

The Dependence on the Monodromy Data of the Isomonodromic Tau Function

M. Bertola^{1,2,*}

¹ Department of Mathematics and Statistics, Concordia University,
1455 de Maisonneuve W., Montréal, Québec, Canada H3G 1M8

² Centre de recherches mathématiques, Université de Montréal, Montréal,
Canada. E-mail: bertola@crm.umontreal.ca; bertola@mathstat.concordia.ca

Received: 27 March 2009 / Accepted: 22 September 2009
Published online: 5 December 2009 – © Springer-Verlag 2009

Abstract: The isomonodromic tau function defined by Jimbo–Miwa–Ueno vanishes on the Malgrange’s divisor of generalized monodromy data for which a vector bundle is non-trivial, or, which is the same, a certain Riemann–Hilbert problem has no solution. In their original work, Jimbo, Miwa, Ueno provided an algebraic construction of its derivatives with respect to isomonodromic times. However the dependence on the (generalized) monodromy data (i.e. monodromy representation and Stokes’ parameters) was not derived. We fill the gap by providing a (simpler and more general) description in which all the parameters of the problem (monodromy-changing and monodromy-preserving) are dealt with at the same level. We thus provide variational formulæ for the isomonodromic tau function with respect to the (generalized) monodromy data. The construction applies more generally: given any (sufficiently well-behaved) family of Riemann–Hilbert problems (RHP) where the jump matrices depend arbitrarily on deformation parameters, we can construct a one-form Ω (not necessarily closed) on the deformation space (Malgrange’s differential), defined off Malgrange’s divisor. We then introduce the notion of discrete Schlesinger transformation: it means that we allow the solution of the RHP to have poles (or zeros) at prescribed point(s). Even if Ω is not closed, its difference evaluated along the original solution and the transformed one, is shown to be the logarithmic differential (on the deformation space) of a function. As a function of the position of the points of the Schlesinger transformation, it yields a natural generalization of the Sato formula for the Baker–Akhiezer vector even in the absence of a tau function, and it realizes the solution of the RHP as such BA vector. Some exemplifications in the setting of the Painlevé II equation and finite Töplitz/Hankel determinants are provided.

* Work supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC).

Contents

- 1. Introduction 540
- 2. Riemann–Hilbert Problems 543
 - 2.1 Deformations and Malgrange’s form 545
 - 2.2 Submanifolds of \mathcal{G} where Ω is closed 551
 - 2.3 “Schlesinger” transformations 552
 - 2.3.1 Generalized Sato formula 558
 - 2.3.2 Hirota bilinear relations 560
 - 2.4 Right gauge equivalence 561
- 3. Rational Differential Equations in Terms of Riemann–Hilbert Data 562
 - 3.1 Monodromy map 562
 - 3.1.1 Forward Birkhoff map. 565
- 4. Inverse Birkhoff Map: Riemann–Hilbert Problem 565
 - 4.1 The set of contours 565
 - 4.2 The jump-matrices 566
 - 4.3 Riemann–Hilbert problem 566
- 5. Tau Function and (Iso)monodromic Deformations 567
 - 5.1 Relationship between ω_M and ω_{JMU} 569
- 6. Applications and Examples 571
 - 6.1 Painlevé II equation 571
 - 6.2 Variation of finite Toeplitz determinants for discontinuous symbols 573
 - 6.3 Hankel and shifted Töplitz determinants 577

1. Introduction

In the eighties Jimbo, Miwa and Ueno [9, 7, 8] derived a set of algebraic (in fact rational) nonlinear equations describing deformations of a rational connection on \mathbb{P}^1 which preserve the generalized monodromy data. They associated to this deformation a closed differential ω_{JMU} on the space of deformation parameters, namely, on the space of “isomonodromic times” which we denote collectively by \vec{t} . In the simplest case of Fuchsian singularities,

$$\Psi'(z) = A(z)\Psi(z), \quad \Psi(\infty) = 1, \quad A(z) = \sum_{j=1}^K \frac{A_j}{z - a_j}, \quad \sum A_j = 0, \quad (1.1)$$

the isomonodromic deformation equations were studied by Schlesinger [15]

$$\delta A_k = - \sum_{j \neq k} [A_k, A_j] \frac{\delta(a_k - a_j)}{a_k - a_j}, \quad \delta := \sum da_j \partial_{a_j}, \quad (1.2)$$

and the Jimbo-Miwa-Ueno differential reads

$$\omega_{JMU} = \frac{1}{2} \sum_{j,k,j \neq k} \text{Tr}(A_j A_k) \frac{\delta(a_j - a_k)}{a_j - a_k}. \quad (1.3)$$

It can be checked directly that if the matrices $A_k(\vec{a})$ depend on the position of the poles as mandated by (1.2) then ω_{JMU} above is a closed differential.

This differential was generalized to an arbitrary (generic) rational connection in [9], to which we refer the reader for details. In the above situation for Schlesinger deformations, the locations of the poles constitute the “isomonodromic” parameters or times and we denote them by \vec{t} . This was an important achievement because of the sweeping applications of isomonodromic deformation to integrable systems (solitons solutions to KP, solutions to Toda, etc.), Painlevé equations and, later, random matrix models. The (exponential) integral of this closed differential is the “isomonodromic tau function”

$$\tau_{JMU}(\vec{t}; \vec{m}) = e^{\int \omega_{JMU}}, \tag{1.4}$$

and the Painlevé property translates to the fact that τ_{JMU} is a holomorphic function of the isomonodromic times that has only zeroes away from an explicit set of times where it has a branching behavior (the set of “diagonals” $a_j = a_k, j \neq k$ in the case of (1.2)). In (1.4) we have indicated that the tau function depends necessarily on the (generalized) monodromy data, denoted generically by \vec{m} ; this dependence is parametric and the present paper addresses precisely the

Question 1 (Naive). *What is the dependence of τ_{JMU} on the monodromy data?*

The question is conceptually simple but slightly ill-posed; since what JMU really defined was only a differential (in symbolic notation)

$$\omega_{JMU} = \sum_j f_j(\vec{t}; \vec{m}) dt_j, \quad \partial_{t_j} f_k = \partial_{t_k} f_j, \tag{1.5}$$

the dependence of τ on \vec{m} is defined only up to multiplication by an arbitrary function of the \vec{m} ’s only. So a better question would be

Question 2 (Refined). *What is the essential dependence of τ_{JMU} on the monodromy data? Can we define an extended **closed** differential Ω on the **total phase space** of the problem that coincides with ω_{JMU} on the isomonodromic submanifold, namely*

$$\omega_{ext} = \sum_j f_j(\vec{t}; \vec{m}) dt_j + \sum_v G_v(\vec{t}, \vec{m}) dm_v, \tag{1.6}$$

$$\partial_{m_v} f_k = \partial_{t_k} G_v, \quad \partial_{m_v} G_\mu = \partial_{m_v} G_\mu? \tag{1.7}$$

The question still admits many solutions as stated, since if we find one such extension we are still free to add any closed differential of the \vec{m} ’s alone. However we may and should understand the problem in a relative setting, where the answer is taken modulo closed forms of the \vec{m} ’s which must be holomorphic on the whole space of monodromy data. The reason for requiring holomorphicity is actually important because of the interpretation of the singularity locus of ω , as we presently explain.

The meaning of the singularity locus of ω_{JMU} . Malgrange (for Fuchsian systems) [11] and later Palmer (for irregular singularities)[14] showed what the meaning of the zero-locus of τ_{JMU} is: when $\tau_{JMU}(\vec{t}; \vec{m}) = 0$ then a vector bundle on \mathbb{P}^1 is nontrivial or –which is equivalent– a Riemann–Hilbert problem is not solvable. This is the equivalent of saying that ω_{JMU} has only simple poles (with “residue” one) away from the non-movable singularity locus (Painlevé property). The divisor where the aforementioned bundle is nontrivial is generally termed **Malgrange Θ divisor** (or simply Malgrange divisor).

It is clear then that whatever extension (1.6) we are looking for, it ought to preserve the singularity locus, that is, the $G_v(\vec{t}, \vec{m})$ may have singularities only where some of the f_k ’s has, and of the same type.

So we arrive to the final formulation of a “sensible” problem, whose solution is the principal aim of the paper.

Problem 1.1. *Formulate a (“natural”) extended closed differential ω_{ext} (1.6) such that its tau function (locally defined up to nonzero multiplicative constants)*

$$\tau_{ext}(\vec{t}, \vec{m}) = e^{\int \omega_{ext}} \quad (1.8)$$

vanishes precisely and only on the Malgrange divisor in the extended phase space of isomonodromic times \vec{t} and monodromy data \vec{m} .

The differential we propose is in fact very natural (see Def. 2.2 and Thm. 5.1): it is the pull-back of the (partial integral of the) generator of the third cohomology of the “loop group” to the submanifold corresponding to our total phase space. In particular it expresses the same cohomology class $c[\Theta]$ [11].

We will briefly indicate some interesting problems which can be addressed and that require the knowledge of the derivative of τ with respect to non-isomonodromic times (hence the knowledge of the G_v ’s).

Organization of the paper. The heart of the paper is in fact Sect. 2 where we introduce (recall) the definition of the Malgrange differential ω_M (Def. 2.1) associated to *any (sufficiently well-behaved) Riemann–Hilbert problem*. Such differential is ill-defined only when the RHP is not solvable: however its exterior differential is (admits a) smooth (extension) over the whole “loop group”, in particular also at the points where the RHP is not solvable [11]. The curvature of ω_M is not zero, but it is so explicit that it is immediate to identify “simple” families of Riemann–Hilbert problems for which ω_M is closed. By adding to ω_M an explicit smooth differential (in particular this does not change its singularity locus) we obtain a new differential Ω (Def. 2.2) whose curvature differs slightly from that of ω_M but within the same cohomology class.

In Sect. 2.3 we investigate the changes in Ω (or ω_M) under modification of the growth conditions for the solution of the RHP (discrete “Schlesinger” transformations); this allows to interpret the matrix solution of *any* Riemann–Hilbert problem as a Baker–Akhiezer function via a Sato-like formula, even if a notion of tau-function is not available (see Thm. 2.1 and Sect. 2.3.1, Sect. 2.3.2).

In the second part we specialize the setting from arbitrary Riemann–Hilbert problems to those that correspond to rational ODEs in the complex plane: Section 3 is a quick reminder to the reader of the classical description of the (generalized) monodromy map, i.e. how to associate to a rational ODE in the complex plane the set of “Birkhoff data” of irregular type, connection matrices, monodromy representation and Stokes’ multipliers. All the material is quite standard [17].

These Birkhoff data can be used viceversa (Sect. 4) to encode an ODE in a Riemann–Hilbert problem (which *may* or may not have a solution, although generically it does [16,17]).

In Sect. 5 we show that Ω is a closed differential in all the deformation parameters, which include • Monodromy representation; • Connection matrices; • Stokes’ matrices.

In Sect. 5.1 we show that the restriction of Ω to the submanifold of isomonodromic times coincides with the differential ω_{JMU} defined in [9]. Therefore Ω realizes the solution to our Problem 1.1.

We conclude the paper with Sect. 6, where applications are provided to Painlevé II, (shifted) Toeplitz and Hankel determinants.

2. Riemann–Hilbert Problems

A Riemann–Hilbert problem consists of the data specified below: we will assume latitude in the smoothness class as this is not our primary focus.

The Riemann–Hilbert data.

1. A finite collection of smooth oriented arcs γ_ν , $j = 1 \dots K$, possibly meeting at a finite number of points but always in non-tangential way (Fig. 5 is a good example). We denote collectively these arcs by the symbol $\Sigma\gamma$,

$$\Sigma\gamma = \bigcup_{\nu} \gamma_{\nu}. \tag{2.1}$$

If any of the contours γ_ν extends to infinity then we shall assume that the contour has an asymptotic direction and that the corresponding jump matrix $M_\nu(z)$ tends to the identity faster than any power $M_\nu(z) = \mathbf{1} + \mathcal{O}(z^{-\infty})$, $z \rightarrow \infty$, with this decay valid in an open sector of nonzero angle around the direction of approach.

2. A collection of $r \times r$ matrices $M_\nu(z)$, each of which is analytic at each interior point of its corresponding arc γ_ν of a locally holomorphic function. We make the assumption that these matrices have unit determinant¹ $\det M_\nu(z) \equiv 1$. We will denote collectively by $M(z)$ the matrix defined on $\Sigma\gamma$ that coincides with $M_\nu(z)$ on γ_ν ,

$$\begin{aligned} M : \Sigma\gamma &\rightarrow SL_r(\mathbb{C}) \\ z &\mapsto \sum_{\nu} M_{\nu}(z)\chi_{\gamma_{\nu}}(z) \end{aligned} \tag{2.2}$$

where, for a set S , χ_S denotes its indicator function.

3. At each point c where several arcs meet, denoting by $\gamma_1, \dots, \gamma_\ell$ the arcs entering a suitably small disk at c , we impose that the jump matrices along its corresponding arc either tend to the identity matrix as $\mathcal{O}((z - c)^\infty)$ (faster than any power) in an open sector containing the direction of approach, or admits a local analytic extension within said disk.

Remark 2.1. The last requirement allows for example the situation depicted on the right in Fig. 1; at the meeting point, c , of the circle and the “stem”, the jump matrices on each local arc admit an analytic continuation to a neighborhood of c . Note that the jump on the circle is actually a multivalued analytic function, but in a neighborhood of c each branch allows an analytic continuation.

Remark 2.2. The condition for the jumps on contours that extend to infinity is perhaps overly restrictive and it is made to avoid case distinctions later on. For example, the RHP for the Airy functions (or for the linear problem associated to the Painlevé one transcendent) does not readily fall within this description. It would however be possible to reduce them to such type of situations with a sequence of transformations. Since the point is of rather technical nature, we leave it to be dealt with on a need-to basis.

¹ This is not really necessary but simplifies some matters. In some situations (typically where the determinant is a rational function) it is necessary (but simple) to relax the condition and allow any matrices in $GL_r(\mathbb{C})$. We face the problem on a need-to basis.

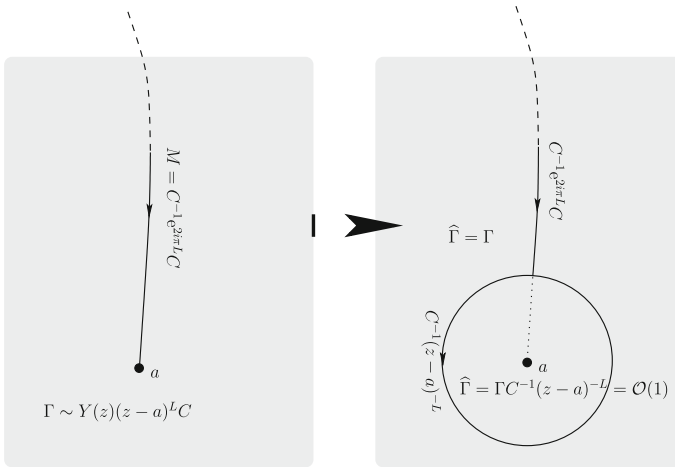


Fig. 1. The process of “cloaking” of a point of growth. The two RHPs are equivalent in the sense that one admits a solution if and only if the other does

The Riemann–Hilbert problem then consists in finding a holomorphic matrix on the complement of the contours

$$\Gamma(z) : \mathbb{C} \setminus \Sigma\gamma \rightarrow SL_r(\mathbb{C}) \tag{2.3}$$

such that it admits (non-tangential) boundary values satisfying

$$\Gamma_+(z) = \Gamma_-(z)M(z) \quad z \in \Sigma\gamma. \tag{2.4}$$

Here and throughout the paper we use the common convention that a subscript \pm denotes the (nontangential) boundary value from the left (plus) or the right (minus) of the given oriented arc. The above data are insufficient to characterize uniquely the matrix Γ (if it exists) and need to be supplemented by

- growth behavior near the endpoints/intersections of the contours γ_ν and at ∞ ;
- an overall normalization.

Therefore we will pose the following

Problem 2.1 (RHP). Find a holomorphic matrix $\Gamma : \mathbb{C} \setminus \Sigma\gamma \rightarrow GL_n(\mathbb{C})$ such that

- $\Gamma_+(x) = \Gamma_-(x)M(x) \quad x \in \Sigma\gamma$;
- $\Gamma(z)$ is uniformly bounded in \mathbb{C} ;
- $\Gamma(z_0) = \mathbf{1}$.

In the following we will always choose $z_0 = \infty$. In case there are contours extending to infinity, this normalization needs a little of explanation: thanks to the requirement that the jump matrices at infinity are exponentially close to the identity one can see that the value of the solution $\Gamma(z)$ is actually continuous at $z = \infty$ and hence it can be chosen as a normalization point. The reason for allowing contours extending to infinity (under our condition for the jump matrices) is because we want to consider the case of Painlevé II

as an example in Sec. 6.1, and also most ODE’s with polynomial coefficients exhibiting Stokes’ phenomenon can be transformed into such a RHP.

In general there are obstructions to the solvability of Problem 2.1: for example let c be a point in $\Sigma\gamma$ where several arcs $\gamma_1, \dots, \gamma_\ell$ meet (at nonzero tangential relative angles) and suppose they are oriented towards c . Let $M_j(z)$, $j = 1, \dots, \ell$ be the corresponding (locally analytic) jump matrices, then

$$M_1(c)M_2(c) \dots M_\ell(c) \neq \mathbf{1} \tag{2.5}$$

is an obstruction. This does not mean that it is impossible to find Γ satisfying the jumps, but it cannot be bounded at c in general and thus a relaxed growth constraint must be allowed at c . However there are situations which are of interest for our applications where a growth behavior can be “traded in” for an extra contour.

