mathfrak{G}_{\diamond}^{\circlearrowright}$ be a family of graphs obtained from $\mathfrak{G}_c^{\circlearrowright}$ by inserting into each input leg and each internal edge of a graph $G \in \mathfrak{G}_c^{\circlearrowright}$ a white rhombic (1, 1)-corolla, $\diamond$, and let us denote the resulting graph by $G_{\diamond}$. Then, by analogy to §2.4.2, we have a canonical degree 0 isomorphism of $\mathbb{S}$-modules,

$$B^c(\mathcal{C}) = \bigoplus_{G_\diamond \in \mathfrak{G}_\diamond^{\circlearrowright}} G_\diamond\langle\mathcal{C}\rangle, \tag{8}$$

where in the r.h.s. we used $s(1)$, s being the isomorphism $\mathbb{K} \to \mathbb{K}[-1]$, to decorate special $\diamond$-vertices. The differential $\partial_{\mathcal{C}}$ is, by definition, equal to zero on the special white rhombic corollas while on ordinary (decorated by $\mathcal{C}$) vertices it is equal to the map $\sum_G \Delta_G : \mathcal{C} \to \sum_{G \in \mathfrak{G}_{gen}^{\circlearrowright}} G\langle\mathcal{C}\rangle$ with the sum running over all possible graphs of the form (4); the unique internal edge in the image of the map $\sum_G \Delta_G$ is then decorated by $\diamond$ so that $\partial_{\mathcal{C}}$ increases the number of rhombic white vertices by one.

In the case when $\mathcal{C}$ is the bar construction, $(\mathcal{C} = B(\mathcal{P}), d + \delta_{\mathcal{P}})$ on some dg properad $(\mathcal{P}, d)$ one has a natural epimorphism of dg properads,

$$\bar{\pi} : \left(B^c(B(\mathcal{P})), \delta := d + \delta_{\mathcal{P}} + \partial_{B(\mathcal{P})}\right) \longrightarrow \left(\mathsf{w}(\mathsf{w}^{-1}\mathcal{P}) = \mathcal{P}, d\right),$$

which is a quasi-isomorphism [MMS]. If we now apply constructions (7) and (8) to $B^c(B(\mathcal{P})$, we shall get decorated graphs whose internal edges are decorated by either black vertices or simultaneously by black and white rhombic vertices. As white rhombic and black corollas placed on the same edge "annihilate" each other with respect to their total impact on graph, we have a degree 0 isomorphism of $\mathbb{S}$-bimodules,

$$B^c(B(\mathcal{P})) = \bigoplus_{G_\diamond \in \mathfrak{G}_{\bullet,-}^{\circlearrowright}} G_{\bullet,-}\langle\mathcal{P}\rangle,$$

where, by definition, $\mathfrak{G}_{\bullet,-}$ is a family of graphs obtained from graphs in $\mathfrak{G}_c^{\circlearrowright}$ by inserting into *some* (possibly, none) internal edges black (1, 1)-corollas; thus every *internal* edge of a graph G from $\mathfrak{G}_{\bullet,-}$ is either straight or equipped with the black (1, 1)-corolla, and every input or output leg of G is straight. Then the differential $\partial_{B(\mathcal{P})}$ gets a very simple interpretation — it eliminates, in accordance with the Leibniz rule, each black corolla making the corresponding edge straight; on the other hand, the differential $\delta_{\mathcal{P}}$ contracts (again in accordance with the Leibniz rule) each internal edge decorated by the black corolla and performs a corresponding to this contraction composition in the original properad $\mathcal{P}$.

2.6. Quadratic wheeled (co)properads and Koszul duality. Koszul duality for ordinary quadratic operads was introduced in [GiKa], for dioperads in [Ga], for ordinary properads in [Va] and for wheeled operads in [MMS]. In this section we extend the idea to arbitrary quadratic wheeled properads.

For a graph $G \in \mathfrak{G}^{\circlearrowright}$ with p vertices and q wheels (that is, closed paths of directed internal edges) set $||G|| := p + q$ and call it the *weight* of G. For an $\mathbb{S}$-module E set $\mathcal{F}_{(\lambda)}^{\circlearrowright}\langle E\rangle$ to be a submodule of the free properad $\mathcal{F}_c^{\circlearrowright}\langle E\rangle$ spanned by decorated graphs of weight λ. Note that properadic compositions $\{\mu_G : G\langle\mathcal{F}_c^{\circlearrowright}\langle E\rangle\rangle \to \mathcal{F}_c^{\circlearrowright}\langle E\rangle\}_{G\in\mathfrak{G}_c^{\circlearrowright}}$ are homogeneous with respect to this weight gradation. Note also that the quadratic subspace $\mathcal{F}_{(2)}^{\circlearrowright}\langle E\rangle \subset \mathcal{F}_c^{\circlearrowright}\langle E\rangle$ is distinguished as it is spanned,

$$\mathcal{F}_{(2)}^{\circlearrowright}\langle E\rangle = \sum_{G\in\mathfrak{G}_{gen}^{\circlearrowright}} G\langle E\rangle,$$

by the minimal set of graphs (4) which generate *all* possible wheeled properadic compositions.

Definition 2.6.1. *A wheeled properad $\mathcal{P}$ is called* ***quadratic*** *if it is the quotient, $\mathcal{P} := \mathcal{F}_c^{\circlearrowright}\langle E\rangle/I$, of a free wheeled properad (generated by some $\mathbb{S}$-bimodule E) by the ideal, I, generated by some subspace $R \subset \mathcal{F}_{(2)}^{\circlearrowright}\langle E\rangle$.*

An obvious dualization of the above definition gives the notion of a *quadratic* coproperad.

Any quadratic (co)properad, $\mathcal{P}$, comes equipped with an induced weight gradation, $\mathcal{P} = \sum_{\lambda\geq 1}\mathcal{P}_{(\lambda)}$, where $\mathcal{P}_{(\lambda)}$ is the image of $\mathcal{F}_{(\lambda)}^{\circlearrowright}\langle E\rangle$ under the natural surjection $\mathcal{F}_c^{\circlearrowright}\langle E\rangle \to \mathcal{P}$. Note that $\mathcal{P}_{(1)} = E$ and $\mathcal{P}_{(2)}$ is given by an exact sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{F}_{(2)}^{\circlearrowright}\langle E\rangle \longrightarrow \mathcal{P}_{(2)} \longrightarrow 0. \tag{9}$$

The subspace $\mathcal{B}(\mathcal{P}_{(1)}) \subset \mathcal{B}(\mathcal{P})$ is obviously a sub-coproperad, but, in general, it is not preserved by the bar differential $\partial_{\mathcal{P}}$. It is not hard to check that an $\mathbb{S}$-bimodule $\mathcal{P}^{¡}$ defined by the exact sequence,

$$0 \longrightarrow \mathcal{P}^{¡} \longrightarrow B(\mathcal{P}_{(1)}) \xrightarrow{\partial_{\mathcal{P}}} B(\mathcal{P})[1],$$

is a sub-coproperad of $\mathcal{B}(\mathcal{P}_{(1)})$ so that the natural composition of inclusions,

$$\imath : \mathcal{P}^{¡} \longrightarrow B(\mathcal{P}_{(1)}) \longrightarrow B(\mathcal{P}), \tag{10}$$

is a monomorphism of *dg* wheeled coproperads.

Definition 2.6.2. *The coproperad $\mathcal{P}^{i}$ is called* ***Koszul dual*** *to a quadratic wheeled properad $\mathcal{P}$.*

Definition 2.6.3. *A quadratic wheeled properad $\mathcal{P}$ is called* ***Koszul****, if the associated morphism of dg coproperads, $\imath : \mathcal{P}^{i} \longrightarrow B(\mathcal{P})$, is a quasi-isomorphism.*

As the cobar construction functor B^c is exact [MMS], the composition

$$\pi : \mathcal{P}_\infty := B^c(\mathcal{P}^{¡}) \xrightarrow{B^c(i)} B^c(B(\mathcal{P})) \xrightarrow{\bar{\pi}} \mathcal{P}$$

is a quasi-isomorphism if $\mathcal{P}$ is Koszul; then the dg free wheeled properad $\mathcal{P}_\infty$ gives us a minimal resolution of $\mathcal{P}$.

2.6.4. Remark on notation In general (i.e. if $\mathcal{P}$ is not Koszul), the dg properad $B^c(\mathcal{P}^{¡})$ is only an approximation to the genuine minimal wheeled resolution of $\mathcal{P}$ (if it exists at all); it is, however, associated *canonically* to $\mathcal{P}$, and, slightly abusing tradition, *we continue denoting it in this paper by $\mathcal{P}_\infty$ even in the cases when $\mathcal{P}$ is not Koszul.*

Note that $B(\mathcal{P}_{(1)})$ is the free co-properad generated by the $\mathbb{S}$-module $\mathsf{w}^{-1}\mathcal{P}_{(1)} = \mathsf{w}^{-1}E$. By the definition of the bar differential $\partial_\mathcal{P}$, the image, I^{co}, of the degree 0 map $\partial_\mathcal{P} : B(\mathcal{P}_{(1)}) \to B(\mathcal{P})[1]$ is spanned by graphs with all (except one!) vertices decorated by the $\mathsf{w}^{-1}\mathcal{P}_{(1)}$ and with the exceptional vertex decorated by $\mathsf{w}^{-1}\mathcal{P}_{(2)}[1]$. Thus we have an exact sequence,

$$0 \longrightarrow \mathcal{P}^{¡} \longrightarrow \mathcal{F}_{co}^{\circlearrowright}\langle \mathsf{w}^{-1}E\rangle \longrightarrow I^{co} \longrightarrow 0.$$

As $\mathsf{w}^{-1}\mathcal{F}_{(2)}^{\circlearrowright}\langle E\rangle = \mathcal{F}_{(2)}^{\circlearrowright}\langle \mathsf{w}^{-1}E\rangle[-1]$, we can rewrite (9) as follows:

$$0 \longrightarrow \mathsf{w}^{-1}\mathcal{R}[1] \longrightarrow \mathcal{F}_{(2)}^{\circlearrowright}\langle \mathsf{w}^{-1}E\rangle \longrightarrow \mathsf{w}^{-1}\mathcal{P}_{(2)}[1] \longrightarrow 0, \tag{11}$$

and conclude that I^{co} is the co-ideal of $\mathcal{F}_{cc}^{\circlearrowright}\langle \mathsf{w}^{-1}E\rangle$ cogenerated by *quadratic* co-relations $\mathsf{w}^{-1}\mathcal{P}_{(2)}[1]$. Hence we proved the following

Proposition 2.6.5. *For any quadratic wheeled properad $\mathcal{P}$ the associated Koszul dual wheeled coproperad $\mathcal{P}^i$ is quadratic.*

Remark 2.6.6. If the $\mathbb{S}$-bimodule $E = \{E(m,n)\}$ is of finite type (i.e. each $E(m,n)$ is finite-dimensional), then it is often easier to work with the wheeled properad $\mathcal{P}^! := (\mathcal{P}^{¡})^*$, the ordinary dual of the coproperad $\mathcal{P}^{¡}$. It is a quadratic wheeled properad generated by the $\mathbb{S}$-bimodule,

$$E^\vee := \left\{E(m,n)^* \otimes sgn_n[-n]\right\},$$

with the relations, $\mathcal{R}^\perp$, given by the exact sequence,

$$0 \longrightarrow \mathcal{R}^\perp \longrightarrow \mathcal{F}_{(2)}^{\circlearrowright}\langle E^\vee\rangle \longrightarrow \mathsf{w}\mathcal{R}^*[-1] \longrightarrow 0, \tag{12}$$

where $\mathcal{R}$ are the quadratic relations for $\mathcal{P}$.

Remark 2.6.7. Definition 2.6.1 implies that there exists a canonical *wheelification functor*,

$$\circlearrowright : \begin{array}{c}\text{Category of quadratic}\\ \text{(co)dioperads}\\ \mathcal{D}\end{array} \begin{array}{c}\longrightarrow\\ \\ \longrightarrow\end{array} \begin{array}{c}\text{Category of quadratic wheeled}\\ \text{(co)properads}\\ \mathcal{D}^{\circlearrowright}\end{array} \tag{13}$$

which is, by definition, identity on the (co)generators and the quadratic (co)relations of the dioperad $\mathcal{D}$. It is worth noting that, in general, $(\mathcal{D}^{¡})^{\circlearrowright} \neq (\mathcal{D}^{\circlearrowright})^{¡}$, implying that $(\mathcal{D}^{\circlearrowright})_\infty$ may be substantially larger than $(\mathcal{D}_\infty)^{\circlearrowright}$, where $\mathcal{D}_\infty$ stands for the cobar construction on the Koszul dual co-dioperad $\mathcal{D}^{¡}$ in the category of dioperads; we refer to [MMS] for explicit examples of this phenomenon for the cases $\mathcal{D} = \mathcal{A}ss$ and $\mathcal{D} = \mathcal{C}omm$, the operads of associative and, respectively, commutative algebras. In the case of the operad of Lie algebras one actually has an equality, $(\mathcal{L}ie^{¡})^{\circlearrowright} = (\mathcal{L}ie^{\circlearrowright})^{¡}$ (see [Me3]).

The wheelification functor does not, in general, preserve Koszulness: a Koszul dioperad, $\mathcal{D}$, may have a non-Koszul wheelification, $\mathcal{D}^{\circlearrowright}$. We give an explicit example of this phenomenon in § 4. It is worth noting in this connection that the functor $^{\circlearrowright}$ applied to the three classical operads $\mathcal{A}ss$, $\mathcal{C}omm$, and $\mathcal{L}ie$ does preserve Koszulness (see [Me3, MMS] for the proofs).

2.7. Homotopy transfer formulae. Let (V, d) and (W, d) be dg vector spaces equipped with linear degree 0 maps of complexes, $i : W \to V$ and $p : V \to W$, such that the composition $i \circ p : V \to V$ is homotopy equivalent to the identity map, $\mathrm{Id} : V \to V$,

$$Id_V = i \circ p + d \circ h + h \circ d, \tag{14}$$

via a fixed homotopy $h : V \to V[-1]$. Without loss of generality we may assume that the data (i, p, h) satisfies the so called *side conditions* [LaSt],

$$p \circ i = \mathrm{Id}_W, \quad p \circ h = 0, \quad h \circ i = 0, \quad h \circ h = 0.$$

When W is the cohomology of the complex V the above data is often called a *cohomological splitting* of (V, d).

Theorem 2.7.1. *Let $\mathcal{P}$ be a quadratic wheeled properad, and $\rho : \mathcal{P} \to \mathcal{E}nd_V$ an arbitrary $\mathcal{P}$-algebra structure on the complex V. For any element $G \in \mathsf{w}\mathcal{P}^i(m, n)$ let*

$$G\langle i, h, p, \rho\rangle \in \mathcal{E}nd_W(m, n),$$

be a linear map $W^{\otimes n} \to W^{\otimes n}$ defined as follows:

(i) *consider the image, $\imath(G)$, of G under the canonical inclusion $\imath : \mathsf{w}\mathcal{P}^i \to \mathsf{w}B_\bullet(\mathcal{P})$;*
(ii) *decorate the input legs of each graph summand in the image $\imath(G)$ with i, the output legs with p, and the special vertices, $\bullet$, lying on the internal edges with h,*
(iii) *replace a decoration, e, of every non-special vertex in $\imath(G)$ by $\rho(e)$, and finally*
(iv) *interpret the resulting decorated graph as a scheme for the composition of maps i, h, $\rho(e)$ and p.*

Then the family of maps,

$$\{G \longrightarrow G\langle i, h, p, \rho\rangle \in \mathcal{E}nd_W\}_{G \in \mathsf{w}\mathcal{P}^i},$$

defines a representation of the dg free wheeled properad $\mathcal{P}_\infty$ in the dg space W.

Proof. Any morphism, $\mathcal{P}_\infty = \mathcal{F}^{\circlearrowright}\langle \mathsf{w}\mathcal{P}^{\mathsf{i}}\rangle \to \mathcal{E}nd_W$, of wheeled properads is uniquely determined by its values on the generators, i.e. by a morphism, $\mathsf{w}^{-1}\mathcal{P}^{\mathsf{i}} \longrightarrow \mathcal{E}nd_W$ of $\mathbb{S}$-modules. Define such a morphism, $\rho_\infty : \mathcal{P}_\infty \to \mathcal{E}nd_W$, by setting

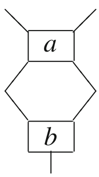
$$\rho_\infty(G) := G\langle i, h, p, \rho\rangle.$$

This morphism gives a representation of the dg properad $\mathcal{P}_\infty$ if and only if it respects the differentials, i.e.

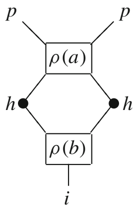
$$\rho_\infty(\partial_{B(\mathcal{P})} G) = d(G\langle i, h, p\rangle),$$

where $\partial_{B(\mathcal{P})}$ is the differential in $B^c(B(\mathcal{P}))$ restricted to the subcomplex $B^c(\mathcal{P}^{\mathsf{i}})$, and d is the differential in $\mathcal{E}nd_W$ (induced by the differential d in W and denoted by the same letter).

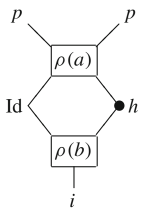
Let us assume, for an illustration, that

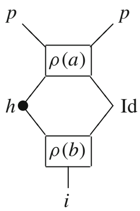
$G =$  $\in \mathsf{w}\mathcal{P}^{\mathsf{i}}$

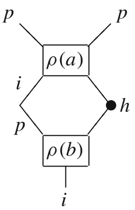
for some $a, b \in \mathcal{P}$. Then

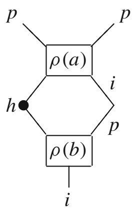
$G\langle i, h, p, \rho\rangle =$ 

and, using (14), we obtain

$d(G\langle i, h, p, \rho\rangle) = (-1)^b$ 

$-\ (-1)^b$ 

$-(-1)^b$ 

$+\ (-1)^b$ 

$$= \rho(\partial_{\mathcal{P}} G) + \rho(\partial_{B(\mathcal{P})} G)$$
$$= \rho(\partial_{B(\mathcal{P})} G).$$

In the above calculation we used

- the identification of $\partial_{\mathcal{P}}$ with a machine deleting the black vertices and contracting the associated internal edge (so that the first two terms in the above sum of 4 graphs are precisely $\rho(\partial_{\mathcal{P}} G\langle i, h, p, \rho\rangle)$),
- the identification of $\partial_{B(\mathcal{P})}$ with a machine deleting the black vertices without subsequent contraction of the associated internal edge (so that the last two terms in the above sum are precisely $\rho(\partial_{B(\mathcal{P})} G\langle i, h, p, \rho\rangle)$), and
- the fact that, by definition of $\mathcal{P}^{¡}$, one has $\rho(\partial_{\mathcal{P}} G) = 0$ for any $G \in \mathsf{w}\mathcal{P}^{¡}$.

The pattern exposed above is universal, i.e. it does not depend on the particularities of G. This simple calculation proves the claim. □

Remarks 2.7.2. (i) The above arguments work for *any* dg sub-coproperad of $B(\mathcal{P})$, not only for $\mathcal{P}^{¡}$.

(ii) Theorem 2.7.1 gives a conceptual explanation of the well-known "experimental" fact that homotopy transfer formulae are given by sums over certain families of *decorated graphs*. Moreover, it follows that these sums describe essentially a *morphism* of coproperads $\mathcal{P}^{¡} \to B(\mathcal{P})$. This fact prompts one to think about the following two closely related problems:

⇒ Given a quadratic (wheeled) properad, construct a quantum field theory whose Feynman perturbation series for the effective action gives precisely the homotopy transfer formulae, i.e. a morphism of (wheeled) coproperads. This idea was first proposed by A. Losev.

$\Leftarrow$ Given a quantum field theory, find dg (wheeled) props such that Feynman's perturbation series for certain expectation values can be interpreted as their morphism.

A simple and beautiful example where both problems have been successfully addressed was constructed by Mnev in [Mn]. Another example is studied in § 6 of this paper. A much less trivial example is given by the works of Kontsevich [Ko] and Cattaneo and Felder [CaFe1] which imply that the quantum Poisson sigma model on the 2-disk describes a morphism of certain dg wheeled props (see [Me4] for their explicit construction).

In the rest of the paper we apply the above theory to a rather simple quadratic wheeled properad, $\mathcal{UL}ie^1\mathcal{B}$, of unimodular Lie 1-bialgebras. Remarkably, representations of the associated dg wheeled prop, $\mathcal{UL}ie^1\mathcal{B}_\infty$, are in one-to-one correspondence with so called *(quasi-classically split) quantum BV manifolds*, interesting structures which one encounters in the Batalin-Vilkovisky quantization of certain gauge systems.

3. Geometry of Quantum Batalin-Vilkovisky Manifolds

3.1. $\mathbb{Z}$-graded formal manifolds. Batalin-Vilkovisky (shortened, BV) formalism [BaVi] is one of most effective and universal methods for perturbative quantization of field theories with gauge symmetries. The first attempt to understand the BV formalism as a *geometric* theory was done by Schwarz who introduced and studied in [Schw] a category of so called *SP-manifolds* to understand BV structures. We adopt, however, in this paper a slightly different picture of BV geometry based on semidensities and Khudaverdian's laplacian [Kh]. When one works in a fixed background the difference between these two pictures is not principal, but we are going to concentrate in this section on *morphisms* and equivalences of BV structures, and in this case the difference becomes decisive.

