Commun. Math. Phys. 295, 639–668 (2010)
Digital Object Identifier (DOI) 10.1007/s00220-010-0985-z

Communications in **Mathematical Physics**

Spinning Q-Balls for the Klein-Gordon-Maxwell Equations

Vieri Benci[1], **Donato Fortunato**[2]

[1] Dipartimento di Matematica Applicata "U. Dini", Università di Pisa, via F. Buonarroti 1/c, 56127 Pisa, Italy. E-mail: benci@dma.unipi.it
[2] Dipartimento di Matematica, Università di Bari and INFN sezione di Bari, Via Orabona 4, 70125 Bari, Italy. E-mail: fortunat@dm.uniba.it

Received: 25 September 2008 / Accepted: 17 November 2009
Published online: 5 February 2010 –

Abstract: The nonlinear Klein-Gordon-Maxwell equations provide models for the interaction between the electromagnetic field and matter. We assume that the nonlinear term W is positive and $W(0) = 0$. This fact makes the theory more suitable for physical models (for example models in supersymmetry theory and in cosmology; see e.g. [16,22,28] and their references).

A three dimensional vortex is a finite energy, stationary solution of the Klein-Gordon-Maxwell equations such that the matter field has nontrivial angular momentum and the magnetic field looks like the field created by a finite solenoid. Under suitable assumptions, we prove the existence of three dimensional vortex-solutions.

Contents

1. Introduction

A *vortex* is a *solitary wave* ψ with non-vanishing angular momentum ($\mathbf{M}(\psi) \neq 0$). Roughly speaking, a *solitary wave* is a solution of a field equation whose energy is localized and which preserves this localization in time. The vortices in the nonlinear Klein-Gordon equation (KG) (with a positive nonlinear term $W(s)$ with $W(0) = 0$) are also considered in the Physics literature with the name of *spinning Q-balls,* even if they do not exhibit spherical symmetry (see e.g. [16,37]).

In this paper we prove the existence of *spinning Q-balls* for the nonlinear Klein-Gordon-Maxwell equations (KGM) (Theorem 3). The KGM represents a basic example of a system of equations exhibiting Poincarè and local gauge symmetries (see e.g. [33] Sect. 2.7 and [38] Sect. 1.4). Various physical phenomena like superconductivity or models for elementary particles and cosmology are described by KGM (see e.g. [16,22,28] and references) or by suitable variants (see e.g. [24] Sect. 8.8, [31] Sect. 3.6, [36] Sect. 4).

Now we will review some results relative to solitary waves and vortices. The KGM can be regarded as a perturbation of the nonlinear Klein-Gordon equation (KG) (see (3)). So first we recall also some existence results of solitary waves and vortices for KG:

- For the case $\mathbf{M}(\psi) = 0$, we recall the pioneering paper of Rosen [32] and [14,17,34]. When the lower order term W is positive and $W(0) = 0$ (see (3)), the spherically symmetric solitary waves have been called *Q-balls* by Coleman in [18] and this is the name used in the physical literature.
- Vortices for KG in two space dimensions have been investigated in [26]; later also three dimensional vortices for KG have been studied (see [3,5,13,16,37]).

Now let us see some literature on KGM. We notice that the peculiarities of the model depend on the lower order term W and it is relevant to distinguish various situations.

- For the case $\mathbf{M}(\psi) = 0$, the existence of solitary waves for KGM was first proved in [7] assuming that

$$W(s) = \frac{1}{2}s^2 - \frac{s^p}{p}, 4 < p < 6, \quad s \geq 0. \tag{1}$$

The existence of solitary waves for KGM in this situation (i.e.with $\mathbf{M}(\psi) = 0$ and W as in (1)) has been studied also in [15,19–21]. In these papers the existence and the non-existence of stationary solutions has been proved under different assumptions.

However the lower order term W defined by (1) is not suitable to model interesting physical models since it is not positive for all s. In fact, in this case, there are configurations with negative energy and since (in relativistic models) energy equals the mass, we have the presence of negative mass which, usually, is not acceptable. So it is relevant to investigate the case $W \geq 0$.

- The case $W \geq 0$ and $\mathbf{M}(\psi) = 0$ has been treated in [8 and 12].

Now let us consider the existence of vortices ($\mathbf{M}(\psi) \neq 0$) for KGM.

- The existence of vortices for Abelian gauge theories in two space dimensions has been discovered in a seminal paper by Abrikosov [1] in the study of the superconductivity. Then, in [30], the planar vortices are studied in the context of elementary particles (see also the books [24,31,33,38] with their references). We point out that, in these cases, the function W that has been considered is of the type

$$W(s) = \left(1 - s^2\right)^2, \tag{2}$$

namely it is a double well shaped and positive function.

- In [11,10] the existence of vortices in 3 space dimensions has been proved assuming (1).
- If W is positive and $W(0) = 0$, the vortices in KGM are called *gauged spinning Q-balls*. As far as we know, no mathematical result exists.

In this paper we prove the existence of gauged spinning Q-balls in 3 space dimensions (Th.3) provided that W satisfies W1), W2), W3) and W4) (see Sect. 2.4) which are the natural assumptions for this kind of problems.

Since the KGM are invariant for the Lorentz group, a Lorentz boost of a vortex creates a travelling and "spinning" solitary wave.

The paper is organized as follows. In Sect. 2 we introduce the KGM-equations, we study some of their general features, we give the definition of three dimensional vortex and finally state the main result in Theorem 3. Section 3 is devoted to the proof of Theorem 3.

Remark 1. In many situations, as in this paper, the existence of stable structure such as solitary waves and/or vortices is obtained by minimising the energy over a class of configurations of a given charge (the charge is defined by (28)). If such a minimizing configuration exists, we may think that there is a force which binds the "matter" (see [6] for details). The relative solitary waves have been called hylomorphic in [4] (see also [12]). This name comes from the Greek words "*hyle*"="*matter* " and "*morphe*"="*form*". For this reason, the spinning Q-balls coud be called "hylomorphic vortices".

2. Statement of the Problem and Results

2.1. The Klein-Gordon-Maxwell system. The nonlinear Klein-Gordon equation for a complex valued field ψ, defined on the spacetime $\mathbb{R}^4$, can be written as follows:

$$\Box\psi + W'(\psi) = 0, \tag{3}$$

where

$$\Box\psi = \frac{\partial^2\psi}{\partial t^2} - \Delta\psi, \quad \Delta\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

and, with some abuse of notation,

$$W'(\psi) = W'(|\psi|)\frac{\psi}{|\psi|}$$

for some smooth function $W : [0, \infty) \to \mathbb{R}$. Hereafter $x = (x_1, x_2, x_3)$ and t will denote the space and time variables. The field $\psi : \mathbb{R}^4 \to \mathbb{C}$ will be called *matter field*. If $W'(s)$ is linear, $W'(s) = m_0^2 s$, $m_0 \neq 0$, Eq. (3) reduces to the Klein-Gordon equation.

Consider the Abelian gauge theory in $\mathbb{R}^4$ equipped with the Minkowski metric and described by the Lagrangian density (see e.g. [9,33,38])

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 - W(|\psi|), \tag{4}$$

where

$$\mathcal{L}_0 = \frac{1}{2}\left[|(\partial_t + iq\phi)\,\psi|^2 - |(\nabla - iq\mathbf{A})\,\psi|^2\right],$$

$$\mathcal{L}_1 = \frac{1}{2}\,|\partial_t\mathbf{A} + \nabla\phi|^2 - \frac{1}{2}\,|\nabla \times \mathbf{A}|^2.$$

Here q denotes a positive parameter, $\nabla\times$ and ∇ denote respectively the curl and the gradient operators with respect to the x variable, $\partial_t + iq\phi$ and $\nabla - iq\mathbf{A}$ are the covariant derivatives, and

$$\mathbf{A} = (A_1, A_2, A_3) \in \mathbb{R}^3 \text{ and } \phi \in \mathbb{R}$$

are the gauge potentials.

Now consider the total action

$$\mathcal{S} = \int (\mathcal{L}_0 + \mathcal{L}_1 - W(|\psi|))\, dxdt. \tag{5}$$

Making the variation of $\mathcal{S}$ with respect to ψ, ϕ and $\mathbf{A}$ we get the system of equations (KGM),

$$(\partial_t + iq\phi)^2\, \psi - (\nabla - iq\mathbf{A})^2\, \psi + W'(\psi) = 0, \tag{6}$$

$$\nabla \cdot (\partial_t \mathbf{A} + \nabla\phi) = q\left(\operatorname{Im}\frac{\partial_t \psi}{\psi} + q\phi\right)|\psi|^2, \tag{7}$$

$$\nabla \times (\nabla \times \mathbf{A}) + \partial_t\,(\partial_t \mathbf{A} + \nabla\phi) = q\left(\operatorname{Im}\frac{\nabla\psi}{\psi} - q\mathbf{A}\right)|\psi|^2. \tag{8}$$

Here $\nabla\cdot$ denotes the divergence operator.

If we make the following change of variables:

$$\mathbf{E} = -\left(\frac{\partial \mathbf{A}}{\partial t} + \nabla\phi\right), \tag{9}$$

$$\mathbf{H} = \nabla \times \mathbf{A}, \tag{10}$$

$$\rho = -q\left(\operatorname{Im}\frac{\partial_t \psi}{\psi} + q\phi\right)|\psi|^2, \tag{11}$$

$$\mathbf{j} = q\left(\operatorname{Im}\frac{\nabla\psi}{\psi} - q\mathbf{A}\right)|\psi|^2, \tag{12}$$

we see that (7) and (8) are the second couple of the Maxwell equations with respect to a matter distribution whose electric charge and current densities are respectively ρ and $\mathbf{j}$:

$$\nabla \cdot \mathbf{E} = \rho, \tag{13}$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j}. \tag{14}$$

Equations (9) and (10) give rise to the first couple of the Maxwell equations:

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{H}}{\partial t} = 0, \tag{15}$$

$$\nabla \cdot \mathbf{H} = 0. \tag{16}$$

If we set

$$\psi\,(t,x) = u\,(t,x)\, e^{iS(t,x)},\ u \in \mathbb{R}^+,\ S \in \frac{\mathbb{R}}{2\pi\mathbb{Z}},$$

Eq. (6) can be split in the two following ones:

$$\Box u + W'(u) + \left[|\nabla S - q\mathbf{A}|^2 - \left(\frac{\partial S}{\partial t} + q\phi \right)^2 \right] u = 0,$$
$$\frac{\partial}{\partial t} \left[\left(\frac{\partial S}{\partial t} + q\phi \right) u^2 \right] - \nabla \cdot \left[(\nabla S - q\mathbf{A})\, u^2 \right] = 0,$$

and these equations, using the variables $\mathbf{j}$ and ρ, become

$$\Box u + W'(u) + \frac{\mathbf{j}^2 - \rho^2}{q^2 u^3} = 0, \tag{17}$$
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0. \tag{18}$$

Equation (18) is the charge continuity equation.

Notice that Eq. (18) is a consequence of (13) and (14).

In conclusion, an Abelian gauge theory, via Eqs. (17 ,13,14,15,16), provides a model of interaction of the matter field ψ with the electromagnetic field $(\mathbf{E}, \mathbf{H})$.

