New Bounds for the Free Energy of Directed Polymers in Dimension 1 + 1 and 1 + 2

Hubert Lacoin

Université Paris Diderot and Laboratoire de Probabilités et Modèles
Aléatoires (CNRS U.M.R. 7599), U.F.R. Mathématiques,
Case 7012 (Site Chevaleret), 75205 Paris cedex 13, France.
E-mail: lacoin@math.jussieu.fr

Abstract: We study the free energy of the directed polymer in a random environment model in dimension 1 + 1 and 1 + 2. For dimension one, we improve the statement of Comets and Vargas in [8] concerning very strong disorder by giving sharp estimates on the free energy at high temperature. In dimension two, we prove that very strong disorder holds at all temperatures, thus solving a long standing conjecture in the field.

1. Introduction

1.1. The model. We study a directed polymer model introduced by Huse and Henley (in dimension 1 + 1) [18] with the purpose of investigating impurity-induced domain-wall roughening in the 2D-Ising model. The first mathematical study of directed polymers in random environment was made by Imbrie and Spencer [19], and was followed by numerous authors [1,3–6,8,9,19,23,26] (for a review on the subject see [7]). Directed polymers in a random environment model, in particular, polymer chains in a solution with impurities.

In our set–up the polymer chain is the graph $\{(i, S_i)\}_{1\le i\le N}$ of a nearest–neighbor path in $\mathbb{Z}^d$, S starting from zero. The equilibrium behavior of this chain is described by a measure on the set of paths: the impurities enter the definition of the measure as *disordered potentials*, given by a typical realization of a field of i.i.d. random variables $\omega = \{\omega_{(i,z)} \; ; \; i \in \mathbb{N}, z \in \mathbb{Z}^d\}$ (with associated law Q). The polymer chain will tend to be attracted by larger values of the environment and repelled by smaller ones. More precisely, we define the Hamiltonian

$$H_N(S) := \sum_{i=1}^{N} \omega_{i,S_i}. \tag{1.1}$$

We denote by P the law of the simple symmetric random walk on $\mathbb{Z}^d$ starting at 0 (in the sequel $Pf(S)$, respectively $Qg(\omega)$, will denote the expectation with respect to P,

respectively Q). One defines the polymer measure of order N at inverse temperature β as

$$\mu_N^{(\beta)}(S) = \mu_N(S) := \frac{1}{Z_N} \exp\left(\beta H_N(S)\right) P(S), \tag{1.2}$$

where Z_N is the normalization factor which makes μ_N a probability measure

$$Z_N := P \exp\left(\beta H_N(S)\right). \tag{1.3}$$

We call Z_N the *partition function* of the system. In the sequel, we will consider the case of $\omega_{(i,z)}$ with zero mean and unit variance and such that there exists $B \in (0, \infty]$ such that

$$\lambda(\beta) = \log Q \exp(\beta \omega_{(1,0)}) < \infty, \quad \text{for } 0 \le \beta \le B. \tag{1.4}$$

Finite exponential moments are required to guarantee that $QZ_N < \infty$. The model can be defined and it is of interest also with environments with heavier tails (see e.g. [26]) but we will not consider these cases here.

1.2. Weak, strong and very strong disorder. In order to understand the role of disorder in the behavior of μ_N, as N becomes large, let us observe that, when $\beta = 0$, μ_N is the law of the simple random walk, so that we know that, properly rescaled, the polymer chain will look like the graph of a d-dimensional Brownian motion. The main questions that arise for our model for $\beta > 0$ are whether or not the presence of disorder breaks the diffusive behavior of the chain for large N, and what the polymer measure looks like when diffusivity does not hold.

Many authors have studied diffusivity in polymer models: in [3], Bolthausen remarked that the renormalized partition function $W_N := Z_N/(QZ_N)$ has a martingale property and proved the following zero-one law:

$$Q\left\{\lim_{N\to\infty} W_N = 0\right\} \in \{0, 1\}. \tag{1.5}$$

A series of papers [1,3,9,19,23] lead to

$$Q\left\{\lim_{N\to\infty} W_N = 0\right\} = 0 \Rightarrow \text{diffusivity}, \tag{1.6}$$

and a consensus in saying that this implication is an equivalence. For this reason, it is natural and it has become customary to say that *weak disorder* holds when W_N converges to some non-degenerate limit and that *strong disorder* holds when W_N tends to zero.

Carmona and Hu [4] and Comets, Shiga and Yoshida [6] proved that strong disorder holds for all β in dimension 1 and 2. The result was completed by Comets and Yoshida [9]: we summarize it here

Theorem 1.1. *There exists a critical value $\beta_c = \beta_c(d) \in [0, \infty]$ (depending on the law of the environment) such that*

- *Weak disorder holds when $\beta < \beta_c$.*
- *Strong disorder holds when $\beta > \beta_c$.*

Moreover:

$$\beta_c(d) = 0 \ for \ d = 1, 2$$
$$\beta_c(d) \in (0, \infty] \ for \ d \geq 3. \tag{1.7}$$

We mention also that the case $\beta_c(d) = \infty$ can only occur when the random variable $\omega_{(0,1)}$ is bounded.

In [4 and 6] a characterization of strong disorder has been obtained in terms of localization of the polymer chain: we cite the following result [6, Theorem 2.1]:

Theorem 1.2. *If $S^{(1)}$ and $S^{(2)}$ are two i.i.d. polymer chains, we have*

$$Q\left\{ \lim_{N \to \infty} W_N = 0 \right\} = Q\left\{ \sum_{N \geq 1} \mu_{N-1}^{\otimes 2}(S_N^{(1)} = S_N^{(2)}) = \infty \right\}. \tag{1.8}$$

Moreover if $Q\{\lim_{N \to \infty} W_N = 0\} = 1$ there exists a constant c (depending on β and the law of the environment) such that for

$$-c \log W_N \leq \sum_{n=1}^{N} \mu_{n-1}^{\otimes 2}(S_n^{(1)} = S_n^{(2)}) \leq -\frac{1}{c} \log W_N. \tag{1.9}$$

One can notice that (1.9) has a very strong meaning in terms of trajectory localization when W_N decays exponentially: it implies that two independent polymer chains tend to share the same endpoint with positive probability. For this reason we introduce now the notion of free energy, we refer to [6, Prop. 2.5] and [9, Theorem 3.2] for the following result:

Proposition 1.3. *The quantity*

$$p(\beta) := \lim_{N \to \infty} \frac{1}{N} \log W_N, \tag{1.10}$$

exists Q-a.s., it is non-positive and non-random. We call it the free energy of the model, and we have

$$p(\beta) = \lim_{N \to \infty} \frac{1}{N} Q \log W_N =: \lim_{N \to \infty} p_N(\beta). \tag{1.11}$$

Moreover $p(\beta)$ is non-increasing in β.

We stress that the inequality $p(\beta) \leq 0$ is the standard *annealing* bound. In view of (1.9), it is natural to say that *very strong disorder* holds whenever $p(\beta) < 0$. One can moreover define $\overline{\beta}_c(d)$ the critical value of β for the free energy i.e. :

$$p(\beta) < 0 \Leftrightarrow \beta > \overline{\beta}_c(d). \tag{1.12}$$

Let us stress that, from the physicists' viewpoint, $\overline{\beta}_c(d)$ is the natural critical point because it is a point of non-analyticity of the free energy (at least if $\overline{\beta}_c(d) > 0$). In view of this definition, we obviously have $\overline{\beta}_c(d) \geq \beta_c(d)$. It is widely believed that $\overline{\beta}_c(d) = \beta_c(d)$, i.e. that there exists no intermediate phase where we have *strong disorder* but not *very strong disorder*. However, this is a challenging question: Comets and Vargas [8] answered it in dimension $1 + 1$ by proving that $\overline{\beta}_c(1) = 0$. In this paper, we make their result more precise. Moreover we prove that $\overline{\beta}_c(2) = 0$.

1.3. Presentation of the results. The first aim of this paper is to sharpen the result of Comets and Vargas on the $1 + 1$-dimensional case. In fact, we are going to give a precise statement on the behavior of $p(\beta)$ for small β. Our result is the following:

Theorem 1.4. *When $d = 1$ and the environment satisfies* (1.4)*, there exist constants c and $\beta_0 < B$ (depending on the distribution of the environment) such that for all $0 \leq \beta \leq \beta_0$ we have*

$$- \frac{1}{c}\beta^4[1 + (\log \beta)^2] \leq p(\beta) \leq -c\beta^4. \tag{1.13}$$

We believe that the logarithmic factor in the lower bound is an artifact of the method. In fact, by using replica-coupling, we have been able to get rid of it in the Gaussian case.

Theorem 1.5. *When $d = 1$ and the environment is Gaussian, there exists a constant c such that for all $\beta \leq 1$,*

$$- \frac{1}{c}\beta^4 \leq p(\beta) \leq -c\beta^4. \tag{1.14}$$

These estimates concerning the free energy give us some idea of the behavior of μ_N for small β. Indeed, Carmona and Hu in [4, Sect. 7] proved a relation between $p(\beta)$ and the overlap (although their notation differs from ours). This relation together with our estimates for $p(\beta)$ suggests that, for low β, the asymptotic contact fraction between independent polymers

$$\lim_{N \to \infty} \frac{1}{N} \mu_N^{\otimes 2} \sum_{n=1}^{N} \mathbf{1}_{\{S_n^{(1)} = S_n^{(2)}\}}, \tag{1.15}$$

behaves like β^2.

The second result we present is that $\overline{\beta}_c(2) = 0$. As for the $1 + 1$-dimensional case, our approach yields an explicit bound on $p(\beta)$ for β close to zero.

Theorem 1.6. *When $d = 2$, there exist constants c and β_0 such that for all $\beta \leq \beta_0$,*

$$- \exp\left(-\frac{1}{c\beta^2}\right) \leq p(\beta) \leq -\exp\left(-\frac{c}{\beta^4}\right), \tag{1.16}$$

so that

$$\overline{\beta}_c(2) = 0, \tag{1.17}$$

and 0 is a point of non-analyticity for $p(\beta)$.

Remark 1.7. After the appearance of this paper as a preprint, the proof of the above result has been adapted by Bertin [2] to prove the exponential decay of the partition function for *Linear Stochastic Evolution* in dimension 2, a model that is a slight generalisation of the directed polymer in a random environment.

Remark 1.8. Unlike in the one dimensional case, the two bounds on the free energy provided by our methods do not match. We believe that the second moment method, that gives the lower bound is quite sharp and gives the right order of magnitude for $\log p(\beta)$. The method developed in [16] to sharpen the estimate on the critical point shift for pinning models at marginality adapted to the context of directed polymer should be able to improve the result, getting $p(\beta) \leq -\exp(-c_\varepsilon \beta^{-(2+\varepsilon)})$ for all $\beta \leq 1$ for any ε.

1.4. Organization of the paper. The various techniques we use have been inspired by ideas used successfully for another polymer model, namely the polymer pinning on a defect line (see [10, 14, 15, 24, 25]). However the ideas we use to establish lower bounds differ sensibly from the ones leading to the upper bounds. For this reason, we present first the proofs of the upper bound results in Sects. 2, 3 and 4. The lower bound results are proven in Sects. 5, 6 and 7.

To prove the lower bound results, we use a technique that combines the so-called *fractional moment method* and change of measure. This approach has been first used for the pinning model in [10] and it has been refined since in [15, 25]. In Sects. 2, we prove a non-optimal upper bound for the free energy in the case of a Gaussian environment in dimension 1 + 1 to introduce the reader to this method. In Sect. 3 we prove the optimal upper bound for an arbitrary environment in dimension 1 + 1, and in Sect. 4 we prove our upper bound for the free energy in dimension 1 + 2 which implies that *very strong disorder* holds for all β. These sections are placed in increasing order of technical complexity, and therefore, should be read in that order.

Concerning the lower–bounds proofs: Sect. 5 presents a proof of the lower bound of Theorem 1.4. The proof combines the second moment method and a directed percolation argument. In Sect. 6 the optimal bound is proven for Gaussian environment, with a specific Gaussian approach similar to what is done in [24]. In Sect. 7 we prove the lower bound for arbitrary environment in dimension 1 + 2. These three parts are completely independent of each other.

2. Some Warm Up Computations

2.1. Fractional moment. Before going into the core of the proof, we want to present here the starting step that will be used repeatedly throughout Sects. 2, 3 and 4. We want to find an upper–bound for the quantity

$$p(\beta) = \lim_{N \to \infty} \frac{1}{N} Q \log W_N. \tag{2.1}$$

However, it is not easy to handle the expectation of a log, for this reason we will use the following trick. Let $\theta \in (0, 1)$, we have (by Jensen inequality)

$$Q \log W_N = \frac{1}{\theta} Q \log W_N^\theta \leq \frac{1}{\theta} \log Q W_N^\theta. \tag{2.2}$$

Hence

$$p(\beta) \leq \liminf_{N \to \infty} \frac{1}{\theta N} \log Q W_N^\theta. \tag{2.3}$$

We are left with showing that the fractional moment $Q W_N^\theta$ decays exponentially which is a problem that is easier to handle.