First we note that

(i) "manifolds", i.e. spaces of fields, used in the BV quantization are often *pointed*; the distinguished point is called a *vacuum* state;
(ii) to make sense of perturbation series around the vacuum state one is only interested in a *formal* neighborhood of that state in the space of fields, not in the global structure of the latter.

Of course, one can try to ignore the formal nature of the perturbation series and accept a genuine smooth supermanifold as a toy model for a space of fields. However, the *formal* nature of the basic notions and operations used in the BV formalism resurrects again when one attempts to make sense of expressions of the type $e^{\frac{\Gamma(x,\hbar)}{\hbar}}$, where the function in the exponent,

$$\Gamma(x,\hbar) = \Gamma_0(x) + \Gamma_1(x)\hbar + \ldots + \Gamma_n(x)\hbar^n + \ldots$$

is a formal power series in a parameter ("Planck constant") $\hbar$. One can try to ignore this issue as well, and set $\hbar = 1$. This is what is often done in many papers on geometric aspects of the BV formalism. We, however, can not afford setting the formal parameter to 1 in the present paper as without $\hbar$ no link between BV manifolds and the homotopy theory of unimodular Lie 1-bialgebras holds true. Therefore right from the beginning we shall be working in the category of *formal* $\mathbb{Z}$-graded manifolds in which one can easily make a coordinate independent sense of functions of the type $e^{\frac{\Gamma(x,\hbar)}{\hbar}}$ by demanding, for example, that $\Gamma_0(x) \in \mathcal{I}$, where $\mathcal{I}$ is the maximal ideal of the distinguished point. Let us give precise definitions.

The category of *finite-dimensional* $\mathbb{Z}$*-graded formal manifolds* over a field $\mathbb{K}$ is, by definition, the category opposite to the category whose

- objects are (isomorphism classes) of completed finitely generated free $\mathbb{Z}$-graded commutative $\mathbb{K}$-algebras; every such a $\mathbb{K}$-algebra $\mathcal{R}$ has a natural translation invariant *adic* topology defined by the condition that the powers, $\{\mathcal{I}^n\}_{n\geq 1}$, of the maximal ideal $\mathcal{I} \subset \mathcal{R}$ form a basis of open neighborhoods of $0 \in \mathcal{R}$;
- morphisms are (isomorphism classes of) continuous morphisms of topological $\mathbb{K}$-algebras.

Thus, every $\mathbb{Z}$-graded formal manifold $\mathcal{M}$ corresponds to a certain isomorphism class, $\mathcal{O}_{\mathcal{M}}$, of completed free finitely generated algebras of the form $\mathbb{K}[[x^1, \ldots, x^n]]$, where formal variables x^a (called *coordinates*) are assigned some degrees $|x^a| \in \mathbb{Z}$. The isomorphism class, $\mathcal{O}_{\mathcal{M}}$, of $\mathbb{K}$-algebras is called the *structure sheaf*[2] of the manifold $\mathcal{M}$. A representation of $\mathcal{O}_{\mathcal{M}}$ in the form $\mathbb{K}[[x_1, \ldots, x_n]]$ is called a *coordinate chart* on $\mathcal{M}$. Such a representation is not canonical: a coordinate chart is defined up to an arbitrary (preserving $\mathbb{Z}$-grading) change of coordinates of the form

$$x^a \longrightarrow \hat{x^a} = \phi^a(x) := \sum_{k=1}^{\infty} \phi^a_{b_1\ldots b_k} x^{b_1} \ldots x^{b_k}, \text{ for some } \phi^a_{b_1\ldots b_k} \in \mathbb{K}, \tag{15}$$

where $\phi^a_{b_1}$ form an invertible matrix. Such changes form a group of formal diffeomorphisms, *Diff* $(\mathcal{M})$, and the Constitution of (formal) Geometry says that every construction on a $\mathbb{Z}$-graded formal manifold $\mathcal{M}$ must be invariant under this group. There are, unfortunately, not that many *Diff* $(\mathcal{M})$-invariant constructions possible in nature, and their study is the major theme of *geometry* rather than *algebra*. This is why we use geometric terminology and intuition throughout this section.

A *smooth map*, $\phi : \mathcal{M} \to \mathcal{N}$, of formal graded manifolds is the same as a morphism, $\phi^* : \mathcal{O}_{\mathcal{N}} \to \mathcal{O}_{\mathcal{M}}$, of their structure sheaves. It is given in local coordinates by formulae of the type (15) with $\phi^a_{b_1}$ not necessarily forming an invertible matrix. A smooth invertible map is called a *diffeomorphism*. The *tangent sheaf*, $\mathcal{T}_{\mathcal{M}}$, of a $\mathbb{Z}$-graded manifold $\mathcal{M}$ is, by definition, the $\mathbb{Z}$-graded $\mathcal{O}_{\mathcal{M}}$-module of derivations of the structure sheaf, that is, the module of linear maps $X : \mathcal{O}_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M}}$ satisfying the Leibnitz condition, $X(fg) = (X(f))g + (-1)^{|X||f|} f X(g)$. It is a free $\mathcal{O}_{\mathcal{M}}$-module generated, in a coordinate chart $\{x^a\}$, by partial derivatives, $\partial/\partial x^a$. Elements of $\mathcal{T}_{\mathcal{M}}$ are called *smooth vector fields* on $\mathcal{M}$. Every vector field $X \in \mathcal{T}_{\mathcal{M}}$ is given in a coordinate chart as a linear combination,

$$X = \sum_a X^a(x) \frac{\partial}{\partial x^a}, \quad X^a(x) \in \mathbb{K}[[x^a]].$$

This representation is not canonical: if $\{\hat{x}^a\}$ is another set of generators of $\mathcal{O}_{\mathcal{M}}$ related to $\{x^a\}$ via (15), then

$$X = \sum_a X^a(x) \frac{\partial}{\partial x^a} = \sum_b \hat{X}^b(\hat{x}) \frac{\partial}{\partial \hat{x}^b}$$

[2] We apologize for using the term *sheaf* in the present formal context as all the *sheaves* the reader encounters in the present section are rather primitive — they are skyscrapers consisting of a single *stalk* over the distinguished point; this terminology helps, however, the geometric intuition (cf. [Me6]).

with

$$\hat{X}^b(\hat{x}) \mid_{\hat{x}^a=\phi^a(x)} = \sum_a X^a(x)\frac{\partial \phi^b(x)}{\partial x^a}.$$

The matrix $(\partial\phi^b(x)/\partial x^a)$ is called the *Jacobian* of the coordinate transformation (15). The $\mathcal{O}_{\mathcal{M}}$-module $\mathcal{T}_{\mathcal{M}}$ has a natural graded Lie algebra structure with respect to the ordinary graded commutator of derivations, $[X_1, X_2] = X_1 \circ X_2 - (-1)^{|X_1||X_2|} X_2 \circ X_1$. The rank of $\mathcal{T}_{\mathcal{M}}$ is equal to the number of generators of the algebra $\mathcal{O}_{\mathcal{M}}$ and is called the *dimension* of the graded manifold $\mathcal{M}$.

Let V be a finite dimensional $\mathbb{Z}$-graded vector space. One can associate to V a $\mathbb{Z}$-graded formal manifold $\mathcal{V}$ by defining $\mathcal{O}_{\mathcal{V}}$ to be the isomorphism class of the $\mathbb{K}$-algebra $\widehat{\odot^\bullet} V^*$, where $V^* := \mathrm{Hom}(V, \mathbb{K})$. The manifold $\mathcal{V}$ is said to be *modeled* on the graded vector space V. Every formal $\mathbb{Z}$-graded manifold $\mathcal{M}$ is modeled by some uniquely defined graded vector space $\mathcal{T}_{\star\in\mathcal{M}} := (\mathcal{I}/\mathcal{I}^2)^*$ called the *tangent vector space at the distinguished point* $\star$ *in* $\mathcal{M}$. Note that every morphism of graded manifolds, $\phi : \mathcal{M} \to \mathcal{N}$, gives rise to a well-defined map, $d\phi_* : \mathcal{T}_{\star\in\mathcal{M}} \to \mathcal{T}_{\star\in\mathcal{N}}$, of tangent vector spaces, but, in general, *not* to a morphism, $d\phi : \mathcal{T}_{\mathcal{M}} \to \mathcal{T}_{\mathcal{N}}$, of tangent sheaves. The latter is well-defined if, for example, ϕ is an isomorphism.

Let $\mathcal{T}^*_{\mathcal{M}} := \mathrm{Hom}_{\mathcal{O}_{\mathcal{M}}}(\mathcal{T}_{\mathcal{M}}, \mathcal{O}_{\mathcal{M}})$ be the dual $\mathcal{O}_{\mathcal{M}}$-module, and let $\Omega^1_{\mathcal{M}} := \mathcal{T}^*_{\mathcal{M}}[1]$ be the same $\mathcal{O}_{\mathcal{M}}$-module $\mathcal{T}^*_{\mathcal{M}}$ but with shifted grading. The latter is called the sheaf of *differential 1-forms* on the graded manifold $\mathcal{M}$. Note that the natural pairing,

$$\begin{array}{rcl} \langle\ ,\ \rangle : \mathcal{T}_{\mathcal{M}} \times \Omega^1_{\mathcal{M}} & \longrightarrow & \mathcal{O}_{\mathcal{M}} \\ X \otimes \tau & \longrightarrow & \langle X, \tau\rangle \end{array}$$

has degree 1. There is a canonical degree -1 $\mathbb{K}$-linear morphism,

$$\begin{array}{rcl} d : \mathcal{O}_{\mathcal{M}} & \longrightarrow & \Omega^1_{\mathcal{M}} \\ f & \longrightarrow & df, \end{array}$$

defined, for arbitrary vector field $X \in \mathcal{T}_{\mathcal{M}}$ by the equality $\langle X, df\rangle = X(f)$. It is clear that $\Omega^1_{\mathcal{M}}$ is a free $\mathcal{O}_{\mathcal{M}}$ module with a basis given, in some coordinate chart $\{x^a\}$, by 1-forms dx^a, i.e. every 1-form τ can be represented in this chart as a linear combination, $\tau = \sum_a dx^a \tau_a(x)$, for some $\tau_a(x) \in \mathbb{K}[[x^a]]$. We also have $df = \sum_a dx^a \partial f/\partial x^a$.

The sheaf of graded commutative algebras, $\Omega^\bullet_{\mathcal{M}} := \odot^\bullet_{\mathcal{O}_{\mathcal{M}}} \Omega^1_{\mathcal{M}}$, generated by 1-forms is called the *De Rham* sheaf on $\mathcal{M}$. Elements of $\Omega^k_M := \odot^k \Omega^1_{\mathcal{M}}$ are called differential k-forms. The morphism $d : \Omega^0_{\mathcal{M}} \to \Omega^1_{\mathcal{M}}$ extends naturally to a morphism $d : \Omega^k_{\mathcal{M}} \to \Omega^{k+1}_{\mathcal{M}}$ for any k making thereby $(\Omega^\bullet_{\mathcal{M}}, d)$ into a sheaf of *differential* algebras, i.e. satisfying $d^2 = 0$ and $d(\tau_1\tau_2) = (d\tau_1)\tau_2 + (-1)^{|\tau_1|}\tau_1 d\tau_2$ for any $\tau_1, \tau_2 \in \Omega_{\mathcal{M}}$. In a local coordinate chart $\{x^a\}$ on $\mathcal{M}$ we have an isomorphism

$$\Omega^\bullet_{\mathcal{M}} \simeq \mathbb{K}[[x^a, dx^a]], \quad |dx^a| = |x^a| - 1,$$

with the de Rham differential given on generators by $d(x^a) := dx^a$, $d(dx^a) := 0$.

3.2. Odd Poisson structure. Let $\mathcal{M}$ be a formal $\mathbb{Z}$-graded manifold. A *odd Poisson structure* on $\mathcal{M}$ is a degree -1 linear map,

$$\begin{array}{rcl} \{\,\bullet\,\}:\mathcal{O}_{\mathcal{M}}\otimes_{\mathbb{K}}\mathcal{O}_{\mathcal{M}} & \longrightarrow & \mathcal{O}_{\mathcal{M}} \\ f\otimes g & \longrightarrow & \{f\bullet g\}, \end{array}$$

such that $\{f\bullet g\}=(-1)^{fg+f+g}\{g\bullet f\}$ and

$$\{f\bullet\{g\bullet h\}\}=\{\{f\bullet g\}\bullet h\}+(-1)^{(|f|+1)(|g|+1}\{g\bullet\{f\bullet h\} \tag{16}$$

$$\{f\bullet gh\}=\{f\bullet g\}h+(-1)^{fg+g}g\{f\bullet h\} \tag{17}$$

for any $f,g,h\in\mathcal{O}_{\mathcal{M}}$. Thus brackets $\{\,\bullet\,\}$ and the ordinary product of functions make the structure sheaf $\mathcal{O}_{\mathcal{M}}$ into a sheaf of so called *Gerstenhaber algebras*.

A $\mathbb{Z}$-graded formal manifold $\mathcal{M}$ equipped with a degree -1 Poisson structure is called an *odd Poisson manifold*. A *Poisson map*, $\phi:(\mathcal{M}_1,\{\,\bullet\,\})\to(\mathcal{M}_2,\{\,\bullet\,\})$, of odd Poisson manifolds is a degree 0 smooth map $\phi:\mathcal{M}_1\to\mathcal{M}_2$ such that

$$\{\phi^*(f)\bullet\phi^*(g)\}=\phi^*\{f\bullet g\}$$

for any $f,g\in\mathcal{O}_{\mathcal{M}_2}$. If one translates brackets $\{\,\bullet\,\}$ from $\mathcal{O}_{\mathcal{M}}$ to its "shifted" version, $\mathcal{O}_{\mathcal{M}}[1]$ via the natural isomorphisms $\mathcal{O}_{\mathcal{M}}\rightleftarrows\mathcal{O}_{\mathcal{M}}[1]$, one obtains an ordinary $\mathbb{Z}$-graded Lie algebra structure on $\mathcal{O}_{\mathcal{M}}[1]$.

An important example of an odd Poisson structure comes from the sheaf of polyvector fields defined next.

3.3. Polyvector fields. For any $\mathbb{Z}$-graded n-dimensional formal manifold M the completed graded commutative algebra $\mathcal{O}_{\mathcal{M}}:=\widehat{\odot^\bullet}(\mathcal{T}_M[-1])$ is free of rank $2n$ and hence defines a $\mathbb{Z}$-graded formal manifold $\mathcal{M}$ which is often called *the total space,* $\mathcal{M}:=\Omega^1_M$, *of the bundle of 1-forms on* M. Elements of its structure sheaf $\mathcal{O}_{\mathcal{M}}$ are called *polyvector fields* on the manifold M and the structure sheaf itself is often denoted by $\mathcal{P}oly(M)$. One sets $\mathcal{P}oly^k(M):=\odot^k(\mathcal{T}_M[-1])$ and call its elements *k-vector fields*. This terminology for $\mathcal{M}$ and its structure sheaf originates from the duality $\mathcal{T}_M[-1]=\mathrm{Hom}_{\mathcal{O}_M}(\Omega^1_M,\mathcal{O}_M)$ and from the natural inclusion of the degree shifted[3] tangent sheaf $\mathcal{T}_M[-1]\subset\mathcal{P}oly(M)$.

A coordinate chart $\{x^a\}$ on M induces a coordinate chart

$$\left\{x^a,\ \psi_a:=\Pi\frac{\partial}{\partial x^a},\ |\psi_a|=-|x^a|+1\right\}$$

on $\mathcal{M}$, where $\Pi:\mathcal{T}_M\to\mathcal{T}_M[-1]$ is the natural isomorphism. A change of coordinates (15) on M induces a change of coordinates,

$$\begin{aligned} x^a &\longrightarrow \hat{x}^a=\phi^a(x)\\ \psi_a &\longrightarrow \hat{\psi}_a=\sum_b\frac{\partial\phi^a(x)}{\partial x^b}\psi_b \end{aligned} \tag{18}$$

on $\mathcal{M}$. In these coordinates we have an isomorphism,

$$\mathcal{O}_{\mathcal{M}}\equiv\mathcal{P}oly(M)\simeq\mathbb{K}[[x^a,\psi_a]].$$

[3] To avoid such a degree shifting the sheaf of polyvector fields $\mathcal{P}oly(M)$ is defined by some authors as $\mathcal{O}_{\mathcal{M}}[-1]$.

It is not hard to check that the degree -1 brackets on $\mathcal{O}_{\mathcal{M}}$ defined in such a coordinate chart by

$$\{f \bullet g\} := \sum_a \left((-1)^{|f||x^a|} \frac{\partial f}{\partial x^a} \frac{\partial g}{\partial \psi_a} + (-1)^{|f|(|x^a|+1)} \frac{\partial f}{\partial \psi_a} \frac{\partial g}{\partial x^a}. \right) \tag{19}$$

satisfy the axioms (16) and (17), and, moreover, are invariant under transformations (18). Hence they define an odd Poisson structure on the manifold $\mathcal{M}$. Brackets (19) on $\mathcal{P}oly(M)$ are often denoted by $[\ \bullet\ ]_S$ and called *Schouten* brackets.

Schouten brackets $[\ \bullet\ ]_S$ restricted to the subsheaf $\mathcal{T}_M[-1] \subset \mathcal{P}oly(M)$ give, modulo the degree shifting, the ordinary commutator of vector fields.

3.4. Odd symplectic structures. Any odd Poisson structure on a graded formal manifold $\mathcal{M}$ defines a homogeneous (of degree 1) section, ν, of the bundle $\mathcal{P}oly^2(\mathcal{M})$ by the formula,

$$\langle \nu, df dg \rangle = \{f \bullet g\} \quad \forall f, g \in \mathcal{O}_{\mathcal{M}},$$

where $\langle\ ,\ \rangle$ stands for the natural duality pairing between $\mathcal{P}oly^2(\mathcal{M}) = \mathrm{Hom}_{\mathcal{O}_{\mathcal{M}}}(\Omega^2_{\mathcal{M}}, \mathcal{O}_{\mathcal{M}})$ and $\Omega^2_{\mathcal{M}}$. An odd Poisson structure on $\mathcal{M}$ is called *non-degenerate* or *odd symplectic* if the associated 2-vector field is non-degenerate in the sense that the induced "raising of indices" morphism of sheaves,

$$\Omega^1_{\mathcal{M}} \xrightarrow{\lrcorner \nu} \mathcal{T}_{\mathcal{M}},$$

is an isomorphism. The inverse map gives rise to a degree -1 differential 2-form, $\omega :=$ "ν^{-1}", on $\mathcal{M}$ which satisfies, due to the Jacobi identity (16), the condition $d\omega = 0$.

3.4.1. Darboux lemma (see, e.g., [Kh, Le, Schw]) Any $\mathbb{Z}$-graded manifold with a non-degenerate odd Poisson structure is locally isomorphic to an odd Poisson manifold $\mathcal{M}$ described in § *3.3*.

Thus any odd symplectic manifold $\mathcal{M}$ admits local coordinates, $\{(x^a, \psi_a), |\psi_a| = -|x^a| + 1\}$, in which the odd Poisson brackets are given by (19). The associated symplectic 2-form is then given by $\omega = \sum_a dx^a d\psi_a$. These coordinates are called *Darboux coordinates*.

3.4.2. Symplectomorphisms and canonical transformations By Lemma 3.4.1, any odd symplectic manifold can be covered by a Darboux coordinate chart (x^a, ψ_a). For future reference we note that a generic change of coordinates

$$\begin{aligned} x^a &\longrightarrow \hat{x}^a = \phi^a(x, \psi) \\ \psi_a &\longrightarrow \hat{\psi}_a = \phi_a(x, \psi) \end{aligned} \tag{20}$$

defines a new Darboux coordinate chart $(\hat{x}^a, \hat{\psi}_a)$ if and only if the equations

$$\sum_a (-1)^{|x^a|(|x^c|+1)} \frac{\partial \phi^a}{\partial x^b} \frac{\partial \phi_a}{\partial x^c} = 0, \quad \sum_a (-1)^{|x^a||x^c|} \frac{\partial \phi^a}{\partial \psi_b} \frac{\partial \phi_a}{\partial \psi_c} = 0,$$
$$\sum_a \left((-1)^{|x^a||x^c|} \frac{\partial \phi^a}{\partial x^b} \frac{\partial \phi_a}{\partial \psi_c} + (-1)^{(|x^a|+1)(|x^b|+1)} \frac{\partial \phi^a}{\partial \psi_c} \frac{\partial \phi_a}{\partial x^b} \right) = \delta^c_b := \begin{cases} 1 \text{ if } b = c, \\ 0 \text{ if } b \neq c, \end{cases} \tag{21}$$

are satisfied. A diffeomorphism (20) satisfying Eqs. (21) is called a *canonical transformation*. It is easy to check that (18) is a canonical transformation for arbitrary functions $\phi^a(x)$ which have the associated Jacobi matrix $\partial\phi^a(x)/\partial x^b|_{x=0}$ invertible.

