Commun. Math. Phys. 292, 667–719 (2009)
Digital Object Identifier (DOI) 10.1007/s00220-009-0886-1

Communications in
Mathematical Physics

Lie Conformal Algebra Cohomology and the Variational Complex

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Received: 22 December 2008 / Accepted: 11 June 2009
Published online: 14 August 2009 –

Dedicated to Corrado De Concini on his 60^{th} *birthday*

Abstract: We find an interpretation of the complex of variational calculus in terms of the Lie conformal algebra cohomology theory. This leads to a better understanding of both theories. In particular, we give an explicit construction of the Lie conformal algebra cohomology complex, and endow it with a structure of a $\mathfrak{g}$-complex. On the other hand, we give an explicit construction of the complex of variational calculus in terms of skew-symmetric poly-differential operators.

1. Introduction

Lie conformal algebras encode the properties of operator product expansions in conformal field theory, and, at the same time, of local Poisson brackets in the theory of integrable evolution equations.

Recall [K] that a *Lie conformal algebra* over a field $\mathbb{F}$ is an $\mathbb{F}[\partial]$-module A, endowed with a λ-*bracket*, that is an $\mathbb{F}$-linear map $A \otimes A \to \mathbb{F}[\lambda] \otimes A$, denoted by $a \otimes b \mapsto [a_\lambda b]$, satisfying the two *sesquilinearity* properties

$$[\partial a_\lambda b] = -\lambda [a_\lambda b], \qquad [a_\lambda \partial b] = (\partial + \lambda)[a_\lambda b], \tag{1}$$

such that the *skew-symmetry*

$$[a_\lambda b] = -[b_{-\partial-\lambda} a] \tag{2}$$

and the *Jacobi identity*

$$[a_\lambda [b_\mu c]] - [b_\mu [a_\lambda c]] = [[a_\lambda b]_{\lambda+\mu} c] \tag{3}$$

hold for any $a, b, c \in A$. It is assumed in (2) that ∂ is moved to the left.

A module over a Lie conformal algebra A is an $\mathbb{F}[\partial]$-module M, endowed with a λ-*action*, that is an $\mathbb{F}$-linear map $A \otimes M \to \mathbb{F}[\lambda] \otimes M$, denoted by $a \otimes b \to a_\lambda b$, such that sesquilinearity (1) holds for $a \in A$, $b \in M$ and the Jacobi identity (3) holds for $a, b \in A$, $c \in M$.

A cohomology theory for Lie conformal algebras was developed in [BKV]. Given a Lie conformal algebra A and an A-module M, one first defines the *basic cohomology complex* $\widetilde{\Gamma}^\bullet(A, M) = \sum_{k \in \mathbb{Z}_+} \widetilde{\Gamma}^k$, where $\widetilde{\Gamma}^k$ consists of $\mathbb{F}$-linear maps $\widetilde{\gamma} : A^{\otimes k} \to \mathbb{F}[\lambda_1, \dots, \lambda_k] \otimes M$, satisfying certain sesquilinearity and skew-symmetry properties, and endows this complex with a differential $\delta : \widetilde{\Gamma}^k \to \widetilde{\Gamma}^{k+1}$, such that $\delta^2 = 0$. This complex is isomorphic to the Lie algebra cohomology complex for the annihilation Lie algebra $\mathfrak{g}_-$ of A with coefficients in the $\mathfrak{g}_-$-module M [BKV, Theorem 6.1].

Next, one endows $\widetilde{\Gamma}^\bullet(A, M)$ with a structure of a $\mathbb{F}[\partial]$-module, such that ∂ commutes with δ, which allows one to define the reduced cohomology complex $\Gamma^\bullet(A, M) = \widetilde{\Gamma}^\bullet(A, M)/\partial \widetilde{\Gamma}^\bullet(A, M)$, and this is the Lie conformal algebra cohomology complex, introduced in [BKV].

Our first contribution to this theory is a more explicit construction of the reduced cohomology complex. Namely, we introduce a new cohomology complex $C^\bullet(A, M) = \oplus_{k \in \mathbb{Z}_+} C^k$, where $C^0 = M/\partial M$, $C^1 = \mathrm{Hom}_{\mathbb{F}[\partial]}(A, M)$, and for $k \geq 2$, C^k consists of poly λ-brackets, namely of $\mathbb{F}$-linear maps $c : A^{\otimes k} \to \mathbb{F}[\lambda_1, \dots, \lambda_{k-1}] \otimes M$, satisfying certain sesquilinearity and skew-symmetry conditions, and we endow $C^\bullet(A, M)$ with a square zero differential d. We construct embeddings of complexes:

$$\Gamma^\bullet(A, M) \subset \bar{C}^\bullet(A, M) \subset C^\bullet(A, M), \tag{4}$$

where $\bar{C}^\bullet(A, M)$ consists of cocycles which vanish if one of the arguments is a torsion element of A. In fact, $\bar{C}^k = C^k$, unless $k = 1$.

We show that $\Gamma^\bullet(A, M) = \bar{C}^\bullet(A, M)$, provided that, as an $\mathbb{F}[\partial]$-module, A is isomorphic to a direct sum of its torsion and a free $\mathbb{F}[\partial]$-module (which is always the case if A is a finitely generated $\mathbb{F}[\partial]$-module). Our opinion is that the slightly larger complex $C^\bullet(A, M)$ is a more correct Lie conformal algebra cohomology complex than the complex $\Gamma^\bullet(A, M)$ of [BKV]. This is illustrated by our Theorem 3.1(c), which says that the $\mathbb{F}[\partial]$-split abelian extensions of A by M are parameterized by $H^2(A, M)$ for the complex $C^\bullet(A, M)$. This holds for the cohomology theory of [BKV] only if A is a free $\mathbb{F}[\partial]$-module.

Following [BKV], we also consider the superspace of basic chains $\widetilde{\Gamma}_\bullet(A, M)$ and its subspace of reduced chains $\Gamma_\bullet(A, M)$ (they are not complexes in general). Corresponding to the embeddings of complexes (4), we introduce the vector superspaces of chains $C_\bullet(A, M)$ and $\bar{C}_\bullet(A, M)$, and the maps:

$$C_\bullet(A, M) \twoheadrightarrow \bar{C}_\bullet(A, M) \to \Gamma_\bullet(A, M). \tag{5}$$

We develop the theory further in the important case for the calculus of variations, when the A-module M is endowed with a commutative associative product, such that ∂ and a_λ for all $a \in A$ are derivations of this product. In this case one can endow the superspace $\widetilde{\Gamma}^\bullet(A, M)$ with a commutative associative product [BKV]. Furthermore, we introduce a Lie algebra bracket on the space $\mathfrak{g} := \Pi \widetilde{\Gamma}_1(A, M)$ (Π, as usual, stands for reversing of the parity). Let $\widehat{\mathfrak{g}} = \eta\mathfrak{g} \oplus \mathfrak{g} \oplus \mathbb{F}\partial_\eta$ be a $\mathbb{Z}$-graded Lie superalgebra extension of $\mathfrak{g}$, where η is an odd indeterminate, $\eta^2 = 0$. We endow $\widetilde{\Gamma}^\bullet(A, M)$ with a structure of a $\mathfrak{g}$-*complex*, which is a $\mathbb{Z}$-grading preserving Lie superalgebra homomorphism $\varphi : \widehat{\mathfrak{g}} \to \mathrm{End}_{\mathbb{F}} \widetilde{\Gamma}^\bullet(A, M)$, such that $\varphi(\partial_\eta) = \delta$. We also show that $\varphi(\widehat{\mathfrak{g}})$ lies in the subalgebra of derivations of the superalgebra $\widetilde{\Gamma}^\bullet(A, M)$. For each $X \in \mathfrak{g}$ we thus have the Lie derivative $L_X = \varphi(X)$ and the contraction operator $\iota_X = \varphi(\eta X)$, satisfying all the usual relations, in particular, the Cartan formula $L_X = \iota_X \delta + \delta \iota_X$.

Denoting by $\mathfrak{g}^\partial$ the centralizer of ∂ in $\mathfrak{g}$, we obtain the induced structure of a $\mathfrak{g}^\partial$-complex for $\Gamma^\bullet(A, M)$, which we, furthermore, extend to the larger complex $C^\bullet(A, M)$. Namely, we introduce a canonical Lie algebra bracket on all spaces of 1-chains with reversed parity (see (5)), so that all the maps $\Pi C_1 \twoheadrightarrow \Pi \bar{C}_1 \to \Pi\Gamma_1 \hookrightarrow \Pi\widetilde{\Gamma}_1$ are Lie algebra homomorphisms, and the embeddings (4) are morphisms of complexes, endowed with a corresponding Lie algebra structure.

What does it all have to do with the calculus of variations? In order to explain this, introduce the notion of an *algebra of differential functions* (in ℓ variables). This is a differential algebra, i.e., a unital commutative associative algebra $\mathcal{V}$ with a derivation ∂, endowed with commuting derivations $\frac{\partial}{\partial u_i^{(n)}}$, $i \in I = \{1, \ldots, \ell\}$, $n \in \mathbb{Z}_+$, such that only a finite number of $\frac{\partial f}{\partial u_i^{(n)}}$ are non-zero for each $f \in \mathcal{V}$, and the following commutation rules with ∂ hold:

$$\left[\frac{\partial}{\partial u_i^{(n)}}, \partial\right] = \frac{\partial}{\partial u_i^{(n-1)}} \quad \text{(the RHS is 0 if } n = 0) \,. \tag{6}$$

An important example is the algebra of differential polynomials $\mathbb{F}[u_i^{(n)} | i \in I, n \in \mathbb{Z}_+]$ with $\partial(u_i^{(n)}) = u_i^{(n+1)}$, $n \in \mathbb{Z}_+$, $i \in I$. Other examples include any localization by a multiplicative subset or any algebraic extension of this algebra.

The basic de Rham complex $\widetilde{\Omega}^\bullet = \widetilde{\Omega}^\bullet(\mathcal{V})$ over $\mathcal{V}$ is defined as an exterior superalgebra over the free $\mathcal{V}$-module $\widetilde{\Omega}^1 = \sum_{i\in I,\, n\in\mathbb{Z}_+} \mathcal{V}\delta u_i^{(n)}$ on generators $\delta u_i^{(n)}$ with odd parity. We have: $\widetilde{\Omega}^\bullet = \bigoplus_{k\in\mathbb{Z}_+} \widetilde{\Omega}^k$, where $\widetilde{\Omega}^0 = \mathcal{V}$, $\widetilde{\Omega}^k = \Lambda^k_{\mathcal{V}} \widetilde{\Omega}^1$. This $\mathbb{Z}$-graded superalgebra is endowed by an odd derivation δ of degree 1, such that $\delta f = \sum_{i\in I,\, n\in\mathbb{Z}_+} \frac{\partial f}{\partial u_i^{(n)}} \delta u_i^{(n)}$ for $f \in \widetilde{\Omega}^0$ and $\delta(\delta u_i^{(n)}) = 0$. One easily checks that $\delta^2 = 0$, so that $\widetilde{\Omega}^\bullet$ is a cohomology complex.

Let $\mathfrak{g}$ be the Lie algebra of derivations of the algebra $\mathcal{V}$ of the form

$$X = \sum_{i\in I,\, n\in\mathbb{Z}_+} P_{i,n} \frac{\partial}{\partial u_i^{(n)}}, \qquad \text{where } \; P_{i,n} \in \mathcal{V}. \tag{7}$$

To any such derivation X we associate an even derivation L_X (Lie derivative) and an odd derivation ι_X (contraction) of the superalgebra $\widetilde{\Omega}^\bullet$ by letting $L_X|_{\mathcal{V}} = X,\ L_X(\delta u_i^{(n)}) = \delta P_{i,n},\ \iota_X|_{\mathcal{V}} = 0,\ \iota_X(\delta u_i^{(n)}) = P_{i,n}$. This provides $\widetilde{\Omega}^\bullet$ with a structure of a $\mathfrak{g}$-complex, by letting $\varphi(X) = L_X$ and $\varphi(\eta X) = \iota_X$. Also, the derivation ∂ extends to an (even) derivation of $\widetilde{\Omega}^\bullet$ by letting $\partial(\delta u_i^{(n)}) = \delta u_i^{(n+1)}$.

It is easy to check, using (6), that ∂ and δ commute, hence we can consider the reduced complex

$$\Omega^\bullet(\mathcal{V}) = \widetilde{\Omega}^\bullet(\mathcal{V}) / \partial\widetilde{\Omega}^\bullet(\mathcal{V}),$$

which is called the *variational complex*. This is, of course, a $\mathfrak{g}^\partial$-complex.

Our main observation is the interpretation of the variational complex $\Omega^\bullet(\mathcal{V})$ in terms of Lie conformal algebra cohomology, given by Theorem 1 below.

Let $R = \bigoplus_{i\in I} \mathbb{F}[\partial]u_i$ be a free $\mathbb{F}[\partial]$-module of rank ℓ, endowed with the trivial λ-bracket $[a_\lambda b] = 0$ for all $a, b \in R$. Let $\mathcal{V}$ be an algebra of differential functions.

We endow $\mathcal{V}$ with the structure of an R-module by letting

$$u_{i_\lambda} f = \sum_{n \in \mathbb{Z}_+} \lambda^n \frac{\partial f}{\partial u_i^{(n)}}, \quad i \in I,$$

and extending to R by sesquilinearity. Let $\mathfrak{g}$ be the Lie algebra of derivations of $\mathcal{V}$ of the form (7), and let $\mathfrak{g}^\partial$ be the subalgebra of $\mathfrak{g}$, consisting of derivations commuting with ∂.

Theorem 1. *The $\mathfrak{g}^\partial$-complexes $C^\bullet(R, \mathcal{V})$ and $\Omega^\bullet(\mathcal{V})$ are isomorphic.*

As a result, we obtain the following interpretation of the complex $\Omega^\bullet(\mathcal{V})$, which explains the name "calculus of variations".

We have: $\Omega^0 = \mathcal{V}/\partial\mathcal{V}$, $\Omega^1 = \mathrm{Hom}_{\mathbb{F}[\partial]}(R, \mathcal{V}) = \mathcal{V}^{\oplus \ell}$. Elements of Ω^0 are called local functionals and the image of $f \in \mathcal{V}$ in Ω^0 is denoted by $\int f$. Elements of Ω^1 are called local 1-forms. The differential $\delta : \Omega^0 \to \Omega^1$ is identified with the variational derivative: $\delta \int f = \left(\frac{\delta \int f}{\delta u_i} \right)_{i \in I} = \frac{\delta f}{\delta u}$, where

$$\frac{\delta f}{\delta u_i} = \sum_{n \in \mathbb{Z}_+} (-\partial)^n \frac{\partial f}{\partial u_i^{(n)}}. \tag{8}$$

Furthermore, the space of 2-cochains C^2 is identified with the space of skew-adjoint differential operators by associating to a λ-bracket $\{\cdot\,_\lambda\,\cdot\} : R^{\otimes 2} \to \mathbb{F}[\lambda] \otimes \mathcal{V}$ the $\ell \times \ell$ matrix $S_{ij}(\partial) = \{u_{j\,\partial} u_i\}_\to$, where the arrow means that ∂ is moved to the right. The differential $\delta : \Omega^1 \to \Omega^2$ is expressed in terms of the Frechet derivative

$$D_F(\partial)_{ij} = \sum_{n \in \mathbb{Z}_+} \frac{\partial F_i}{\partial u_j^{(n)}} \partial^n, \qquad i, j \in I, \tag{9}$$

which defines an $\mathbb{F}$-linear map: $\mathcal{V}^\ell \to \mathcal{V}^{\oplus \ell}$. Namely: $\delta F = D_F(\partial) - D_F(\partial)^*$. The subspace of closed 2-cochains in C^2 is identified with the space of symplectic differential operators.

A 2-cochain, which is a skew-adjoint differential operator $S_{ij}(\partial)$, can be identified with the corresponding $\mathbb{F}$-linear map $(\mathcal{V}^\ell)^2 \to \mathcal{V}/\partial\mathcal{V}$, of "differential type", given by

$$S(P, Q) = \int \sum_{i,j \in I} Q_i S_{ij}(\partial) P_j.$$

Skew-adjointness of S translates to the skew-symmetry condition $S(P, Q) = -S(Q, P)$.

More generally, the space of k-cochains C^k for $k \geq 2$ is identified with the space of all skew-symmetric $\mathbb{F}$-linear maps $S : (\mathcal{V}^\ell)^k \to \mathcal{V}/\partial\mathcal{V}$, of "differential type":

$$S(P^1, \ldots, P^k) = \int \sum_{\substack{i_1, \ldots, i_k \in I \\ n_1, \ldots, n_k \in \mathbb{Z}_+}} f_{i_1, \ldots, i_k}^{n_1, \ldots, n_k} (\partial^{n_1} P_{i_1}^1) \cdots (\partial^{n_k} P_{i_k}^k),$$

where $f_{i_1, \ldots, i_k}^{n_1, \ldots, n_k} \in \mathcal{V}$. The skew-symmetry condition is simply $S(P^1, \ldots, P^k) = \mathrm{sign}(\sigma) S(P^{\sigma(1)}, \ldots, P^{\sigma(k)})$, for every $\sigma \in S_k$. The subspace of closed k-cochains for $k \geq 2$ is the subspace of "symplectic" k-differential operators.

We prove in [BDK] that the cohomology H^j of the complex $\Omega^\bullet(\mathcal{V})$ is zero for $j \geq 1$ and $H^0 = \mathcal{C}/(\mathcal{C} \cap \partial\mathcal{V})$, where $\mathcal{C} := \{f \in \mathcal{V} \mid \frac{\partial f}{\partial u_i^{(n)}} = 0 \ \forall i \in I, n \in \mathbb{Z}_+\}$, provided that $\mathcal{V}$ is normal, as defined in Sect. 5.6. (Any algebra of differential functions can be included in a normal one.) As a corollary, we obtain (cf. [D]) that Ker $\frac{\delta}{\delta u} = \partial\mathcal{V} + \mathcal{C}$, and $F \in$ Im $\frac{\delta}{\delta u}$ iff $D_F(\partial)$ is a self-adjoint differential operator, provided that $\mathcal{V}$ is normal. The first result can be found in [D] (see also [Di] and [Vi], where it is proved under stronger conditions on $\mathcal{V}$), but it is certainly much older. The second result, at least under stronger conditions on $\mathcal{V}$, goes back to [H], [V]. We also obtain the classification of symplectic differential operators (cf. [D]) and of symplectic poly-differential operators for normal $\mathcal{V}$, which seems to be a new result.

Thus, the interaction between the Lie conformal algebra cohomology and the variational calculus has led to progress in both theories. On the one hand, the variational calculus motivated some of our constructions in the Lie conformal algebra cohomology. On the other hand, the Lie conformal algebra cohomology interpretation of the variational complex has led to a better understanding of this complex and to a classification of symplectic differential operators.

The ground field is an arbitrary field $\mathbb{F}$ of characteristic 0.

We wish to thank Bojko Bakalov for very valuable comments, in particular, for the observation that our complex $C^\bullet(A, M)$ is isomorphic to the complex in [BDAK, Sect. 15.1], in the case when the Hopf algebra H is $\mathbb{F}[\partial]$.

2. Lie Conformal Algebra Cohomology Complexes

2.1. The basic cohomology complex $\widetilde{\Gamma}^\bullet$ and the reduced cohomology complex $\Gamma^\bullet$. Let us review, following [BKV], the definition of the basic and reduced cohomology complexes associated to a Lie conformal algebra A and an A-module M. A *k-cochain* of A with coefficients in M is an $\mathbb{F}$-linear map

$$\widetilde{\gamma} : A^{\otimes k} \to \mathbb{F}[\lambda_1, \ldots, \lambda_k] \otimes M, \quad a_1 \otimes \cdots \otimes a_k \mapsto \widetilde{\gamma}_{\lambda_1,\ldots,\lambda_k}(a_1, \ldots, a_k),$$

satisfying the following two conditions:

A1. $\widetilde{\gamma}_{\lambda_1,\ldots,\lambda_k}(a_1, \ldots, \partial a_i, \ldots, a_k) = -\lambda_i \widetilde{\gamma}_{\lambda_1,\ldots,\lambda_k}(a_1, \ldots, a_k)$ for all i,
A2. $\widetilde{\gamma}$ is skew-symmetric w.r.t. simultaneous permutations of the a_i's and the λ_i's.

Remark 1. Note that Condition A1 implies that $\widetilde{\gamma}_{\lambda_1,\ldots,\lambda_k}(a_1, \ldots, a_k)$ is zero if one of the elements a_i is a torsion element of the $\mathbb{F}[\partial]$-module A.

We let $\widetilde{\Gamma}^k = \widetilde{\Gamma}^k(A, M)$ be the space of all k-cochains, and $\widetilde{\Gamma}^\bullet = \widetilde{\Gamma}^\bullet(A, M) = \bigoplus_{k \geq 0} \widetilde{\Gamma}^k$. The differential δ of a k-cochain $\widetilde{\gamma}$ is defined by the following formula:

$$\begin{aligned}(\delta\widetilde{\gamma})_{\lambda_1,\ldots,\lambda_{k+1}}(a_1, \ldots, a_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i+1} a_{i\,\lambda_i} \Big(\widetilde{\gamma}_{\lambda_1, \stackrel{i}{\check{\cdots}}, \lambda_{k+1}}(a_1, \stackrel{i}{\check{\cdots}}, a_{k+1}) \Big) \\ &+ \sum_{\substack{i,j=1 \\ i<j}}^{k+1} (-1)^{k+i+j+1} \widetilde{\gamma}_{\lambda_1, \stackrel{i}{\check{\cdots}} \stackrel{j}{\check{\cdots}}, \lambda_{k+1}, \lambda_i + \lambda_j}(a_1, \stackrel{i}{\check{\cdots}} \stackrel{j}{\check{\cdots}}, a_{k+1}, [a_{i\,\lambda_i} a_j]). \end{aligned} \tag{10}$$

One checks that δ maps $\widetilde{\Gamma}^k$ to $\widetilde{\Gamma}^{k+1}$, and that $\delta^2 = 0$. The $\mathbb{Z}$-graded space $\widetilde{\Gamma}^\bullet(A, M)$ with the differential δ is called the *basic cohomology complex* associated to A and M.

Define the structure of an $\mathbb{F}[\partial]$-module on $\widetilde{\Gamma}^\bullet$ by letting

$$(\partial\widetilde{\gamma})_{\lambda_1,\dots,\lambda_k}(a_1,\dots,a_k) = (\partial^M + \lambda_1 + \dots + \lambda_k)\left(\widetilde{\gamma}_{\lambda_1,\dots,\lambda_k}(a_1,\dots,a_k)\right), \qquad (11)$$

where ∂^M denotes the action of ∂ on M. One checks that δ and ∂ commute, and therefore $\partial\widetilde{\Gamma}^\bullet \subset \widetilde{\Gamma}^\bullet$ is a subcomplex. We can consider the *reduced cohomology complex* $\Gamma^\bullet(A,M) = \widetilde{\Gamma}^\bullet(A,M)/\partial\widetilde{\Gamma}^\bullet(A,M) = \bigoplus_{k\in\mathbb{Z}_+}\Gamma^k(A,M)$. For example, $\Gamma^0 = M/\partial^M M$, and we denote, as in the calculus of variations, by $\int m$ the image of $m \in M$ in $M/\partial^M M$. As before we let, for brevity, $\Gamma^\bullet = \Gamma^\bullet(A,M)$ and $\Gamma^k = \Gamma^k(A,M)$, $k \in \mathbb{Z}_+$.

In the following sections we will find a simpler construction of the reduced cohomology complex $\Gamma^\bullet$, in terms of poly λ-brackets.

2.2. Poly λ-brackets. Let A and M be $\mathbb{F}[\partial]$-modules, and, as before, denote by ∂^M the action of ∂ on M. For $k \geq 1$, a *k-λ-bracket* on A with coefficients in M is, by definition, an $\mathbb{F}$-linear map $c : A^{\otimes k} \to \mathbb{F}[\lambda_1,\dots,\lambda_{k-1}] \otimes M$, denoted by

$$a_1 \otimes \dots \otimes a_k \mapsto \{a_1{}_{\lambda_1} \cdots a_{k-1}{}_{\lambda_{k-1}} a_k\}_c,$$

satisfying the following conditions:

B1. $\{a_1{}_{\lambda_1} \cdots (\partial a_i)_{\lambda_i} \cdots a_{k-1}{}_{\lambda_{k-1}} a_k\}_c = -\lambda_i \{a_1{}_{\lambda_1} \cdots a_{k-1}{}_{\lambda_{k-1}} a_k\}_c$, for $1 \leq i \leq k-1$;
B2. $\{a_1{}_{\lambda_1} \cdots a_{k-1}{}_{\lambda_{k-1}} (\partial a_k)\}_c = (\lambda_1 + \dots + \lambda_{k-1} + \partial^M)\{a_1{}_{\lambda_1} \cdots a_{k-1}{}_{\lambda_{k-1}} a_k\}_c$;
B3. c is skew-symmetric with respect to simultaneous permutations of the a_i's and the λ_i's in the sense that, for every permutation σ of the indices $\{1,\dots,k\}$, we have:

$$\{a_1{}_{\lambda_1} \cdots a_{k-1}{}_{\lambda_{k-1}} a_k\}_c = \mathrm{sign}(\sigma)\{a_{\sigma(1)}{}_{\lambda_{\sigma(1)}} \cdots a_{\sigma(k-1)}{}_{\lambda_{\sigma(k-1)}} a_{\sigma(k)}\}_c \Big|_{\lambda_k \mapsto \lambda_k^\dagger}.$$

The notation in the RHS means that λ_k is replaced by $\lambda_k^\dagger = -\sum_{j=1}^{k-1}\lambda_j - \partial^M$, if it occurs, and ∂^M is moved to the left.