2.2. A non-optimal upper–bound in dimension 1+1. To introduce the reader to the general method used in this paper, combining fractional moment and change of measure, we start by proving a non–optimal result for the free–energy, using a finite volume criterion. As a more complete result is to be proved in the next section, we restrict to the Gaussian case here. The method used here is based on the one of [8], majorizing the free energy

of the directed polymer by the one of multiplicative cascades. Let us mention that is has been shown recently by Liu and Watbled [22] that this majorization is in a sense optimal, they obtained this result by improving the concentration inequality for the free energy.

The idea of combining the fractional moment with a change of measure and finite volume criterion has been used with success for the pinning model in [10].

Proposition 2.1. *There exists a constant c such that for all $\beta \leq 1$,*

$$p(\beta) \leq -\frac{c\beta^4}{(|\log \beta| + 1)^2}. \tag{2.4}$$

Proof of Proposition 2.1 in the case of Gaussian environment. For β sufficiently small, we choose n to be equal to $\left\lceil \frac{C_1 |\log \beta|^2}{\beta^4} \right\rceil$ for a fixed constant C_1 (here and throughout the paper for $x \in \mathbb{R}$, $\lceil x \rceil$, respectively $\lfloor x \rfloor$, will denote the upper, respectively the lower, integer part of x) and define $\theta := 1 - (\log n)^{-1}$. For $x \in \mathbb{Z}$ we define

$$W_n(x) := P \exp\left(\sum_{i=1}^{n} [\beta\omega_{(i,S_i)} - \beta^2/2]\right) \mathbf{1}_{\{S_n = x\}}. \tag{2.5}$$

Note that $\sum_{x \in \mathbb{Z}} W_n(x) = W_n$. We use a statement which can be found in the proof of Theorem 3.3. in [8]:

$$\log Q W_{nm}^{\theta} \leq m \log Q \sum_{x \in \mathbb{Z}} [W_n(x)]^{\theta} \quad \forall m \in \mathbb{N}. \tag{2.6}$$

This combined with (2.3) implies that

$$p(\beta) \leq \frac{1}{\theta n} \log Q \sum_{x \in \mathbb{Z}} [W_n(x)]^{\theta}. \tag{2.7}$$

Hence, to prove the result, it is sufficient to show that

$$Q \sum_{x \in \mathbb{Z}} [W_n(x)]^{\theta} \leq e^{-1}, \tag{2.8}$$

for our choice of θ and n.

In order to estimate $Q[W_n(x)]^{\theta}$ we use an auxiliary measure $\widetilde{Q}$. The region where the walk $(S_i)_{0 \leq i \leq n}$ is likely to go is $J_n = \left([1, n] \times [-C_2\sqrt{n}, C_2\sqrt{n}]\right) \cap \mathbb{N} \times \mathbb{Z}$, where C_2 is a big constant.

We define $\widetilde{Q}$ as the measure under which the $\omega_{i,x}$ are still independent Gaussian variables with variance 1, but such that $\widetilde{Q}\omega_{i,x} = -\delta_n \mathbf{1}_{(i,x) \in J_n}$, where $\delta_n = 1/(n^{3/4} \sqrt{2C_2 \log n})$. This measure is absolutely continuous with respect to Q and

$$\frac{\mathrm{d}\widetilde{Q}}{\mathrm{d}Q} = \exp\left(-\sum_{(i,x) \in J_n} \left[\delta_n \omega_{i,x} + \frac{\delta_n^2}{2}\right]\right). \tag{2.9}$$

Then we have for any $x \in \mathbb{Z}$, using the Hölder inequality we obtain,

$$Q\left[W_n(x)^{\theta}\right] = \widetilde{Q}\left[\frac{\mathrm{d}Q}{\mathrm{d}\widetilde{Q}} (W_n(x))^{\theta}\right] \leq \left(\widetilde{Q}\left[\left(\frac{\mathrm{d}Q}{\mathrm{d}\widetilde{Q}}\right)^{\frac{1}{1-\theta}}\right]\right)^{1-\theta} \left(\widetilde{Q} W_n(x)\right)^{\theta}. \tag{2.10}$$

The first term on the right-hand side can be computed explicitly and is equal to

$$\left(Q \left(\frac{dQ}{d\tilde{Q}} \right)^{\frac{\theta}{1-\theta}} \right)^{1-\theta} = \exp \left(\frac{\theta \delta_n^2}{2(1-\theta)} \# J_n \right) \leq e, \tag{2.11}$$

where the last inequality is obtained by replacing δ_n and θ by their values (recall $\theta = 1 - (\log n)^{-1}$). Therefore combining (2.10) and (2.11) we get that

$$Q \sum_{x \in \mathbb{Z}} (W_n(x))^\theta \leq e \sum_{|x| \leq n} \left(\tilde{Q} W_n(x) \right)^\theta. \tag{2.12}$$

To bound the right–hand side, we first get rid of the exponent θ in the following way:

$$\sum_{|x| \leq n} n^{-3\theta} \left(\tilde{Q} W_n(x) \right)^\theta \leq n^{-3\theta} \# \{ x \in \mathbb{Z}, \ |x| \leq n \text{ such that } \tilde{Q} W_n(x) \leq n^{-3} \}$$

$$+ \sum_{|x| \leq n} \mathbf{1}_{\{ \tilde{Q} W_n(x) > n^{-3} \}} \tilde{Q} W_n(x) n^{3(1-\theta)}. \tag{2.13}$$

If n is sufficiently large (i.e., β sufficiently small) the first term on the right-hand side is smaller than $1/n$ so that

$$\sum_{|x| \leq n} \left(\tilde{Q} W_n(x) \right)^\theta \leq \exp(3) \tilde{Q} W_n + \frac{1}{n}. \tag{2.14}$$

We are left with showing that the expectation of W_n with respect to the measure $\tilde{Q}$ is small. It follows from the definition of $\tilde{Q}$ that

$$\tilde{Q} W_n = P \exp \left(-\beta \delta_n \# \{ i \mid (i, S_i) \in J_n \} \right), \tag{2.15}$$

and therefore

$$\tilde{Q} W_n \leq P \{ \text{the trajectory } S \text{ goes out of } J_n \} + \exp(-n\beta\delta_n). \tag{2.16}$$

One can choose C_2 such that the first term is small, and the second term is equal to $\exp(-\beta n^{1/4}/\sqrt{2C_2 \log n}) \leq \exp(-C_1^{1/4}/4\sqrt{C_2})$ that can be arbitrarily small by choosing C_1 large compared to $(C_2)^{1/2}$. In that case (2.8) is satisfied and we have

$$p(\beta) \leq \frac{1}{\theta n} \log e^{-1} \leq -\frac{\beta^4}{2C_1 |\log \beta|^2} \tag{2.17}$$

for small enough β. $\quad \square$

3. Proof of the Upper Bound of Theorem 1.4 and 1.5

The upper bound we found in the previous section is not optimal, and can be improved by replacing the finite volume criterion (2.8) by a more sophisticated coarse graining method. The technical advantage of the coarse graining we use is that we will not have to choose the θ of the fractional moment close to 1 as we did in the previous section and this is the way we get rid of the extra log factor we had. The idea of using this type of coarse graining for the copolymer model appeared in [25] and this has been a substantial source of inspiration for this proof.

We will prove the following result first in the case of the Gaussian environment, and then adapt the proof to the general environment.

Proof in the case of Gaussian environment. Let n be the smallest squared integer bigger than $C_3 \beta^{-4}$ (if β is small we are sure that $n \leq 2C_3 \beta^{-4}$). The number n will be used in the sequel of the proof as a scaling factor. Let $\theta < 1$ be fixed (say $\theta = 1/2$). We consider a system of size $N = nm$ (where m is meant to tend to infinity).

Let I_k denote the interval $I_k = [k\sqrt{n}, (k+1)\sqrt{n})$. In order to estimate QW_N^θ we decompose W_N according to the contribution of different families path:

$$W_N = \sum_{y_1, y_2, \ldots, y_m \in \mathbb{Z}} \widetilde{W}_{(y_1, y_2, \ldots, y_m)}, \tag{3.1}$$

where

$$\widetilde{W}_{(y_1, y_2, \ldots, y_m)} = P \exp \left[\sum_{i=1}^N \left(\beta \omega_{i, S_i} - \frac{\beta^2}{2} \right) \mathbf{1}_{\{S_{in} \in I_{y_i}, \forall i = 1, \ldots, m\}} \right]. \tag{3.2}$$

Then, we apply the inequality $\left(\sum a_i \right)^\theta \leq \sum a_i^\theta$ (which holds for any finite or countable collection of positive real numbers) to this decomposition and average with respect to Q to get

$$QW_{nm}^\theta \leq \sum_{y_1, y_2, \ldots, y_m \in \mathbb{Z}} Q\widetilde{W}_{(y_1, y_2, \ldots, y_m)}^\theta. \tag{3.3}$$

In order to estimate $Q\widetilde{W}_{(y_1, y_2, \ldots, y_m)}^\theta$, we use an auxiliary measure as in the previous section. The additional idea is to make the measure change depend on $y_1, \ldots, y_m$.

For every $Y = (y_1, \ldots, y_m)$ we define the set J_Y as

$$J_Y := \left\{ (km + i, y_k \sqrt{n} + z), \ k = 0, \ldots, m-1, \ i = 1, \ldots, n, \ |z| \leq C_4 \sqrt{n} \right\}, \tag{3.4}$$

where y_0 is equal to zero. Note that for big values of n and m,

$$\#J_Y \sim 2C_4 mn^{3/2}. \tag{3.5}$$

We define the measure $\widetilde{Q}_Y$ to be the measure under which the $\omega_{(i,x)}$ are independent Gaussian variables with variance 1 and mean $\widetilde{Q}_Y \omega_{(i,x)} = -\delta_n \mathbf{1}_{\{(i,x) \in J_Y\}}$, where $\delta_n = n^{-3/4} C_4^{-1/2}$. The law $\widetilde{Q}_Y$ is absolutely continuous with respect to Q and its density is equal to
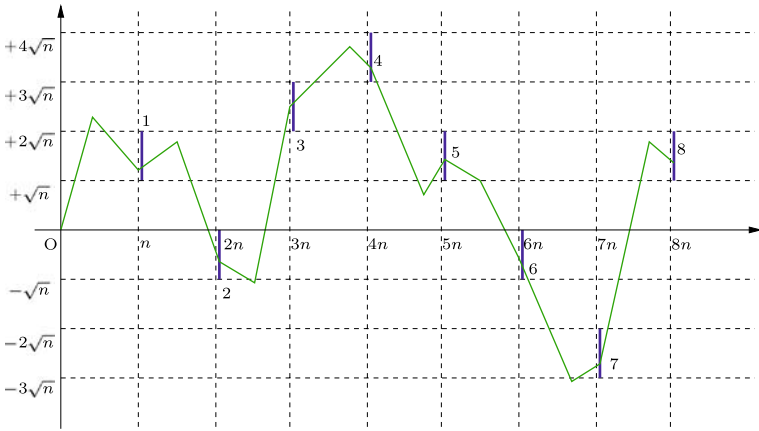
Fig. 1. The partition of W_{nm} into $\widetilde{W}^{(y_1,\dots,y_m)}$ is to be viewed as a coarse graining. For $m = 8$, $(y_1, \dots, y_8) = (1, -1, 2, 3, 1, -1, -3, 1)$, $\widetilde{W}_n^{(y_1,\dots,y_m)}$ corresponds to the contribution to W_N of the path going through the thick barriers on the figure

$$\frac{d\widetilde{Q}_Y}{dQ}(\omega) = \exp\left(-\sum_{(i,x)\in J_Y}\left[\delta_n\omega_{(i,x)} + \delta_n^2/2\right]\right). \tag{3.6}$$

Using Hölder inequality with this measure as we did in the previous section, we obtain

$$Q\left[\widetilde{W}^{\theta}_{(y_1,y_2,\dots,y_m)}\right] = \widetilde{Q}_Y\left[\frac{dQ}{d\widetilde{Q}_Y}\widetilde{W}^{\theta}_{(y_1,y_2,\dots,y_m)}\right]$$

$$\leq \widetilde{Q}_Y\left(\left[\left(\frac{dQ}{d\widetilde{Q}_Y}\right)^{\frac{1}{1-\theta}}\right]\right)^{1-\theta}\left(\widetilde{Q}_Y\widetilde{W}_{(y_1,\dots,y_m)}\right)^{\theta}. \tag{3.7}$$

The value of the first term can be computed explicitly,

$$\left(Q\left[\left(\frac{dQ}{d\widetilde{Q}_Y}\right)^{\frac{\theta}{1-\theta}}\right]\right)^{1-\theta} = \exp\left(\frac{\#J_Y\theta\delta_n^2}{2(1-\theta)}\right) \leq \exp(3m), \tag{3.8}$$

where the upper bound is obtained by using the definition of δ_n, (3.5) and the fact that $\theta = 1/2$.