A Poisson diffeomorphism of odd symplectic manifolds, $\phi : (\mathcal{M}, \omega) \to (\hat{\mathcal{M}}, \hat{\omega})$, is called a *symplectomorphism*. This is the same as a diffeomorphism $\phi : \mathcal{M} \to \hat{\mathcal{M}}$ of smooth $\mathbb{Z}$-graded manifolds such that $\phi^*(\hat{\omega}) = \omega$. It is the assumption on the non-degeneracy of the odd symplectic forms which forces one to define symplectomorphisms as special cases of *diffeomorphisms*. In Darboux coordinate charts, (x^a, ψ_a) and $(\hat{x}^a, \hat{\psi}_a)$, on $\mathcal{M}$ and, respectively, $\hat{\mathcal{M}}$ any symplectomorphism is given by functions (20) satisfying Eqs. (21); for that reason a symplectomorphism is also often called a canonical transformation.

Remark 3.4.3. The collection of Poisson morphisms of odd symplectic manifolds is much richer than the collection of symplectomorphisms. For example, if $\mathcal{M}_1$ and $\mathcal{M}_2$ are odd symplectic manifolds, then $\mathcal{M}_1 \times \mathcal{M}_2$ is naturally an odd symplectic manifold and the natural projection $\mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{M}_1$ is a well defined Poisson morphism, which is *not* a symplectomorphism.

3.4.4. Hamiltonian vector fields For any function $\Phi \in \mathcal{O}_{\mathcal{M}}$ on an odd Poisson manifold $\mathcal{M}$, the associated map

$$\begin{array}{rcl} H_\Phi : \mathcal{O}_M & \longrightarrow & \mathcal{O}_M \\ g & \longrightarrow & \{\Phi \bullet g\} \end{array}$$

is a derivation of the structure ring $\mathcal{O}_{\mathcal{M}}$ and hence is a smooth vector field on $\mathcal{M}$ called the *Hamiltonian vector field associated with a function* Φ. It is not hard to check that $[H_{\Phi_1}, H_{\Phi_2}] = H_{\{\Phi_1 \bullet \Phi_2\}}$ for any $\Phi_1, \Phi_2 \in \mathcal{O}_M$.

If the Poisson structure is non-degenerate, then in a local Darboux coordinate chart one has,

$$H_\Phi = \sum_a \left((-1)^{|\Phi||x^a|} \frac{\partial \Phi}{\partial x^a} \frac{\partial}{\partial \psi_a} + (-1)^{\Phi|(|x^a|+1)} \frac{\partial \Phi}{\partial \psi_a} \frac{\partial}{\partial x^a} \right).$$

Note that if the function $\Phi(x^a, \psi_a)$ has degree 1, then the associated hamiltonian vector field H_Φ has degree zero, and it makes sense to consider a system of ordinary differential equations,

$$\begin{aligned} \frac{d\phi^a(x,\psi,t)}{dt} &= -(-1)^{|x^a|} \frac{\partial \Phi(\hat{x},\hat{\psi})}{\partial \hat{\psi}_a} \,|_{\hat{x}^a = \phi^a(x,\psi,t),\, \hat{\psi}_a = \phi_a(x,\psi,t)}, \\ \frac{d\phi_a(x,\psi,t)}{dt} &= (-1)^{|x^a|} \frac{\partial \Phi(\hat{x},\hat{\psi})}{\partial \hat{x}^a} \,|_{\hat{x}^a = \phi^a(x,\psi,t),\, \hat{\psi}_a = \phi_a(x,\psi,t)}, \\ \phi^a(x,\psi,t)|_{t=0} &= x^a, \\ \phi_a(x,\psi,t)|_{t=0} &= \psi_a, \end{aligned} \tag{22}$$

for the unknown functions $\phi^a(x, \psi, t)$ and $\phi_a(x, \psi, t)$ of degrees $|x^a|$ and, respectively, $|\psi_a|$. Moreover, a classical theorem from the theory of systems of ordinary differential equations guarantees that, for a sufficiently small strictly positive $\varepsilon \in \mathbb{R}$, its solution,

$$\{\phi_t^a = \phi^a(x,\psi,t), \phi_{t\,a} = \phi_a(x,\psi,t)\},$$

exists and is unique for all t in the interval $[0, \varepsilon)$. Moreover, the solution is real analytic with respect to the parameter t. Using the above equations it is easy to check that

$$\frac{d}{dt}\sum_a (-1)^{|x^a|(|x^c|+1)} \frac{\partial \phi_t^a}{\partial x^b}\frac{\partial \phi_{t\,a}}{\partial x^c} = 0, \quad \frac{d}{dt}\sum_a (-1)^{|x^a||x^c|}\frac{\partial \phi^a}{\partial \psi_b}\frac{\partial \phi_a}{\partial \psi_c} = 0,$$
$$\frac{d}{dt}\sum_a \left((-1)^{|x^a||x^c|}\frac{\partial \phi^a}{\partial x^b}\frac{\partial \phi_a}{\partial \psi_c} + (-1)^{(|x^a|+1)(|x^b|+1)}\frac{\partial \phi^a}{\partial \psi_c}\frac{\partial \phi_a}{\partial x^b}\right) = 0,$$

implying, in view of the boundary $t = 0$ conditions on ϕ_t, that, for any $t \in [0, \varepsilon)$ the map

$$x^a \longrightarrow \hat{x}^a = \phi^a(x, \psi, t)$$
$$\psi_a \longrightarrow \hat{\psi}_a = \phi_a(x, \psi, t)$$

satisfies Eq. (21) and hence defines a canonical transformation. Thus any degree 1 function Φ on $\mathcal{M}$ gives rise naturally to a 1-parameter family of local symplectomorphisms $\phi_t : \mathcal{M} \to \mathcal{M}$.

3.4.5. Lagrangian submanifolds Let $\mathcal{M}$ be a $\mathbb{Z}$-graded manifold and $I \subset \mathcal{O}_\mathcal{M}$ an ideal such that the quotient ring $\mathcal{O}_\mathcal{S} \subset \mathcal{O}_\mathcal{M}/I$ is free; this ring corresponds, therefore, to a $\mathbb{Z}$-graded manifold $\mathcal{S}$ which is called a *submanifold* of $\mathcal{M}$; the natural epimorphism $\mathcal{O}_\mathcal{M} \to \mathcal{O}_\mathcal{S}$ is called an *embedding* $\mathcal{S} \hookrightarrow \mathcal{M}$.

Lemma 3.4.1 implies that any odd symplectic manifold $(\mathcal{M}, \omega)$ is even dimensional, say $\dim \mathcal{M} = 2n$. An n-dimensional submanifold $\mathcal{L} \hookrightarrow \mathcal{M}$ is called *Lagrangian* if $\omega|_\mathcal{L} = 0$, i.e. the induced map $\omega|_\mathcal{L} : \odot^2(\mathcal{T}_\mathcal{L}[-1]) \to \mathcal{O}_\mathcal{L}$ is zero. The *normal* sheaf, $N_{\mathcal{L}|\mathcal{M}}$, of the submanifold $\mathcal{L} \hookrightarrow \mathcal{M}$ is defined by the short exact sequence of sheaves of $\mathcal{O}_\mathcal{L}$-modules,

$$0 \longrightarrow \mathcal{T}_\mathcal{L} \stackrel{i}{\longrightarrow} \mathcal{T}_\mathcal{M}\,|_\mathcal{L} \longrightarrow N_{\mathcal{L}|\mathcal{M}} \longrightarrow 0$$

so that its dualization (and degree shifting) gives

$$0 \longrightarrow N^*_{\mathcal{L}|\mathcal{M}}[1] \longrightarrow \Omega^1_\mathcal{M}\,|_\mathcal{L} \stackrel{p}{\longrightarrow} \Omega^1_\mathcal{L} \longrightarrow 0.$$

The odd symplectic form ω provides us with a *degree* 0 isomorphism of the middle terms of the short exact sequences above,

$$\mathcal{T}_\mathcal{M}\,|_\mathcal{L} \stackrel{\lrcorner\omega}{\longrightarrow} \Omega^1_\mathcal{M}.$$

The condition $\omega|_\mathcal{L} = 0$ is equivalent to saying that the composition

$$\mathcal{T}_\mathcal{L} \stackrel{i}{\longrightarrow} \mathcal{T}_\mathcal{M}\,|_\mathcal{L} \stackrel{\lrcorner\omega}{\longrightarrow} \Omega^1_\mathcal{M} \stackrel{p}{\longrightarrow} \Omega^1_\mathcal{L}$$

vanishes. Hence we get a canonical monomorphism of sheaves,

$$\lrcorner\omega \circ i : \mathcal{T}_\mathcal{L} \longrightarrow N^*_{\mathcal{L}|\mathcal{M}}[1],$$

which is an isomorphism because both sheaves have the same rank as locally free $\mathcal{O}_\mathcal{L}$-modules. Hence, for any Lagrangian submanifold $\mathcal{L} \hookrightarrow \mathcal{M}$, $N_{\mathcal{L}|\mathcal{M}} = (\mathcal{T}_\mathcal{L})^*[1] = \Omega^1_\mathcal{L}$ and there is a canonically associated exact sequence,

$$O \longrightarrow \mathcal{T}_\mathcal{L} \stackrel{i}{\longrightarrow} \mathcal{T}_\mathcal{M}\,|_\mathcal{L} \longrightarrow \Omega^1_\mathcal{L} \longrightarrow 0 \tag{23}$$

of sheaves.

3.5. Densities and semidensities. If V is a $\mathbb{Z}$-graded free module over a $\mathbb{Z}$-graded ring R, then $\mathcal{B}er(V)$ is, by definition, a degree 0 rank 1 free module over R equipped with a distinguished family of bases $\{D_e\}$ defined as follows [Ma1] (see also [Ca2]):

(i) for any base $e = \{e_\alpha\}$ of the module V, there is an associated basis vector, D_e, of $\mathcal{B}er(V)$;
(ii) if $e = \{e_\alpha\}$ and $\hat{e} = \{\hat{e}_\beta\}$ are two bases of V with the relation $\hat{e}_\beta = \sum_\alpha e_\alpha A^\alpha_\beta$ for some non-degenerate matrix $A^\beta_\alpha \in R$, then $D_{\hat{e}} = Ber(A) D_e$, where $Ber(A)$ is the Berezinian of the matrix A.

If $\mathcal{M}$ is a $\mathbb{Z}$-graded manifold, then

$$\mathcal{B}er(\mathcal{M}) := \mathcal{B}er(\mathcal{T}^*_{\mathcal{M}}) = \left(\mathcal{B}er(\Omega^1_{\mathcal{M}})\right)^*$$

is a rank 1 locally free sheaf of $\mathcal{O}_{\mathcal{M}}$-modules. Its elements which do not vanish at the distinguished point, are called *densities* or *volume forms* on the manifold $\mathcal{M}$. Let, for concreteness, $\mathcal{M}$ be an odd symplectic manifold. To every Darboux coordinate chart (x^b, ψ_b) on $\mathcal{M}$ there corresponds, by definition, a basis section, $D_{x,\psi}$, of $\mathcal{B}er(\mathcal{M})$; if (x^b, ψ_b) and $(\hat{x}^a, \hat{\psi}_a)$ are two Darboux coordinate charts related to each other by a canonical transformation (20), then

$$D_{\hat{x},\hat{\psi}} = Ber\left(\frac{\partial(\hat{x}, \hat{\psi})}{\partial(x, \psi)}\right) D_{x,\psi},$$

where $\frac{\partial(\hat{x},\hat{\psi})}{\partial(x,\psi)}$ stands for the Jacobi matrix of the natural transformation (20).

One can show (see, e.g., [Schw, KhVo]) that for any odd symplectic manifold $\mathcal{M}$ the sheaf $\mathcal{B}er(\mathcal{M})$ admits a square root, that is, there exists a sheaf $\mathcal{B}er^{1/2}(\mathcal{M})$ of so called *semidensities* such that

$$\mathcal{B}er(\mathcal{M}) = \left(\mathcal{B}er^{1/2}(\mathcal{M})\right)^{\otimes 2}. \tag{24}$$

An element Θ of the $\mathcal{O}_{\mathcal{M}}$-module $\mathcal{B}er^{1/2}(\mathcal{M})$ which does not vanish at the distinguished point, is called a *semidensity* on $\mathcal{M}$. In a Darboux coordinate chart on $\mathcal{M}$ a semidensity, Θ, can be represented in the form $\Theta = \Theta_{x,\psi}\sqrt{D}_{x,\psi}$ for some degree zero formal power series $\Theta_{x,\psi} \in \mathbb{K}[[x^a, \psi_a]]$ with $\Theta_{x,\psi}|_{x=0,\psi=0} \in \mathbb{K}^*$. Under a canonical transformation this representation changes as follows:

$$\Theta_{\hat{x},\hat{\psi}} = \left(Ber\left(\frac{\partial(\hat{x}, \hat{\psi})}{\partial(x, \psi)}\right)\right)^{-1/2} \Theta_{x,\psi}. \tag{25}$$

3.5.1. Odd Laplacian on semidensities Let $(\mathcal{M}, \omega)$ be an odd symplectic manifold. The odd symplectic structure on $\mathcal{M}$ gives canonically rise to a differential operator on semidensities,

$$\begin{array}{rcl}\Delta_\omega : \mathcal{B}er^{1/2}(\mathcal{M}) & \longrightarrow & \mathcal{B}er^{1/2}(\mathcal{M}) \\ \Theta & \longrightarrow & \Delta_\omega\Theta,\end{array}$$

defined in an arbitrary Darboux coordinate system as follows [Kh]:

$$\Delta_\omega\Theta := \left(\sum_a \frac{\partial^2\Theta_{x,\psi}}{\partial x^a \partial\psi_a}\right)\sqrt{D}_{x,\omega}.$$

A remarkable fact is that Δ_ω is well-defined, i.e. does not depend on a particular choice of Darboux coordinates used in the definition, as under arbitrary canonical transformations (20) one has [Kh],

$$\sum_a \frac{\partial^2\Theta_{\hat{x},\hat{\psi}}}{\partial\hat{x}^a\partial\hat{\psi}_a}\,|_{\substack{\hat{x}^a=\phi^a(x,\psi)\\ \hat{\psi}_a=\phi_a(x,\psi)}} = \left(Ber\left(\frac{\partial(\hat{x},\hat{\psi})}{\partial(x,\psi)}\right)\right)^{-1/2}\sum_a \frac{\partial^2\Theta_{x,\psi}}{\partial x^a\partial\psi_a}. \tag{26}$$

The operator Δ_ω is uniquely determined by the underlying odd symplectic structure and is called the *odd Laplacian*. This is an odd analogue of the modular vector field in ordinary Poisson geometry [Wein]. Its invariant definition can be found in [Se]; that definition is a bit tricky and involves Manin's beautiful description of the Berezinian $\mathcal{B}er(\mathcal{M})$ as a cohomology class in a certain complex (see Chap. 3, §4.7 in [Ma1]).

For an arbitrary Darboux coordinate chart (x^a, ψ_a) the second order operator $\sum_a \frac{\partial^2}{\partial x^a\partial\psi_a}$ is denoted from now on by $\Delta_{x,\psi}$ or, when a particular choice of Darboux coordinates is implicitly assumed, simply by Δ_0. This operator has an invariant meaning only when applied to (coordinate representatives of) semidensities, not to ordinary functions.

Lemma 3.5.2. $\Delta_\omega^2 = 0$.

Proof is evident when one uses Darboux coordinates.

3.6. Batalin-Vilkovisky manifolds. A *Batalin-Vilkovisky structure* (or, shortly, *BV-structure*) on an odd symplectic manifold $(\mathcal{M},\omega)$ is a semidensity $\Theta \in \mathcal{B}er^{1/2}(\mathcal{M})$ satisfying an equation,

$$\Delta_\omega\Theta = 0. \tag{27}$$

Equation (27) is called a *master equation*, while its solution $\Theta \in \mathcal{B}er^{1/2}(\mathcal{M})$ a *master semidensity*.

Such structures first emerged in the powerful Batalin-Vilkovisky approach to the quantization of field theories with gauge symmetries (see, e.g., [BaVi,Schw,Ca1,Ca2, CKTB] and references cited there). A concrete example of such a BV quantization machine is considered in § 6 below.

The automorphism group, $Aut(\mathcal{M},\omega)$, of the odd symplectic manifold (that is, the group of symplectomorphisms $(\mathcal{M},\omega) \to (\mathcal{M},\omega)$) acts naturally on the set of BV-structures on $(\mathcal{M},\omega)$: if Θ is a master semidensity, then for any $\phi \in Aut(\mathcal{M},\omega)$, its pullback $\phi^*(\Theta)$ defined by (25) is again a master semidensity.

Remark 3.6.1. Following Schwarz [Schw], BV-structures on odd symplectic manifolds are defined in many papers in a different way: one first fixes an extra structure, a volume form ρ on $\mathcal{M}$, and then one defines an odd Laplacian, $\Delta_{\omega,\rho}$, on *functions* as a map

$$\begin{aligned}\Delta_{\omega,\rho} : \mathcal{O}_\mathcal{M} &\longrightarrow \mathcal{O}_\mathcal{M}\\ f &\longrightarrow \frac{\mathcal{L}_{H_f}\rho}{\rho},\end{aligned}$$

where $\mathcal{L}_{H_f}$ stands for the Lie derivative along the Hamiltonian vector field H_f associated with a function $f \in \mathcal{O}_{\mathcal{M}}$. The data (M, ω, ρ) is called in [Schw] an *SP-manifold*, and a BV-structure on an SP-manifold is defined as a function $f \in \mathcal{O}_{\mathcal{M}}$ satisfying the equation $\Delta_{\omega,\rho} f = 0$. In fact, the volume form ρ can not be arbitrary but must satisfy an extra condition [Schw] which assures that in some Darboux coordinate the equation for f takes the form $\Delta_{x,\psi} f = 0$ making it completely equivalent to the above semidensity approach via an association

$$f \rightleftarrows \frac{\Theta}{\sqrt{\rho}}.$$

The SP-manifold approach to the BV-geometry does not seem to be a natural one for the following two reasons:

(i) it is often accompanied with an undue restriction of the gauge group of the set of BV-structures on $\mathcal{M}$ from arbitrary symplectomorphisms to *volume preserving* symplectomorphisms (in contrast to ordinary symplectic geometry, in *odd* symplectic geometry a generic symplectomorphism is *not* necessarily volume preserving);
(ii) in applications of the BV formalism to quantizations one never integrates over the "phase space" $\mathcal{M}$ itself but rather over its Lagrangian submanifolds $\mathcal{L} \hookrightarrow \mathcal{M}$ (see § 6 for a concrete example) depending on a gauge fixing. Thus what one needs in applications is not a volume form on $\mathcal{M}$ but rather a global object on that odd symplectic manifold which restricts to a volume form on its arbitrary Lagrangian submanifold $\mathcal{L} \hookrightarrow \mathcal{M}$. Extension (23) implies,

$$\mathcal{B}er(\mathcal{M}) \mid_{\mathcal{L}} = \mathcal{B}er(\mathcal{L}) \otimes \mathcal{B}er(\Omega^1_{\mathcal{L}})^* = \mathcal{B}er(\mathcal{L})^{\otimes 2}, \tag{28}$$

which in turn implies that it is an appropriately chosen semidensity on $\mathcal{M}$ (rather than a volume form on $\mathcal{M}$) which might restrict to a volume form on the Lagrangian submanifold.

In fact we have no choice as to adopt a definition of BV structures via semidensities rather than via SP-manifolds as in our approach Definition 3.6 (as well as definition of a morphism of BV-manifolds, see § 3.9 below) *follows* from the homotopy theory of Lie 1-bialgebras and the associated homotopy transfer formulae.

Definition 3.6.2. *A data* $(\mathcal{M}, \omega, \Theta)$ *consisting of an odd symplectic manifold* $(\mathcal{M}, \omega)$ *and a master semidensity* $\Theta \in \mathcal{B}er^{1/2}(\mathcal{M})$ *is called a* ***Batalin-Vilkovisky manifold****, or simply a* ***BV-manifold****.*

3.6.3. Dilation group action and pointed manifolds If Θ is a BV structure on an odd symplectic manifold $\mathcal{M}$ then $\lambda\Theta$ is again a BV structure for any non-zero constant $\lambda \in \mathbb{K}$. From now on we identify such BV structures, i.e. we understand a master semidensity Θ as an element of the projective space $\mathbb{P}\mathcal{B}er^{1/2}(\mathcal{M})$.