Remark 2. A structure of a Lie conformal algebra on A is a 2-λ-bracket on A with coefficients in A, satisfying the Jacobi identity (3).

We let $C^0 = M/\partial^M M$ and, for $k \geq 1$, we denote by $C^k = C^k(A,M)$ the space of all k-λ-brackets on A with coefficients in M. For example, C^1 is the space of all $\mathbb{F}[\partial]$-module homomorphisms $c : A \to M$. We let $C^\bullet = \bigoplus_{k\in\mathbb{Z}_+} C^k$, the space of all *poly λ-brackets*.

We also define $\bar{C}^\bullet = \bigoplus_{k\in\mathbb{Z}_+}\bar{C}^k$, where $\bar{C}^0 = C^0 = M/\partial^M M$, and $\bar{C}^k \subset C^k$ is the subspace of k-λ-brackets c with the following additional property: $\{a_1{}_{\lambda_1} \cdots a_{k-1}{}_{\lambda_{k-1}} a_k\}_c$ is zero if one of the elements a_i is a torsion element in A. Clearly, $\bar{C}^1$ needs not be equal to C^1. On the other hand, it is easy to check, using the sesquilinearity Conditions B1 and B2, that $\bar{C}^k = C^k$ for $k \geq 2$.

2.3. The complex of poly λ-brackets. We next define a differential d on the space $C^\bullet$ of poly λ-brackets such that $d(C^k) \subset C^{k+1}$ and $d^2 = 0$, thus making $C^\bullet$ a cohomology complex.

For $\int m \in C^0 = M/\partial^M M$, we let $d\int m \in C^1$ be the following $\mathbb{F}[\partial]$-module homomorphism:

$$\left(d\textstyle\int m\right)(a) \left(= \{a\}_{d\int m}\right) := a_{-\partial^M} m. \tag{12}$$

This is well defined since, if $m \in \partial^M M$, the RHS is zero due to sesquilinearity. For $c \in C^k$, with $k \geq 1$, we let $dc \in C^{k+1}$ be the following poly λ-bracket:

$$\begin{aligned}
\{a_{1\lambda_1} \cdots a_{k\lambda_k} a_{k+1}\}_{dc} &:= \sum_{i=1}^{k} (-1)^{i+1} a_{i\lambda_i} \left\{ a_{1\lambda_1} \overset{i}{\check{\cdots}} a_{k\lambda_k} a_{k+1} \right\}_c \\
&+ \sum_{\substack{i,j=1 \\ i<j}}^{k} (-1)^{k+i+j+1} \left\{ a_{1\lambda_1} \overset{i}{\check{\cdots}}\overset{j}{\check{\cdots}} a_{k\lambda_k} a_{k+1\,\lambda_{k+1}^\dagger} [a_{i\lambda_i} a_j] \right\}_c \\
&+ (-1)^k a_{k+1\,\lambda_{k+1}^\dagger} \left\{ a_{1\lambda_1} \cdots a_{k-1\,\lambda_{k-1}} a_k \right\}_c \\
&+ \sum_{i=1}^{k} (-1)^i \left\{ a_{1\lambda_1} \overset{i}{\check{\cdots}} a_{k\lambda_k} [a_{i\lambda_i} a_{k+1}] \right\}_c,
\end{aligned} \tag{13}$$

where, as before, $\lambda_{k+1}^\dagger = -\sum_{j=1}^k \lambda_j - \partial^M$, and ∂^M is moved to the left.

For example, for an $\mathbb{F}[\partial]$-module homomorphism $c : A \to M$, we have

$$\{a_\lambda b\}_{dc} = a_\lambda c(b) - b_{-\lambda-\partial} c(a) - c([a_\lambda b]). \tag{14}$$

Proposition 1. *(a) For $c \in C^k$, we have $d(c) \in C^{k+1}$ and $d^2(c) = 0$. This makes $(C^\bullet, d)$ a cohomology complex.*
(b) $d(\bar{C}^k) \subset \bar{C}^{k+1}$ for all $k \geq 0$. Hence $(\bar{C}^\bullet, d)$ is a cohomology subcomplex of $(C^\bullet, d)$.

Proof. We prove part (b) first. For $k \geq 1$ there is nothing to prove. For $k = 0$ just notice that, if $\int m \in M/\partial^M M$ and $a \in A$ is a torsion element, then, by (12), we have $\left(d\int m\right)(a) = 0$, since torsion elements of A act trivially in any module [K]. Hence $d\int m \in \bar{C}^1$. In order to prove part (a) we have to check that, if $c \in C^k$, then dc, defined by (12) and (13), satisfies Conditions B1, B2, B3, and $d(dc) = 0$. To simplify the arguments, we rewrite (13) in a concise form:

$$\begin{aligned}
\{a_{1\lambda_1} \cdots a_{k\lambda_k} a_{k+1}\}_{dc} &:= \left(\sum_{i=1}^{k+1} (-1)^{i+1} a_{i\lambda_i} \left\{ a_{1\lambda_1} \overset{i}{\check{\cdots}} a_{k\lambda_k} a_{k+1} \right\}_c \right. \\
&+ \left. \sum_{\substack{i,j=1 \\ i<j}}^{k+1} (-1)^{k+i+j+1} \left\{ a_{1\lambda_1} \overset{i}{\check{\cdots}}\overset{j}{\check{\cdots}} a_{k+1\,\lambda_{k+1}} [a_{i\lambda_i} a_j] \right\}_c \right) \Bigg|_{\lambda_{k+1} = \lambda_{k+1}^\dagger},
\end{aligned} \tag{15}$$

where the RHS is evaluated at $\lambda_{k+1} = \lambda_{k+1}^\dagger = -\sum_{j=1}^k \lambda_j - \partial^M$, with ∂^M acting from the left. The above equation should be interpreted by saying that, in the first term in the RHS, for $i = k+1$, the last index λ_k does not appear in the poly λ-bracket. Let us replace a_h by ∂a_h in Eq. (15). It is not hard to check, using Conditions B1 and B2 for c and the sesquilinearity of the λ-action of A on M, that, for $1 \leq h \leq k$, each term in the RHS of (15) gets multiplied by $-\lambda_h$, while, for $h = k+1$, each term gets multiplied by $-\lambda_{k+1}^\dagger = \sum_{j=1}^k \lambda_j + \partial^M$. Hence dc satisfies Conditions B1 and B2. In order to prove B3., let σ be a permutation of the set $\{1, \dots, k+1\}$. A basic observation is that, if we *first* replace $\lambda_{\sigma(k+1)}$ by $\lambda_{\sigma(k+1)}^\dagger = -\lambda_1 - \overset{\sigma(k+1)}{\check{\cdots}} - \lambda_{k+1} - \partial^M$, and *then* λ_{k+1} by $\lambda_{k+1}^\dagger = -\lambda_1 \cdots - \lambda_k - \partial^M$, as a result $\lambda_{\sigma(k+1)}$ stays unchanged. Notice, moreover, that, for $1 \leq i \leq k+1$, $\{\sigma(1), \overset{i}{\check{\cdots}}, \sigma(k+1)\}$ is a permutation of $\{1, \overset{\sigma(i)}{\check{\cdots}}, k+1\}$, and its sign is $(-1)^{i+\sigma(i)}\text{sign}(\sigma)$. Hence, using the assumption B3 on c, we get

$$
\begin{aligned}
& a_{\sigma(i)}{}_{\lambda_{\sigma(i)}} \left\{ a_{\sigma(1)}{}_{\lambda_{\sigma(1)}} \overset{i}{\check{\cdots}} a_{\sigma(k)}{}_{\lambda_{\sigma(k)}} a_{\sigma(k+1)} \right\}_c \Big|_{\lambda_{k+1}=\lambda_{k+1}^\dagger} \\
&\quad = \text{sign}(\sigma)(-1)^{i+\sigma(i)} a_{\sigma(i)}{}_{\lambda_{\sigma(i)}} \left\{ a_{1\lambda_1} \overset{\sigma(i)}{\check{\cdots}} a_{k\lambda_k} a_{k+1} \right\}_c \Big|_{\lambda_{k+1}=\lambda_{k+1}^\dagger} .
\end{aligned}
\tag{16}
$$

Similarly, for the second term in (15), we notice that $\{\sigma(1), \overset{i}{\check{\cdot}} \cdots \overset{j}{\check{\cdot}}, \sigma(k+1)\}$ is a permutation of $\{1, \overset{\sigma(i)\sigma(j)}{\check{\cdots}\check{\cdots}}, k+1\}$, and its sign is $(-1)^{i+j+\sigma(i)+\sigma(j)}\text{sign}(\sigma)$ if $\sigma(i) < \sigma(j)$, and it is $(-1)^{i+j+\sigma(i)+\sigma(j)+1}\text{sign}(\sigma)$ if $\sigma(i) > \sigma(j)$. Hence, for $\sigma(i) < \sigma(j)$ we have

$$
\begin{aligned}
& \left\{ a_{\sigma(1)}{}_{\lambda_{\sigma(1)}} \overset{i}{\check{\cdots}} \overset{j}{\check{\cdots}} a_{\sigma(k+1)}{}_{\lambda_{\sigma(k+1)}} [a_{\sigma(i)}{}_{\lambda_{\sigma(i)}} a_{\sigma(j)}] \right\}_c \Big|_{\lambda_{k+1}=\lambda_{k+1}^\dagger} \\
&\quad = \text{sign}(\sigma)(-1)^{i+j+\sigma(i)+\sigma(j)} \\
&\qquad \times \left\{ a_{1\lambda_1} \overset{\sigma(i)\sigma(j)}{\check{\cdots}\check{\cdots}} a_{k+1}{}_{\lambda_{k+1}} [a_{\sigma(i)}{}_{\lambda_{\sigma(i)}} a_{\sigma(j)}] \right\}_c \Big|_{\lambda_{k+1}=\lambda_{k+1}^\dagger} ,
\end{aligned}
\tag{17}
$$

while for $\sigma(i) > \sigma(j)$ we have, by the skew-symmetry of the λ-bracket in A,

$$
\begin{aligned}
& \left\{ a_{\sigma(1)}{}_{\lambda_{\sigma(1)}} \overset{i}{\check{\cdots}} \overset{j}{\check{\cdots}} a_{\sigma(k+1)}{}_{\lambda_{\sigma(k+1)}} [a_{\sigma(i)}{}_{\lambda_{\sigma(i)}} a_{\sigma(j)}] \right\}_c \Big|_{\lambda_{k+1}=\lambda_{k+1}^\dagger} \\
& = \text{sign}(\sigma)(-1)^{i+j+\sigma(i)+\sigma(j)} \\
&\qquad \times \left\{ a_{1\lambda_1} \overset{\sigma(j)\sigma(i)}{\check{\cdots}\check{\cdots}} a_{\sigma(k+1)}{}_{\lambda_{k+1}} [a_{\sigma(j)}{}_{-\lambda_{\sigma(i)}-\partial} a_{\sigma(i)}] \right\}_c \Big|_{\lambda_{k+1}=\lambda_{k+1}^\dagger} \\
& = \text{sign}(\sigma)(-1)^{i+j+\sigma(i)+\sigma(j)} \\
&\qquad \times \left\{ a_{1\lambda_1} \overset{\sigma(j)\sigma(i)}{\check{\cdots}\check{\cdots}} a_{\sigma(k+1)}{}_{\lambda_{k+1}} [a_{\sigma(j)}{}_{\lambda_{\sigma(j)}} a_{\sigma(i)}] \right\}_c \Big|_{\lambda_{k+1}=\lambda_{k+1}^\dagger} .
\end{aligned}
\tag{18}
$$

In the last identity we used the assumption that c satisfies condition B2. Clearly, Eqs. (16), (17) and (18), together with the definition (15) of dc, imply that dc satisfies condition B3. We are left to prove that $d^2c = 0$. We have, by (15),

$$\{a_{1\lambda_1}\cdots a_{k+1\,\lambda_{k+1}}a_{k+2}\}_{d^2c} = \Bigg(\sum_{i=1}^{k+2}(-1)^{i+1}a_{i\,\lambda_i}\left\{a_{1\lambda_1}\overset{i}{\check{\cdots}}a_{k+1\,\lambda_{k+1}}a_{k+2}\right\}_{dc}$$
$$+\sum_{\substack{i,j=1\\ i<j}}^{k+2}(-1)^{k+i+j}\left\{a_{1\lambda_1}\overset{i}{\check{\cdots}}\overset{j}{\check{\cdots}}a_{k+2\,\lambda_{k+2}}[a_{i\,\lambda_i}a_j]\right\}_{dc}\Bigg)\Bigg|_{\lambda_{k+2}=\lambda_{k+2}^{\dagger}}, \tag{19}$$

where, in the RHS, we replace λ_{k+2} by $\lambda_{k+2}^{\dagger} = -\sum_{j=1}^{k+1}\lambda_j - \partial^M$, and ∂^M is moved to the left. Again by (15) and by sesquilinearity of the λ-action of A on M, the first term in the RHS of (19) is

$$\Bigg(\sum_{\substack{i,j=1\\ i\neq j}}^{k+2}(-1)^{i+j}\epsilon(i,j)a_{j\,\lambda_j}\left(a_{i\,\lambda_i}\left\{a_{1\lambda_1}\overset{i}{\check{\cdots}}\overset{j}{\check{\cdots}}a_{k+1\,\lambda_{k+1}}a_{k+2}\right\}_c\right)$$
$$+\sum_{\substack{i,j,h=1\\ i<j\\ i,j\neq h}}^{k+2}(-1)^{k+i+j+h}\epsilon(i,h)\epsilon(j,h)$$
$$\times\, a_{h\,\lambda_h}\left\{a_{1\lambda_1}\overset{i}{\check{\cdots}}\overset{j}{\check{\cdots}}\overset{h}{\check{\cdots}}a_{k+2\,\lambda_{k+2}}[a_{i\,\lambda_i}a_j]\right\}_c\Bigg)\Bigg|_{\lambda_{k+2}=\lambda_{k+2}^{\dagger}}, \tag{20}$$

where $\epsilon(i,j)$ is +1 if $i<j$ and -1 if $i>j$. Similarly, by (13) the second term in the RHS of (19) is

$$\Bigg(\sum_{\substack{i,j,h=1\\ i<j\\ i,j\neq h}}^{k+2}(-1)^{k+i+j+h+1}\epsilon(h,i)\epsilon(h,j)a_{h\,\lambda_h}\left\{a_{1\lambda_1}\overset{h}{\check{\cdots}}\overset{i}{\check{\cdots}}\overset{j}{\check{\cdots}}a_{k+2\,\lambda_{k+2}}[a_{i\,\lambda_i}a_j]\right\}_c$$
$$+\sum_{\substack{i,j=1\\ i<j}}^{k+2}(-1)^{i+j}[a_{i\,\lambda_i}a_j]_{\lambda_i+\lambda_j}\left\{a_{1\lambda_1}\overset{i}{\check{\cdots}}\overset{j}{\check{\cdots}}a_{k+1\,\lambda_{k+1}}a_{k+2}\right\}_c$$
$$+\sum_{\substack{i,j,p,q=1\\ i<j,\,p<q\\ \{i,j\}\cap\{p,q\}=\emptyset}}^{k+2}(-1)^{i+j+p+q}\epsilon(p,i)\epsilon(p,j)\epsilon(q,i)\epsilon(q,j) \tag{21}$$
$$\times\left\{a_{1\lambda_1}\overset{p}{\check{\cdots}}\overset{q}{\check{\cdots}}\overset{i}{\check{\cdots}}\overset{j}{\check{\cdots}}a_{k+2\,\lambda_{k+2}}[a_{i\,\lambda_i}a_j]_{\lambda_i+\lambda_j}[a_{p\,\lambda_p}a_q]\right\}_c$$

$$+\sum_{\substack{i,j,h=1\\ i<j\\ i,j\neq h}}^{k+2}(-1)^{k+i+j+h}\epsilon(h,i)\epsilon(h,j)$$

$$\left.\left.\times\left\{a_{1\lambda_1}\overset{h}{\check{\cdots}}\overset{i}{\check{\cdots}}\overset{j}{\check{\cdots}}a_{k+2\,\lambda_{k+2}}[a_{h\,\lambda_h}[a_{i\,\lambda_i}a_j]]\right\}_c\right)\right|_{\lambda_{k+2}=\lambda_{k+2}^\dagger}.$$

Notice that the first term in (20) is the negative of the second term in (21), and the second term in (20) is the negative of the first term in (21). Moreover, it is not hard to check, using the Jacobi identity for the λ-bracket on A, that the last term in (21) is identically zero, and, using the skew-symmetry Condition B3 on c, that also the third term in (21) is zero. In conclusion, $d^2c = 0$, as we wanted. □

In the next section we shall embed the cohomology complex $\Gamma^\bullet$, introduced in Sect. 1.1, in the cohomology complex $\bar{C}^\bullet$, and we shall prove that, if the $\mathbb{F}[\partial]$-module A decomposes as a direct sum of the torsion and a free submodule, then this embedding is an isomorphism. We believe that the (slightly) bigger cohomology complex $C^\bullet$ is a more natural and a more correct definition for the Lie conformal algebra cohomolgy complex. This will be clear when interpreting in Sect. 3 the cohomology $H(C^\bullet, d)$ in terms of abelian Lie conformal algebra extensions of A by the module M.

2.4. Isomorphism of the cohomology complexes $\Gamma^\bullet$ and $\bar{C}^\bullet$. We define, for $k \geq 1$, an $\mathbb{F}$-linear map $\psi^k : \widetilde{\Gamma}^k \to C^k$, as follows. Given $\widetilde{\gamma} \in \widetilde{\Gamma}^k$, we define $\psi^k(\widetilde{\gamma}) : A^{\otimes k} \to \mathbb{F}[\lambda_1, \dots, \lambda_{k-1}] \otimes M$, by:

$$\{a_{1\lambda_1}\cdots a_{k-1\,\lambda_{k-1}}a_k\}_{\psi^k(\widetilde{\gamma})} = \widetilde{\gamma}_{\lambda_1,\dots,\lambda_{k-1},\lambda_k^\dagger}(a_1,\dots,a_k), \tag{22}$$

where, as before, $\lambda_k^\dagger = -\sum_{j=1}^{k-1}\lambda_j - \partial^M$, and ∂^M is moved to the left.

Lemma 1. *(a) For $\widetilde{\gamma} \in \widetilde{\Gamma}^k$, we have $\psi^k(\widetilde{\gamma}) \in \bar{C}^k$.*

(b) We have Ker $(\psi^k) = \partial\widetilde{\Gamma}^k$. *Hence ψ^k induces an injective $\mathbb{F}$-linear map $\psi^k : \Gamma^k = \widetilde{\Gamma}^k/\partial\widetilde{\Gamma}^k \hookrightarrow \bar{C}^k \subset C^k$.*

(c) Suppose that the Lie conformal algebra A decomposes, as $\mathbb{F}[\partial]$-module, as

$$A = T \oplus (\mathbb{F}[\partial] \otimes U)\,, \tag{23}$$

where T is the torsion of A and $\bar{A} = \mathbb{F}[\partial] \otimes U$ is a complementary free submodule. Then $\psi^k(\widetilde{\Gamma}^k) = \bar{C}^k$, hence ψ^k induces a bijective $\mathbb{F}$-linear map $\psi^k : \Gamma^k \xrightarrow{\sim} \bar{C}^k$.

Proof. Let $\widetilde{\gamma} \in \widetilde{\Gamma}^k$, and consider $c = \psi^k(\widetilde{\gamma})$. We want to prove that $c \in \bar{C}^k$. It is clear that c satisfies Conditions B1 and B2. Let us check that it also satisfies Condition B3. Let σ be a permutation of the set $\{1, \dots, k\}$, and let $i = \sigma(k)$. Since $\widetilde{\gamma}$ satisfies the skew-symmetry Condition A2, we have

$$\begin{aligned}\{a_{\sigma(1)\,\lambda_{\sigma(1)}}\cdots a_{\sigma(k-1)\,\lambda_{\sigma(k-1)}}a_{\sigma(k)}\}_c &= \widetilde{\gamma}_{\lambda_{\sigma(1)},\dots,\lambda_{\sigma(k-1)},\lambda_{\sigma(k)}^\dagger}(a_{\sigma(1)},\dots,a_{\sigma(k)})\\ &= \mathrm{sign}(\sigma)\widetilde{\gamma}_{\lambda_1,\dots,\lambda_i^\dagger,\dots,\lambda_k}(a_1,\dots,a_k).\end{aligned} \tag{24}$$

If we then replace λ_k by $\lambda_k^\dagger$, as prescribed by Condition B3, we get

$$\lambda_i^\dagger \mapsto -\lambda_1 - \stackrel{i}{\check{\cdots}} - \lambda_{k-1} - \lambda_k^\dagger - \partial^M = \lambda_i. \tag{25}$$

Therefore the RHS of (24) becomes $\mathrm{sign}(\sigma)\{a_{1\lambda_1}\cdots a_{k-1\lambda_{k-1}} a_k\}_c$, as required. It is also clear that c vanishes on the torsion of A, thanks to Remark 1, so that $c \in \bar{C}^k$. This proves part (a).

By the definition (11) of the action of ∂ on $\widetilde{\Gamma}^k$, and the definition (22) of ψ^k, we have

$$\{a_{1\lambda_1}\cdots a_{k-1\lambda_{k-1}} a_k\}_{\psi^k \partial\widetilde{\gamma}} = (\partial\widetilde{\gamma})_{\lambda_1,\dots,\lambda_{k-1},\lambda_k^\dagger}(a_1,\dots,a_k) = 0,$$

since $-\lambda_1 - \cdots - \lambda_{k-1} - \lambda_k^\dagger - \partial^M = 0$. Hence $\partial\widetilde{\Gamma}^k \subset \mathrm{Ker}\,(\psi^k)$. For the opposite inclusion, let $\widetilde{\gamma} \in \mathrm{Ker}\,(\psi^k)$. Namely, $\widetilde{\gamma}_{\lambda_1,\dots,\lambda_{k-1},\lambda_k^\dagger}(a_1,\dots,a_k) = 0$. By Taylor expanding in $\lambda_k^\dagger - \lambda_k$, we have

$$\sum_{n=0}^{\infty} \frac{1}{n!}(-\Lambda - \partial^M)^n \frac{d^n}{d\lambda_k^n}\widetilde{\gamma}_{\lambda_1,\dots,\lambda_k}(a_1,\dots,a_k) = 0, \tag{26}$$

where $\Lambda = \sum_{j=1}^k \lambda_j$. We denote by $\widetilde{\vartheta} : A^{\otimes k} \to \mathbb{F}[\lambda_1,\dots,\lambda_k] \otimes M$ the following $\mathbb{F}$-linear map

$$\widetilde{\vartheta}_{\lambda_1,\dots,\lambda_k}(a_1,\dots,a_k) = \sum_{n=1}^{\infty} \frac{1}{n!}(-\Lambda - \partial^M)^{n-1} \frac{d^n}{d\lambda_k^n}\widetilde{\gamma}_{\lambda_1,\dots,\lambda_k}(a_1,\dots,a_k).$$

Equation (26) can then be rewritten as

$$\widetilde{\gamma}_{\lambda_1,\dots,\lambda_k}(a_1,\dots,a_k) = (\partial^M + \lambda_1 + \cdots + \lambda_k)\widetilde{\vartheta}_{\lambda_1,\dots,\lambda_k}(a_1,\dots,a_k). \tag{27}$$

It follows from Eq. (27) that $\widetilde{\vartheta}$ satisfies Conditions A1 and A2, since $\widetilde{\gamma}$ does. Hence $\widetilde{\vartheta} \in \widetilde{\Gamma}^k$. Equation (27) then implies that $\widetilde{\gamma} = \partial\widetilde{\vartheta} \in \partial\widetilde{\Gamma}^k$, thus proving (b).