Example 2.1. A typical example is depicted in Fig. 1; here on the contour that terminates at $z = a$ there is a constant jump of the form $M = C^{-1}e^{2i\pi L}C$, where L is upper triangular (in Jordan form, for example). It is clear that the diagonal entries of L (the **exponents of formal monodromy**) are defined only up to addition of integers. This arbitrariness in fact corresponds to the *choice* of a growth behavior at the endpoint for the solution of the RHP. Once this choice has been made, we can always recast the problem into an equivalent one where the solution Γ is bounded. This is achieved by adding a small circle at a and re-defining $\tilde{\Gamma}(z) = \Gamma(z)C^{-1}(z - a)^{-L}$ inside, with the cut of $(z - a)^{-L}$ along the dotted line on the right panel of Fig. 1: in the new formulation for the problem, $\tilde{\Gamma}(z)$ will be bounded in a neighborhood of $z = a$ and the two problems are equivalent, in the sense that one has solution if and only if the other does, and the relation between the two solutions is also very simple.

(Loop) group structure. Although we will not make any explicit use of the following fact, we mention that the matrices $M : \Sigma\gamma \rightarrow SL_r(\mathbb{C})$ that satisfy all the conditions above form a (infinite dimensional) **Lie group**, akin to the usual loop group, were it not for the fact that the contour is not a (collection of) circle(s). It is convenient to introduce a symbol for this manifold (group),

$$\mathcal{G} := \left\{ M : \bigcup \gamma_v \mapsto SL_r(\mathbb{C}), M|_{\gamma_v}(z) \text{ locally analytic} \right\}. \tag{2.6}$$

Since we will only consider finite dimensional (analytic) submanifolds of \mathcal{G} we will not dwell on the infinite-dimensional differential-geometric issues. Most notably the tangent bundle $T\mathcal{G}$ will be used in a rather naive form where possible issues of topological nature (arising from the infinite dimensionality) are disposed of. In fact only the tangent space to the finite dimensional submanifolds of interest will appear, and hence the point is not relevant.

2.1. Deformations and Malgrange’s form. Suppose now that $M(z) = M(z; \vec{s})$ depends holomorphically on additional parameters \vec{s} ; the reader should think of this as an explicit dependence, dictated by the problem under consideration. The parameters \vec{s} could be thought of as coordinates on a manifold of deformations. When -eventually- we specialize the setting these \vec{s} ’s will be the isomonodromic times together with the monodromy data. On this manifold we define the one-form (differential) already used by Malgrange [11].

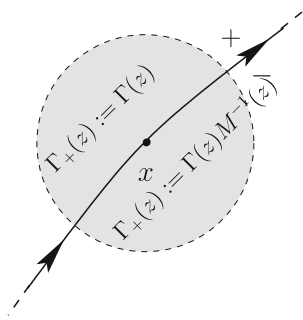


Fig. 2. The local extension of Γ_+ across a contour

Definition 2.1. Let ∂ denote the derivative w.r.t. one of the parameters s and assume that the Riemann–Hilbert Problem 2.1 admits a solution in an open subset of the s -parameter space.² Then we define Malgrange’s form ω_M ,

$$\omega_M(\partial) = \omega_M(\partial; [\Gamma]) := - \int_{\Sigma\gamma} \text{Tr} \left(\Gamma_-^{-1}(x) \Gamma'_-(x) \Xi_{\partial}(x) \right) \frac{dx}{2i\pi} \tag{2.7}$$

$$\Xi_{\partial}(z) := \partial M(z) M^{-1}(z).$$

Remark 2.3. The minus sign is –of course– conventional, but it is important because with the minus the tau-function (when it exists) has a zero and not a pole on the Malgrange divisor.

Remark 2.4. The definition would hold identically also for any RHP formulated on a Riemann surface.

Remark 2.5. The reader may frown upon the boundary value of the derivative of Γ : however, since our assumptions mandate that $M_\nu(z)$ is analytic in a neighborhood of the contour γ_ν , it is simple to show that the derivative Γ' also admits (bounded) boundary values. In fact more is true: since the matrices $M(z)$ are analytic at each interior point x of the contours, both boundary values of the solution can be analytically continued within a neighborhood of x to (germs of) analytic functions that we denote again $\Gamma_{\pm}(z)$. For example, if we want to analytically continue $\Gamma_+(x)$ to an analytic function $\Gamma_+(z)$ within a small disk of $x \in \Sigma\gamma$ not containing any other arc, we simply define $\Gamma_+(z) := \Gamma(z)$ for z on the left half disk (relative to the orientation of the arc), and $\Gamma_+(z) := \Gamma(z)M^{-1}(z)$ on the right side (see Fig. 2).

Remark 2.6. In the second notation for ω_M we have indicated the dependence on Γ , which depends on the choice of jump matrices **and** specifications of the growth behaviors. When no ambiguity can arise, we will understand such a dependence.

² The small–norm theorem for Riemann–Hilbert problems implies that if a RHP is solvable, then any sufficiently small deformation (in L^2 and L^∞ norms) of the jump matrices leads to a solvable RHP. With our assumptions on the s -dependence of the jump matrices this implies that the subset of solvable RHP is an open set (if non-empty).

Curvature of ω_M . The first issue is the computation of the exterior differential: the computation can be found in [11] but in rather abstract terms and we prefer to give a direct derivation here. We have first

Lemma 2.1. *Let ∂ denote any vector field in the parameters of the jump matrices. Then*

$$\partial\Gamma(z) = \int_{\Sigma\gamma} \frac{\Gamma_-(x)\Xi_\partial(x)\Gamma_-^{-1}(x)}{x-z} \frac{dx}{2i\pi} \Gamma(z) \tag{2.8}$$

and, consequently

$$\partial\left(\frac{\Gamma^{-1}(z)\Gamma(w)}{z-w}\right) = \int_{\Sigma\gamma} \frac{\Gamma^{-1}(z)\Gamma_-(x)}{z-x} \Xi_\partial(x) \frac{\Gamma^{-1}(x)\Gamma(w)}{x-w} \frac{dx}{2i\pi}, \tag{2.9}$$

$$\partial\left(\Gamma^{-1}(z)\Gamma'(z)\right) = \int_{\Sigma\gamma} \frac{\Gamma^{-1}(z)\Gamma_-(x)\Xi_\partial(x)\Gamma_-^{-1}(x)\Gamma(z)}{(z-x)^2} \frac{dx}{2i\pi}. \tag{2.10}$$

Proof. First of all we have

$$\partial\Gamma_+(z) = \partial\Gamma_-(z)M(z) + \Gamma_-(z)\partial M(z), \quad z \in \Sigma\gamma. \tag{2.11}$$

This is a non-homogeneous Riemann–Hilbert problem: since the problem is normalized at infinity, $\Gamma(\infty) = \mathbf{1}$, it follows that $\partial\Gamma(z) = \mathcal{O}(z^{-1})$. It is seen that the proposed expression fulfills (2.11) and this last condition. The uniqueness follows from the fact that the homogeneous part has only the trivial solution that tends to zero at ∞ . The formula (2.9) follows by direct application of (2.8) and the Leibnitz rule. Finally, formula (2.10) can be obtained as a limit $w \rightarrow z$ of (2.9) as follows:

$$\partial\left(\Gamma^{-1}(z)\Gamma'(z)\right) = -\lim_{w \rightarrow z} \partial\left(\frac{\Gamma^{-1}(z)\Gamma(w) - \mathbf{1}}{z-w}\right) = -\lim_{w \rightarrow z} \partial\left(\frac{\Gamma^{-1}(z)\Gamma(w)}{z-w}\right). \tag{2.12}$$

□

Proposition 2.1. *Denoting with $\partial, \tilde{\partial}$ the derivatives w.r.t to two of the parameters \vec{s} , the exterior differential of ω_M is*

$$\eta(\partial, \tilde{\partial}) := \partial\omega_M(\tilde{\partial}) - \tilde{\partial}\omega_M(\partial) = \frac{1}{2} \int_{\Sigma\gamma} \text{Tr} \left(\Xi_{\tilde{\partial}}(x) \frac{d}{dx} \Xi_\partial(x) - \frac{d}{dx} \Xi_{\tilde{\partial}}(x) \Xi_\partial(x) \right) \frac{dx}{2i\pi}. \tag{2.13}$$

Proof. Let $\partial, \tilde{\partial}$ be two commuting vector fields so that

$$\delta\omega(\partial, \tilde{\partial}) = \partial\omega(\tilde{\partial}) - \tilde{\partial}\omega(\partial). \tag{2.14}$$

We have, from Definition 2.1,

$$\begin{aligned} \partial\omega_M(\tilde{\partial}) &= - \int_{\Sigma\gamma} \text{Tr} \left(\partial \left(\Gamma_-^{-1}(x)\Gamma'_-(x) \right) \Xi_{\tilde{\partial}}(x) \right) \frac{dx}{2i\pi} \\ &\quad - \int_{\Sigma\gamma} \text{Tr} \left(\Gamma_-^{-1}(x)\Gamma'_-(x) \partial \left(\Xi_{\tilde{\partial}}(x) \right) \right) \frac{dx}{2i\pi} \\ &= - \int_{\Sigma\gamma} \text{Tr} \left(\int_{\Sigma\gamma} \frac{\mathcal{K}(x,y)\Xi_\partial(y)\mathcal{K}(y,x)\Xi_{\tilde{\partial}}(x)}{(x-y)^2} \frac{dy}{2i\pi} \right) \Big|_{x=x_-} \frac{dx}{2i\pi} \\ &\quad - \int_{\Sigma\gamma} \text{Tr} \left(\Gamma_-^{-1}(x)\Gamma'_-(x) \partial \left(\Xi_{\tilde{\partial}}(x) \right) \right) \frac{dx}{2i\pi}, \end{aligned} \tag{2.15}$$

where we have used (2.10) and introduced the convenience notation

$$\mathcal{K}(x, y) = \Gamma_-^{-1}(x)\Gamma_-(y). \tag{2.16}$$

Subtracting the term $\tilde{\partial}\omega_M(\partial)$ to (2.15) we have

$$\begin{aligned} & \partial\omega_M(\tilde{\partial}) - \tilde{\partial}\omega_M(\partial) \\ &= - \int_{\Sigma_\gamma} \left[\int_{\Sigma_\gamma} \text{Tr} \left(\frac{\mathcal{K}(x, y)\Xi_\partial(y)\mathcal{K}(y, x)\Xi_{\tilde{\partial}}(x)}{(x-y)^2} \frac{dy}{2i\pi} \right) \right] \Big|_{x=x_-} \frac{dx}{2i\pi} \\ & \quad + \int_{\Sigma_\gamma} \left[\int_{\Sigma_\gamma} \text{Tr} \left(\frac{\mathcal{K}(x, y)\Xi_{\tilde{\partial}}(y)\mathcal{K}(y, x)\Xi_\partial(x)}{(x-y)^2} \frac{dy}{2i\pi} \right) \right] \Big|_{x=x_-} \frac{dx}{2i\pi} \\ & \quad + \int_{\Sigma_\gamma} \text{Tr} \left(\Gamma_-^{-1}(x)\Gamma'_-(x) [\Xi_{\tilde{\partial}}(x), \Xi_\partial(x)] \right) \frac{dx}{2i\pi} \end{aligned} \tag{2.17}$$

(the subscripts $x=x_\pm$ or $y=y_\pm$ indicate which boundary value is taken after the integration), where we have used the zero curvature relation for $\Xi_\partial(x) = \partial M(x)M^{-1}(x)$, namely

$$\partial\Xi_{\tilde{\partial}}(x) - \tilde{\partial}\Xi_\partial(x) = [\Xi_\partial(x), \Xi_{\tilde{\partial}}(x)]. \tag{2.18}$$

In the first two terms the order of integration is important since the kernel is singular due to the denominator $(x-y)^2$. Let us examine the first two terms in the right-hand side of Eq. (2.17): if we denote temporarily by $F(x, y)$ the expression

$$F(z, w) := \text{Tr} \left(\Gamma_-^{-1}(z)\Gamma_-(w)\Xi_\partial(w)\Gamma_-^{-1}(w)\Gamma_-(z)\Xi_{\tilde{\partial}}(z) \right) \tag{2.19}$$

the numerators are simply $F(x, y)$ and $F(y, x)$ respectively, due to the cyclicity of the trace; note that by Remark 2.5 the function $\Gamma_-(z)$ has been extended to a local analytic function across the contours of the jumps, so that $F(z, w)$ can be taken as locally analytic around the contours of integration in both variables. If we interchange the order of integration we must evaluate the boundary value of the singular kernel $\frac{1}{(x-y)^2}$ on the opposite side, namely

$$\begin{aligned} \int_{\Sigma_\gamma} \left[\int_{\Sigma_\gamma} \frac{F(x, y)}{(x-y)^2} \frac{dy}{2i\pi} \right] \Big|_{x=x_-} \frac{dx}{2i\pi} &= \int_{\Sigma_\gamma} \left[\int_{\Sigma_\gamma} \frac{F(x, y)}{(x-y)^2} \frac{dx}{2i\pi} \right] \Big|_{y=y_+} \frac{dy}{2i\pi} \\ &= \int_{\Sigma_\gamma} \left[\int_{\Sigma_\gamma} \frac{F(y, x)}{(x-y)^2} \frac{dy}{2i\pi} \right] \Big|_{x=x_+} \frac{dx}{2i\pi}, \end{aligned} \tag{2.20}$$

where in the last equality we simply have renamed the dummy variables. We thus have

$$\begin{aligned}
 & \int_{\Sigma_\gamma} \left[\int_{\Sigma_\gamma} \operatorname{Tr} \left(\frac{\mathcal{K}(x, y) \Xi_{\bar{\theta}}(y) \mathcal{K}(y, x) \Xi_{\theta}(x)}{(x - y)^2} \right) \frac{dy}{2i\pi} \right]_{x=x_-} \frac{dx}{2i\pi} \\
 & \quad - \int_{\Sigma_\gamma} \left[\int_{\Sigma_\gamma} \operatorname{Tr} \left(\frac{\mathcal{K}(x, y) \Xi_{\theta}(y) \mathcal{K}(y, x) \Xi_{\bar{\theta}}(x)}{(x - y)^2} \right) \frac{dy}{2i\pi} \right]_{x=x_-} \frac{dx}{2i\pi} \\
 & = \int_{\Sigma_\gamma} \left[\int_{\Sigma_\gamma} \frac{F(y, x)}{(x - y)^2} \frac{dy}{2i\pi} \right]_{x=x_-} \frac{dx}{2i\pi} - \int_{\Sigma_\gamma} \left[\int_{\Sigma_\gamma} \frac{F(x, y)}{(x - y)^2} \frac{dy}{2i\pi} \right]_{x=x_-} \frac{dx}{2i\pi} \\
 & = \int_{\Sigma_\gamma} \left[\int_{\Sigma_\gamma} \frac{F(y, x)}{(x - y)^2} \frac{dy}{2i\pi} \right]_{x=x_-} \frac{dx}{2i\pi} \\
 & \quad - \frac{1}{2} \int_{\Sigma_\gamma} \left[\int_{\Sigma_\gamma} \frac{F(y, x)}{(x - y)^2} \frac{dy}{2i\pi} \right]_{x=x_+} \frac{dx}{2i\pi} - \frac{1}{2} \int_{\Sigma_\gamma} \left[\int_{\Sigma_\gamma} \frac{F(x, y)}{(x - y)^2} \frac{dy}{2i\pi} \right]_{x=x_-} \frac{dx}{2i\pi} \\
 & = -\frac{1}{2} \int_{\Sigma_\gamma} \partial_y F(y, x) \Big|_{y=x} \frac{dx}{2i\pi} + \frac{1}{2} \int_{\Sigma_\gamma} \left[\int_{\Sigma_\gamma} \frac{F(y, x)}{(x - y)^2} \frac{dy}{2i\pi} \right]_{x=x_-} \frac{dx}{2i\pi} \\
 & \quad - \frac{1}{2} \int_{\Sigma_\gamma} \left[\int_{\Sigma_\gamma} \frac{F(x, y)}{(x - y)^2} \frac{dy}{2i\pi} \right]_{x=x_-} \frac{dx}{2i\pi} \\
 & = -\frac{1}{2} \int_{\Sigma_\gamma} \partial_y F(y, x) \Big|_{y=x} \frac{dx}{2i\pi} + \frac{1}{2} \int_{\Sigma_\gamma} \left[\int_{\Sigma_\gamma} \frac{F(x, y)}{(x - y)^2} \frac{dy}{2i\pi} \right]_{x=x_+} \frac{dx}{2i\pi} \\
 & \quad - \frac{1}{2} \int_{\Sigma_\gamma} \left[\int_{\Sigma_\gamma} \frac{F(x, y)}{(x - y)^2} \frac{dy}{2i\pi} \right]_{x=x_-} \frac{dx}{2i\pi} \\
 & = -\frac{1}{2} \int_{\Sigma_\gamma} \partial_y F(y, x) \Big|_{y=x} \frac{dx}{2i\pi} + \frac{1}{2} \int_{\Sigma_\gamma} \partial_y F(x, y) \Big|_{y=x} \frac{dx}{2i\pi} \\
 & = \frac{1}{2} \int_{\Sigma_\gamma} \left(\partial_y F(x, y) \Big|_{y=x} - \partial_y F(y, x) \Big|_{y=x} \right) \frac{dx}{2i\pi}. \tag{2.21}
 \end{aligned}$$

We have used, on the fourth and fifth line of (2.21) the derivative form of Cauchy integral formula. Computing explicitly the derivatives involved in this last expression we find

$$\begin{aligned}
 & \frac{1}{2} \int_{\Sigma_\gamma} \left(\partial_y F(y, x) \Big|_{y=x} - \partial_y F(x, y) \Big|_{y=x} \right) \frac{dx}{2i\pi} \\
 & \quad = \frac{1}{2} \int_{\Sigma_\gamma} \operatorname{Tr} \left(\Xi'_{\bar{\theta}}(x) \Xi_{\bar{\theta}}(x) - \Xi_{\theta}(x) \Xi'_{\bar{\theta}}(x) \right) \frac{dx}{2i\pi} \\
 & \quad \quad - \int_{\Sigma_\gamma} \operatorname{Tr} \left(\Gamma_-^{-1}(x) \Gamma'_-(x) \left[\Xi_{\bar{\theta}}(x), \Xi_{\theta}(x) \right] \right) \frac{dx}{2i\pi}. \tag{2.22}
 \end{aligned}$$

Replacing this last expression into (2.17) yields the proof. \square

The Theta divisor. The reader should observe and compare the expressions (2.13) and (2.7): the crucial point is that ω_M (2.7) is not defined whenever the RHP does not admit a

solution. On the contrary, as a sort of “miracle” the expression (2.13) for its curvature is defined and holomorphic for any group-valued matrix $M(z)$, whether or not the RHP is solvable. In other words, η is a closed two-form which is smooth also on the Malgrange divisor, whereas ω_M is undefined at those points. Malgrange proved [11] (and Palmer generalized [14]) that ω_M has a “pole” on such divisor: evidently, in the curvature such a pole disappears.