Formal $\mathbb{Z}$-graded manifolds $\mathcal{M}$ are always *pointed*, i.e. have a distinguished point $* \in \mathcal{M}$ corresponding to the unique maximal ideal in $\mathcal{O}_{\mathcal{M}}$ which is often called (in the quantization context) *the vacuum state*. In a Darboux coordinate system (x^a, ψ_a) centered at $*$ one can always normalize a master semidensity $\Theta = \Theta_{x,\psi} D_{x,\psi} \in \mathbb{P}\mathcal{B}er^{1/2}(\mathcal{M})$ in such a way that $\Theta_{x,\psi}|_{x=\psi=0} = 1 \in \mathbb{K}$, and this normalization is invariant under formal canonical transformations. It is often suitable to represent such a normalized semidensity in the form $\Theta = e^{\Gamma(x,\psi)}\sqrt{D_{x,\psi}}$ for some smooth formal function $\Gamma(x,\psi)$ *vanishing at zero* (so that its exponent is well-defined as a formal power series); the master equation takes then the form

$$\Delta_\omega \Theta = \left(\Delta_0 \Gamma + \frac{1}{2}\{\Gamma \bullet \Gamma\}\right)\Theta = 0.$$

As Θ is, by assumption, non-vanishing, the latter equation is equivalent to

$$\Delta_0 \Gamma + \frac{1}{2}\{\Gamma \bullet \Gamma\} = 0, \tag{29}$$

where $\{\bullet\}$ are the odd Poisson brackets on $\mathcal{M}$. The normalization $\Gamma|_* = 0$ is assumed from now on.

3.7. Sheaves of Gerstenhaber-Batalin-Vilkovisky (GBV) algebras. A $\mathbb{Z}$-graded commutative unital algebra $\mathcal{A}$ equipped with a degree -1 linear map $\Delta : \mathcal{A} \to \mathcal{A}$ satisfying

(i) $\Delta^2 = 0$,
(ii) and, for any $a, b, c \in \mathcal{A}$,

$$\begin{aligned}\Delta(abc) = {} & \Delta(ab)c + (-1)^{|b|(|a|+1)} b\Delta(ac) + (-1)^{|a|} a\Delta(bc) \\ & -\Delta(a)bc - (-1)^{|a|} a\Delta(b)c - (-1)^{|a|+|b|} ab\Delta(c)\end{aligned}$$

is called a *GBV-algebra* (see, e.g., [Ma2]). Note that $\Delta(1) = 0$. One can check [Ma2] that the linear map

$$\begin{array}{rcl}[\bullet] : \mathcal{A} \otimes \mathcal{A} & \longrightarrow & \mathcal{A} \\ a \otimes b & \longrightarrow & [a \bullet b] := (-1)^{|a|}\Delta(ab) - (-1)^{|a|}\Delta(a) \circ b - a\Delta(b)\end{array} \tag{30}$$

makes $\mathcal{A}$ into an odd Lie superalgebra, i.e. the Jacobi identities of type (16) are satisfied. Moreover, the odd Poisson identity (17) also holds true,

$$[a \bullet (bc)] = [a \bullet b]c + (-1)^{|a|(|b|+1)} b[a \bullet c],$$

for any $a, b \in \mathcal{A}$. The operator Δ is called a BV-operator of the GBV-algebra $\mathcal{A}$.

Lemma 3.7.1. *Let $(\mathcal{M}, \omega, \Theta)$ be a BV-manifold. Then its structure sheaf is naturally a sheaf of GBV-algebras with the BV operator given by*

$$\begin{array}{rcl}\Delta_{\omega,\Theta} : \mathcal{O}_{\mathcal{M}} & \longrightarrow & \mathcal{O}_{\mathcal{M}} \\ f & \longrightarrow & \Delta_{\omega,\Theta} f := \frac{\Delta_\omega(f\Theta)}{\Theta}.\end{array}$$

Proof. Representing Θ in a local Darboux coordinate system as $e^\Gamma\sqrt{D_{x,\psi}}$, we get

$$\Delta_{\omega,\Theta} f = \frac{\Delta_0(fe^\Gamma)D_{x,\psi}}{\Theta} = \frac{(\Delta_0 f + \{\Gamma \bullet f\})\,\Theta}{\Theta} = \Delta_0 f + \{\Gamma \bullet f\}.$$

Then, for any $f \in \mathcal{O}_{\mathcal{M}}$,

$$\begin{aligned}(\Delta_{\omega,\Theta})^2 f &= \Delta_{\omega,\Theta}\left(\Delta_0 f + \{\Gamma \bullet f\}\right) \\ &= \Delta_0\left(\Delta_0 f + \{\Gamma \bullet f\}\right) + \{\Gamma \bullet (\Delta_0 f + \{\Gamma \bullet f\})\} \\ &= \left\{\left(\Delta_0 \Gamma + \frac{1}{2}\{\Gamma \bullet \Gamma\}\right) \bullet f\right\} \\ &= 0,\end{aligned}$$

so that condition (i) in the definition of a GBV algebra is satisfied. Condition (ii) can be checked analogously. □

It is easy to see that the odd Lie brackets induced on the structure sheaf $\mathcal{O}_{\mathcal{M}}$ by formula (30) coincide precisely with the Poisson brackets of the underlying odd symplectic structure.

3.8. Quantum master equation. Let $\hbar$ be a formal parameter of degree 2 and let $\mathbb{K}[[\hbar]] := \{\sum_{n\geq 0} a_n\hbar^n,\ a_n \in \mathbb{R}\}$ be the associated graded commutative ring of formal power series. The latter defines a $\mathbb{Z}$-graded manifold which we denote by $\mathbb{K}[[\hbar]]^{\vee}$. We are interested in considering $\hbar$-twisted formal smooth manifolds $\mathcal{M}^{\hbar}$ whose structure sheaves $\mathcal{O}_{\mathcal{M}^{\hbar}}$ are non-canonically isomorphic to $\mathcal{O}_{\mathcal{M}}[[\hbar]] := \mathcal{O}_{\mathcal{M}} \otimes_{\mathbb{R}} \mathbb{R}[[\hbar]]$ where $\mathcal{O}_{\mathcal{M}}$ is the structure sheaf of some $\mathbb{Z}$-graded smooth manifold $\mathcal{M}$. Such an $\hbar$-twisted manifold $\mathcal{M}^{\hbar}$ is best understood as a formal *family* of manifolds, $\pi : \mathcal{M}_{\hbar} \to \mathbb{K}[[\hbar]]^{\vee}$, over the 1-dimensional formal $\mathbb{Z}$-graded manifold $\mathbb{K}[[\hbar]]^{\vee}$. The fiber, $\mathcal{M}^0 := \pi^{-1}(\star)$, over the distinguished point $\star \in \mathbb{K}[[\hbar]]^{\vee}$ is a $\mathbb{Z}$-graded formal manifold called the *classical limit of* $\mathcal{M}^{\hbar}$.

More precisely, let us consider a category, $\mathcal{C}_{\hbar}$, whose objects are (isomorphism classes) of completed $\mathbb{Z}$-graded free $\mathbb{K}[[\hbar]]$-algebras; they are equipped with a natural monomorphism, $\pi^* : \mathbb{K}[[\hbar]] \to \mathcal{O}_{\mathcal{M}_{\hbar}}$, of $\mathbb{K}[[\hbar]]$-algebras; morphisms in this category are defined as *continuous* morphisms of topological $\mathbb{K}[[\hbar]]$-algebras commuting with the monomorphism π^*. The quotient of an algebra $\mathcal{O}_{\mathcal{M}^{\hbar}}$ by the ideal generated by $\hbar$ is denoted by $\mathcal{O}_{\mathcal{M}^0}$; this is the structure sheaf of the classical limit $\mathcal{M}^0 = \pi^{-1}(\star)$.

The opposite category $\mathcal{C}_{\hbar}^{\vee}$ is called the category of $\hbar$-twisted manifolds. As in § 3.1–3.4 on can define natural relative versions of all basic concepts — tangent sheaves, De Rham sheaves, odd Poisson structures and odd symplectic structures. For example, an $\hbar$-twisted odd symplectic manifold can be defined as an equivalence class of Darboux coordinate charts (x^a, ψ_a) modulo canonical transformation of the form,

$$\begin{aligned} x^a &\longrightarrow \hat{x}^a = \phi^a(x, \psi, \hbar), \\ \psi_a &\longrightarrow \hat{\psi}_a = \phi_a(x, \psi, \hbar), \end{aligned} \tag{31}$$

where $\phi^a(x, \psi, \hbar)$ and $\phi_a(x, \psi, \hbar)$ are formal power series from $\mathbb{K}[[x^a, \psi_a, \hbar]]$ such that Eqs. (21) hold and the Jacobian $\frac{\partial(\hat{x},\hat{\psi})}{\partial(x,\psi)}$ gives an invertible matrix at the point $(x^a = 0, \psi_a = 0, \hbar = 0)$.

Let $\mathcal{M}_{\hbar}$ be an $\hbar$-twisted $\mathbb{Z}$-graded manifold. We need a singular (with respect to $\hbar$) extension of its structure sheaf $\mathcal{O}_{\mathcal{M}^{\hbar}}$. Let us fix an isomorphism $i : \mathcal{O}_{\mathcal{M}^{\hbar}} \simeq \mathcal{O}_{\mathcal{M}^0} \otimes_{\mathbb{K}} \mathbb{K}[[\hbar]]$, and use it to extend $\mathcal{O}_{\mathcal{M}^{\hbar}}$ as follows,

$$\mathcal{O}_{\mathcal{M}^{\hbar,\hbar^{-1}}} := \left\{ \sum_{n=-\infty}^{\infty} f_n\hbar^n \in \mathcal{O}_{\mathcal{M}^0} \otimes_{\mathbb{K}} \mathbb{K}[[\hbar, \hbar^{-1}]] : f_{-n} \in I^n \text{ for } n \geq 1 \right\}, \tag{32}$$

where I is the maximal ideal in $\mathcal{O}_{\mathcal{M}^0}$. The resulting vector space has natural a $\mathbb{K}[[\hbar]]$-algebra structure extending that of $\mathcal{O}_{\mathcal{M}_{\hbar}}$; moreover, the isomorphism class of this extension is independent of a particular choice of a map i used in the definition.

Let $\mathcal{M}^{\hbar}$ be an $\hbar$-twisted odd symplectic manifold. An invertible element Θ of the sheaf $\mathcal{B}er^{1/2}(\mathcal{M}^{\hbar}) \otimes_{\mathcal{O}_{\mathcal{M}^{\hbar}}} \mathcal{O}_{\mathcal{M}^{\hbar,\hbar^{-1}}}$ is called *regular* if in some Darboux coordinate charts, (x^a, ψ_a) it can be represented in the form

$$\Theta = e^{\frac{\Gamma}{\hbar}} \sqrt{D_{x,\psi}}$$

for some function $\Gamma(x, \psi, \hbar) \in \mathcal{I}$ whose classical limit, $\Gamma|_{\hbar=0}$, lies in I. (Here and elsewhere $\mathcal{I}$ stands for the maximal ideal in the $\mathbb{K}$-algebra $\mathcal{O}_{\mathcal{M}_\hbar}$, and I for the maximal ideal of $\mathcal{O}_{\mathcal{M}^0}$.) It is clear that this notion does not depend on the choice of a Darboux coordinate chart used in the definition.

Definition 3.8.1. *A* ***quantum Batalin-Vilkovisky structure*** *on an $\hbar$-twisted odd symplectic manifold $(\mathcal{M}_\hbar, \omega)$ is a regular element $\Theta \in \mathcal{B}er^{1/2}(\mathcal{M}^\hbar) \otimes_{\mathcal{O}_{\mathcal{M}^\hbar}} \mathcal{O}_{\mathcal{M}^{\hbar,\hbar^{-1}}}$ satisfying an equation,*

$$\Delta_\omega \Theta = 0. \tag{33}$$

This equation is called a quantum master equation*, while its solution Θ a* quantum master semidensity. *In a Darboux coordinate system the quantum master equation has the form,*

$$\hbar\Delta_0\Gamma + \frac{1}{2}\{\Gamma \bullet \Gamma\} = 0. \tag{34}$$

The structure sheaf $\mathcal{O}_{\mathcal{M}^\hbar}$ can be made into a sheaf of GBV algebras with respect to the operator,

$$\begin{array}{rcl}\Delta_{\omega,\Theta} : \mathcal{O}_{\mathcal{M}^\hbar} & \longrightarrow & \mathcal{O}_{\mathcal{M}^\hbar} \\ f & \longrightarrow & \Delta_{\omega,\Theta} f := \hbar\frac{\Delta_\omega(f\Theta)}{\Theta} = \hbar\Delta_0 f + \{\Gamma \bullet f\}.\end{array} \tag{35}$$

3.8.2. Fact-definition Let $\phi : (\mathcal{M}^\hbar, \omega) \to (\hat{\mathcal{M}}^\hbar, \hat{\omega})$ be a symplectomorphism of odd symplectic manifolds, and Θ a quantum BV structure on $\mathcal{M}^\hbar$. Then $\hat{\Theta} := (\phi^{-1})^*\Theta$ (with the pullback map $(\phi^{-1})^*$ given in local coordinates by (25)) is a quantum BV structure on $\hat{\mathcal{M}}^\hbar$; such a pair, $(\mathcal{M}^\hbar, \omega, \Theta)$ and $(\hat{\mathcal{M}}^\hbar, \hat{\omega}, \hat{\Theta})$, of quantum BV structures is called *symplectomorphic*.

3.9. Quantum BV manifolds. Let V be a $\mathbb{Z}$-graded finite-dimensional vector space. Slightly abusing notations, the formal $\hbar$-twisted $\mathbb{Z}$-graded manifold corresponding to the isomorphism class of the $\mathbb{K}[[\hbar]]$-algebra $\widehat{\odot^\bullet}(V \oplus V^*[-1])^* \otimes_{\mathbb{K}} \mathbb{K}[[\hbar]]$ is denoted from now on by $\mathcal{M}_V^\hbar$ rather than by $\mathcal{M}_{V\oplus V^*[-1]}^\hbar$; it has a natural odd symplectic structure induced by the pairing between V and $V^*[-1]$. Any $\hbar$-twisted odd symplectic manifold is isomorphic to $\mathcal{M}_V^\hbar$ for some non-canonically defined vector space V. We shall consider next an extra structure — a *ordered* pair of transversal Lagrangian submanifolds in the classical $\hbar \to 0$ limit, $\mathcal{M}_V^0$, of $\mathcal{M}_V^\hbar$ — which will make the correspondence $\mathcal{M}_V^\hbar \rightleftarrows V$ canonical.

The inclusions $V \subset V \oplus V^*[-1]$ and $V^*[-1] \subset V \oplus V^*[-1]$ correspond to two transversal Lagrangian submanifolds in $\mathcal{M}_V^0$ which we denote by the symbols $\mathcal{L}_V$ and, respectively, $\mathcal{L}_V^\perp$ and consider from now as an extra part of the definition of an $\hbar$ twisted odd symplectic manifold $\pi : \mathcal{M}_V^\hbar \to \mathbb{K}[[\hbar]]^\vee$. The automorphism group of $\mathcal{M}_V^\hbar$ consists, therefore, of those symplectomorphisms $\phi : \mathcal{M}_V^\hbar \to \mathcal{M}_V^\hbar$ which leave Lagrangian submanifolds $\mathcal{L}_V$ and $\mathcal{L}_V^\perp$ in the fiber $\mathcal{M}_V^0$ over $\hbar = 0$ invariant. We can always find an *adopted* Darboux coordinate chart, (x^a, ψ_a) on $\mathcal{M}_V^\hbar$ such that the Lagrangian submanifold $\mathcal{L}_V \hookrightarrow \mathcal{M}_V^0$ is given by the equations $\psi_a = 0$ and the Lagrangian submanifold

$\mathcal{L}_V^\perp \hookrightarrow \mathcal{M}_V^0$ by the equations $x^a = 0$. Then $Aut(\mathcal{M}_V^\hbar)$ consists of canonical transformations (31) satisfying the conditions,

$$\phi^a(x,\psi,\hbar)\mid_{x=\hbar=0}= 0, \quad \phi_a(x,\psi,\hbar)\mid_{\psi=\hbar=0}= 0. \tag{36}$$

Note that the vector space V is canonically isomorphic to the Lagrangian subspace $\mathcal{T}_{\star\in\mathcal{L}_V}$ of the tangent space $\mathcal{T}_{\star\in\mathcal{M}_V}$ and hence has an invariant meaning. Moreover, the tangent space $\mathcal{T}_{\star\in\mathcal{M}_V}$ is canonically decomposed into a direct sum of Lagrangian subspaces, $\mathcal{T}_{\star\in\mathcal{L}_V}\oplus\mathcal{T}^\perp_{\star\in\mathcal{L}_V}$, so that the odd symplectic structure on $\mathcal{M}_V^0$ does indeed coincide with the one which is induced from the natural parings between $\mathcal{T}_{\star\in\mathcal{L}_V}$ and $\mathcal{T}^\perp_{\star\in\mathcal{L}_V} = \mathrm{Hom}_K(\mathcal{T}_{\star\in\mathcal{L}_V},\mathbb{K})[-1]$.

3.9.1. Fact (cf. [Schw]) Any odd symplectic manifold $\mathcal{M}^\hbar$ equipped with an ordered pair of transversal Lagrangian submanifolds $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\mathcal{M}^0$ is symplectomorphic to the odd symplectic manifold $\mathcal{M}_V^\hbar$ for some uniquely defined vector space $V := \mathcal{T}_{\star\in\mathcal{L}_1}$.

Definition 3.9.2. (i) *A* ***quantum BV manifold*** *is an $\hbar$-twisted odd symplectic manifold $\mathcal{M}_V^\hbar$ associated with some graded vector space V equipped with a quantum Batalin-Vilkovisky structure Θ such that in an adopted Darboux coordinate chart (x^a,ψ_a) one has $\Theta = e^{\frac{\Gamma}{\hbar}}D_{x,\psi}$ with "classical" and "semiclassical" parts of $\Gamma(x,\psi,\hbar) = \sum_{k\geq 0}\Gamma_k(x,\psi)\hbar^k$ satisfying the boundary conditions,*

$$\Gamma_0(x,\psi)\in I_V I_{V^\perp} \text{ and } \Gamma_1(x,\psi)\in I_V + I_{V^\perp}, \tag{37}$$

where $I_V \simeq \psi\mathbb{K}[[x,\psi]]$ and $I_{V^\perp}\simeq x\mathbb{K}[[x,\psi]]$ are the ideals of the Lagrangian submanifolds $\mathcal{L}_V$ and, respectively, $\mathcal{L}_{V^\perp}$ in the classical limit $\mathcal{M}_V^0$.

(ii) *A* symplectomorphism, $\phi : (\mathcal{M}^\hbar_{V_1},\Theta_1)\to(\mathcal{M}^\hbar_{V_2},\Theta_2)$, *of quantum BV manifolds is a symplectomorphism ϕ of the associated quantum BV structures (see §3.8.2) which respects Lagrangian submanifolds in the fibre over $\hbar = 0$, i.e.* $\lim_{\hbar\to 0}\phi(\mathcal{L}_{V_1})\subset\mathcal{L}_{V_2}$ *and* $\lim_{\hbar\to 0}\phi(\mathcal{L}^\perp_{V_1})\subset\mathcal{L}^\perp_{V_2}$.

Remarks 3.9.3. (i) The second boundary condition, $\Gamma_1(x,\psi)\in I_V + I_{V^\perp}$, says only that Γ_1 has no constant term, i.e. Γ itself has no term proportional to $\hbar$; as the quantum master equation is invariant under translations $\Gamma\to\Gamma+\mathbb{K}[[\hbar]]$, the second boundary condition is only a partial normalization condition on the quantum master function. The first boundary condition in Definition 3.9.2(i) is quite restrictive, but still allows many interesting examples such as, e.g., BF theory and its various generalizations (see, e.g., [CaRo,Mn] and also § 6).

(ii) In view of the presence of boundary conditions, the structure in §3.9.2 should be more precisely called a *quantum BV manifold with split quasi-classical limit*. We abbreviate it to simply a *quantum BV manifolds* in this paper.