Assume next that A decomposes as in (23). We need to prove that, for $c \in \bar{C}^k$, we can find $\widetilde{\gamma} \in \widetilde{\Gamma}^k$ such that

$$\psi^k(\widetilde{\gamma}) = c. \tag{28}$$

Such a k-cochain can be constructed as follows. For $u_1,\dots,u_k \in U$, we let

$$\widetilde{\gamma}_{\lambda_1,\dots,\lambda_k}(u_1,\dots,u_k) = \{u_{1\,\lambda_1 - \frac{\Lambda+\partial^M}{k}} \cdots u_{k-1\,\lambda_{k-1} - \frac{\Lambda+\partial^M}{k}} u_k\}_c, \tag{29}$$

where $\Lambda = \sum_{i=0}^{k-1} \lambda_i$, and we extend it to $(\mathbb{F}[\partial] \otimes U)^{\otimes k}$ by the sesquilinearity Condition A1, and to $A^{\otimes k}$ letting it be zero if one of the arguments is in the torsion T. We need to check that $\widetilde{\gamma}$ satisfies Conditions A1, A2 and (28). Condition A1 is obvious. It suffices to check Condition A2 for elements $a_i = u_i \in U$, $i = 1,\dots,k$. Let σ be a permutation of the indices $\{1,\dots,k\}$. We have,

$$\begin{aligned}&\widetilde{\gamma}_{\lambda_{\sigma(1)},\dots,\lambda_{\sigma(k)}}(u_{\sigma(1)},\dots,u_{\sigma(k)})\\ &= \{u_{\sigma(1)\,\lambda_{\sigma(1)} - \frac{\Lambda+\partial^M}{k}} \cdots u_{\sigma(k-1)\,\lambda_{\sigma(k-1)} - \frac{\Lambda+\partial^M}{k}} u_{\sigma(k)}\}_c.\end{aligned} \tag{30}$$

We then observe that

$$\left(\lambda_k - \frac{\Lambda + \partial^M}{k}\right)^\dagger = -\sum_{i=1}^{k-1}\left(\lambda_i - \frac{\Lambda + \partial^M}{k}\right) - \partial^M = \lambda_k - \frac{\Lambda + \partial^M}{k}.$$

Hence, since c satisfies Condition B3, the RHS of (30) is equal to

$$\mathrm{sign}(\sigma)\{u_1{}_{\lambda_1 - \frac{\Lambda+\partial^M}{k}} \cdots u_{k-1}{}_{\lambda_{k-1} - \frac{\Lambda+\partial^M}{k}} u_k\}_c = \mathrm{sign}(\sigma)\widetilde{\gamma}_{\lambda_1,\dots,\lambda_k}(u_1,\dots,u_k).$$

Finally, we prove that (28) holds. We have, for $u_1,\dots,u_k \in U$,

$$\{u_1{}_{\lambda_1} \cdots u_{k-1}{}_{\lambda_{k-1}} u_k\}_{\psi^k(\widetilde{\gamma})} = \widetilde{\gamma}_{\lambda_1,\dots,\lambda_{k-1},\lambda_k^\dagger}(u_1,\dots,u_k). \tag{31}$$

Note that, if we replace λ_k by $\lambda_k^\dagger$, $\Lambda + \partial^M$ becomes 0. Hence, by the definition (29) of $\widetilde{\gamma}$, the RHS of (31) is equal to $\{u_1{}_{\lambda_1} \cdots u_{k-1}{}_{\lambda_{k-1}} u_k\}_c$. This proves that (28) holds for elements of U. Clearly both sides of (28) are zero if one of the elements a_i is in T. Since both $\psi^k(\widetilde{\gamma})$ and c satisfy the sesquilinearity Conditions B1 and B2, we conclude that (28) holds for every $a_i \in A$. $\square$

Theorem 2. *The identity map on $M/\partial M$ and the maps ψ^k, $k \geq 1$, induce an embedding of cohomology complexes $\Gamma^\bullet \hookrightarrow \bar{C}^\bullet$. If, moreover, the Lie conformal algebra A decomposes, as $\mathbb{F}[\partial]$-module, in a direct sum of a free module and the torsion, this map is an isomorphism of complexes: $\Gamma^\bullet \simeq \bar{C}^\bullet$.*

Proof. By Lemma 1 we already know that ψ^k factors through an injective $\mathbb{F}$-linear map $\psi^k : \Gamma^k \hookrightarrow \bar{C}^k$, and that, if A decomposes as in (23), this map is bijective. Hence, in order to prove the theorem, we only have to prove that the following diagrams are commutative:

$$\begin{tikzcd} & \bar{C}^1 \\ M/\partial^M M \arrow[ur, "d"] \arrow[r, "\delta"'] & \Gamma^1 \arrow[u, "\psi^1"'] \end{tikzcd} \quad , \quad \begin{tikzcd} \bar{C}^k \arrow[r, "d"] & \bar{C}^{k+1} \\ \Gamma^k \arrow[u, "\psi^k"] \arrow[r, "\delta"'] & \Gamma^{k+1} \arrow[u, "\psi^{k+1}"'] \end{tikzcd} \quad , \ \forall k \geq 1. \tag{32}$$

First, given $\int m \in M/\partial^M M$, we have $(\delta m)_\lambda(a) = a_\lambda m$, so that $\left(\psi^1 \delta m\right)(a) = a_{-\partial^M} m = \left(d\int m\right)(a)$, namely the first diagram in (32) is indeed commutative. Next, given $k \geq 1$, let $\widetilde{\gamma} \in \widetilde{\Gamma}^k$ be a representative of $\gamma \in \Gamma^k$. We need to prove that

$$d\psi^k(\widetilde{\gamma}) = \psi^{k+1}(\delta\widetilde{\gamma}). \tag{33}$$

From (13) and (22), we have

$$\{a_1{}_{\lambda_1} \cdots a_k{}_{\lambda_k} a_{k+1}\}_{d\psi^k(\widetilde{\gamma})} = \sum_{i=1}^k (-1)^{i+1} a_i{}_{\lambda_i}\left\{a_1{}_{\lambda_1} \overset{i}{\check{\cdots}} a_k{}_{\lambda_k} a_{k+1}\right\}_{\psi^k(\widetilde{\gamma})}$$
$$+ (-1)^k a_{k+1}{}_{\lambda_{k+1}^\dagger}\left\{a_1{}_{\lambda_1} \cdots a_{k-1}{}_{\lambda_{k-1}} a_k\right\}_{\psi^k(\widetilde{\gamma})}$$

$$+\sum_{\substack{i,j=1\\i<j}}^{k}(-1)^{k+i+j+1}\left\{a_{1\lambda_1}\stackrel{i}{\check{\cdot}}\cdots\stackrel{j}{\check{\cdot}}\,a_{k\lambda_k}a_{k+1\lambda_{k+1}^\dagger}[a_{i\lambda_i}a_j]\right\}_{\psi^k(\widetilde{\gamma})}$$

$$+\sum_{i=1}^{k}(-1)^i\left\{a_{1\lambda_1}\stackrel{i}{\check{\cdots}}\,a_{k\lambda_k}[a_{i\lambda_i}a_{k+1}]\right\}_{\psi^k(\widetilde{\gamma})}$$

$$=\sum_{i=1}^{k}(-1)^{i+1}a_{i\lambda_i}\widetilde{\gamma}_{\lambda_1,\stackrel{i}{\check{\cdots}},\lambda_k,\lambda_{k+1}^\dagger}(a_1,\stackrel{i}{\check{\cdots}},a_{k+1})$$

$$+(-1)^k a_{k+1\lambda_{k+1}^\dagger}\widetilde{\gamma}_{\lambda_1,\dots,\lambda_k}(a_1,\dots,a_k)$$

$$+\sum_{\substack{i,j=1\\i<j}}^{k}(-1)^{k+i+j+1}\widetilde{\gamma}_{\lambda_1,\stackrel{i}{\check{\cdot}}\cdots\stackrel{j}{\check{\cdot}},\lambda_k,\lambda_{k+1}^\dagger,\lambda_i+\lambda_j}(a_1,\stackrel{i}{\check{\cdot}}\cdots\stackrel{j}{\check{\cdot}},a_{k+1},[a_{i\lambda_i}a_j])$$

$$+\sum_{i=1}^{k}(-1)^i\widetilde{\gamma}_{\lambda_1,\stackrel{i}{\check{\cdots}},\lambda_k,\lambda_i+\lambda_{k+1}^\dagger}(a_1,\stackrel{i}{\check{\cdots}},a_k,[a_{i\lambda_i}a_{k+1}]). \tag{34}$$

In the last equality we used the sesquilinearity of the λ-action of A on M. Clearly, the RHS of (34) is the same as $\{a_{1\lambda_1}\cdots a_{k\lambda_k}a_{k+1}\}_{\psi^{k+1}(\delta\widetilde{\gamma})}$. This proves Eq. (33) and the theorem. □

2.5. Exterior multiplication on $\widetilde{\Gamma}^\bullet$. To complete the section, we review the definition of the wedge product on the basic Lie conformal algebra cohomology complex $\widetilde{\Gamma}^\bullet(A,M)$ (cf. [BKV]). We assume that A is a Lie conformal algebra and M is an A-module endowed with a commutative, associative product such that $\partial^M: M\to M$, and $a_\lambda^M: M\to\mathbb{F}[\lambda]\otimes M$, are (ordinary) derivations of this product.

Consider the basic Lie conformal algebra cohomology complex $\widetilde{\Gamma}^\bullet(A,M)$ introduced in Sect. 2.1. Given two cochains $\widetilde{\alpha}\in\widetilde{\Gamma}^h$ and $\widetilde{\beta}\in\widetilde{\Gamma}^k$, we define their *exterior multiplication* $\widetilde{\alpha}\wedge\widetilde{\beta}\in\widetilde{\Gamma}^{k+h}$ by the following formula:

$$(\widetilde{\alpha}\wedge\widetilde{\beta})_{\lambda_1,\dots,\lambda_{h+k}}(a_1,\dots,a_{h+k})=\sum_{\sigma\in S_{h+k}}\frac{\operatorname{sign}(\sigma)}{h!k!}\widetilde{\alpha}_{\lambda_{\sigma(1)},\dots,\lambda_{\sigma(h)}}(a_{\sigma(1)},\dots,a_{\sigma(h)})$$
$$\widetilde{\beta}_{\lambda_{\sigma(h+1)},\dots,\lambda_{\sigma(h+k)}}(a_{\sigma(h+1)},\dots,a_{\sigma(h+k)}),$$

where the sum is over the set S_{h+k} of all permutations of $\{1,\dots,h+k\}$.

Proposition 2. *(a) The exterior multiplication (35) makes $\widetilde{\Gamma}^\bullet$ into a $\mathbb{Z}$-graded commutative associative superalgebra, generated by $\widetilde{\Gamma}^0\oplus\widetilde{\Gamma}^1$, $M=\widetilde{\Gamma}^0$ being an even subalgebra.*
(b) The operator ∂, acting on $\widetilde{\Gamma}^\bullet$ by (11), is an even derivation of the superalgebra $\widetilde{\Gamma}^\bullet$.
(c) The differential δ, defined by (10), is an odd derivation of the superalgebra $\widetilde{\Gamma}^\bullet$.

Proof. It follows from the analogous well known results for the Lie algebra cohomology, using the isomorphism of the complex $\widetilde{\Gamma}^\bullet$ to the Lie algebra cohomology complex for the annihilation Lie algebra of A (see [BKV, Theorem 5.1]). □

3. Cohomology and Extensions

In this section we interpret the cohomology of the complex $(C^\bullet, d)$ in terms of extensions of the Lie conformal algebra A and its modules.

We start by reviewing the notions of extensions of a module over a Lie conformal algebra and of a Lie conformal algebra. Let A be a Lie conformal algebra and let $M,\ N$ be A-modules. We denote by ∂^M and ∂^N the $\mathbb{F}[\partial]$-module structure on M and N respectively, and by a^M_λ and a^N_λ the λ-action of $a \in A$ on M and N respectively. An *extension* of M by N is, by definition, an A-module E together with a short exact sequence of A-modules

$$0 \to N \to E \to M \to 0.$$

We can fix a splitting $E = M \oplus N$ as $\mathbb{F}$-vector spaces. This space is an A-module extension of M by N if it is endowed with:

1. an endomorphism ∂^E of $M \oplus N$, such that $\partial^E|_N = \partial^N$ and $\partial^E m - \partial^M m \in N$ for every $m \in M$, which makes $M \oplus N$ into an $\mathbb{F}[\partial]$-module;
2. a λ-action of A on $M \oplus N$ such that $a^E_\lambda|_N = a^N_\lambda$ for every $a \in A$ and $a^E_\lambda m - a^M_\lambda m \in \mathbb{F}[\lambda] \otimes N$ for every $a \in A$ and $m \in M$, which makes $M \oplus N$ into an A-module.

In this setting, two structures of A-module extensions E and E' on $M \oplus N$ are *isomorphic* if there is an A-module isomorphism $\sigma : E \to E'$ such that $\sigma|_N = \mathbb{I}_N$ and $\sigma(m) - m \in N$ for every $m \in M$. An extension E is *split* if it is isomorphic to $M \oplus N$ as an A-module, and it is said to be $\mathbb{F}[\partial]$-*split* if it is isomorphic to $M \oplus N$ as an $\mathbb{F}[\partial]$-module, namely if we can choose the $\mathbb{F}$-vector space splitting $E = M \oplus N$ such that $\partial^E = \partial^M \oplus \partial^N$.

We can also talk about extensions of a Lie conformal algebra. Let $A,\ B$ be two Lie conformal algebras, and assume that the $\mathbb{F}[\partial]$-module B is endowed with a structure of an A-module. We denote by ∂^A and ∂^B the $\mathbb{F}[\partial]$-module structure on A and B respectively, and by $[\cdot\,_\lambda\,\cdot]^A$ and $[\cdot\,_\lambda\,\cdot]^B$ the λ-brackets on A and B respectively. An *extension* of A by B is, by definition, a Lie conformal algebra E together with a short exact sequence of Lie conformal algebras

$$0 \to B \to E \to A \to 0.$$

In other words, if we fix a splitting $E = A \oplus B$ as $\mathbb{F}$-vector spaces, the structure of a Lie conformal algebra extension on E consists of:

1. an endomorphism ∂^E of $A \oplus B$, such that $\partial^E|_B = \partial^B$ and $\partial^E a - \partial^A a \in B$ for every $a \in A$, which makes $A \oplus B$ into an $\mathbb{F}[\partial]$-module,
2. a λ-bracket on $A \oplus B$ such that $[\cdot\,_\lambda\,\cdot]^E|_B = [\cdot\,_\lambda\,\cdot]^B$, $[a_\lambda b]^E = a_\lambda b$ for every $a \in A$ and $b \in B$, and $[a_\lambda a']^E - [a_\lambda a']^A \in \mathbb{F}[\lambda] \otimes B$ for every $a, a' \in A$, which makes $A \oplus B$ into a Lie conformal algebra.

As before, two structures of Lie conformal algebra extensions E and E' on $A \oplus B$ are *isomorphic* if there is a Lie conformal algebra isomorphism $\sigma : E \to E'$ such that $\sigma|_B = \mathbb{I}_B$ and $\sigma(a) - a \in B$ for every $a \in A$. E is a *split* extension if it is isomorphic, as a Lie conformal algebra, to the semi-direct sum of A and B, and it is said to be $\mathbb{F}[\partial]$-*split* if it is isomorphic to $A \oplus B$ as an $\mathbb{F}[\partial]$-module, namely if we can choose the $\mathbb{F}$-vector space splitting $E = A \oplus B$ such that $\partial^E = \partial^A \oplus \partial^B$.

We next review the construction of the module $\mathrm{Chom}(M, N)$ of conformal homomorphisms [K]. A *conformal homomorphism* from the $\mathbb{F}[\partial]$-module M to the $\mathbb{F}[\partial]$-module N is an $\mathbb{F}$-linear map $\varphi_\lambda : M \to \mathbb{F}[\lambda] \otimes N$ such that

$$\varphi_\lambda(\partial^M m) = (\partial^N + \lambda)\varphi_\lambda(m).$$

We denote by $\mathrm{Chom}(M, N)$ the space of all conformal homomorphisms from M to N. It has the structure of an $\mathbb{F}[\partial]$-module given by

$$(\partial\varphi)_\lambda = -\lambda\varphi_\lambda.$$

If, moreover, M and N are modules over the Lie conformal algebra A, then $\mathrm{Chom}(M, N)$ has the structure of an A-module, given by

$$(a_\lambda\varphi)_\mu \;=\; a_\lambda^N \circ \varphi_{\mu-\lambda} - \varphi_{\mu-\lambda} \circ a_\lambda^M .$$

In the following theorem, we denote by $H^\bullet(A, M) = \bigoplus_{k\in\mathbb{Z}_+} H^k(A, M)$ the cohomology of the complex $(C^\bullet, d)$ associated to the Lie conformal algebra A and the A-module M (see Sect. 2.3).

Theorem 3. *(a) $H^0(A, M)$ is naturally identified with the set of isomorphism classes of extensions of $\mathbb{F}$, considered as A-module with trivial action of ∂ and trivial λ-action of A, by the A-module M.*
(b) $H^1(A, \mathrm{Chom}(M, N))$ is identified with the set of isomorphism classes of $\mathbb{F}[\partial]$-split extensions of the A-module M by the A-module N.
(c) $H^2(A, M)$ is naturally identified with the set of isomorphism classes of $\mathbb{F}[\partial]$-split extensions of the Lie conformal algebra A by the A-module M, viewed as a Lie conformal algebra with the zero λ-bracket.

Proof. By definition, $H^0(A, M)$ consists of elements $\int m \in M/\partial^M M$ in the kernel of d, namely such that $a_{-\partial^M} m = 0$ for every $a \in A$. In other words,

$$H^0(A, M) = \left\{m \in M \,|\, a_{-\partial^M} m = 0,\ \forall a \in A\right\} / \partial^M M. \tag{35}$$

On the other hand, as discussed above, a structure of an A-module extension E of $\mathbb{F}$ by M on the space $M \oplus \mathbb{F}$ is uniquely defined by an element $\partial^E 1 = m \in M$ such that $a_\lambda^M m \in (\partial^M + \lambda)\mathbb{F}[\lambda] \otimes M$ (or, equivalently, such that $a_{-\partial^M} m = 0$) for every $a \in M$. Indeed, the corresponding λ-action of $a_\lambda^E 1 \in M[\lambda]$, is then uniquely defined by the equation $(\partial^M + \lambda)\left(a_\lambda^E 1\right) = a_\lambda^M m$, imposed by sesquilinearity. It is immediate to check that this construction makes $E = M \oplus \mathbb{F}$ into an A-module. Furthermore, let $m,\ m' \in M$ be such that $a_{-\partial^M} m = a_{-\partial^M} m' = 0$ for every $a \in A$, and consider the corresponding structures of A-module extensions E and E' on $M \oplus \mathbb{F}$. An isomorphism of A-module extensions $\sigma : E \to E'$ is completely defined by an element $\sigma(1) - 1 = n \in M$, such that $\partial^E \sigma(1) = \sigma(\partial^{E'} 1)$, or, equivalently, $m = m' + \partial n$. Hence, m and m' correspond to isomorphic extensions if and only if they differ by an element of ∂M. This proves part (a).

By definition, $C^1(A, \mathrm{Chom}(M, N))$ is the space of $\mathbb{F}[\partial]$-linear maps $c : A \to \mathrm{Chom}(M, N)$. It can be identified, letting $c(a)_\lambda(m) = a_\lambda^c m$, with the space of $\mathbb{F}$-linear maps $A \otimes M \to \mathbb{F}[\lambda] \otimes N$, satisfying the following sesquilinearity conditions (for $a \in A,\ m \in M$):

$$(\partial a)_\lambda^c m = -\lambda a_\lambda^c m, \qquad a_\lambda^c(\partial m) = (\lambda + \partial^N)(a_\lambda^c m). \tag{36}$$

The equation $dc = 0$ for c to be closed can then be rewritten, recalling (13) and using the above notation, as follows ($a, b \in A$, $m \in M$):

$$a_\lambda^N(b_\mu^c m) + a_\lambda^c(b_\mu^M m) - b_\mu^N(a_\lambda^c m) - b_\mu^c(a_\lambda^M m) - [a_\lambda b]_{\lambda+\mu}^c m = 0. \tag{37}$$

Notice that, if $\varphi_\lambda \in \mathrm{Chom}(M, N)$, then φ_0 is an $\mathbb{F}[\partial]$-linear map from M to N, and conversely, any $\mathbb{F}[\partial]$-linear map $\varphi : M \to N$ can be thought of as an element of $\mathrm{Chom}(M, N)$ which is independent of λ. It follows that any element in $d(C^0(A, \mathrm{Chom}(M, N)))$, when written in the above notation, is of the form

$$a_\lambda^{(d\varphi)} m = a_\lambda^N \varphi(m) - \varphi(a_\lambda^M m), \tag{38}$$

for an $\mathbb{F}[\partial]$-linear map $\varphi : M \to N$. In conclusion,

$$\begin{aligned} &H^1(A, \mathrm{Chom}(M, N)) \\ &= \left\{ c : A \otimes M \to \mathbb{F}[\lambda] \otimes N \,\middle|\, (36) - (37) \text{ hold} \right\} \Big/ \{c \text{ of the form } (38)\}. \end{aligned} \tag{39}$$

On the other hand, as discussed at the beginning of the section, a structure of $\mathbb{F}[\partial]$-split extension E of M by N on the space $M \oplus N$ is uniquely determined by the elements $a_\lambda^E m - a_\lambda^M m =: a_\lambda^c m \in \mathbb{F}[\lambda] \otimes N$, and the requirement that $E = M \oplus N$ is an A-module exactly says that $a_\lambda^c m$ satisfies conditions (36) and (37). Furthermore, let E and E' be two such extensions, associated to the closed elements c and c' respectively. An isomorphism $\sigma : E \to E'$ is uniquely determined by the elements $\sigma(m) - m =: \varphi(m) \in N$. The condition that σ commutes with the action of $\partial = \partial^M \oplus \partial^N$, i.e. $\sigma(\partial^M m) = (\partial^M \oplus \partial^N)\sigma(m)$, is equivalent to $\varphi(\partial^M m) = \partial^N \varphi(m)$, namely $\varphi : M \to N$ is an $\mathbb{F}[\partial]$-linear map. The condition that σ commutes with the λ-action of A, i.e. $\sigma(a_\lambda^E m) = a_\lambda^{E'} \sigma(m)$, is equivalent to

$$a_\lambda^c m + \varphi(a_\lambda^M m) = a_\lambda^{c'} m + a_\lambda^N \varphi(m),$$

which means that c and c' differ by an exact element. This proves part (b).

We are left to prove part (c). The space $C^2(A, M)$ consists of $\mathbb{F}$-linear maps $c : A^{\otimes 2} \to \mathbb{F}[\lambda] \otimes M$, denoted by $a \otimes b \mapsto \{a_\lambda b\}_c$, satisfying the conditions of sesquilinearity

$$\{\partial a_\lambda b\}_c = -\lambda \{a_\lambda b\}_c, \qquad \{a_\lambda \partial b\}_c = (\lambda + \partial^M)\{a_\lambda b\}_c, \tag{40}$$

and skew-symmetry

$$\{b_\lambda a\}_c = -\{a_{-\lambda-\partial^M} b\}_c. \tag{41}$$

Recalling the definition (13) of d and using the skew-symmetry of the λ-bracket on A, the equation $dc = 0$ for c to be closed can be written as follows:

$$\begin{aligned} &a_\lambda \{b_\mu z\}_c - b_\mu \{a_\lambda z\}_c + z_{-\lambda-\mu-\partial^M} \{a_\lambda b\}_c \\ &+ \{a_\lambda [b_\mu z]\}_c - \{b_\mu [a_\lambda z]\}_c + \{z_{-\lambda-\mu-\partial^M} [a_\lambda b]\}_c = 0, \end{aligned} \tag{42}$$

for every $a, b, z \in A$. Recall that $C^1(A, M)$ consists of $\mathbb{F}[\partial]$-linear maps $\varphi : A \to M$. Hence exact elements $c = d\varphi$ are of the form

$$\{a_\lambda b\}_{d\varphi} = a_\lambda \varphi(b) - b_{-\lambda-\partial^M} \varphi(a) - \varphi([a_\lambda b]). \tag{43}$$

We thus have

$$H^2(A, M) = \left\{ c : A^{\otimes 2} \to \mathbb{F}[\lambda] \otimes M \,\middle|\, (40) - (42) \text{ hold} \right\} \Big/ \{c \text{ of the form } (43)\}. \tag{44}$$

Once we fix an $\mathbb{F}[\partial]$-splitting $E = M \oplus A$, an "abelian" extension E of the Lie conformal algebra A by the A-module M is determined by a λ-bracket $[\cdot\,_\lambda\,\cdot]^E : (M \oplus A)^{\otimes 2} \to \mathbb{F}[\lambda] \otimes (M \oplus A)$, satisfying the axioms of a Lie conformal algebra, and such that $[m_\lambda m']^E = 0$ for $m, m' \in M$, $[a_\lambda m]^E = a_\lambda m$ for $a \in A$ and $m \in M$, and $[a_\lambda b]^E - [a_\lambda b] \in \mathbb{F}[\lambda] \otimes M$. Let

$$\{a_\lambda b\}_c := [a_\lambda b]^E - [a_\lambda b] \in \mathbb{F}[\lambda] \otimes M.$$

It is not hard to check that the axioms of sesquilinearity, skew-symmetry and Jacobi identity for $[\cdot\,_\lambda\,\cdot]^E$ become Eqs. (40), (41) and (42) respectively. Namely $[\cdot\,_\lambda\,\cdot]^E$ defines a structure of Lie conformal algebra extension on E if and only if c is a closed element of $C^2(A, M)$. Let E and E' be two such extensions, associated to the closed elements c and c' respectively. An isomorphism $\sigma : E \to E'$ is uniquely determined by the elements $\sigma(a) - a =: \varphi(a) \in M$. It is easy to check that σ commutes with the action of $\partial = \partial^M \oplus \partial^A$ if and only if $\varphi(\partial a) = \partial^M \varphi(a)$, namely $\varphi : A \to M$ is an $\mathbb{F}[\partial]$-linear map. Finally, σ defines a Lie conformal algebra isomorphism, i.e. $\sigma([a_\lambda b]^E) = [\sigma(a)_\lambda \sigma(b)]^{E'}$, if and only if

$$\{a_\lambda b\}_c + \varphi([a_\lambda b]) = \{a_\lambda b\}_{c'} + a_\lambda \varphi(b) - b_{-\lambda-\partial^M} \varphi(a),$$

which means that c and c' differ by an exact element. $\square$

Remark 3. Part (a) of Theorem 3 is the same statement as [BKV, Theorem 3.1-2]. Part (b) is equivalent to [BKV, Theorem 3.1-3]. This is due to Theorem 2 and the fact that $\mathrm{Chom}(M, N)$ is free as an $\mathbb{F}[\partial]$-module, hence $C^\bullet(A, \mathrm{Chom}(M, N)) = \bar{C}^\bullet(A, \mathrm{Chom}(M, N))$. However, [BKV, Theorem 3.1-4] is false, unless A is free as an $\mathbb{F}[\partial]$-module. Part (c) of Theorem 3 is the corrected version of it. This lends some support to our opinion that the cohomology complex $(C^\bullet, d)$ is a more correct definition of a Lie conformal algebra cohomology complex. Moreover, as it appears from the proof of Theorem 3, the identification of the cohomology of the complex $(C^\bullet, d)$ with the extensions of Lie conformal algebras and their representations is more direct and natural than for the complex $(\Gamma^\bullet, \delta)$. As B. Bakalov pointed out, Theorem 3 is a special case of Theorem 5.1 from [BDAK], valid for any Lie pseudoalgebra.