Tau function. It is also apparent that ω_M is in general not a closed one-form on the whole (infinite-dimensional) manifold \mathcal{G} : however, it may become closed when restricted to suitable submanifolds $\mathcal{L} \hookrightarrow \mathcal{G}$. Since the curvature η (2.13) is explicitly computable, it is easy to verify for a given explicit submanifold \mathcal{L} whether $\omega_M|_{\mathcal{L}}$ is closed or not. When such restriction turns out to be closed we can define (locally) a function on \mathcal{L} by

$$\tau_{\mathcal{L}} = e^{\int \omega_M}|_{\mathcal{L}}. \tag{2.23}$$

By Malgrange’s (Palmer’s) results, this function (defined up to nonzero multiplicative constant) will vanish at the intersection with the Θ divisor,

$$\tau_{\mathcal{L}}(\vec{s}) = 0 \Leftrightarrow \vec{s} \in \mathcal{L} \cap \Theta. \tag{2.24}$$

Suppose, however that $\omega_M|_{\mathcal{L}}$ is not closed; we may still seek another differential ϑ that “cures” the curvature

$$d(\omega_M + \vartheta)|_{\mathcal{L}} \equiv 0 \tag{2.25}$$

and then proceed with the construction of the tau function as before.

Ideally such differential should be smooth on the whole \mathcal{G} so that it does not change the cohomology class of ω_M and also does not change the singularity structure; this way the tau function will still vanish only at $\mathcal{L} \cap \Theta$ and our goal is met.

Although it is not a pedagogical approach, since we know what \mathcal{L} will be for our purposes and we have already found ϑ , we will describe it directly here.

Changing the curvature of ω_M . Consider the one-form ϑ that, evaluated on a vector ∂ yields

$$\vartheta(\partial) := -\frac{1}{2} \int_{\Sigma_{\mathcal{Y}}} \text{Tr} \left(M'(x)M^{-1}(x)\partial M(x)M^{-1}(x) \right) \frac{dx}{2i\pi}. \tag{2.26}$$

Its curvature is the two-form

$$\begin{aligned} \delta\vartheta(\partial, \tilde{\partial}) &= \partial\vartheta(\tilde{\partial}) - \tilde{\partial}\vartheta(\partial) \\ &= \frac{1}{2} \int_{\Sigma_{\mathcal{Y}}} \text{Tr} \left(\tilde{\partial}M'(x)M^{-1}(x)\partial M(x)M^{-1}(x) - \partial M'(x)M^{-1}(x)\tilde{\partial}M(x)M^{-1}(x) \right) \frac{dx}{2i\pi} \end{aligned} \tag{2.27}$$

which the reader can verify using Leibnitz rule; here the prime denotes (as always) the derivative with respect to x . The important (but trivial) additional observation is that ϑ is a smooth differential on the whole \mathcal{G} , since it does not require the solution of a RHP.

Definition 2.2. *The modified Malgrange differential is defined as $\Omega := \omega_M + \vartheta$.*

Remark 2.7. A direct computation shows that

$$\begin{aligned} & \int_{\Sigma\gamma} \operatorname{Tr} \left(\Gamma_{-}^{-1}(x) \Gamma'_{-}(x) \partial M(x) M^{-1}(x) \right) \frac{dx}{2i\pi} \\ & - \int_{\Sigma\gamma} \operatorname{Tr} \left(\Gamma_{+}^{-1}(x) \Gamma'_{+}(x) M^{-1}(x) \partial M(x) \right) \frac{dx}{2i\pi} \\ & = - \int_{\Sigma\gamma} \operatorname{Tr} \left(M'(x) M^{-1}(x) \partial M(x) M^{-1}(x) \right) \frac{dx}{2i\pi} = 2\vartheta(\partial), \end{aligned} \tag{2.28}$$

so that Ω can be written in the more symmetric form

$$\begin{aligned} \Omega(\partial; [\Gamma]) &= -\frac{1}{2} \int_{\Sigma\gamma} \operatorname{Tr} \left(\Gamma_{-}^{-1}(x) \Gamma'_{-}(x) \partial M(x) M^{-1}(x) \right. \\ & \quad \left. + \Gamma_{+}^{-1}(x) \Gamma'_{+}(x) M^{-1}(x) \partial M(x) \right) \frac{dx}{2i\pi}. \end{aligned} \tag{2.29}$$

Combining the computation of the curvature of ω_M (Prop. 2.1) with the curvature of ϑ above we have the simple

Proposition 2.2. *The curvature of the modified Malgrange form is*

$$\begin{aligned} \delta\Omega(\partial, \tilde{\partial}) &= \partial\Omega(\tilde{\partial}) - \tilde{\partial}\Omega(\partial) \\ &= \frac{1}{2} \int_{\Sigma\gamma} \operatorname{Tr} \left(M'(x) M^{-1}(x) \left[\partial M(x) M^{-1}(x), \tilde{\partial} M(x) M^{-1}(x) \right] \right) \frac{dx}{2i\pi}. \end{aligned} \tag{2.30}$$

Remark 2.8. Referring to ([11] Thm. 5.5) the curvature $\delta\Omega$ clearly lies in the same cohomology class as $\delta\omega_M$ since the two differ by a smooth exact differential. If we define γ as the total Maurer–Cartan form $\gamma(x) = M'(x)M^{-1}(x) dx + \sum \partial_{s_j} M(x)M^{-1}(x) ds_j$ one has the expression

$$\delta\Omega = \frac{1}{12\pi i} \int_{\Sigma\gamma} \operatorname{Tr}(\gamma \wedge \gamma \wedge \gamma). \tag{2.31}$$

2.2. Submanifolds of \mathcal{G} where Ω is closed. Looking at the formula of the curvature form of Prop. 2.2 we can ask ourselves what kind of jump-matrices give rise to a closed instance of the form Ω ; we will consider cases where the integrand is identically zero. This can happen for many reasons but for our purposes and for all the applications that we could find, it is sufficient to consider those jump matrices $M(z)$ that on each arc of $\Sigma\gamma$ reduce to one of the following forms:

- **Piecewise triangular.** Matrices of the forms

$$M_\nu(z) = P_\nu(\mathbf{1} + N_\nu(z))P_\nu^{-1}, \quad z \in \gamma_\nu, \tag{2.32}$$

where $N_\nu(z)$ are upper-triangular analytic matrices and P_ν is any **constant** permutation matrix (i.e. an element of the Weyl group for SL_r).

- **Constants.** Matrices independent of z .
- **Torals.** Matrices $M_\nu = D_\nu(z)$ with $D_\nu(z)$ diagonal matrices or any conjugation thereof by an arbitrarily chosen but fixed matrix.

We see in Sect. 3 that any (generic) rational ODE can be encoded in a Riemann–Hilbert problem with jumps of the form indicated here above. Therefore Ω on these submanifolds yields a closed differential.

We will also show that its restriction to the *isomonodromic* submanifolds coincides with ω_{JMU} .

2.3. “Schlesinger” transformations. The aim of this section is to compare the differential ω_M on two RHP for $\Gamma, \tilde{\Gamma}$ defined by the **same jumps** but allowing poles (or requiring vanishing) in the columns of the solution at specified points. We note immediately that two such solutions differ (if both exist) by a **left multiplication** by a rational matrix $R(z)$; indeed –having $\Gamma, \tilde{\Gamma}$ the same jumps by assumption– the matrix

$$R(z) := \tilde{\Gamma}(z)\Gamma^{-1}(z) \tag{2.33}$$

is an analytic function in \mathbb{C} taking away the points where $\tilde{\Gamma}$ has poles. Since $\tilde{\Gamma}$ is allowed to have poles, the matrix $R(z)$ can have a pole of order at most equal to that of $\tilde{\Gamma}$ and hence it is forced to be rational by Liouville’s theorem.

We now make the following observation: let $R(z)$ be a rational matrix such that the divisor of all poles of both $R(z), R^{-1}(z)$ consists of the points $c_1, \dots, c_K \notin \Sigma_\gamma$. Define

$$\tilde{\Gamma}(z) := R(z)\Gamma(z). \tag{2.34}$$

Quite clearly $\tilde{\Gamma}$ solve a different RHP with the same jumps but possibly poles at those of R . The difference between the ω_M evaluated using Γ or $\tilde{\Gamma}$ is given by

$$\begin{aligned} \omega_M(\partial; [\tilde{\Gamma}]) - \omega_M(\partial; [\Gamma]) &= - \int_{\Sigma_\gamma} \text{Tr} \left(\tilde{\Gamma}^{-1}(x)\tilde{\Gamma}'_-(x)\partial M(x)M^{-1}(x) \right) \frac{dx}{2i\pi} \\ &\quad + \int_{\Sigma_\gamma} \text{Tr} \left(\Gamma^{-1}(x)\Gamma'_-(x)\partial M(x)M^{-1}(x) \right) \frac{dx}{2i\pi} \\ &= - \int_{\Sigma_\gamma} \text{Tr} \left(\Gamma^{-1}(x)R^{-1}(x)R'(x)\Gamma_-(x)\partial M(x)M^{-1}(x) \right) \frac{dx}{2i\pi} \\ &= - \int_{\Sigma_\gamma} \text{Tr} \left(R^{-1}(x)R'(x)\Gamma_-(x)\partial M(x)M^{-1}(x)\Gamma^{-1}(x) \right) \frac{dx}{2i\pi} \\ &= \int_{\Sigma_\gamma} \text{Tr} \left(R^{-1}(x)R'(x) \left(\partial\Gamma_+(x)\Gamma_+^{-1}(x) - \partial\Gamma_-(x)\Gamma_-(x) \right) \right) \frac{dx}{2i\pi} \\ &= \sum_{j=1}^K \text{res}_{z=c_j} \text{Tr} \left(R^{-1}(z)R'(z)\partial\Gamma(z)\Gamma^{-1}(z) \right) dz. \end{aligned} \tag{2.35}$$

Therefore we see that –while ω_M is an object of transcendental nature, the above difference requires only the evaluation of a finite number of residues. Note also that –since Ω (Def. 2.2) differ by ω_M only in an explicit term ϑ that depends *only* on the jump matrices, we have

$$\omega_M(\partial; [\tilde{\Gamma}]) - \omega_M(\partial; [\Gamma]) = \Omega(\partial; [\tilde{\Gamma}]) - \Omega(\partial; [\Gamma]). \tag{2.36}$$

We now proceed with the definitions in the title of the section and specialize the class of rational matrices $R(z)$.

Definition 2.3. Given two distinct points $\xi \neq \eta$ and two (possibly formal) series

$$Y_\xi(z) = G_\xi \left(\mathbf{1} + \sum_{\ell=1}^\infty Y_{\xi;\ell} z_\xi^\ell \right), \quad Y_\eta(z) = G_\eta \left(\mathbf{1} + \sum_{\ell=1}^\infty Y_{\eta;\ell} z_\eta^\ell \right), \quad (2.37)$$

$$z_\xi := (z - \xi), \quad z_\infty := \frac{1}{z}, \quad (2.38)$$

an elementary Schlesinger transformation at two distinct points $\xi \neq \eta$ is a rational matrix $R(z)$ such that

$$R(z)Y_\xi(z) = \widehat{Y}_\xi(z)z_\xi^{E_i}, \quad R(z)Y_\eta(z) = \widehat{Y}_\eta(z)z_\eta^{-E_j}, \quad (2.39)$$

where \widehat{Y}_\bullet denote formal series of the same form. In the case of ∞ we have $G_\infty = \mathbf{1} = \widehat{G}_\infty$. If neither $\xi \neq \infty \neq \eta$ then we impose also $R(\infty) = 1$.

[For the case $\xi = \eta$ the definition should be modified in an obvious way: please see below].

The problem is purely of algebraic nature, and not a very difficult one: the computation is contained in [[7], App. A] and the derivation will not be reported (we will give below the relevant results). Suppose now we have a RHP for Γ with jump matrices M as in Prob. 2.1: at points $\xi \in \Sigma\gamma$ the solution Γ to Prob. 2.1 yields a (convergent) power series which can be used as input for the above procedures.

If $\xi \in \Sigma\gamma$ we cannot have a (even formal) series since –in general– not even the value of Γ is well defined at $z = \xi$. In the applications to ODEs the following situation occurs; let $\xi \in \Sigma\gamma$ be a point where $\gamma_1, \dots, \gamma_\ell$ meet ($\ell \geq 1$). Suppose that the jump matrices are analytic in small sectors centered at ξ containing the direction of approach and that –in said sector– (see Fig. 3)

$$M_j(z) = \mathbf{1} + \mathcal{O}((z - \xi)^\infty). \quad (2.40)$$

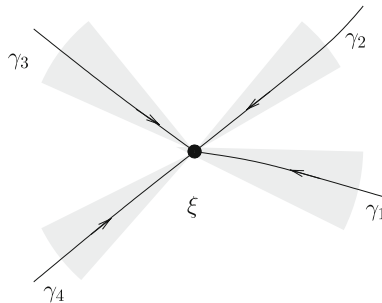


Fig. 3. Illustration of the sectors of analyticity for M_j where the decay (2.40) should be valid

Then it is not hard to see³ that the solution $\Gamma(z)$ has the **same** asymptotic expansion in each of the sectors at ξ separated by the incoming arcs

$$\Gamma(z) \sim G_\xi \left(\mathbf{1} + \sum_1^\infty Y_{\xi;j}(z - \xi)^j \right) =: \widehat{Y}_\xi(z). \tag{2.41}$$

We will allow to perform Schlesinger transformations involving either points ξ not in $\Sigma\gamma$ or points where the condition (2.40) is met, so that the solution Γ defines unambiguously a (formal) analytic series centered at the point.

Definition 2.4. *The elementary Schlesinger transformation $\left\{ \begin{smallmatrix} \xi & \eta \\ i & j \end{smallmatrix} \right\}$ for the solution Γ of Problem 2.1 is the solution (if it exists) of the following new RHP (where $z_\xi = (z - \xi)$ if $\xi \neq \infty$ and $z_\infty := \frac{1}{z}$)*

$\xi \neq \eta$	$\xi = \eta \ (i \neq j)$	(2.42)
$\widetilde{\Gamma}_+ = \widetilde{\Gamma}_- M$	$\widetilde{\Gamma}_+ = \widetilde{\Gamma}_- M$	
$\widetilde{\Gamma}(z) = \mathcal{O}(1)z_\xi^{-E_i}, \ z \sim \xi$	$\widetilde{\Gamma}(z) = \mathcal{O}(1)z_\xi^{-E_i+E_j}, \ z \sim \xi$	
$\widetilde{\Gamma}(z) = \mathcal{O}(1)z_\eta^{E_j}, \ z \sim \eta$		

where the normalization is fixed by requiring that $\widetilde{\Gamma}(z) \sim \mathbf{1} + \mathcal{O}(z^{-1})$ if $\xi \neq \infty \neq \eta$ or –if either ξ or η are infinity– that the term indicated by $\mathcal{O}(1)$ above is actually $\mathbf{1} + \mathcal{O}(z^{-1})$.

Here and below we use the notation E_{ij} for the elementary matrices (with a 1 on the i^{th} row, j^{th} column) and $E_i := E_{ii}$. It is immediately seen that

$$\widetilde{\Gamma}(z) = R(z)\Gamma(z) \tag{2.43}$$

with $R(z)$ rational: indeed the ratio $\widetilde{\Gamma}(z)\Gamma^{-1}(z)$ for the transform $\left\{ \begin{smallmatrix} \xi & \eta \\ i & j \end{smallmatrix} \right\}$ does not have jumps and may have at most a simple pole at $z = \xi$. The matrix $R(z)$ can be computed directly from the (possibly formal) series-expansion of Γ at ξ, η (see [7]). We give below a more compact formula that the reader can check autonomously.

³ By moving slightly the jump –which can be done due to the local analyticity of the jump matrix– one sees that the solution Γ can be “continued” analytically across the jump from the left and from the right. The ratio of these two extensions in the common sector of analyticity differs from the identity by exponentially small terms, which have no bearing on the asymptotic expansion.