Observe that the Lagrangian (4) is invariant with respect to the gauge transformations

$$\psi \to e^{iq\chi} \psi, \tag{19}$$
$$\phi \to \phi - \partial_t \chi, \tag{20}$$
$$\mathbf{A} \to \mathbf{A} + \nabla \chi, \tag{21}$$

where $\chi \in C^\infty\left(\mathbb{R}^4\right)$.

So, our equations are gauge invariant; if we use the variable $u, \rho, \mathbf{j}, \mathbf{E}, \mathbf{H}$, this fact can be checked directly since these variables are gauge invariant.

In fact, Eqs. (13–17) are the gauge invariant formulation of Eqs. (6–8).

2.2. Conservation laws. Noether's theorem states that any invariance for a one-parameter group of the Lagrangian implies the existence of an integral of motion (see e.g. [25]).

Here there are the integrals which are relevant for this paper.

- *Energy*. Energy, by definition, is the quantity which is preserved by the time invariance of the Lagrangian; using the gauge invariant variables, it takes the following form:

$$\mathcal{E} = \mathcal{E}_m + \mathcal{E}_f, \tag{22}$$

where

$$\mathcal{E}_m = \frac{1}{2} \int \left[\left(\frac{\partial u}{\partial t} \right)^2 + |\nabla u|^2 + W(u) + \frac{\rho^2 + \mathbf{j}^2}{2q^2 u^2} \right] dx,$$
$$\rho = -q \left(\frac{\partial S}{\partial t} + q\phi \right) u^2, \tag{23}$$
$$\mathbf{j} = q\,(\nabla S - q\mathbf{A})\, u^2, \tag{24}$$

and

$$\mathcal{E}_f = \frac{1}{2}\int \left(\mathbf{E}^2 + \mathbf{H}^2\right) dx$$

(for the computation of $\mathcal{E}$, see e.g. ([9])).

- *Momentum.* Momentum, by definition, is the quantity which is preserved by the space invariance of the Lagrangian; using the gauge invariant variables, it takes the following form:

$$\mathbf{P} = \mathbf{P}_m + \mathbf{P}_f, \tag{25}$$

where

$$\mathbf{P}_m = \int \left[-\left(\partial_t u \,\nabla u\right) + \frac{\rho \mathbf{j}}{q^2 u^2} \right] dx$$

and

$$\mathbf{P}_f = \int \mathbf{E} \times \mathbf{H}\, dx.$$

- *Angular momentum.* The angular momentum, by definition, is the quantity which is preserved by virtue of the invariance under space rotations of the Lagrangian with respect to the origin. Using the gauge invariant variables, we get:

$$\mathbf{M} = \mathbf{M}_m + \mathbf{M}_f, \tag{26}$$

where

$$\mathbf{M}_m = \int \left[-\mathbf{x} \times \left(\nabla u \,\partial_t u\right) + \mathbf{x} \times \frac{\rho \mathbf{j}}{q^2 u^2} \right] dx \tag{27}$$

and

$$\mathbf{M}_f = \int \mathbf{x} \times \left(\mathbf{E} \times \mathbf{H}\right)\, dx.$$

Notice that each of the integrals $\mathcal{E}$, $\mathbf{P}$, $\mathbf{M}$ can be split in two parts (see (22), (25), (26)). The first one refers to the "matter field" and the second to the "electromagnetic field".

- *Electric charge.* The electric charge is the quantity which is preserved by the gauge action (19, 20, 21). Using (18), we see that it has the following expression:

$$Q = \int \rho dx = -q \int \left(\partial_t S + q\phi\right) u^2 dx. \tag{28}$$

2.3. Stationary solutions and vortices. We look for stationary solutions of (6), (7), (8), namely solutions of the form

$$\psi\left(t,x\right)=u\left(x\right)e^{iS(x,t)},\ u\in\mathbb{R}^{+},\ \omega\in\mathbb{R},\ S=S_0(x)-\omega t\in\frac{\mathbb{R}}{2\pi\mathbb{Z}},\tag{29}$$

$$\partial_t\mathbf{A}=0,\ \partial_t\phi=0.\tag{30}$$

Substituting (29) and (30) in (6), (7), (8), we get the following equations:

$$-\Delta u+\left[\left|\nabla S_0-q\mathbf{A}\right|^2-(\omega-q\phi)^2\right]u+W'\left(u\right)=0,\tag{31}$$

$$-\nabla\cdot\left[\left(\nabla S_0-q\mathbf{A}\right)u^2\right]=0,\tag{32}$$

$$-\Delta\phi=q\left(\omega-q\phi\right)u^2,\tag{33}$$

$$\nabla\times\left(\nabla\times\mathbf{A}\right)=q\left(\nabla S_0-q\mathbf{A}\right)u^2.\tag{34}$$

Observe that Eq. (32) easily follows from Eq. (34). Then we are reduced to study the system (31), (33), (34). The energy of a solution of equations (31), (33), (34) has the following expression:

$$\mathcal{E}=\frac{1}{2}\int\left(|\nabla u|^2+|\nabla\phi|^2+|\nabla\times\mathbf{A}|^2+(|\nabla S_0-q\mathbf{A}|^2+(\omega-q\phi)^2)\,u^2\right)+\int W(u).\tag{35}$$

Moreover the (electric) charge (see (28)) is given by

$$Q=q\sigma,\tag{36}$$

where

$$\sigma=\int\left(\omega-q\phi\right)u^2dx.\tag{37}$$

For a possible interpretation of σ see [6].

Clearly, when $u=0$, the only finite energy gauge potentials which solve (33), (34) are the trivial ones $\mathbf{A}=0$, $\phi=0$.

It is possible to have three types of finite energy stationary non trivial solutions:

- electrostatic solutions: $\mathbf{A}=0$, $\phi\neq0$;
- magnetostatic solutions: $\mathbf{A}\neq0$, $\phi=0$;
- electro-magneto-static solutions: $\mathbf{A}\neq0$, $\phi\neq0$.

Under suitable assumptions, all these types of solutions exist. The existence and the non existence of electrostatic solutions for Eqs. (31), (33) have been proved under different assumptions on W. In [7,15,19–21] lower order terms W like (1) have been taken into account. In [8 and 12] the existence of electrostatic solutions has been studied for a class of positive lower order terms W. In particular the existence of radially symmetric, electrostatic solutions has been analyzed. These solutions have zero angular momentum.

Here we are interested in electro-magneto-static solutions, in particular we shall study the existence of vortices, which are solutions with nonvanishing angular momentum.

We set

$$\Sigma = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = x_2 = 0 \right\},$$

and we define the map

$$\theta : \mathbb{R}^3 \backslash \Sigma \to \frac{\mathbb{R}}{2\pi\mathbb{Z}},$$
$$\theta(x_1, x_2, x_3) = \operatorname{Im} \log(x_1 + i x_2).$$

In (29) we take $S_0 = \ell\theta$ (ℓ integer) and give the following definition.

Definition 2. *A finite energy solution* $(u, S_0, \phi, \mathbf{A})$ *of Eq.* (31), (33), (34) *is called vortex if* $S_0 = \ell\theta(x)$ *with* $\ell \neq 0$.

In this case, ψ has the following form:

$$\psi(t, x) = u(x)\, e^{i(\ell\theta(x) - \omega t)};\ \ \ell \in \mathbb{Z} - \{0\}. \tag{38}$$

We shall see (Proposition 7) that the angular momentum $\mathbf{M}_m$ of the matter field of a vortex does not vanish; this fact justifies the name "vortex".

Observe that $\theta \in C^\infty \left(\mathbb{R}^3 \backslash \Sigma, \frac{\mathbb{R}}{2\pi\mathbb{Z}} \right)$. We set with abuse of notation

$$\nabla\theta(x) = \frac{x_2}{x_1^2 + x_2^2}\mathbf{e}_1 - \frac{x_1}{x_1^2 + x_2^2}\mathbf{e}_2,$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is the canonical base in $\mathbb{R}^3$.

Using the ansatz (38), Eqs. (31), (33), (34) become

$$-\Delta u + \left[|\ell\nabla\theta - q\mathbf{A}|^2 - (\omega - q\phi)^2 \right] u + W'(u) = 0, \tag{39}$$

$$-\Delta\phi = q\,(\omega - q\phi)\, u^2, \tag{40}$$

$$\nabla \times (\nabla \times \mathbf{A}) = q\,(\ell\nabla\theta - q\mathbf{A})\, u^2. \tag{41}$$

2.4. The main existence result. Let W satisfy the following assumptions:

- W1) $\forall s \geq 0 :\ \ W(s) \geq 0$,
- W2) W is C^2 with $W(0) = W'(0) = 0$, $W''(0) = m^2 > 0$,
- W3) $\inf_{s>0} \left(\frac{W(s)}{\frac{m^2}{2}s^2} \right) < 1$,
- W4) There exist positive constants c_1, c_2, p, q, with $2 < q \leq p < 6$, such that for $s \geq 0$,

$$\left| N'(s) \right| \leq c_1 s^{q-1} + c_1 s^{p-1}.$$

We shall set

$$W(s) = \frac{m^2}{2}s^2 + N(s). \tag{42}$$

Clearly assumption W3) is equivalent to require that there exists $s_0 > 0$ such that

$$N(s_0) < 0. \tag{43}$$

By rescaling time and space we can assume without loss of generality

$$m^2 = 1.$$

Moreover, for technical reasons it is useful to assume that W is defined for all $s \in \mathbb{R}$ just setting

$$W(s) = W(-s) \quad for\ s < 0.$$

Now we can state the main existence result.

Theorem 3. *Assume that the function W satisfies assumptions W1),W2),W3),W4). Then for all $\ell \in \mathbb{Z}$ there exists $\bar{q} > 0$ such that for every $0 \leq q \leq \bar{q}$, Eqs. (39)–(41) admit a finite energy solution in the sense of distributions $(u, \omega, \phi, \mathbf{A})$, $u \neq 0$, $\omega > 0$. The maps u, ϕ depend only on the variables $r = \sqrt{x_1^2 + x_2^2}$ and x_3*

$$u = u(r, x_3), \ \phi = \phi(r, x_3),$$

and the magnetic potential $\mathbf{A}$ has the following form:

$$\mathbf{A} = a(r, x_3)\nabla\theta = a(r, x_3)\left(\frac{x_2}{r^2}\mathbf{e}_1 - \frac{x_1}{r^2}\mathbf{e}_2\right). \tag{44}$$

If $q = 0$, then $\phi = 0$, $\mathbf{A} = 0$. If $q > 0$ then $\phi \neq 0$. Moreover $\mathbf{A} \neq 0$ if and only if $\ell \neq 0$.

Remark 4. When there is no coupling with the electromagnetic field, i.e. $q = 0$, Eqs. (39)–(41) reduce to find vortices to the nonlinear Klein-Gordon equation and an analogous result has been obtained in [3].

Remark 5. When $\ell = 0$ and $q > 0$ the last part of Theorem 3 states the existence of electrostatic solutions, namely finite energy solutions with $u \neq 0$, $\phi \neq 0$ and $\mathbf{A} = 0$. This result is a variant of a recent theorem (see [12]).

Remark 6. By the presence of the term $\nabla\theta$ Eqs. (39), (41) are not invariant under the $O(3)$ group action as it happens for Eqs. (6)–(8) we started from. Indeed there is a breaking of radial symmetry and the solutions $u, \phi, \mathbf{A}$ in Theorem 3 have only an S^1 symmetry.