Example 1. Consider the centerless Virasoro Lie conformal algebra $\mathrm{Vir}^0 = \mathbb{F}[\partial]L$, with λ-bracket given by $[L_\lambda L]^0 = (\partial + 2\lambda)L$. We have $C^1(\mathrm{Vir}^0, \mathbb{F}) = \mathbb{F}a$, where $a : \mathrm{Vir}^0 \to \mathbb{F}[\lambda]$ is determined by $a(L) = 1$, and $C^2(\mathrm{Vir}^0, \mathbb{F}) = \mathbb{F}\alpha \oplus \mathbb{F}\beta$, where $\alpha, \beta : {\mathrm{Vir}^0}^{\otimes 2} \to \mathbb{F}[\lambda]$ are determined by $\{L_\lambda L\}_\alpha = \lambda$ and $\{L_\lambda L\}_\beta = \lambda^3$. In particular $da = 2\alpha$. Therefore $H^2(\mathrm{Vir}^0, \mathbb{F})$ is one-dimensional, meaning that, up to isomorphism, there is a unique 1-dimensional central extension of Vir^0, namely $\mathrm{Vir} = \mathbb{F}[\partial]L \oplus \mathbb{F}F$, with C central and $[L_\lambda L] = (\partial + 2\lambda)L + \frac{\lambda^3}{12}C$. Note that, since Vir^0 is free as an $\mathbb{F}[\partial]$-module, this is the same answer that we get if we consider the cohomology complex $\Gamma^\bullet \simeq \bar{C}^\bullet$.

On the other hand, we have $C^1(\mathrm{Vir}, \mathbb{F}) = \mathbb{F}a \oplus \mathbb{F}b$, where $a, b : \mathrm{Vir} \to \mathbb{F}$ are determined by $a(L) = 1,\ a(C) = 0,\ b(L) = 0,\ b(C) = 1$, which is strictly bigger than $\bar{C}^1(\mathrm{Vir}, \mathbb{F}) = \mathbb{F}a$. The 2-cochains are as before: $C^2(\mathrm{Vir}, \mathbb{F}) = \mathbb{F}\alpha \oplus \mathbb{F}\beta$, with α and β

determined by $\{L_\lambda L\}_\alpha = \lambda$ and $\{L_\lambda L\}_\beta = \lambda^3$. In particular $da = 2\alpha$ and $db = \frac{1}{12}\beta$. Therefore $H^2(\mathrm{Vir}, \mathbb{F}) = 0$, which corresponds to the fact that there are no non-trivial 1-dimensional central extensions of Vir. On the contrary, the second cohomology of the complex $\Gamma^\bullet \simeq \bar{C}^\bullet$ is one-dimensional.

Remark 4. Since $\bar{C}^k = C^k$ for $k \neq 1$, the corresponding cohomologies $H^n(\bar{C}^\bullet)$ and $H^n(C^\bullet)$ are isomorphic unless $n = 1$ or 2. In particular, it follows from [BKV, Theorem 7.1], that for the complex $C^\bullet(\mathrm{Vir}, \mathbb{F})$ we have: $H^n(\mathrm{Vir}, \mathbb{F}) = \mathbb{F}$ for $n = 0$ or 3, and $H^n(\mathrm{Vir}, \mathbb{F}) = 0$ otherwise.

4. The Space of *k*-Chains, Contractions and Lie Derivatives

4.1. $\mathfrak{g}$*-complexes.* Recall that a cohomology complex is a $\mathbb{Z}$-graded vector space $B^\bullet$, endowed with an endomorphism d, such that $d(B^k) \subset B^{k+1}$ and $d^2 = 0$. We view $B^\bullet$ as a vector superspace, where elements of B^k have the same parity as k in $\mathbb{Z}/2\mathbb{Z}$, so that d is an odd operator.

Let $\mathfrak{g}$ be a Lie algebra, and let $\widehat{\mathfrak{g}} = \eta\mathfrak{g} \oplus \mathfrak{g} \oplus \mathbb{F}\partial_\eta$, where η is odd such that $\eta^2 = 0$, be the associated $\mathbb{Z}$-graded Lie superalgebra. A $\mathfrak{g}$-*structure* on the complex $B^\bullet$ is a $\mathbb{Z}$-grading preserving Lie algebra homomorphism

$$\varphi : \widehat{\mathfrak{g}} \to \mathrm{End}\, B^\bullet,$$

such that $\varphi(\partial_\eta) = d$. A complex with a given $\mathfrak{g}$-structure is called a $\mathfrak{g}$-*complex*.

Given $X \in \mathfrak{g}$, the operator $\iota_X = \varphi(\eta X)$ on $B^\bullet$ is called the *contraction*, and the operator $L_X = \varphi(X)$ is called the *Lie derivative* (along X). Note that we have Cartan's formula

$$L_X = [d, \iota_X], \tag{45}$$

and the commutation relations

$$[d, L_X] = 0,\ \ [\iota_X, \iota_Y] = 0,\ \ [L_X, \iota_Y] = [\iota_X, L_Y] = \iota_{[X,Y]},\ \ [L_X, L_Y] = L_{[X,Y]}. \tag{46}$$

Remark 5. In order to construct a $\mathfrak{g}$-structure on a complex $(B^\bullet, d)$, it suffices to construct commuting odd operators ι_X on $B^\bullet$, depending linearly on X, such that $\iota_X(B^k) \subset B^{k-1}$, and

$$[[d, \iota_X], \iota_Y] = \iota_{[X,Y]},\ \ \forall X, Y \in \mathfrak{g}. \tag{47}$$

Indeed, if we define L_X by (45), all commutation relations (46) hold.

Let ∂ be an endomorphism of the complex $(B^\bullet, d)$, i.e. such that $\partial(B^k) \subset B^k$ and $[d, \partial] = 0$. Let

$$\mathfrak{g}^\partial = \{X \in \mathfrak{g} \,|\, [\iota_X, \partial] = 0\} \subset \mathfrak{g}.$$

Notice that $[L_X, \partial] = 0$ for all $X \in \mathfrak{g}^\partial$. It follows that $\mathfrak{g}^\partial$ is a Lie subalgebra of $\mathfrak{g}$, and that $(\partial B^\bullet, d)$ is a subcomplex of $(B^\bullet, d)$ with a $\mathfrak{g}^\partial$-structure. The corresponding quotient complex

$$(B^\bullet/\partial B^\bullet, d),$$

has an induced $\mathfrak{g}^\partial$-structure, and it is called the *reduced* $\mathfrak{g}^\partial$*-complex*.

A *morphism* of a $\mathfrak{g}$-complex $(B^\bullet, \varphi)$ to an $\mathfrak{h}$-complex $(C^\bullet, \psi)$ is a Lie algebra homomorphism $\pi : \mathfrak{g} \to \mathfrak{h}$ and a $\mathbb{Z}$-grading preserving linear map $\rho : B^\bullet \to C^\bullet$, such that

$$\rho(\varphi(g)b) = \psi(\pi(g))\rho(b),$$

for all $b \in B^\bullet$ and $g \in \widehat{\mathfrak{g}}$, where π is extended to a Lie superalgebra homomorphism $\widehat{\mathfrak{g}} \to \widehat{\mathfrak{h}}$ by letting $\pi(\eta X) = \eta\pi(X)$ and $\pi(\partial_\eta) = \partial_\eta$. Such a morphism is an isomorphism if both π and ρ are isomorphisms.

4.2. The basic and reduced spaces of chains $\widetilde{\Gamma}_\bullet$ *and* $\Gamma_\bullet$. The definitions of the basic and reduced spaces of k-chains are obtained, following [BKV], by dualizing, respectively, the definitions of the spaces $\widetilde{\Gamma}^k$ and Γ^k introduced in Sect. 2.1. In particular, the basic space $\widetilde{\Gamma}_k(A, M)$ of *k-chains* of the Lie conformal algebra A with coefficients in the A-module M is, by definition, the quotient of the space $A^{\otimes k} \otimes \mathrm{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_k], M)$, where $\mathrm{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_k], M)$ is the space of $\mathbb{F}$-linear maps from $\mathbb{F}[\lambda_1, \dots, \lambda_k]$ to M, by the following relations:

C1. $a_1 \otimes \cdots \partial a_i \cdots \otimes a_k \otimes \phi = -a_1 \otimes \cdots \otimes a_k \otimes (\lambda_i^* \phi)$, where we denote $\lambda_i^*\phi \in \mathrm{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_k], M)$ is defined by

$$(\lambda_i^*\phi)(f(\lambda_1, \dots, \lambda_k)) = \phi(\lambda_i f(\lambda_1, \dots, \lambda_k))\,; \tag{48}$$

C2. $a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k)} \otimes (\sigma^*\phi) = \mathrm{sign}(\sigma) a_1 \otimes \cdots \otimes a_k \otimes \phi$, for every permutation $\sigma \in S_k$, where $\sigma^*\phi \in \mathrm{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_k], M)$ is defined by

$$(\sigma^*\phi)(f(\lambda_1, \dots, \lambda_k)) = \phi(f(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(k)})). \tag{49}$$

We let, for brevity, $\widetilde{\Gamma}_k = \widetilde{\Gamma}_k(A, M)$ and $\widetilde{\Gamma}_\bullet = \widetilde{\Gamma}_\bullet(A, M) = \bigoplus_{k \in \mathbb{Z}_+} \widetilde{\Gamma}_k$.

The following statement is the analogue of Remark 1 for the space of k-chains.

Lemma 2. *If one of the elements a_i is a torsion element of the $\mathbb{F}[\partial]$-module A, we have $a_1 \otimes \cdots \otimes a_k \otimes \phi = 0$ in $\widetilde{\Gamma}_k$. In particular, $\widetilde{\Gamma}_k$ can be identified with the quotient of the space $\bar{A}^{\otimes k} \otimes \mathrm{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_k], M)$ by the relations C1 and C2 above, where $\bar{A} = A/\mathrm{Tor}\,A$ denotes the quotient of the $\mathbb{F}[\partial]$-module A by its torsion.*

Proof. If $P(\partial)a_i = 0$ for some polynomial P, we have, by the relation C1.,

$$0 = a_1 \otimes \cdots (P(\partial)a_i) \cdots \otimes a_k \otimes \phi \equiv a_1 \otimes \cdots a_i \cdots \otimes a_k \otimes (P(-\lambda_i^*)\phi).$$

To conclude the lemma we are left to prove that the linear endomorphism $P(-\lambda_i^*)$ of the space $\mathrm{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_k], M)$ is surjective. For this, consider the subspace $P(-\lambda_i)\mathbb{F}[\lambda_1, \dots, \lambda_k] \subset \mathbb{F}[\lambda_1, \dots, \lambda_k]$, and fix a complementary subspace $U \subset \mathbb{F}[\lambda_1, \dots, \lambda_k]$, so that $\mathbb{F}[\lambda_1, \dots, \lambda_k] = P(-\lambda_i)\mathbb{F}[\lambda_1, \dots, \lambda_k] \oplus U$. Given $\phi \in \mathrm{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_k], M)$, we define the linear map $\psi : \mathbb{F}[\lambda_1, \dots, \lambda_k] \to M$ by letting $\psi|_U = 0$ and $\psi(P(-\lambda_i)f(\lambda_1, \dots, \lambda_k)) = \phi(f(\lambda_1, \dots, \lambda_k))$ for every $f \in \mathbb{F}[\lambda_1, \dots, \lambda_k]$. Clearly, $P(-\lambda_i^*)\psi = \phi$. $\square$

The space $\widetilde{\Gamma}_\bullet$ is endowed with a structure of a $\mathbb{Z}$-graded $\mathbb{F}[\partial]$-module, with the action of ∂ induced by the natural action on $A^{\otimes k} \otimes \mathrm{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_k], M)$:

$$\begin{aligned}\partial\,(a_1 \otimes \cdots \otimes a_k \otimes \phi) &= \sum_{i=1}^{k} a_1 \otimes \cdots (\partial a_i) \cdots \otimes a_k \otimes \phi + a1 \otimes \cdots \otimes a_k \otimes (\partial \phi) \\ &= a_1 \otimes \cdots \otimes a_k \otimes \left((-\lambda_1^* - \cdots - \lambda_k^* + \partial)\phi\right), \end{aligned} \tag{50}$$

where $\partial\phi \in \mathrm{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_k], M)$ is defined by $(\partial\phi)(f) = \partial^M(\phi(f))$. The *reduced* space of chains $\Gamma_\bullet = \bigoplus_{k \in \mathbb{Z}_+} \Gamma_k$ is, by definition, the subspace of ∂-invariant chains: $\Gamma_k = \{\xi \in \widetilde{\Gamma}_k \mid \partial\xi = 0\} \subset \widetilde{\Gamma}_k$.

For example, for $k = 0$ we have $\widetilde{\Gamma}_0 = M$ and $\Gamma_0 = \{m \in M \mid \partial m = 0\}$. Next, consider the case $k = 1$ and suppose that the $\mathbb{F}[\partial]$-module A admits the decomposition (23), as a direct sum of Tor A and a complementary free submodule $\mathbb{F}[\partial] \otimes U$. We already pointed out in Lemma 2 that $a \otimes \phi = 0$ in $\widetilde{\Gamma}_1$ if $a \in \mathrm{Tor}(A)$. Moreover, by the sesquilinearity Condition C1 we have $(P(\partial)u) \otimes \phi = u \otimes (P(-\lambda^*)\phi)$ in $\widetilde{\Gamma}_1$, for every $u \in U$, $\phi \in \mathrm{Hom}(\mathbb{F}[\lambda], M)$ and every polynomial P. Hence we can identify $\widetilde{\Gamma}_1 \simeq U \otimes \mathrm{Hom}(\mathbb{F}[\lambda], M)$. Under this identification, an element $u \otimes \phi \in \widetilde{\Gamma}_1$ is annihilated by ∂ if and only if the map $\phi : \mathbb{F}[\lambda] \to M$, satisfies the equation $(-\lambda^* + \partial)\phi = 0$, namely if $\phi(\lambda^n) = \partial^n\phi(1)$ for every $n \in \mathbb{Z}_+$. Clearly, there is a bijective correspondence between such maps and the elements of M, given by $\phi \mapsto \phi(1) \in M$. In conclusion, we have an isomorphism $\Gamma_1 \simeq U \otimes M$.

Remark 6. Apparently, there is no natural way to define a differential δ on k-chains, making $\widetilde{\Gamma}_\bullet$ and $\Gamma_\bullet$ homology complexes. The one given in [BKV, Sect. 4] is divergent, unless any $m \in M$ is annihilated by a power of ∂^M.

4.3. Contraction operators acting on $\widetilde{\Gamma}^\bullet$ and $\Gamma^\bullet$. Assume, as in Sect. 2.5, that A is a Lie conformal algebra and M is an A-module endowed with a commutative, associative product $\mu : M \otimes M \to M$, such that $\partial^M : M \to M$, and $a_\lambda : M \to \mathbb{C}[\lambda] \otimes M$, satisfy the Leibniz rule. Given an h-chain $\xi \in \widetilde{\Gamma}_h$, we define the *contraction operator* $\iota_\xi : \widetilde{\Gamma}^k \to \widetilde{\Gamma}^{k-h}$, $k \geq h$, as follows. If $a_1 \otimes \cdots \otimes a_h \otimes \phi \in A^{\otimes h} \otimes \mathrm{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_h], M)$ is a representative of $\xi \in \widetilde{\Gamma}_h$, and $\widetilde{\gamma} \in \widetilde{\Gamma}^k$, we let

$$(\iota_\xi \widetilde{\gamma})_{\lambda_{h+1},\dots,\lambda_k}(a_{h+1}, \dots, a_k) = \phi^\mu\left(\widetilde{\gamma}_{\lambda_1,\dots,\lambda_k}(a_1, \dots, a_k)\right), \tag{51}$$

where, in the RHS, ϕ^μ denotes the composition of the maps, commuting with $\lambda_{h+1}, \dots, \lambda_k$,

$$\mathbb{F}[\lambda_1, \dots, \lambda_h] \otimes M \xrightarrow{\phi \otimes \mathbb{1}} M \otimes M \xrightarrow{\mu} M. \tag{52}$$

We extend the definition of ι_ξ to all elements $\xi \in \widetilde{\Gamma}_h$ by linearity on ξ, and we let $\iota_\xi(\widetilde{\gamma}) = 0$ if $k < h$. We also define the Lie derivative L_ξ by Cartan's formula: $L_\xi = [\delta, \iota_\xi]$.

It is immediate to check, using the sesquilinearity and skew-symmetry Conditions A1 and A2 for $\widetilde{\gamma}$ (cf. Sect. 2.1), that the RHS in (51) does not depend on the choice of the representative for ξ in $A^{\otimes h} \otimes \mathrm{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_h], M)$. Moreover, if $\widetilde{\gamma} \in \widetilde{\Gamma}^k$, it follows that $\iota_\xi(\widetilde{\gamma})$ satisfies both Conditions A1 and A2, namely $\iota_\xi(\widetilde{\gamma}) \in \widetilde{\Gamma}^{k-h}$.

Proposition 3. *The contraction operators on the superspace $\widetilde{\Gamma}^\bullet$ commute, i.e. for $\xi \in \widetilde{\Gamma}_h$ and $\zeta \in \widetilde{\Gamma}_j$ we have*

$$\iota_\xi \iota_\zeta \;=\; (-1)^{hj} \iota_\zeta \iota_\xi .$$

Proof. Let $a_1 \otimes \cdots \otimes a_h \otimes \phi \in A^{\otimes h} \otimes \mathrm{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_h], M)$ be a representative for $\xi \in \widetilde{\Gamma}_h$, $b_1 \otimes \cdots \otimes b_j \otimes \psi \in A^{\otimes j} \otimes \mathrm{Hom}(\mathbb{F}[\mu_1, \dots, \mu_j], M)$ be a representative for $\zeta \in \widetilde{\Gamma}_j$, and let $\widetilde{\gamma} \in \widetilde{\Gamma}^k$. By the definition (51) of the contraction operators, we have

$$\begin{aligned}&(\iota_\zeta \iota_\xi \widetilde{\gamma})_{\nu_1,\dots,\nu_{k-h-j}}(c_1, \dots, c_{k-h-j})\\ &\quad = \psi^\mu \left(\phi^\mu \left(\widetilde{\gamma}_{\lambda_1,\dots,\lambda_h,\mu_1\cdots,\mu_j,\nu_1,\dots,\nu_{k-h-j}}(a_1, \dots, a_h, b_1, \dots, b_j, c_1, \dots, c_{k-h-j}) \right)\right).\end{aligned}$$

Since obviously ϕ^μ and ψ^μ commute, the proposition follows from Condition A2 for $\widetilde{\gamma}$. $\square$

Proposition 4. *For every basic h-chain $\xi \in \widetilde{\Gamma}_h$, we have*

$$[\partial, \iota_\xi] = \partial \circ \iota_\xi - \iota_\xi \circ \partial = \iota_{\partial\xi}. \tag{53}$$

In particular, if $\xi \in \Gamma_h$ is a reduced h-chain, then ι_ξ commutes with ∂, and it induces a well-defined contraction operator on the reduced cohomology complex: $\iota_\xi : \Gamma^k \to \Gamma^{k-h}$.

Proof. Let $a_1 \otimes \cdots \otimes a_h \otimes \phi$ be a representative of $\xi \in \widetilde{\Gamma}_h$, and let $\widetilde{\gamma} \in \widetilde{\Gamma}^k$. By the definition (11) of the action of ∂ on $\widetilde{\Gamma}^k$, we have

$$\begin{aligned}&\left(\partial \iota_\xi \widetilde{\gamma}\right)_{\lambda_{h+1},\dots,\lambda_k}(a_{h+1}, \dots, a_k)\\ &= (\partial^M + \lambda_{h+1} + \cdots + \lambda_k)\phi^\mu \left(\widetilde{\gamma}_{\lambda_1,\dots,\lambda_k}(a_1, \dots, a_k)\right),\end{aligned} \tag{54}$$

and, similarly,

$$\begin{aligned}&\left(\iota_\xi \partial \widetilde{\gamma}\right)_{\lambda_{h+1},\dots,\lambda_k}(a_{h+1}, \dots, a_k)\\ &= \phi^\mu \left((\partial^M + \lambda_1 + \cdots + \lambda_k)\widetilde{\gamma}_{\lambda_1,\dots,\lambda_k}(a_1, \dots, a_k)\right).\end{aligned} \tag{55}$$

On the other hand, by the definition (50) of the action of ∂ on $\widetilde{\Gamma}_h$, we have

$$\begin{aligned}\left(\iota_{\partial\xi} \widetilde{\gamma}\right)_{\lambda_{h+1},\dots,\lambda_k}(a_{h+1}, \dots, a_k) &= (\partial\phi)^\mu \left(\widetilde{\gamma}_{\lambda_1,\dots,\lambda_k}(a_1, \dots, a_k)\right)\\ &- \phi^\mu \left((\lambda_1 + \cdots + \lambda_h)\widetilde{\gamma}_{\lambda_1,\dots,\lambda_k}(a_1, \dots, a_k)\right).\end{aligned} \tag{56}$$

Equation (53) then follows by (54), (55), (56), and by the following result.

Lemma 3. *For every linear map $\phi : \mathbb{F}[\lambda_1, \dots, \lambda_h] \to M$, we have*

$$[\partial^M, \phi^\mu] = \partial^M \circ \phi^\mu - \phi^\mu \circ (id \otimes \partial^M) = (\partial\phi)^\mu, \tag{57}$$

where $\phi^\mu : \mathbb{F}[\lambda_1, \dots, \lambda_h] \otimes M \to M$ is defined in (52).

Proof. Given $f \otimes m \in \mathbb{F}[\lambda_1, \dots, \lambda_h] \otimes M$ we have

$$\begin{aligned}\left(\partial^M \circ \phi^\mu\right)(f \otimes m) \;&=\; \partial^M \left(\phi(f) \cdot m\right),\\ \left(\phi^\mu \circ (\mathbb{1} \otimes \partial^M)\right)(f \otimes m) \;&=\; \phi(f) \cdot (\partial^M m),\\ (\partial\phi)^\mu(f \otimes m) \;&=\; (\partial^M \phi(f)) \cdot m.\end{aligned}$$

Equation (57) follows since, by assumption, ∂^M is a derivation of M. $\square$

For example, for $h=0$, the contraction by $m\in M=\widetilde{\Gamma}_0$ is given by the commutative associative product in M, namely we have, $(\iota_m\widetilde{\gamma})_{\lambda_1,\dots,\lambda_k}(a_1,\dots,a_k) = m\widetilde{\gamma}_{\lambda_1,\dots,\lambda_k}(a_1,\dots,a_k)$ (which is the same as the exterior multiplication by $m\in\widetilde{\Gamma}^0=M$). If, moreover, $m\in M$ is such that $\partial m=0$, we have $\iota_m\partial\widetilde{\gamma}=\partial\iota_m\widetilde{\gamma}$, so that ι_m induces a well-defined map $\Gamma^k\to\Gamma^k$. Next, consider the case $h=1$. Recall from the previous section that, if A decomposes as in (23), we have $\widetilde{\Gamma}_1\simeq U\otimes\mathrm{Hom}(\mathbb{F}[\lambda],M)$. The contraction operator associated to $\xi=u\otimes\phi\in\widetilde{\Gamma}_1$ is given by $(\iota_\xi\widetilde{\gamma})_{\lambda_2,\dots,\lambda_k}(a_2,\dots,a_k)=\phi^\mu\left(\widetilde{\gamma}_{\lambda,\lambda_2,\dots,\lambda_k}(u,a_2,\dots,a_k)\right)$. Moreover, we have $\Gamma_1\simeq U\otimes M$, and the contraction operator associated to $\xi=u\otimes m$ is given by

$$(\iota_\xi\widetilde{\gamma})_{\lambda_2,\dots,\lambda_k}(a_2,\dots,a_k)=\widetilde{\gamma}_{\partial^M,\lambda_2,\dots,\lambda_k}(u,a_2,\dots,a_k)_{\to}m, \tag{58}$$

where the arrow in the RHS means that ∂^M should be moved to the right. Clearly, $\iota_\xi\partial\widetilde{\gamma}=\partial\iota_\xi\widetilde{\gamma}$, and ι_ξ induces a well-defined map $\Gamma^k\to\Gamma^{k-1}$.