Proposition 2.3 (cf. App. A, B in [7]). *The left-multiplier matrix $R(z)$ that implements the elementary Schlesinger transform $\left\{ \begin{smallmatrix} \xi & \eta \\ i & j \end{smallmatrix} \right\}$ is given by*

$\left\{ \begin{smallmatrix} \xi & \eta \\ i & j \end{smallmatrix} \right\} \quad (\xi \neq \eta)$ $R(z) = \mathbf{1} + \frac{\xi - \eta}{(\Gamma^{-1}(\xi)\Gamma(\eta))_{ij}} \frac{\Gamma(\eta)E_{ji}\Gamma^{-1}(\xi)}{z - \xi}$ $R^{-1}(z) = \mathbf{1} - \frac{\xi - \eta}{(\Gamma^{-1}(\xi)\Gamma(\eta))_{ij}} \frac{\Gamma(\eta)E_{ji}\Gamma^{-1}(\xi)}{z - \eta}$ $\det R(z) = \frac{z - \eta}{z - \xi}$	$\left\{ \begin{smallmatrix} \xi & \xi \\ i & j \end{smallmatrix} \right\} \quad (i \neq j)$ $R(z) = \mathbf{1} + \frac{1}{(\Gamma^{-1}(\xi)\Gamma'(\xi))_{ij}} \frac{\Gamma(\xi)E_{ji}\Gamma^{-1}(\xi)}{z - \xi}$ $R^{-1}(z) = \mathbf{1} - \frac{1}{(\Gamma^{-1}(\xi)\Gamma'(\xi))_{ij}} \frac{\Gamma(\xi)E_{ji}\Gamma^{-1}(\xi)}{z - \xi}$ $\det R = 1$
---	--

(2.44)

$\left\{ \begin{smallmatrix} \xi & \infty \\ i & j \end{smallmatrix} \right\} \quad (\xi \neq \infty)$ $R(z) = \mathbf{1} - E_{jj} + \frac{R_1}{z - \xi}$ $R_1 = \frac{-1}{(\Gamma(\xi)^{-1})_{ij}} [(\mathbf{1} - E_{jj})\Gamma'(\infty) - \mathbf{1}] E_{ji}\Gamma^{-1}(\xi)$ $R^{-1}(z) = E_{jj}(z - \xi) + R_0$ $R_0 = \left(\mathbf{1} - \frac{E_{ji}\Gamma^{-1}(\xi)}{\Gamma_{ij}^{-1}(\xi)} \right) (\mathbf{1} + \Gamma'(\infty)E_{jj})$ $\det R(z) = \frac{1}{z - \xi}$	$\left\{ \begin{smallmatrix} \infty & \eta \\ i & j \end{smallmatrix} \right\} \quad (\eta \neq \infty)$ $R(z) = E_{ii}(z - \eta) + R_0$ $R_0 = (\mathbf{1} - E_{ii}\Gamma'(\infty)) \left(\mathbf{1} - \frac{\Gamma(\eta)E_{ji}}{\Gamma_{ij}(\eta)} \right)$ $R^{-1}(z) = \mathbf{1} - E_{ii} + \frac{R_1}{z - \eta}$ $R_1 = \frac{1}{\Gamma(\eta)_{ij}} \Gamma(\eta)E_{ji} [\Gamma'(\infty)(\mathbf{1} - E_{ii}) + \mathbf{1}]$ $\det R(z) = (z - \eta)$
---	---

(2.45)

where we have denoted by $\Gamma'(\infty)$ the derivative in the local parameter, that is

$$\Gamma(z) = \mathbf{1} + \frac{1}{z}\Gamma'(\infty) + \dots \tag{2.46}$$

(The formulae for $\left\{ \begin{smallmatrix} \infty & \infty \\ i & j \end{smallmatrix} \right\}$ can be found in loc cit.)

If ξ (or η or both) belong to $\Sigma\gamma$ where condition (2.40) is in place, the computation leading to (2.35) can be still carried out with minor modifications in the process but not in the result. First of all note that the integral $\omega_M(\partial, [\tilde{\Gamma}])$ is still convergent since $\Xi_\partial = \partial MM^{-1} = \mathcal{O}((x - c)^\infty)$ (recall that we assumed $M = \mathbf{1} + \mathcal{O}((x - c)^\infty)$) and hence the additional algebraic growth of $\tilde{\Gamma}^{-1}\tilde{\Gamma}'$ along the contours incident at c is still integrable when multiplied by Ξ_∂ .

Let $\Sigma\gamma_\epsilon$ denote the contours that lie outside of ϵ disks centered at the points $z = \xi, \eta$ (possibly the same) of the Schlesinger transform; we then have

$$\begin{aligned} \omega_M(\partial; [\tilde{\Gamma}]) - \omega_M(\partial; [\Gamma]) &= -\lim_{\epsilon \rightarrow 0} \int_{\Sigma\gamma_\epsilon} \text{Tr} \left(\left(\tilde{\Gamma}_-^{-1}(x)\tilde{\Gamma}'_-(x) - \Gamma_-^{-1}(x)\Gamma'_-(x) \right) \Xi_\partial(x) \right) \frac{dx}{2i\pi} \\ &= -\lim_{\epsilon \rightarrow 0} \int_{\Sigma\gamma_\epsilon} \text{Tr} \left(\Gamma_-^{-1}(x)R^{-1}(x)R'(x)\Gamma_-(x) \partial M(x)M^{-1}(x) \right) \frac{dx}{2i\pi} \end{aligned}$$

$$\begin{aligned}
 &= - \lim_{\epsilon \rightarrow 0} \int_{\Sigma\gamma_\epsilon} \text{Tr} \left(R^{-1}(x)R'(x)\Gamma_-(x) \partial M(x)M^{-1}(x)\Gamma_-^{-1}(x) \right) \frac{dx}{2i\pi} \\
 &= \lim_{\epsilon \rightarrow 0} \int_{\Sigma\gamma_\epsilon} \text{Tr} \left(R^{-1}(x)R'(x) \left(\partial\Gamma_+(x)\Gamma_+^{-1}(x) - \partial\Gamma_-(x)\Gamma_-(x) \right) \right) \frac{dx}{2i\pi} \\
 &= \lim_{\epsilon \rightarrow 0} \oint_{|x-\xi|=|x-\eta|=\epsilon} \text{Tr} \left(R^{-1}(x)R'(x)\partial\Gamma(x)\Gamma^{-1}(x) \right) \frac{dx}{2i\pi}. \tag{2.47}
 \end{aligned}$$

In the computation of the limit (2.47) we can replace $\partial\Gamma(x)\Gamma^{-1}(x)$ by a suitable truncation of the asymptotic series at $z = \xi, \eta$ (which does not depend on the direction of approach under our assumptions for M), committing an $o(1)$ error as $\epsilon \rightarrow 0$. Then the limit equals the formal residue

$$\begin{aligned}
 \omega_M(\partial; [\tilde{\Gamma}]) - \omega_M(\partial; [\Gamma]) &= \text{res}_{z=\xi} \text{Tr} \left(R^{-1}(z)R'(z)\partial\widehat{Y}_\xi(z)\widehat{Y}_\xi(z)^{-1} \right) dz \\
 &\quad + \text{res}_{z=\eta} \text{Tr} \left(R^{-1}(z)R'(z)\partial\widehat{Y}_\eta(z)\widehat{Y}_\eta(z)^{-1} \right) dz. \tag{2.48}
 \end{aligned}$$

Here the residue means simply the coefficient of the power -1 of the local parameter (note that $R^{-1}(z)R'(z)$ has at most a double pole and hence the formal residue involves at most the first two terms in the formal series $\widehat{Y}(z)$). In case only one point is involved in the Schlesinger transformation we have simply only one residue at the end.

Remark 2.9. In the case of a Schlesinger transformation involving two distinct points the determinant of the solution $\det \Gamma$ cannot remain constant, since $R(z)$ has non-constant determinant. This does not pose any significant problem as we explain presently. To account for the different power-law of the columns at the two points $z = \xi, z = \eta$, small counterclockwise circles around those points should be added to $\Sigma\gamma$ imposing additional jumps of the form

$$M_1(z) = (z - \xi)^{-E_i}, \quad |z - \xi| = \epsilon, \quad M_2(z) = (z - \eta)^{E_j}, \quad |z - \eta| = \epsilon, \tag{2.49}$$

conjugating by the same matrices any jump of a contour that passes within said circles. Of course these jumps do not have unit determinant and hence the *uniqueness* of the solution must be argued in a different way from the one used in the unimodular case: given a solution $\Gamma(z)$ of the new RHP we see that $\det \Gamma(z)$ is analytic and bounded everywhere, except for jumps on the new circles where

$$\begin{aligned}
 \det \Gamma_+(z) &= (z - \xi)^{-1} \det \Gamma_-(z) \\
 \text{on } |z - \xi| = \epsilon \text{ and } \det \Gamma_+(z) &= (z - \eta) \det \Gamma_-(z), \quad \text{on } |z - \eta| = \epsilon.
 \end{aligned} \tag{2.50}$$

This means that $\det \Gamma(z)$ admits analytic continuation in the interior of the two disks, with a simple pole at $z = \eta$ and a simple zero at $z = \xi$, plus the condition $\det \Gamma(\infty) = 1$. This forces $\det \Gamma(z) \equiv \frac{z-\xi}{z-\eta}$ outside of the disks, $\det \Gamma(z) = (z - \xi)$ for $|z - \eta| < \epsilon$ and viceversa $\det \Gamma(z) = \frac{1}{z-\eta}$ for $|z - \xi| < \epsilon$. Any solution will have the same determinant and hence the uniqueness is established along the same way used previously.

A direct computation based on (2.35) yields the following theorem, which is simply a rephrasing of an homologous theorem in [7], with the proper extension of understanding to the setting of RHPs.

Theorem 2.1 (Thm. 4.1 in [7]). *Given two RHPs related by the elementary Schlesinger transformation $\begin{Bmatrix} \xi & \eta \\ i & j \end{Bmatrix}$ (Def. 2.4), the difference of the Malgrange differential on the two solutions is a closed differential on the deformation manifold given by*

$$\omega_M(\partial; [\tilde{\Gamma}]) - \omega_M(\partial; [\Gamma]) = \partial \ln H \begin{Bmatrix} \xi & \eta \\ i & j \end{Bmatrix}, \tag{2.51}$$

where

$$H \begin{Bmatrix} \xi & \eta \\ i & j \end{Bmatrix} = \begin{cases} (\Gamma^{-1}(\xi)\Gamma'(\xi))_{ij} & \text{for } \begin{Bmatrix} \xi & \xi \\ i & j \end{Bmatrix} \ (i \neq j) \\ (\Gamma(\eta))_{ij} & \text{for } \begin{Bmatrix} \infty & \eta \\ i & j \end{Bmatrix}, \ \eta \neq \infty \\ (\Gamma(\xi)^{-1})_{ij} & \text{for } \begin{Bmatrix} \xi & \infty \\ i & j \end{Bmatrix}, \ \xi \neq \infty \\ \frac{(\Gamma^{-1}(\xi)\Gamma(\eta))_{ij}}{\xi - \eta} & \text{for } \begin{Bmatrix} \xi & \eta \\ i & j \end{Bmatrix}, \ \xi \neq \infty \neq \eta \end{cases}, \tag{2.52}$$

where the notation $\Gamma'(\infty)$ –as previously– denotes the derivative in the local parameter

$$\Gamma(z) =: \mathbf{1} + \frac{1}{z}\Gamma'(\infty) + \dots \tag{2.53}$$

For the reader’s convenience we verify Thm. 2.1 for the case $\begin{Bmatrix} \xi & \eta \\ i & j \end{Bmatrix}$ with distinct $\xi \neq \infty \neq \eta$ and both $\xi, \eta \notin \Sigma\gamma$. Let ∂ be a variation of the jump-matrices M (i.e. not moving ξ, η). We have from Prop. 2.3,

$$R^{-1}(z)R'(z) = \frac{\Gamma(\eta)E_{ji}\Gamma^{-1}(\xi)}{(\Gamma^{-1}(\xi)\Gamma(\eta))_{ij}} \left(\frac{1}{z - \eta} - \frac{1}{z - \xi} \right). \tag{2.54}$$

We then have to compute the (possibly formal) residue (2.35)

$$\begin{aligned} \operatorname{res}_{z=\xi, \eta} \operatorname{Tr} & \left\{ \frac{\Gamma(\eta)E_{ji}\Gamma^{-1}(\xi)}{(\Gamma^{-1}(\xi)\Gamma(\eta))_{ij}} \left(\frac{1}{z - \eta} - \frac{1}{z - \xi} \right) \partial \Gamma(z)\Gamma^{-1}(z) \right\} dz \\ &= \frac{(\Gamma^{-1}(\xi)\partial\Gamma(\eta))_{ij}}{(\Gamma^{-1}(\xi)\Gamma(\eta))_{ij}} - \frac{(\Gamma^{-1}(\xi)\partial\Gamma(\xi)\Gamma^{-1}(\xi)\Gamma(\eta))_{ij}}{(\Gamma^{-1}(\xi)\Gamma(\eta))_{ij}} \\ &= \frac{(\Gamma^{-1}(\xi)\partial\Gamma(\eta))_{ij}}{(\Gamma^{-1}(\xi)\Gamma(\eta))_{ij}} + \frac{(\partial(\Gamma^{-1}(\xi))\Gamma(\eta))_{ij}}{(\Gamma^{-1}(\xi)\Gamma(\eta))_{ij}} \\ &= \partial \ln \left(\frac{(\Gamma^{-1}(\xi)\Gamma(\eta))_{ij}}{\xi - \eta} \right). \end{aligned} \tag{2.55}$$

To verify the formula for $\partial_\xi, \partial_\eta$ as well, we must add a small circle around them and a new jump $M(z) = (z - \xi)^{E_i}, M(z) = (z - \eta)^{-E_j}$ respectively. Since ξ, η did not exist as deformation parameters in the RHP for Γ , from the definition of ω_M we need to compute (we do it only for ∂_ξ , leaving the verification for ∂_η to the reader)

$$\omega_M(\partial_\xi; [\tilde{\Gamma}]) = - \oint_{|x-\xi|=\epsilon} \operatorname{Tr} \left(\tilde{\Gamma}_-^{-1}(x)\tilde{\Gamma}'_-(x) \frac{E_i}{x - \xi} \right) \frac{dx}{2i\pi}. \tag{2.56}$$

Note that $\tilde{\Gamma}_-(z) = R(z)\Gamma(z)$ and $\tilde{\Gamma}_+(z) = R(z)\Gamma(z)(z - \xi)^{-E_j}$ when z belongs to the circle $|z - \xi| = \epsilon$ so that

$$\begin{aligned} \omega_M(\partial_\xi; [\tilde{\Gamma}]) &= - \operatorname{res}_{z=\xi} \operatorname{Tr} \left(\frac{R^{-1}(z)R'(z)\Gamma(z)E_i\Gamma^{-1}(z)}{z - \xi} + \frac{\Gamma^{-1}(z)\Gamma'(z)E_i}{z - \xi} \right) dz \\ &= - \frac{1}{\xi - \eta} - \frac{(\Gamma^{-1}(\xi)\Gamma'(\xi)\Gamma^{-1}(\xi)\Gamma(\eta))_{ij}}{(\Gamma^{-1}(\xi)\Gamma(\eta))_{ij}} = \partial_\xi \ln \left(\frac{(\Gamma^{-1}(\xi)\Gamma(\eta))_{ij}}{\xi - \eta} \right). \end{aligned} \tag{2.57}$$

This proves completely the case considered. It appears quite obviously that

Proposition 2.4. *An elementary Schlesinger transformation exists if and only if $H \left\{ \begin{smallmatrix} \xi & \eta \\ i & j \end{smallmatrix} \right\} \neq 0$.*

Remark 2.10. The zeroes of the matrix entries of the solution of the RHP acquire therefore the meaning of intersection of the Malgrange divisor with the space of the parameter added to the problem, namely, the position of the Schlesinger transform $\left\{ \begin{smallmatrix} \infty & \eta \\ i & j \end{smallmatrix} \right\}$.

We are not going to dwell at length on the algebra of iterated elementary Schlesinger transformations and on the general transformation since the formulæ are contained in [7]; we only point out that in loc. cit. the transformations were applied to solution of either isomonodromic or isospectral deformation problems and not a general Riemann–Hilbert problem. Thus, we are mainly shifting the perspective (and compactifying some notation) of [7].

Remark 2.11. The fact that ω evaluated on two solutions of RHPs with the same jumps is a closed differential is immediate from the fact that the curvature of ω does not depend on the growth behavior of the solution Γ but only on the jump matrices. When we specialize the setting to the case relevant to ODEs, Thm. 2.1 will hold for differentiations with respect to the monodromy data as well.

2.3.1. Generalized Sato formula Let $\eta \notin \Sigma\gamma$: denote by $\Gamma_{\eta;ij}(z)$ the Schlesinger transform $\left\{ \begin{smallmatrix} \infty & \eta \\ i & j \end{smallmatrix} \right\}$ of the solution of the RHP 2.1. Then the second formula (2.52) reads

$$\Gamma_{ij}(\eta) \propto \exp \int^{\bar{s}} \omega_M(\bullet; [\Gamma_{\eta;ij}]) - \omega_M(\bullet; [\Gamma]), \tag{2.58}$$

where the one form under integration is closed by the above remark. This is nothing but the Sato formula for the Baker Akhiezer vector; of course, at this level of generality we do not have a “tau” function because—in general— ω_M will not be a closed differential of the deformation parameters of the problem. **If** the problem admits a τ -function, that is, **if** the differential ω_M (or Ω) is closed on the submanifold of Riemann–Hilbert problems under consideration, then we have a honest version of the Sato formula,

$$\Gamma_{ij}(\eta) = \frac{\tau \left\{ \begin{smallmatrix} \infty & \eta \\ i & j \end{smallmatrix} \right\}}{\tau}, \tag{2.59}$$

where $\tau \left\{ \begin{smallmatrix} \infty & \eta \\ i & j \end{smallmatrix} \right\}$ stands for the τ -function of the problem with the “insertion” of the Schlesinger transform.