Proposition 7. *Let $(u, \omega, \phi, \mathbf{A})$ be a non trivial, finite energy solution of Eqs. (39)–(41) as in Theorem 3. Then the angular momentum $\mathbf{M}_m$ (see (27)) has the following expression:*

$$\mathbf{M}_m = -\left[\int (\ell - qa)(\omega - q\phi)\, u^2 dx\right]\mathbf{e}_3, \tag{45}$$

and, if $\ell \neq 0$, it does not vanish.

Proof. By (27), (38), (23), (24) and (44), we have that

$$\mathbf{M}_m = \int \mathbf{x} \times \nabla\theta\, (\ell - qa)(\omega - q\phi)\, u^2 dx.$$

Let us compute

$$\begin{aligned}\mathbf{x}\times\nabla\theta &= (x_1\mathbf{e}_1+x_2\mathbf{e}_2+x_3\mathbf{e}_3)\times\left(\frac{x_2}{r^2}\mathbf{e}_1-\frac{x_1}{r^2}\mathbf{e}_2\right)\\ &= -\frac{x_1^2}{r^2}\mathbf{e}_3-\frac{x_2^2}{r^2}\mathbf{e}_3+\frac{x_2x_3}{r^2}\mathbf{e}_2+\frac{x_1x_3}{r^2}\mathbf{e}_1\\ &= \frac{x_1x_3}{r^2}\mathbf{e}_1+\frac{x_2x_3}{r^2}\mathbf{e}_2-\mathbf{e}_3.\end{aligned}$$

Then

$$\mathbf{M}_m(\psi)=\int\left(\frac{x_1x_3}{r^2}\mathbf{e}_1+\frac{x_2x_3}{r^2}\mathbf{e}_2-\mathbf{e}_3\right)(\ell-qa)(\omega-q\phi)\,u^2dx. \tag{46}$$

On the other hand, since the functions $x_1x_3\frac{(\ell-qa)(\omega-q\phi)u^2}{r^2}$ and $x_2x_3\frac{(\ell-qa)(\omega-q\phi)u^2}{r^2}$ are odd in x_1 and x_2 respectively, we have

$$\int x_1x_3\frac{(\ell-qa)(\omega-q\phi)\,u^2}{r^2}=\int x_2x_3\frac{(\ell-qa)(\omega-q\phi)\,u^2}{r^2}=0. \tag{47}$$

Then (45) follows from (46) and (47). Now let $\ell\neq 0$. In order to see that $\mathbf{M}_m\neq 0$, it is sufficient to prove that

$$(\ell-qa)(\omega-q\phi)>0, \tag{48}$$

or that

$$(\ell-qa)(\omega-q\phi)<0. \tag{49}$$

Clearly, since $\ell,\omega\neq 0$ (48) or (49) are satisfied when $q=0$. Now let $q>0$. Assume that $\ell>0$ and we show that (48) is verified. The case $\ell<0$ can be treated analogously.

By (33) we have that

$$-\Delta\phi+q^2u^2\phi=q\omega u^2.$$

Since ω/q is a supersolution, by the maximum principle, $\phi<\omega/q$ and hence $\omega-q\phi>0$. So, in order to prove (48), it remains to show that

$$\ell-qa>0. \tag{50}$$

By (34) we have that

$$\nabla\times(\nabla\times\mathbf{A})=q(\ell\nabla\theta-q\mathbf{A})\,u^2. \tag{51}$$

Now a straight computation shows that,

$$\nabla\times(\nabla\times a\nabla\theta)=b\,\nabla\theta, \tag{52}$$

where

$$b=-\frac{\partial^2 a}{\partial r^2}+\frac{1}{2}\frac{\partial a}{\partial r}-\frac{\partial^2 a}{\partial x_3^2}.$$

Then, setting $\mathbf{A}=a\nabla\theta$ in (51) and using (52), we have

$$-\frac{\partial^2 a}{\partial r^2}+\frac{1}{2}\frac{\partial a}{\partial r}-\frac{\partial^2 a}{\partial x_3^2}=q(\ell-qa)\,u^2.$$

Since ℓ/q is a supersolution, by the maximum principle, $a<\ell/q$ and hence (50) is proved. □

Finally let us observe that under general assumptions on W, magnetostatic solutions (i.e. with $\omega = \phi = 0$) do not exist. In fact the following proposition holds:

Proposition 8. *Assume that W satisfies the assumptions $W(0) = 0$ and $W'(s)s \geq 0$. Then (39), (40), (41) has no solutions with $\omega = \phi = 0$.*

Proof. Set $\omega = 0$, $\phi = 0$ in (39) and we get

$$-\Delta u + |\ell\nabla\theta - q\mathbf{A}|^2 \, u + W'(u) = 0.$$

Then, multiplying by u and integrating, we get

$$\int |\nabla u|^2 + |\ell\nabla\theta - q\mathbf{A}|^2 \, u^2 + W'(u)u = 0.$$

So, since $W'(s)s \geq 0$, we get $u = 0$. □

3. The Existence Proof

3.1. The functional setting. Let H^1 denote the usual Sobolev space with norm

$$\|u\|_{H^1}^2 = \int (|\nabla u|^2 + u^2)dx;$$

moreover we need to use also the weighted Sobolev space $\hat{H}^1$ whose norm is given by

$$\|u\|_{\hat{H}^1}^2 = \int \left[|\nabla u|^2 + \left(1 + \frac{\ell^2}{r^2}\right) u^2 \right] dx, \;\; \ell \in \mathbb{Z},$$

where $r = \sqrt{x_1^2 + x_2^2}$. Clearly $\hat{H}^1 = H^1$ when $\ell = 0$.

We set $\mathcal{D} = C_0^\infty(\mathbb{R}^3)$ and we denote by $\mathcal{D}^{1,2}$ the completion of $\mathcal{D}$ with respect to the inner product

$$(v \mid w)_{\mathcal{D}^{1,2}} = \int \nabla v \cdot \nabla w dx. \tag{53}$$

Here and in the following the dot $\cdot$ will denote the Euclidean inner product in $\mathbb{R}^3$.

We set

$$\begin{aligned} &H = \hat{H}^1 \times \mathcal{D}^{1,2} \times \left(\mathcal{D}^{1,2}\right)^3, \\ &\|(u, \phi, \mathbf{A})\|_H^2 = \int |\nabla u|^2 + \left(1 + \frac{\ell^2}{r^2}\right) u^2 + |\nabla\phi|^2 + |\nabla\mathbf{A}|^2. \end{aligned} \tag{54}$$

We shall denote by $u = u(r, x_3)$ the real maps in $\mathbb{R}^3$ which depend only on $r = \sqrt{x_1^2 + x_2^2}$ and x_3. We set

$$\mathcal{D}_r = \{u \in \mathcal{D} : u = u(r, x_3)\}, \tag{55}$$

and we shall denote by $\mathcal{D}_r^{1,2}$ (respectively $\hat{H}_r^1$) the closure of $\mathcal{D}_r$ in the $\mathcal{D}^{1,2}$ (respectively $\hat{H}^1$) norm.

Now we consider the functional

$$J(u,\phi,\mathbf{A}) = \frac{1}{2}\int |\nabla u|^2 - |\nabla\phi|^2 + |\nabla\times\mathbf{A}|^2 + \frac{1}{2}\int\left[|\ell\nabla\theta - q\mathbf{A}|^2 - (\omega - q\phi)^2\right]u^2 + \int W(u), \tag{56}$$

where $(u,\phi,\mathbf{A}) \in H$. Equations (39), (40) and (41) are the Euler-Lagrange equations of the functional J. Standard computations show that the following lemma holds:

Lemma 9. *Assume that W satisfies W1),..., W4). Then the functional J is C^1 on H.*

By the above lemma it follows that the critical points $(u,\phi,\mathbf{A}) \in H$ of J (with $u \geq 0$) are weak solutions of Eq. (39), (40) and (41), namely

$$\int \nabla u\cdot\nabla v + \left[|\ell\nabla\theta - q\mathbf{A}|^2 - (\omega - q\phi)^2\right]uv + W'(u)\,v = 0,\ \forall v\in\hat{H}^1, \tag{57}$$

$$\int \nabla\phi\cdot\nabla w - qu^2(\omega - q\phi)\,w = 0,\ \forall w\in\mathcal{D}^{1,2}, \tag{58}$$

$$\int (\nabla\times\mathbf{A})\cdot(\nabla\times\mathbf{V}) - qu^2(\ell\nabla\theta - q\mathbf{A})\cdot\mathbf{V} = 0,\ \forall\mathbf{V}\in(\mathcal{D}^{1,2})^3. \tag{59}$$

3.2. Solutions in the sense of distributions. Since $\mathcal{D}$ is not contained in $\hat{H}^1$, a solution $(u,\phi,\mathbf{A}) \in H$ of (57), (58), (59) need not to be a solution of (39), (40), (41) in the sense of distributions on $\mathbb{R}^3$. In fact, since $\nabla\theta(x)$ is singular on Σ, it might be that for some test function $v\in\mathcal{D}$, when $\ell\neq 0$, the integral $\int|\ell\nabla\theta - q\mathbf{A}|^2\,uv$ diverges, unless u is sufficiently small as $x\to\Sigma$.

In this section we will show that this fact does not occur, namely the singularity is removable in the sense of the following theorem:

Theorem 10. *Let $(u_0,\phi_0,\mathbf{A}_0)\in H$, $u_0\geq 0$ be a solution of (57), (58), (59) (i.e. a critical point of J). Then $(u_0,\phi_0,\mathbf{A}_0)$ is a solution of Eqs. (39), 40) and (41) in the sense of distribution, namely*

$$\int \nabla u_0\cdot\nabla v + \left[|\ell\nabla\theta - q\mathbf{A}_0|^2 - (\omega - q\phi_0)^2\right]u_0 v + W'(u_0)\,v = 0,\ \forall v\in\mathcal{D}, \tag{60}$$

$$\int \nabla\phi_0\cdot\nabla w - qu_0^2(\omega - q\phi_0)\,w = 0,\ \forall w\in\mathcal{D}, \tag{61}$$

$$\int (\nabla\times\mathbf{A}_0)\cdot(\nabla\times\mathbf{V}) - qu_0^2(\ell\nabla\theta - q\mathbf{A}_0)\cdot\mathbf{V} = 0,\ \forall\mathbf{V}\in\mathcal{D}^3. \tag{62}$$

Let χ_n (n positive integer) be a family of smooth functions depending only on $r=\sqrt{x_1^2+x_2^2}$ and x_3 and which satisfy the following assumptions:

- $\chi_n(r,x_3) = 1$ for $r\geq\frac{2}{n}$,
- $\chi_n(r,x_3) = 0$ for $r\leq\frac{1}{n}$,
- $|\chi_n(r,x_3)|\leq 1$,
- $|\nabla\chi_n(r,x_3)|\leq 2n$,
- $\chi_{n+1}(r,x_3)\geq\chi_n(r,x_3)$.