Now we compute the second term

$$\widetilde{Q}_Y\widetilde{W}_{(y_1,\dots,y_m)} = P\exp\left(-\beta\delta_n\#\{i\,|\,(i, S_i) \in J_Y\}\right)\mathbf{1}_{\{S_{kn}\in I_{y_k},\ \forall k\in[1,m]\}}. \tag{3.9}$$

We define

$$J := \{(i, x),\ i = 1, \dots, n,\ |x| \leq C_4\sqrt{n}\},$$
$$\overline{J} := \{(i, x),\ i = 1, \dots, n,\ |x| \leq (C_4 - 1)\sqrt{n}\}. \tag{3.10}$$

Equation (3.9) implies that (recall that P_x is the law of the simple random walk starting from x, and that we set $y_0 = 0$)

$$\widetilde{Q}_Y \widetilde{W}_{(y_1,\ldots,y_m)} \leq \prod_{k=1}^{m} \max_{x \in I_0} P_x \exp\left(-\beta \delta_n \# \{i \ : \ (i, S_i) \in J\}\right) \mathbf{1}_{\{S_n \in I_{y_k - y_{k-1}}\}}. \quad (3.11)$$

Combining this with (3.1), (3.7) and (3.8) we have

$$\log Q W_N^\theta \leq m \left[3 + \log \sum_{y \in \mathbb{Z}} \left(\max_{x \in I_0} P_x \exp\left(-\beta \delta_n \# \{i \ : \ (i, S_i) \in J\}\right) \mathbf{1}_{\{S_n \in I_y\}} \right)^\theta \right]. \quad (3.12)$$

If the quantity in the square brackets is smaller than -1, by Eq. (2.3) we have $p(\beta) \leq -1/n$. Therefore, to complete the proof it is sufficient to show that

$$\sum_{y \in \mathbb{Z}} \left(\max_{x \in I_0} P_x \exp\left(-\beta \delta_n \# \{i \ : \ (i, S_i) \in J\}\right) \mathbf{1}_{\{S_n \in I_y\}} \right)^\theta \quad (3.13)$$

is small. To reduce the problem to the study of a finite sum, we observe (using some well known result on the asymptotic behavior of a random walk) that given $\varepsilon > 0$ we can find R such that

$$\sum_{|y| \geq R} \left(\max_{x \in I_0} P_x \exp\left(-\beta \delta_n \# \{i \ : \ (i, S_i) \in J\}\right) \mathbf{1}_{\{S_n \in I_y\}} \right)^\theta \leq \sum_{|y| \geq R} \max_{x \in I_0} \left(P_x \{S_n \in I_y\} \right)^\theta$$

$$\leq \varepsilon. \quad (3.14)$$

To estimate the remainder of the sum we use the following trivial bound:

$$\sum_{|y| < R} \left(\max_{x \in I_0} P_x \exp\left(-\beta \delta_n \# \{i \ : \ (i, S_i) \in J\}\right) \mathbf{1}_{\{S_n \in I_y\}} \right)^\theta$$

$$\leq 2R \left(\max_{x \in I_0} P_x \exp\left(-\beta \delta_n \# \{i \ : \ (i, S_i) \in J\}\right) \right)^\theta. \quad (3.15)$$

Then we get rid of the max in the sum by observing that if a walk starting from x makes a step in J, the walk with the same increments starting from 0 will make the same step in $\overline{J}$ (recall (3.10)),

$$\max_{x \in I_0} P_x \exp\left(-\beta \delta_n \# \{i \ : \ (i, S_i) \in J\}\right) \leq P \exp\left(-\beta \delta_n \# \{i | (i, S_i) \in \overline{J}\}\right). \quad (3.16)$$

Now we are left with something similar to what we encountered in the previous section

$$P \exp\left(-\beta \delta_n \# \{i \ : \ (i, S_i) \in \overline{J}\}\right) \leq P\{ \text{ the random walk goes out of } \overline{J} \} + \exp(-n\beta\delta_n). \quad (3.17)$$

If C_4 is chosen large enough, the first term can be made arbitrarily small by choosing C_4 large, and the second is equal to $\exp(-C_3^{-1/4}/\sqrt{C_4})$ and can be made also arbitrarily

small if C_3 is chosen large enough once C_4 is fixed. An appropriate choice of constant and the use of (3.16) and (3.17) can lead then to

$$2R \left(\max_{x \in I_0} P_x \exp\left(-\beta \delta_n \# \{i \; : \; (i, S_i) \in J\}\right) \right)^\theta \leq \varepsilon. \tag{3.18}$$

This combined with (3.14) completes the proof. □

Proof of the general case. In the case of a general environment, some modifications have to be made in the proof above, but the general idea remains the same. In the change of measure one has to change the shift of the environment in J_Y (3.6) by an exponential tilt of the measure as follows:

$$\frac{\mathrm{d}\widetilde{Q}_Y}{\mathrm{d}Q}(\beta) = \exp\left(- \sum_{(i,z) \in J_Y} \left[\delta_n \omega_{(i,z)} + \lambda(-\delta_n) \right] \right). \tag{3.19}$$

The formula estimating the cost of the change of measure (3.8) becomes

$$\left(Q \left(\frac{\mathrm{d}Q}{\mathrm{d}\widetilde{Q}_Y} \right)^{\frac{\theta}{1-\theta}} \right)^{1-\theta} = \exp\left(\#J_Y \left[(1-\theta)\lambda \left(\frac{\theta \delta_n}{1-\theta} \right) + \theta\lambda(-\delta_n) \right] \right)$$

$$\leq \exp(2m), \tag{3.20}$$

where the last inequality is true if β_n is small enough if we consider that $\theta = 1/2$ and use the fact that $\lambda(x) \overset{x \to 0}{\sim} x^2/2$ (ω has 0 mean and unit variance). The next thing we have to do is to compute the effect of this change of measure in this general case, i.e. find an equivalent for (3.9). When computing $\widetilde{Q}_Y \widetilde{W}_{(y_1,\ldots,y_m)}$, the quantity

$$\widetilde{Q}_Y \exp(\beta \omega_{1,0} - \lambda(\beta)) = \exp\left[\lambda(\beta - \delta_n) - \lambda(-\delta_n) - \lambda(\beta) \right] \tag{3.21}$$

appears instead of $\exp(-\beta \delta_n)$. Using twice the mean value theorem, one gets that there exists h and h' in $(0, 1)$ such that

$$\lambda(\beta - \delta_n) - \lambda(-\delta_n) - \lambda(\beta) = \delta_n \left[\lambda'(-h\delta_n) - \lambda'(\beta - h\delta_n) \right]$$

$$= -\beta \delta_n \lambda''(-h\delta_n + h'\beta). \tag{3.22}$$

And as ω has unit variance, $\lim_{x \to 0} \lambda''(x) = 1$. Therefore if β and δ_n are chosen small enough, the right-hand side of the above is less than $-\beta \delta_n/2$. So (3.9) can be replaced by

$$\widetilde{Q}_Y \widetilde{W}_{(y_1,\ldots,y_m)} \leq P \exp\left(-\frac{\beta \delta_n}{2} \# \{i \,|\, (i, S_i) \in J_Y\} \right) \mathbf{1}_{\{S_{kn} \in I_{y_k}, \; \forall k \in [1,m]\}}. \tag{3.23}$$

The remaining steps follow closely the argument exposed for the Gaussian case. □

4. Proof of the Upper Bound in Theorem 1.6

In this section, we prove the main result of the paper: very strong disorder holds at all temperatures in dimension 2.

The proof is technically quite involved. It combines the tools of the two previous sections with a new idea for the change of measure: changing the covariance structure of the environment. We mention that this idea was introduced recently in [15] to deal with the *marginal disorder* case in the pinning model. We choose to present first a proof for the Gaussian case, where the idea of the change of measure is easier to grasp.

Before starting, we sketch the proof and how it should be decomposed in different steps:

(a) We reduce the problem by showing that it is sufficient to show that for some real number $\theta < 1$, $Q W_N^\theta$ decays exponentially with N.
(b) We use a coarse graining decomposition of the partition function by splitting it into different contributions that correspond to trajectories that stay in a large corridor. This decomposition is similar to the one used in Sect. 3.
(c) To estimate the fractional moment terms appearing in the decomposition, we change the law of the environment around the corridors corresponding to each contribution. More precisely, we introduce negative correlations into the Gaussian field of the environment. We do this change of measure in such a way that the new measure is not very different from the original one.
(d) We use some basic properties of the random walk in $\mathbb{Z}^2$ to compute the expectation under the new measure.

Proof for the Gaussian environment. We fix n to be the smallest squared integer bigger than $\exp(C_5/\beta^4)$ for some large constant C_5 to be defined later, for small β we have $n \le \exp(2C_5/\beta^4)$. The number n will be used in the sequel of the proof as a scaling factor. For $y = (a, b) \in \mathbb{Z}^2$ we define $I_y = [a\sqrt{n}, (a+1)\sqrt{n}-1] \times [b\sqrt{n}, (b+1)\sqrt{n}-1]$ so that I_y are disjoint and cover $\mathbb{Z}^2$. For $N = nm$, we decompose the normalized partition function W_N into different contributions, very similarly to what is done in dimension one (i.e. decomposition (3.3)), and we refer to Fig. 2 to illustrate how the decomposition looks like:

$$W_N = \sum_{y_1,\ldots,y_m \in \mathbb{Z}^2} \widetilde{W}_{(y_1,\ldots,y_m)}, \qquad (4.1)$$

where

$$\widetilde{W}_{(y_1,\ldots,y_m)} = P \exp\left(\sum_{i=1}^N \left[\beta\omega_{i,S_i} - \beta^2/2\right]\right) \mathbf{1}_{\{S_{in} \in I_{y_i}, \forall i=1,\ldots,m\}}. \qquad (4.2)$$

We fix $\theta < 1$ and apply the inequality $(\sum a_i)^\theta \le \sum a_i^\theta$ (which holds for any finite or countable collection of positive real numbers) to get

$$Q W_N^\theta \le \sum_{y_1,\ldots,y_m \in \mathbb{Z}^2} Q\widetilde{W}_{(y_1,\ldots,y_m)}^\theta. \qquad (4.3)$$

In order to estimate the different terms in the sum of the right–hand side in (4.3), we define some auxiliary measures $\widetilde{Q}_Y$ on the the environment for every $Y = (y_0, y_1, \ldots, y_m) \in \mathbb{Z}^{d+1}$ with $y_0 = 0$. We will choose the measures Q_Y absolutely continuous with respect to Q. We use the Hölder inequality to get the following upper bound:
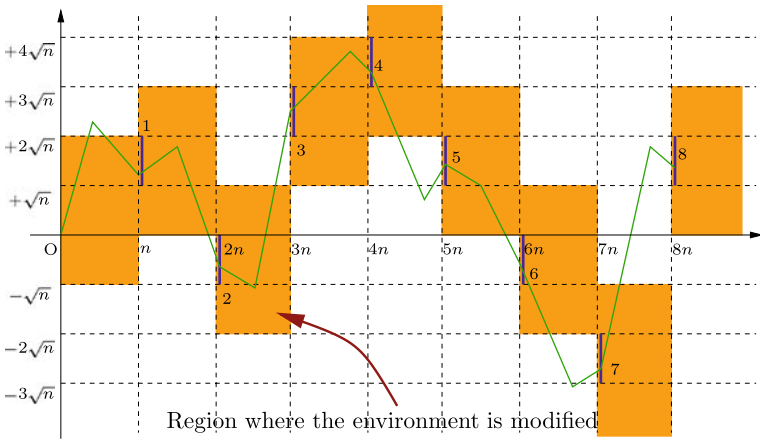
Fig. 2. This figure represents in a rough way the change of measure Q_Y. The region where the mean of $\omega_{(i,x)}$ is lowered (the shadow region on the figure) corresponds to the region where the simple random walk is likely to go, given that it goes through the thick barriers

$$Q\widetilde{W}^{\theta}_{(y_1,\dots,y_m)} \leq \left(Q\left(\frac{dQ}{d\widetilde{Q}_Y}\right)^{\frac{\theta}{1-\theta}}\right)^{1-\theta} \left(\widetilde{Q}_Y\widetilde{W}_{(y_1,\dots,y_m)}\right)^{\theta}. \qquad (4.4)$$

Now, we describe the change of measure we will use. Recall that for the 1-dimensional case we used a shift of the environment along the corridor corresponding to Y. The reader can check that this method would not give the exponential decay of W_N in this case. Instead we change the covariance function of the environment along the corridor on which the walk is likely to go by introducing some negative correlation.