4.4. The Lie algebra structure on $\mathfrak{g}=\Pi\widetilde{\Gamma}_1$ and the $\mathfrak{g}$-structure on the complex $(\widetilde{\Gamma}^\bullet,\delta)$. In this section we want to define a Lie algebra structure on the space of 1-chains $\widetilde{\Gamma}_1$, thus making $\widetilde{\Gamma}^\bullet$ a $\Pi\widetilde{\Gamma}_1$-complex (recall the definition in Sect. 4.1), where Π means that we take opposite parity, namely we consider $\widetilde{\Gamma}_1$ as an even vector space. We start by describing the space of 1-chains in a slightly different form. Recall that $\widetilde{\Gamma}_1$ is the quotient of the space $A\otimes\mathrm{Hom}(\mathbb{F}[\lambda],M)$ by the image of the operator $\partial\otimes 1+1\otimes\lambda^*$. We shall identify $\mathrm{Hom}(\mathbb{F}[\lambda],M)$ with $M[[x]]$ via the map

$$\phi\ \mapsto\ \sum_{n\in\mathbb{Z}_+}\frac{1}{n!}\phi(\lambda^n)x^n.$$

It is immediate to check that, under this identification, the action of ∂ on $\mathrm{Hom}(\mathbb{F}[\lambda],M)$ corresponds to the natural action of ∂ on $M[[x]]$, while the operator λ^* acting on $\mathrm{Hom}(\mathbb{F}[\lambda],M)$ corresponds to the operator $\partial_x=\frac{d}{dx}$ on $M[[x]]$. Thus, the space of 1-chains is

$$\widetilde{\Gamma}_1\ =\ (A\otimes M[[x]])\big/(\partial\otimes 1+1\otimes\partial_x)(A\otimes M[[x]]).$$

Recalling (50), the corresponding action of ∂ on $\widetilde{\Gamma}_1$ is given by

$$\partial(a\otimes m(x))\ =\ a\otimes(\partial-\partial_x)m(x), \tag{59}$$

and the reduced space of 1-chains is $\Gamma_1=\{\xi=a\otimes m(x)\in\widetilde{\Gamma}_1\,|\,\partial\xi=0\}$. In particular, if A admits a decomposition (23) as a direct sum of Tor A and a complementary free submodule $\mathbb{F}[\partial]\otimes U$, we have $\widetilde{\Gamma}_1\simeq U\otimes M[[x]]$, and the reduced subspace $\Gamma_1\subset\widetilde{\Gamma}_1$ consists of elements of the form

$$\xi=u\otimes(e^{x\partial}m),\ \ u\in U,\ m\in M. \tag{60}$$

Given $\xi\in\widetilde{\Gamma}_1$, we can write the action of the contraction operator $\iota_\xi:\widetilde{\Gamma}^k\to\widetilde{\Gamma}^{k-1}$, defined by (51). Consider the pairing $M[[x]]\otimes\mathbb{F}[\lambda]\to M$ given by

$$\langle x^m,\lambda^n\rangle=n!\,\delta_{m,n},\ \ m,n\in\mathbb{Z}_+. \tag{61}$$

It induces a pairing $\langle\,,\,\rangle:\ M[[x]]\otimes(\mathbb{F}[\lambda]\otimes M)\to M$, given by

$$M[[x]]\otimes\mathbb{F}[\lambda]\otimes M\xrightarrow{\langle,\rangle\otimes\mathbb{1}}M\otimes M\xrightarrow{\mu}M, \tag{62}$$

where μ in the last step denotes the commutative associative product on M. Then, if $a_1 \otimes m(x_1) \in A \otimes M[[x_1]]$ is a representative of $\xi \in \widetilde{\Gamma}_1$, the contraction operator $\iota_\xi : \widetilde{\Gamma}^k \to \widetilde{\Gamma}^{k-1}$ acts as follows:

$$(\iota_\xi \widetilde{\gamma})_{\lambda_2,\dots,\lambda_k}(a_2,\dots,a_k) = \left\langle m(x_1), \widetilde{\gamma}_{\lambda_1,\lambda_2,\dots,\lambda_k}(a_1,a_2,\dots,a_k)\right\rangle, \tag{63}$$

where, in the RHS, $\langle , \rangle$ denotes the contraction of x_1 with λ_1 defined in (62). Clearly, if ξ is as in (60), Eq. (63) reduces to (58).

We can also write down the formula for the Lie derivative $L_\xi = \delta \circ \iota_\xi + \iota_\xi \circ \delta$. Let $a_1 \otimes m(x_1) \in A \otimes M[[x_1]]$ be a representative of $\xi \in \widetilde{\Gamma}_1$. Recalling the expression (10) of the differential δ, we have

$$\begin{aligned}
&(\delta \iota_\xi \widetilde{\gamma})_{\lambda_2,\dots,\lambda_{k+1}}(a_2,\dots,a_{k+1}) \\
&= \textstyle\sum_{i=2}^{k+1} (-1)^i a_{i\,\lambda_i} \left\langle m(x_1), \widetilde{\gamma}_{\lambda_1,\lambda_2,\overset{i}{\check{\cdots}},\lambda_{k+1}}(a_1,a_2,\overset{i}{\check{\cdots}},a_{k+1})\right\rangle \\
&+ \textstyle\sum_{\substack{i,j=2\\ i<j}}^{k+1} (-1)^{k+i+j} \left\langle m(x_1), \widetilde{\gamma}_{\lambda_1,\lambda_2,\overset{i}{\check{\cdots}}\overset{j}{\check{\cdots}},\lambda_{k+1},\lambda_i+\lambda_j}(a_1,a_2,\overset{i}{\check{\cdots}}\overset{j}{\check{\cdots}},a_{k+1},[a_{i\,\lambda_i} a_j])\right\rangle,
\end{aligned}$$

and

$$\begin{aligned}
&(\iota_\xi \delta \widetilde{\gamma})_{\lambda_2,\dots,\lambda_{k+1}}(a_2,\dots,a_{k+1}) \\
&\quad= \textstyle\sum_{i=1}^{k+1} (-1)^{i+1} \left\langle m(x_1), a_{i\,\lambda_i}\left(\widetilde{\gamma}_{\lambda_1,\overset{i}{\check{\cdots}},\lambda_{k+1}}(a_1,\overset{i}{\check{\cdots}},a_{k+1})\right)\right\rangle \\
&\qquad+ \textstyle\sum_{\substack{i,j=1\\ i<j}}^{k+1} (-1)^{k+i+j+1} \left\langle m(x_1), \widetilde{\gamma}_{\lambda_1,\overset{i}{\check{\cdots}}\overset{j}{\check{\cdots}},\lambda_{k+1},\lambda_i+\lambda_j}(a_1,\overset{i}{\check{\cdots}}\overset{j}{\check{\cdots}},a_{k+1},[a_{i\,\lambda_i} a_j])\right\rangle.
\end{aligned}$$

We then use the assumption that the λ-action of A on M is by derivations of the commutative associative product of M, to get, from the above two equations,

$$\begin{aligned}
(L_\xi \widetilde{\gamma})_{\lambda_2,\dots,\lambda_{k+1}}(a_2,\dots,a_{k+1}) &= \left\langle m(x_1), a_{1\,\lambda_1}\left(\widetilde{\gamma}_{\lambda_2,\dots,\lambda_{k+1}}(a_2,\dots,a_{k+1})\right)\right\rangle \\
&+ \sum_{i=2}^{k+1} (-1)^i \left\langle \left(a_{i\,\lambda_i} m(x_1)\right), \widetilde{\gamma}_{\lambda_1,\lambda_2,\overset{i}{\check{\cdots}},\lambda_{k+1}}(a_1,a_2,\overset{i}{\check{\cdots}},a_{k+1})\right\rangle \\
&- \sum_{j=2}^{k+1} \left\langle m(x_1), \widetilde{\gamma}_{\lambda_2,\dots,\lambda_1+\lambda_j,\dots,\lambda_{k+1}}(a_2,\dots,[a_{1\,\lambda_1} a_j],\dots,a_{k+1})\right\rangle.
\end{aligned} \tag{64}$$

We next introduce a Lie algebra structure on $\mathfrak{g} = \Pi \widetilde{\Gamma}_1$ and the corresponding $\mathfrak{g}$-structure on the complex $(\widetilde{\Gamma}^\bullet, \delta)$. Define the following bracket on the space $A \otimes M[[x]]$:

$$\begin{aligned}
[a \otimes m(x), b \otimes n(x)] &= [a_{\partial_{x_1}} b] \otimes m(x_1) n(x)\,\Big|_{x_1=x} \\
&- a \otimes \left\langle n(x_1), b_{\lambda_1} m(x)\right\rangle + b \otimes \left\langle m(x_1), a_{\lambda_1} n(x)\right\rangle,
\end{aligned} \tag{65}$$

where, as before, $\langle , \rangle$ in the RHS denotes the contraction of x_1 with λ_1 defined in (62).

Lemma 4. *(a) The bracket (65) on $A \otimes M[[x]]$ induces a well-defined Lie algebra bracket on the space $\mathfrak{g} = \Pi \widetilde{\Gamma}_1$.*

(b) The operator $\partial \otimes 1 + 1 \otimes \partial$ on $A \otimes M[[x]]$ is a derivation of the bracket (65). In particular, ∂ defined in (59) is a derivation of the Lie algebra $\mathfrak{g} = \Pi \widetilde{\Gamma}_1$, and $\mathfrak{g}^\partial = \Pi \Gamma_1 \subset \mathfrak{g}$ is a Lie subalgebra.

Proof. Notice that, by the definition (62) of the inner product $\langle\ ,\ \rangle$,

$$\langle f(x_1), \lambda_1 g(x_1)\rangle = \langle \partial_{x_1} f(x_1), g(\lambda_1)\rangle. \tag{66}$$

Hence, by (65) and the sesquilinearity conditions, we have

$$[(\partial\otimes 1+1\otimes\partial_x)\,(a\otimes m(x))\,, b\otimes n(x)] = -(\partial\otimes 1+1\otimes\partial_x)\,\big(a\otimes\big\langle n(x_1), b_{\lambda_1}m(x)\big\rangle\big),$$

and

$$\begin{aligned}&[a\otimes m(x), (\partial\otimes 1+1\otimes\partial_x)\,(b\otimes n(x))]\\ &= (\partial\otimes 1+1\otimes\partial_x)\,\Big([a_{\partial_{x_1}}b]\otimes m(x_1)n(x)\,\big|_{x_1=x} + b\otimes\big\langle m(x_1), a_{\lambda_1}n(x)\big\rangle\Big)\,.\end{aligned}$$

It follows that $(\partial\otimes 1+1\otimes\partial_x)$ is a derivation of the bracket (65), and that (65) induces a well-defined bracket on the quotient $\widetilde{\Gamma}_1 = A\otimes M[[x]]\big/(\partial\otimes 1+1\otimes\partial_x)(A\otimes M[[x]])$. Next, let us prove skew-symmetry. We have

$$\begin{aligned}&[a\otimes m(x), b\otimes n(x)] + [b\otimes n(x), a\otimes m(x)]\\ &= \Big(\Big([a_{\partial_{x_1}}b] + [b_{\partial_x}a]\Big)\otimes m(x_1)n(x)\Big)\,\Big|_{x_1=x},\end{aligned}$$

and the RHS belongs to $(\partial\otimes 1+1\otimes\partial_x)(A\otimes M[[x]])$, due to the skew-symmetry of the λ-bracket on A. For part (a), we are left to prove the Jacobi identity. Applying twice (65), we have

$$\begin{aligned}&[a\otimes m(x), [b\otimes n(x), c\otimes p(x)]]\\ &= [a_{\partial_{x_1}}[b_{\partial_{x_2}}c]]\otimes m(x_1)n(x_2)p(x)\,\Big|_{x_1=x_2=x}\\ &\quad - [a_{\partial_{x_1}}b]\otimes m(x_1)\big\langle p(x_2), c_{\lambda_2}n(x)\big\rangle\,\big|_{x_1=x}\\ &\quad + [a_{\partial_{x_1}}c]\otimes m(x_1)\big\langle n(x_2), b_{\lambda_2}p(x)\big\rangle\,\big|_{x_1=x}\\ &\quad + [b_{\partial_{x_2}}c]\otimes\big\langle m(x_1), a_{\lambda_1}(n(x_2)p(x))\big\rangle\,\big|_{x_2=x}\\ &\quad - a\otimes\big\langle\big\langle n(x_1)p(x_2), [b_{\lambda_1}c]_{\lambda_1+\lambda_2}m(x)\big\rangle\big\rangle\\ &\quad + a\otimes\big\langle\big\langle p(x_2), c_{\lambda_2}n(x_1)\big\rangle, b_{\lambda_1}m(x)\big\rangle\\ &\quad - a\otimes\big\langle\big\langle n(x_2), b_{\lambda_2}p(x_1)\big\rangle, c_{\lambda_1}m(x)\big\rangle\\ &\quad - b\otimes\big\langle m(x_1), a_{\lambda_1}\big\langle p(x_2), c_{\lambda_2}n(x)\big\rangle\big\rangle\\ &\quad + c\otimes\big\langle m(x_1), a_{\lambda_1}\big\langle n(x_2), b_{\lambda_2}p(x)\big\rangle\big\rangle\,.\end{aligned} \tag{67}$$

For the fifth term in the RHS we used (66) and the following obvious identity:

$$\langle f(x_1)g(x_2)\big|_{x_1=x_2}, h(\lambda_2)\rangle = \langle\langle f(x_1)g(x_2), h(\lambda_1+\lambda_2)\rangle\rangle, \tag{68}$$

where, in the RHS, we denote by $\langle\langle\ ,\ \rangle\rangle$ the pairing of $\mathbb{F}[[x_1,x_2]]$ and $\mathbb{F}[\lambda_1,\lambda_2]$, defined by contracting x_1 with λ_1 and x_2 with λ_2, as in (61). Similarly, we have

$$\begin{aligned}&[b\otimes n(x), [a\otimes m(x), c\otimes p(x)]]\\ &= [b_{\partial_{x_2}}[a_{\partial_{x_1}}c]]\otimes m(x_1)n(x_2)p(x)\,\Big|_{x_1=x_2=x}\\ &\quad - [b_{\partial_{x_1}}a]\otimes n(x_1)\big\langle p(x_2), c_{\lambda_2}m(x)\big\rangle\,\big|_{x_1=x}\end{aligned}$$

$$
\begin{aligned}
&+ [b_{\partial_{x_2}} c] \otimes n(x_2) \left\langle m(x_1), a_{\lambda_1} p(x) \right\rangle \big|_{x_2=x} \\
&+ [a_{\partial_{x_1}} c] \otimes \left\langle n(x_2), b_{\lambda_2} (m(x_1) p(x)) \right\rangle \big|_{x_1=x} \\
&- b \otimes \left\langle\left\langle m(x_1) p(x_2), [a_{\lambda_1} c]_{\lambda_1+\lambda_2} n(x) \right\rangle\right\rangle \\
&+ b \otimes \left\langle\left\langle p(x_2), c_{\lambda_2} m(x_1) \right\rangle, a_{\lambda_1} n(x) \right\rangle \\
&- b \otimes \left\langle\left\langle m(x_1), a_{\lambda_1} p(x_2) \right\rangle, c_{\lambda_2} n(x) \right\rangle \\
&- a \otimes \left\langle n(x_1), b_{\lambda_1} \left\langle p(x_2), c_{\lambda_2} m(x) \right\rangle\right\rangle \\
&+ c \otimes \left\langle n(x_2), b_{\lambda_2} \left\langle m(x_1), a_{\lambda_1} p(x) \right\rangle\right\rangle,
\end{aligned} \tag{69}
$$

and, for the third term of Jacobi identity,

$$
\begin{aligned}
&[[a \otimes m(x), b \otimes n(x)], c \otimes p(x)] \\
&\quad = [[a_{\partial_{x_1}} b]_{\partial_{x_1}+\partial_{x_2}} c] \otimes m(x_1) n(x_2) p(x) \big|_{x_1=x_2=x} \\
&\qquad - [a_{\partial_{x_1}} b] \otimes \left\langle p(x_2), c_{\lambda_2} (m(x_1) n(x)) \right\rangle \big|_{x_1=x} \\
&\qquad - [a_{\partial_{x_1}} c] \otimes \left\langle n(x_2), b_{\lambda_2} m(x_1) \right\rangle p(x) \big|_{x_1=x} \\
&\qquad + [b_{\partial_{x_2}} c] \otimes \left\langle m(x_1), a_{\lambda_1} n(x_2) \right\rangle p(x) \big|_{x_2=x} \\
&\qquad + c \otimes \left\langle\left\langle m(x_1) n(x_2), [a_{\lambda_1} b]_{\lambda_1+\lambda_2} p(x) \right\rangle\right\rangle \\
&\qquad - c \otimes \left\langle\left\langle n(x_2), b_{\lambda_2} m(x_1) \right\rangle, a_{\lambda_1} p(x) \right\rangle \\
&\qquad + c \otimes \left\langle\left\langle m(x_1), a_{\lambda_1} n(x_2) \right\rangle, b_{\lambda_2} p(x) \right\rangle \\
&\qquad + a \otimes \left\langle p(x_2), c_{\lambda_2} \left\langle n(x_1), b_{\lambda_1} m(x) \right\rangle\right\rangle \\
&\qquad - b \otimes \left\langle p(x_2), c_{\lambda_2} \left\langle m(x_1), a_{\lambda_1} n(x) \right\rangle\right\rangle .
\end{aligned} \tag{70}
$$

We now combine Eqs. (67), (69) and (70), to get the Jacobi identity. In particular, the first terms in the RHS of (67), (69) and (70) combine to zero, due to the Jacobi identity for the λ-bracket on A. For the second terms in the RHS of (67), (69) and (70), we use the skew-symmetry of the λ-bracket on A and the Leibniz rule for the λ-action of A on M, to conclude that their combination belongs to $(\partial \otimes 1 + 1 \otimes \partial_x)(A \otimes M[[x]])$. The third term in the RHS of (67) combines with the fourth term in the RHS of (69) and the third term in the RHS of (70) to give zero, and similarly for the combination of the fourth term in the RHS of (67), the third term in the RHS of (69) and the fourth term in the RHS of (70). Furthermore, the combination of the fifth, sixth and seventh terms in the RHS of (67), the eighth term in the RHS of (69) and the eighth term in the RHS of (70) give

$$
a \otimes \left\langle\left\langle n(x_1) p(x_2), \left\{ -[b_{\lambda_1} c]_{\lambda_1+\lambda_2} m(x) + b_{\lambda_1} c_{\lambda_2} m(x) - c_{\lambda_2} b_{\lambda_1} m(x) \right\}\right\rangle\right\rangle,
$$

which is zero due to the Jacobi identity for the λ-action of A on M, and similarly for the remaining terms in (67), (69) and (70). We are left to prove part (b). We have

$$
\begin{aligned}
&[(\partial \otimes 1 + 1 \otimes \partial)(a \otimes m(x)), b \otimes n(x)] \\
&\quad = [\partial a_{\partial_{x_1}} b] \otimes m(x_1) n(x) \big|_{x_1=x} + [a_{\partial_{x_1}} b] \otimes (\partial m(x_1)) n(x) \big|_{x_1=x} \\
&\qquad - (\partial a) \otimes \left\langle n(x_1), b_{\lambda_1} m(x) \right\rangle - a \otimes \left\langle n(x_1), b_{\lambda_1} (\partial m(x)) \right\rangle \\
&\qquad + b \otimes \left\langle m(x_1), (\partial a)_{\lambda_1} n(x) \right\rangle + b \otimes \left\langle (\partial m(x_1)), a_{\lambda_1} n(x) \right\rangle,
\end{aligned} \tag{71}
$$

and

$$\begin{aligned}
&[a \otimes m(x), (\partial \otimes 1 + 1 \otimes \partial)(b \otimes n(x))] \\
&\quad = [a_{\partial_{x_1}} \partial b] \otimes m(x_1)n(x)\,\Big|_{x_1=x} + [a_{\partial_{x_1}} b] \otimes m(x_1)(\partial n(x))\,\Big|_{x_1=x} \\
&\qquad - a \otimes \big\langle n(x_1), (\partial b)_{\lambda_1} m(x)\big\rangle - a \otimes \big\langle (\partial n(x_1)), b_{\lambda_1} m(x)\big\rangle \\
&\qquad + (\partial b) \otimes \big\langle m(x_1), a_{\lambda_1} n(x)\big\rangle + b \otimes \big\langle m(x_1), a_{\lambda_1} (\partial n(x))\big\rangle . \qquad (72)
\end{aligned}$$

Putting Eqs. (71) and (72) together we get

$$\begin{aligned}
(\partial \otimes 1 + 1 \otimes \partial)\,[a \otimes m(x), b \otimes n(x)] &= [(\partial \otimes 1 + 1 \otimes \partial)(a \otimes m(x)), b \otimes n(x)] \\
&\quad + [a \otimes m(x), (\partial \otimes 1 + 1 \otimes \partial)(b \otimes n(x))] .
\end{aligned}$$

This completes the proof of the lemma. □

Proposition 5. *The basic cohomoloy complex* $(\widetilde{\Gamma}^\bullet, \delta)$ *admits a* $\mathfrak{g}$*-structure,* $\varphi : \widehat{\mathfrak{g}} \to \mathrm{End}\, \widetilde{\Gamma}^\bullet$*, where* $\mathfrak{g} = \Pi \widetilde{\Gamma}_1$ *is the Lie algebra with the Lie bracket induced by* (65)*, given by* $\varphi(\partial_\eta) = \delta$*,* $\varphi(\eta\xi) = \iota_\xi$*,* $\varphi(\xi) = L_\xi$*,* $\xi \in \widetilde{\Gamma}_1$*. The corresponding reduced (by* ∂*)* $\mathfrak{g}^\partial$*-complex is* $(\Gamma^\bullet, \delta)$*.*

Proof. In view of Remark 5 and Proposition 3, we only have to check that

$$[L_{\xi_1}, \iota_{\xi_2}] = \iota_{[\xi_1, \xi_2]}, \qquad (73)$$

where, for $\xi \in \widetilde{\Gamma}_1$, ι_ξ is given by (63) and L_ξ is given by (64). For $i = 1, 2$, let then $a_i \otimes m_i(x) \in A \otimes M[[x]]$ be a representative of $\xi_i \in \widetilde{\Gamma}_1$. We have

$$\begin{aligned}
&(L_{\xi_1} \iota_{\xi_2} \widetilde{\gamma})_{\lambda_3, \dots, \lambda_{k+1}}(a_3, \dots, a_{k+1}) \\
&\quad = \big\langle m_1(x_1), a_{1\,\lambda_1} \big\langle m_2(x_2), \widetilde{\gamma}_{\lambda_2, \lambda_3, \dots, \lambda_{k+1}}(a_2, a_3, \dots, a_{k+1})\big\rangle\big\rangle \\
&\qquad + \sum_{i=3}^{k+1} (-1)^i \Big\langle (a_{i\,\lambda_i} m_1(x_1)), \Big\langle m_2(x_2), \widetilde{\gamma}_{\lambda_1, \lambda_2, \lambda_3, \overset{i}{\check{\dots}}, \lambda_{k+1}}(a_1, a_2, a_3, \overset{i}{\check{\dots}}, a_{k+1})\Big\rangle\Big\rangle \\
&\qquad - \sum_{j=3}^{k+1} \big\langle m_1(x_1), \big\langle m_2(x_2), \widetilde{\gamma}_{\lambda_2, \dots, \lambda_1 + \lambda_j, \dots, \lambda_{k+1}}(a_2, \dots, [a_{1\,\lambda_1} a_j], \dots, a_{k+1})\big\rangle\big\rangle , \qquad (74)
\end{aligned}$$

where, as in (61), with $\langle , \rangle$ we contract x_1 with λ_1 and x_2 with λ_2. Similarly, we have

$$\begin{aligned}
&(\iota_{\xi_2} L_{\xi_1} \widetilde{\gamma})_{\lambda_3, \cdots, \lambda_{k+1}}(a_3, \dots, a_{k+1}) \\
&\quad = \big\langle m_2(x_2), \big\langle m_1(x_1), a_{1\,\lambda_1} \big(\widetilde{\gamma}_{\lambda_2, \dots, \lambda_{k+1}}(a_2, \dots, a_{k+1})\big)\big\rangle\big\rangle \\
&\qquad + \sum_{i=2}^{k+1} (-1)^i \Big\langle m_2(x_2), \Big\langle (a_{i\,\lambda_i} m_1(x_1)), \widetilde{\gamma}_{\lambda_1, \lambda_2, \overset{i}{\check{\dots}}, \lambda_{k+1}}(a_1, a_2, \overset{i}{\check{\dots}}, a_{k+1})\Big\rangle\Big\rangle \\
&\qquad - \sum_{j=2}^{k+1} \big\langle m_2(x_2), \big\langle m_1(x_1), \widetilde{\gamma}_{\lambda_2, \dots, \lambda_1 + \lambda_j, \dots, \lambda_{k+1}}(a_2, \dots, [a_{1\,\lambda_1} a_j], \dots, a_{k+1})\big\rangle\big\rangle . \qquad (75)
\end{aligned}$$

Combining Eqs. (74) and (75), we get

$$\begin{aligned}
&([L_{\xi_1}, \iota_{\xi_2}]\widetilde{\gamma})_{\lambda_3,\cdots,\lambda_{k+1}}(a_3,\ldots,a_{k+1}) \\
&\quad = \left\langle\!\left\langle m_1(x_1), (a_{1\,\lambda_1} m_2(x_2))\right\rangle, \widetilde{\gamma}_{\lambda_2,\lambda_3,\ldots,\lambda_{k+1}}(a_2,a_3,\ldots,a_{k+1})\right\rangle \\
&\qquad - \left\langle\!\left\langle m_2(x_2), (a_{2\,\lambda_2} m_1(x_1))\right\rangle, \widetilde{\gamma}_{\lambda_1,\lambda_3,\ldots,\lambda_{k+1}}(a_1,a_3,\ldots,a_{k+1})\right\rangle \\
&\qquad + \left\langle\!\left\langle m_1(x_1)m_2(x_2), \widetilde{\gamma}_{\lambda_1+\lambda_2,\lambda_3,\ldots,\lambda_{k+1}}([a_{1\,\lambda_1} a_2], a_3,\ldots,a_{k+1})\right\rangle\!\right\rangle,
\end{aligned} \tag{76}$$

where, for the first term, we used the fact that the λ-action of A on M is by derivations of the product on M. To conclude, we use Eqs. (66) and (68) to rewrite the RHS of (76) as $(\iota_\xi\widetilde{\gamma})_{\lambda_3,\ldots,\lambda_{k+1}}(a_3,\ldots,a_{k+1})$, where

$$\begin{aligned}
\xi &= a_2 \otimes \left\langle m_1(x_1), (a_{1\,\lambda_1} m_2(x))\right\rangle - a_1 \otimes \left\langle m_2(x_2), (a_{2\,\lambda_2} m_1(x))\right\rangle \\
&\quad + [a_{1\,\partial_{x_1}} a_2] \otimes m_1(x_1)m_2(x)\Big|_{x_1=x} = [\xi_1,\xi_2].
\end{aligned}$$

□

4.5. The space of chains $C_\bullet$. Recall from Theorem 2 that the cohomology complex $\Gamma^\bullet$ is a subcomplex of the cohomology complex $C^\bullet$ defined in Sect. 2.3. One may ask whether, for a reduced h-chain $\xi \in \Gamma_h$, there is a natural way to extend the definition of the contraction operator ι_ξ to the complex $C^\bullet$. In order to formulate the statement, in Theorem 4 below, we first define a new space of chains, obtained by dualizing the definition of the complex $C^\bullet$.