To make the remark a bit more concrete, let us pick a point $z = a \notin \Sigma\gamma$ and a small disk \mathbb{D} . For simplicity in writing the formulæ we will simply set $a = 0$. Let $T(z)$ be a diagonal matrix defined on $\partial\mathbb{D}$ that admits analytic continuation on $\mathbb{P}^1 \setminus \mathbb{D}$,

$$T(z) = \sum_{k=1}^{\infty} \frac{T_k}{z^k}, \quad T_k = \text{diag}(t_{k;1}, \dots, t_{k;r}), \tag{2.60}$$

where the Laurent series is supposed to be actually convergent on $\mathbb{P}^1 \setminus \mathbb{D}$. Let $\Gamma(z; [T])$ denote the solution of the RHP,

$$\Gamma_+(z) = \Gamma_-(z)M(z), \quad z \in \Sigma\gamma, \tag{2.61}$$

$$\Gamma_+(z) = \Gamma_-(z)e^{T(z)}, \quad z \in \partial\mathbb{D}, \tag{2.62}$$

$$\Gamma(\infty) = \mathbf{1}. \tag{2.63}$$

If the $L^\infty \cap L^2$ norm of $e^{T(z)} - \mathbf{1}$ on $\partial\mathbb{D}$ is sufficiently small, then the solvability of the problem is guaranteed by standard perturbation theorems. Therefore $\Gamma(z; [T])$ is defined at least in a ball around $T(z) \equiv 0$. Since the jump matrix $e^{T(z)}$ is diagonal (and analytic in the complement of the disk), it is immediately seen from Prop. 2.2 that $\omega_M = \Omega^4$ and that they are closed as differentials on the (infinite dimensional) manifold of T 's. Thus there is a locally defined function such that

$$\delta \ln \tau(T) := \oint_{\partial\mathbb{D}} \text{Tr} \left(\Gamma_-^{-1}(x) \Gamma'_-(x) \delta T(x) \right) \frac{dx}{2i\pi}. \tag{2.64}$$

Denote now by $\tau \left\{ \begin{smallmatrix} \xi & \eta \\ i & j \end{smallmatrix} \right\} (\vec{T})$ ⁵ the tau-function resulting after the elementary Schlesinger transformation $\left\{ \begin{smallmatrix} \xi & \eta \\ i & j \end{smallmatrix} \right\}$. Then the content of Thm. 2.1 can be rephrased as

$$\Gamma_{ij}(\zeta) = \frac{\tau \left\{ \begin{smallmatrix} \infty & \zeta \\ i & j \end{smallmatrix} \right\} (\vec{T})}{\tau(\vec{T})}, \quad \Gamma^{-1}_{ij}(\zeta) = \frac{\tau \left\{ \begin{smallmatrix} \zeta & \infty \\ i & j \end{smallmatrix} \right\} (\vec{T})}{\tau(\vec{T})}. \tag{2.65}$$

In fact, more is true: if ζ falls within the disk \mathbb{D} then the RHP for the Schlesinger transforms $\left\{ \begin{smallmatrix} \infty & \zeta \\ i & j \end{smallmatrix} \right\}, \left\{ \begin{smallmatrix} \zeta & \infty \\ i & j \end{smallmatrix} \right\}$ can be formulated as (**exercise**)

$\begin{array}{l} \Gamma_+(z) = \Gamma_-(z)e^{T(z)} \left(1 - \frac{\zeta}{z}\right)^{-E_j} z^{E_i - E_j}, \quad z \in \partial\mathbb{D} \\ \Gamma_+(z) = \Gamma_-(z)z^{-E_i} M z^{E_i} \quad z \in \Sigma\gamma \\ \Gamma(\infty) = \mathbf{1} \end{array}$	$\begin{array}{l} \Gamma_+(z) = \Gamma_-(z)e^{T(z)} \left(1 - \frac{\zeta}{z}\right)^{E_i} z^{E_i - E_j}, \quad z \in \partial\mathbb{D} \\ \Gamma_+(z) = \Gamma_-(z)z^{E_j} M(z)z^{-E_j} \quad z \in \Sigma\gamma \\ \Gamma(\infty) = \mathbf{1} \end{array}$
---	--

(2.66)

The jumps on $\partial\mathbb{D}$ can be written

$$\begin{aligned} e^{T(z)} \left(1 - \frac{\zeta}{z}\right)^{-E_j} &= \exp \sum_{k=1}^{\infty} \frac{T_k + E_j \zeta^k / k}{z^k}; \quad e^{T(z)} \left(1 - \frac{\zeta}{z}\right)^{E_i} \\ &= \exp \sum_{k=1}^{\infty} \frac{T_k - E_i \zeta^k / k}{z^k}. \end{aligned} \tag{2.67}$$

⁴ This follows by observing that ϑ will be identically zero due to the analyticity of T outside of the disk.

⁵ The notation \vec{T} stands for the (infinite) vector of the (matrix) coefficients of $T(z)$, $\vec{T} = (T_1, T_2, \dots)$.

This leads to the following identities

$$\Gamma_{ij}(\zeta; [T]) = \Gamma_{ij}(0; [T - E_j[\zeta]]), \quad \Gamma^{-1}_{ij}(\zeta; [T]) = \Gamma^{-1}_{ij}(0; [T + E_i[\zeta]]), \tag{2.68}$$

and hence

$$\Gamma_{ij}(\zeta; [T]) = \frac{\tau \left\{ \begin{smallmatrix} \infty & 0 \\ i & j \end{smallmatrix} \right\} (\vec{T} - E_j[\zeta])}{\tau(\vec{T})}; \quad \Gamma^{-1}_{ij}(\zeta; [T]) = \frac{\tau \left\{ \begin{smallmatrix} 0 & \infty \\ i & j \end{smallmatrix} \right\} (\vec{T} + E_i[\zeta])}{\tau(\vec{T})}. \tag{2.69}$$

Here we have used the standard notation $[\zeta] = (\zeta, \frac{\zeta^2}{2}, \dots, \frac{\zeta^k}{k}, \dots)$.

2.3.2. Hirota bilinear relations We will not go into much depth here, since all is well-known, but it may give an *analytic* perspective on the relations, which are usually taken only formally.

As in the previous Sect. 2.3.1, let \mathcal{C} be a counterclockwise circle around a point $z = a \notin \Sigma_\gamma$ ($a = 0$ for simplicity) and let $T(z) : \mathcal{C} \rightarrow gl_r(\mathbb{C})$ be as (2.60). We have

$$\oint_{\mathcal{C}} \Gamma_+(\zeta; [T]) e^{-T(\zeta) + \vec{T}(\zeta)} \Gamma_+^{-1}(\zeta; [\vec{T}]) \frac{d\zeta}{2i\pi\zeta^2} = \oint_{\mathcal{C}} \Gamma_-(\zeta; [T]) \Gamma_-^{-1}(\zeta; [\vec{T}]) \frac{d\zeta}{2i\pi\zeta^2}. \tag{2.70}$$

Now, the matrix $\Gamma(x; [T])\Gamma^{-1}(x; [\vec{T}])$ has **no jumps outside** \mathcal{C} since the other jumps of the problem have been left unmodified; thus it is analytic on the complement of the disk \mathbb{D} and goes to the identity at ∞ . Thus we have the identity

$$\oint_{\mathcal{C}} \Gamma_+(\zeta; [T]) e^{-T(\zeta) + \vec{T}(\zeta)} \Gamma_+^{-1}(\zeta; [\vec{T}]) \frac{d\zeta}{2i\pi\zeta^2} = 0 \tag{2.71}$$

which is valid *identically* in T, \vec{T} in a neighborhood of $T \equiv 0$.

In view of the interpretation of Γ as the Baker–Akhiezer vector for the τ -function (Sato-formula above), the reader may regard (2.71) as the generating function of an infinity of bilinear identities between matrix-valued τ functions;

$$\sum_{\ell=1}^r \oint \tau \left\{ \begin{smallmatrix} \infty & 0 \\ i & \ell \end{smallmatrix} \right\} (\vec{T} - E_\ell[\zeta]) e^{\vec{T}_\ell(\zeta) - T_\ell(\zeta)} \tau \left\{ \begin{smallmatrix} 0 & \infty \\ \ell & j \end{smallmatrix} \right\} (\vec{T} + E_\ell[\zeta]) \frac{d\zeta}{\zeta^2} \equiv 0. \tag{2.72}$$

The identity (2.72) should be used as a generating function of an infinite hierarchy of PDEs for the matrix-valued tau-function $[\tau(\vec{T})]_{ij} = \tau \left\{ \begin{smallmatrix} \infty & 0 \\ i & j \end{smallmatrix} \right\} (\vec{T})$ when expanding it in Taylor series with respect to \vec{T} around the diagonal $\vec{T} = \vec{T}$. This generates a sort of “addition theorem” for tau-functions (see Remark 2 in [8]).

The variational formula (2.9) in Lemma 2.1 takes on an added significance in view of the identity

$$\frac{1}{\tau} \tau \left\{ \begin{smallmatrix} \xi & \eta \\ i & j \end{smallmatrix} \right\} = \left[\frac{\Gamma^{-1}(\xi)\Gamma(\eta)}{\xi - \eta} \right]_{ij}. \tag{2.73}$$

To explain this in a simple situation we now consider **two disks** \mathbb{D}_0 and \mathbb{D}_1 centered at two points $-a = 0, 1$; on the boundaries of these disks we introduce diagonal jumps exactly as in Sect. 2.3.1,

$$T^{(0)}(z) = \sum_{k=1}^{\infty} \frac{T_k^{(0)}}{z^k}, \quad T^{(1)}(z) = \sum_{k=1}^{\infty} \frac{T_k^{(1)}}{(z-1)^k}. \tag{2.74}$$

We denote by $\tau(\vec{T}^{(0)}, \vec{T}^{(1)})$ the tau–function as a function of the two (infinite) sets of times and $\tau \begin{Bmatrix} 0 & 1 \\ i & j \end{Bmatrix}(\vec{T}^{(0)}, \vec{T}^{(1)})$ the Schlesinger–transformed one. Let $\zeta \in \mathbb{D}_0$ and $\eta \in \mathbb{D}_1$; retracing the steps that lead to (2.69) we find

$$\frac{1}{\tau(\vec{T}^{(0)}, \vec{T}^{(1)})} \tau \begin{Bmatrix} 0 & 1 \\ i & j \end{Bmatrix}(\vec{T}^{(0)} + E_i[\xi], \vec{T}^{(1)} - E_j[\eta]) = \left[\frac{\Gamma^{-1}(\xi)\Gamma(\eta)}{\xi - \eta} \right]_{ij}. \tag{2.75}$$

This formula is the content (in different notation) of (3.11) in Thm. 3.2 of [8].

The variational equation (2.9) in our Lemma 2.1 applied to derivatives with respect to some directions in $T^{(0)}, T^{(1)}$ then become the generating functions for the Hirota bilinear relations that appear in Thm. 3.4 of [8]. They all boil down to the following identity, to be understood as generating functions of PDEs when evaluating its Taylor expansion on the diagonal $T^{(j)} = \tilde{T}^{(j)}, j = 0, 1$,

$$\begin{aligned} 0 &= \left(\oint_{\partial\mathbb{D}_0} + \oint_{\partial\mathbb{D}_1} \right) \frac{\Gamma^{-1}(\xi; T)\Gamma_{-}(x; T)}{\xi - x} \frac{\Gamma_{-}^{-1}(x; \tilde{T})\Gamma(\eta; \tilde{T})}{x - \eta} \frac{dx}{2i\pi} \\ &= \oint_{\partial\mathbb{D}_0} \frac{\Gamma^{-1}(\xi; T)\Gamma_{+}(x; T)}{\xi - x} e^{\tilde{T}^{(0)}(x) - T^{(0)}(x)} \frac{\Gamma_{+}^{-1}(x; \tilde{T})\Gamma(\eta; \tilde{T})}{x - \eta} \frac{dx}{2i\pi} + 0 \\ &\quad + \oint_{\partial\mathbb{D}_1} \frac{\Gamma^{-1}(\xi; T)\Gamma_{+}(x; T)}{\xi - x} e^{\tilde{T}^{(1)}(x) - T^{(1)}(x)} \frac{\Gamma_{+}^{-1}(x; \tilde{T})\Gamma(\eta; \tilde{T})}{x - \eta} \frac{dx}{2i\pi}. \tag{2.76} \end{aligned}$$

The interested reader should compare this with Theorem 3.4 in [8]. Since it is not the primary focus of this paper (and it is certainly not a new result), we will not pursue the issue here, also because it has been dealt with at length in [8], even though in the context of isomonodromic and isospectral deformations only.

2.4. Right gauge equivalence. We will say that the two problems are **(right) gauge equivalent** if there exists an analytic function

$$G : \mathbb{C} \setminus \Sigma_{\gamma} \rightarrow GL_r(\mathbb{C}) \tag{2.77}$$

admitting boundary values (also for its derivative) at Σ_{γ} , with $G(z) = \mathbf{1} + \mathcal{O}(z^{-1})$ as $z \rightarrow \infty$ and such that the jump matrices stand in the relation

$$\tilde{M}(z) = G_{-}^{-1}(z)M(z)G_{+}(z), \quad z \in \Sigma_{\gamma}. \tag{2.78}$$

It is immediate then that the two solutions are related by $\tilde{\Gamma}(z) = \Gamma(z)G(z)$. It is then seen that the difference of ω_M (or Ω) along the two solutions $\Gamma, \tilde{\Gamma}$ differ only in terms that do not involve Γ or $\tilde{\Gamma}$ and depend only and explicitly on M, G . Thus this equivalence

will not modify the singularity locus of ω_M and –if both deformation families admit a tau function– both tau functions will differ only by multiplication by a smooth nonzero factor. Indeed a direct computation yields

$$\begin{aligned} &\omega_M(\partial; [\tilde{\Gamma}]) - \omega_M(\partial; [\Gamma]) \\ &= \int_{\Sigma\gamma} \text{Tr} \left(\Gamma_-^{-1}(x)\Gamma'_-(x)G_-^{-1}(x)\partial G_-(x) - \Gamma_+^{-1}(x)\Gamma'_+(x)G_+^{-1}(x)\partial G_+(x) \right) \frac{dx}{2i\pi} \\ &\quad + \int_{\Sigma\gamma} \text{Tr} \left(M^{-1}(x)M'(x)G_+^{-1}(x)\partial G_+(x) - G_-^{-1}(x)G'_-(x)G_-^{-1}(x)\partial G_-(x) \right. \\ &\quad \left. - G_-^{-1}(x)G'_-(x)\partial M(x)M^{-1}(x) \right. \\ &\quad \left. + G_-^{-1}(x)G'_-(x)M(x)G_+^{-1}(x)\partial G_+(x)M^{-1}(x) \right) \frac{dx}{2i\pi}. \end{aligned} \tag{2.79}$$

The first term in the right hand side of (2.79) vanishes by the Cauchy theorem, since it amounts to the (boundary value of) the integral of $\text{Tr}(\Gamma^{-1}\Gamma'G^{-1}\partial G)$ on a collection of contours surrounding $\Sigma\gamma$ and contractible. Thus the difference will not involve the solution of the RHP and hence be a smooth differential in the parameters (i.e. gauge equivalence cannot modify the Malgrange divisor). Such equivalence does play a role in some cases (see [1] for examples where this happens, although not phrased in these terms).

3. Rational Differential Equations in Terms of Riemann–Hilbert Data

We now describe the class of ODE’s with rational coefficients

$$\Psi'(z) = A(z)\Psi(z) \tag{3.1}$$

in an unconventional way: we will start from the formulation of a RHP and then indicate how this problem (when solvable) is equivalent to an ODE. The class of matrices $A(z)$ that will be eventually described has poles at points a_1, \dots, a_k with orders $n_1 + 1, \dots, n_k + 1$. If $n_j = 0$ then the pole is simple.

One may take the point of view that we are providing a different (transcendental) coordinate system on the finite–dimensional vector space of rational matrices with fixed polar divisor.

The forward problem, namely, the construction of the RHP from the matrix $A(z)$ is more standard and we only sketch the main points, since it does not really play a direct rôle here. This procedure is often called the **(extended) monodromy map**. The standard reference for many assertions below is Wasow’s book [17] but also the paper [9] provides a concise recall.

3.1. *Monodromy map.* Given a rational matrix

$$A(z) = \sum_{j=1}^K \sum_{\ell=1}^{n_j+1} \frac{A_{j,\ell-1}}{(z - a_j)^\ell} + \sum_{\ell=0}^{n_0-1} A_{0,\ell+1}z^\ell, \tag{3.2}$$

we consider the ODE $\Psi'(z) = A(z)\Psi(z)$. Without loss of generality we will assume $\text{Tr}A(z) \equiv 0$ so that any solution has constant determinant that we can assume to be unity $\det \Psi \equiv 1$. We make the usual assumption:

Assumption 3.1 (Genericity). *The leading-coefficient matrices A_{j,n_j} for $n_j > 0$ have distinct simple eigenvalues. The diagonal matrix of eigenvalues (with an arbitrarily chosen order) will be denoted by T_{j,n_j} . If $n_j = 0$ (simple pole) then $T_{j,0}$ has eigenvalues which are also distinct modulo \mathbb{Z} (i.e. no pair of eigenvalues differ by an integer).*

To simplify some issues in the general description we will assume that in fact there is no pole of the connection $\partial_z - A(z)$ at $z = \infty$; this can always be achieved without loss of generality by a Möbius transformation that maps ∞ to a finite point (without mapping any of the other poles to infinity!).

Monodromy representation. In order to fix the normalization of the initial value problem we choose a finite basepoint $z_0 \in \mathbb{C}$ (not coinciding with any of the poles) and we will denote by \mathcal{D} the simply connected domain of \mathbb{P}^1 obtained by dissecting \mathbb{P}^1 along nonintersecting smooth arcs joining z_0 with each of the poles (Fig. 4).

We consider the initial value problem $\Psi(\infty) = \mathbf{1}$ (recall that ∞ is now a regular point of our ODE). By analytic continuation of the solution around a loop (based at ∞) that “goes around a_j ” (i.e. has index one relative to a_j and zero relative to z_0 and all other poles) we obtain $\Psi(z) \mapsto \Psi(z)M_j^{-1}$, $\det M_j = 1$. These loops generate the fundamental group $\pi_1(\mathbb{C} \setminus \{a_1, \dots\}, \infty)$ and provide a representation of this fundamental group

$$\pi(\mathbb{P}^1 \setminus \{a_1, \dots\}, \infty) \mapsto SL_r(\mathbb{C}). \tag{3.3}$$

Therefore, on the simply connected domain \mathcal{D} defined here above, the matrix $\Psi(z)$ is analytic, but on the boundary of the cuts from z_0 to the pole a_j we have a jump M_j .