We introduce the change of measure that we use for this case. Given $Y = (y_0, y_1, \dots, y_m)$ we define m blocks $(B_k)_{k \in [1,m]}$ and J_Y their union (here and in the sequel, $|z|$ denotes the l^{∞} norm on $\mathbb{Z}^2$):

$$B_k := \left\{(i, z) \in \mathbb{N} \times \mathbb{Z}^2 : \lceil i/n \rceil = k \text{ and } |z - \sqrt{n}y_{k-1}| \leq C_6\sqrt{n}\right\},$$

$$J_Y := \bigcup_{k=1}^{m} B_k. \qquad (4.5)$$

We fix the covariance of the field ω under the law $\widetilde{Q}_Y$ to be equal to

$$\widetilde{Q}_Y\left(\omega_{i,z}\omega_{i,z'}\right) = \mathcal{C}^Y_{(i,z),(j,z')}$$

$$:= \begin{cases} \mathbf{1}_{\{(i,z)=(j,z')\}} - V_{(i,z),(j,z')} & \text{if } \exists\, k \in [1, m] \\ & \text{such that } (i, z) \text{ and } (j, z') \in B_k \\ \mathbf{1}_{\{(i,z)=(j,z')\}} & \text{otherwise,} \end{cases} \qquad (4.6)$$

where

$$V_{(i,z),(j,z')} := \begin{cases} 0 & \text{if } (i, z) = (j, z') \\ \dfrac{\mathbf{1}_{\{|z-z'|\leq C_7\sqrt{|j-i|}\}}}{100C_6C_7n\sqrt{\log n}|j-i|} & \text{otherwise.} \end{cases} \qquad (4.7)$$

We define

$$\widehat{V} := (V_{(i,z),(j,z')})_{(i,z),(j,z')\in B_1}. \tag{4.8}$$

One remarks that the so-defined covariance matrix $\mathcal{C}^Y$ is block diagonal with m identical blocks which are copies of $I - \widehat{V}$ corresponding to the B_k, $k \in [1, m]$, and just ones on the diagonal elsewhere. Therefore, the change of measure we describe here exists if and only if $I - \widehat{V}$ is definite positive.

The largest eigenvalue for $\widehat{V}$ is associated to a positive vector and therefore is smaller than

$$\max_{(i,z)\in B_1} \sum_{(j,z')\in B_1} \left| V_{(i,z),(j,z')} \right| \leq \frac{C_7}{C_6\sqrt{\log n}}. \tag{4.9}$$

For the sequel we choose n such that the spectral radius of $\widehat{V}$ is less than $(1 - \theta)/2$ so that $I - \widehat{V}$ is positive definite. With this setup, $\widetilde{Q}_Y$ is well defined.

The density of the modified measure $\widetilde{Q}_Y$ with respect to Q is given by

$$\frac{\mathrm{d}\widetilde{Q}_Y}{\mathrm{d}Q}(\omega) = \frac{1}{\sqrt{\det \mathcal{C}^Y}} \exp\left(-\frac{1}{2}{}^t\omega((\mathcal{C}^Y)^{-1} - I)\omega\right), \tag{4.10}$$

where

$${}^t\omega M\omega = \sum_{(i,z),(j,z')\in\mathbb{N}\times\mathbb{Z}^2} \omega_{(i,z)} M_{(i,z),(j,z')}\omega_{(j,z')}, \tag{4.11}$$

for any matrix M of $(\mathbb{N} \times \mathbb{Z}^2)^2$ with finite support.

Then we can compute explicitly the value of the second term in the right-hand side of (4.4),

$$\left(Q\left(\frac{\mathrm{d}Q}{\mathrm{d}\widetilde{Q}_Y}\right)^{\frac{\theta}{1-\theta}}\right)^{1-\theta} = \sqrt{\frac{\det \mathcal{C}^Y}{\det\left(\frac{\mathcal{C}^Y}{1-\theta} - \frac{\theta I}{1-\theta}\right)^{1-\theta}}}. \tag{4.12}$$

Note that the above computation is right if and only if $\mathcal{C}^Y - \theta I$ is a definite positive matrix. Since its eigenvalues are the same of those of $(1 - \theta)I - \widehat{V}$, this holds for large n thanks to (4.9). Using again the fact that $\mathcal{C}^Y$ is composed of m blocks identical to $I - \widehat{V}$, we get from (4.12),

$$\left(Q\left(\frac{\mathrm{d}Q}{\mathrm{d}\widetilde{Q}}\right)^{\frac{\theta}{1-\theta}}\right)^{1-\theta} = \left(\frac{\det(I - \widehat{V})}{\det(I - \widehat{V}/(1 - \theta))^{1-\theta}}\right)^{m/2}. \tag{4.13}$$

In order to estimate the determinant in the denominator, we compute the Hilbert-Schmidt norm of $\widehat{V}$. One can check that for all n,

$$\|\widehat{V}\|^2 = \sum_{(i,z),(j,z')\in B_1} V^2_{(i,z),(j,z')} \leq 1. \tag{4.14}$$

We use the inequality $\log(1 + x) \geq x - x^2$ for all $x \geq -1/2$ and the fact that the spectral radius of $\widehat{V}/(1 - \theta)$ is bounded by $1/2$ (cf. (4.9)) to get that

$$\det\left[I - \frac{\widehat{V}}{1 - \theta}\right] = \exp\left(\text{Trace}\left(\log\left(I - \frac{\widehat{V}}{1 - \theta}\right)\right)\right) \geq \exp\left(-\frac{\|\widehat{V}\|^2}{(1 - \theta)^2}\right)$$
$$\geq \exp\left(-\frac{1}{(1 - \theta)^2}\right). \tag{4.15}$$

For the numerator, Trace $\widehat{V} = 0$ implies that $\det(I - \widehat{V}) \leq 1$. Combining this with (4.13) and (4.15) we get

$$\left(Q\left(\frac{dQ}{d\widetilde{Q}_Y}\right)^{\frac{\theta}{1-\theta}}\right)^{1-\theta} \leq \exp\left(\frac{m}{2(1 - \theta)}\right). \tag{4.16}$$

Now that we have computed the term corresponding to the change of measure, we estimate $\widetilde{W}_{(y_1,\ldots,y_m)}$ under the modified measure (just by computing the variance of the Gaussian variables in the exponential, using (4.6)) :

$$\widetilde{Q}_Y \widetilde{W}_{(y_1,\ldots,y_m)} = P\,\widetilde{Q}_Y \exp\left(\sum_{i=1}^N\left(\beta\omega_{i,S_i} - \frac{\beta^2}{2}\right)\right)\mathbf{1}_{\{S_{kn}\in I_{y_k},\forall k=1,\ldots,m\}}$$

$$= P\exp\left(\frac{\beta^2}{2}\sum_{\substack{1\leq i,\ j\leq N\\z,z'\in\mathbb{Z}^2}}\left(\mathcal{C}^Y_{(i,z),(j,z')} - \mathbf{1}_{\{(i,z)=(j,z')\}}\right)\mathbf{1}_{\{S_i=z,S_j=z'\}}\right)$$
$$\times \mathbf{1}_{\{S_{kn}\in I_{y_k},\forall k=1,\ldots,m\}}. \tag{4.17}$$

Replacing $\mathcal{C}^Y$ by its value we get that

$$\widetilde{Q}_Y \widetilde{W}_{(y_1,\ldots,y_m)} = P\exp\left(-\frac{\beta^2}{2}\sum_{\substack{1\leq i\neq j\leq N\\1\leq k\leq m}}\frac{\mathbf{1}_{\{((i,S_i),(j,S_j))\in B_k^2,\ |S_i-S_j|\leq C_7\sqrt{|i-j|}\}}}{100C_6C_7n\sqrt{\log n}\,|j - i|}\right)$$
$$\mathbf{1}_{\{S_{kn}\in I_{y_k},\forall k=1,\ldots,m\}}. \tag{4.18}$$

Now we do something similar to (3.11): for each "slice" of the trajectory $(S_i)_{i\in[(m-1)k,mk]}$, we bound the contribution of the above expectation by maximizing over the starting point (recall that P_x denotes the probability distribution of a random walk starting at x). Thanks to the conditioning, the starting point has to be in I_{y_k}. Using the translation invariance of the random walk, this gives us the following ($\vee$ stands for maximum):

$$\widetilde{Q}_Y \widecheck{W}^{(y_1,\ldots,y_m)} \leq \prod_{i=k}^m \max_{x\in I_0}$$
$$\times P_x\left[\exp\left(-\frac{\beta^2}{2}\sum_{1\leq i\neq j\leq n}\frac{\mathbf{1}_{\{|S_i|\vee|S_j|\leq C_6\sqrt{n},\ |S_i-S_j|\leq C_7\sqrt{|i-j|}\}}}{100C_6C_7n\sqrt{\log n}\,|j - i|}\right)\mathbf{1}_{\{S_n\in I_{y_k-y_{k-1}}\}}\right]. \tag{4.19}$$

For trajectories S of a directed random-walk of n steps, we define the quantity

$$G(S) := \sum_{1 \leq i \neq j \leq n} \frac{\mathbf{1}_{\{|S_i| \vee |S_j| \leq C_6\sqrt{n}, \ |S_i - S_j| \leq C_7\sqrt{|i-j|}\}}}{100 C_6 C_7 n \sqrt{\log n}|j - i|}. \tag{4.20}$$

Combining (4.19) with (4.16), (4.4) and (4.3), we finally get

$$QW_N^\theta \leq \exp\left(\frac{m}{2(1-\theta)}\right)\left[\sum_{y \in \mathbb{Z}} \max_{x \in I_0}\left(P_x \exp\left(-\frac{\beta^2}{2}G(S)\right)\mathbf{1}_{\{S_n \in I_y\}}\right)^\theta\right]^m. \tag{4.21}$$

The exponential decay of QW_N^θ (with rate n) is guaranteed if we can prove that

$$\sum_{y \in \mathbb{Z}} \max_{x \in I_0}\left(P_x \exp\left(-\frac{\beta^2}{2}G(S)\right)\mathbf{1}_{\{S_n \in I_y\}}\right)^\theta \tag{4.22}$$

is small. The rest of the proof is devoted to that aim.

We fix some $\varepsilon > 0$. Asymptotic properties of the simple random walk, guarantees that we can find $R = R_\varepsilon$ such that

$$\sum_{|y| \geq R} \max_{x \in I_0}\left(P_x \exp\left(-\frac{\beta^2}{2}G(S)\right)\mathbf{1}_{\{S_n \in I_y\}}\right)^\theta \leq \sum_{|y| \geq R} \max_{x \in I_0}\left(P_x\{S_n \in I_y\}\right)^\theta \leq \varepsilon. \tag{4.23}$$

To estimate the rest of the sum, we use the following trivial and rough bound

$$\sum_{|y| < R} \max_{x \in I_0}\left[P_x \exp\left(-\frac{\beta^2}{2}G(S)\right)\mathbf{1}_{\{S_n \in I_y\}}\right]^\theta \leq R^2\left[\max_{x \in I_0} P_x \exp\left(-\frac{\beta^2}{2}G(S)\right)\right]^\theta. \tag{4.24}$$

Then we use the definition of $G(S)$ to get rid of the max by reducing the width of the zone where we have negative correlation:

$$\max_{x \in I_0} P_x \exp\left(-\frac{\beta^2}{2}G(S)\right)$$

$$\leq P \exp\left(-\frac{\beta^2}{2}\sum_{1 \leq i \neq j \leq n} \frac{\mathbf{1}_{\{|S_i| \vee |S_j| \leq (C_6-1)\sqrt{n}, \ |S_i - S_j| \leq C_7\sqrt{|i-j|}\}}}{100 C_6 C_7 n \sqrt{\log n}|j - i|}\right). \tag{4.25}$$

We define $\overline{B} := \{(i, z) \in \mathbb{N} \times \mathbb{Z}^2 \ : \ i \leq m, |z| \leq (C_6 - 1)\sqrt{n}\}$. We get from the above that

$$\max_{x \in I_0} P_x \exp\left(-\frac{\beta^2}{2}G(S)\right) \leq P\{\text{the RW goes out of } \overline{B}\}$$

$$+ P \exp\left(-\frac{\beta^2}{2}\sum_{1 \leq i \neq j \leq n} \frac{\mathbf{1}_{\{|S_i - S_j| \leq C_7\sqrt{|i-j|}\}}}{100 C_6 C_7 n \sqrt{\log n}|j - i|}\right). \tag{4.26}$$