Lemma 11. *Let φ be a function in $H^1 \cap L^\infty$ with bounded support and set $\varphi_n = \varphi \cdot \chi_n$. Then, up to a subsequence, we have that*

$$\varphi_n \to \varphi \text{ weakly in } H^1.$$

Proof. Clearly $\varphi_n \to \varphi$ $a.e.$ Then, by standard arguments, the conclusion holds if we show that $\{\varphi_n\}$ is bounded in H^1. Clearly $\{\varphi_n\}$ is bounded in L^2. Let us now prove that

$$\left\{\int |\nabla\varphi_n|^2\right\} \text{ is bounded.}$$

We have

$$\begin{aligned}\int |\nabla\varphi_n|^2 &\leq 2\int |\nabla\varphi \cdot \chi_n|^2 + |\varphi \cdot \nabla\chi_n|^2 \\ &\leq 2\int |\nabla\varphi|^2 + 2\int_{\Gamma_\varepsilon} |\varphi \cdot \nabla\chi_n|^2,\end{aligned}$$

where

$$\Gamma_\varepsilon = \left\{x \in \mathbb{R}^3 : \varphi \neq 0 \text{ and } |\nabla\chi_n(r,z)| \neq 0\right\}.$$

By our construction, $|\Gamma_\varepsilon| \leq c/n^2$, where c depends only on φ. Thus

$$\begin{aligned}\int |\nabla\varphi_n|^2 &\leq 2\int |\nabla\varphi|^2 + 2\|\varphi\|_{L^\infty}^2 \int_{\Gamma_\varepsilon} |\nabla\chi_n|^2 \\ &\leq 2\int |\nabla\varphi|^2 + 2\|\varphi\|_{L^\infty}^2 \cdot |\Gamma_\varepsilon| \cdot \|\nabla\chi_n\|_{L^\infty}^2 \\ &\leq 2\int |\nabla\varphi|^2 + 8c\|\varphi\|_{L^\infty}^2.\end{aligned}$$

Thus φ_n is bounded in H^1 and $\varphi_n \to \varphi$ weakly in H^1. □

Now we are ready to prove Theorem 10.

Proof. Clearly (61) and (62) immediately follow by (58) and (59). Let us prove (60). The case $\ell = 0$ is trivial. So assume $\ell \neq 0$. We take any $v \in \mathcal{D}$ and set $\varphi_n = v^+\chi_n$, where $v^+ = \frac{|v|+v}{2}$. Then, taking φ_n as a test function in Eq. (57), we have

$$\int \nabla u_0 \cdot \nabla\varphi_n + \left[|q\mathbf{A}_0 - \ell\nabla\theta|^2 - (q\phi_0 - \omega)^2\right] u_0\varphi_n + W'(u_0)\,\varphi_n = 0. \tag{63}$$

Equation (63) can be written as follows

$$A_n + B_n + C_n + D_n = 0, \tag{64}$$

where

$$A_n = \int \nabla u_0 \cdot \nabla\varphi_n,\ B_n = \int \left(q^2\mathbf{A}_0^2 u_0 - (q\phi_0 - \omega)^2 u_0 + W'(u_0)\right)\varphi_n, \tag{65}$$

$$C_n = -2\int q\mathbf{A}_0 \cdot \ell\nabla\theta\, u_0\varphi_n,\ D_n = \int |\ell\nabla\theta|^2 u_0\varphi_n. \tag{66}$$

By Lemma 11,

$$\varphi_n \rightarrow v^+ \text{ weakly in } H^1. \tag{67}$$

Then we have

$$A_n \rightarrow \int \nabla u_0 \cdot \nabla v^+. \tag{68}$$

Now

$$\left(q^2 \mathbf{A}_0^2 u_0 - (q\phi_0 - \omega)^2 u_0 + W'(u_0)\right) \in L^{6/5} = \left(L^6\right)'.$$

Then, using again (67) and by the embedding $H^1 \subset L^6$, we have

$$B_n \rightarrow \int \left(q^2 \mathbf{A}_0^2 u_0 - (q\phi_0 - \omega)^2 u_0 + W'(u_0)\right) v^+ < \infty. \tag{69}$$

Now we shall prove that

$$C_n \rightarrow -2 \int q \mathbf{A}_0 \cdot \ell \nabla \theta \; u_0 v^+ < \infty. \tag{70}$$

Set

$$C = B_R \times [-d, d], \; B_R = \left\{(x_1, x_2) \in \mathbb{R}^2 : r^2 = x_1^2 + x_2^2 < R\right\},$$

where $d, \; R > 0$ are so large that the cylinder C contains the support of v^+.
Then

$$\int \left(\frac{\varphi_n}{r}\right)^{\frac{3}{2}} dx = \int_C \left(\frac{v^+ \chi_n}{r}\right)^{\frac{3}{2}} dx \tag{71}$$

$$\leq c_1 \int_{-d}^{d} \int_0^R \left(\frac{1}{r}\right)^{\frac{3}{2}} r dr dx_3 = M < \infty, \tag{72}$$

where $c_1 = 2\pi \sup \left(v^+\right)^{\frac{3}{2}}$. By (72) we have

$$\int |\mathbf{A}_0 \cdot \nabla \theta \; u_0 \varphi_n| \, dx \leq \|u_0 \mathbf{A}_0\|_{L^3} \left\|\frac{\varphi_n}{r}\right\|_{L^{\frac{3}{2}}} \leq \|u_0 \mathbf{A}_0\|_{L^3} M^{\frac{2}{3}}. \tag{73}$$

Now

$$|\mathbf{A}_0 \cdot \nabla \theta \; u_0 \varphi_n| \rightarrow \left|\mathbf{A}_0 \cdot \nabla \theta \; u_0 v^+\right| \text{ a.e. in } \mathbb{R}^3$$

and the sequence $\{|\mathbf{A}_0 \cdot \nabla \theta \; u_0 \varphi_n|\}$ is monotone . Then, by the monotone convergence theorem, we get

$$\int |\mathbf{A}_0 \cdot \ell \nabla \theta \; u_0 \varphi_n| \, dx \rightarrow \int \left|\mathbf{A}_0 \cdot \ell \nabla \theta \; u_0 v^+\right| dx. \tag{74}$$

By (73) and (74) we deduce that

$$\int \left|\mathbf{A}_0 \cdot \ell \nabla \theta \; u_0 v^+\right| dx < \infty. \tag{75}$$

Then, since

$$|\mathbf{A}_0 \cdot \nabla\theta \; u_0\varphi_n| \le \left|\mathbf{A}_0 \cdot \nabla\theta \; u_0 v^+\right| \in L^1,$$

by the dominated convergence theorem, we get (70). Finally we prove that

$$D_n \to \int |\ell\nabla\theta|^2 \, u_0 v^+ < \infty. \tag{76}$$

By (64), (68), (69) and (70) we have that

$$D_n = \int |\ell\nabla\theta|^2 \, u_0\varphi_n \text{ is bounded.} \tag{77}$$

Then the sequence $|\nabla\theta|^2 \; u_0\varphi_n$ is monotone and it converges a.e. to $|\nabla\theta|^2 \; u_0 v^+$. Then, by the monotone convergence theorem, we get

$$\int |\ell\nabla\theta|^2 \; u_0\varphi_n dx \to \int |\ell\nabla\theta|^2 \; u_0 v^+ dx. \tag{78}$$

By (77) and (78) we get (76).

Taking the limit in (64) and by using (68), (69), (70), (76) we have

$$\int \nabla u_0 \cdot \nabla v^+ + \left[|q\mathbf{A}_0 - \ell\nabla\theta|^2 - (q\phi_0 - \omega)^2\right] u_0 v^+ + W'(u_0)\, v^+ = 0.$$

Taking $\varphi_n = v^- \chi_n$ and arguing in the same way as before, we get

$$\int \nabla u_0 \cdot \nabla v^- + \left[|q\mathbf{A}_0 - \ell\nabla\theta|^2 - (q\phi_0 - \omega)^2\right] u_0 v^- + W'(u_0)\, v^- = 0.$$

Then

$$\int \nabla u_0 \cdot \nabla v + \left[|q\mathbf{A}_0 - \ell\nabla\theta|^2 - (q\phi_0 - \omega)^2\right] u_0 v + W'(u_0)\, v = 0.$$

Since $v \in \mathcal{D}$ is arbitrary, we get that Eq. (60) is solved. □

The presence of the term $-\int |\nabla\phi|^2$ gives to the functional J a strong indefiniteness, namely any critical point of J has infinite Morse index: this fact is a great obstacle to a direct study of the critical points. To avoid this difficulty we shall introduce a *reduced functional*

3.3. The reduced functional. Equation (40) can be written as follows

$$-\Delta\phi + q^2u^2\phi = q\omega u^2, \tag{79}$$

and it can be easily verified (see [7], Lemma 3.3) that for any $u \in H^1(\mathbb{R}^3)$, there exists a unique solution $\phi \in \mathcal{D}^{1,2}$ of (79).

Clearly, if $u \in \hat{H}^1_r(\mathbb{R}^3)$, the solution $\phi = \phi_u$ of (79) belongs to $\mathcal{D}^{1,2}_r$. Then we can define the map

$$u \in \hat{H}^1_r(\mathbb{R}^3) \to \; Z_\omega(u) = \phi_u \in \mathcal{D}^{1,2}_r \text{ solution of (79).} \tag{80}$$

Since ϕ_u solves (79), clearly we have

$$d_\phi J(u, Z_\omega(u), \mathbf{A}) = 0, \tag{81}$$

where J is defined in (56) and $d_\phi J$ denotes the partial differential of J with respect to ϕ.

Proposition 12. *The map defined in (80) is C^1.*

Proof. Observe that $d_\phi J = J'_\phi$ does not depend on $\mathbf{A}$. By (81), the points $(u, Z_\omega(u))$ of the graph of the map (80) are the zeros of J'_ϕ. On the other hand a straightforward calculation shows that the derivatives $J''_{\phi\phi}$, $J''_{\phi u}$ of J'_ϕ are

$$J''_{\phi\phi}(u,\phi)[\xi,\eta] = \int \nabla\xi \cdot \nabla\eta dx + \int q^2u^2\xi\eta dx,$$
$$J''_{\phi u}(u,\phi)[\xi,\nu] = \int \nabla\xi \cdot \nabla\nu dx + \int q^2u^2\xi\nu dx,$$

where $\xi, \eta \in \mathcal{D}_r^{1,2}$ and $\nu \in \hat{H}_r^1(\mathbb{R}^3)$. Standard calculations show that $J''_{\phi\phi}$ and $J''_{\phi u}$ are continuous and $J''_{\phi\phi}$ is invertible. Then the conclusion follows by using the implicit function theorem. □

For $u \in H^1(\mathbb{R}^3)$, let $\Phi = \Phi_u$ be the solution of Eq. (79) with $\omega = 1$, then Φ_u solves the equation

$$-\Delta\Phi_u + q^2u^2\Phi_u = qu^2. \tag{82}$$

Clearly

$$\phi_u = \omega\Phi_u. \tag{83}$$

Now let $q > 0$, then, by maximum principle arguments, it is easy to show that for any $u \in H^1(\mathbb{R}^3)$ the solution Φ_u of (82) satisfies

$$0 \leq \Phi_u \leq \frac{1}{q}. \tag{84}$$

Now, if $(u, \mathbf{A}) \in \hat{H}^1 \times \left(\mathcal{D}^{1,2}\right)^3$, we set

$$\tilde{J}(u,\mathbf{A}) = J(u, Z_\omega(u), \mathbf{A}),$$

where J is defined in (56). Observe that, since the functional J and the map $u \to Z_\omega(u) = \phi_u$ are C^1 (see Lemma 9 and Proposition 12), also the functional $\tilde{J}$ is C^1. Now, by using the chain rule and Eq. (81), it can be shown (see the first part of the proof of Theorem 16 in [11] or the Proposition 3.5 in [7]) that

$$\left((u,\mathbf{A}) \text{ critical point of } \tilde{J}\right) \Longrightarrow ((u, Z_\omega(u), \mathbf{A}) \text{ critical point of } J). \tag{85}$$