Stokes’ phenomenon in brief. Consider a higher order pole a_j , with $n_j \geq 1$ and denote as follows the distinct eigenvalues of the leading coefficient matrix A_{j,n_j} ,

$$T_{j,n_j} = \text{diag}(\Lambda_{j,1}, \dots, \Lambda_{j,r}), \quad \Lambda_{j,\ell} \neq \Lambda_{j,s}, \quad \ell \neq s, \quad \zeta := (z - a_j). \tag{3.4}$$

One can find at each pole a_j $2n_j$ directions (the **anti-Stokes** directions) angularly separated by $\frac{\pi}{n_j}$ such that along each of them there exists a permutation σ (uniquely defined) yielding the definite ordering below:

$$\Re \left(\frac{\Lambda_{j,\sigma(1)}}{\zeta^{n_j}} \right) > \Re \left(\frac{\Lambda_{j,\sigma(2)}}{\zeta^{n_j}} \right) > \dots > \Re \left(\frac{\Lambda_{j,\sigma(r)}}{\zeta^{n_j}} \right). \tag{3.5}$$

Note that if on a direction we have the above ordering, on the next (counter)clockwise we have the exact reversed (with the same permutation).

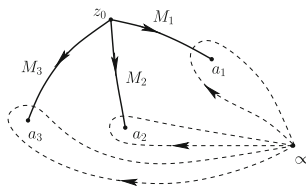


Fig. 4. The schematic arrangement of the basepoint z_0 and the dissection of \mathbb{P}^1 into a simply connected domain \mathcal{D} by means of cuts (thick arcs). Also the generators of the homotopy group and the corresponding jumps on the arcs of the dissection

The theorem which can be found in several places (e.g. [17]) is the following: given our IVP $\Psi(z)$ on the simply connected domain \mathcal{D} , there are $2n_j$ matrices $S_{j,\nu}$, $\nu = 1 \dots 2n_j$ (the **Stokes' matrices**) which are of the form

$$S_{j,\nu} = P_\sigma(\mathbf{1} + N_{j,\nu})P_\sigma^{-1}, \tag{3.6}$$

where $N_{j,\nu}$ is upper-triangular on the directions where the ordering is as in (3.5) and lower-triangular on the directions where the ordering is reversed, and P_σ is the permutation matrix corresponding to the permutation σ appearing in (3.5). There is an invertible matrix $C_j \in SL_r(\mathbb{C})$ (the **connection matrix**) and **diagonal** matrix

$$H_j(\zeta) := e^{T_j(\zeta)}\zeta^{L_j}, \quad T_j(\zeta) = \sum_{\ell=1}^{n_j} T_{j,\ell}\zeta^{-\ell}, \quad \zeta = (z - a_j) \tag{3.7}$$

in which T_{j,n_j} is the matrix of eigenvalues of the leading coefficient used in the definition of the anti-Stokes directions. This matrix will be called the **toral element** (for lack of a better name, since it belongs to the complex toral subalgebra of $SL_r(\mathbb{C})$) and the diagonal matrix L_j is called the **exponent of formal monodromy**.

These matrices (Stokes' connections and toral elements) are uniquely determined by the following set of conditions

1. In one (arbitrarily chosen and then fixed) sector separated by the two consecutive anti-Stokes directions we have the asymptotic expansion

$$\Psi(z) \sim \widehat{Y}_j(z)e^{T_j(\zeta)}\zeta^{L_j}C_j, \tag{3.8}$$

$$\widehat{Y}_j(z) = G_j \left(\mathbf{1} + \sum_{\ell=0}^{\infty} Y_{j,\ell}\zeta^\ell \right), \quad \det G_j = 1, \tag{3.9}$$

where the (constant in z) matrix G_j depends on which pole we are considering.

2. In the next sector counterclockwise we have

$$\Psi(z) \sim \widehat{Y}_j(z)e^{T_j(\zeta)}\zeta^{L_j}S_{j,1}C_j, \tag{3.10}$$

where we have labeled by 1 the anti-Stokes direction separating the two sectors, and in general

$$\Psi(z) \sim \widehat{Y}_j(z)e^{T_j(\zeta)}\zeta^{L_j}S_{j,\nu} \cdots S_{j,2} \cdot S_{j,1}C_j, \tag{3.11}$$

where we are in the sector between two anti-Stokes and we have crossed ν such lines.

3. The monodromy around the pole under consideration, the connection matrix and the Stokes matrices satisfy the condition

$$M_j = C_j^{-1}S_{j,2n_j} \cdots S_{j,1}C_j. \tag{3.12}$$

Note that at a simple pole we simply have $n_j = 0$ and hence there are no anti-Stokes' lines but only the exponents of formal monodromy L_j and the connection matrix C_j .

Remark 3.1. If we had allowed a pole at infinity we would need to make only technical modifications: in that case –of course– one cannot set-up an initial value problem at ∞ , but can require a specified normalized asymptotic behavior in a chosen sector. The description of the Stokes' phenomenon is identical to the one provided above, using the coordinate $\zeta = \frac{1}{z}$. We feel that there is no sufficient added value to this more general setting to justify entering into these confusing details, especially given that the properties of a rational ODE are invariant under Möbius transformations.

Since these are standard facts about ODEs we will not dwell on other subtleties.

3.1.1. *Forward Birkhoff map.* Given the rational matrix $A(z)$ of our form, a chosen basepoint and dissection \mathcal{D} , anti-Stokes lines and base-sector from where to start the counting we have associated the collection of all the data

$$\mathcal{M} := \{a_j, T_j(\zeta_j), L_j, C_j, \{S_{\nu,j} : \nu = 1 \dots 2n_j\}, \}_{j=1,\dots,k}. \tag{3.13}$$

The statement is that the map is (locally) injective but not surjective in general; there are some choices of data in \mathcal{M} for which there is no corresponding ODE of the specified form. As the reader may suspect (or know), these data constitute the Malgrange divisor.

There are two logically (and historically) distinct types of data in the above: the **isomonodromic times**

$$\mathcal{T} := \{a_j, T_j(\zeta_j)\}_{j=1\dots k} \tag{3.14}$$

(the positions a_j and the coefficients of the matrices $T_j(\zeta_j)$). On the other hand there are the genuine generalized monodromy data

$$\mathcal{S} := \{L_j, C_j, \{S_{j,\nu} : \nu = 1 \dots 2n_j\}, \}_{j=1\dots k} \tag{3.15}$$

and \mathcal{M} is locally the product of the two. Of course this separation can be only done locally since the anti-Stokes directions depend on the leading coefficients of $T_j(\zeta)$ and the dissection \mathcal{D} depends on the position of the poles, and hence some care is in order –if global questions are at issue– in describing the patching of these local descriptions (see for example [5]).

4. Inverse Birkhoff Map: Riemann–Hilbert Problem

Given the Birkhoff data described above, the question arises as to whether one can invert the map: starting from a concrete dissection, anti-Stokes lines, etc., together with all the matrices appearing in \mathcal{M} , can one *reconstruct* $A(z)$? To this end it is necessary to specify a Riemann–Hilbert problem in the same spirit as Sect. 2.

4.1. *The set of contours.* The set of contours $\Sigma\gamma$ consists of (see Fig. 5)

1. For each pole we draw a circle not containing any other pole: we will call this the **connection circle**.
2. For each pole a_j a smaller circle is chosen, called the **formal monodromy circle**. On this circle a point β_j is chosen.
3. Each β_j on the formal monodromy circle is connected with a set of mutually nonintersecting paths (**stems**) to the basepoint.
4. At the higher poles we choose a third smaller circle called the **toral circle**.
5. A point (arbitrarily chosen) on the *toral circle* is connected finally to the pole by $2n_j$ smooth curves that approach the singularity along the directions mentioned in Sect. 3.1;

The word description is awkward but Fig. 5 should clarify all the elements.

4.2. *The jump-matrices.* Rather than describing the matrices in words, we refer to the picture (Fig. 5) where we depict a situation with only two points a_1, a_2 with $n_1 = 0$ (simple pole). The general picture is quite simply a repetition of several copies of the basic elements already manifest here. Since there is no monodromy around the basepoint z_0 chosen for the dissection, we have to impose the constraint

$$C_K^{-1} e^{2i\pi L_K} C_K \dots C_1^{-1} e^{2i\pi L_1} C_1 = \mathbf{1}. \tag{4.1}$$

It will be understood that the matrices $(z - a_j)^{L_j}$ are defined on the formal monodromy circles as continuous functions taken away from the point of insertion of the points β_j (Fig. 5).

4.3. *Riemann–Hilbert problem.*

Problem 4.1. Find a piecewise analytic matrix-valued function $\Gamma(z)$ on the complement of the indicated contours as per Fig. 5 so that

- on each arc it solves $\Gamma_+(z) = \Gamma_-(z)M(z)$ with the matrix $M(z)$ as indicated in Fig. 5;
- it is bounded on \mathbb{P}^1 ;
- it satisfies the normalization $\Gamma(\infty) = \mathbf{1}$.

Some properties follow immediately:

- Any solution satisfies $\det \Gamma(z) \equiv 1$. Indeed $\det M(z) \equiv 1$ implies that $\det(\Gamma(z))$ has no jumps across the contours. Since $\Gamma(z)$ is bounded, so must $\det \Gamma$ be, and hence it is an entire function, bounded everywhere, hence a constant. Since $\Gamma(\infty) = \mathbf{1}$ then $\det \Gamma(z) = \det \Gamma(\infty) = 1$.

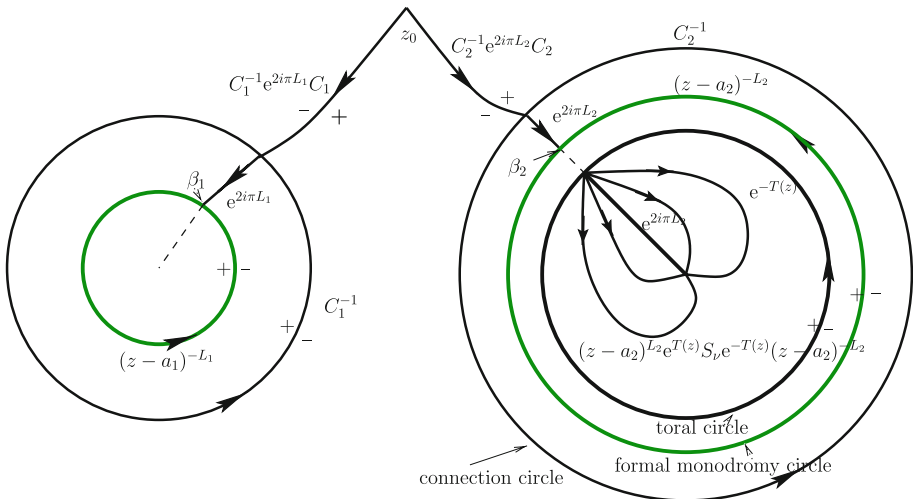


Fig. 5. The typical set of contours. The dashed lines within the formal monodromy circles are not jumps, but the domain where the determination of the powers $(z - a)^{-L}$ is continuously defined so as to give a precise meaning to the expressions for the various jump matrices

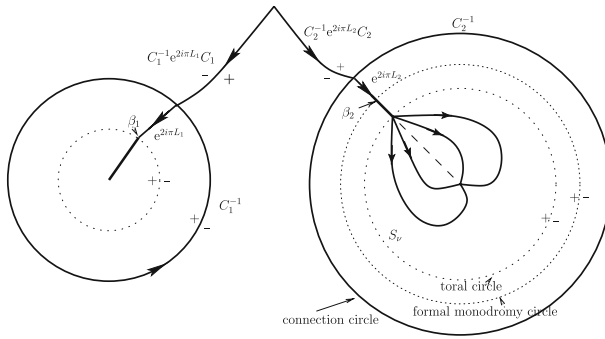


Fig. 6. The constant-jump Riemann–Hilbert problem for $\Psi(z)$

- If a solution exists, it is unique: indeed any two solutions $\Gamma(z), \tilde{\Gamma}(z)$ are analytically invertible by the above remark. Hence $\tilde{\Gamma}(z)\Gamma^{-1}(z)$ is promptly seen to have no jumps, be uniformly bounded, and thus it is a constant. The normalization forces $\Gamma(z) \equiv \tilde{\Gamma}(z)$.

The connection with the ODE is as follows; define the piecewise analytic matrix $\Psi(z)$ related to the solution $\Gamma(z)$ as follows:

- $\Psi(z) = \Gamma(z)$ outside the formal monodromy circles;
- $\Psi(z) = \Gamma(z)(z - a_j)^{L_j}$ in the annulus between toral and formal-monodromy circles;
- $\Psi(z) = \Gamma(z)e^{T_j(z)}(z - a_j)^{L_j}$ inside the formal-monodromy circles.

The cut of the logarithm is taken where the dotted line is traced in Fig. 5. It is promptly seen that $\Psi(z)$ solves a new RHP where the jump matrices are piecewise constant and unimodular (see Fig. 6 for an example). Thus

$$A(z) := \Psi'(z)\Psi^{-1}(z) \tag{4.2}$$

is a meromorphic (traceless) matrix function with isolated singularities at the a_j 's. A local inspection shows that it has poles of finite order there, and hence it is rational.

5. Tau Function and (Iso)monodromic Deformations

The goal now is to show that the modified Malgrange form Ω is closed when evaluated along the manifold $\mathcal{L} \subset \mathcal{G}$ consisting of jump-matrices $M(z)$'s described in the previous section. We emphasize that the parameters are

- the toral data $T_j(z)$'s,
- the connection matrices C_j ,
- the Stokes' matrices $S_{j,v}$,
- the exponents of formal monodromy L_j .

The reader acquainted with the literature about isomonodromic deformations should now realize that we allow many more directions of deformations. The computation of the closure of Ω relies directly on Prop. 2.2:

$$\delta\Omega(\partial, \tilde{\partial}) = \frac{1}{2} \int_{\Sigma_\gamma} \text{Tr} \left(M'(x)M^{-1}(x) \left[\partial M(x)M^{-1}(x), \tilde{\partial} M(x)M^{-1}(x) \right] \right) \frac{dx}{2i\pi}. \tag{5.1}$$

It is clear that no contribution to (5.1) can come from the arcs where $M(z)$ is independent of z . This leaves only the contributions coming from the toral circles, the formal monodromy circles and the Stokes’ lines. Let us consider each type separately

- **Stokes’ lines;** up to a constant conjugation by a permutation matrix the jumps are of the form $\mathbf{1} + N(z)$ with a strictly triangular $N(z)$, and hence they do not contribute to (5.1) (the trace is identically zero).
- **Toral circles;** on the Toral circles the jumps are diagonal and hence the commutator in (5.1) is identically zero.
- **Formal monodromy circles;** once more, since the jumps are diagonal, no contribution to (5.1) comes from them. Note that at a simple pole the matrix L may actually be upper triangular: in this case the trace vanishes identically.

Therefore we have

Theorem 5.1. *The differential Ω restricted to the submanifold \mathcal{L} of group-valued jump-matrices described above is closed and defines a local function via the formula*

$$\tau(\vec{T}, \vec{a}, \vec{L}, \vec{S}, \widehat{}) = e^{\int \Omega}. \tag{5.2}$$

This function is defined up to a nonzero multiplicative constant and it vanishes precisely when the Riemann–Hilbert problem is not solvable, namely, on the Malgrange Θ divisor.

The fact that τ has zeroes (and not a branching singularity) does not follow from our construction, it follows from [14 and 11].

We would like to stress that, had we chosen the Malgrange form ω_M directly, the closure would have failed. Indeed the reader may check that

$$\delta\omega_M(\partial, \tilde{\partial}) = \frac{1}{2} \sum_j \left(\partial a_j \tilde{\partial} \text{Tr}(L_j^2) - \tilde{\partial} a_j \partial \text{Tr}(L_j^2) \right) K_j, \tag{5.3}$$

$$K_j = \oint_{\beta_j} \frac{\ln(x - a_j)}{(x - a_j)^2} \frac{dx}{2i\pi} = \frac{1}{\beta_j - a_j} \neq 0, \tag{5.4}$$

where the integrals are on the toral circles with basepoint at the necks β_j .

Aside from this, the correction term ϑ in (2.26) does not contribute anything for most of the contours in $\Sigma\gamma$; in fact the only contribution comes –not surprisingly in view of the above computation– from the formal monodromy circles

$$\begin{aligned} \vartheta(\partial) &= \frac{1}{2} \int_{\Sigma\gamma} \text{Tr} \left(M'(x)M^{-1}(x)\partial M(x)M^{-1}(x) \right) \frac{dx}{2i\pi} \\ &= \frac{1}{2} \sum_j \text{Tr}(L_j\partial L_j) \oint_{\beta_j} \frac{\ln(x - a_j)}{x - a_j} \frac{dx}{2i\pi} \end{aligned} \tag{5.5}$$

$$= \frac{1}{2} \sum_j \text{Tr}(L_j\partial L_j)(2i\pi + 2 \ln(\beta_j - a_j)). \tag{5.6}$$

Note that this “problem” is invisible if we allow only isomonodromic deformations, in which case using ω_M or Ω yields the same differential.

Before we turn to a list of applications of the above result to explicit examples, we make the connection with the definition of [9].