3.10. Homotopy classification of quantum BV manifolds. Let $(\mathcal{M}_V^\hbar,\Theta)$ be a quantum BV manifold associated with a $\mathbb{Z}$-graded vector space V. In an adopted Darboux coordinate chart we have $\Theta = e^{\frac{\Gamma}{\hbar}}\sqrt{D_{x,\psi}}$, where, in view of boundary conditions (37), the formal power series must have the form

$$\Gamma(x,\psi,\hbar) = \underbrace{\sum_{a,b}\Gamma_{(0)}{}^a_b x^b\psi_a}_{\Gamma_0} + \underbrace{\sum_{\substack{n\geq 1,\, p+q+2n\geq 3\\ p+n\geq 2,\, q+n\geq 2}}\frac{1}{p!q!}\Gamma_{(n)}{}^{b_1\ldots b_q}_{a_1\ldots a_p}x^{a_1}\ldots x^{a_p}\psi_{b_1}\ldots\psi_{b_q}\hbar^n}_{\Gamma},$$

for some $\Gamma_{(n)\,a_1\ldots a_p}^{\quad\; b_1\ldots b_q} \in \mathbb{K}$. Quantum master equation (34) immediately implies

$$\{\Gamma_0 \bullet \Gamma_0\} = 0,$$

or, equivalently,

$$\sum_c \Gamma_{(0)\,c}^{\;\;a} \Gamma_{(0)\,b}^{\;\;c} = 0.$$

The linear functions $x^a \bmod \mathcal{I}_\hbar^2$, where $\mathcal{I}_\hbar$ is the maximal ideal in the $\mathbb{K}[[\hbar]]$-algebra $\mathbb{K}[[x^a, \psi_a, \hbar]]$, form a basis of the vector space V^*; let $\{e_a\}$ be the associated dual basis of V, and define a degree 1 map

$$\begin{array}{rcl} d : V & \longrightarrow & V \\ e_a & \longrightarrow & d(e_a) := \sum_c e_c \Gamma_{(0)\,a}^{\;\;c} \end{array}.$$

Clearly, $d^2 = 0$. Moreover, the map d does not depend on the choice of an adopted Darboux coordinate chart (x^a, ψ_a) used in its definition as the third equation in (21) implies that under a generic canonical transformation (31),

$$x^a \longrightarrow \hat{x}^a = \phi^a(x, \psi, \hbar) = \sum_b \mathcal{A}_b^a x^b + \text{other terms}, \quad \mathcal{A}_b^a \in \mathbb{K},$$

$$\psi_a \longrightarrow \hat{\psi}_a = \phi_a(x, \psi, \hbar) = \sum_b \mathcal{B}_a^b \psi_b + \text{other terms}, \quad \mathcal{B}_b^a \in \mathbb{K},$$

the leading matrices $\mathcal{A}$ and $\mathcal{B}$ must be inverse to each other. Hence the differential d on the vector space V is defined canonically and is called *the differential induced by the master semidensity* or simply *induced differential*. In this situation we say that the quantum BV manifold $(\mathcal{M}_V^\hbar, \Theta)$ *is modeled on a dg vector space* (V, d). Setting $d := \{\Gamma_0 \bullet \ldots\}$ we can rewrite the quantum master equation in the form,

$$d\Gamma + \hbar\Delta_0\Gamma + \frac{1}{2}\{\Gamma \bullet \Gamma\} = 0. \tag{38}$$

Definition 3.10.1. *A quantum BV-manifold* $(\mathcal{M}_V^\hbar, \Theta)$ *is called* ***minimal*** *if in some (and hence any) adopted Darboux coordinate system* (x^a, ψ_a) *one has* $\Theta = e^{\frac{\Gamma}{\hbar}}\sqrt{D_{x,\psi}}$ *with the quadratic part,* Γ_0*, of* Γ *vanishing, i.e. with* $\Gamma = \Gamma$.

Definition 3.10.2. *A quantum BV-manifold* $(\mathcal{M}_V^\hbar, \Theta)$ *is called* ***contractible*** *if the associated complex* (V, d) *is acyclic and there exists an adopted Darboux coordinate system* (x^a, ψ_a) *in which* $\Theta = e^{\frac{\Gamma}{\hbar}}\sqrt{D_{x,\psi}}$ *with* $\Gamma = \Gamma_0$*, i.e. with* $\Gamma = 0$.

3.10.3. Symplectomorphisms and morphisms of tangent complexes. A symplectomorphism, $\phi : (\mathcal{M}_V^\hbar, \Theta) \to (\mathcal{M}_{\hat{V}}^\hbar, \hat{\Theta})$, of quantum BV manifolds induces a linear map, $\mathcal{T}_{\star\in\mathcal{M}_V^\hbar} \to \mathcal{T}_{\star\in\mathcal{M}_{\hat{V}}^\hbar}$, of tangent spaces at the distinguished points and, as $\phi(\mathcal{L}_V) \subset \mathcal{L}_{\hat{V}}$ and $\phi(\mathcal{L}_V^\perp) \subset \mathcal{L}_{\hat{V}}^\perp$, the linear maps,

$$d\phi_\star : \mathcal{T}_{\star\in\mathcal{L}_V} = V \longrightarrow \mathcal{T}_{\star\in\mathcal{L}_{\hat{V}}} = \hat{V},$$

and

$$d\phi_\star^\perp : \mathcal{T}_{\star\in\mathcal{L}_V^\perp} = V^*[-1] \longrightarrow \mathcal{T}_{\star\in\mathcal{L}_{\hat{V}^\perp}} = \hat{V}^*[-1]$$

of the associated subspaces. The differentials d in V and $\hat{d}$ in $\hat{V}$ induce, respectively, dual differentials d^* in $V^*[-1]$ and $\hat{d}^*$ in $\hat{V}^*[-1]$.

Lemma 3.10.4. *The maps* $d\phi_\star : V \longrightarrow \hat{V}$ *and* $d\phi_\star^\perp : V^*[-1] \to \hat{V}^*[-1]$ *respect the induced differentials.*

Proof. Let (x^a, ψ_a) and $(\hat{x}^A, \hat{\psi}_A)$ be arbitrary adopted Darboux coordinate charts on odd symplectic manifolds $\mathcal{M}$ and, respectively, $\hat{\mathcal{M}}$. The map ϕ is given in these coordinates by

$$\begin{aligned} \hat{x}^A &= \textstyle\sum_a \mathcal{A}_a^A x^a + \text{higher order terms}, \quad \mathcal{A}_a^A \in \mathbb{K}, \\ \hat{\psi}_A &= \textstyle\sum_a \mathcal{B}_A^a \psi_a + \text{higher order terms}, \quad \mathcal{B}_A^a \in \mathbb{K}. \end{aligned}$$

The invertible matrix $\mathcal{A}_a^B$ (resp. $\mathcal{B}_A^a$) is a coordinate representative of the map $d\phi_\star$ (resp., $d\phi_\star^\perp$) in the associated (to a choice of Darboux coordinates) bases of V and $\hat{V}$ (resp., of $V^*[-1]$ and $\hat{V}^*[-1]$). Equality (21) implies that matrices $\mathcal{A}$ and $\mathcal{B}$ are inverse to each other; then equality (25) in the limit $\hbar \to 0$ implies

$$\sum_b \mathcal{A}_b^A \Gamma_{(0)}{}^b{}_a = \sum_B \Gamma_{(0)}{}^A{}_B \mathcal{A}_a^B, \quad \sum_b \Gamma_{(0)}{}^A{}_b \mathcal{B}_A^b = \sum_B \mathcal{B}_B^a \Gamma_{(0)}{}^B{}_A,$$

which in turn implies the required claims. □

Main Theorem 3.10.5. *Every quantum BV-manifold* $(\mathcal{M}_V^\hbar, \Theta)$ *is symplectomorphic to the product,* $(\mathcal{M}_{V_1}^\hbar, \Theta_1) \times (\mathcal{M}_{V_2}^\hbar, \Theta_2)$*, of a minimal quantum BV manifold* $(\mathcal{M}_{V_1}^\hbar, \Theta_1)$ *and a contractible one,* $(\mathcal{M}_{V_2}^\hbar, \Theta_2)$.

Proof. We shall construct by induction an adopted Darboux coordinate chart $(x^a, \psi_a)_{a\in I}$ on $\mathcal{M}_V$ in which the quantum density Θ is represented by $e^{\frac{\Gamma(x,\psi,\hbar)}{\hbar}}\sqrt{D_{x,\psi}}$ with

$$\Gamma(x,\psi,\hbar) = \underbrace{\sum_{A,B\in I'} \Gamma_A^B x^A \psi_B}_{\Gamma_0} + \underbrace{\sum_{N=3}^{\infty} \sum_{\substack{N=p+q+2n \\ p,q\geq 1, n\geq 0}} \sum_{\mathbf{b}_\bullet,\mathbf{c}_\bullet\in I''} \frac{1}{p!q!} \Gamma_{(n)\mathbf{b}_1\ldots\mathbf{b}_p}^{\mathbf{c}_1\ldots\mathbf{c}_q} x^{\mathbf{b}_1}\ldots x^{\mathbf{b}_p}\psi_{\mathbf{c}_1}\ldots\psi_{\mathbf{c}_q}\hbar^n}_{\Gamma}, \qquad (39)$$

for some partition of the labeling set $I = \{1, 2, \ldots, \dim_{\mathbb{K}} V\}$ into *disjoint* subsets $I = I' \coprod I''$. Then the data,

$$\left(\mathbb{K}[[x^A, \psi_A, \hbar]], \;\; \Theta_2 := e^{\frac{\Gamma_0}{\hbar}}\sqrt{D}_{x^A,\psi_A}\right)_{A\in I'},$$

defines a contractible BV manifold $(\mathcal{M}^{\hbar}_{V_2}, \Theta_2)$ while the data,

$$\left(\mathbb{K}[[x^{\mathbf{a}}, \psi_{\mathbf{a}}, \hbar]],\ \ \Theta_1 := e^{\frac{\Gamma}{\hbar}}\sqrt{D}_{x^{\mathbf{a}},\psi_{\mathbf{a}}}\right)_{\mathbf{a}\in I''},$$

defines a minimal BV manifold $(\mathcal{M}^{\hbar}_{V_1}, \Theta_1)$ proving thereby the Main Theorem. The complete separation of variables in (39) assures that Θ_1 and Θ_2 satisfy the corresponding quantum master equations.

The required separation of variables (39) will be achieved by induction on an integer valued parameter N starting with $N = 2$. From now on the *order* of a monomial

$$x^{a_1}\ldots x^{a_p}\psi_{b_1}\ldots\psi_{b_q}\hbar^n \in \mathbb{K}[[x,\psi,\hbar]]$$

is assumed to be $p+q+2n$ and, for an natural number N, we denote by $\mathcal{O}(N)$ the subset of $\mathbb{K}[[x,\psi,\hbar]]$ consisting of formal power series spanned by monomials of order $\geq N$. For formal power series $f, g \in \mathbb{K}[[x,\psi,\hbar]]$ the equality $f = g \mod \mathcal{O}(N)$ means equality of their polynomial parts of order strictly less than N.

As we already know, the lowest second order polynomial part,

$$\Gamma_0 := \sum_{a,b} \Gamma_{(0)}{}^a_b x^b \psi_a$$

of the master function $\Gamma(x,\psi,\hbar)$ defines a differential d in the vector space V (and hence in $V[-1]$). A choice of an adopted Darboux coordinate system (x^a, ψ_a) determines the associated basis, $\{\psi_a \mod \mathcal{I}^2_{\hbar}\}$, of $V[-1]$ and the (dual) basis, $\{x^a \mod \mathcal{I}^2_{\hbar}\}$, of V^*. As we are working over a field of characteristic zero, it is always possible to (non-canonically) represent the complex $(V[-1], d)$ as a direct sum,

$$V[-1] = H(V,d)[-1] \oplus B \oplus B[-1]$$

with the differential d given by $d(a\oplus b\oplus c) = b[-1]$. Let $\{\psi_{\mathbf{a}}\}_{\mathbf{a}\in I'}$, be a basis of $H(V,d)[-1]$, $\{\psi_\alpha\}_{\alpha\in J}$ a basis of B and $\{\psi_{\bar\alpha} := d\psi_\alpha\}_{\alpha\in J}$ the associated basis of $B[-1]$. In the basis $\{\psi_{\mathbf{a}}, \psi_\alpha, \psi_{\bar\alpha}\}$ of $V[-1]$ the differential d is given by the block-matrix

$$d = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & \mathrm{Id} & 0 \end{pmatrix}. \tag{40}$$

The above splitting induces an associated splitting of V^*, and hence an associated dual base, $\{x^{\mathbf{a}}, x^\alpha, x^{\bar\alpha}\}$ of V^*. Thus we can always find an adopted Darboux coordinate chart

$$\left(x^a, \psi_a\right) = \left(\underbrace{(x^\alpha, x^{\bar\alpha}, \psi_\alpha, \psi_{\bar\alpha})}_{(x^A,\psi_A)}, (x^{\mathbf{a}}, \psi_{\mathbf{a}})\right) \tag{41}$$

in which the master semidensity is given by

$$\begin{aligned}\Gamma(x,\psi,\hbar) &= \sum_{a,b\in I} \Gamma_{(0)}{}^a_b x^b\psi_a \mod \mathcal{O}(3)\\ &= \sum_{\alpha\in J} x^\alpha \psi_{\bar\alpha} \mod \mathcal{O}(3).\end{aligned} \tag{42}$$

Assume now that we have constructed an adopted Darboux chart (41) in which $\Gamma(x, \psi, \hbar)$ is given by (39) modulo terms of order $N+1 \geq 3$, i.e.,

$$\Gamma(x, \psi, \hbar) = \sum_{\alpha \in J} x^{\alpha} \psi_{\bar{\alpha}} + \sum_{k=3}^{N} \Gamma_k(x^{\mathbf{a}}, \psi_{\mathbf{a}}, \hbar) \quad \mathrm{mod}\ \mathcal{O}(N+1)$$

holds true for some $N \geq 2$. Here Γ_k stands for a sum of monomials of degree k. It follows from the quantum master equation (34) that the next term, $\Gamma_{N+1}(x^{\mathbf{a}}, x^A, \psi_{\mathbf{a}}, \psi_A, \hbar)$, in the Taylor expansion of Γ must satisfy an equation,

$$\{\sum_{\alpha \in J} x^{\bar{\alpha}} \psi_{\alpha} \bullet \Gamma_{N+1}\} + \hbar \Delta_0 \Gamma_{N+1} + \frac{1}{2} \sum_{\substack{p+q=N+3 \\ p,q \geq 3}} \left\{\Gamma_p \bullet \Gamma_q\right\} = 0. \tag{43}$$

The map

$$\begin{aligned} d : \mathbb{K}[[x^{\mathbf{a}}, x^A, \psi_{\mathbf{a}}, \psi_A, \hbar]] &\longrightarrow \mathbb{K}[[x^{\mathbf{a}}, x^A, \psi_{\mathbf{a}}, \psi_A, \hbar]] \\ f &\longrightarrow df := \{\sum_{\alpha \in J} x^{\alpha} \psi_{\bar{\alpha}} \bullet f\} \end{aligned}$$

is a differential which can be equivalently represented as

$$d = \sum_{\alpha \in J} \left(\psi_{\bar{\alpha}} \frac{\partial}{\partial \psi_{\alpha}} + (-1)^{|x^{\alpha}|} x^{\alpha} \frac{\partial}{\partial x^{\bar{\alpha}}} \right), \quad |\psi_{\bar{\alpha}}| = |\psi_{\alpha}| + 1,\ |x^{\alpha}| = |x^{\bar{\alpha}}| + 1.$$

As $d\Delta_0 + \Delta_0 d = 0$ and $\Delta_0^2 = 0$, the map $d + \hbar\Delta_0$ is also a differential in $\mathbb{K}[[x^{\mathbf{a}}, x^A, \psi_{\mathbf{a}}, \psi_A, \hbar]]$. Thus we can rewrite the master equation (43) in the form,

$$(d + \hbar\Delta_0)\Gamma_{N+1} = \mathbf{F}_{N+1}, \tag{44}$$

where $\mathbf{F}_{N+1} := -\frac{1}{2} \sum_{\substack{p+q=N+3 \\ p,q \geq 3}} \{\Gamma_p \bullet \Gamma_q\}$ does *not* depend (by the induction assumption) on the variables (x^A, ψ_A).

Lemma A. *The vector subspace*

$$i : \mathbb{K}[[x^{\mathbf{a}}, \psi_{\mathbf{a}}, \hbar]] \subset \mathbb{K}[[x^{\mathbf{a}}, x^A, \psi_{\mathbf{a}}, \psi_A, \hbar]]$$

is a subcomplex of the complex $(\mathbb{K}[[x^{\mathbf{a}}, x^A, \psi_{\mathbf{a}}, \psi_A, \hbar]], d + \hbar\Delta_0)$ *with the induced differential* δ *being equal to* $\hbar \sum_{\mathbf{a} \in I''} \frac{\partial^2}{\partial x^{\mathbf{a}} \partial \psi_{\mathbf{a}}}$. *The inclusion* i *is a quasi-isomorphism of complexes.*[4]

[4] It is worth pointing out that the homology of the complex $(\mathbb{K}[[x^{\mathbf{a}}, \psi_{\mathbf{a}}]], \frac{\partial^2}{\partial x^{\mathbf{a}} \partial \psi_{\mathbf{a}}})$ is a one dimensional vector space spanned over $\mathbb{K}$ by the product, $\eta = x^{\mathbf{a}'} \cdots \psi_{\mathbf{a}''}$, of all those elements of the set, $(x^{\mathbf{a}}, \psi_{\mathbf{a}})$, of generators which have degrees in $2\mathbb{Z}+1$. Hence the cohomology of the complex $(\mathbb{K}[[x^{\mathbf{a}}, \psi_{\mathbf{a}}, \hbar]], \delta)$ is equal to the direct sum $A \oplus \hbar\mathbb{K}[[\hbar]] \otimes \eta$, where A is the kernel of the operator $\frac{\partial^2}{\partial x^{\mathbf{a}} \partial \psi_{\mathbf{a}}}$ in $\mathbb{K}[[x^{\mathbf{a}}, \psi_{\mathbf{a}}]]$.

Proof of Lemma A. The inclusion i respects the filtrations,

$$\mathbb{K}[[x^{\mathbf{a}}, \psi_{\mathbf{a}}, \hbar]] \supset \hbar\mathbb{K}[[x^{\mathbf{a}}, \psi_{\mathbf{a}}, \hbar]] \supset \hbar^2\mathbb{K}[[x^{\mathbf{a}}, \psi_{\mathbf{a}}, \hbar]] \supset \ldots,$$
$$\mathbb{K}[[x^{\mathbf{a}}, x^A, \psi_{\mathbf{a}}, \psi_A, \hbar]] \supset \hbar\mathbb{K}[[x^{\mathbf{a}}, x^A, \psi_{\mathbf{a}}, \psi_A, \hbar]] \supset \hbar^2\mathbb{K}[[x^{\mathbf{a}}, x^A, \psi_{\mathbf{a}}, \psi_A, \hbar]] \supset \ldots,$$

and hence induces maps,

$$i_r : (E_r, \delta_r) \longrightarrow (\mathcal{E}_r, D_r) \;\; r \geq 0,$$

of the associated spectral sequences. The differential δ_0 vanishes while the differential D_0 is equal to d. The Poincaré Lemma (see, e.g., §3.4.5 in [Ma1]) says that the cohomology of the complex ($\mathcal{E}_0 = \mathbb{K}[[x^{\mathbf{a}}, x^A, \psi_{\mathbf{a}}, \psi_A]], D_0 = d$) is equal $\mathbb{K}[[x^{\mathbf{a}}, \psi_{\mathbf{a}}]] =: \mathcal{E}_1$. Hence the map $i_1 : (E_1, \delta_1) \to (\mathcal{E}_1, D_1)$ is obviously an isomorphism. Both spectral sequences are regular (terminating at $r = 2$), and the filtrations are complete and exhaustive. Hence by the classical Complete Convergence Theorem 5.5.10 (see p.139 in [Weib]) they both converge. Then Comparison Theorem 5.2.12 in [Weib] says that the inclusion i is a quasi-isomorphism completing the proof of Lemma A. □

Lemma B. *Every solution, $A = \sum_{k=0}^{\infty} A_k \hbar^k$, $A_k \in \mathbb{K}[[x^{\mathbf{a}}, x^A, \psi_{\mathbf{a}}, \psi_A]]$, of the equation,*

$$(d + \hbar\Delta_0)A = 0,$$

can be represented in the form

$$A = \mathbf{B} + (d + \hbar\Delta_0)C$$

for some $\mathbf{B} \in \mathbb{K}[[x^{\mathbf{a}}, \psi_{\mathbf{a}}, \hbar]]$ and $C \in \mathbb{K}[[x^{\mathbf{a}}, x^A, \psi_{\mathbf{a}}, \psi_A, \hbar]]$. Moreover, if A satisfies the boundary conditions (37), that is,

$$A_0|_{x=0} = 0, \quad A_0|_{\psi=0} = 0, \quad A_1|_{x=0} \cdot A_1|_{\psi=0} = 0,$$

and has polynomial order $N + 1 \geq 3$, then $\mathbf{B}$ can be chosen to satisfy (37) and have order $N + 1 \geq 3$ as well.