One can make the first term of the right-hand side arbitrarily small by choosing C_6 large, in particular one can choose C_6 such that

$$P\left\{\max_{i\in[0,n]}|S_n| \geq (C_6 - 1)\sqrt{n}\right\} \leq (\varepsilon/R^2)^{\frac{1}{\theta}}. \tag{4.27}$$

To bound the other term, we introduce the quantity

$$D(n) := \sum_{1\leq i\neq j\leq n} \frac{1}{n\sqrt{\log n}|j - i|}, \tag{4.28}$$

and the random variable X,

$$X := \sum_{1\leq i\neq j\leq n} \frac{\mathbf{1}_{\{|S_i - S_j|\leq C_7\sqrt{|i-j|}\}}}{n\sqrt{\log n}|j - i|}. \tag{4.29}$$

For any $\delta > 0$, we can find C_7 such that $P(X) \geq (1 - \delta)D(n)$. We fix C_7 such that this holds for some good δ (to be chosen soon), and by remarking that $0 \leq X \leq D(n)$ almost surely, we obtain (using the Markov inequality)

$$P\{X > D(n)/2\} \geq 1 - 2\delta. \tag{4.30}$$

Moreover we can estimate $D(n)$, getting that for n large enough,

$$D(n) \geq \sqrt{\log n}. \tag{4.31}$$

Using (4.30) and (4.31) we get

$$P\exp\left(-\frac{\beta^2}{2}\sum_{1\leq i\neq j\leq n}\frac{\mathbf{1}_{\{|S_i - S_j|\leq C_7\sqrt{|i-j|}\}}}{100C_6C_7 n\sqrt{\log n}|j - i|}\right) = P\exp\left(-\frac{\beta^2}{200C_6C_7}X\right)$$
$$\leq 2\delta + \exp\left(-\frac{\beta^2\sqrt{\log n}}{200C_6C_7}\right). \tag{4.32}$$

Due to the choice of n we have made (recall $n \geq \exp(C_5/\beta^4)$), the second term is less than $\exp\left(-\beta^2 C_5^{1/2}/(200C_6C_7)\right)$. We can choose δ, C_7 and C_5 such that the right-hand side is less than $(\varepsilon/R^2)^{\frac{1}{\theta}}$. This combined with (4.27), (4.26), (4.24) and (4.23) allow us to conclude that

$$\sum_{y\in\mathbb{Z}}\max_{x\in I_0}\left(P_x\exp\left(-\frac{\beta^2}{2}G(S)\right)\mathbf{1}_{\{S_n\in I_y\}}\right)^{\theta} \leq 3\varepsilon. \tag{4.33}$$

So that with a right choice for ε, (4.21) implies

$$QW_N^{\theta} \leq \exp(-m). \tag{4.34}$$

Then (2.3) allows us to conclude that $p(\beta) \leq -1/n$. $\square$

Proof for the general environment. The case of the general environment does not differ very much from the Gaussian case, but one has a different approach for the change of measure in (4.4). In this proof, we will largely refer to what has been done in the Gaussian case, whose proof should be read first.

Let K be a large constant. One defines the function f_K on $\mathbb{R}$ to be

$$f_K(x) = -K\mathbf{1}_{\{x > \exp(K^2)\}}.$$

Recall the definitions (4.5) and (4.7), and define the g_Y function of the environment as

$$g_Y(\omega) = \exp\left(\sum_{k=1}^{m} f_K\left(\sum_{(i,z),(j,z')\in B_k} V_{(i,z),(j,z')}\omega_{i,z}\omega_{j,z'}\right)\right).$$

Multiplying by g_Y penalizes by a factor $\exp(-K)$ the environment for which there is too much correlation in one block. This is a way of producing negative correlation in the environment. For the rest of the proof we use the notation

$$U_k := \sum_{(i,z),(j,z')\in B_k} V_{(i,z),(j,z')}\omega_{i,z}\omega_{j,z'}. \tag{4.35}$$

We do a computation similar to (4.4) to get

$$Q\left[\widetilde{W}^{\theta}_{(y_1,\ldots,y_m)}\right] \leq \left(Q\left[g_Y(\omega)^{-\frac{\theta}{1-\theta}}\right]\right)^{1-\theta}\left(Q\left[g_Y(\omega)\widetilde{W}_{(y_1,\ldots,y_n)}\right]\right)^{\theta}. \tag{4.36}$$

The block structure of g_Y allows to express the first term as a power of m,

$$Q\left[g_Y(\omega)^{-\frac{\theta}{1-\theta}}\right] = \left(Q\left[\exp\left(-\frac{\theta}{1-\theta}f_K(U_1)\right)\right]\right)^m. \tag{4.37}$$

Equation (4.14) says that

$$\mathrm{Var}_Q(U_1) \leq 1. \tag{4.38}$$

So that

$$P\left\{U_1 \geq \exp(K^2)\right\} \leq \exp(-2K^2), \tag{4.39}$$

and hence

$$Q\left[\exp\left(-\frac{\theta}{1-\theta}f_K(U_1)\right)\right]$$
$$\leq 1 + \exp\left(-2K^2 + \frac{\theta}{1-\theta}K\right) \leq 2, \tag{4.40}$$

if K is large enough. We are left with estimating the second term

$$Q\left[g_Y(\omega)\widetilde{W}_{(y_1,\ldots,y_n)}\right] = P\,Qg_Y(\omega)\exp\left(\sum_{i=1}^{nm}[\beta\omega_{i,S_i} - \lambda(\beta)]\right)\mathbf{1}_{\{S_{kn}\in I_{y_k},\forall k=1\ldots m\}}. \tag{4.41}$$

For a fixed trajectory of the random walk S, we consider $\overline{Q}_S$ the modified measure on the environment with density

$$\frac{d\overline{Q}_S}{dQ} := \exp\left(\sum_{i=1}^{nm} [\beta\omega_{i,S_i} - \lambda(\beta)]\right). \tag{4.42}$$

Under this measure

$$\overline{Q}_S\omega_{i,z} = \begin{cases} 0 & \text{if } z \neq S_i \\ Q\omega e^{\beta\omega_{0,1}-\lambda(\beta)} := m(\beta) & \text{if } z = S_i. \end{cases} \tag{4.43}$$

As $\omega_{1,0}$ has zero-mean and unit variance under Q, (1.4) implies $m(\beta) = \beta + o(\beta)$ around zero and that $\mathrm{Var}_{\overline{Q}_S}\,\omega_{i,z} \leq 2$ for all (i, z) if β is small enough. Moreover $\overline{Q}_S$ is a product measure, i.e. the $\omega_{i,z}$ are independent variables under Q_S. With this notation (4.41) becomes

$$P\overline{Q}_S\,[g_Y(\omega)]\,\mathbf{1}_{\{S_{kn}\in I_{y_k},\forall k=1,\ldots,m\}}. \tag{4.44}$$

As in the Gaussian case, one wants to bound this by a product using the block structure. Similarly to (4.19), we use translation invariance to get the following upper bound:

$$\prod_{k=1}^{m} \max_{x\in I_0} P_x\overline{Q}_S \exp\left(f_K\,(U_1)\right) \mathbf{1}_{\{S_n\in I_{y_k-y_{k-1}}\}}. \tag{4.45}$$

Using this in (4.36) with the bound (4.40) we get the inequality

$$QW_N^\theta \leq 2^{m(1-\theta)} \left(\sum_{y\in\mathbb{Z}^2}\left[\max_{x\in I_0} P_x\overline{Q}_S \exp\left(f_K\,(U_1)\right) \mathbf{1}_{\{S_n\in I_y\}}\right]^\theta\right)^m. \tag{4.46}$$

Therefore to prove exponential decay of QW_N^θ, it is sufficient to show that

$$\sum_{y\in\mathbb{Z}^2}\left[\max_{x\in I_0} P_x\overline{Q}_S \exp\left(f_K\,(U_1)\right) \mathbf{1}_{\{S_n\in I_y\}}\right]^\theta \tag{4.47}$$

is small. As seen in the Gaussian case (cf. (4.23),(4.24)), the contribution of y far from zero can be controlled and therefore it is sufficient for our purpose to check

$$\max_{x\in I_0} P_x\overline{Q}_S \exp\left(f_K\,(U_1)\right) \leq \delta, \tag{4.48}$$

for some small δ. Similarly to (4.26), we force the walk to stay in the zone where the environment is modified by writing

$$\max_{i\in I_0} P_x\overline{Q}_S \exp\left(f_K\,(U_1)\right) \leq P\{\max_{i\in[0,n]} |S_i| \geq (C_6 - 1)\sqrt{n}\}$$
$$+\max_{x\in I_0} P_x\overline{Q}_S \exp\left(f_K\,(U_1)\right) \mathbf{1}_{\{|S_n-S_0|\leq(C_6-1)\sqrt{n}\}}. \tag{4.49}$$

The first term is smaller than $\delta/6$ if C_6 is large enough. To control the second term, we will find an upper bound for

$$P_x \overline{Q}_S \exp\left(f_K\left(U_1\right)\right) \mathbf{1}_{\{\max_{i \in [0,n]} |S_i - S_0| \le (C_6-1)\sqrt{n}\}}, \tag{4.50}$$

which is uniform in $x \in I_0$.

What we do is the following: we show that for most trajectories S the term in f_K has a large mean and a small variance with respect to Q_S so that $f_K(\dots) = -K$ with large $\overline{Q}_S$ probability. The rest will be easy to control as the term in the expectation is at most one.

The expectation of U_1 under $\overline{Q}_S$ is equal to

$$m(\beta)^2 \sum_{1 \le i,j \le n} V_{(i,S_i),(j,S_j)}. \tag{4.51}$$

When the walk stays in the block B_1 we have (using definition (4.29))

$$\sum_{1 \le i,j \le n} V_{(i,S_i),(j,S_j)} = \frac{1}{100 C_6 C_7} X. \tag{4.52}$$

The distribution of X under P_x is the same for all $x \in I_0$. It has been shown earlier (cf. (4.30) and (4.31)), that if C_7 is chosen large enough,

$$P\left\{\frac{m(\beta)^2}{100 C_6 C_7} X \le \frac{\sqrt{\log n}}{200 C_6 C_7}\right\} \le \frac{\delta}{6}. \tag{4.53}$$

As $m(\beta) \ge \beta/2$ if β is small, if C_5 is large enough (recall $n \ge \exp(C_5/\beta^4)$), this together with (4.52) gives

$$P_x\left\{m(\beta)^2 \overline{Q}_S\left(U_1\right) \le 2\exp(K^2); \max_{i \in [0,n]} |S_i - S_0| \le (C_6 - 1)\sqrt{n}\right\} \le \frac{\delta}{6}. \tag{4.54}$$

To bound the variance of U_1 under $\overline{Q}_S$, we decompose the sum

$$U_1 = \sum_{(i,z),(j,z') \in B_1} V_{(i,z),(j,z')} \omega_{i,z} \omega_{j,z} = m(\beta)^2 \sum_{1 \le i,j \le n} V_{(i,S_i),(j,S_j)}$$

$$+ 2m(\beta) \sum_{\substack{1 \le i \le n \\ (j,z') \in B_1}} V_{(i,S_i),(j,z')}(\omega_{j,z'} - m(\beta)\mathbf{1}_{\{z'=S_j\}})$$

$$+ \sum_{(i,z),(j,z') \in B_1} V_{(i,z),(j,z')}(\omega_{i,z} - m(\beta)\mathbf{1}_{\{z=S_i\}})(\omega_{j,z} - m(\beta)\mathbf{1}_{\{z'=S_j\}}). \tag{4.55}$$

And hence (using the fact that $(x + y)^2 \le 2x^2 + 2y^2$),

$$\text{Var}_{\overline{Q}_S} U_1 \le 16 m(\beta)^2 \sum_{(j,z') \in B_1} \left(\sum_{1 \le i \le n} V_{(i,S_i),(j,z')}\right)^2 + 8 \sum_{((i,z),(j,z')) \in B_1} V^2_{(i,z),(j,z')}, \tag{4.56}$$

where we used that $\mathrm{Var}_{\overline{Q}_S} \, \omega_{i,z} \leq 2$ (which is true for β small enough). The last term is less than 8 thanks to (4.14), so that we just have to control the first one. Independently of the choice of (j, z') we have the bound

$$\sum_{1 \leq i \leq n} V_{(i,S_i),(j,z')} \leq \frac{\sqrt{\log n}}{C_6 C_7 n}. \tag{4.57}$$

Moreover it is also easy to check that

$$\sum_{(j,z') \in B_1} \sum_{1 \leq i \leq n} V_{(i,S_i),(j,z')} \leq \frac{C_7 n}{C_6 \sqrt{\log n}}, \tag{4.58}$$

(these two bounds follow from the definition of $V_{(i,z),(j,z')}$: (4.7)). Therefore

$$\sum_{(j,z') \in B_1} \left(\sum_{1 \leq i \leq n} V_{(i,S_i),(j,z')} \right)^2 \leq \left[\sum_{(j,z') \in B_1} \sum_{1 \leq i \leq n} V_{(i,S_i),(j,z')} \right]$$
$$\times \max_{(j,z) \in B_1} \sum_{1 \leq i \leq n} V_{(i,S_i),(j,z')} \leq 1. \tag{4.59}$$

Injecting this into (4.56) guarantees that for β small enough,

$$\mathrm{Var}_{\overline{Q}_S} \, U_1 \leq 10. \tag{4.60}$$

With Chebyshev inequality, if K has been chosen large enough and

$$\overline{Q}_S U_1 \geq 2 \exp(K^2), \tag{4.61}$$

we have

$$\overline{Q}_S \left\{ U_1 \leq \exp(K^2) \right\} \leq \delta/6. \tag{4.62}$$

Hence combining (4.62) with (4.54) gives

$$P_x \overline{Q}_S \left\{ U_1 \leq \exp(K^2); \max_{i \in [0,n]} |S_i - S_0| \leq (C_6 - 1)\sqrt{n} \right\} \leq \delta/3. \tag{4.63}$$

We use this in (4.49) to get

$$\max_{x \in I_0} P_x \overline{Q}_S \exp\left(f_K \left(U_1 \right) \right) \leq \frac{\delta}{2} + e^{-K}. \tag{4.64}$$

So our result is proved provided that K has been chosen large enough. $\square$

5. Proof of the Lower Bound in Theorem 1.4

In this section we prove the lower bound for the free-energy in dimension 1 in an arbitrary environment. To do so we apply the second moment method to some quantity related to the partition function, and combine it with a percolation argument. The idea of the proof was inspired by a study of a polymer model on a hierarchical lattice [21] where this type of coarse-graining appears naturally.