We will refer to $\tilde{J}(u,\mathbf{A})$ as the *reduced action functional*. From (82) we have

$$\int qu^2\Phi_u dx = \int |\nabla\Phi_u|^2 dx + q^2 \int u^2\Phi_u^2 dx. \tag{86}$$

Now, by (83), (86), we have:

$$\begin{aligned}\tilde{J}(u,\mathbf{A}) = J(u, Z_\omega(u), \mathbf{A}) &= \frac{1}{2}\int |\nabla u|^2 - |\nabla \phi_u|^2 + |\nabla\times\mathbf{A}|^2 \\ &+\frac{1}{2}\int\left[|\ell\nabla\theta - q\mathbf{A}|^2 - (q\phi_u - \omega)^2\right]u^2 + \int W(u) \\ &= \frac{1}{2}\int\left(|\nabla u|^2 + |\nabla\times\mathbf{A}|^2 + |\ell\nabla\theta - q\mathbf{A}|^2 u^2\right) \\ &-\frac{1}{2}\omega^2\int\left(|\nabla\Phi_u|^2 + q^2u^2\Phi_u^2 + u^2 - 2qu^2\Phi_u\right) + \int W(u) \\ &= \frac{1}{2}\int |\nabla u|^2 + |\nabla\times\mathbf{A}|^2 + |\ell\nabla\theta - q\mathbf{A}|^2 u^2 + \int W(u) \\ &-\frac{\omega^2}{2}\int([1 - q\Phi_u])u^2. \end{aligned} \tag{87}$$

Then

$$\tilde{J}(u,\mathbf{A}) = I(u,\mathbf{A}) - \frac{\omega^2}{2}K_q(u), \tag{88}$$

where

$$I(u,\mathbf{A}) = \frac{1}{2}\int |\nabla u|^2 + |\nabla\times\mathbf{A}|^2 + |\ell\nabla\theta - q\mathbf{A}|^2 u^2 + \int W(u)$$

and

$$K_q(u) = \int([1 - q\Phi_u])u^2. \tag{89}$$

Now, following the same lines as before, we can define the *reduced energy functional* as follows:

$$\tilde{\mathcal{E}}(u,\mathbf{A}) = \mathcal{E}(u, Z_\omega(u), \mathbf{A}),$$

where (see (35))

$$\mathcal{E} = \frac{1}{2}\int\left(|\nabla u|^2 + |\nabla\phi|^2 + |\nabla\times\mathbf{A}|^2 + (|\ell\nabla\theta - q\mathbf{A}|^2 + (\omega - q\phi)^2)u^2\right) + \int W(u). \tag{90}$$

It can be shown as for (88) that

$$\tilde{\mathcal{E}}(u,\mathbf{A}) = I(u,\mathbf{A}) + \frac{\omega^2}{2}K_q(u). \tag{91}$$

Observe that

$$Q = q\sigma = q\omega K_q(u)$$

represents the (electric) charge (see (36) and (37)), so that we can write for $u \neq 0$,

$$\tilde{\mathcal{E}}(u,\mathbf{A}) = I(u,\mathbf{A}) + \frac{\omega^2}{2}K_q(u) = I(u,\mathbf{A}) + \frac{\sigma^2}{2K_q(u)}.$$

Then for any $\sigma \neq 0$, the functional defined by

$$E_{\sigma,q}(u, \mathbf{A}) = I(u, \mathbf{A}) + \frac{\sigma^2}{2K_q(u)}, \quad (u, \mathbf{A}) \in \hat{H}^1 \times \left(\mathcal{D}^{1,2}\right)^3, \ u \neq 0 \tag{92}$$

represents the energy on the configuration $(u, \omega\Phi_u, \mathbf{A})$ having charge $Q = q\sigma$ or, equivalently, frequency $\omega = \frac{\sigma}{K_q(u)}$.

The following lemma holds

Lemma 13. *The functional*

$$\hat{H}^1 \in u \to K(u) = \int u^2(1 - q\Phi_u)dx$$

is differentiable and for any $u \in \hat{H}^1$ we have

$$K'(u) = 2u(1 - q\Phi_u)^2. \tag{93}$$

Proof. Set

$$\mathcal{A}(u, \Phi) = \int |\nabla\Phi|^2 \, dx + \int u^2(1 - q\Phi)^2 dx.$$

By (86) clearly we have

$$\mathcal{A}(u, \Phi_u) = K(u).$$

Then

$$K'(u) = \frac{\partial \mathcal{A}}{\partial u}(u, \Phi_u) + \frac{\partial \mathcal{A}}{\partial \Phi}(u, \Phi_u)\Phi_u', \tag{94}$$

where $\frac{\partial \mathcal{A}}{\partial u}$, $\frac{\partial \mathcal{A}}{\partial \Phi}$ denote the partial derivatives of $\mathcal{A}$ with respect to u and Φ respectively. Since Φ_u solves (82), we have

$$\frac{\partial \mathcal{A}}{\partial \Phi}(u, \Phi_u) = 0.$$

Then (94) gives

$$K'(u) = \frac{\partial \mathcal{A}}{\partial u}(u, \Phi_u) = 2u(1 - q\Phi_u)^2.$$

□

The following proposition holds

Proposition 14. *Let $\sigma \neq 0$ and let $(u, \mathbf{A}) \in \hat{H}^1 \times \left(\mathcal{D}^{1,2}\right)^3$, $u \neq 0$ be a critical point of $E_{\sigma,q}$ (see (92)). Then, if we set $\omega = \frac{\sigma}{K_q(u)}$, $(u, Z_\omega(u), \mathbf{A})$ is a critical point of J.*

Proof. Since $(u, \mathbf{A}) \in \hat{H}^1 \times \left(\mathcal{D}^{1,2}\right)^3$, $u \neq 0$ is a critical point of $E_{\sigma,q}$, we have

$$0 = E'_{\sigma,q}(u, \mathbf{A}) = I'(u, \mathbf{A}) - \frac{\sigma^2 K'_q(u)}{2K_q(u)^2} = I'(u, \mathbf{A}) - \frac{\omega^2 K'_q(u)}{2}, \ \omega = \frac{\sigma}{K_q(u)}.$$

Hence $(u, \mathbf{A})$ is a critical point of the functional

$$\tilde{J}(u, \mathbf{A}) = I(u, \mathbf{A}) - \frac{\omega^2 K_q(u)}{2}.$$

So by (85) $(u, Z_\omega(u), \mathbf{A})$ is a critical point of J. □

By Proposition 14 and Theorem 10 we are reduced to study the critical points of $E_{\sigma,q}$ which is a functional bounded from below.

However $E_{\sigma,q}$ contains the term $\int |\nabla \times \mathbf{A}|^2$ which is not a Sobolev norm.

In order to avoid this difficulty we introduce a suitable manifold $V \subset \hat{H}^1 \times \left(\mathcal{D}^{1,2}\right)^3$ such that:

- the critical points of J restricted to V satisfy Eq. (39), (40), 41); namely V is a "natural constraint" for J.
- The components $\mathbf{A}$ of the elements in V are divergence free, then the term $\int |\nabla \times \mathbf{A}|^2$ can be replaced by $\|\mathbf{A}\|^2_{(\mathcal{D}^{1,2})^3} = \int |\nabla \mathbf{A}|^2$.

We set

$$\mathcal{A}_0 := \left\{ \mathbf{X} \in \mathcal{C}_0^\infty(\mathbb{R}^3 \backslash \Sigma, \mathbb{R}^3) : \mathbf{X} = b(r, x_3)\nabla\theta; \ b \in \mathcal{C}_0^\infty\left(\mathbb{R}^3 \backslash \Sigma, \mathbb{R}\right) \right\}. \tag{95}$$

Let $\mathcal{A}$ denote the closure of $\mathcal{A}_0$ with respect to the norm of $\left(\mathcal{D}^{1,2}\right)^3$. We shall consider the following space:

$$V := \hat{H}^1_r \times \mathcal{A}, \tag{96}$$

where $\hat{H}^1_r$ has been defined in Sect. 3.1. We shall set $U = (u, \mathbf{A})$ and

$$\|U\|_V = \|(u, \mathbf{A})\|_V = \|u\|_{\hat{H}^1_r} + \|\mathbf{A}\|_{(\mathcal{D}^{1,2})^3}.$$

Lemma 15. *If* $\mathbf{A} \in \mathcal{A}$*, then*

$$\int |\nabla \times \mathbf{A}|^2 = \int |\nabla \mathbf{A}|^2.$$

Proof. Let $\mathbf{A} = b\nabla\theta \in \mathcal{A}_0$. Since b depends only on r and x_3, it is easy to check that

$$\nabla b \cdot \nabla\theta = 0.$$

Since θ is harmonic in $\mathbb{R}^3 \backslash \Sigma$ and b has support in $\mathbb{R}^3 \backslash \Sigma$,

$$b\Delta\theta = 0.$$

Then

$$\nabla \cdot \mathbf{A} = \nabla \cdot (b\nabla\theta) = \nabla b \cdot \nabla\theta + b\Delta\theta = 0.$$

Thus, by continuity, we get

$$\int (\nabla \cdot \mathbf{A})^2 = 0 \text{ for any } \mathbf{A} \in \mathcal{A}.$$

Then

$$\int |\nabla \times \mathbf{A}|^2 = \int (\nabla \cdot \mathbf{A})^2 + \int |\nabla \times \mathbf{A}|^2 = \int |\nabla \mathbf{A}|^2 .$$

□

3.4. Analysis of the minimizing sequences. The ratio energy/charge is a crucial quantity for the following lemmas. For a charge $\sigma > 0$ this ratio is defined as a function of u and $\mathbf{A}$ in the following way;

$$\Lambda_{\sigma,q}(u, \mathbf{A}) = \frac{E_{\sigma,q}(u, \mathbf{A})}{\sigma} = \frac{I(u, \mathbf{A})}{\sigma} + \frac{\sigma}{2K_q(u)}, \quad (u, \mathbf{A}) \in \hat{H}^1 \times \left(\mathcal{D}^{1,2}\right)^3, \ u \neq 0,$$

where

$$K_q(u) = \int ([1 - q\Phi_u])\, u^2. \tag{97}$$

In the following we shall always assume that the W satisfies W1),W2),W3), W4). First we state the following continuity lemma:

Lemma 16. *Let $u \in H^1$, then*

$$\int (1 - q\Phi_u)u^2 \to \int u^2 \text{ as } q \to 0.$$

Proof. Clearly it is enough to show that

$$q \int \Phi_u u^2 \to 0 \text{ as } q \to 0. \tag{98}$$

Since Φ_u depends on q a little work is needed to prove (98). Since Φ_u solves (82), we have

$$\begin{aligned} \|\Phi_u\|^2_{\mathcal{D}^{1,2}} + q^2 \int u^2 \Phi_u^2 &= q \int u^2 \Phi_u \\ &\leq q \, \|u\|^2_{L^{\frac{12}{5}}} \, \|\Phi_u\|_{L^6} , \end{aligned} \tag{99}$$

and then, if $u \neq 0$, we have

$$\frac{\|\Phi_u\|^2_{\mathcal{D}^{1,2}}}{\|\Phi_u\|_{L^6}} \leq q \, \|u\|^2_{L^{\frac{12}{5}}} .$$