Proof of Lemma B. The first part of this lemma follows, of course, from lemma A, but we show another explicit proof which makes the second part of the Lemma immediate. We have,

$$dA_0 = 0, \;\; dA_1 = -\Delta_0 A_0, \;\; \ldots \;\; , dA_i = -\Delta_0 A_{i+1}, \;\; \ldots.$$

The Poincaré Lemma says that the cohomology of the complex $(\mathbb{K}[[x^{\mathbf{a}}, x^A, \psi_{\mathbf{a}}, \psi_A]], d)$ is equal $\mathbb{K}[[x^{\mathbf{a}}, \psi_{\mathbf{a}}]]$. Hence we get,

$$A_0 = \mathbf{B}_0 + dC_0, \;\; A_1 - \Delta_0 C_0 = \mathbf{B}_1 + dC_1, \;\; \ldots \;\; , \;\; A_i - \Delta_0 C_{i-1} = \mathbf{B}_i + dC_i, \;\; \ldots$$

for some $\mathbf{B}_i \in \mathbb{K}[[x^{\mathbf{a}}, \psi_{\mathbf{a}}]]$ and $C_i \in \mathbb{K}[[x^{\mathbf{a}}, x^A, \psi_{\mathbf{a}}, \psi_A]]$, $i = 0, 1, 2, \ldots$. Thus

$$A = \underbrace{\sum_{k=0}^{\infty} \mathbf{B}_k \hbar^k}_{\mathbf{B}} + (d + \hbar\Delta_0) \underbrace{\sum_{k=0}^{\infty} C_k \hbar^k}_{C},$$

proving the first half of Lemma B. The differentials $d + \hbar\Delta_0$ and δ preserve the polynomial order, and the splitting homotopy in the proof of the Poincare' Lemma (see p. 171

in [Ma1]) can also be chosen to be degree preserving. Thus if A has order $N+1$, then $\mathbf{B}$ and C can also be chosen to have order $N+1$ (or be zero). The boundary conditions for A imply $\mathbf{B}_0|_{x^{\mathbf{a}}=0} + d(C_0|_{x^{\mathbf{a}}}) = 0$ which, by the Poincaré Lemma, in turn implies $\mathbf{B}_0|_{x^{\mathbf{a}}=0}$. Analogously, $\mathbf{B}_0|_{\psi_{\mathbf{a}}=0}$. Note that equation $A_1|_{x=0} \cdot A_1|_{\psi=0} = 0$ says that the formal power series $A_1 \in \mathbb{K}[[x, \psi]]$ has no constant (i.e. belonging to $\mathbb{K}$) term. As C_0 is of order $N+1 \geq 3$ in x and ψ, the power series $\Delta_0 C_0$ has not a constant term as well. Then $\mathbf{B}_1$, being the part of $A_1 - \Delta_0 C_0$ which is independent of (x^A, ψ_A), has no constant term either and hence $\mathbf{B}_1|_{x=0} \cdot \mathbf{B}_1|_{\psi=0} = 0$. The proof is completed. □

Lemma C. *Every solution of Eq.* (44) *can be represented in the form,*

$$\Gamma_{N+1} = \Gamma_{N+1} + (d + \hbar\Delta_0)\Psi_{N+1}$$

for some $\Gamma_{N+1} \in \mathbb{K}[[x^{\mathbf{a}}, \psi_{\mathbf{a}}, \hbar]]$ *and* $\Psi_{N+1} \in \mathbb{K}[[x^{\mathbf{a}}, x^A, \psi_{\mathbf{a}}, \psi_A, \hbar]]$*. Moreover, if* Γ_{N+1} *satisfies the boundary conditions* (37) *and* $N+1 \geq 3$*, then* Γ_{N+1} *also satisfies boundary conditions* (37)*.*

Proof of Lemma C. If an element $\mathbf{F}_{N+1} \in \mathbb{K}[[x^{\mathbf{a}}, \psi_{\mathbf{a}}, \hbar]]$ is $(d+\hbar\Delta_0)$-exact in $\mathbb{K}[[x^{\mathbf{a}}, x^A, \psi_{\mathbf{a}}, \psi_A, \hbar]]$, then, by Lemma A, it is δ-exact, i.e.

$$\mathbf{F}_{N+1} = \delta \mathbf{G}_{N+1}$$

for some $\mathbf{G}_{N+1} \in \mathbb{K}[[x^{\mathbf{a}}, \psi_{\mathbf{a}}, \hbar]]$, and we can rewrite (44) in the form,

$$(d + \hbar\Delta_0)(\Gamma_{N+1} - \mathbf{G}_{N+1}) = 0.$$

Then the claim follows from Lemma B. □

We continue with an inductive proof of the Main Theorem. Our task now is to show that one can further adjust a Darboux coordinate chart (41) in such a way that decomposition (39) holds true $\bmod \mathcal{O}(N+2)$, i.e.

$$\Gamma(x, \psi, \hbar) = \sum_{\alpha \in J} x^\alpha \psi_{\bar{\alpha}} + \sum_{k=3}^{N+1} \Gamma_k(x^{\mathbf{a}}, \psi_{\mathbf{a}}, \hbar) \mod \mathcal{O}(N+2). \tag{45}$$

Let $\Phi_{N+1} \in \mathcal{O}_{\mathcal{M}_V} \simeq \mathbb{K}[[x^{\mathbf{a}}, x^A, \psi_{\mathbf{a}}, \psi_A, \hbar]]$ have degree 1 and order $N+1$. The associated degree 0 Hamiltonian vector field $H_{\Phi_{N+1}}$ on $\mathcal{M}_V$ generates a one parameter family of canonical transformations (see §3.4.4) which makes sense at $t=1$,

$$x^a \longrightarrow \hat{x}^a = \phi^a(x, \psi, \hbar), \quad \psi_a \longrightarrow \hat{\psi}_a = \phi_a(x, \psi, \hbar)$$

and induces the following change of the coordinate representation of the master function,

$$e^{\frac{\hat{\Gamma}(\hat{x}, \hat{\psi}, \hbar)}{\hbar}}\Big|_{\substack{\hat{x}=\phi(x,\psi,\hbar) \\ \hat{\psi}=\phi(x,\psi,\hbar)}} = \left(Ber\left(\frac{\partial(\hat{x}, \hat{\psi})}{\partial(x, \psi)} \right) \right)^{-1/2} e^{\frac{\Gamma(x,\psi,\hbar)}{\hbar}}. \tag{46}$$

Equations (22) for the symplectomorphism generated by $H_{\Phi_{N+1}}$ imply,

$$\hat{x}^a = x^a - (-1)^{|x^a|} \frac{\partial \Phi_{N+1}}{\partial \psi_a} \mod \mathcal{O}(N+1),$$
$$\hat{\psi}_a = \psi_a + (-1)^{|x^a|} \frac{\partial \Phi_{N+1}}{\partial x^a} \mod \mathcal{O}(N+1)$$

so that

$$\begin{aligned}\hat{\Gamma}(\hat{x}, \hat{\psi}, \hbar) &= \hat{\Gamma}(x, \psi, \hbar) + \{\Phi_{N+1} \bullet \hat{\Gamma}\} \mod \mathcal{O}(N+2) \\ &= \hat{\Gamma}(x, \psi, \hbar) - \{\sum_{\alpha \in J} x^{\alpha} \psi_{\bar{\alpha}} \bullet \Phi_{N+1}\} \mod \mathcal{O}(N+2) \\ &= \hat{\Gamma}(x, \psi, \hbar) - d\Phi \mod \mathcal{O}(N+2),\end{aligned}$$

and

$$\begin{aligned}Ber\left(\frac{\partial(\hat{x}, \hat{\psi})}{\partial(x, \psi)}\right) &= 1 + \sum_a \left((-1)^{|x^a|} \frac{\partial}{\partial x^a}\left(-(-1)^{|x^a|}\frac{\partial \Phi}{\partial \psi_a}\right)\right. \\ &\quad \left. + (-1)^{|\psi_a|}\frac{\partial}{\partial \psi_a}\left((-1)^{|x^a|}\frac{\partial \Phi}{\partial x^a}\right)\right), \\ &= 1 - 2\Delta_0 \Phi,\end{aligned}$$

where we used a well-known fact that $Ber(1+X) = 1+Str(X)$ modulo higher order polynomials in entries of X. Thus Eq. (46) says that $\hat{\Gamma}(x, \psi, \hbar) = \Gamma(x, \psi, \hbar) \mod \mathcal{O}(N+1)$ and

$$\hat{\Gamma}_{N+1} - d\Phi_{N+1} = \Gamma_{N+1} + \hbar\Delta_0\Phi.$$

Representing Γ_{N+1} as in Lemma C, we obtain,

$$\hat{\Gamma}_{N+1} = \Gamma_{N+1} + (d + \hbar\Delta_0)(\Phi_{N+1} + \Psi_{N+1}),$$

and conclude that by choosing $\Phi_{N+1} = -\Psi_{N+1}$ we can always adjust the adopted Darboux coordinate system in such a way that separation of variables (39) holds true $\mathrm{mod}\,\mathcal{O}(N+2)$. The induction completes proof of the Main Theorem. □

3.11. Quantum morphisms of BV manifolds. A *quantum morphism*,

$$\phi_\hbar : \left(\mathcal{M}_V^\hbar, \omega, \Theta\right) \longrightarrow \left(\mathcal{M}_{\hat{V}}^\hbar, \hat{\omega}, \hat{\Theta}\right)$$

of quantum BV manifolds is, by definition, a morphism of dg $\mathbb{K}[[\hbar]]$-modules (see (35)),

$$\phi_\hbar^* : \left(\mathcal{O}_{\mathcal{M}_{\hat{V}}^\hbar}, \Delta_{\hat{\omega},\hat{\Theta}}\right) \longrightarrow \left(\mathcal{O}_{\mathcal{M}_V^\hbar}, \Delta_{\omega,\Theta}\right)$$

inducing in the classical limit $\hbar \to 0$ a morphism of algebras, $\phi_0^* : \mathcal{O}_{\mathcal{M}_{\hat{V}}^0} \to \mathcal{O}_{\mathcal{M}_V^0}$ which preserves the ideals of the distinguished Lagrangian submanifolds in $\mathcal{M}_V^0$ and $\mathcal{M}_{\hat{V}}^0$.

It is easy to see that any quantum morphism $\phi_\hbar : \left(\mathcal{M}_V^\hbar, \omega, \Theta\right) \longrightarrow \left(\mathcal{M}_{\hat{V}}^\hbar, \hat{\omega}, \hat{\Theta}\right)$ induces a morphism, $d\phi_0 : (V, d) \to (\hat{V}, \hat{d})$, of the associated tangent complexes; such a morphism is called a *quasi-isomorphism* if the map $d\phi_0$ induces an isomorphism of the associated cohomology groups. Note that a quantum morphism is a morphism of algebras only in the classical limit; therefore, in general, it is not a morphism of smooth manifolds and can not be characterized in local coordinates (i.e. in terms of generators of the structure sheaves). Let us denote by $\widehat{Cat}(BV)$ the category of quantum BV manifolds associated to the above definition of quantum morphisms.

Examples 3.11.1. (i) *Symplectomorphisms* of quantum BV manifolds are obviously quantum morphisms.

(ii) Natural *projections*,

$$\phi_\hbar : \mathcal{M}_V^\hbar \times \mathcal{M}_{\hat{V}}^\hbar \longrightarrow \mathcal{M}_V^\hbar$$

are obviously quantum morphisms.
The above two examples are special in the sense that the associated maps of dg $\mathbb{K}[[\hbar]]$-modules, $\phi_\hbar^* : (\mathcal{O}_{\mathcal{M}_{\hat{V}}^\hbar}, \Delta_{\hat{\omega},\hat{\Theta}}) \to (\mathcal{O}_{\mathcal{M}_V^\hbar}, \Delta_{\omega,\Theta})$, are maps of $\mathbb{K}[[\hbar]]$-algebras. The next example does *not* have this property in general.

(iii) Let $\left(\mathcal{M}_V^\hbar, \omega, \Theta\right)$ and $\left(\mathcal{M}_{\hat{V}}^\hbar, \hat{\omega}, \hat{\Theta}\right)$ be quantum BV manifolds. It is a well-known and very useful fact [Schw] that, for a Lagrangian submanifold $\mathcal{L}^\hbar \subset \mathcal{M}_{\hat{V}}^\hbar$, the associated integration map

$$\begin{array}{rcl}\phi_\hbar^* : \mathcal{O}_{\mathcal{M}_V^\hbar \times \mathcal{M}_{\hat{V}}^\hbar} & \longrightarrow & \mathcal{O}_{\mathcal{M}_V^\hbar} \\ f & \longrightarrow & \int_{\mathcal{L}^\hbar} f\hat{\Theta}\end{array}$$

satisfies,

$$\begin{aligned}\phi_\hbar^*\left((\Delta_{\omega,\Theta} + \Delta_{\hat{\omega},\hat{\Theta}})f\right) &= \int_{\mathcal{L}^\hbar} \left((\Delta_{\omega,\Theta} + \Delta_{\hat{\omega},\hat{\Theta}})f\right)\hat{\Theta} \\ &= \hbar \int_{\mathcal{L}^\hbar} \frac{\Delta_\omega (f\Theta)}{\Theta}\hat{\Theta} + \hbar \int_{\mathcal{L}^\hbar} \Delta_{\hat{\omega}}\left(f\hat{\Theta}\right) \\ &= \hbar \frac{\Delta_\omega\left(\int_{\mathcal{L}^\hbar} f\hat{\Theta}\right)}{\Theta} \\ &= \Delta_{\omega,\Theta}\phi_h^*(f),\end{aligned}$$

and is, therefore, a quantum morphism provided the integral exists as a perturbative series in $\hbar$. We shall see § 6 that such a *quantum embedding*

$$\phi_h : \mathcal{M}_V^\hbar \longrightarrow \mathcal{M}_V^\hbar \times \mathcal{M}_{\hat{V}}^\hbar$$

can always be constructed (as a well-defined formal power series in $\hbar$ satisfying the algebra morphism condition in the limit $\hbar \to 0$) in the case when the quantum manifold $\mathcal{M}_{\hat{V}}^\hbar$ is contractible; in the latter case the quantum embedding is also called *contractible*; such quantum embeddings are often given by Feynman type sums over decorated graphs.

Proposition 3.11.2. *For any quantum BV manifold $\mathcal{M}_V^\hbar$ and any decomposition, $\mathcal{M}_V^\hbar \simeq \mathcal{M}_{min}^\hbar \times \mathcal{M}_{cntr}^\hbar$, into a product of a minimal quantum BV manifold and a contractible one, there exists a contractible quantum embedding,*

$$\phi_h : \mathcal{M}_{min}^\hbar \longrightarrow \mathcal{M}_V^\hbar,$$

such that $\pi_\hbar \circ \phi_\hbar = \mathrm{Id}$, *where* $\pi_\hbar$ *is the composition* $\mathcal{M}_V^\hbar \xrightarrow{\simeq} \mathcal{M}_{min}^\hbar \times \mathcal{M}_{cntr}^\hbar \to \mathcal{M}_{min}^\hbar$.

We shall prove this statement in § 6 below by giving an explicit formula for $\phi_\hbar$. This fact has an important corollary which we discuss next.

Let $Cat(BV)$ be the full subcategory of $\widehat{Cat}(BV)$ whose class of morphisms consists, by definition, of all possible compositions of symplectomorphisms, projections and contractible quantum embeddings. Then Theorem 3.10.5 and Proposition 3.11.2 imply that quasi-isomorphisms in this category are equivalence relations. Therefore, in the homotopy theory sense, the category $Cat(BV)$ is as good as, for example, the famous category of strong homotopy Lie algebras [Ko, St].

4. From Unimodular Lie 1-Bialgebras to Quantum BV Manifolds

4.1. Lie n-bialgebras. [Me2, Me4]. A *Lie n-bialgebra* is a graded vector space V, equipped with linear maps,

$$\Delta : V \to V \wedge V \quad \text{and} \quad [\,\bullet\,] : \wedge^2(V[-n]) \to V[-n],$$

such that

- the data (V, δ) is a Lie coalgebra;
- the data $(V[-n], [\,\bullet\,])$ is a Lie algebra;
- the compatibility condition,

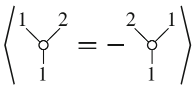
$$\begin{aligned}\Delta[a \bullet b] = \sum a_1 \otimes [a_2 \bullet b] + [a \bullet b_1] \otimes b_2 \\ + (-1)^{|a||b|+n|a|+n|b|}([b \bullet a_1] \otimes a_2 + b_1 \otimes [b_2 \bullet a]),\end{aligned}$$

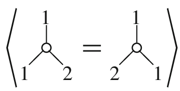
  holds for any $a, b \in V$. Here $\Delta a =: \sum a_1 \otimes a_2$, $\Delta b =: \sum b_1 \otimes b_2$.

The case $n = 0$ gives us the ordinary definition of Lie bialgebra [Dr]. The case $n = 1$ is of most interest to us in this paper as it controls Poisson geometry [Me2] and, with unimodularity conditions added, controls the category of quantum BV manifolds (see § 4.3 below). Note that in this case one has $\wedge^2(V[-1]) = (\odot^2 V)[-2]$ so that the brackets $[\,\bullet\,]$ describe a degree 1 linear map $\odot^2 V \to V$.

4.1.1. Wheeled prop(erad) of Lie 1-bialgebras. This is a wheeled prop(erad), $\mathcal{L}ie^1\mathcal{B}^{\circlearrowright} := \mathcal{F}^{\circlearrowright}\langle E\rangle / < \mathcal{R} >$, defined as the quotient of the free wheeled prop(erad) generated by an $\mathbb{S}$-bimodule

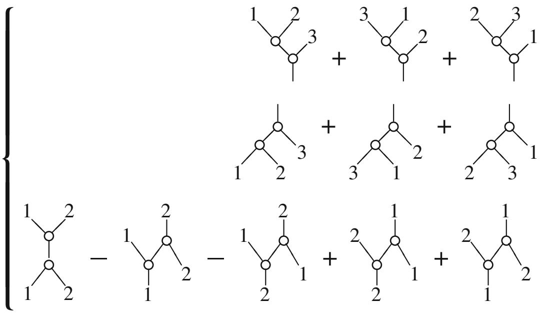
$E(m,n) :=$ (47)

- $sgn_2 \otimes \mathbf{1}_1 \equiv$ span  if $m = 2, n = 1$,

- $\mathbf{1}_1 \otimes \mathbf{1}_2[-1] \equiv$ span  if $m = 1, n = 2$,

- 0 otherwise,

by the ideal generated by the relations

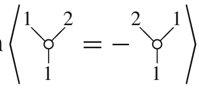
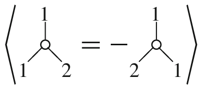
$\mathcal{R}:$

$\in \mathcal{F}^{\circlearrowright}_{(2)}\langle E\rangle(3,1)$
$\in \mathcal{F}^{\circlearrowright}_{(2)}\langle E\rangle(1,3)$
$\in \mathcal{F}^{\circlearrowright}_{(2)}\langle E\rangle(2,2).$ (48)

It is clear from the association

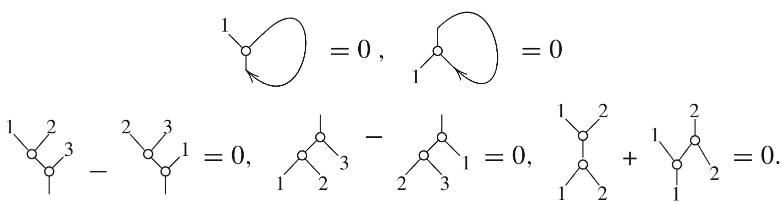
$\Delta \leftrightarrow$  , $[\bullet] \leftrightarrow$ 

that there is a one-to-one correspondence between representations of $\mathcal{L}ie^1\mathcal{B}^{\circlearrowright}$ in a finite dimensional space V and Lie 1-bialgebra structures in V.

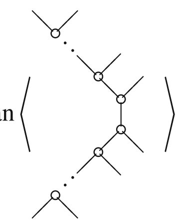
4.1.2. Cobar construction on the Koszul dual coproperad $(\mathcal{L}ie^1\mathcal{B}^{\circlearrowright})^{¡}$ It follows from the exact sequence (12) that the Koszul dual wheeled properad $(\mathcal{L}ie^1\mathcal{B}^{\circlearrowright})^!$ is the quotient, $\mathcal{F}^{\circlearrowright}\langle E^\vee\rangle / < \mathcal{R}^\perp >$, of the free wheeled prop(erad) generated by the $\mathbb{S}$-bimodule,

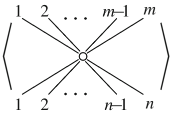
$E^\vee(m,n) :=$
$sgn_2 \otimes \mathbf{1}_1[1] \equiv \mathrm{span}$  if $m=2, n=1,$
$\mathbf{1}_1 \otimes sgn_2[1] \equiv \mathrm{span}$  if $m=1, n=2,$ (49)
0 otherwise,

by the ideal generated by relations

$\mathcal{R}^\perp:$


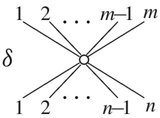
Thus

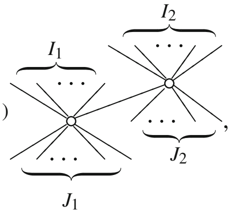
$(\mathcal{L}ie^1\mathcal{B}^{\circlearrowright})^{¡}(m,n) \simeq (\mathcal{L}ie^1\mathcal{B}^{\circlearrowright})^!(m,n) = sgn_m \otimes sgn_n[m+n-2] = \mathrm{span}$  , and, in accordance with § 2.6, the dg free wheeled prop $\mathcal{L}ie^1\mathcal{B}_\infty^{\circlearrowright} := B^c((\mathcal{L}ie^1\mathcal{B}^{\circlearrowright})^{¡})$ is generated by the $\mathbb{S}$-bimodule,

$$\mathsf{w}(\mathcal{L}ie^1\mathcal{B}^{\circlearrowright})^{¡}(m,n) = sgn_m \otimes 1\!\!1_n[m-2] = \text{span}\Big\langle$$ 

$$\Big\rangle, \quad m,n \geq 1, m+n \geq 3, \qquad (50)$$

and its differential is given on the generating corollas by (cf. [Me2])

$$\delta$$ 

$$= \sum_{\substack{[1,\ldots,m]=I_1\sqcup I_2\\ |I_1|\geq 0, |I_2|\geq 1}} \sum_{\substack{[1,\ldots,n]=J_1\sqcup J_2\\ |J_1|\geq 1, |J_2|\geq 1}} (-1)^{\sigma(I_1\sqcup I_2)+|I_1|(|I_2|+1)}$$ 

$$, \qquad (51)$$

where $\sigma(I_1 \sqcup I_2)$ is the sign of the shuffle $[1,\ldots,m] = I_1 \sqcup I_2$. It is easy to see that representations of $\mathcal{L}ie^1\mathcal{B}_\infty^{\circlearrowright}$-algebras in a finite-dimensional vector space V are in one-to-one correspondence with graded pointed formal Poisson structures on V, that is, total degree 2 polyvector fields, $\pi \in \wedge^{\bullet\geq 1}\mathcal{T}_V$, which satisfy the Schouten equations $[\pi,\pi]_S = 0$ and vanish at the distinguished point $0 \in V$ (cf. [Me2,Me4] and § 4.3 below).