We let $C_\bullet = \bigoplus_{k\in\mathbb{Z}_+} C_k$, where $C_0 = \{m \in M \mid \partial m = 0\}$ $(= \Gamma_0)$, and, for $k \geq 1$, we define the space C_k of *k-chains* of A with coefficients in M as the quotient of the space $A^{\otimes k} \otimes \mathrm{Hom}(\mathbb{F}[\lambda_1,\ldots,\lambda_{k-1}], M)$ by the following relations:

D1. $a_1 \otimes \cdots \partial a_i \cdots \otimes a_k \otimes \phi \equiv -a_1 \otimes \cdots \otimes a_k \otimes (\lambda_i^*\phi)$, for every $1 \leq i \leq k-1$;
D2. $a_1 \otimes \cdots \otimes a_{k-1} \otimes (\partial a_k) \otimes \phi \equiv a_1 \otimes \cdots \otimes a_k \otimes ((\lambda_1^* + \cdots + \lambda_{k-1}^* - \partial)\phi)$;
D3. $a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k)} \otimes (\sigma^*\phi) \equiv \mathrm{sign}(\sigma) a_1 \otimes \cdots \otimes a_k \otimes \phi$, for every permutation $\sigma \in S_k$, where $\sigma^*\phi \in \mathrm{Hom}(\mathbb{F}[\lambda_1,\ldots,\lambda_{k-1}], M)$ is defined by

$$(\sigma^*\phi)(f(\lambda_1,\ldots,\lambda_{k-1})) = \phi\left(f(\lambda_{\sigma(1)},\ldots,\lambda_{\sigma(k-1)})\big|_{\lambda_k\mapsto\lambda_{k\dagger}}\right), \tag{77}$$

where in the RHS we have to replace λ_k by $\lambda_{k\dagger} = -\lambda_1 - \cdots - \lambda_{k-1} + \partial^M$ and move ∂^M to the left of ϕ.

For example, $C_1 = (A \otimes M)/\partial(A \otimes M)$. In particular, in C_1 it is not necessarily true that $a \otimes m$ is equivalent to zero for every torsion element a of the $\mathbb{F}[\partial]$-module A. On the other hand the analogue of Lemma 2 holds for $k \geq 2$:

Lemma 5. *If $k \geq 2$ and $a_i \in \mathrm{Tor}\, A$ for some i, we have $a_1 \otimes \cdots \otimes a_k \otimes \phi = 0$ in C_k.*

Proof. For $1 \leq i \leq k-1$, relation D1 is the same as relation C1, hence the same argument as in the proof of Lemma 2 works. Similarly, for $i = k$, if $P(\partial)a_k = 0$, we have by the relation D2,

$$0 = a_1 \otimes \cdots \otimes a_{k-1} \otimes (P(\partial)a_k) \otimes \phi = a_1 \otimes \cdots \otimes a_k \otimes (P(\lambda_1^* + \cdots + \lambda_{k-1}^* - \partial)\phi),$$

and to conclude the lemma we need to prove that the linear endomorphism $P(\lambda_1^* + \cdots + \lambda_{k-1}^* - \partial)$ of $\mathrm{Hom}(\mathbb{F}[\lambda_1,\ldots,\lambda_{k-1}], M)$ is surjective. In other words, given an element

$\phi \in \mathrm{Hom}(\mathbb{F}[\lambda_1,\dots,\lambda_{k-1}],M)$, we want to find an element $\psi \in \mathrm{Hom}(\mathbb{F}[\lambda_1,\dots,\lambda_{k-1}],M)$ such that $P(\lambda_1^*+\cdots+\lambda_{k-1}^*-\partial)\psi=\phi$. Suppose, for simplicity, that the polynomial P is monic of degree N. Hence

$$P(\lambda_1^*+\cdots+\lambda_{k-1}^*-\partial)=(\lambda_1^*)^N+\sum_{n=0}^N \partial^n R_n(\lambda_1^*,\dots,\lambda_{k-1}^*),$$

where the polynomials $R_n \in \mathbb{F}[\lambda_1^*,\dots,\lambda_{k-1}^*]$, considered as polynomials in λ_1, have degree strictly less than N. Then ψ can be constructed recursively by saying that $\psi(\lambda_1^{n_1}\lambda_2^{n_2}\cdots\lambda_{k-1}^{n-1})=0$ for $n_1<N$, and

$$\begin{aligned}&\psi(\lambda_1^{N+n_1}\lambda_2^{n_2}\dots\lambda_{k-1}^{n-1})\\&\quad=\phi(\lambda_1^{n_1}\cdots\lambda_{k-1}^{n-1})-\textstyle\sum_{n=0}^N \partial^{M^n}\psi\left(R_n(\lambda_1,\dots,\lambda_{k-1})\lambda_1^{n_1}\cdots\lambda_{k-1}^{n-1}\right).\end{aligned}$$

Since the RHS only depends on $\psi(\lambda_1^{m_1}\lambda_2^{m_2}\cdots\lambda_{k-1}^{m-1})$ with $m_1<N+n_1$, the above equation defines ψ by induction on n_1. Clearly, $P(\lambda_1^*+\cdots+\lambda_{k-1}^*-\partial)\psi=\phi$. $\square$

In analogy with the notation used in Sect. 2.2, we introduce the space $\bar{C}_\bullet=\bigoplus_{k\in\mathbb{Z}_+}\bar{C}_k$, by taking the quotient of the space $C_\bullet$ by the torsion of A. More precisely, let $\bar{C}_0=\{m\in M\,|\,\partial m=0\}=C_0$ and, for $k\geq 1$, $\bar{C}_k$ is the quotient of the space $\bar{A}^{\otimes k}\otimes \mathrm{Hom}(\mathbb{F}[\lambda_1,\dots,\lambda_{k-1}],M)$, where $\bar{A}=A/\operatorname{Tor}A$, by the relations D1, D2 and D3 above. In particular, by Lemma 5, $\bar{C}_k=C_k$ for $k\neq 1$, and there is a natural surjective map $C_1\twoheadrightarrow\bar{C}_1$.

We next want to describe the relation between the spaces C_k and Γ_k. In particular, we are going to define a canonical map $\chi_k: C_k\to\Gamma_k$, and we will prove in Proposition 7 that, if the $\mathbb{F}[\partial]$-module A decomposes as a direct sum of its torsion and a free submodule, χ_k factors through an isomorphism $\bar{C}_k\simeq\Gamma_k$.

For $k\geq 1$, let $\rho_k:\mathrm{Hom}(\mathbb{F}[\lambda_1,\dots,\lambda_k],M)\twoheadrightarrow\mathrm{Hom}(\mathbb{F}[\lambda_1,\dots,\lambda_{k-1}],M)$, be the restriction map associated to the inclusion $\mathbb{F}[\lambda_1,\dots,\lambda_{k-1}]\subset\mathbb{F}[\lambda_1,\dots,\lambda_k]$. Let

$$\chi_k:\ \mathrm{Hom}(\mathbb{F}[\lambda_1,\dots,\lambda_{k-1}],M)\hookrightarrow\mathrm{Hom}(\mathbb{F}[\lambda_1,\dots,\lambda_k],M),$$

be the injective linear map defined by

$$(\chi_k\phi)\left(f(\lambda_1,\dots,\lambda_k)\right)=\phi\left(f(\lambda_1,\dots,\lambda_{k-1},\lambda_{k\dagger})\right),\tag{78}$$

where in the RHS we let $\lambda_{k\dagger}=-\sum_{j=1}^{k-1}\lambda_j+\partial^M$ and we move ∂^M to the left.

Lemma 6. (a) *We have* $\rho_k\circ\chi_k=\mathbb{1}$ *on* $\mathrm{Hom}(\mathbb{F}[\lambda_1,\dots,\lambda_{k-1}],M)$. *Hence* $\chi_k\circ\rho_k$ *is a projection operator on* $\mathrm{Hom}(\mathbb{F}[\lambda_1,\dots,\lambda_k],M)$*, whose image is naturally isomorphic to* $\mathrm{Hom}(\mathbb{F}[\lambda_1,\dots,\lambda_{k-1}],M)$.

(b) *The image of* χ_k *consists of the elements* $\phi\in\mathrm{Hom}(\mathbb{F}[\lambda_1,\dots,\lambda_k],M)$ *such that*

$$(\lambda_1^*+\cdots+\lambda_k^*)\phi\;=\;\partial\phi.\tag{79}$$

(c) *We have the commutation relations*

$$\lambda_i^*\circ\chi_k=\chi_k\circ\lambda_i^*\ \ \forall 1\leq i\leq k-1,\quad \lambda_k^*\circ\chi_k=\chi_k\circ(-\lambda_1^*-\cdots-\lambda_{k-1}^*+\partial),\tag{80}$$

where λ_i^* *is the linear endomorphism of* $\mathrm{Hom}(\mathbb{F}[\lambda_1,\dots,\lambda_k],M)$ *defined by* (48).

(d) *For every permutation* $\sigma \in S_k$ *we have*

$$\sigma^* \circ \chi_k = \chi_k \circ \sigma^*, \tag{81}$$

where σ^* *in the LHS denotes the endomorphism of* $\mathrm{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_k], M)$ *defined by Eq.* (49)*, while in the RHS it denotes the endomorphism of* $\mathrm{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_{k-1}], M)$ *defined by* (77)*.*

Proof. Part (a) is obvious. Given $\phi \in \mathrm{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_{k-1}], M)$, we have, by the definition (78) of χ_k,

$$\begin{aligned}&\big((\lambda_1^* + \cdots + \lambda_k^* - \partial)\chi_k\phi\big)\left(f(\lambda_1, \dots, \lambda_k)\right)\\&\quad = (\chi_k\phi)\left((\lambda_1 + \cdots + \lambda_k - \partial^M) f(\lambda_1, \dots, \lambda_k)\right) = 0,\end{aligned}$$

namely $\chi_k\phi$ satisfies Eq. (79). Conversely, if $\phi \in \mathrm{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_k], M)$ solves Eq. (79), we have, by Taylor expanding in $\lambda_{k\dagger} - \lambda_k = -\lambda_1 - \cdots - \lambda_k + \partial^M$,

$$\begin{aligned}&(\chi_k\rho_k\phi)\left(f(\lambda_1, \dots, \lambda_k)\right) = \phi\left(f(\lambda_1 \cdots, \lambda_{k-1}, \lambda_{k\dagger})\right)\\&= \sum_{n \in \mathbb{Z}_+} \frac{1}{n!}\left((-\lambda_1^* - \cdots - \lambda_k^* + \partial)^n\phi\right)\left((\partial_{\lambda_k}^n f)(\lambda_1, \dots, \lambda_k)\right) = \phi\left(f(\lambda_1, \dots, \lambda_k)\right).\end{aligned}$$

Hence, ϕ is in the image of χ_k, as we wanted. This proves part (b). For part (c), the first equation in (80) is clear. The second equation follows by part (b). We are left to prove part (d). Given $\phi \in \mathrm{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_{k-1}], M)$ we have, for every permutation $\sigma \in S_k$,

$$(\sigma^*\chi_k\phi)\left(f(\lambda_1, \dots, \lambda_k)\right) = \phi\left(f(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(k)})\Big|_{\lambda_k \mapsto \lambda_{k\dagger}}\right),$$

and

$$\begin{aligned}&(\chi_k\sigma^*\phi)\left(f(\lambda_1, \dots, \lambda_k)\right)\\&\quad = \phi\left(f(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(k-1)}, -\lambda_{\sigma(1)} - \cdots - \lambda_{\sigma(k-1)} + \partial^M)\Big|_{\lambda_k \mapsto \lambda_{k\dagger}}\right).\end{aligned}$$

Equation (81) follows by the fact that, for $\sigma(k) \neq k$, when we replace λ_k by $\lambda_{k\dagger} = -\lambda_1 - \cdots - \lambda_{k-1} + \partial^M$, the expression $-\lambda_{\sigma(1)} - \cdots - \lambda_{\sigma(k-1)} + \partial^M$ is replaced by $\lambda_{\sigma(k)}$. □

We extend χ_k to an injective map $\chi_k : A^{\otimes k} \otimes \mathrm{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_{k-1}], M) \hookrightarrow A^{\otimes k} \otimes \mathrm{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_k], M)$, given by

$$\chi_k(a_1 \otimes \cdots \otimes a_k \otimes \phi) = a_1 \otimes \cdots \otimes a_k \otimes \chi_k(\phi). \tag{82}$$

Moreover, we denote by $\langle C1, C2\rangle \subset A^{\otimes k} \otimes \mathrm{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_k], M)$ the subspace generated by the relations C1 and C2 from Sect. 4.2, and by $\langle D1, D2, D3\rangle \subset A^{\otimes k} \otimes \mathrm{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_{k-1}], M)$ the subspace generated by the relations D1, D2 and D3.

Proposition 6. (a) $\chi_k\left(\langle D1, D2, D3\rangle\right) \subset \langle C1, C2\rangle$.
(b) *For every* $x \in A^{\otimes k} \otimes \mathrm{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_{k-1}], M)$*, we have* $\partial\chi_k(x) \in \langle C1, C2\rangle$*.*
(c) χ_k *induces a well-defined linear map* $\chi_k : C_k \to \Gamma_k$*.*

Proof. For $1 \le i \le k-1$, we have

$$\begin{aligned}&\chi_k\left(a_1 \otimes \cdots (\partial a_i) \cdots \otimes a_k \otimes \phi + a_1 \otimes \cdots \otimes a_k \otimes (\lambda_i^* \phi)\right)\\&\quad = a_1 \otimes \cdots (\partial a_i) \cdots \otimes a_k \otimes \chi_k(\phi) + a_1 \otimes \cdots \otimes a_k \otimes \chi_k(\lambda_i^* \phi)\right) ,\end{aligned}$$

and this is in $\langle C1, C2\rangle$ thanks to Lemma 6(c). Similarly, by the second equation in (80),

$$\begin{aligned}&\chi_k\left(a_1 \otimes \cdots \otimes a_{k-1} \otimes (\partial a_k) \otimes \phi - a_1 \otimes \cdots \otimes a_k \otimes (\lambda_1^* + \cdots + \lambda_{k-1}^* - \partial)\phi)\right)\\&\quad = a_1 \otimes \cdots \otimes a_{k-1} \otimes (\partial a_k) \otimes \chi_k(\phi) + a_1 \otimes \cdots \otimes a_k \otimes \lambda_k^* \chi_k(\phi) \in \langle C1, C2\rangle .\end{aligned}$$

Furthermore, by Lemma 6(d), we have, for every permutation $\sigma \in S_k$,

$$\begin{aligned}&\chi_k\left(a_1 \otimes \cdots \otimes a_k \otimes \phi - \operatorname{sign}(\sigma) a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k)} \otimes (\sigma^* \phi)\right)\\&\quad = a_1 \otimes \cdots \otimes a_k \otimes \chi_k(\phi) - \operatorname{sign}(\sigma) a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k)} \otimes (\sigma^* \chi_k(\phi)) \in \langle C1, C2\rangle .\end{aligned}$$

This proves part (a). From (50) and Lemma 6(b), we have

$$\partial \chi_k(a_1 \otimes \cdots \otimes a_k \otimes \phi) \equiv a_1 \otimes \cdots \otimes a_k \otimes (-\lambda_1^* - \cdots - \lambda_k^* + \partial)\chi_k(\phi) = 0,$$

thus proving (b). Part (c) follows from (a) and (b). □

Proposition 7. *If the Lie conformal algebra A decomposes, as an $\mathbb{F}[\partial]$-module, in a direct sum of the torsion and a complementary free $\mathbb{F}[\partial]$-submodule, the identity map on $C_0 = \{m \in M \mid \partial m = 0\}$ and the maps $\chi_k : C_k \to \Gamma_k$, $k \ge 1$, factor through a bijective map $\bar{C}_\bullet \simeq \Gamma_\bullet$.*

Proof. Suppose that the $\mathbb{F}[\partial]$-module A decomposes as in (23). By definition, in the space $\bar{C}_k$ we have that $a_1 \otimes \cdots \otimes a_k \otimes \phi \equiv 0$ if one of the elements a_i is in $T = \operatorname{Tor} A$. The same is true in the space $\widetilde{\Gamma}_k$ by Lemma 2. It follows that χ_k induces a well-defined map

$$\chi_k : \bar{C}_k \to \Gamma_k \subset \widetilde{\Gamma}_k. \tag{83}$$

Moreover, in the space $\widetilde{\Gamma}_k$ we have, using relation C1, that

$$(P_1(\partial)u_1) \otimes \cdots \otimes (P_k(\partial)u_k) \otimes \phi \equiv u_1 \otimes \cdots \otimes u_k \otimes \left(P_1(-\lambda_1^*) \cdots P_k(-\lambda_k^*)\phi\right),$$

for every $u_i \in U$ and $\phi \in \operatorname{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_k], M)$. Hence, we can identify the space $\widetilde{\Gamma}_k$ with the quotient of the space $U^{\otimes k} \otimes \operatorname{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_k], M)$ by the relation C2. Similarly, in the space $\bar{C}_k$ we have, using the relations D1 and D2, that

$$\begin{aligned}&(P_1(\partial)u_1) \otimes \cdots \otimes (P_k(\partial)u_k) \otimes \phi\\&\quad \equiv u_1 \otimes \cdots \otimes u_k \otimes \left(P_1(-\lambda_1^*) \cdots P_{k-1}(-\lambda_{k-1}^*) P_k(\lambda_1^* + \cdots + \lambda_{k-1}^* - \partial)\phi\right),\end{aligned}$$

for every $u_i \in U$ and $\phi \in \operatorname{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_{k-1}], M)$. Hence, we can identify the space $\bar{C}_k$ with the quotient of the space $U^{\otimes k} \otimes \operatorname{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_{k-1}], M)$ by the relation D3. The map χ_k in (83) is then induced by the map $U^{\otimes k} \otimes \operatorname{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_{k-1}], M) \to U^{\otimes k} \otimes \operatorname{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_k], M)$, given by

$$u_1 \otimes \cdots \otimes u_k \otimes \phi \mapsto u_1 \otimes \cdots \otimes u_k \otimes \chi_k(\phi),$$

for every $u_i \in U$ and $\phi \in \mathrm{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_{k-1}], M)$. Recalling (50), the action of ∂ on $\widetilde{\Gamma}_k$ is induced by the map $U^{\otimes k} \otimes \mathrm{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_k], M) \to U^{\otimes k} \otimes \mathrm{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_k], M)$, given by

$$u_1 \otimes \cdots \otimes u_k \otimes \phi \mapsto u_1 \otimes \cdots \otimes u_k \otimes \left((-\lambda_1^* - \cdots - \lambda_k^*)\phi\right).$$

Hence, the subspace $\Gamma_k \subset \widetilde{\Gamma}_k$ is spanned by elements of the form $u_1 \otimes \cdots \otimes u_k \otimes \phi$, such that $(-\lambda_1^* - \cdots - \lambda_k^*)\phi = 0$. By Lemma 6(b), this is the same as the image of χ_k. Therefore the map (83) is surjective. Finally, injectiveness of (83) is clear since, by Lemma (6)(d), relation D3 corresponds, via χ_k, to relation C2. $\square$

4.6. Contraction operators acting on $C^\bullet$. Assume, as in Sect. 4.3, that A is a Lie conformal algebra and M is an A-module endowed with a commutative, associative product $\mu : M \otimes M \to M$, such that $\partial^M : M \to M$, and $a_\lambda : M \to \mathbb{C}[\lambda] \otimes M$, satisfy the Leibniz rule. Given an h-chain $x \in C_h$, we define the *contraction operator* $\iota_x : C^k \to C^{k-h}$, $k \geq h$, in the same way as we defined, in Sect. 4.3, the contraction operator associated to an element of $\widetilde{\Gamma}_h$. If $a_1 \otimes \cdots \otimes a_h \otimes \phi \in A^{\otimes h} \otimes \mathrm{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_{h-1}], M)$ is a representative of $x \in C_h$, and $c \in C^k$, we let, for $h < k$,

$$\begin{aligned} &\{a_{h+1\,\lambda_{h+1}} \cdots a_{k-1\,\lambda_{k-1}} a_k\}_{\iota_x c} \\ &= (\chi_h \phi)^\mu \left(\{a_{1\,\lambda_1} \cdots a_{h\,\lambda_h} a_{h+1\,\lambda_{h+1}} \cdots a_{k-1\,\lambda_{k-1}} a_k\}_c\right), \end{aligned} \tag{84}$$

where, in the RHS, ϕ^μ is defined by (52) and χ_h is given by (78). For $h = k$, equation (84) has to be modified as follows:

$$\iota_x c = \int \phi^\mu \left(\{a_{1\,\lambda_1} \cdots a_{k-1\,\lambda_{k-1}} a_k\}_c\right) \in M/\partial^M M = C^0. \tag{85}$$

Lemma 7. *(a) For $c \in C^k$, the RHS of (84) does not depend on the choice of the representative for x in $A^{\otimes h} \otimes \mathrm{Hom}(\mathbb{F}[\lambda_1, \dots, \lambda_{h-1}], M)$. Hence the contraction operator ι_x is well defined for $x \in C_h$.*
(b) For $c \in C^k$, the RHS of (84) satisfies Conditions B1, B2 and B3. Hence $\iota_x c \in C^{k-h}$.