So, since $\mathcal{D}^{1,2}$ is continuously embedded into L^6, we easily get

$$\|\Phi_u\|_{\mathcal{D}^{1,2}} \leq c_1 q \, \|u\|^2_{L^{\frac{12}{5}}} , \tag{100}$$

where c_1 is a positive constant. Then we get

$$q \int u^2 \Phi_u \leq q \, \|u\|^2_{L^{\frac{12}{5}}} \, \|\Phi_u\|_{L^6} \leq c_1 q^2 \, \|u\|^4_{L^{\frac{12}{5}}} ,$$

from which we deduce (98). □

Lemma 17. *There exist* $\sigma, \bar{q} > 0$, *such that for all* $0 \leq q < \bar{q}$ *there exists* $u \in \hat{H}_r^1$ *such that*

$$\Lambda_{\sigma,q}(u,0) < 1.$$

Proof. For $0 < \mu < \lambda$ we set:

$$T_{\lambda,\mu} = \left\{(r,x_3) : (r-\lambda)^2 + {x_3}^2 \leq \mu\right\}$$

and, for $\lambda > 2$, we consider a smooth function u_λ with cylindrical symmetry such that

$$u_\lambda(r,x_3) = \begin{cases} s_0 & if \quad (r,x_3) \in T_{\lambda,\lambda/2} \\ 0 & if \quad (r,x_3) \notin T_{\lambda,\lambda/2+1} \end{cases},$$

where s_0 is such that $N(s_0) < 0$ (see (43)). Moreover we may assume that

$$|\nabla u_\lambda\,(r,x_3)| \leq 2 \text{ for } (r,x_3) \in T_{\lambda,\lambda/2+1} \backslash T_{\lambda,\lambda/2}.$$

We have that for all $\sigma \neq 0$,

$$\Lambda_{\sigma,q}(u_\lambda,0) = \frac{1}{\sigma}\int\left[\frac{1}{2}|\nabla u_\lambda|^2 + \frac{\ell^2}{2}\frac{u_\lambda^2}{r^2} + W(u_\lambda)\right]dx + \frac{\sigma}{2K_q(u_\lambda)}$$
$$= \frac{\int\left[|\nabla u_\lambda|^2 + \frac{\ell^2 u_\lambda^2}{r^2}\right]dx}{2\sigma} + \frac{\int u_\lambda^2}{2\sigma} + \frac{\int N(u_\lambda)dx}{\sigma} + \frac{\sigma}{2K_q(u_\lambda)}$$

(remember that W has the form (42) and $m^2 = 1$). Now take

$$\sigma = \sigma_\lambda = \int u_\lambda^2;$$

in this case we get

$$\Lambda_{\sigma_\lambda,q}(u_\lambda,0) = \frac{1}{2} + \frac{\sigma_\lambda}{2K_q(u_\lambda)} + \frac{\int\left[|\nabla u_\lambda|^2 + \frac{\ell^2 u_\lambda^2}{r^2}\right]dx}{2\int u_\lambda^2} + \frac{\int N(u_\lambda)dx}{\int u_\lambda^2}. \tag{101}$$

By a direct computation we have that

$$\int |\nabla u_\lambda|^2 \leq c_1 meas(T_{\lambda,\lambda/2+1} \backslash T_{\lambda,\lambda/2}) = c_2\lambda^2 \tag{102}$$

$$\int \frac{u_\lambda^2}{r^2} \leq \frac{c_3}{\lambda^2} meas(T_{\lambda,\lambda/2+1}) = c_4\lambda \tag{103}$$

$$\int u_\lambda^2 \geq c_5 meas(T_{\lambda,\lambda/2+1}) = c_6\lambda^3 \tag{104}$$

so that

$$\frac{\int\left[|\nabla u_\lambda|^2 + \frac{\ell^2 u_\lambda^2}{r^2}\right]dx}{2\int u_\lambda^2} = O\left(\frac{1}{\lambda}\right). \tag{105}$$

Moreover

$$\int N(u_\lambda)dx \leq N(s_0)meas(T_{\lambda,\lambda/2}) + c_7 meas(T_{\lambda,\lambda/2+1} \backslash T_{\lambda,\lambda/2})$$
$$\leq c_8 N(s_0)\lambda^3 + c_9\lambda^2. \tag{106}$$

From (106) and (104) we get

$$\frac{\int N(u_\lambda)dx}{\int u_\lambda^2} \leq c_{10}\frac{N(s_0)}{s_0^2} + O\left(\frac{1}{\lambda}\right) = g(s_0, \lambda). \tag{107}$$

From (101), (105) and (107) we get

$$\Lambda_{\sigma_\lambda,q}(u_\lambda, 0) \leq \frac{1}{2} + \frac{\sigma_\lambda}{2K_q(u_\lambda)} + g(s_0, \lambda). \tag{108}$$

Since $N(s_0) < 0$, we can take λ_0 so large that

$$g(s_0, \lambda_0) < 0. \tag{109}$$

Now we take

$$\sigma = \sigma_{\lambda_0} = \int u_{\lambda_0}^2, \quad \text{and} \quad u = u_{\lambda_0}.$$

Now, by Lemma 16, we have

$$K_q(u) \to K_0(u) = \sigma \text{ for } q \to 0.$$

So

$$\frac{\sigma}{2K_q(u)} \to \frac{1}{2} \text{ for } q \to 0. \tag{110}$$

Then, by (108), (109) and (110), there is $\bar{q} > 0$ so small that, for all $0 \leq q < \bar{q}$, we have

$$\Lambda_{\sigma,q}(u, 0) \leq \frac{1}{2} + \frac{\sigma}{2K_q(u)} + g(s_0, \lambda_0) < 1.$$

□

Now the following a priori estimate on the minimizing sequences can be obtained

Lemma 18. *Any minimizing sequence* $(u_n, \mathbf{A}_n) \subset V$ *for* $E_{\sigma,q}\ |_V$ *is bounded in* $\hat{H}^1 \times \left(\mathcal{D}^{1,2}\right)^3$.

Proof. Let $(u_n, \mathbf{A}_n) \subset V$ be a minimizing sequence for $E_{\sigma,q}\ |_V$. Clearly

$$\|\mathbf{A}_n\|_{(\mathcal{D}^{1,2})^3} \text{ is bounded.}$$

So it remains to prove that

$$\|u_n\|_{\hat{H}_r^1} \text{ is bounded.} \tag{111}$$

To this end we shall first show that

$$\|u_n\|_{L^2} \text{ is bounded.} \tag{112}$$

Since $(u_n, \mathbf{A}_n)$ is a minimizing sequence for $E_{\sigma,q}|_V$ we get

$$\int W(u_n) \quad \text{and} \quad \int |\nabla u_n|^2 \text{ are bounded.} \tag{113}$$

Then we have also that

$$\int u_n^6 \text{ is bounded.} \tag{114}$$

Let $\varepsilon > 0$ and set

$$\Omega_n = \left\{ x \in \mathbb{R}^3 : |u_n(x)| > \varepsilon \right\} \quad \text{and} \quad \Omega_n^c = \mathbb{R}^3 \backslash \Omega_n.$$

By (113) and since $W \geq 0$ we have

$$\int_{\Omega_n^c} W(u_n) \text{ is bounded.} \tag{115}$$

By W_2) we can write

$$W(s) = \frac{1}{2}s^2 + \circ(s^2).$$

Then, if ε is small enough, there is a constant $c > 0$ such that

$$\int_{\Omega_n^c} W(u_n) \geq c \int_{\Omega_n^c} u_n^2. \tag{116}$$

By (115) and (116) we get that

$$\int_{\Omega_n^c} u_n^2 \text{ is bounded.} \tag{117}$$

On the other hand

$$\int_{\Omega_n} u_n^2 \leq \left(\int_{\Omega_n} u_n^6 \right)^{\frac{1}{3}} meas(\Omega_n)^{\frac{2}{3}}. \tag{118}$$

By (114) we have that

$$meas(\Omega_n) \text{ is bounded.} \tag{119}$$

By (118), (119) and again by (114) we get

$$\int_{\Omega_n} u_n^2 \text{ is bounded.} \tag{120}$$

So (112) follows from (117) and (120).

Let us finally prove (111).

Clearly

$$
\begin{aligned}
E_{\sigma,q}(u_n,\mathbf{A}_n) &\geq I(u_n,\mathbf{A}_n)\\
&\geq \frac{1}{2}\int \left(|\nabla u_n|^2 + |\nabla \mathbf{A}_n|^2 + q^2 |\mathbf{A}_n|^2 u_n^2 + \ell^2 \frac{u_n^2}{r^2} - 2q\frac{\ell}{r} |\mathbf{A}_n| \, |u_n|^2 \right) dx\\
&\geq \frac{1}{2} \|u_n\|^2_{\hat{H}^1_r} - q \int \frac{\ell}{r} |\mathbf{A}_n| \, |u_n|^2 - \sup \|u_n\|_{L^2}. \qquad (121)
\end{aligned}
$$

Also we have

$$
\begin{aligned}
\int \frac{q\ell}{r} |\mathbf{A}_n| \, |u_n|^2 &\leq \frac{1}{2}\int \left(4q^2\ell^2 |\mathbf{A}_n|^2 + \frac{1}{4r^2} \right) |u_n|^2\\
&\leq \frac{1}{8} \|u_n\|^2_{\hat{H}^1_r} + 2q^2\ell^2 \int |\mathbf{A}_n|^2 |u_n|^2. \qquad (122)
\end{aligned}
$$

Since $E_{\sigma,q}(u_n,\mathbf{A}_n)$ is bounded, by (121) and (122) we deduce that

$$
c_1 \geq \left(\frac{1}{2} - \frac{1}{8}\right) \|u_n\|^2_{\hat{H}^1_r} - 2q^2\ell^2 \int |\mathbf{A}_n|^2 |u_n|^2. \qquad (123)
$$

Here c_1, c_2 will denote suitable constants.

Now, since $\|u_n\|_{L^2}$ and $\|u_n\|_{L^6}$ are bounded, also $\|u_n\|_{L^3}$ is bounded.

Then, by using also the boundedness of $\|\mathbf{A}_n\|_{L^6}$, we get

$$
\int |\mathbf{A}_n|^2 |u_n|^2 \leq \left(\|\mathbf{A}_n\|_{L^6}\right)^{\frac{1}{3}} \left(\|u_n\|_{L^3}\right)^{\frac{2}{3}} \leq c_2. \qquad (124)
$$

From (123) and (124) we deduce the boundedness of $\|u_n\|^2_{\hat{H}^1_r}$. □

By Lemma 18 any minimizing sequence $U_n := (u_n, \mathbf{A}_n) \subset V$ of $E_{\sigma,q}|_V$ weakly converges (up to a subsequence). Observe that $E_{\sigma,q}$ is invariant for translations along the x_3-axis, namely for $U \in V$ and $L \in \mathbb{R}$ we have

$$
E_{\sigma,q}(T_L U) = E_{\sigma,q}(U),
$$

where

$$
T_L(U)(x_1,x_2,x_3) = U(x_1,x_2,x_3+L). \qquad (125)
$$

As a consequence of this invariance we have that $(u_n,\mathbf{A}_n)$ does not contain in general a (strongly) convergent subsequence. So we argue as follows: we prove that for suitable σ, q there exists a minimizing sequence $(u_n,\mathbf{A}_n)$ of $E_{\sigma,q}|_V$ which, up to translations along the x_3-direction, weakly converges to a non-trivial limit $(u_0,\mathbf{A}_0)$. This limit will be actually a critical point of $E_{\sigma_0,q}$ for some charge σ_0.