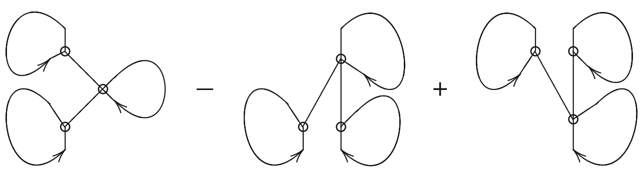
4.1.3. Non-Koszulnes of $\mathcal{L}ie^1\mathcal{B}^{\circlearrowright}$. Let $\mathcal{L}ie^1\mathcal{B}_\infty$ be a subcomplex of the complex $\mathcal{L}ie^1\mathcal{B}_\infty^{\circlearrowright}$ spanned by graphs with no closed directed paths, i.e with no wheels. This subset has an obvious structure of an ordinary prop and, in fact, is a minimal resolution of the ordinary prop, $\mathcal{L}ie^1\mathcal{B}$, of Lie 1-bialgebras (which is defined by the same generators (47) and relations (48) as $\mathcal{L}ie^1\mathcal{B}^{\circlearrowright}$ but in the category of ordinary props). The natural epimorphism,

$$\pi : (\mathcal{L}ie^1\mathcal{B}_\infty, \delta) \longrightarrow (\mathcal{L}ie^1\mathcal{B}, 0)$$

which sends to zero all generating (m,n)-corollas (50) except those with $m+n=3$, is a quasi-isomorphism [Me2,Me3]. This means that the prop $\mathcal{L}ie^1\mathcal{B}$ is Koszul in the category of *ordinary* props. The wheelification functor from the category of ordinary props to the category of wheeled props [MMS] sends these two props into precisely $\mathcal{L}ie^1\mathcal{B}^{\circlearrowright}$ and $\mathcal{L}ie^1\mathcal{B}_\infty^{\circlearrowright}$, and the above morphism π into the associated morphism of dg wheeled props,

$$\pi^{\circlearrowright} : (\mathcal{L}ie^1\mathcal{B}_\infty^{\circlearrowright}, \delta) \longrightarrow (\mathcal{L}ie^1\mathcal{B}^{\circlearrowright}, 0).$$

The morphism $\pi^{\circlearrowright}$ is *not*, however, a quasi-isomorphism: the following element [Me3]:

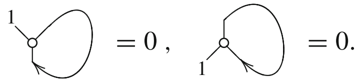

$$\in \mathcal{L}ie^1\mathcal{B}_\infty^{\circlearrowright} \qquad (52)$$

gives a non-trivial cohomology class in $H(\mathcal{L}ie^1\mathcal{B}^{\circlearrowright}_\infty, \delta)$ which is, however, sent to zero under $\pi^{\circlearrowright}$. This means that the wheeled prop of Lie 1-bialgebras is *not* Koszul, and its minimal resolution, $(\mathcal{L}ie^1\mathcal{B}^{\circlearrowright})_\infty$ is larger than $\mathcal{L}ie^1\mathcal{B}^{\circlearrowright}_\infty$. Representations of $(\mathcal{L}ie^1\mathcal{B}^{\circlearrowright})_\infty$ in a vector space V are called formal *wheeled Poisson structures*; these (at present mysterious) structures are Maurer-Cartan elements of a certain L_∞ algebra[5] which, in accordance with the general theory of [MeVa], is canonically associated to $(\mathcal{L}ie^1\mathcal{B}^{\circlearrowright})_\infty$ and which involve not only Schouten brackets but also divergence operators; it was proven in [Me4] that wheeled Poisson structures can be deformation quantized over $\mathbb{Q}$.

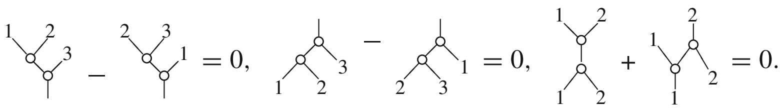
4.2. Wheeled prop, $\mathcal{UL}ie^1\mathcal{B}$, of unimodular Lie 1-bialgebras. A finite dimensional Lie 1-bialgebra V is called unimodular if, for any $e \in V$ and $e^* \in V^*$, the supertraces of linear maps,

$$Ad_e : \begin{array}{ccc} V & \longrightarrow & V \\ v & \longrightarrow & [e \bullet v] \end{array} \quad \text{and} \quad Ad_{e^*} : \begin{array}{ccc} V^* & \longrightarrow & V^* \\ v^* & \longrightarrow & [e^*, v^*] \end{array},$$

are zero. Here [,] are the Lie brackets on V^* induced by Lie coalgebra structure on V. The wheeled prop(erad), $\mathcal{UL}ie^1\mathcal{B}$ of unimodular Lie 1-bialgebras is a quotient of the free wheeled prop(erda) generated by the $\mathbb{S}$-bimodule (47) by the ideal generated by relations (48) and the following ones:

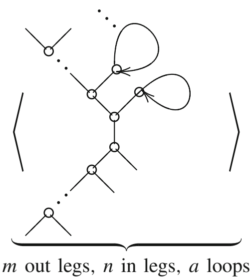


Hence the Koszul dual properad, $(\mathcal{UL}ie^1\mathcal{B})^!$, is a quadratic wheeled properad generated by the $\mathbb{S}$-bimodule (49) modulo the relations,



Therefore,

$$(\mathcal{UL}ie^1\mathcal{B})^!(m,n) = \bigoplus_{a=0}^{\infty} sgn_m \otimes sgn_n[m+n-2-2a] = \text{span}\left\langle \quad \right\rangle.$$



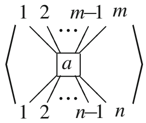
Note that the graph on the r.h.s. above is zero unless $\mathbb{Z}_{\geq 0}$-valued parameters m, n and a satisfy inequalities,

$$m + n + 2a \geq 3, m + a \geq 1, n + a \geq 1.$$

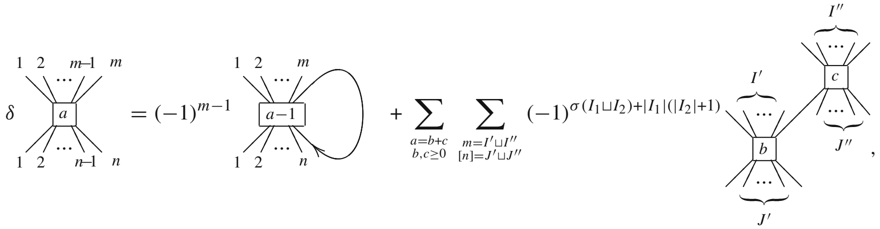
[5] Graph (52) gives, in fact, an explicit formula for a particular μ_3 composition in that L_∞ algebra.

Hence the dg free wheeled prop $\mathcal{U}\mathcal{L}ie^1\mathcal{B}_\infty := B^c((\mathcal{U}\mathcal{L}ie^1\mathcal{B})^{¡})$ is generated by an $\mathbb{S}$-bimodule,

$$\mathsf{w}(\mathcal{U}\mathcal{L}ie^1\mathcal{B})^{¡}(m,n) = \bigoplus_{a\geq 0}^{\infty} sgn_m \otimes \mathbf{1}_n[m-2-2a] = \mathrm{span}\left\langle \quad \right\rangle_{\substack{m+n+2a\geq 3\\ m+a\geq 1, n+a\geq 1}} \tag{53}$$



Definition of the cobar construction given in § 2.5 gives, after straightforward computations, the following formula for the differential in $\mathcal{U}\mathcal{L}ie^1\mathcal{B}_\infty$:



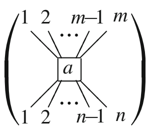
(54)

where $\sigma(I_1 \sqcup I_2)$ is the sign of the shuffle $[1,\ldots,m] = I_1 \sqcup I_2$.

4.3. Representations of $\mathcal{U}\mathcal{L}ie^1\mathcal{B}_\infty$ and quantum BV manifolds. Let (V,d) be a finite-dimensional dg vector space, and $\mathcal{M}^{\hbar}_{V^*}$ the formal $\hbar$-twisted odd symplectic manifold corresponding to the graded commutative ring $\widehat{\odot^\bullet}(V^*\oplus V[-1])[[\hbar]]$, $\hbar$ being the formal variable of degree 2 (see § 3.9).

An arbitrary morphism $\rho: \mathcal{U}\mathcal{L}ie^1\mathcal{B}_\infty \to \mathcal{E}nd_V$ is uniquely defined by its values on the generators,

$$\rho^{(a)}_{m,n} := \rho\left(\quad\right) \in \mathrm{Hom}(\odot^n V, \wedge^m V[2-m-2a])$$



$$= \odot^n V^* \otimes \wedge^m V[2-m-2a]) \subset \mathcal{E}nd_V(m,n).$$

We assemble the collection of linear maps, $\{\rho^{(a)}_{m,n}\}$, into one "generating" degree 0 function,

$$\Gamma := \sum_{a,m,n\geq 0} \rho^{(a)}_{m,n}\hbar^a \in \mathcal{O}_{\mathcal{M}_V}.$$

Let $\{e_a\}$ be an arbitrary basis in V, and $\{x^a, \psi_a\}$ the associated basis in $V^*\oplus V[-1]$, $|\psi_a| = -|x^a| - 1 = |e_a| - 1$. Then

$$\rho^{(a)}_{m,n}(e_{b_1},\ldots,e_{b_n}) = \sum_{a_1,\ldots,a_m} \mu^{\alpha_1\ldots\alpha_m}_{b_1\ldots b_n}\psi_{a_1}\ldots\psi_{a_m},$$

$$d(e_b) = \sum_b d^a_b\psi_a,$$

for some $\mu^{\alpha_1...\alpha_m}_{b_1...b_n} \in \mathbb{K}$, $d^a_b \in \mathbb{K}$, and we set

$$\Gamma := d + \Gamma = \sum_{a,b} d^a_b x^b \psi_b + \sum_{m+n\geq 3} \frac{1}{m!n!} \sum_{\substack{a_1,\ldots,a_m\\ b_1,\ldots,b_n}} \mu^{a_1...a_m}_{b_1...b_n} x^{b_1} \cdots x^{b_n} \psi_{a_1} \ldots \psi_{a_m} \in \mathcal{O}_{\mathcal{M}_V}.$$

It is a straightforward calculation to check using formula (54) that the compatibility of the morphism ρ with the differentials,

$$\rho \circ \delta = d \circ \rho$$

is equivalent to the equation,

$$\Delta_0 \Gamma + \frac{1}{2}[\Gamma \bullet \Gamma]_S = 0,$$

where $\Delta_0 = \sum \partial^2/\partial x^a \partial \psi_a$ and $[\ \bullet\ ]_S$ stand for the odd Poisson brackets on $\mathcal{M}^{\hbar}_{V^*}$. The function Γ satisfies the boundary conditions (37) of the definition §3.9.2. Hence we have proven the following

Proposition 4.3.1. *There is a one-to-one correspondence between representations of the dg wheeled prop* $(\mathcal{ULie}^1\mathcal{B}_\infty, \delta)$ *in a dg vector space* V *and quantum BV structures on the formal odd symplectic manifold* $\mathcal{M}^{\hbar}_{V^*}$.

Thus the category of quantum BV manifolds is controlled by a surprisingly simple quadratic wheeled prop, $\mathcal{ULie}^1\mathcal{B}$, of unimodular Lie 1-bialgebras.

Remark 4.3.2. We do not know at present whether or not the wheeled properad $\mathcal{ULie}^1\mathcal{B}$ is Koszul, i.e. whether or not the natural epimorphism,

$$\pi^{\circlearrowright} : (\mathcal{ULie}^1\mathcal{B}_\infty, \delta) \longrightarrow (\mathcal{ULie}^1\mathcal{B}, 0),$$

is a quasi-isomorphism. Our study of the category of quantum BV manifolds in § 3 was partly motivated by this open problem. If it is Koszul, then *unimodular* Poisson structures can be deformation quantized over $\mathbb{Q}$ with the help of the wheeled prop quantization machine developed in [Me4].

5. Wheeled dg Prop of Unimodular Poisson Structures

5.1. Modular volume form. Let M be a $\mathbb{Z}$-graded manifold. A Poisson structure on M is a Maurer-Cartan element, $\pi \in \wedge^\bullet \mathcal{T}_M$, in the Schouten Lie algebra on M, that is, a total degree 2 polyvector field, satisfying the equation $[\pi \bullet \pi]_S = 0$. If M is concentrated in degree 0, then π must be a bivector field, but in general π might have non-zero summands lying in $\wedge^n \mathcal{T}_M$ with $n \neq 2$. Let $\mathcal{M}$ be the total space of the bundle, Ω^1_M, of 1-forms on M. Then a polyvector field π defines a function on $\mathcal{M}$ which we denote by the same letter; the Schouten equations translate into $\{\pi \bullet \pi\} = 0$, where $\{\ \bullet\ \}$ are the odd Poisson brackets associated with the canonical odd symplectic structure on $\mathcal{M}$ (see § 3.3). The Poisson structure π gives rise to the associated degree 1 hamiltonian vector field, H_π, on $\mathcal{M}$ which is homological, i.e. $[H_\pi, H_\pi] = H_{\{\pi\bullet\pi\}} = 0$. Any volume form $\nu \in Ber(M)$, induces, via the canonical isomorphism $Ber(\mathcal{M}) = (Ber(M))^{\otimes 2}$, a volume form on $\mathcal{M}$ which we denote by $\hat{\nu}$.

Definition 5.1.1. *[Wein]. Let (M, π) be a $\mathbb{Z}$-graded Poisson manifold. A volume form $\nu \in Ber(M)$ is called* modular *if the equation,*

$$\mathcal{L}_{H_\pi} \hat{\nu} = 0,$$

*is satisfied. In this case π is called a **unimodular Poisson structure on** (M, ν).*

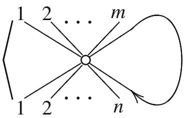
Any vector space V (viewed as a linear formal manifold) admits a translation invariant Berezin volume form, ν_0, which is defined uniquely up to multiplication by a non-zero constant. A formal Poisson structure π on (V, ν_0) is called a *unimodular Poisson structure on V*. If $\{x^a\}$ are linear coordinates on V, then a unimodular Poisson structure on V is given by an ordinary Poisson structure on V,

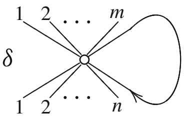
$$\pi := \sum_{n \geq 1} \sum_{a_1, \ldots, a_n} \pi^{a_1, \ldots, a_n}(x) \psi_{a_1} \ldots \psi_{a_n} \in \mathcal{O}_{\Omega^1_V},$$

with coefficients $\pi^{a_1, \ldots, a_n}(x)$ satisfying an extra condition

$$\sum_b \frac{\partial \pi^{b a_2, \ldots, a_n(x)}}{\partial x^b} = 0, \quad \forall n \geq 1.$$

5.2. Wheeled dg prop of unimodular Poisson structures. Let $I^{\circlearrowright}$ be the ideal in the dg wheeled prop $\mathcal{U}\mathcal{L}ie^1\mathcal{B}_\infty$ (see §4.1.2) generated by loops,

$I^{\circlearrowright} := \Big\langle$  $\Big\rangle$.

Lemma 5.2.1. δ  $\in I^{\circlearrowright}$.

Proof is a straightforward calculation based on formula (51).
Thus $I^{\circlearrowright}$ is a *dg* ideal in $\mathcal{U}\mathcal{L}ie^1\mathcal{B}_\infty$, and the quotient prop,

$$\mathcal{U}\mathcal{P}oisson := \mathcal{U}\mathcal{L}ie^1\mathcal{B}_\infty / I^{\circlearrowright},$$

is a *dg* wheeled prop whose representations in a dg vector space V are in one-to-one correspondence with formal unimodular Poisson structures, $\pi \in \wedge^\bullet \mathcal{T}_V$, which vanish at $O \in V$.

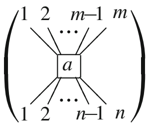
Remark 5.2.2. Every free wheeled prop has a natural filtration by the number of vertices. For applications to homological algebra and differential geometry one is often interested in *completed* (with respect to this filtration) *topological* props, and in *continuous* morphisms between them [Me4, Me7, MeVa]).

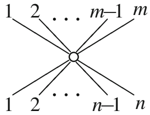
In the next section we shall assume that both dg props $\mathcal{U}\mathcal{L}ie^1\mathcal{B}_\infty$ and $\mathcal{U}\mathcal{P}oisson$ are completed with respect to the filtration by the number of vertices.

5.3. Quasi-isomorphism theorem. A continuous morphism of dg wheeled topological props,

$$F : \mathcal{U}\mathcal{L}ie^1\mathcal{B}_\infty \longrightarrow \mathcal{U}\mathcal{P}oisson,$$

given on the generators by the formula

$F\Big($ 

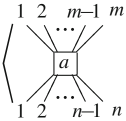
$\Big) = \begin{cases}$ 

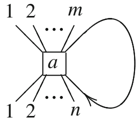
for $a = 0$, $\\ 0$ *otherwise*, $\end{cases}$

is a quasi-isomorphism.

Proof. The prop $\mathcal{U}\mathcal{L}ie^1\mathcal{B}_\infty$ is generated by the $\mathbb{S}$-module (53). Let us enlarge the latter non-differential $\mathbb{S}$-bimodule to a *dg* $\mathbb{S}$-bimodule, $(E = \{E(m,n)\}, d_0)$, given by

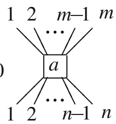
$$E(m,n) := \bigoplus_{a\geq 0}^{\infty} (sgn_m \otimes \mathbf{1}_n[m-2-2a] \oplus sgn_m \otimes \mathbf{1}_n[m-1-2a])$$

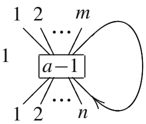
$= \text{span}\Big\langle$ 

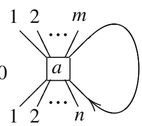
, 

$\Big\rangle$

with the direct summand zero unless $m+n+2a \geq 3$, $m+a \geq 2$ and $n+a \geq 2$, and with differential d_0 given on the generators of E by

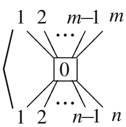
d_0 

$= (-1)^{m-1}$ 

d_0 

$= 0.$

It is clear that the cohomology, $H(E) = \{H(E)(m,n)\}$, of this dg $\mathbb{S}$-bimodule is equal to

$H(E)(m,n) = \text{span}\Big\langle$ 

$\Big\rangle.$

Consider next the decreasing filtrations

$$\mathcal{U}\mathcal{L}ie^1\mathcal{B}_\infty = F_0\mathcal{U}\mathcal{L}ie^1\mathcal{B}_\infty \supset F_1\mathcal{U}\mathcal{L}ie^1\mathcal{B}_\infty \supset \ldots \supset F_p\mathcal{U}\mathcal{L}ie^1\mathcal{B}_\infty \supset \ldots,$$
$$\mathcal{U}\mathcal{P}oisson = F_0\mathcal{U}\mathcal{P}oisson \supset F_1\mathcal{U}\mathcal{P}oisson \supset \ldots \supset F_p\mathcal{U}\mathcal{P}oisson \supset \ldots,$$

of dg props $\mathcal{ULie}^1\mathcal{B}_\infty$ and $\mathcal{UPoisson}$ by the number of vertices: the subspaces F_p spanned, by definition, by decorated graphs with at least p vertices. The morphism F respects the filtrations and hence induces the morphism, $\{F_r : (\mathcal{E}_r\mathcal{ULie}^1\mathcal{B}_\infty, d_r) \to (\mathcal{E}_r\mathcal{UPoisson}, \delta_r\}$, of the associated spectral sequence, in particular, a morphism,

$$F_0 : (\mathcal{E}_0\mathcal{ULie}^1\mathcal{B}_\infty, d_0) \to (\mathcal{E}_r\mathcal{UPoisson}, d_0\}$$

of the initial terms. The dg $\mathbb{S}$-bimodule $(\mathcal{E}_0\mathcal{ULie}^1\mathcal{B}_\infty, d_0)$ is canonically isomorphic to the following one:

$$\mathcal{F}^{\circlearrowright}_{no\ loops}\langle E\rangle := \sum_{G\in\mathfrak{G}^{\circlearrowright}_{no\ loops}} G\langle E\rangle,$$

with the differential induced from d_0 on E (hence the same notation). As we work over a field of characteristic zero, by Kunneth and Mashke theorems the functor $\mathcal{F}^{\circlearrowright}_{no\ loops}$ on the category of dg $\mathbb{S}$-bimodules is exact, i.e.

$$H\left(\mathcal{F}^{\circlearrowright}_{no\ loops}\langle E\rangle\right) = \mathcal{F}^{\circlearrowright}_{no\ loops}\langle H\,(E)\rangle.$$

Therefore, the morphism F_0 is an isomorphism. By assumptions on $\mathcal{ULie}^1\mathcal{B}_\infty$ and $\mathcal{UPoisson}$, both filtrations are complete, exhaustive and regular (degenerating at the 1st term). Hence the associated spectral sequences are convergent by the classical Complete Convergence Theorem 5.5.10 (see p.139 in [Weib]). Then, by the classical Comparison Theorem 5.2.12 (see p. 126 [Weib]), the morphism F is a quasi-isomorphism. □

6. *BF* Theory of Quantum BV Manifolds

6.1. Introduction. This section is inspired by the work of Mnev [Mn] on a remarkable approach to the homotopy transfer formulae of unimodular L_∞-algebras which is based on the BV quantization of an extended BF theory and the associated Feynman integrals. We apply here Losev-Mnev ideas to unimodular Lie 1-bialgebras and show that the Feynman integrals technique provides us with exactly the same formulae for the homotopy transfer of $\mathcal{ULie}^1\mathcal{B}_\infty$-structures as the ones which one obtains with the help of the Koszul duality technique in the wheeled props approach to quantum BV manifolds (see §§ 2–4). We believe that the established interrelation,

Feynman integrals $\leftrightarrows$ Morphisms of dg wheeled (co)props

is quite general.