Proof. If $x = a_1 \otimes \cdots (\partial a_i) \cdots \otimes a_h \otimes \phi + a_1 \cdots \otimes a_h \otimes (\lambda_i^* \phi)$, for $1 \leq i \leq h-1$, we have

$$\begin{aligned} \{a_{h+1\,\lambda_{h+1}} \cdots a_{k-1\,\lambda_{k-1}} a_k\}_{\iota_x c} = (\chi_h \phi)^\mu \Big(&\{a_{1\,\lambda_1} \cdots (\partial a_i)_{\lambda_i} \cdots a_{k-1\,\lambda_{k-1}} a_k\}_c \\ &+ \lambda_i \{a_{1\,\lambda_1} \cdots a_{k-1\,\lambda_{k-1}} a_k\}_c\Big), \end{aligned}$$

and this is zero since, by assumption, c satisfies Condition B1. Similarly, if $x = a_1 \otimes \cdots \otimes a_{h-1} \otimes (\partial a_h) \otimes \phi - a_1 \cdots \otimes a_h \otimes ((\lambda_1^* + \cdots + \lambda_{h-1}^* - \partial)\phi)$, we have

$$\begin{aligned} &\{a_{h+1\,\lambda_{h+1}} \cdots a_{k-1\,\lambda_{k-1}} a_k\}_{\iota_x c} \\ &\quad = (\chi_h \phi)^\mu \Big(\{a_{1\,\lambda_1} \cdots (\partial a_h)_{\lambda_h} \cdots a_{k-1\,\lambda_{k-1}} a_k\}_c \\ &\qquad - (\lambda_1 + \cdots + \lambda_{h-1}) \{a_{1\,\lambda_1} \cdots a_{k-1\,\lambda_{k-1}} a_k\}_c\Big) \\ &\qquad + (\chi_h \partial \phi)^\mu \left(\{a_{1\,\lambda_1} \cdots a_{k-1\,\lambda_{k-1}} a_k\}_c\right). \end{aligned} \tag{86}$$

Using Condition B1 for c, we can rewrite the RHS of (86) as

$$(\chi_h\phi)^\mu\left(-(\lambda_1+\cdots+\lambda_{h-1}+\lambda_h)\{a_{1\lambda_1}\cdots a_{k-1\lambda_{k-1}}a_k\}_c\right)$$
$$+(\chi_h\partial\phi)^\mu\left(\{a_{1\lambda_1}\cdots a_{k-1\lambda_{k-1}}a_k\}_c\right),$$

which is zero thanks to Lemma 6(c). Furthermore, if $x = a_1\otimes\cdots\otimes a_h\otimes\phi - \mathrm{sign}(\sigma)a_{\sigma(1)}\otimes\cdots\otimes a_{\sigma(h)}\otimes(\sigma^*\phi)$, for a permutation $\sigma\in S_h$, we have

$$\begin{aligned}
&\{a_{h+1\lambda_{h+1}}\cdots a_{k-1\lambda_{k-1}}a_k\}_{\iota_x c}\\
&= (\chi_h\phi)^\mu\left(\{a_{1\lambda_1}\cdots a_{k-1\lambda_{k-1}}a_k\}_c\right)\\
&\qquad -\mathrm{sign}(\sigma)(\chi_h\sigma^*\phi)^\mu\left(\{a_{1\lambda_1}\cdots a_{k-1\lambda_{k-1}}a_k\}_c\right)\\
&= (\chi_h\phi)^\mu\Big(\{a_{1\lambda_1}\cdots a_{k-1\lambda_{k-1}}a_k\}_c\\
&\qquad -\mathrm{sign}(\sigma)\{a_{\sigma(1)\lambda_{\sigma(1)}}\cdots a_{\sigma(h)\lambda_{\sigma(h)}}a_{h+1\lambda_{h+1}}\cdots a_{k-1\lambda_{k-1}}a_k\}_c\Big),
\end{aligned}\tag{87}$$

where, in the second equality, we used Lemma (6)(d) and the definition (49) of σ^* acting on $\mathrm{Hom}(\mathbb{F}[\lambda_1,\ldots,\lambda_h],M)$. Clearly, the RHS of (87) is zero since, by assumption, c satisfies Condition B3. This proves part (a). For part (b), Condition B1 for $\iota_x c$ follows immediately from the same condition on c. We have

$$\begin{aligned}
\{a_{h+1\lambda_{h+1}}\cdots a_{k-1\lambda_{k-1}}(\partial a_k)\}_{\iota_x c} &= (\chi_h\phi)^\mu\left(\{a_{1\lambda_1}\cdots a_{k-1\lambda_{k-1}}(\partial a_k)\}_c\right)\\
&= (\chi_h\phi)^\mu\left((\lambda_1+\cdots+\lambda_{k-1}+\partial^M)\{a_{1\lambda_1}\cdots a_{k-1\lambda_{k-1}}a_k\}_c\right).
\end{aligned}\tag{88}$$

By Lemmas 3 and 6(c), the RHS of (88) is the same as

$$\begin{aligned}
&(\lambda_{h+1}+\cdots+\lambda_{k-1}+\partial^M)(\chi_h\phi)^\mu\left(\{a_{1\lambda_1}\cdots a_{k-1\lambda_{k-1}}a_k\}_c\right)\\
&\quad=(\lambda_{h+1}+\cdots+\lambda_{k-1}+\partial^M)\{a_{h+1\lambda_{h+1}}\cdots a_{k-1\lambda_{k-1}}a_k\}_{\iota_x c},
\end{aligned}$$

namely $\iota_x c$ satisfies Condition B2. Similarly, for Condition B3, let σ be a permutation of the set $\{h+1,\ldots,k\}$. We have

$$\begin{aligned}
&\{a_{\sigma(h+1)\lambda_{\sigma(h+1)}}\cdots a_{\sigma(k-1)\lambda_{\sigma(k-1)}}a_{\sigma(k)}\}_{\iota_x c}\\
&\quad= (\chi_h\phi)^\mu\left(\{a_{1\lambda_1}\cdots a_{h\lambda_h}a_{\sigma(h+1)\lambda_{\sigma(h+1)}}\cdots a_{\sigma(k-1)\lambda_{\sigma(k-1)}}a_{\sigma(k)}\}_c\right).
\end{aligned}\tag{89}$$

We then observe that, replacing in the above equation λ_k by $-\sum_{j=h+1}^{k-1}\lambda_j-\partial^M$, ∂^M acting from the left, is the same as replacing it, inside the argument of $(\chi_h\phi)^\mu$ in the RHS, by $-\sum_{j=1}^{k-1}\lambda_j-\partial^M$. For this we use Lemmas 3 and 6(c). After this substitution, the RHS of (89) becomes, using Condition B3 for c, $\square$

$$\mathrm{sign}(\sigma)(\chi_h\phi)^\mu\left(\{a_{1\lambda_1}\cdots a_{k-1\lambda_{k-1}}a_k\}_c\right) = \{a_{h+1\lambda_{h+1}}\cdots a_{k-1\lambda_{k-1}}a_k\}_{\iota_x c}.$$

Proposition 8. *The contraction operators on the superspace $C^\bullet$ commute, i.e. for $x\in C_h$ and $y\in C_j$ we have*

$$\iota_x\iota_y = (-1)^{hj}\iota_y\iota_x.$$

Proof. Let $a_1 \otimes \cdots \otimes a_h \otimes \phi \in A^{\otimes h} \otimes \mathrm{Hom}(\mathbb{F}[\lambda_1, \ldots, \lambda_{h-1}], M)$ be a representative for $x \in C_h$, $b_1 \otimes \cdots \otimes b_j \otimes \psi \in A^{\otimes j} \otimes \mathrm{Hom}(\mathbb{F}[\mu_1, \ldots, \mu_{j-1}], M)$ be a representative for $y \in C_j$, and let $c \in C^k$. For $k > h + j$, the proof is similar to that of Proposition 3. Thus we only have to consider the case $k = h + j$. Recalling (84) and (85), we have

$$\iota_y(\iota_x c) = \int \psi^\mu \Big((\chi_h \phi)^\mu \Big(\{a_{1\lambda_1} \cdots a_{h-1\lambda_{h-1}} a_{h\lambda_h} b_{1\mu_1} \cdots b_{j-1\mu_{j-1}} b_j\}_c \Big)\Big).$$

Applying the skew-symmetry Condition B3 for c and using the definition (78) of χ_h, we get, after integration by parts, that the RHS is $\square$

$$(-1)^{hj} \int (\chi_j \psi)^\mu \Big(\phi^\mu \Big(\{b_{1\mu_1} \cdots b_{j-1\mu_{j-1}} b_{j\mu_j} a_{1\lambda_1} \cdots a_{h-1\lambda_{h-1}} a_h\}_c \Big)\Big),$$

which is the same as $(-1)^{hj} \iota_x(\iota_y c)$.

For example, given an element $m \in C_0 = \{m' \in M \mid \partial m' = 0\}$, we have $\{a_{1\lambda_1} \cdots a_{k-1\lambda_{k-1}} a_k\}_{\iota_m c} = m\{a_{1\lambda_1} \cdots a_{k-1\lambda_{k-1}} a_k\}_c$. Recall also that $C_1 = A \otimes M/\partial(A \otimes M)$. The contraction operators associated to 1-chains are given by the following formulas: if $c \in C^1 = \mathrm{Hom}_{\mathbb{F}[\partial]}(A, M)$, then

$$\iota_{a\otimes m} c = \int mc(a), \tag{90}$$

while if $c \in C^k$, with $k \geq 2$, then

$$\{a_{2\lambda_2} \cdots a_{k-1\lambda_{k-1}} a_k\}_{\iota_{a_1\otimes m} c} = \{a_{1\partial^M} a_{2\lambda_2} \cdots a_{k-1\lambda_{k-1}} a_k\}_{c_{\rightarrow}} m, \tag{91}$$

where the arrow in the RHS means, as usual, that ∂^M should be moved to the right.

Also we have the following formulas for the Lie derivative $L_x = [d, \iota_x]$ by a 1-chain $x \in C_1$ acting on $C^0 = M/\partial^M M$ and $C^1 = \mathrm{Hom}_{\mathbb{F}[\partial]}(A, M)$:

$$\begin{aligned} L_{a\otimes m} \textstyle\int n &= \textstyle\int (a_{\partial^M} n)_{\rightarrow} m, \\ (L_{a\otimes m} c)(b) &= \big(a_{\partial^M} c(b)\big)_{\rightarrow} m +_{\leftarrow} \big((b_{-\partial^M} m) c(a)\big) - c\left([a_{\partial^M} b]\right)_{\rightarrow} m, \end{aligned} \tag{92}$$

where the left arrow in the RHS means, as usual, that ∂^M should be moved to the left.

The definitions of the contraction operators associated to elements of $\Gamma_\bullet$ and $C_\bullet$ are "compatible". This is stated in the following:

Theorem 4. *For $x \in C_h$ and $\gamma \in \Gamma^k$, with $k \geq h$, we have*

$$\iota_x(\psi^k(\gamma)) = \psi^{k-h}(\iota_{\chi_h(x)}(\gamma)),$$

where $\psi^k : \Gamma^k \hookrightarrow C^k$, denotes the injective linear map defined in Theorem 2, and $\chi_h : C_h \to \Gamma_h$, denotes the linear map defined in Proposition 6. In other words, there is a commutative diagram of linear maps:

$$\begin{tikzcd} C^k \arrow[r, "\iota_x"] & C^{k-h} \\ \Gamma^k \arrow[u, hook, "\psi^k"] \arrow[r, "\iota_\xi"] & \Gamma^{k-h} \arrow[u, hook, "\psi^{k-h}"'] \end{tikzcd}, \tag{93}$$

provided that $\xi \in \Gamma_h$ and $x \in C_h$ are related by $\xi = \chi_h(x)$.

Proof. Let $\widetilde{\gamma} \in \widetilde{\Gamma}^k$ be a representative of $\gamma \in \Gamma^k$, and let $a_1 \otimes \cdots \otimes a_h \otimes \phi \in A^{\otimes h} \otimes \mathrm{Hom}(\mathbb{F}[\lambda_1, \ldots, \lambda_{h-1}], M)$ be a representative of $x \in C_h$. Recalling the definition (22) of ψ^k and the definition (84) of ι_x, we have

$$\{a_{h+1\,\lambda_{h+1}} \cdots a_{k-1\,\lambda_{k-1}} a_k\}_{\iota_x \psi^k(\widetilde{\gamma})} = (\chi_h \phi)^\mu \left(\widetilde{\gamma}_{\lambda_1, \ldots, \lambda_{k-1}, \lambda_k^\dagger}(a_1, \ldots, a_k)\right), \tag{94}$$

where, in the RHS, $\lambda_k^\dagger$ stands for $-\sum_{j=1}^{k-1} \lambda_j - \partial^M$, with ∂^M acting on the argument of $(\chi_h \phi)^\mu$. By Lemmas 3 and (6)(c), we can replace $\lambda_k^\dagger$ by $-\sum_{j=h+1}^{k-1} \lambda_j - \partial^M$, where now ∂^M is moved to the left of $(\chi_h \phi)^\mu$. Hence, the RHS of (94) is the same as

$$\begin{aligned} &(\chi_h \phi)^\mu \left(\widetilde{\gamma}_{\lambda_1, \lambda_2, \ldots, \lambda_k}(a_1, \ldots, a_k)\right) \big|_{\lambda_k \mapsto \lambda_k^\dagger} \\ &\quad = \{a_{h+1\,\lambda_{h+1}} \cdots a_{k-1\,\lambda_{k-1}} a_k\}_{\psi^{k-h}(\iota_{\chi_h(x)}(\widetilde{\gamma}))}, \end{aligned}$$

thus completing the proof of the theorem. □

4.7. Lie conformal algebroids. A Lie conformal algebroid is an analogue of a Lie algebroid.

Definition 1. *A* **Lie conformal algebroid** *is a pair (A, M), where A is a Lie conformal algebra, M is a commutative associative differential algebra with derivative ∂^M, such that A is a left M-module and M is a left A-module, satisfying the following compatibility conditions ($a, b \in A,\ m, n \in M$):*

L1. $\partial(ma) = (\partial^M m)a + m(\partial a)$,
L2. $a_\lambda(mn) = (a_\lambda m)n + m(a_\lambda n)$,
L3. $[a_\lambda mb] = (a_\lambda m)b + m[a_\lambda b]$.

It follows from Condition L3 and skew-symmetry (2) of the λ-bracket, that

L3'. $[ma_\lambda b] = \left(e^{\partial^M \partial_\lambda} m\right)[a_\lambda b] + (a_{\lambda+\partial} m)_{\to} b$,

where the first term in the RHS is $\sum_{i=0}^\infty \frac{1}{i!}\left((\lambda + \partial^M)^i m\right)(a_{(i)} b)$, and in the second term the arrow means, as usual, that ∂ should be moved to the right, acting on b.

We next give two examples analogous to those in the Lie algebroid case. Let M be, as above, a commutative associative differential algebra. Recall from Sect. 3 that a conformal endomorphism on M is an $\mathbb{F}$-linear map $\varphi(=\varphi_\lambda) : M \to \mathbb{F}[\lambda] \otimes M$ satisfying $\varphi_\lambda(\partial^M m) = (\partial^M + \lambda)\varphi_\lambda(m)$. The space $\mathrm{Cend}(M)$ of conformal endomorphism is then a Lie conformal algebra with the $\mathbb{F}[\partial]$-module structure given by $(\partial\varphi)_\lambda = -\lambda\varphi_\lambda$, and the λ-bracket given by

$$[\varphi_\lambda \psi]_\mu = \varphi_\lambda \circ \psi_{\mu-\lambda} - \psi_{\mu-\lambda} \circ \varphi_\lambda.$$

Example 2. Let $\mathrm{Cder}(M)$ be the subalgebra of the Lie conformal algebra $\mathrm{Cend}(M)$ consisting of all conformal derivations on M, namely of the the conformal endomorphisms satisfying the Leibniz rule: $\varphi_\lambda(mn) = \varphi_\lambda(m)n + m\varphi_\lambda(n)$. Then the pair $(\mathrm{Cder}(M), M)$ is a Lie conformal algebroid, where M carries the tautological $\mathrm{Cder}(M)$-module structure, and $\mathrm{Cder}(M)$ carries the following M-module structure:

$$(m\varphi)_\lambda = \left(e^{\partial^M \partial_\lambda} m\right)\varphi_\lambda. \tag{95}$$

This is indeed an M-module, since $e^{x\partial^M}(mn) = (e^{x\partial^M}m)(e^{x\partial^M}n)$. Furthermore, Condition L1 holds thanks to the obvious identity $e^{\partial^M\partial_\lambda}\lambda = (\lambda + \partial^M)e^{\partial^M\partial_\lambda}$. Condition L2. holds by definition. Finally, for Condition L3 we have

$$\begin{aligned}
[\varphi_\lambda m\psi]_\mu(n) &= \varphi_\lambda\left((m\psi)_{\mu-\lambda}(n)\right) - (m\psi)_{\mu-\lambda}\left(\varphi_\lambda(n)\right) \\
&= \varphi_\lambda\left(\left(e^{\partial^M\partial_\mu}m\right)\psi_{\mu-\lambda}(n)\right) - \left(e^{\partial^M\partial_\mu}m\right)\psi_{\mu-\lambda}\left(\varphi_\lambda(n)\right) \\
&= \left(e^{(\lambda+\partial^M)\partial_\mu}\varphi_\lambda(m)\right)\psi_{\mu-\lambda}(n) + \left(e^{\partial^M\partial_\mu}m\right)\left(\varphi_\lambda\left(\psi_{\mu-\lambda}(n)\right) - \psi_{\mu-\lambda}\left(\varphi_\lambda(n)\right)\right) \\
&= \left(e^{\partial^M\partial_\mu}\varphi_\lambda(m)\right)\psi_\mu(n) + \left(e^{\partial^M\partial_\mu}m\right)[\varphi_\lambda\psi]_\mu(n) = \left(\varphi_\lambda(m)\psi + m[\varphi_\lambda\psi]\right)_\mu(n).
\end{aligned}$$

Example 3. Assume, as in Sect. 4.3, that A is a Lie conformal algebra and M is an A-module endowed with a commutative, associative product, such that $\partial^M : M \to M$, and $a_\lambda : M \to \mathbb{C}[\lambda] \otimes M$, for $a \in A$, satisfy the Leibniz rule. The space $M \otimes A$ has a natural structure of $\mathbb{F}[\partial]$-module, where ∂ acts as

$$\widetilde{\partial}(m \otimes a) = (\partial^M m) \otimes a + m \otimes (\partial a). \tag{96}$$

Clearly, $M \otimes A$ is a left M-module via multiplication on the first factor. We define a left λ-action of $M \otimes A$ on M by

$$(m \otimes a)_\lambda n = \left(e^{\partial^M\partial_\lambda}m\right)(a_\lambda n), \tag{97}$$

and a λ-bracket on $M \otimes A$ by

$$\begin{aligned}
&[(m \otimes a)_\lambda(n \otimes b)] \\
&= \left(\left(e^{\partial^M\partial_\lambda}m\right)n\right) \otimes [a_\lambda b] + ((m \otimes a)_\lambda n) \otimes b - e^{\widetilde{\partial}\partial_\lambda}\left((n \otimes b)_{-\lambda}m \otimes a\right).
\end{aligned} \tag{98}$$

We claim that (96) and (98) make $M \otimes A$ a Lie conformal algebra, (97) makes M an $M \otimes A$-module, and the pair $(M \otimes A, M)$ is a Lie conformal algebroid. This will be proved in Proposition 9, using Lemmas 8 and 9.

Lemma 8. (a) *The following λ-bracket defines a Lie conformal algebra structure on the $\mathbb{C}[\partial]$-module $M \otimes A$:*

$$[(m \otimes a)_\lambda(n \otimes b)]_0 = \left(\left(e^{\partial^M\partial_\lambda}m\right)n\right) \otimes [a_\lambda b]. \tag{99}$$

(b) *For $x, y \in M \otimes A$ and $m \in M$, we have*

$$[mx_\lambda y]_0 = \left(e^{\partial^M\partial_\lambda}m\right)[x_\lambda y]_0, \quad [x_\lambda my]_0 = m[x_\lambda y]_0. \tag{100}$$

Proof. For the first sesquilinearity condition, we have

$$\begin{aligned}
\left[\widetilde{\partial}(m \otimes a)_\lambda(n \otimes b)\right]_0 &= \left(\left(e^{\partial^M\partial_\lambda}\partial^M m\right)n\right) \otimes [a_\lambda b] - \left(\left(e^{\partial^M\partial_\lambda}m\right)n\right) \otimes \lambda[a_\lambda b] \\
&= -\lambda\,[(m \otimes a)_\lambda(n \otimes b)]_0\,.
\end{aligned}$$

The second sesquilinearity condition and skew-symmetry can be proved in a similar way, and they are left to the reader. Let us check the Jacobi identity. We have

$$\left[(m \otimes a)_\lambda\left[(n \otimes b)_\mu(p \otimes c)\right]_0\right]_0 = \left(e^{\partial^M\partial_\lambda}m\right)\left(e^{\partial^M\partial_\mu}n\right)p \otimes [a_\lambda[b_\mu c]].$$

Exchanging $a \otimes m$ with $b \otimes n$ and λ with μ, we get

$$\left[(n \otimes b)\,{}_\mu\ [(m \otimes a)\,{}_\lambda\,(p \otimes c)]_0\right]_0 = \left(e^{\partial^M \partial_\lambda} m\right)\left(e^{\partial^M \partial_\mu} n\right) p \otimes [b_\mu[a_\lambda c]].$$

Furthermore, we have

$$[[m \otimes a\,{}_\lambda\, n \otimes b]_0\,{}_\nu\ p \otimes c]_0 = \left(e^{\partial^M \partial_\nu}\left(e^{\partial^M \partial_\lambda} m\right) n\right) p \otimes [[a_\lambda b]_\nu c].$$

Putting $\nu = \lambda + \mu$, the RHS becomes

$$\left(e^{\partial^M \partial_\lambda} m\right)\left(e^{\partial^M \partial_\mu} n\right) p \otimes [[a_\lambda b]_{\lambda+\mu} c].$$

Hence, the Jacobi identity for the λ-bracket (99) follows immediately from the Jacobi identity for the λ-bracket on A. This proves part (a). Part (b) is immediate. □

We define another λ-product on $M \otimes A$:

$$(m \otimes a)_\lambda (n \otimes b) = ((m \otimes a)_\lambda n) \otimes b. \tag{101}$$

Notice that the λ-bracket (98) can be nicely written in terms of the λ-bracket (99) and the λ-product (101):

$$[x_\lambda y] = [x_\lambda y]_0 + x_\lambda y - y_{-\lambda-\widetilde{\partial}} x. \tag{102}$$

Lemma 9. (a) *The λ-product (101) satisfies both sesquilinearity conditions (for $x, y \in M \otimes A$):*

$$(\widetilde{\partial} x)_\lambda y = -\lambda\, x_\lambda y, \quad x_\lambda(\widetilde{\partial} y) = (\lambda + \widetilde{\partial})(x_\lambda y). \tag{103}$$

(b) *For $x \in M \otimes A$, $m \in M$ and y either in $M \otimes A$ or in M, we have*

$$(mx)_\lambda y = \left(e^{\partial^M \partial_\lambda} m\right) x_\lambda y, \quad x_\lambda(my) = (x_\lambda m)y + m(x_\lambda y). \tag{104}$$

(c) *We have the following identity for $x, y, z \in M \otimes A$:*

$$x_\lambda[y_\mu z]_0 = [(x_\lambda y)_{\lambda+\mu} z]_0 + [y_\mu (x_\lambda z)]_0. \tag{105}$$

(d) *We have the following identity for $x, y \in M \otimes A$ and z either in M or in $M \otimes A$:*

$$x_\lambda (y_\mu z) - y_\mu (x_\lambda z) = [x_\lambda y]_{\lambda+\mu} z. \tag{106}$$

Proof. We have

$$(\widetilde{\partial}(m \otimes a))_\lambda (n \otimes b) = \left(e^{\partial^M \partial_\lambda}(\partial^M - \lambda) m\right)(a_\lambda n) \otimes b.$$

The first sesquilinearity condition follows from the obvious identity $e^{\partial^M \partial_\lambda}(\partial^M - \lambda) = -\lambda e^{\partial^M \partial_\lambda}$. The second sesquilinearity condition can be proved in a similar way. This proves part (a). Part (b) is immediate. For part (c) and (d), let $x = a \otimes m,\ y = b \otimes n,\ z = c \otimes p\ \in A \otimes M$. We have

$$x_\lambda[y_\mu z]_0 = \left(e^{\partial^M \partial_\lambda} m\right)\left(a_\lambda \left(e^{\partial^M \partial_\mu} n\right) p\right) \otimes [b_\mu c]. \tag{107}$$

Similarly,

$$[(x_\lambda y)_\nu z]_0 = \left(e^{\partial^M \partial_\nu}\left(e^{\partial^M \partial_\lambda} m\right)(a_\lambda n)\right) p \otimes [b_\nu c]. \tag{108}$$

Hence, if we put $\nu = \lambda + \mu$, the RHS becomes

$$\left(e^{\partial^M \partial_\lambda} m\right)\left(e^{\partial^M \partial_\mu}(a_\lambda n)\right) p \otimes [b_{\lambda+\mu} c] = \left(e^{\partial^M \partial_\lambda} m\right)\left(a_\lambda\left(e^{\partial^M \partial_\mu} n\right)\right) p \otimes [b_\mu c], \tag{109}$$

where we used the sesquilinearity of the λ-bracket on A. Furthermore, we have

$$[y_\mu(x_\lambda z)]_0 = \left(e^{\partial^M \partial_\mu} n\right)\left(e^{\partial^M \partial_\lambda} m\right)(a_\lambda p) \otimes [b_\mu c]. \tag{110}$$

Combining Eqs. (107), (109) and (110), we immediately get (105), thanks to the assumption that the λ-action of A on M is a derivation of the commutative associative product on M. We are left to prove part (d). We have

$$\begin{aligned} x_\lambda(y_\mu p) &= \left(e^{\partial^M \partial_\lambda} m\right) a_\lambda\left(\left(e^{\partial^M \partial_\mu} n\right)(b_\mu p)\right) \\ &= \left(e^{\partial^M \partial_\lambda} m\right)\left(e^{\partial^M \partial_\mu} n\right) a_\lambda(b_\mu p) + \left(e^{\partial^M \partial_\lambda} m\right)\left(e^{\partial^M \partial_\mu}(a_\lambda n)\right)(b_{\lambda+\mu} p). \end{aligned} \tag{111}$$

For the second equality, we used the Leibniz rule and the sesquilinearity condition for the λ-action of A on M. Exchanging x with y and λ with μ, we have

$$y_\mu(x_\lambda p) = \left(e^{\partial^M \partial_\lambda} m\right)\left(e^{\partial^M \partial_\mu} n\right) b_\mu(a_\lambda p) + \left(e^{\partial^M \partial_\mu} n\right)\left(e^{\partial^M \partial_\lambda}(b_\mu m)\right)(a_{\lambda+\mu} p). \tag{112}$$

By similar computations, we get

$$(x_\lambda y)_{\lambda+\mu} z = \left(e^{\partial^M \partial_\lambda} m\right)\left(e^{\partial^M \partial_\mu}(a_\lambda n)\right)(b_{\lambda+\mu} p), \tag{113}$$

and

$$(y_{-\lambda-\partial} x)_{\lambda+\mu} p = \left(e^{\partial^M \partial_\mu} n\right)\left(e^{\partial^M \partial_\lambda}(b_\mu m)\right)(a_{\lambda+\mu} p). \tag{114}$$

Finally, it follows by a straightforward computation that

$$[x_\lambda y]_{0\,\lambda+\mu} z = \left(e^{\partial^M \partial_\lambda} m\right)\left(e^{\partial^M \partial_\mu} n\right)[a_\lambda b]_{\lambda+\mu} p. \tag{115}$$

Equation (106) is obtained combining Eqs. (111), (112), (113), (114) and (115). □

Proposition 9. (a) *The λ-bracket (98) defines a Lie conformal algebra structure on the $\mathbb{F}[\partial]$-module $M \otimes A$.*
(b) *The λ-action (97) defines a structure of a $M \otimes A$-module on M.*
(c) *The pair $(M \otimes A, M)$ is a Lie conformal algebroid.*
(d) *We have a Lie conformal algebroid homomorphism $(M \otimes A, M) \to (\mathrm{Cder}(M), M)$, given by the identity map on M and the following Lie conformal algebra homomorphism from $M \otimes A$ to $\mathrm{Cder}(M)$:*

$$m \otimes a \mapsto \left(e^{\partial^M \partial_\lambda} m\right) a_\lambda.$$

Proof. It immediately follows from Lemma 8 and Lemma 9(a) that the λ-bracket (102) satisfies sesquilinearity and skew-symmetry. Furthermore, the Jacobi identity for the λ-bracket (98) follows from Lemma 8 and Eqs. (105) and (106). This proves part (a). Part (b) is Lemma 103(c), in the case $z \in M$. For part (c) we need to check Conditions L1, L2 and L3. The first two conditions are immediate. The last one follows from Eqs. (100) and (104). Finally, part (d) is straightforward and is left to the reader. □

4.8. The Lie algebra structure on ΠC_1 and the ΠC_1-structure on the complex $(C^\bullet, d)$. Recall that the space of 1-chains of the complex $(C^\bullet, d)$ is $C_1 = (A \otimes M)/\partial(A \otimes M)$ with odd parity. We want to define a Lie algebra structure on ΠC_1, where, as usual, Π denotes parity reversing, making $C^\bullet$ into a ΠC_1-complex. By Proposition 9(a), we have a Lie conformal algebra structure on $M \otimes A$. Hence, if we identify $M \otimes A$ with $A \otimes M$ by exchanging the two factors, we get a structure of a Lie algebra on the quotient space $(A \otimes M)/\partial(A \otimes M)$, induced by the λ-bracket at $\lambda = 0$ [K].