To follow the above program we first prove the following lemma

Lemma 19. *Let $U_n = (u_n,\mathbf{A}_n) \subset V$ be a minimizing sequence of $E_{\sigma,q}|_V$, $\sigma > 0$. Then there exist δ, $M > 0$ such that*

$$
\delta \leq \omega_n \leq M,
$$

where

$$
\omega_n = \frac{\sigma}{K_q(u_n)}.
$$

Proof. Since $(u_n, \mathbf{A}_n) \subset V$ is a minimizing sequence of the functional $E_{\sigma,q}|_V$ defined by

$$E_{\sigma,q}(u, \mathbf{A}) = I(u, \mathbf{A}) + \frac{\sigma^2}{2K_q(u)},$$

we have that for some constant $c_1 > 0$,

$$c_1 \leq K_q(u_n). \tag{126}$$

Also for some constant $c_2 > 0$ we have

$$K_q(u_n) \leq c_2. \tag{127}$$

In fact, arguing by contradiction, we assume that, up to a subsequence,

$$K_q(u_n) = \int (\left[1 - q\Phi_{u_n}\right]) u_n^2 \to \infty,$$

then by (84) also we get

$$\int u_n^2 \to \infty,$$

contradicting (112).

Finally the conclusion immediately follows from (126) and (127). □

Now we shall prove the following proposition

Proposition 20. *There exist σ, $\bar{q} > 0$ such that for all $0 \leq q < \bar{q}$, for any minimizing sequence $(u_n, \mathbf{A}_n) \subset V$ of $E_{\sigma,q}|_V$ we have*

$$\int |N(u_n)| \geq c > 0 \text{ for } n \text{ large.}$$

Proof. Let σ and q be chosen as required in Lemma 17. Now let $(u_n, \mathbf{A}_n) \subset V$ be a minimizing sequence of $E_{\sigma,q}$ and hence of $\Lambda_{\sigma,q}$. Then by Lemma 17 we get for n sufficiently large,

$$\Lambda_{\sigma,q}(u_n, \mathbf{A}_n) \leq 1 - \delta, \ \ \delta > 0. \tag{128}$$

Then we have also

$$\frac{\int \left[|\nabla u_n|^2 + \frac{\ell^2 u_n^2}{r^2}\right] dx}{2\sigma} + \frac{\int u_n^2}{2\sigma} + \frac{\int N(u_n) dx}{\sigma} + \frac{\sigma}{2\int u_n^2} \leq 1 - \delta.$$

Thus

$$\frac{\int N(u_n) dx}{\sigma} \leq 1 - \delta - \left(\frac{\int u_n^2}{2\sigma} + \frac{\sigma}{2\int u_n^2}\right) \leq -\delta.$$

This implies that

$$\int N(u_n) dx \leq -\delta\sigma.$$

Then

$$\int |N(u_n)|\,dx \geq \delta\sigma.$$

□

Proposition 21. *For any* $\sigma, q \geq 0$ *there exists a minimizing sequence* $(u_n, \mathbf{A}_n)$ *of* $E_{\sigma,q}|_V$, *with* $u_n \geq 0$ *and which is also a P.S. sequence for* $E_{\sigma,q}$, *i.e.*

$$E'_{\sigma,q}\,(u_n, \mathbf{A}_n) \to 0.$$

Proof. Let $(u_n, \mathbf{A}_n) \subset V$ be a minimizing sequence for $E_{\sigma,q}\,|_V$. It is not restrictive to assume that $u_n \geq 0$, in fact, if not, we can replace u_n with $|u_n|$ (see (90)). By standard variational arguments we can also assume that $(u_n, \mathbf{A}_n)$ is a P.S. sequence for $E_\sigma\,|_V$, namely we can assume that

$$E'_{\sigma,q}\,|_V\,(u_n, \mathbf{A}_n) \to 0.$$

By using the same arguments used in proving Theorem 16 in [11], it can be shown that $(u_n, \mathbf{A}_n)$ is a P.S. sequence also for $E_{\sigma,q}$, i.e.

$$E'_{\sigma,q}\,(u_n, \mathbf{A}_n) \to 0. \tag{129}$$

□

Proposition 22. *There exist* $\sigma, \bar{q} > 0$ *such that for all* $0 \leq q < \bar{q}$ *there exists a P.S. sequence* $U_n = (u_n, \mathbf{A}_n)$ *for* $E_{\sigma,q}$ *which weakly converges to* $(u_0, \mathbf{A}_0)$, $u_0 \geq 0$ *and* $u_0 \neq 0$..

Proof. Take σ, q as in Proposition 20. By Proposition 21 there exists a minimizing sequence $U_n = (u_n, \mathbf{A}_n)$ of $E_{\sigma,q}\,|_V$ with $u_n \geq 0$ and which is also a P.S. sequence for $E_{\sigma,q}$, i.e.

$$E'_{\sigma,q}\,(U_n) \to 0.$$

By Proposition 20 and assumption W4), we can assume that

$$c_1\,\|u_n\|^q_{L^q} + c_2\,\|u_n\|^p_{L^p} \geq c > 0 \text{ for } n \text{ large.} \tag{130}$$

By Lemma 18 the sequence $\{U_n\}$ is bounded in $\hat{H}^1 \times \left(\mathcal{D}^{1,2}\right)^3$ so we can assume that it weakly converges. However the weak limit could be trivial. We will show that there is a sequence of integers j_n such that (see (125)) $V_n := T_{j_n}U_n \rightharpoonup U_0 = (u_0, \mathbf{A}_0)$, $u_0 \neq 0$, weakly in $H^1 \times \left(\mathcal{D}^{1,2}\right)^3$.

We set

$$\Omega_j = \{(x_1, x_2, x_3) : j \leq x_3 < j+1\},\; j \text{ integer.}$$

In the following $c_3, \ldots, c_6$ denote positive constants.

We have for all n,

$$\begin{aligned}\|u_n\|_{L^q}^q &= \sum_j \int_{\Omega_j} |u_n|^q = \sum_j \left(\int_{\Omega_j} |u_n|^q\right)^{1/q} \cdot \left(\int_{\Omega_j} |u_n|^q\right)^{\frac{q-1}{q}} \\ &\leq \sup_j \|u_n\|_{L^q(\Omega_j)} \sum_j \left(\int_{\Omega_j} |u_n|^q\right)^{\frac{q-1}{q}} \leq c_3 \cdot \sup_j \|u_n\|_{L^q(\Omega_j)} \cdot \sum_j \|u_n\|_{H^1(\Omega_j)}^{q-1} \\ &\leq c_3 \cdot \sup_j \|u_n\|_{L^q(\Omega_j)} \cdot \|u_n\|_{H^1(\mathbb{R}^3)}^{q-1} \leq (\text{since } \|u_n\|_{H^1(\mathbb{R}^3)}^{q-1} \text{ is bounded}) \\ &\leq c_4 \sup_j \|u_n\|_{L^q(\Omega_j)} . \end{aligned} \tag{131}$$

Analogously we get

$$\|u_n\|_{L^p}^p \leq c_5 \sup_j \|u_n\|_{L^p(\Omega_j)} . \tag{132}$$

Then by (130), (131) and (132) it is easy to deduce that, for n large, there exists an integer j_n such that

$$\|u_n\|_{L^q(\Omega_{j_n})} + \|u_n\|_{L^p(\Omega_{j_n})} \geq c_6 > 0. \tag{133}$$

Now set

$$\left(u_n', \mathbf{A}_n'\right) = U_n'(x_1, x_2, x_3) = U_n(x_1, x_2, x_3 + j_n) = T_{j_n}(U_n) .$$

By Lemma 18 the sequence u_n' is bounded $\hat{H}^1\left(\mathbb{R}^3\right)$, then (up to a subsequence) it converges weakly to $u_0 \in \hat{H}^1\left(\mathbb{R}^3\right)$. Clearly $u_0 \geq 0$, since $u_n' \geq 0$. We want to show that $u_0 \neq 0$. Now, let $\varphi = \varphi(x_3)$ be a nonnegative, C^∞-function whose value is 1 for $0 < x_3 < 1$ and 0 for $|x_3| > 2$. Then, the sequence $\varphi u_n'$ is bounded in $H_0^1(\mathbb{R}^2 \times (-2, 2))$, moreover $\varphi u_n'$ has cylindrical symmetry. Then, using the compactness result proved in [23], we have that, up to a subsequence,

$\varphi u_n'$ converges strongly both in $L^q(\mathbb{R}^2 \times (-2, 2))$ and in $L^p(\mathbb{R}^2 \times (-2, 2))$.

On the other hand

$$\varphi u_n' \to \varphi u_0 \; a.e. \tag{134}$$

Then

$$\varphi u_n' \to \varphi u_0 \text{ strongly both in } L^q(\mathbb{R}^2 \times (-2, 2)) \text{ and in } L^p(\mathbb{R}^2 \times (-2, 2)). \tag{135}$$

Moreover for $r = p, q$ we clearly have

$$\left\|\varphi u_n'\right\|_{L^r(\mathbb{R}^2\times(-2,2))} \geq \left\|u_n'\right\|_{L^r(\Omega_0)} = \|u_n\|_{L^r(\Omega_{j_n})} . \tag{136}$$

Then by (135), (136) and (133) we have

$$\|\varphi u_0\|_{L^q(\mathbb{R}^2\times(-2,2))} + \|\varphi u_0\|_{L^p(\mathbb{R}^2\times(-2,2))} \geq c_6 > 0.$$

Thus we have that $u_0 \neq 0$. □

Proposition 23. *There exists* $\bar{q} > 0$ *such that, for all* $0 \leq q < \bar{q}$, *for some charge* $\sigma_0 > 0$, $E_{\sigma_0,q}$ *has a critical point* $(u_0, \mathbf{A}_0)$ $u_0 \neq 0$, $u_0 \geq 0$.

Proof. Let $\sigma, q > 0$ be as in Proposition 22, then there exists a sequence $U_n = (u_n, \mathbf{A}_n)$ in V, with $u_n \geq 0$ and such that

$$E'_{\sigma,q}(u_n, \mathbf{A}_n) \to 0 \tag{137}$$

and

$$(u_n, \mathbf{A}_n) \to (u_0, \mathbf{A}_0) \text{ weakly, } u_0 \neq 0.$$

Since $u_n \geq 0$ we have $u_0 \geq 0$.

Let us show that $U_0 = (u_0, \mathbf{A}_0)$ is a critical point of $E_{\sigma_0,q}$ for some charge $\sigma_0 > 0$.