6.2. BF-theory of unimodular Lie 1-bialgebras. Let V be finite-dimensional, and assume that its dual space V^* is equipped with a structure of unimodular dg Lie 1-bialgebra, i.e. with degree 1 Lie brackets $[\ \bullet\ ] : \odot^2 V^* \to V^*[1]$ and degree 0 Lie co-brackets $\Delta^{CoLie} : V^* \to \wedge^2 V^*$ (see § 4.2). The dualization and degree shifting of the latter gives a map $[\ ,\ ] : \odot^2(V[-1]) \to V[-2]$ which makes $V[-1]$ into a degree 1 Lie algebra. Consider a degree 2 polynomial function (called *action*) on the vector space $V^* \oplus V[-1]$,

$$\begin{array}{rcl} S : V^* \oplus V[-1] & \longrightarrow & \mathbb{K} \\ p \oplus \omega & \longrightarrow & S(p,\omega) := \langle p, d\omega\rangle\ +\ \frac{1}{2}\langle p, [\omega,\omega]\rangle + \frac{1}{2}\langle [p\bullet p], \omega\rangle, \end{array}$$

where $\langle\ ,\ \rangle$ stand for the natural pairing. A choice of a basis $\{e_a\}$ in V induces linear coordinates $\{p_a : |p_a| = |e_a|\}$ on V^* and linear coordinates $\{\omega^a : |\omega_a| = 1 - |e_a|\}$ on $V[-1]$ in which the function S takes the form

$$S(p,\omega) = \sum_{a,b}\left(p_a D^a_b \omega^b \pm \sum_c \frac{1}{2}\left(p_b p_c C^{bc}_a \omega_a \pm p_a \Phi^a_{bc}\omega^b\omega^c\right)\right),$$

where D^a_b, C^{bc}_a and Φ^a_{bc} are the structure constants of, respectively, the differential, the odd Lie brackets and Lie cobrackets in the chosen basis.

Let $\mathcal{M}^{\hbar}_V$ be an odd symplectic manifold corresponding to the completed graded commutative ring $\widehat{\odot}(V \oplus V^*[1])[[\hbar]] \simeq \mathbb{K}[[p_a, \omega^a, \hbar]]$.

Lemma 6.2.1. *The semidensity $e^{\frac{S(p,\omega)}{\hbar}}\sqrt{D}_{p,\omega}$ makes $\mathcal{M}^{\hbar}_V$ into a quantum BV manifold.*

Proof. The boundary conditions $S|_{p=0} = 0$ and $S|_{\omega=0} = 0$ are obvious so that, by Defintion 3.9.2(i), one should only check the equation $\hbar\Delta_0 S + \frac{1}{2}\{S \bullet S\} = 0$, where $\Delta_0 = \sum \frac{\partial^2}{\partial p_a \partial\omega^a}$. As S is independent of $\hbar$, this is equivalent to two equations,

$$\{S \bullet S\} = 0 \ \text{ and } \ \Delta_0 S = 0.$$

The first equation follows from relations (48). Equations $\Delta_0\langle p, [\omega \bullet \omega]\rangle = 0$ and $\Delta_0\langle [p,p], \omega\rangle = 0$ are equivalent to unimodularity of $[\ \bullet\ ]$ and δ^{CoLie}. Finally, equation $\Delta_0\langle p, d\omega\rangle = 0$ follows from the well-known fact that, for an arbitrary differential d, there exist a basis in V in which d is given by a matrix (40) with zero supertrace. □

The quadratic form $S_{(2)} := \langle p, d\omega\rangle$ is degenerate on the vector space $V^* \oplus V[-1]$. We shall next specify a subspace, $W \subset V^* \oplus V[-1]$, on which $S_{(2)}$ is non-degenerate so that one can develop a perturbative quantization of the action $S = S_{(2)} + S_{(3)}$ with $S_{(2)}$ determining the "propagator" of the quantum theory and with the cubic part, $S_{(3)} := \frac{1}{2}\langle p, [\omega \bullet \omega]\rangle + \frac{1}{2}\langle [p,p], \omega\rangle$, playing the role of "interactions" between "fields" p and ω. With this purpose we fix an arbitrary cohomological splitting,

$$V = H(V) \oplus B \oplus B[-1], \tag{55}$$

of the complex V. Let $p_a = \{p'_{\mathbf{a}}, p''_\alpha, p'''_\alpha\}$ be adopted to this splitting basis of V in which the differential is given by the matrix (40). Put another way, $\{p'_{\mathbf{a}}\}_{\mathbf{a}\in I'}$ is a basis of the cohomology group $H(V,d)$, $\{p''_\alpha\}_{\alpha\in J}$ a basis of B, $\{p'''_\alpha\}_{\alpha\in J}$ a basis of $B[-1]$ and the differential d is given by

$$dp'_{\mathbf{a}} = 0, \quad dp''_\alpha = p'''_\alpha, \quad dp'''_\alpha = 0.$$

This splitting of V induces associated splitting of $V^*[1]$ and hence the associated split base of the direct sum $V \oplus V^*[1]$ which we denote as follows,

$$\underbrace{V}_{p} \oplus \underbrace{V^*[1]}_{\omega} = \underbrace{H(V)}_{p'_{\mathbf{a}}} \oplus \underbrace{B}_{p''_\alpha} \oplus \underbrace{B[-1]}_{p'''_\alpha} \oplus \underbrace{H(V)^*[1]}_{\omega'^{\mathbf{a}}} \oplus \underbrace{B^*[1]}_{\omega'''^\alpha} \oplus \underbrace{B^*[2]}_{\omega''^\alpha},$$

so that

$$d\omega'^{\mathbf{a}} = 0, \quad d\omega''^\alpha = -\omega'''^\alpha, \quad d\omega'''^\alpha = 0.$$

The linear functions on the space $V^* \oplus V[-1]$ corresponding to the above basis vectors of $V \oplus V^*[1]$ we denote by the same letters $p'_{\mathbf{a}}, p''_\alpha, p'''_\alpha, \omega'^{\mathbf{a}}, \omega''^\alpha, \omega'''^\alpha$. Then the quadratic term of the action takes the form (cf. (42))

$$S_{(2)} = \langle p, d\omega\rangle = - < p''', \omega''' >= -\sum_{\alpha\in J} p'''_\alpha \omega'''^\alpha, \tag{56}$$

where $<\,,\,>$ is the natural degree 2 pairing between B and $B^*[2]$.

Let now $\mathcal{M}^\hbar_{B\oplus B[-1]}$ be the formal odd symplectic manifold corresponding to a graded commutative algebra

$$\widehat{\odot^\bullet}\left(B \oplus B[-1] \oplus B^*[1] \oplus B^*[2]\right) \otimes \mathbb{K}[[\hbar]] \simeq \mathbb{K}[[p'', p''', \omega'', \omega''', \hbar]],$$

and $\mathcal{M}^\hbar_{H(V)}$ the odd symplectic manifold corresponding to

$$\widehat{\odot^\bullet}\left(H(V) \oplus H(V)^*[1]\right)[[\hbar]] \simeq \mathbb{K}[[p', \omega', \hbar]].$$

Cohomological splitting (55) induces an isomorphism of odd Poisson manifolds,

$$\mathcal{M}^\hbar_V = \mathcal{M}^\hbar_{H(V)} \times \mathcal{M}^\hbar_{B\oplus B[-1]}.$$

Following [Mn] we shall show next how a perturbative Feynman type integration along a Lagrangian submanifold $\mathcal{L}$ in the odd symplectic manifold $\mathcal{M}^\hbar_{B\oplus B[-1]}$ transforms a simple quantum BV structure on $\mathcal{M}^\hbar_V$ given by Lemma 6.2.1 into a rather non-trivial quantum BV structure on $\mathcal{M}^\hbar_{H(V)}$ (in full accordance with Theorem 2.7.1). Let $\sqrt{D}_{B\oplus B[-1]}$ be the semidensity on $\mathcal{M}^\hbar_{B\oplus B[-1])}$ associated with the choice of linear Darboux coordinates made above.

Lemma 6.2.2. *For any Lagrangian submanifold $\mathcal{L}$ in $\mathcal{M}^\hbar_{B\oplus B[-1]}$ and any function $f \in \mathcal{O}_{\mathcal{M}_V}$ one has,*

$$\bar{\Delta}_0 \int_{\mathcal{L}} f\sqrt{D}_{B\oplus B[-1]}|_{\mathcal{L}} = \int_{\mathcal{L}} (\Delta_0 f)\sqrt{D}_{B\oplus B[-1]}|_{\mathcal{L}},$$

provided the integral exists. Here $\Delta_0 = \sum_a \frac{\partial^2}{\partial p_a \partial\omega^a}$ is the odd Laplacian on $\mathcal{M}^\hbar_V$, $\bar{\Delta}_0 = \sum_{\mathbf{a}} \frac{\partial^2}{\partial p_{\mathbf{a}} \partial\omega^{\mathbf{a}}}$ is the odd Laplacian on $\mathcal{M}^\hbar_{H(V)}$ and $\sqrt{D}_{B\oplus B[-1]}|_{\mathcal{L}}$ stands for the restriction (in accordance with (28)) of the semidensity $\sqrt{D}_{B\oplus B[-1]}$ to a volume form on $\mathcal{L}$.

This lemma is in fact a classical Stokes theorem in disguise. We refer to [Schw] or [CaFe2] for its simple proof. Thus, if we can find a Lagrangian submanifold $\mathcal{L} \subset \mathcal{M}_{B\oplus B[-1]}$ such that the integral $\int_{\mathcal{L}} f\sqrt{D}_{B\oplus B[-1]}|_{\mathcal{L}}$ exists for $f = e^{\frac{S(p,\omega)}{\hbar}}$ given by Lemma 6.2.1, then we obtain a quantum BV structure on the $\hbar$-twisted odd symplectic manifold $\mathcal{M}^\hbar_{H(V)}$ from the unimodular Lie 1-bialgebra structure on V^*. Formula (56) suggests a natural choice: let $\mathcal{L}$ be the formal $\mathbb{Z}$-graded manifold associated with the vector subspace $B[-1] \oplus B^*[1] \subset B \oplus B[-1] \oplus B^*[1] \oplus B^*[2]$. It is a submanifold of $\mathcal{M}_{B\oplus B[-1]}$ given by the equations $p'' = \omega'' = 0$. The semidensity $\sqrt{D}_{B\oplus B[-1]}$ restricts to $\mathcal{L}$ as an ordinary translation invariant Berezin volume $dp'''d\omega''' = \prod_\alpha dp'''_\alpha d\omega'''^\alpha$

(see [Be]). As the quadratic volume form $S_{(2)} = - < p''', \omega''' >$ is obviously non-degenerate on $B[-1] \oplus B^*[1]$, the integral,

$$N := \int_{\mathcal{L}} e^{\frac{S_2(p,\omega)}{\hbar}} \sqrt{D}_{B\oplus B[-1]}|_{\mathcal{L}} = \int e^{-\frac{<p''',\omega'''>}{\hbar}} dp''' d\omega''',$$

is a well-defined constant[6]. Moreover,

$$\begin{aligned} e^{\frac{S_{eff}(p',\omega',\hbar)}{\hbar}} &:= N^{-1} \int e^{\frac{S(p',p''',\omega',\omega''')}{\hbar}} dp''' d\omega''' \\ &= N^{-1} \int e^{\frac{-<p''',\omega'''>+S_{(3)}(p',p''',\omega',\omega''')}{\hbar}} dp''' d\omega''' \\ &= N^{-1} \sum_{k\geq 0} \frac{\hbar^{-k}}{k!} \int e^{-\frac{<p''',\omega'''>}{\hbar}} \left(S_{(3)}(p',p''',\omega',\omega''')\right)^k dp''' d\omega''' \end{aligned}$$

is well-defined as an element of the algebra (32). It can be computed via the classical Wick theorem (see. e.g., [CKTB],) with the propagator $\langle\langle\omega''', p'''\rangle\rangle$ (which is, by definition equal to the quadratic form inverse to $S_{(2)}$) given by the matrix[7]

$$\langle\langle\omega'''^{\alpha}, p'''_{\beta}\rangle\rangle_0 := -\hbar\delta^{\alpha}_{\beta}.$$

As

$$\begin{aligned} S_{(3)}(p',p''',\omega',\omega''') &= \frac{1}{2}\left\langle p'+p''', \left[(\omega'+\omega''')\bullet(\omega'+\omega''')\right]\right\rangle + \frac{1}{2}\left\langle [p'+p''', p'+p'''], \omega'+\omega'''\right\rangle \\ &= S_3(p',\omega') + \langle p', [\omega'\bullet\omega''']\rangle + \frac{1}{2}\left\langle p', \left[\omega'''\bullet\omega'''\right]\right\rangle + \langle p''', [\omega'\bullet\omega''']\rangle \\ &\quad + \frac{1}{2}\left\langle p''', \left[\omega'''\bullet\omega'''\right]\right\rangle + \langle [p',p'''],\omega'\rangle + \langle [p',p'''],\omega'''\rangle + \frac{1}{2}\left\langle [p''',p'''],\omega'\right\rangle \\ &\quad + \frac{1}{2}\left\langle [p''',p'''],\omega''\right\rangle, \end{aligned}$$

we conclude by the Wick theorem that this integral is equal to the formal power series,

$$e^{\frac{S_{eff}(p',\omega',\hbar)}{\hbar}} = \sum_{G\in\tilde{G}^{\circlearrowright}} G(p',\omega',\hbar),$$

where the sum runs over all possible graphs built from corollas of two types,

[,] ↔  , [•] ↔ .

It is well-known (see, e.g., Ch. 4, §3 in [Ma2] or Prop. 2.10 in [Po]) that

$$\log \sum_{G\in\tilde{G}^{\circlearrowright}} G(p',\omega',\hbar) = \sum_{G\in\tilde{G}^{\circlearrowright}_c} G(p',\omega',\hbar),$$

6 This is a "Gaussian" integral of special type 1.2.1.2 according to Cattaneo's review [CKTB] of Gaussian integrals. Strictly speaking, we should view here the formal parameter $\hbar$ as a purely imaginary complex number ih with h being an arbitrary positive real number; such "Gaussian" integrals can be made well-defined via a real analytic continuation of ordinary Gauss integrals for positive definitive quadratic forms, see [CKTB].

7 This matrix (up to the factor $\hbar^{-1}$) is precisely the coordinate representation of the homotopy operator $h: V \to V$ (see § 2.7).

where the sum on the r.h.s. runs over the subset, $\tilde{G}_c^{\circlearrowright} \subset \tilde{G}^{\circlearrowright}$, consisting of *connected* graphs. Thus the effective action can be written finally as

$$S_{eff} = \sum_{G\in\tilde{G}_c^{\circlearrowright}} G(p',\omega',\hbar) = \sum_{g\geq 0}\sum_{G\in\tilde{G}_{g,c}^{\circlearrowright}} \hbar^g G(p',\omega'), \tag{57}$$

where

– the second sum runs over the subset, $\tilde{G}_{g,c}^{\circlearrowright} \subset \tilde{G}_{g,c}^{\circlearrowright}$, consisting of all possible connected trivalent directed graphs of genus g;
– $G(p',\omega')$ is a linear map $H(V)^{\otimes\bullet} \to H(V)^{\otimes\bullet}$ obtained from the graph G by decorating it exactly as in Theorem 2.7.1: vertices are decorated by the structure constants, C_{ab}^c and Φ_a^{bc}, of the Lie and co-Lie operations in V, and internal edges are decorated with the homotopy operator h; legs are now decorated with p'' and ω''.

By Lemmas 6.2.2 and 6.2.1, the effective action satisfies the equation,

$$\Delta_0 e^{\frac{S_{eff}(p',\omega',\hbar)}{\hbar}} = 0, \quad \text{i.e. } \hbar\Delta_0 S_{eff} + \frac{1}{2}\{S_{eff}\bullet S_{eff}\} = 0,$$

and hence makes $\mathcal{M}_{H(V)}$ into a quantum BV manifold.

Proposition 6.2.3. *For any dg Lie 1-bialgebra on V and any cohomological splitting of V there is a canonically associated structure of quantum BV manifold on the cohomology, $H(V)$, given by the quantum master function (57). Moreover, there exists a natural quasi-isomorphism of quantum BV manifolds,*

$$\phi_\hbar : \left(\mathcal{M}_{H(V)}, e^{\frac{S_{eff}(p',\omega',\hbar)}{\hbar}}\sqrt{D}_{p',\omega'}\right) \longrightarrow \left(\mathcal{M}_V, e^{\frac{S(p,\omega)}{\hbar}}\sqrt{D}_{p,\omega}\right).$$

Proof. It remains to construct a morphism $\phi_\hbar$, which, by Definition 3.9.11, is a topological morphism of $\mathbb{K}[[\hbar]]$-modules,

$$\phi^* : \mathbb{K}[[p,\omega,\hbar]]. \longrightarrow \mathbb{K}[[p',\omega',\hbar]],$$

which in the limit $\hbar \to 0$ induces a morphism of algebras and satisfies the equation

$$e^{\frac{-S_{eff}(p',\omega',\hbar)}{\hbar}}\bar{\Delta}_0\left(\phi_\hbar^*(f)e^{\frac{S_{eff}(p',\omega',\hbar)}{\hbar}}\right) = \phi^*\left(e^{\frac{-S(p,\omega)}{\hbar}}\Delta_0\left(fe^{\frac{S(p,\omega)}{\hbar}}\right)\right), \tag{58}$$

for any $f \in \mathbb{K}[[p,\omega,\hbar]]$. In view of Lemma 6.2.2, the map (cf. [Mn])

$$\phi_\hbar^*(f) := N^{-1}e^{\frac{-S_{eff}(p',\omega',\hbar)}{\hbar}}\int_{p''=0,\omega''=0} f(p,\omega,\hbar)e^{\frac{S(p,\omega)}{\hbar}}dp'''d\omega''' \tag{59}$$

does satisfy Eq. (58):

$$\begin{aligned}
&e^{\frac{-S_{eff}(p',\omega',\hbar)}{\hbar}}\Delta_0'\left(\phi_\hbar^*(f)e^{\frac{S_{eff}(p',\omega',\hbar)}{\hbar}}\right)\\
&= N^{-1}e^{\frac{-S_{eff}(p',\omega',\hbar)}{\hbar}}\int_{p''=0,\omega''=0}\Delta_0\left(f(p,\omega,\hbar)e^{\frac{S(p,\omega)}{\hbar}}\right)dp'''d\omega'''\\
&= \phi^*\left(e^{\frac{-S(p,\omega)}{\hbar}}\Delta_0\left(fe^{\frac{S(p,\omega)}{\hbar}}\right)\right).
\end{aligned}$$

Moreover, in the limit $\hbar \to 0$ formula (58) gives simply the evaluation map,

$$\lim_{\hbar \to 0} \phi_\hbar^*(f) = f|_{\hbar=0,\, p''=0,\, p'''=0, \omega''=0, \omega'''=0},$$

and hence defines a morphism of algebras $\mathcal{O}_{\mathcal{M}_V^0} \to \mathcal{O}_{\mathcal{M}_{H(V)}^0}$. □

Formula (58) proves Proposition 3.11.2 in the special case when the quantum master function $S(\omega, p)$ is associated with a unimodular Lie 1-bialgebra structure on a finite-dimensional vector space. However the same formula (59) gives obviously a well-defined perturbative power series in $\hbar$ for an arbitrary (quasi-classically) split quantum master function $S(p, \omega, \hbar)$ and proves thereby Proposition 3.11.2 in general.

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