5.1. *Relationship between ω_M and ω_{JMU} .* In order to establish the relationship we must “freeze” all the monodromy parts of the Birkhoff data, that is, the connection matrices, the Stokes matrices and the exponents of formal monodromy. As noted above, in this case $\Omega = \omega_M$. Then the observation that ω_M and ω_{JMU} coincide can be dug out [14] but we re-derive it here for the sake of being self-contained. If now ∂ is a derivative along the isomonodromic submanifold of the Birkhoff data, the differential reduces to an integral only on the Stokes’ lines, the toral circles and the formal-monodromy circles. Since the expression is repeated for each pole, we consider only the localization at one pole $a = a_j$, i.e. we vary the toral data/position only of one pole. Let ∂ denote one such deformation involving only data at $a = a_j$. Then

$$\Omega(\partial) = \omega_M(\partial) = - \int \text{Tr} \left(\Gamma_{-}^{-1}(x) \Gamma'_{-}(x) \partial M(x) M^{-1}(x) \right) \frac{dx}{2i\pi}. \tag{5.7}$$

There are two types of integrals here: the integrals along the Stokes lines and the integral around the toral circle. Along each Stokes contour the integrand can be equivalently written as follows:

$$\begin{aligned} & \text{Tr} \left(\Gamma_{-}^{-1}(x) \Gamma'_{-}(x) \left(\partial \widehat{T}_j(x) - \overbrace{e^{\widehat{T}_j(x)} S_{j,v} e^{-\widehat{T}_j(x)}}^{M(x)} \partial \widehat{T}_j(x) e^{\widehat{T}_j(x)} \overbrace{S_{j,v}^{-1} e^{-\widehat{T}_j(x)}}^{M^{-1}(x)} \right) \right) \\ &= \text{Tr} \left(\Gamma_{-}^{-1}(x) \Gamma'_{-}(x) \partial \widehat{T}_j(x) - \Gamma_{+}^{-1}(x) \Gamma'_{+}(x) \partial \widehat{T}_j(x) \right) + \text{Tr} \left(M^{-1}(x) M'(x) \partial \widehat{T}_j(x) \right) \end{aligned} \tag{5.8}$$

$$\widehat{T}_j(z) := T_j(z) + L_j \ln(z - a_j), \tag{5.9}$$

0 (traceless)

where we have used the cyclicity of the trace on the second term followed by

$$M^{-1}(x) \Gamma_{-}^{-1}(x) \Gamma'_{-}(x) M(x) = \Gamma_{+}^{-1}(x) \Gamma'_{+}(x) - M^{-1}(x) M'(x). \tag{5.10}$$

Recall also that $\partial \widehat{T}_j(z)$ does not contain a logarithmic term because $\partial L_j = 0$ (only isomonodromic deformations are allowed). As a consequence of (5.8) the resulting sum of integrals along the Stokes tentacles (refer to Fig. 7) can be evaluated by excising an ϵ circle⁶ around the pole $z = a_j$ and integrating $\text{Tr}(\Gamma^{-1}(z) \Gamma'(z) \partial \widehat{T}_j(z))$ on the solid black contours indicated in Fig. 7, followed by a limit $\epsilon \rightarrow 0$. For fixed ϵ the integral along these solid “tentacles” is equal (by Cauchy’s theorem) to the counterclockwise integral along the ϵ circle and the clockwise integral along the bigger dashed circle indicated in Fig. 7, on the + side (inside) of the toral circle.

We now have two integrals of $\text{Tr}(\Gamma_{-}^{-1}(x) \Gamma'_{-}(x) \partial T_j(x))$ on the right (outside) of the toral circle, coming from the definition of ω_M minus the integral of $\text{Tr}(\Gamma_{+}^{-1}(x) \Gamma'_{+}(x) \partial \widehat{T}_j(x))$ on the right (inside) coming from the above contour deformation:

$$- \oint_{\text{toral}} \text{Tr}(\Gamma_{-}^{-1}(x) \Gamma'_{-}(x) \partial T_j(x) - \Gamma_{+}^{-1}(x) \Gamma'_{+}(x) \partial \widehat{T}_j(x)) dx. \tag{5.11}$$

Given the jump condition on the toral circle $\Gamma_{+}(z) = \Gamma_{-}(z) e^{-T_j(z)}$, we have

$$\text{Tr}(\Gamma_{+}^{-1}(z) \Gamma'_{+}(z) \partial \widehat{T}_j(z)) = \text{Tr}(\Gamma_{-}^{-1}(z) \Gamma'_{-}(z) \partial \widehat{T}_j(z)) - T'_j(z) \partial \widehat{T}_j(z). \tag{5.12}$$

⁶ Note that $\Gamma(z)$ has a jump of the form $e^{2i\pi L_j}$ in the straight ray from a to the toral circle (Fig. 5), but the expression $\text{Tr}(\Gamma^{-1}(x) \Gamma'(x) \partial \widehat{T}_j(x))$ does **not have a jump** there, since L_j and \widehat{T}_j are both diagonal. This is the reason why we did not draw the corresponding solid line in Fig. 7.

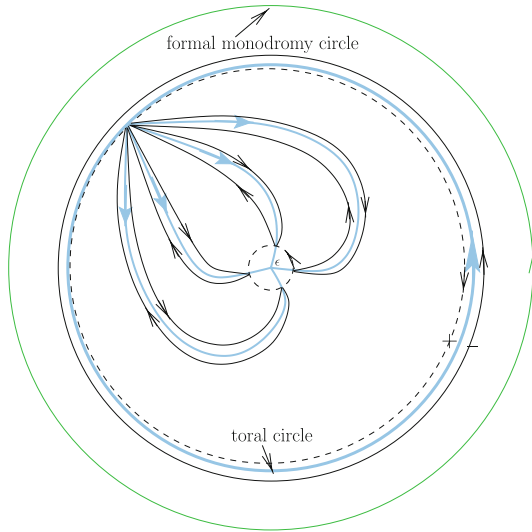


Fig. 7. The set of contours for the computation of ω_{JMU}

Hence (5.11) reduces to (recall from (5.9 that $\partial T_j(z) - \partial \widehat{T}_j(z) = -\frac{L_j \partial a_j}{z - a_j}$)

$$= - \overbrace{\oint_{\text{toral}} \text{Tr}(T'_j(x) \partial \widehat{T}_j(x))}^0 + \oint_{\text{toral}} \text{Tr} \left(\Gamma_-^{-1}(x) \Gamma'_-(x) \frac{L_j \partial a_j}{x - a_j} \right) \frac{dx}{2i\pi}, \quad (5.13)$$

where the first integral is zero because $T'_j(z) \partial \widehat{T}_j(z)$ is a Laurent polynomial starting from the term $(z - a_j)^{-2}$. The remaining can be deformed within the annulus between the toral and formal-monodromy circle (and the Γ_- of the toral circle becomes the Γ_+ of the local monodromy circle) and combines with the outer integral therein to give

$$\begin{aligned} & \oint_{\text{for.mon.}} \text{Tr} \left(\left(\Gamma_+^{-1}(x) \Gamma'_+(x) - \Gamma_-^{-1}(x) \Gamma'_-(x) \right) \frac{L_j \partial a_j}{a_j - x} \right) \frac{dx}{2i\pi} \\ &= - \oint_{\text{for.mon.}} \frac{\text{Tr}(L_j^2) \partial a_j^2}{(x - a_j)^2} \frac{dx}{2i\pi} = 0. \end{aligned} \quad (5.14)$$

The remaining integrals can therefore be retracted to an integral around the ϵ circle:

$$\oint_{|x - a_j| = \epsilon} \text{Tr} \left(\Gamma^{-1}(x) \Gamma'(x) \partial \widehat{T}_j(x) \right) \frac{dx}{2i\pi}. \quad (5.15)$$

In each sector we have the expansion valid for any N ,

$$\Gamma(z) = G_j \left(\mathbf{1} + \sum_0^N Y_{j,k}(z - a_j)^k \right) + \mathcal{O}((z - a_j)^{N+1}) \sim G_j \left(\mathbf{1} + \sum_0^\infty Y_{j,k}(z - a_j)^k \right), \quad (5.16)$$

where the coefficient matrices G, Y_k are the same irrespective of the sector. If we replace $\Gamma(z)$ in (5.15) by a suitably high truncation of the formal series, we commit an error that it is easily estimated to vanish as ϵ tends to zero. On the other hand the new integral is independent of ϵ and reduces to the formal residue,

$$\Omega(\partial; [\Gamma]) = \omega_M(\partial; [\Gamma]) = \operatorname{res}_{z=a_j} \operatorname{Tr} \left(\widehat{Y}_j^{-1}(z) \widehat{Y}'_j(z) \partial \widehat{T}_j(z) \right) dz ,$$

$$\widehat{Y}_j(z) := G_j \left(\mathbf{1} + \sum_0^\infty Y_{j,k}(z - a_j)^k \right), \tag{5.17}$$

to be understood simply as the coefficient of $(z - a_j)^{-1}$ of the above formal series with a *finite* Laurent tail. The reader acquainted with the definition of ω_{JMU} will recognize that the expression is precisely the same defining ω_{JMU} in [9] (of course one should repeat the residue computation at each pole and sum up).

In conclusion we have shown

Proposition 5.1. *The (modified) Malgrange differential Ω restricted to the manifold of isomonodromic deformations coincides with the Jimbo-Miwa-Ueno differential.*

6. Applications and Examples

6.1. Painlevé II equation. We single out the second Painlevé for its relative simplicity in formulæ and as an illustration of potential applications to other Painlevé equations. We follow the setup in [6]. The Riemann–Hilbert description is shown in the picture, with the notations and condition

$$L(s) := \begin{bmatrix} 1 & 0 \\ s e^{2i\left(\frac{4}{3}z^3+tz\right)} & 1 \end{bmatrix}, \quad U(s) := \begin{bmatrix} 1 s e^{-2i\left(\frac{4}{3}z^3+tz\right)} \\ 0 & 1 \end{bmatrix}, \tag{6.1}$$

$$s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0. \tag{6.2}$$

The matrices $U(s), L(s)$ (where the letters stand mnemonically to remind one of the “upper” or “lower” triangularity) are the jump matrices for the RHP, and the parameter s depends on the ray under consideration as in Fig. 8. Condition 6.2 is the condition that the products of the jumps at the origin is the identity. The Riemann–Hilbert problem is then that of finding $\Gamma(z)$ uniformly bounded, with the indicated jumps and the normalization condition

$$\Gamma(z) \sim \mathbf{1} + \mathcal{O}(z^{-1}). \tag{6.3}$$

The solution has the symmetry

$$\Gamma(z) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Gamma(-z) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{6.4}$$

which is seen by showing that if $\Gamma(z)$ solves the problem then so does the right hand side of (6.4). The matrix $T(z)$ in the notation of Sect. 3 is given by

$$T(z) = - \left(\frac{4i}{3} z^3 + itz \right) \sigma_3, \quad \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{6.5}$$

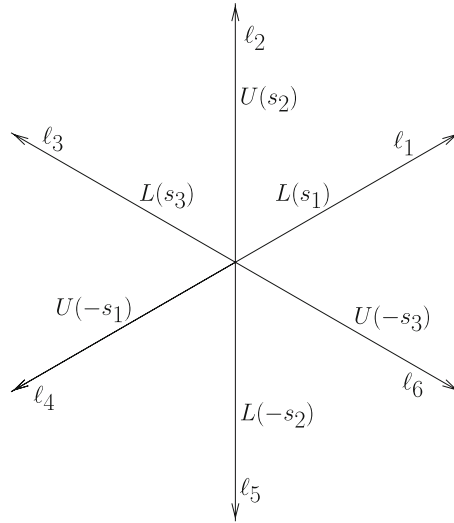


Fig. 8. The contours (ℓ_1, \dots, ℓ_6) and jump matrices (with the notation in (6.1)) of the RHP associated to the second Painlevé transcendent

and the matrix $\Psi(z) := \Gamma(z)e^T$ has constant jumps (on the anti-Stokes lines) and satisfies

$$\begin{aligned} \Psi'(z)\Psi(z)^{-1} &= -i(4z^2 + t + 2u^2)\sigma_3 - 4uz\sigma_2 - 2v\sigma_1 \\ &= \begin{pmatrix} -4i & 0 \\ 0 & 4i \end{pmatrix} z^2 + \begin{pmatrix} 0 & 4iu \\ -4iu & 0 \end{pmatrix} z + \begin{pmatrix} -it - 2iu^2 & -2v \\ -2v & it + 2iu^2 \end{pmatrix}, \end{aligned} \tag{6.6}$$

$$\sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad v = q'(t). \tag{6.7}$$

Under an isomonodromic deformation the coefficients u, v in (6.6) must evolve as functions of t and –in particular–

$$u = u(t; \vec{s}) = 2 \lim_{z \rightarrow \infty} z \Gamma_{12}(z; t, \vec{s}) \tag{6.8}$$

(the limit does not depend on the sector we choose) solves the second Painlevé equation

$$\frac{d^2u}{dt^2} = 2u^3 + tu. \tag{6.9}$$

By direct computations we have

$$\begin{aligned} \omega_{JMU} &= \left(\left(\frac{du}{dt} \right)^2 - tu^2 - u^4 \right) dt = \partial_t \ln \tau(t; \vec{s}) dt \\ \partial_t^2 \ln \tau(t; \vec{s}) &= u(t; \vec{s})^2. \end{aligned} \tag{6.10}$$

Generically we can solve the condition (6.2) for s_3 and use s_1, s_2 as independent variables: more appropriately one should consider the parameter space as the algebraic manifold specified by (6.2). Introduce the matrix kernel

$$\mathbf{K}(x, y) := \frac{\Gamma(x)^{-1}\Gamma(y)}{x - y}. \tag{6.11}$$

We then have

$$\begin{aligned} \partial_{s_1} \ln \tau(t; \vec{s}) &= -2 \int_{\ell_1} \text{Tr} \left(\Gamma^{-1}(x)\Gamma'(x) \begin{bmatrix} 0 & 0 \\ e^{\frac{i4}{3}x^3+itx} & 0 \end{bmatrix} \right) \frac{dx}{2i\pi} \\ &\quad + 2 \frac{1+s_2^2}{(1+s_1s_2)^2} \int_{\ell_3} \text{Tr} \left(\Gamma^{-1}(x)\Gamma'(x) \begin{bmatrix} 0 & e^{-\frac{i4}{3}x^3-itx} \\ 0 & 0 \end{bmatrix} \right) \frac{dx}{2i\pi} \\ &= -2 \int_{\ell_1} \mathbf{K}_{12}(x, x) e^{\frac{i4}{3}x^3+itx} \frac{dx}{2i\pi} \\ &\quad + 2 \frac{1+s_2^2}{(1+s_1s_2)^2} \int_{\ell_3} \mathbf{K}_{21}(x, x) e^{-\frac{i4}{3}x^3-itx} \frac{dx}{2i\pi}, \end{aligned} \tag{6.12}$$

where the boundary-value indication is irrelevant since the indicated matrix element does not have a jump on the corresponding line and we have used the symmetry (6.4) which causes the factor of 2. Using formula (2.15) for the second derivatives and then (2.9), one may derive an integral formula for the derivative $\partial_t^2 \partial_{s_1} \ln \tau$; however, in view of (6.8) it is simpler to use the variational formula (2.8) directly, thus yielding

$$\begin{aligned} \partial_{s_1} u(t; \vec{s}) &= 2 \lim_{z \rightarrow \infty} z \partial_{s_1} \Gamma_{12}(z) \\ &= 4 \lim_{z \rightarrow \infty} \int_{\ell_1} \frac{z \left(\Gamma(x) \begin{bmatrix} 0 & 0 \\ e^{\frac{i4}{3}x^3+itx} & 0 \end{bmatrix} \Gamma^{-1}(x)\Gamma(z) \right)_{12} dx}{x - z} \frac{dx}{2i\pi} \\ &\quad - 4 \lim_{z \rightarrow \infty} \frac{1+s_2^2}{(1+s_1s_2)^2} \int_{\ell_3} \frac{z \left(\Gamma(x) \begin{bmatrix} 0 & e^{-\frac{i4}{3}x^3-itx} \\ 0 & 0 \end{bmatrix} \Gamma^{-1}(x)\Gamma(z) \right)_{12} dx}{x - z} \frac{dx}{2i\pi}, \end{aligned} \tag{6.13}$$

where we have used again the symmetry (6.4) to simplify the expression. Since $\Gamma(z) \rightarrow \mathbf{1}$ we have

$$\partial_{s_1} u(t; \vec{s}) = 4 \frac{1+s_2^2}{(1+s_1s_2)^2} \int_{\ell_3} \Gamma_{11}(x)^2 e^{-\frac{i4}{3}x^3-itx} \frac{dx}{2i\pi} - 4 \int_{\ell_1} \Gamma_{12}(x)^2 e^{\frac{i4}{3}x^3+itx} \frac{dx}{2i\pi}. \tag{6.14}$$

6.2. Variation of finite Toeplitz determinants for discontinuous symbols. Let $S^1 = \{|z| = 1\}$. Suppose that we have the following data:

- A finite partition of S^1 in sub-arcs γ_j , with β_j being the separating points;
- A collection of functions $a_j(z) : \gamma_j \rightarrow \mathbb{C}$ which are analytic in a neighborhood of γ_j .

Define $a(z) : S^1 \rightarrow \mathbb{C}$ as the piecewise analytic function that coincides with a_j on each (interior of) γ_j . The n^{th} Toeplitz determinant is defined as

$$T_n[a] := \det[\mu_{j-i}]_{i,j=1\dots n}, \quad \mu_j := \oint_{|z|=1} z^j a(z) \frac{dz}{2i\pi z}, \quad T_0[a] := 1. \quad (6.15)$$

It was shown in [1] that $T_n[a]$ (and more general objects) are isomonodromic tau functions for the following RHP;

$$\begin{aligned} \tilde{\Gamma}_+(z) &= \tilde{\Gamma}_-(z) \begin{bmatrix} 1 & w(z) \\ 0 & 1 \end{bmatrix}, \quad z \in S^1, \\ \tilde{\Gamma}(z) &= (\mathbf{1} + \mathcal{O}(z^{-1})) z^{n\sigma_3}, \quad z \rightarrow \infty, \\ \tilde{\Gamma}(z) &= \mathcal{O}(\ln(z - \beta_j)), \quad z \rightarrow \beta_j, \\ w(z) &:= z^{n-1} a(z). \end{aligned} \quad (6.16)$$

Note that $\tilde{\Gamma}$ depends on the integer n as well, and we will write $\tilde{\Gamma}_n$ when needed to specify the value of this integer. In order to reduce to the general setting of Sect. 2 we should add a big counterclockwise circle (the ‘‘formal monodromy’’ circle) with jump $z^{n\sigma_3}$ and replace the asymptotics with $\Gamma(\infty) = \mathbf{1}$; to take care of the logarithmic growth at the endpoints β_j we take ϵ -circles centered at the β_j (with ϵ sufficiently small) and define $\Gamma(z)$ as $\Gamma(z) = \tilde{\Gamma}(z)$ for z outside of the added circles, while inside we set

$$\Gamma(z) := \tilde{\Gamma}(z) \begin{bmatrix} 1 & -\mathcal{C}[w](z) \\ 0 & 1 \end{bmatrix}, \quad |z - \beta_j| < \epsilon, \quad \mathcal{C}[w](z) := \oint_{|x|=1} \frac{w(x) dx}{x - z} \frac{1}{2i\pi}. \quad (6.17)$$

Thus the problem (6.16) is recast to the equivalent Riemann–Hilbert problem (see Fig. 9),

$$\begin{aligned} \Gamma_+(z) &= \Gamma_-(z) \begin{bmatrix} 1 & z^n a(z) \\ 0 & 1 \end{bmatrix}, \quad |z| = 1, \quad \min_j |z - \beta_j| > \epsilon, \\ \Gamma_+(z) &= \Gamma_-(z) \begin{bmatrix} 1 & -\mathcal{C}[w](z) \\ 0 & 1 \end{bmatrix}, \quad |z - \beta_j| = \epsilon, \\ \Gamma_+(z) &= \Gamma_-(z) z^{n\sigma_3}, \quad |z| = R > 1, \\ \Gamma(z) &= (\mathbf{1} + \mathcal{O}(z^{-1})), \quad z \rightarrow \infty, \end{aligned} \quad (6.18)$$

and $\Gamma(z)$ uniformly bounded in \mathbb{C} . When necessary to specify the integer n , we will simply write $\Gamma_n(z)$.