Proposition 5.1. *There exists a constant C such that for all $\beta \leq 1$ we have*

$$p(\beta) \geq -C\beta^4((\log \beta)^2 + 1). \tag{5.1}$$

We use two technical lemmas to prove the result. The first is just a statement about scaling of the random walk, the second is more specific to our problem.

Lemma 5.2. *There exists a constant c_{RW} such that for large even squared integers n,*

$$P\{S_n = \sqrt{n}, 0 < S_i < \sqrt{n} \text{ for } 0 < i < n\} = c_{RW}n^{-3/2} + o(n^{-3/2}). \tag{5.2}$$

Lemma 5.3. *For any $\varepsilon > 0$ we can find a constant c_ε and β_0 such that for all $\beta \leq \beta_0$, for every even squared integer $n \leq c_\varepsilon/(\beta^4|\log \beta|)$ we have*

$$\mathrm{Var}_Q\left[P\left(\exp\left(\sum_{i=1}^{n-1} (\beta\omega_{i,S_i} - \lambda(\beta)) \right) \, \bigg| \, S_n = \sqrt{n}, 0 < S_i < \sqrt{n} \text{ for } 0 < i < n \right) \right] < \varepsilon. \tag{5.3}$$

Proof of Proposition 5.1 from Lemma 5.2 and 5.3. Let n be some fixed integer and define

$$\overline{W} := P \exp\left(\sum_{i=1}^{n-1} (\beta\omega_{i,S_i} - \lambda(\beta)) \right) \mathbf{1}_{\{S_n=\sqrt{n}, 0<S_i<\sqrt{n} \text{ for } 0<i<n\}}, \tag{5.4}$$

which corresponds to the contribution to the partition function W_n of paths with fixed end point $\sqrt{n}$ staying within a cell of width $\sqrt{n}$, with the specification the environment on the last site is not taken into account. $\overline{W}$ depends only on the value of the environment ω in this cell (see Fig. 3).

One also defines the following quantities for $(i, y) \in \mathbb{N} \times \mathbb{Z}$:

$$\overline{W}_i^{(y,y+1)} := P_{\sqrt{n}y}\left[e^{\sum_{k=1}^{n-1}[\beta\omega_{in+k,S_k} - \lambda(\beta)]} \mathbf{1}_{\{S_n=\sqrt{(y+1)n}, 0<S_i-y\sqrt{n}<\sqrt{n} \text{ for } 0<i<n\}} \right],$$

$$\overline{W}_i^{(y,y-1)} := P_{\sqrt{n}y}\left[e^{\sum_{k=1}^{n-1}[\beta\omega_{in+k,S_k} - \lambda(\beta)]} \mathbf{1}_{\{S_n=(y-1)\sqrt{n}, -\sqrt{n}<S_i-y\sqrt{n}<0 \text{ for } 0<i<n\}} \right], \tag{5.5}$$

which are random variables that have the same law as $\overline{W}$. Moreover because of independence of the environment in different cells, one can see that

$$\left(\overline{W}_i^{(y,y\pm1)}; \; (i, y) \in \mathbb{N} \times \mathbb{Z} \text{ such that } i - y \text{ is even} \right),$$

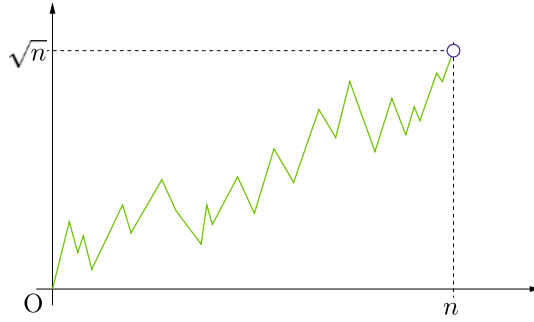is a family of independent variables.

Fig. 3. We consider a resticted partition function $\overline{W}$ by considering only paths going from one to the other corner of the cell, without going out. This restriction will give us the independence of a random variable corresponding to different cells which will be crucial to make the proof work

Let $N = nm$ be a large integer. We define $\Omega = \Omega_N$ as the set of path

$$\Omega := \{S \ : \ \forall i \in [1, m], \ |S_{in} - S_{(i-1)n}| = \sqrt{n}, \ \forall j \in [1, n-1],$$
$$S_{(i-1)n+j} \in \left(S_{(i-1)n}, S_{in}\right)\},$$

$$(5.6)$$

where the interval $\left(S_{i(n-1)}, S_{in}\right)$ is to be seen as $\left(S_{in}, S_{i(n-1)}\right)$ if $S_{in} < S_{i(n-1)}$, and

$$\mathcal{S} := \left\{ s = (s_0, s_1, \ldots, s_m) \in \mathbb{Z}^{m+1} \ : \ s_0 = 0 \text{ and } |s_i - s_{i-1}| = 1, \ \forall i \in [1, m] \right\}. \quad (5.7)$$

We use the trivial bound

$$W_N \geq P\left[\exp\left(\sum_{i=1}^{nm} (\beta \omega_{i, S_i} - \lambda(\beta)) \right) \mathbf{1}_{\{S \in \Omega\}} \right], \quad (5.8)$$

to get that

$$W_N \geq \sum_{s \in \mathcal{S}} \prod_{i=0}^{m-1} \overline{W}_i^{(s_i, s_{i+1})} \exp\left(\beta \omega_{(i+1)n, s_{i+1}\sqrt{n}} - \lambda(\beta) \right) \quad (5.9)$$

(the exponential term is due to the fact that $\overline{W}$ does not take into account the site in the top corner of each cell).

The idea of the proof is to find a value of n (depending on β) such that we are sure that for any value of m we can find a path s such that along the path the values of $(\overline{W}_i^{(s_i, s_{i+1})})$ are not too low (i.e. close to the expectation of $\overline{W}$) and to do so, it seems natural to seek for a percolation argument.

Let p_c be the critical exponent for directed percolation in dimension $1 + 1$ (for an account on directed percolation see [17, Sect. 12.8] and references therein). From Lemma 5.3 and the Chebyshev inequality, one can find a constant C_8 and β_0 such that for all $n \leq \frac{C_8}{\beta^4 |\log \beta|}$ and $\beta \leq \beta_0$,

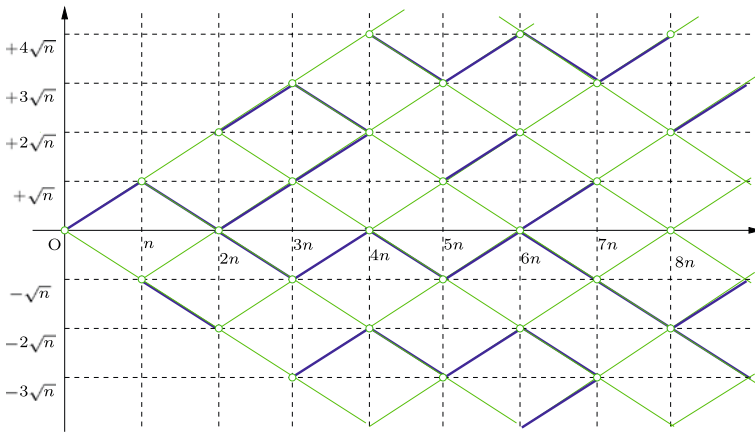$$Q\{\overline{W} \geq Q\overline{W}/2\} \geq \frac{p_c + 1}{2}. \quad (5.10)$$

Fig. 4. This figure illustrates the percolation argument used in the proof. To each cell is naturally associated a random variable $\overline{W}_i^{y, y\pm 1}$, and these random variables are i.i.d. When $\overline{W}_i^{y, y\pm 1} \geq 1/2 Q \overline{W}$ we open the edge in the corresponding cell (thick edges on the picture). As this happens with a probability strictly superior to p_c, we have a positive probability to have an infinite path linking 0 to infinity

We choose n to be the biggest squared even integer that is less than $\frac{C_8}{\beta^4 |\log \beta|}$. (In particular $n \geq \frac{C_8}{2\beta^4 |\log \beta|}$ if β is small enough.)

As shown in Fig. 4, we associate to our system the following directed percolation picture. For all $(i, y) \in \mathbb{N} \times \mathbb{Z}$ such that $i - y$ is even:

- If $\overline{W}_i^{(y, y\pm 1)} \geq (1/2) Q \overline{W}$, we say that the edge linking the opposite corners of the corresponding cell is open.
- If $\overline{W}_i^{(y, y\pm 1)} < (1/2) Q \overline{W}$, we say that the same edge is closed.

Equation (5.10) and the fact the considered random variables are independent assures that with positive probability there exists an infinite directed path starting from zero.

When there exists an infinite open path linking zero to infinity, we can define the highest open path in an obvious way. Let $(s_i)_{i=1}^m$ denote this highest path. If m is large enough, by law of large numbers we have that with a probability close to one,

$$\sum_{i=1}^{m} \left[\beta \omega_{ni, \sqrt{n}s_i} - \lambda(\beta) \right] \geq -2m\lambda(\beta). \tag{5.11}$$

Using this and the percolation argument with (5.9) we finally get that with a positive probability which does not depend on m we have

$$W_{nm} \geq \left[(1/2) e^{-2\lambda(\beta)} Q \overline{W} \right]^m. \tag{5.12}$$

Taking the log and making m tend to infinity this implies that

$$p(\beta) \geq \frac{1}{n} \left[-2\lambda(\beta) - \log 2 + \log Q \overline{W} \right] \geq -\frac{c}{n} \log n. \tag{5.13}$$

For some constant c, if n is large enough (we used Lemma 5.2 to get the last inequality. The result follows by replacing n by its value. $\quad\square$

Proof of Lemma 5.2. Let n be square and even. T_k, $k \in \mathbb{Z}$ denote the first hitting time of k by the random walk S (when $k = 0$ it denotes the return time to zero). We have

$$P\{S_n = \sqrt{n}, 0 < S_i < \sqrt{n}, \text{ for all } 1 < i < n\}$$

$$= \sum_{k=1}^{n-1} P\{T_{\sqrt{n}/2} = k, \ S_j > 0 \text{ for all } j < n \text{ and } T_{\sqrt{n}} = n\} \qquad (5.14)$$

$$= P\{T_{\sqrt{n}/2} < n, \ S_j < \sqrt{n} \text{ for all } j < n \text{ and } T_0 = n)\},$$

where the second equality is obtained with the strong Markov property used for $T = T_{\sqrt{n}/2}$, and the reflexion principle for the random walk. The last line is equal to

$$P\{\max_{k \in [0,n]} S_k \in [\sqrt{n}/2, \sqrt{n}) | T_0 = n\} P\{T_0 = n\}. \qquad (5.15)$$

We use here a variant of Donsker's Theorem, for a proof see [20, Theorem 2.6].

Lemma 5.4. *The process*

$$t \mapsto \left\{ \frac{S_{\lceil nt \rceil}}{\sqrt{n}} \ \middle| \ T_0 = n \right\}, \quad t \in [0, 1] \qquad (5.16)$$

converges in distribution to the normalized Brownian excursion in the space $D([0, 1], \mathbb{R})$.