By (137) we get that

$$dE_{\sigma,q}(U_n)[w, 0] \to 0 \quad \text{and} \quad dE_{\sigma,q}(U_n)[0, \mathbf{w}] \to 0 \text{ for any } (w, \mathbf{w}) \in \hat{H}^1 \times \left(\mathcal{D}^{1,2}\right)^3.$$

Then for any $w \in C_0^\infty(\mathbb{R}^3 \setminus \Sigma)$ and $\mathbf{w} \in \left(C_0^\infty(\mathbb{R}^3)\right)^3$ we have

$$d_u I(U_n)[w] + d_u \left(\frac{\sigma^2}{2K_q(u_n)}\right)[w] \to 0 \tag{138}$$

and

$$d_{\mathbf{A}} I(U_n)[\mathbf{w}] \to 0, \tag{139}$$

where d_u and $d_{\mathbf{A}}$ denote the partial differentials of I with respect u and $\mathbf{A}$. So from (138) we get for any $w \in C_0^\infty(\mathbb{R}^3 \setminus \Sigma)$,

$$d_u I(U_n)[w] - \frac{\sigma^2 K'_q(u_n)}{2\left(K_q(u_n)\right)^2}[w] \to 0$$

which can be written as follows:

$$d_u I(U_n)[w] - \frac{\omega_n^2 K'_q(u_n)}{2}[w] \to 0, \tag{140}$$

where

$$\omega_n = \frac{\sigma}{K_q(u_n)}.$$

By Lemma 19 we have (up to a subsequence)

$$\omega_n \to \omega_0 > 0.$$

Then by (140) we get for any $w \in C_0^\infty(\mathbb{R}^3 \setminus \Sigma)$

$$d_u I(U_n)[w] - \frac{\omega_0^2 K'_q(u_n)}{2}[w] \to 0. \tag{141}$$

Now let Φ_n be the solution in $\mathcal{D}^{1,2}$ of the equation

$$-\Delta\Phi_n + q^2 u_n^2 \Phi_n = q u_n^2. \tag{142}$$

Since $\{u_n\}$ is bounded in H^1 (see (111) and (112)) and since Φ_n solves (142), standard Sobolev estimates show that $\{\Phi_n\}$ is bounded in $\mathcal{D}^{1,2}$ and that its weak limit (up to subsequence) Φ_0 is a weak solution of

$$-\Delta\Phi_0 + q^2 u_0^2 \Phi_0 = q u_0^2. \tag{143}$$

Then, by Lemma 13, we have

$$K_q'(u_n) = 2u_n(1 - q\Phi_n)^2 \quad \text{and} \quad K_q'(u_0) = 2u_0(1 - q\Phi_0)^2. \tag{144}$$

By standard calculations we have:

$$\begin{gathered}\text{for any } w \in C_0^\infty(\mathbb{R}^3 \setminus \Sigma) \\ \int u_n(1 - q\Phi_n)^2 w \to \int u_0(1 - q\Phi_0)^2 w.\end{gathered} \tag{145}$$

Then, by (144) and (145), we get for any $w \in C_0^\infty(\mathbb{R}^3 \setminus \Sigma)$,

$$K_q'(u_n)\,[w] \to K_q'(u_0)\,[w]\,. \tag{146}$$

Similar standard estimates show that for any $w \in C_0^\infty(\mathbb{R}^3 \setminus \Sigma)$,

$$d_u I(U_n)\,[w] \to d_u I(U_0)\,[w]\,. \tag{147}$$

Then, passing to the limit in (141), by (146) and (147), we get

$$d_u I(U_0)\,[w] - \frac{\omega_0^2 K_q'(u_0)}{2}\,[w] = 0 \text{ for any } w \in C_0^\infty(\mathbb{R}^3 \setminus \Sigma). \tag{148}$$

On the other hand similar arguments show that we can pass to the limit also in $d_{\mathbf{A}} I(U_n)\,[\mathbf{w}]$ and have

$$\begin{gathered}\text{for all } \mathbf{w} \in \left(C_0^\infty(\mathbb{R}^3)\right)^3 \\ d_{\mathbf{A}} I(U_n)\,[\mathbf{w}] \to d_{\mathbf{A}} I(U_0)\,[\mathbf{w}]\,.\end{gathered} \tag{149}$$

From (139) and (149) we get

$$d_{\mathbf{A}} I(U_0)\,[\mathbf{w}] = 0 \text{ for all } \mathbf{w} \in \left(C_0^\infty(\mathbb{R}^3)\right)^3. \tag{150}$$

By (148) and (150) we deduce, by using density and continuity arguments, that $U_0 = (u_0, \mathbf{A}_0)$ is a critical point of $E_{\sigma_0,q}$ with $\sigma_0 = \omega_0 K_q(u_0) > 0$. □

Proof of Theorem 3. The first part of Theorem 3 immediately follows from Propositions 23, 14 and Theorem 10. In fact, if u_0, $\mathbf{A}_0$ are like in Proposition 23, by Proposition 14 and Theorem 10 we deduce that $(u_0, \omega_0, \phi_0, \mathbf{A}_0)$ with $\omega_0 = \frac{\sigma_0}{K_q(u_0)}$, $\phi_0 = Z_{\omega_0}(u_0)$ solves (39), (40), (41).

Now assume $q = 0$, then, by (40) and (41), we easily deduce that $\phi_0 = 0$ and $\mathbf{A}_0 = 0$. Finally assume that $q > 0$. Then, since $\omega_0 > 0$, by (40) we deduce that $\phi_0 \neq 0$. Moreover by (41) we easily deduce that $\mathbf{A}_0 \neq 0$ if and only if $\ell \neq 0$. □

References

1. Abrikosov, A.A.: On the magnetic properties of superconductors of the second group. Sov. Phys. JETP **5**, 1174–1182 (1957)
2. Anagnostopoulos, K.N., Axenides, M., Floratos, E.G., Tetradis, N.: Large gauged Q-Balls. Phys. Rev. **D64** (2001)

3. Badiale, M., Benci, V., Rolando, S.: *Three dimensional vortices in the nonlinear wave equation*. Boll. Unione Mat. Ital., Ser. IX, in press
4. Bellazzini, J., Benci, V., Bonanno, C., Sinibaldi, E.: *Hylomorphic solitons in the nonlinear Klein-Gordon equation*. http://arxiv.org/abs/0810.5079v1[math.Ap], 2008
5. Bellazzini, J., Bonanno, C.: *Nonlinear Schrödinger equations with strongly singular potentials.* http://arxiv.org/abs/0903.3301v1[math-ph], 2009
6. Benci, V.: Hylomorphic solitons. Milan J. Math. **77**, 271–332 (2009)
7. Benci, V., Fortunato, D.: Solitary waves of the nonlinear Klein-Gordon field equation coupled with the Maxwell equations. Rev. Math. Phys. **14**, 409–420 (2002)
8. Benci, V., Fortunato, D.: Solitary waves in Abelian gauge theories. Adv. Nonlinear Stud. **3**, 327–352 (2008)
9. Benci, V., Fortunato, D.: Solitary waves in the nolinear wave equation and in Gauge theories. J. Fixed Point Th and Appl. **1**(1), 61–86 (2007)
10. Benci, V., Fortunato, D.: Existence of 3D-vortices in abelian Gauge theories. Med. J. Math. **3**, 409–418 (2006)
11. Benci, V., Fortunato, D.: Three dimensional vortices in abelian Gauge theories. Nonlinear Analysis **70**, 4402–4421 (2009)
12. Benci, V., Fortunato, D.: Existence of hylomorphic solitary waves in Klein-Gordon and in Klein-Gordon-Maxwell equations. Rend. Accad. Naz. Lincei, Mat. Appl. **20**, 243–279 (2009)
13. Benci, V., Visciglia, N.: Solitary waves with non vanishing angular momentum. Adv. Nonlinear Stud. **3**, 151–160 (2003)
14. Berestycki, H., Lions, P.L.: Nonlinear scalar field equations, I - Existence of a ground state. Arch. Rat. Mech. Anal. **82**, 313–345 (1983)
15. Cassani, D.: Existence and non-existence of solitary waves for the critical Klein-Gordon equation coupled with Maxwell's equations. Nonlinear Anal. **58**, 733–747 (2004)
16. Campanelli, L., Ruggieri, M.: Spinning supersymmetric Q balls. Phys. Rev. D **80**, 036006 (2009)
17. Coleman, S., Glaser, V., Martin, A.: Action minima among solutions to a class of Euclidean Scalar field equation. Commun. Math. Phys. **58**, 211–221 (1978)
18. Coleman, S.: Q-Balls. Nucl. Phys. **B262**, 263–283 (1985); erratum: **B269**, 744–745 (1986)
19. D'Aprile, T., Mugnai, D.: Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger -Maxwell equations. Proc. of Royal Soc. of Edinburgh, Sect. A Math. **134**, 893–906 (2004)
20. D'Aprile, T., Mugnai, D.: Non-existence results for the coupled Klein-Gordon- Maxwell equations. Adv. Nonlinear Stud. **4**, 307–322 (2004)
21. D'Avenia, P., Pisani, L.: Nonlinear Klein-Gordon equations coupled with Born-Infeld equations. Electronics J. Diff. Eqs. **26**, 1–13 (2002)
22. Enqvist, K., McDonald, J.: Q-Balls and Baryogenesis in the MSSM. Phys. Lett. B **425**, 309–321 (1998)
23. Esteban, M., Lions, P.L.: A compactness lemma. Nonlinear Anal. **7**, 381–385 (1983)
24. Felsager, B.: *Geometry, Particles and Fields*. Odense: Odense University Press, 1981
25. Gelfand, I.M., Fomin, S.V.: *Calculus of Variations*. Englewood Cliffs, NJ: Prentice-Hall, 1963
26. Kim, C., Kim, S., Kim, Y.: Global nontopological vortices. Phys. Rev. D **47**, 5434–5443 (1985)
27. Lee, K., Stein-Schabes, J.A., Watkins, R., Widrow, L.M.: Gauged Q balls. Phys. Rev. D **39**, 1665–1673 (1989)
28. Kusenko, A., Shaposhnikov, M.: Supersymmetric Q-balls as dark matter. Phys. Lett. B **418**, 46–54 (1998)
29. Landau, L., Lifchitz, E.: *Théorie du Champ*. Moscow: Editions Mir, 1966
30. Nielsen, H., Olesen, P.: Vortex-line models for dual strings. Nucl. Phys. B **61**, 45–61 (1973)
31. Rajaraman, R.: *Solitons and Instantons*. Amsterdam: North-Holland, 1989
32. Rosen, G.: Particlelike solutions to nonlinear complex scalar field theories with positive-definite energy densities. J. Math. Phys. **9**, 996–998 (1968)
33. Rubakov, V.: *Classical Theory of Gauge Fields*. Princeton, NJ: Princeton University Press, 2002
34. Strauss, W.A.: Existence of solitary waves in higher dimensions. Commun. Math. Phys. **55**, 149–162 (1977)
35. Struwe, M.: *Variational Methods, Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*. NewYork-Berlin: Springer, 1996
36. Vilenkin, A., Shellard, E.P.S.: *Cosmic Strings and other Topological Defects*. Cambridge: Cambrige University Press, 1994
37. Volkov, M.S., Wöhnert, E.: Spinning Q-balls. Phys. Rev. D **66**, 085003 (2002)
38. Yang, Y.: *Solitons in Field Theory and Nonlinear Analysis*. NewYork-Berlin: Springer, 2000

Communicated by G. Gallavotti