Note that the additional jump on $|z| = R$ (oriented counterclockwise) is independent of the symbol and hence undergoes no deformation. Therefore this has no impact in the definition of the differential $\Omega_n(\bullet) = \Omega(\bullet; [\Gamma_n])$.

Note that the first column of the solution $\Gamma(z)$ to problem (6.18) consists of **polynomials** (of degree $\leq n$) for $|z| < R$; indeed it has no jumps on $|z| = 1$ and the added small circles (due to the triangularity of the jump-matrices). Outside the circle $|z| = R$, said column divided by z^n is bounded at infinity, which proves the assertion.

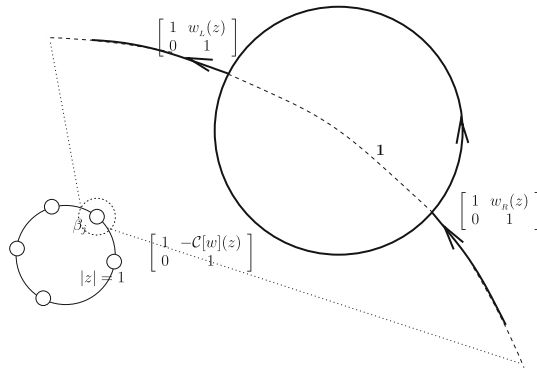


Fig. 9. The modification to the RHP to guarantee that the solution is bounded everywhere: here $C[w](z) := \oint \frac{w(x) dx}{x-z} \frac{1}{2i\pi}$ and the new RHP has no jump within the small disk centered at β_j

Similarly, the second row of $\Gamma^{-1}(z)$ (which solves a RHP with the jump matrix on the left) also consists of polynomials, by parallel arguments.

Thus the integrand in the definition of the differential Ω is in fact the integral of a polynomial kernel⁷

$$\mathbf{K}_{21}(z, w) := \frac{[\Gamma^{-1}(z)\Gamma(w)]_{21}}{z - w} \tag{6.19}$$

of degree $\leq n - 1$ in both variables (it has no jumps, it is regular on the diagonal $z = w$ and bounded by a power at infinity separately in each variable).

Now, the computations in [1] were done for an *isomonodromic* tau function, however the proof passes through without change to the extended Ω -differential. Namely, the proof consisted in showing that

$$\Omega_{n+1} - \Omega_n = \delta \ln \left(\frac{T_{n+1}[a]}{T_n[a]} \right), \tag{6.20}$$

where Ω_n denotes the differential evaluated on the solution Γ_n of (6.18). Since the two RHPs differ by an elementary Schlesinger transformation of the type discussed in Sect. 2.3, this part is identical. Moreover, $\Omega_0 \equiv 0$ because the integrands in (2.26) and (2.7) are identically zero (all the matrices are upper triangular and Ξ_∂ is strictly upper triangular), which means that $\tau_0 = \exp \int \Omega_0$ is a constant independent of the symbol, and can be taken to be 1. Since $T_0[a] = 1$ as well, this implies that

$$\delta \ln T_n[a] = \Omega_n, \tag{6.21}$$

where δ is the total differential w.r.t. any parameters and may appear in the symbol and the positions of the endpoints.

⁷ A fact which is well known and key to matrix model computations.

Dependence on the β_j . The dependence on each of the β_j is only in the jumps around the corresponding circle, and we have

$$\Xi_{\partial\beta_j} = \begin{bmatrix} 0 & \frac{-\Delta_j(\vec{\beta})}{2i\pi} \frac{1}{z-\beta_j} \\ 0 & 0 \end{bmatrix}, \quad \Delta_j(\vec{\beta}) := w_R(\beta_j) - w_L(\beta_j), \quad (6.22)$$

where the subscripts $_{L,R}$ indicate the boundary value on the Left/Right of w along the contour, relative to the orientation of the contour, namely, Δ is the jump of the symbol. This shows –a well-known fact– that

$$\partial_{\beta_j} T_n[a] = - \oint_{|x-\beta_j|=\epsilon} \mathbf{K}_{21}(x, x) \frac{-\Delta_j(\vec{\beta})}{2i\pi} \frac{1}{x-\beta_j} \frac{dx}{2i\pi} = \frac{\Delta_j(\vec{\beta})}{2i\pi} \mathbf{K}_{21}(\beta_j, \beta_j). \quad (6.23)$$

Dependence on the symbol. Let ∂ denote a deformation of one of the functions a_j ; then the jump matrices $M(z)$ deforms only on the arc γ_j and the small circles we have added, on a “dumb-bell” contour (Fig. 10). In particular, on the two circles the integral for Ω reduces to

$$\oint_{|z-\beta_j|=\epsilon} \mathbf{K}_{21}(x, x) \mathcal{C}[\partial w](x) \frac{dx}{2i\pi}, \quad (6.24)$$

$$\mathcal{C}[\partial w](z) = \int_{\gamma_j} \frac{x^n \partial a_j(x; \vec{s})}{x-z} \frac{dx}{2i\pi}. \quad (6.25)$$

Since, $\mathbf{K}_{21}(x, x)$ is analytic in the interior of the ϵ –circle (in fact, it is a polynomial!) then we can perform the integration by “contour deformation” leaving an integral of $\mathbf{K}_{21}(x, x)$ against the jump of $\mathcal{C}[\partial w]$. In conclusion, taking into consideration all the contours we have simply

$$\partial T_n[a] = \int_{\gamma_j} \mathbf{K}_{21}(x, x) x^n \partial a_j(x; \vec{s}) \frac{dx}{2i\pi}. \quad (6.26)$$

In general we have simply

$$\partial \ln T_n[a] = \oint_{|z|=1} \text{Tr} \left(\Gamma^{-1}(x) \Gamma'(x) \begin{bmatrix} 0 & x^n \partial_s a(x; \vec{s}) \\ 0 & 0 \end{bmatrix} \right) \frac{dx}{2i\pi} \quad (6.27)$$

$$= \oint \mathbf{K}_{21}(x, x) x^n \partial_s a(x; \vec{s}) \frac{dx}{2i\pi} \quad (6.28)$$

$$\mathbf{K}(x, y) := \frac{\Gamma^{-1}(x) \Gamma(y)}{x-y}. \quad (6.29)$$

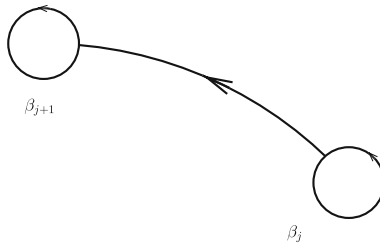


Fig. 10. The “dumbbell” contour

Example 6.1. A particular case is if $a_j(x; s) = s_j$ are constants, in which case the above reduces to an integral on the sub-arc γ_j ,

$$\partial_{s_j} \ln T_n[a] = \int_{\gamma_j} \mathbf{K}_{21}(x, x) x^n \frac{dx}{2i\pi}. \tag{6.30}$$

6.3. Hankel and shifted Töplitz determinants. The notion of *semiclassical symbols* was defined in [12] and it was shown in [1] that the corresponding Hankel/Toeplitz determinant was an isomonodromic tau function, to within an explicit non-vanishing factor.

The weights we are considering are all of the *semiclassical type* as defined in [2, 4, 10, 12, 13]. This means that they are of the form $\mu(x) = e^{-V(x)}$ with $V'(x)$ an arbitrary rational function.

They are integrated over contours γ_j which can be arbitrary contours in the complex plane as long as all integrals $\int_{\gamma_j} x^k \mu(x) dx$ are convergent integrals. The range for k will be either \mathbb{N} or \mathbb{Z} , depending on the situation; a detailed description of the contours can be found in [2, 3] and we refer thereto for a more detailed discussion.

We will choose arbitrary complex constants \varkappa_j for each contour γ_j and use the notation

$$\int_{\kappa} x^k \mu(x) dx := \sum \varkappa_j \int_{\gamma_j} x^k \mu(x) dx = \mu_{\kappa}, \tag{6.31}$$

We will also use the notation $\kappa : \mathbb{C} \rightarrow \mathbb{C}$ to indicate the locally constant function that takes the (constant) value \varkappa_j on the corresponding contour γ_j . The Markov function (sometimes referred to as a Weyl function) for these semiclassical weights is simply defined as the locally analytic function on $\mathbb{C} \setminus \cup \gamma_j$ given by

$$W(x) := \int_{\kappa} \frac{\mu(\zeta) d\zeta}{\zeta - x}, \quad \kappa := \sum_j \varkappa_j \chi_{\gamma_j}(x). \tag{6.32}$$

The function $W(x)$ has logarithmic growth at the **hard-edges**, i.e. the endpoints of contours γ_j , where μ is $\mathcal{O}(1)$. In this case one verifies that $W(x) = \mathcal{O}(\ln|x - a|)$, where a is the hard-edge point.

Consider the jump matrix supported on the contours $\cup \gamma_j$,

$$M(z) := \begin{bmatrix} 1 & \varkappa_j e^{-V(z)} \\ 0 & 1 \end{bmatrix}, \quad z \in \gamma_j. \tag{6.33}$$

Consider the following Riemann–Hilbert problems for a matrix $\Gamma = \Gamma^{(n, \ell)}$ parametrized by two integers n, ℓ

$$\Gamma_+(z) = \Gamma_-(z) M(z)$$

$$\Gamma(z) = \begin{cases} \mathcal{O}(1) \begin{bmatrix} 1 & 0 \\ 0 & z^{n-\ell-1} \end{bmatrix} & z \rightarrow 0 \\ (\mathbf{1} + \mathcal{O}(z^{-1})) \begin{bmatrix} z^n & 0 \\ 0 & z^{-\ell-1} \end{bmatrix} & z \rightarrow \infty \end{cases}, \tag{6.34}$$

and $\Gamma(z) = \mathcal{O}(\ln(z - a))$ at any “hard-edge”.

By definition of Schlesinger transformations (Sect. 2.3) we see that all these problems are successive Schlesinger transformations of the problem $n = 0, \ell = 1$, which has the immediate solution

$$\Gamma^{(0,1)}(z) = \begin{bmatrix} 1 & \int_{\mathcal{X}} \frac{e^{-V(x)} dx}{x-z} \\ 0 & 1 \end{bmatrix} \tag{6.35}$$

It is also apparent that the differentials ω_M (or Ω , they coincide here because the term ϑ in Def. 2.2 vanishes identically, being the jumps $\mathbf{1}$ +upper triangular) are closed differentials in the deformations of V , the endpoints and in the \mathcal{X}_j 's.

Moreover, comparing the change $\omega_M(\partial; [\Gamma^{(n,\ell)}])$ between two problems with $n \mapsto n \pm 1$ or $\ell \mapsto \ell \pm 1$ one sees exactly as in [1] that

$$\partial \ln \Delta_n^\ell = \omega_M(\partial; [\Gamma^{(n,\ell)}]), \tag{6.36}$$

where Δ_n^ℓ are the shifted Toeplitz determinants

$$\Delta_n^\ell := \det \begin{pmatrix} \mu_\ell & \mu_{\ell+1} & \cdots & \mu_{\ell+n-1} \\ \mu_{\ell-1} & \mu_\ell & \cdots & \mu_{\ell+n-2} \\ & \ddots & \ddots & \\ \mu_{\ell-n+1} & \mu_{\ell-n+2} & \cdots & \mu_\ell \end{pmatrix} \quad \Delta_0^\ell \equiv 1, \quad \Delta_{-n}^\ell \equiv 0.$$

Here the case of Hankel determinants corresponds to $\ell = n - 1$; then Δ_n^{n-1} is –up to a permutation of columns, hence a sign– a Hankel determinant.

In particular the derivatives w.r.t. the parameters \mathcal{X}_j are

$$\partial_{\mathcal{X}_j} \ln \Delta_n^\ell = \int_{\gamma_j} K^{(n,\ell)}(x, x) e^{-V(x)} \frac{dx}{2i\pi}. \tag{6.37}$$

$$K^{(n,\ell)}(x, y) = \frac{[\Gamma^{(n,\ell)}(x)^{-1} \Gamma^{(n,\ell)}(y)]_{21}}{x - y}. \tag{6.38}$$

This kernel is the usual Christoffel–Darboux kernel: for example, if $\ell = n - 1$ (Hankel determinants) then

$$K^{(n,n-1)}(x, y) = \frac{1}{h_{n-1}} \frac{p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)}{x - y} = \sum_{j=0}^{n-1} \frac{1}{h_j} p_j(x)p_j(y), \tag{6.39}$$

where p_j are the monic orthogonal polynomials of exact degree j for the moment-functional, namely,

$$\int_{\mathcal{X}} p_j(x)p_k(x)e^{-V(x)} dx = h_j \delta_{jk}. \tag{6.40}$$

Remark 6.1. Note that the RHP for $\Gamma^{(n,\ell)}$ can be converted to a RHP with constant jumps for $\Psi := \Gamma e^{-\frac{V}{2}\sigma_3}$ and hence reduced to a rational ODE $\Psi'\Psi^{-1} = D(z)$. The resulting *isomonodromic* tau function is not exactly equal to Δ_n^ℓ because the RHP for $\Gamma^{(n,\ell)}$ and the RHP in the canonical form described in Sect. 4 differ by a gauge in the sense of Sect. 2.4. However the corresponding factor is easy to compute and this was accomplished in [1].

We conclude with the remark that the variational formulæ for Δ_n^ℓ are valid for very general weights, not necessarily of semiclassical type. Indeed we are really using only the results of Sect. 2, where the connection to rational ODE is not necessary.

Acknowledgements. The author thanks Jacques Hurtubise and John Harnad for helpful discussion. The work was partially supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

References

1. Bertola, M.: Moment determinants as isomonodromic tau functions. *Nonlinearity* **22**(1), 29–50 (2009)
2. Bertola, M., Eynard, B., Harnad, J.: Semiclassical orthogonal polynomials, matrix models and isomonodromic tau functions. *Commun. Math. Phys.* **263**(2), 401–437 (2006)
3. Bertola, M., Gekhtman, M.: Biorthogonal Laurent polynomials, Töplitz determinants, minimal Toda orbits and isomonodromic tau functions. *Constr. Approx.* **26**(3), 383–430 (2007)
4. Bertola, M.: Bilinear semiclassical moment functionals and their integral representation. *J. Approx. Theory* **121**(1), 71–99 (2003)
5. Boalch, P.: Symplectic manifolds and isomonodromic deformations. *Adv. Math.* **163**(2), 137–205 (2001)
6. Its, A., Kapaev, A.: *The Nonlinear Steepest Descent Approach to the Asymptotics of the Second Painlevé Transcendent in the Complex Domain*. Volume **23** of *Progr. Math. Phys.* Boston, MA: Birkhäuser Boston, 2002
7. Jimbo, M., Miwa, T.: Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II. *Phys. D* **2**(3), 407–448 (1981)
8. Jimbo, M., Miwa, T.: Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. III. *Phys. D* **4**(1), 26–46 (1981)
9. Jimbo, M., Miwa, T., Ueno, K.: Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I. General theory and τ -function. *Phys. D* **2**(2), 306–352 (1981)
10. Magnus, A.P.: Painlevé-type differential equations for the recurrence coefficients of semi-classical orthogonal polynomials. In: *Proceedings of the Fourth International Symposium on Orthogonal Polynomials and their Applications* (Evian-Les-Bains, 1992), Volume **57**, London: Elsevier, 1995, pp. 215–237
11. Malgrange, B.: Sur les déformations isomonodromiques. I. Singularités régulières. In: *Mathematics and Physics (Paris, 1979/1982)*, Volume **37** of *Progr. Math.*, Boston, MA: Birkhäuser Boston, 1983, pp. 401–426
12. Marcellán, F., Rocha, I.A.: On semiclassical linear functionals: integral representations. In: *Proceedings of the Fourth International Symposium on Orthogonal Polynomials and their Applications* (Evian-Les-Bains, 1992), Volume **57**, London: Elsevier, 1995, pp. 239–249
13. Marcellán, F., Rocha, I.A.: Complex path integral representation for semiclassical linear functionals. *J. Approx. Theory* **94**(1), 107–127 (1998)
14. Palmer, J.: Zeros of the Jimbo, Miwa, Ueno tau function. *J. Math. Phys.* **40**(12), 6638–6681 (1999)
15. Schlesinger, L.J.: Über die Lösungen gewisser Differentialgleichungen als Funktionen der singulären Punkte. *J. Reine Angew. Math.* **141**, 96–145 (1912)
16. Sibuya, Y.: *Linear Differential Equations in the Complex Domain: Problems of Analytic Continuation*, Volume **82** of *Translations of Mathematical Monographs*. Providence, RI: Amer. Math. Soc., 1990, Translated from the Japanese by the author
17. Wasow, W.: *Asymptotic Expansions for Ordinary Differential Equations*. New York: Dover Publications Inc., 1987, reprint of the 1976 edition

Communicated by M. Aizenman