We also know that (see for example [13, Prop. A.10]) for n even $P(T_0 = n) = \sqrt{2/\pi} n^{-3/2} + o(n^{-3/2})$. Therefore, from (5.15) we have

$$P\{S_n = \sqrt{n}, 0 < S_i < \sqrt{n}, \text{ for all } 1 < i < n\}$$

$$= \sqrt{2/\pi} n^{-3/2} \mathbb{P}\left[\max_{t \in [0,1]} \mathbf{e}_t \in (1/2, 1)\right] + o(n^{-3/2}), \qquad (5.17)$$

where $\mathbf{e}$ denotes the normalized Brownian excursion, and $\mathbb{P}$ its law. $\square$

Proof of Lemma 5.3. Let β be fixed and small enough, and n be some squared even integer which is less than $c_\varepsilon/(\beta^4 |\log \beta|)$. We will fix the value c_ε independently of β later in the proof, and always consider that β is sufficiently small. By a direct computation the variance of

$$P\left[\exp\left(\sum_{i=1}^{n-1} [\beta \omega_{i,S_i} - \lambda(\beta)]\right) \ \middle| \ S_n = \sqrt{n}, 0 < S_i < \sqrt{n} \text{ for } 0 < i < n\right] \qquad (5.18)$$

is equal to

$$P^{\otimes 2}\left[\exp\left(\sum_{i=1}^{n-1} \gamma(\beta) \mathbf{1}_{\{S_i^{(1)} = S_i^{(2)}\}}\right) \ \middle| \ A_n\right] - 1, \qquad (5.19)$$

where

$$A_n = \left\{ S_n^{(j)} = \sqrt{n}, 0 < S_i^{(j)} < \sqrt{n} \text{ for } 0 < i < n, \ j = 1, 2 \right\}, \qquad (5.20)$$

and $\gamma(\beta) = \lambda(2\beta) - 2\lambda(\beta)$ (recall that $\lambda(\beta) = \log Q \exp(\beta \omega_{(1,0)})$), and $S_n^{(j)}$, $j = 1, 2$ denotes two independent random walks with law denoted by $P^{\otimes 2}$. From this it follows

that if n is small the result is quite straightforward. We will therefore only be interested in the case of large n (i.e. bounded away from zero by a large fixed constant).

We define $\tau = (\tau_k)_{k \geq 0} = \{S_i^{(1)} = S_i^{(2)}, i \geq 0\}$ the set where the walks meet (it can be written as an increasing sequence of integers). By the Markov property, the random variables $\tau_{k+1} - \tau_k$ are i.i.d. , and we say that τ is a *renewal sequence*.

We want to bound the probability that the renewal sequence τ has too many returns before times $n - 1$, in order to estimate (5.19). To do so, we make the usual computations with the Laplace transform.

From [11, p. 577] , we know that

$$1 - P^{\otimes 2} \exp(-x\tau_1) = \frac{1}{\sum_{n \in \mathbb{N}} \exp(-xn) P\{S_n^{(1)} = S_n^{(2)}\}}. \tag{5.21}$$

Thanks to the local central limit theorem for the simple random walk, we know that for large n,

$$P\{S_n^{(1)} = S_n^{(2)}\} = \frac{1}{\sqrt{\pi n}} + o(n^{-1/2}). \tag{5.22}$$

So we can get from (5.21) that when x is close to zero,

$$\log P^{\otimes 2} \exp(-x\tau_1) = -\sqrt{x} + o(\sqrt{x}). \tag{5.23}$$

We fix x_0 such that $\log P \exp(-x\tau_1) \leq \sqrt{x}/2$ for all $x \leq x_0$. For any $k \leq n$ we have

$$P^{\otimes 2}\{|\tau \cap [1, n - 1]| \geq k\} = P^{\otimes 2}\{\tau_k \leq n - 1\} \leq \exp((n - 1)x) P^{\otimes 2} \exp(-\tau_k x)$$
$$\leq \exp\left[nx + k \log P^{\otimes 2} \exp(-x\tau_1)\right]. \tag{5.24}$$

For any $k \leq \lfloor 4n\sqrt{x_0} \rfloor = k_0$ one can choose $x = (k/4n)^2 \leq x_0$ in the above and use the definition of x_0 to get that

$$P^{\otimes 2}\{|\tau \cap [1, n - 1]| \geq k\} \leq \exp\left(-k^2/(32n)\right). \tag{5.25}$$

In the case where $k > k_0$ we simply bound the quantity by

$$P^{\otimes 2}\{|\tau \cap [1, n - 1]| \geq k\} \leq \exp\left(k_0^2/(32n)\right) \leq \exp\left(-nx_0/4\right). \tag{5.26}$$

By Lemma (5.2), if n is large enough,

$$P^{\otimes} A_n \geq 1/2c_{RW}^2 n^{-3}. \tag{5.27}$$

A trivial bound on the conditioning gives us

$$P^{\otimes 2}\left(|\tau \cap [1, n - 1]| \geq k \mid A_n\right) \leq \min\left(1, 2c_{RW}^{-2} n^3 \exp\left(-k^2/(32n)\right)\right) \quad \text{if } k \leq k_0,$$
$$P^{\otimes 2}\left(|\tau \cap [1, n - 1]| \geq k \mid A_n\right) \leq 2c_{RW}^{-2} n^3 \exp\left(-nx_0/4\right) \quad \text{otherwise.} \tag{5.28}$$

We define $k_1 := \lceil 16\pi \sqrt{n \log(2c_{RW}^{-2} n^3)} \rceil$. The above implies that for n large enough we have

$$P^{\otimes 2} \left(|\tau \cap [1, n - 1]| \geq k \mid A_n \right) \leq 1 \quad \text{if } k \leq k_1,$$

$$P^{\otimes 2} \left(|\tau \cap [1, n - 1]| \geq k \mid A_n \right) \leq \exp\left(-k^2/(64n) \right) \quad \text{if } k_1 \leq k \leq k_0, \quad (5.29)$$

$$P^{\otimes 2} \left(|\tau \cap [1, n - 1]| \geq k \mid A_n \right) \leq \exp\left(-nx_0/8 \right) \quad \text{otherwise.}$$

Now we are ready to bound (5.19). Integration by part gives,

$$P^{\otimes 2} \left[\exp\left(\gamma\beta |\tau \cap [1, n - 1]| \right) \mid A_n \right] - 1$$
$$= \gamma(\beta) \int_0^\infty \exp(\gamma(\beta)x) P^{\otimes 2} \left(|\tau \cap [1, n - 1]| \geq x \mid A_n \right) dx. \quad (5.30)$$

We split the right-hand side in three parts corresponding to the three different bounds we have in (5.28): $x \in [0, k_1]$, $x \in [k_1, k_0]$ and $x \in [k_0, n]$. It suffices to show that each part is less than $\varepsilon/3$ to finish the proof. The first part is

$$\gamma(\beta) \int_0^{k_1} \exp(\gamma(\beta)x) P^{\otimes 2} \left(|\tau \cap [1, n-1]| \geq x \mid A_n \right) dx \leq \gamma(\beta) k_1 \exp(\gamma(\beta)k_1). \quad (5.31)$$

One uses that $n \leq \frac{c_\varepsilon}{\beta^4 |\log \beta|}$ and $\gamma(\beta) = \beta^2 + o(\beta^2)$ to get that for β small enough and n large enough if c_ε is well chosen we have

$$k_1 \gamma(\beta) \leq 100\beta^2 \sqrt{n \log n} \leq \varepsilon/4, \quad (5.32)$$

so that $\gamma(\beta)k_1 \exp(\gamma(\beta)k_1) \leq \varepsilon/3$.

We use our bound for the second part of the integral to get

$$\gamma(\beta) \int_{k_1}^{k_0} \exp(\gamma(\beta)x) P^{\otimes 2} \left(|\tau \cap [1, n - 1]| \geq x \mid A_n \right) dx$$
$$\leq \gamma(\beta) \int_0^\infty \exp\left(\gamma(\beta)x - x^2/(64n) \right) dx = \int_0^\infty \exp\left(x - \frac{x^2}{64n\gamma(\beta)^2} \right) dx. \quad (5.33)$$

Replacing n by its value, we see that the term that goes with x^2 in the exponential can be made arbitrarily large, provided that c_ε is small enough. In particular we can make the left-hand side less than $\varepsilon/3$.

Finally, we estimate the last part

$$\gamma(\beta) \int_{k_0}^n \exp(\gamma(\beta)^2 x) P^{\otimes 2} \left(|\tau \cap [1, n - 1]| \geq x \mid A_n \right) dx$$
$$\leq \gamma(\beta) \int_0^n \exp(\gamma(\beta)x - nx_0/8) dx = n \exp(-[\gamma(\beta) - x_0/8]n). \quad (5.34)$$

This is clearly less than $\varepsilon/3$ if n is large and β is small. $\quad \square$

6. Proof of the Lower Bound of Theorem 1.5

In this section we use the method of replica-coupling that is used for the disordered pinning model in [24] to derive a lower bound on the free energy. The proof here is an adaptation of the argument used there to prove disorder irrelevance.

The main idea is the following: Let $W_N(\beta)$ denote the renormalized partition function for inverse temperature β. A simple Gaussian computation gives

$$\left.\frac{\mathrm{d}Q \log W_N(\sqrt{t})}{\mathrm{d}t}\right|_{t=0} = -\frac{1}{2}P^{\otimes 2}\sum_{i=1}^{N}\mathbf{1}_{\{S_i^{(1)}=S_i^{(2)}\}}, \tag{6.1}$$

where $S^{(1)}$ and $S^{(2)}$ are two independent random walks under the law $P^{\otimes 2}$. This implies that for small values of β (by the equality of derivative at $t=0$),

$$Q \log W_N(\beta) \approx -\log P^{\otimes 2}\exp\left(\beta^2/2\sum_{i=1}^{N}\mathbf{1}_{\{S_N^{(1)}=S_N^{(2)}\}}\right). \tag{6.2}$$

This tends to make us believe that

$$p(\beta) = -\lim_{N\to\infty}\log P^{\otimes 2}\exp\left(\beta^2/2\sum_{i=1}^{N}\mathbf{1}_{\{S_N^{(1)}=S_N^{(2)}\}}\right). \tag{6.3}$$

However, things are not that simple because (6.2) is only valid for fixed N, and one needs some more work to get something valid when N tends to infinity. The proofs aim to use a convexity argument and simple inequalities to be able to get the inequality

$$p(\beta) \geq -\lim_{N\to\infty}\log P^{\otimes 2}\exp\left(2\beta^2\sum_{i=1}^{N}\mathbf{1}_{\{S_N^{(1)}=S_N^{(2)}\}}\right). \tag{6.4}$$

The fact that convexity is used in a crucial way makes it quite hopeless to get the other inequality using this method.

Proof. Let us define for β fixed and $t \in [0,1]$,

$$\Phi_N(t,\beta) := \frac{1}{N}Q \log P \exp\left(\sum_{i=1}^{N}\left[\sqrt{t}\beta\omega_{i,S_i} - \frac{t\beta^2}{2}\right]\right), \tag{6.5}$$

and for $\lambda \geq 0$,

$$\Psi_N(t,\lambda,\beta) := \frac{1}{2N}Q \log P^{\otimes 2}\exp\left(\sum_{i=1}^{N}\left[\sqrt{t}\beta(\omega_{i,S_i^{(1)}}+\omega_{i,S_i^{(2)}})-t\beta^2+\lambda\beta^2\mathbf{1}_{\{S_i^{(1)}=S_i^{(2)}\}}\right]\right). \tag{6.6}$$

One can notice that $\Phi_N(0,\beta) = 0$ and $\Phi_N(1,\beta) = p_N(\beta)$ (recall the definition of p_N (1.11)), so that Φ_N is an interpolation function. Via the Gaussian integration by par formula,

$$Q\omega f(\omega) = Qf'(\omega), \tag{6.7}$$

valid (if ω is a centered standard Gaussian variable) for every differentiable functions such that $\lim_{|x| \to \infty} \exp(-x^2/2) f(x) = 0$, one finds

$$\frac{d}{dt} \Phi_N(t, \beta) = -\frac{\beta^2}{2N} \sum_{j=1}^{N} \sum_{z \in \mathbb{Z}} Q \left(\frac{P \exp\left(\sum_{i=1}^{N} \left[\sqrt{t}\beta \omega_{i,S_i} - \frac{t\beta^2}{2} \right] \right) \mathbf{1}_{\{S_j = z\}}}{P \exp\left(\sum_{i=1}^{N} \left[\sqrt{t}\beta \omega_{i,S_i} - \frac{t\beta^2}{2} \right] \right)} \right)^2$$

$$= -\frac{\beta^2}{2N} Q \left(\mu_n^{(\sqrt{t}\beta)} \right)^{\otimes 2} \left[\sum_{i=1}^{N} \mathbf{1}_{\{S_i^{(1)} = S_i^{(2)}\}} \right]. \tag{6.8}$$

This is (up to the negative multiplicative constant $-\beta^2/2$) the expected overlap fraction of two independent replicas of the random–walk under the polymer measure for the inverse temperature $\sqrt{t}\beta$. This result has been using the Itô formula in [4, Sect. 7].

For notational convenience, we define

$$H_N(t, \lambda, S^{(1)}, S^{(2)}) = \sum_{i=1}^{N} \left[\sqrt{t}\beta(\omega_{i,S_i^{(1)}} + \omega_{i,S_i^{(2)}}) - t\beta^2 + \lambda\beta^2 \mathbf{1}_{\left\{ S_i^{(1)} = S_i^{(2)} \right\}} \right]. \tag{6.9}$$

We use Gaussian integration by part again, for Ψ_N:

$$\frac{d}{dt} \Psi_N(t, \lambda, \beta) = \frac{\beta^2}{2N} \sum_{j=1}^{N} Q \frac{P^{\otimes 2} \exp\left(H_N(t, \lambda, S^{(1)}, S^{(2)}) \right) \mathbf{1}_{\{S_j^{(1)} = S_j^{(2)}\}}}{P^{\otimes 2} \exp\left(H_N(t, \lambda, S^{(1)}, S^{(2)}) \right)}$$

$$- \frac{\beta^2}{4N} \sum_{j=1}^{N} \sum_{z \in \mathbb{Z}} Q \left(\frac{P^{\otimes 2} \left(\mathbf{1}_{\{S_j^{(1)} = z\}} + \mathbf{1}_{\{S_j^{(2)} = z\}} \right) \exp\left(H_N(t, \lambda, S^{(1)}, S^{(2)}) \right)}{P^{\otimes 2} \exp\left(H_N(t, \lambda, S^{(1)}, S^{(2)}) \right)} \right)^2$$

$$\leq \frac{\beta^2}{2N} \sum_{j=1}^{N} Q \frac{P^{\otimes 2} \exp\left(H_N(t, \lambda, S^{(1)}, S^{(2)}) \right) \mathbf{1}_{\{S_j^{(1)} = S_j^{(2)}\}}}{P^{\otimes 2} \exp\left(H_N(t, \lambda, S^{(1)}, S^{(2)}) \right)}$$

$$= \frac{d}{d\lambda} \Psi_N(t, \lambda, \beta). \tag{6.10}$$

The above implies that for every $t \in [0, 1]$ and $\lambda \geq 0$,

$$\Psi_N(t, \lambda, \beta) \leq \Psi_N(0, \lambda + t, \beta). \tag{6.11}$$

Comparing (6.8) and (6.10), and using convexity and monotonicity of $\Psi_N(t, \lambda, \beta)$ with respect to λ, and the fact that $\Psi_N(t, 0, \beta) = \Phi_N(t, \beta)$ one gets

$$-\frac{d}{dt} \phi_N(t, \beta) = \frac{d}{d\lambda} \Psi_N(t, \lambda, \beta) \Big|_{\lambda=0}$$

$$\leq \frac{\Psi_N(t, 2 - t, \beta) - \Phi_N(t, \beta)}{2 - t} \leq \Psi_N(0, 2, \beta) - \Phi_N(t, \beta), \tag{6.12}$$

where in the last inequality we used $(2 - t) \geq 1$ and (6.11). Integrating this inequality between 0 and 1 and recalling $\Phi_N(1, \beta) = p_N(\beta)$ we get

$$p_N(\beta) \geq (1 - e)\Psi_N(0, 2, \beta). \tag{6.13}$$

On the right-hand side of the above we recognize something related to pinning models. More precisely

$$\Psi_N(0, 2, \beta) = \frac{1}{2N} \log Y_N, \tag{6.14}$$

where

$$Y_N = P^{\otimes 2} \exp\left(2\beta^2 \sum_{i=1}^{N} \mathbf{1}_{\left\{S_N^{(1)} = S_N^{(2)}\right\}}\right) \tag{6.15}$$

is the partition function of a homogeneous pinning system of size N and parameter $2\beta^2$ with underlying renewal process the sets of zero of the random walk $S^{(2)} - S^{(1)}$. This is a well known result in the study of the pinning model (we refer to [13, Sect. 1.2] for an overview and the results we cite here) that

$$\lim_{N \to \infty} \frac{1}{N} \log Y_N = \mathrm{F}(2\beta^2), \tag{6.16}$$

where F denotes the free energy of the pinning model. Moreover, it is also stated

$$\mathrm{F}(h) \overset{h \to 0+}{\sim} h^2/2. \tag{6.17}$$

Then passing to the limit in (6.14) ends the proof of the result for any constant strictly bigger than 4. $\quad\square$

7. Proof the Lower Bound in Theorem 1.6

The technique used in the two previous sections could be adapted here to prove the results but in fact it is not necessary. Because of the nature of the bound we want to prove in dimension 2 (we do not really track the best possible constant in the exponential), it will be sufficient here to control the variance of W_n up to some value, and then the concentration properties of $\log W_n$ to get the result. The reader can check than using the same method in dimension 1 does not give the right power of β.

First we prove a technical result to control the variance of W_n which is the analog of (5.3) in dimension 1. Recall that $\gamma(\beta) := \lambda(2\beta) - 2\lambda(\beta)$ with $\lambda(\beta) := \log Q \exp(\beta\omega_{(1,0)})$.

Lemma 7.1. *For any $\varepsilon < 0$, one can find a constant $c_\varepsilon > 0$ and $\beta_0 > 0$ such that for any $\beta \leq \beta_0$, for any $n \leq \exp\left(c_\varepsilon/\beta^2\right)$ we have*

$$\mathrm{Var}_Q \, W_n \leq \varepsilon. \tag{7.1}$$

Proof. A straightforward computation shows that the the variance of W_n is given by

$$\mathrm{Var}_Q \, W_n = P^{\otimes 2} \exp\left(\gamma(\beta) \sum_{i=1}^{n} \mathbf{1}_{\left\{S_i^{(1)} = S_i^{(2)}\right\}}\right) - 1, \tag{7.2}$$

where $S^{(i)}$, $i = 1, 2$ are two independent 2–dimensional random walks.

As the above quantity is increasing in n, it will be enough to prove the result for n large. For technical convenience we choose to prove the result for $n = \lfloor \exp(-c_\varepsilon/\gamma(\beta)) \rfloor$ (recall $\gamma(\beta) = \lambda(2\beta) - 2\lambda(\beta)$) which does not change the result since $\gamma(\beta) = \beta^2 + o(\beta^2)$.

The result we want to prove seems natural since we know that $(\sum_{i=1}^{n} \mathbf{1}_{\{S_i^{(1)} = S_i^{(2)}\}}) / \log n$ converges to an exponential variable (see e.g. [12]), and $\gamma(\beta) \sim c_\varepsilon \log n$. However, convergence of the right–hand side of (7.2) requires the use of the Dominated Convergence Theorem, and the proof of the domination hypothesis is not straightforward. It could be extracted from the proof of the large deviation result in [12], however we include a full proof of convergence here for the sake of completeness.

We define $\tau = (\tau_k)_{k \geq 0} = \{S_i^{(1)} = S_i^{(2)}, i \geq 0\}$ the set where the walks meet (it can be written as an increasing sequence). By the Markov property, the random variables $\tau_{k+1} - \tau_k$ are i.i.d.

To prove the result, we compute bounds on the probability of having too many points before n in the renewal τ. As in the 1 dimensional case, we use the Laplace transform to do so. From [11, p. 577] , we know that

$$1 - P^{\otimes 2} \exp(-x\tau_1) = \frac{1}{\sum_{n \in \mathbb{N}} \exp(-xn) P\{S_n^{(1)} = S_n^{(2)}\}}. \tag{7.3}$$

The local central limit theorem says that for large n,

$$P^{\otimes 2}\{S_n^{(1)} = S_n^{(2)}\} \sim \frac{1}{\pi n}. \tag{7.4}$$

Using this in (7.3) we get that when x is close to zero,

$$\log P^{\otimes 2} \exp(-x\tau_1) \sim -\frac{\pi}{|\log x|}. \tag{7.5}$$

We use the following estimate:

$$P^{\otimes 2}\{|\tau \cap [1, n]| \geq k\} = P^{\otimes 2}\{\tau_k \leq n\} \leq \exp(nx) P^{\otimes 2} \exp(-\tau_k x)$$
$$= \exp\left[nx + k \log P^{\otimes 2} \exp(-x\tau_1)\right]. \tag{7.6}$$

Let x_0 be such that for any $x \leq x_0$, $\log P^{\otimes 2} \exp(-x\tau_1) \geq -3/|\log x|$. For k such that $k/(n \log(n/k)) \leq x_0$, we replace x by $k/(n \log(n/k))$ in (7.6) to get

$$P^{\otimes 2}\{|\tau \cap [1, n]| \geq k\} \leq \exp\left(\frac{k}{\log(n/k)} - \frac{3k}{\log\left[k/(n \log n/k)\right]}\right)$$
$$\leq \exp\left(-\frac{k}{\log(n/k)}\right), \tag{7.7}$$

where the last inequality holds if k/n is small enough. We fix $k_0 = \delta n$ for some small δ. We get that

$$P^{\otimes 2}\{|\tau \cap [1, n]| \geq k\} \leq \exp\left(-\frac{k}{\log(n/k)}\right) \quad \text{if } k \leq k_0,$$
$$P^{\otimes 2}\{|\tau \cap [1, n]| \geq k\} \leq \exp\left(-\frac{k_0}{\log(n/k_0)}\right) = \exp\left(-\frac{\delta n}{\log(1/\delta)}\right) \quad \text{if } k \geq k_0. \tag{7.8}$$

We are ready to bound (7.2). We remark that using integration by part we obtain

$$P \exp\left(\gamma(\beta)|\tau \cap [1, n]|\right) - 1 = \int_0^n \gamma(\beta) \exp(\gamma(\beta)x) P^{\otimes 2}(\tau \cap [1, n]| \geq x) \, dx. \tag{7.9}$$

To bound the right–hand side, we use the bounds we have concerning τ: (7.8). We have to split the integral in three parts.

The integral between 0 and 1 can easily be made less than $\varepsilon/3$ by choosing β small. Using $n \leq \exp(c_\varepsilon/\gamma(\beta))$, we get that

$$\int_1^{\delta n} \gamma(\beta) \exp(\gamma(\beta)x) P^{\otimes 2}(\tau \cap [1,n]| \geq x)\,dx \leq \int_1^{\delta n} \gamma(\beta) \exp\left(\gamma(\beta)x - \frac{x}{\log(n/x)}\right)dx$$

$$\leq \int_1^{\delta n} \gamma(\beta) \exp\left(\gamma(\beta)x - \frac{\gamma(\beta)\beta x}{c_\varepsilon}\right)$$

$$\leq \frac{c_\varepsilon}{1 - c_\varepsilon}. \tag{7.10}$$

This is less than $\varepsilon/3$ if c_ε is chosen appropriately. The last part to bound is

$$\int_{\delta n}^n \gamma(\beta) \exp(\gamma(\beta)x) P^{\otimes 2}(\tau \cap [1,n]| \geq x) \leq n\gamma(\beta) \exp\left(\gamma(\beta)n - \frac{\delta n}{\log 1/\delta}\right) \leq \varepsilon/3, \tag{7.11}$$

where the last inequality holds if n is large enough, and β is small enough. $\square$

Proof of the lower bound in Theorem 1.6. By a martingale method that one can find a constant c_9 such that

$$\mathrm{Var}_Q \log W_n \leq C_9 n, \qquad \forall n \geq 0, \forall \beta \leq 1. \tag{7.12}$$

(See [6, Prop. 2.5] and its proof for more details).

Therefore Chebyshev inequality gives

$$Q\left\{\left|\frac{1}{n}\log W_n - \frac{1}{n}Q\log W_n\right| \geq n^{-1/4}\right\} \leq C_9 n^{-1/2}. \tag{7.13}$$

Using Lemma 7.1 and Chebyshev inequality again, we can find a constant C_{10} such that for small β and $n = \lceil \exp(C_{10}/\beta^2) \rceil$ we have

$$Q\{W_n < 1/2\} \leq 1/2. \tag{7.14}$$

This combined with (7.13) implies that

$$\frac{-\log 2}{n} \leq n^{-1/4} + Q\frac{1}{n}\log W_n \leq n^{-1/4} + p(\beta). \tag{7.15}$$

Replacing n by its value we get

$$p(\beta) \geq -n^{-1/4} - \frac{\log 2}{n} \geq -\exp(-C_{10}/5\beta^2). \tag{7.16}$$

$\square$

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