Explicitly, we get the following well-defined Lie algebra bracket on $\Pi C_1 = (A \otimes M)/\partial(A \otimes M)$:

$$[a \otimes m, b \otimes n] = [a_{\partial_1^M} b]_\to \otimes mn + b \otimes \left(a_{\partial^M} n\right)_\to m - a \otimes \left(b_{\partial^M} m\right)_\to n, \tag{116}$$

where in the RHS, as usual, the right arrow means that ∂^M should be moved to the right, and in the first summand ∂_1^M denotes ∂^M acting only on the first factor m.

Recall from Sect. 4.4 that $\widetilde{\Gamma}_1 = (A \otimes M[[x]])\big/(\partial \otimes 1 + 1 \otimes \partial_x)(A \otimes M[[x]])$, and $\Gamma_1 = \left\{\xi \in \widetilde{\Gamma}_1 \mid \partial\xi = 0\right\}$, where the action of ∂ on $\widetilde{\Gamma}_1$ is given by (59). Under this identification, the map $\chi_1 : C_1 \to \Gamma_1$ defined by (78) and (82) is given by

$$\chi_1(a \otimes m) = a \otimes e^{x\partial^M} m. \tag{117}$$

Proposition 10. *The map $\chi_1 : C_1 \to \Gamma_1$ is a Lie algebra homomorphism, which factors through a Lie algebra isomorphism $\chi_1 : \bar{C}_1 \to \Gamma_1$, provided that A decomposes as in (23).*

Proof. We have, by (116) and (117) that

$$\begin{aligned}\chi_1\left([a \otimes m, b \otimes n]\right) &= [a_{\partial_1^M} b]_\to \otimes \left(e^{x\partial^M} m\right)\left(e^{x\partial^M} n\right)\\ &+ b \otimes e^{x\partial^M}\left((a_{\partial^M} n)_\to m\right) - a \otimes e^{x\partial^M}\left((b_{\partial^M} m)_\to n\right).\end{aligned} \tag{118}$$

Recalling formula (65) for the Lie bracket on $\widetilde{\Gamma}_1$, we have

$$\begin{aligned}[\chi_1(a \otimes m), \chi_1(b \otimes n)]) &= [a_{\partial_{x_1}} b] \otimes \left(e^{x_1\partial^M} m\right)\left(e^{x\partial^M} n\right)\Big|_{x_1=x}\\ &+ b \otimes \left\langle m(x_1), a_{\lambda_1} n(x)\right\rangle - a \otimes \left\langle n(x_1), b_{\lambda_1} m(x)\right\rangle.\end{aligned} \tag{119}$$

Clearly, the first term in the RHS of (118) is the same as the first term in the RHS of (119). Recalling the definition (62) of the pairing $\langle\, ,\, \rangle$, and using the sesquilinearity of the λ-action of A on M, we have that the second term in the RHS of (118) is the same as the second term in the RHS of (119), and similarly for the third terms. The last statement follows from Proposition 7. □

Proposition 11. *The complex* $(C^\bullet, d)$ *has a* ΠC_1*-structure* $\varphi : \widehat{\Pi C_1} \to \operatorname{End} C^\bullet$*, given by* $\varphi(\partial_\eta) = d$*,* $\varphi(\eta x) = \iota_x$*,* $\varphi(x) = L_x = [d, \iota_x]$*. Moreover,* $(\bar{C}^\bullet, d)$ *is a* ΠC_1*-subcomplex.*

Proof. Due to Remark 5 and Proposition 8, we only need to check that, for $x, y \in \Pi C_1$, we have

$$[L_x, \iota_y] = \iota_{[x,y]}. \tag{120}$$

This follows from a long but straightforward computation, using the explicit formulas (13) and (91) for the differential and the contraction operators. It is left to the reader.

Notice though that, in the special case when A decomposes as in (23), Eq. (120) is a corollary of Proposition 5, Theorem 2 and Theorem 4 for $h = 1$. Indeed, due to these results, it suffices to check that both sides of (120) coincide when acting on $C^1 = \operatorname{Hom}_{\mathbb{F}[\partial]}(A, M)$. In the latter case, using Eqs. (12), (14), (90), (91), (92) and (116), we have, for $c \in C^1$,

$$\begin{aligned} L_{a\otimes m}(\iota_{b\otimes n} c) &= \textstyle\int c(b)\left(a_{\partial^M} n\right)_\to m + \int n\left(a_{\partial^M} c(b)\right)_\to m, \\ \iota_{b\otimes n}(L_{a\otimes m} c) &= \textstyle\int n\left(a_{\partial^M} c(b)\right)_\to m + \int c(a)\left(b_{\partial^M} m\right)_\to n - \int nc\left([a_{\partial^M} b]\right)_\to m, \\ \iota_{[a\otimes m, b\otimes n]} c &= \textstyle\int nc\left([a_{\partial^M} b]\right)_\to m + \int c(b)\left(a_{\partial^M} n\right)_\to m - \int c(a)\left(b_{\partial^M} m\right)_\to n. \end{aligned}$$

It follows that (120) holds when applied to elements of C^1. □

The above results imply the following

Theorem 5. *The maps* $\psi^\bullet : \Gamma^\bullet \to \bar{C}^\bullet \subset C^\bullet$ *and* $\chi_1 : C_1 \to \Gamma_1$ *define a homomorphism of* $\mathfrak{g}$*-complexes. Provided that A decomposes as in* (23)*, we obtain an isomorphism of* $\Pi C_1 \simeq \Pi\Gamma_1$*-complexes* $\psi^\bullet : \Gamma^\bullet \xrightarrow{\sim} \bar{C}^\bullet$*.*

Proof. It follows from Theorem 2, Proposition 7, Theorem 4 and Proposition 10. □

4.9. Pairings between 1-chains and 1-cochains. Recall that $\widetilde{\Gamma}^0 = M$. Hence, the contraction operators of 1-chains, restricted to the space of 1-cochains, define a natural pairing $\widetilde{\Gamma}_1 \times \widetilde{\Gamma}^1 \to M$, which, to $\xi \in \widetilde{\Gamma}_1$ and $\widetilde{\gamma} \in \widetilde{\Gamma}^1$, associates

$$\iota_\xi \widetilde{\gamma} = \phi^\mu(\widetilde{\gamma}_\lambda(a)) \in M, \tag{121}$$

where $a \otimes \phi \in A \otimes \operatorname{Hom}(\mathbb{F}[\lambda], M)$ is a representative of ξ.

When we consider the reduced spaces, we have $\Gamma^0 = M/\partial M$, and the above map induces a natural pairing $\Gamma_1 \times \Gamma^1 \to M/\partial M$, which, to $\xi \in \Gamma_1$ and $\gamma \in \Gamma^1$, associates

$$\iota_\xi \gamma = \textstyle\int \phi^\mu(\widetilde{\gamma}_\lambda(a)) \in M/\partial M, \tag{122}$$

where again $a \otimes \phi \in A \otimes \operatorname{Hom}(\mathbb{F}[\lambda], M)$ is a representative of ξ, and $\widetilde{\gamma} \in \widetilde{\Gamma}^1$ is a representative of γ.

A similar pairing can be defined for 1-chains in C_1 and 1-cochains in C^1. Recall that $C^0 = M/\partial M$, C^1 is the space of $\mathbb{F}[\partial]$-module homomorphisms $c : A \to M$, and $C_1 = A \otimes M/\partial(A \otimes M)$. The corresponding pairing $C_1 \times C^1 \to M/\partial M$, is obtained as follows. To $x \in C_1$ and $c \in C^1$, we associate, recalling (85),

$$\iota_x(c) = \textstyle\int m \cdot c(a) \in M/\partial M, \tag{123}$$

where $a \otimes m \in A \otimes M$ is a representative of x.

Recalling Theorems 2 and 4, the above pairings (122) and (123) are compatible in the sense that $\iota_x(c) = \iota_\xi(\gamma)$, provided that $\gamma \in \Gamma^1$ and $c \in C^1$ are related by $c = \psi^1(\gamma)$, and $\xi \in \Gamma_1$ and $x \in C_1$ are related by $\xi = \chi_1(x)$.

4.10. Contraction by a 1*-chain as an odd derivation of* $\widetilde{\Gamma}^\bullet$. Recall that, if the A-module M has a commutative associative product, and ∂^M and a_λ^M are even derivations of it, then the basic cohomology complex $\widetilde{\Gamma}^\bullet$ is a $\mathbb{Z}$-graded commutative associative superalgebra with respect to the exterior product (35), and the differential δ is an odd derivation of degree +1.

Proposition 12. *The contraction operator* ι_ξ*, associated to a 1-chain* $\xi \in \widetilde{\Gamma}_1$*, is an odd derivation of the superalgebra* $\widetilde{\Gamma}^\bullet$ *of degree -1.*

Proof. Let $a_1 \otimes \phi$, with $a_1 \in A$ and $\phi \in \mathrm{Hom}(\mathbb{F}[\lambda_1], M)$, be a representative of $\xi \in \widetilde{\Gamma}_1$. By the definition (35) of the exterior product, we have

$$\begin{aligned}
&(\iota_\xi(\widetilde{\alpha} \wedge \widetilde{\beta}))_{\lambda_2,\dots,\lambda_{h+k}}(a_2,\dots,a_{h+k}) \\
&\quad = \sum_{\sigma \in S_{h+k}} \frac{\mathrm{sign}(\sigma)}{h!k!} \phi^\mu \left(\widetilde{\alpha}_{\lambda_{\sigma(1)},\dots,\lambda_{\sigma(h)}}(a_{\sigma(1)},\dots,a_{\sigma(h)})\times\right. \\
&\qquad\qquad \left.\times \widetilde{\beta}_{\lambda_{\sigma(h+1)},\dots,\lambda_{\sigma(h+k)}}(a_{\sigma(h+1)},\dots,a_{\sigma(h+k)})\right). \qquad (124)
\end{aligned}$$

By the skew-symmetry condition A2 for $\widetilde{\alpha}$ and $\widetilde{\beta}$, we can rewrite the RHS of (124) as

$$\begin{aligned}
&\sum_{i=1}^{h} \sum_{\sigma \mid \sigma(i)=1} \frac{\mathrm{sign}(\sigma)}{h!k!} (-1)^{i+1} \phi^\mu \left(\widetilde{\alpha}_{\lambda_1,\lambda_{\sigma(1)},\overset{i}{\check{\cdots}},\lambda_{\sigma(h)}}(a_1, a_{\sigma(1)}, \overset{i}{\check{\cdots}}, a_{\sigma(h)}) \right) \\
&\qquad \times \widetilde{\beta}_{\lambda_{\sigma(h+1)},\dots,\lambda_{\sigma(h+k)}}(a_{\sigma(h+1)},\dots,a_{\sigma(h+k)}) \\
&+ \sum_{i=h+1}^{h+k} \sum_{\sigma \mid \sigma(i)=1} \frac{\mathrm{sign}(\sigma)}{h!k!} (-i)^{i-h+1} \widetilde{\alpha}_{\lambda_{\sigma(1)},\dots,\lambda_{\sigma(h)}}(a_{\sigma(1)},\dots,a_{\sigma(h)}) \times \\
&\qquad \times \phi^\mu \left(\widetilde{\beta}_{\lambda_1,\lambda_{\sigma(h+1)},\overset{i}{\check{\cdots}},\lambda_{\sigma(h+k)}}(a_1, a_{\sigma(h+1)}, \overset{i}{\check{\cdots}}, a_{\sigma(h+k)}) \right). \qquad (125)
\end{aligned}$$

The set of all permutations $\sigma \in S_{h+k}$ such that $\sigma(i) = 1$, is naturally in bijection with the set of all permutations τ of $\{2,\dots,h+k\}$, and the correspondence between the signs is $\mathrm{sign}(\tau) = (-1)^{i+1}\mathrm{sign}(\sigma)$. Hence, (125) can be rewritten as □

$$\begin{aligned}
&\sum_\tau \frac{\mathrm{sign}(\tau)}{h!k!} \left(h(\iota_\xi \widetilde{\alpha})_{\lambda_{\tau(2)},\dots,\lambda_{\tau(h)}}(a_{\tau(2)},\dots,a_{\tau(h)})\times \right. \\
&\qquad \times \widetilde{\beta}_{\lambda_{\tau(h+1)},\dots,\lambda_{\tau(h+k)}}(a_{\tau(h+1)},\dots,a_{\tau(h+k)}) \\
&\quad + k(-1)^h \widetilde{\alpha}_{\lambda_{\tau(2)},\dots,\lambda_{\tau(h+1)}}(a_{\tau(2)},\dots,a_{\tau(h+1)})\times \\
&\qquad \left. \times (\iota_\xi \widetilde{\beta})_{\lambda_{\tau(h+2)},\dots,\lambda_{\tau(h+k)}}(a_{\tau(h+2)},\dots,a_{\tau(h+k)}) \right) \\
&= (\iota_\xi(\widetilde{\alpha}) \wedge \widetilde{\beta})_{\lambda_2,\dots,\lambda_{h+k}}(a_2,\dots,a_{h+k}) \\
&\quad + (-1)^h (\widetilde{\alpha} \wedge \iota_\xi(\widetilde{\beta}))_{\lambda_2,\dots,\lambda_{h+k}}(a_2,\dots,a_{h+k}).
\end{aligned}$$

Remark 7. One can show that the $\mathfrak{g}$-structure of all our complexes $\widetilde{\Gamma}^\bullet$, $\Gamma^\bullet$ and $C^\bullet$ can be extended to a structure of a calculus algebra, as defined in [DTT]. Namely, one can extend the Lie algebra bracket from the space of 1-chains to the whole space of chains (with reverse parity), and define there a commutative superalgebra structure, which extends our $\mathfrak{g}$-structure and satisfies all the properties of a calculus algebra.

5. The Complex of Variational Calculus as a Lie Conformal Algebra Cohomology Complex

5.1. Algebras of differential functions. An *algebra of differential functions* $\mathcal{V}$ in the variables u_i, indexed by a finite set $I = \{1, \dots, \ell\}$, is, by definition, a differential algebra (i.e. a unital commutative associative algebra with a derivation ∂), endowed with commuting derivations $\frac{\partial}{\partial u_i^{(n)}} : \mathcal{V} \to \mathcal{V}$, for all $i \in I$ and $n \in \mathbb{Z}_+$, such that, given $f \in \mathcal{V}$, $\frac{\partial}{\partial u_i^{(n)}} f = 0$ for all but finitely many $i \in I$ and $n \in \mathbb{Z}_+$, and the following commutation rules with ∂ hold:

$$\left[\frac{\partial}{\partial u_i^{(n)}}, \partial \right] = \frac{\partial}{\partial u_i^{(n-1)}}, \tag{126}$$

where the RHS is considered to be zero if $n = 0$. As in the previous sections, we denote by $f \mapsto \int f$ the canonical quotient map $\mathcal{V} \to \mathcal{V}/\partial\mathcal{V}$.

Denote by $\mathcal{C} \subset \mathcal{V}$ the subspace of constant functions, i.e.

$$\mathcal{C} = \left\{ f \in \mathcal{V} \,\Big|\, \frac{\partial f}{\partial u_i^{(n)}} = 0 \ \forall i \in I,\ n \in \mathbb{Z}_+ \right\}. \tag{127}$$

It follows from (126) by downward induction that

$$\mathrm{Ker}\,(\partial) \subset \mathcal{C}. \tag{128}$$

Also, clearly, $\partial\mathcal{C} \subset \mathcal{C}$.

Typical examples of algebras of differential functions are: the ring of polynomials

$$R_\ell = \mathbb{F}[u_i^{(n)} \,|\, i \in I, n \in \mathbb{Z}_+], \tag{129}$$

where $\partial(u_i^{(n)}) = u_i^{(n+1)}$, any localization of it by some multiplicative subset $S \subset R$, such as the whole field of fractions $Q = \mathbb{F}(u_i^{(n)} \,|\, i \in I, n \in \mathbb{Z}_+)$, or any algebraic extension of the algebra R or of the field Q obtained by adding a solution of a certain polynomial equation. In all these examples the action of $\partial : \mathcal{V} \to \mathcal{V}$ is given by $\partial = \sum_{i \in I, n \in \mathbb{Z}_+} u_i^{(n+1)} \frac{\partial}{\partial u_i^{(n)}}$. Another example of an algebra of differential functions is the ring $R_\ell[x] = \mathbb{F}[x, u_i^{(n)} \,|\, i \in I, n \in \mathbb{Z}_+]$, where $\partial = \frac{\partial}{\partial x} + \sum_{i \in I, n \in \mathbb{Z}_+} u_i^{(n+1)} \frac{\partial}{\partial u_i^{(n)}}$.

The *variational derivative* $\frac{\delta}{\delta u} : \mathcal{V} \to \mathcal{V}^{\oplus \ell}$ is defined by

$$\frac{\delta f}{\delta u_i} := \sum_{n \in \mathbb{Z}_+} (-\partial)^n \frac{\partial f}{\partial u_i^{(n)}}. \tag{130}$$

It follows immediately from (126) that

$$\frac{\delta}{\delta u_i}(\partial f) = 0, \tag{131}$$

for every $i \in I$ and $f \in \mathcal{V}$, namely, $\partial\mathcal{V} \subset \mathrm{Ker}\, \frac{\delta}{\delta u}$.

A *vector field* is, by definition, a derivation of $\mathcal{V}$ of the form

$$X = \sum_{i \in I, n \in \mathbb{Z}_+} P_{i,n} \frac{\partial}{\partial u_i^{(n)}}, \quad P_{i,n} \in \mathcal{V}. \tag{132}$$

We let $\mathfrak{g}$ be the Lie algebra of all vector fields. The subalgebra of *evolutionary vector fields* is $\mathfrak{g}^\partial \subset \mathfrak{g}$, consisting of the vector fields commuting with ∂. By (126), a vector field X is evolutionary if and only if it has the form

$$X_P = \sum_{i \in I, n \in \mathbb{Z}_+} (\partial^n P_i) \frac{\partial}{\partial u_i^{(n)}}, \quad \text{where } P = (P_i)_{i \in I} \in \mathcal{V}^\ell. \tag{133}$$

5.2. Normal algebras of differential functions. Let $\mathcal{V}$ be an algebra of differential functions in the variables u_i, $i \in I = \{1, \dots, \ell\}$. For $i \in I$ and $n \in \mathbb{Z}_+$ we let

$$\mathcal{V}_{n,i} := \left\{ f \in \mathcal{V} \,\middle|\, \frac{\partial f}{\partial u_j^{(m)}} = 0 \ \text{ if } (m, j) > (n, i) \text{ in lexicographic order} \right\}. \tag{134}$$

We also let $\mathcal{V}_{n,0} = \mathcal{V}_{n-1,\ell}$.

A natural assumption on $\mathcal{V}$ is to contain elements $u_i^{(n)}$, for $i \in I$, $n \in \mathbb{Z}_+$, such that

$$\frac{\partial u_i^{(n)}}{\partial u_j^{(m)}} = \delta_{ij}\delta_{mn}. \tag{135}$$

Clearly, such elements are uniquely defined up to adding constant functions. Moreover, choosing these constants appropriately, we can assume that $\partial u_i^{(n)} = u_i^{(n+1)}$. Thus, under this assumption $\mathcal{V}$ is an algebra of differential functions extension of the algebra R_ℓ in (129).

Lemma 10. *Let $\mathcal{V}$ be an algebra of differential functions extension of the algebra R_ℓ. Then:*

(a) *We have $\partial = \partial_R + \partial'$, where*

$$\partial_R = \sum_{i \in I, n \in \mathbb{Z}_+} u_i^{(n+1)} \frac{\partial}{\partial u_i^{(n)}}, \tag{136}$$

and ∂' is a derivation of $\mathcal{V}$ which commutes with all $\frac{\partial}{\partial u_i^{(n)}}$ and which vanishes on $R_\ell \subset \mathcal{V}$. In particular, $\partial' \mathcal{V}_{n,i} \subset \mathcal{V}_{n,i}$.

(b) *If $f \in \mathcal{V}_{n,i} \backslash \mathcal{V}_{n,i-1}$, then $\partial f \in \mathcal{V}_{n+1,i} \backslash \mathcal{V}_{n+1,i-1}$, and it has the form*

$$\partial f = \sum_{j \le i} h_j u_j^{(n+1)} + r, \tag{137}$$

where $h_j \in \mathcal{V}_{n,i}$ for all $j \le i$, $r \in \mathcal{V}_{n,i}$, and $h_i \neq 0$.

(c) *For $f \in \mathcal{V}$, $\int fg = 0$ for every $g \in \mathcal{V}$ if and only if $f = 0$.*

Proof. Part (a) is clear. By part (a), we have that ∂f is as in (137), where $h_j = \frac{\partial f}{\partial u_j^{(n)}} \in \mathcal{V}_{n,i}$, and $r = \sum_{j\in I, m\leq n} u_j^{(m)} \frac{\partial f}{\partial u_j^{(m-1)}} + \partial' f \in \mathcal{V}_{n,i}$. We are left to prove part (c). Suppose $f \neq 0$ is such that $\int fg = 0$ for every $g \in \mathcal{V}$. By taking $g = 1$, we have that $f \in \partial\mathcal{V}$. Hence f has the form (137) for some $i \in I$ and $n \in \mathbb{Z}_+$. But then $u_i^{(n+1)} f$ does not have this form, so that $\int u_i^{(n+1)} f \neq 0$. $\square$

Definition 2. *The algebra of differential functions $\mathcal{V}$ is called* **normal** *if we have $\frac{\partial}{\partial u_i^{(n)}}\left(\mathcal{V}_{n,i}\right) = \mathcal{V}_{n,i}$ for all $i \in I, n \in \mathbb{Z}_+$. Given $f \in \mathcal{V}_{n,i}$, we denote by $\int du_i^{(n)} f \in \mathcal{V}_{n,i}$ a preimage of f under the map $\frac{\partial}{\partial u_i^{(n)}}$. This integral is defined up to adding elements from $\mathcal{V}_{n,i-1}$.*

Proposition 13. *Any normal algebra of differential functions $\mathcal{V}$ is an extension of R_ℓ.*

Proof. As pointed out above, we need to find elements $u_i^{(n)} \in \mathcal{V}$, for $i \in I, n \in \mathbb{Z}_+$, such that (135) holds. By the normality assumption, there exists $v_i^n \in \mathcal{V}_{n,i}$ such that $\frac{\partial v_i^n}{\partial u_i^{(n)}} = 1$. Note that $\frac{\partial}{\partial u_i^{(n)}} \frac{\partial v_i^n}{\partial u_{i-1}^{(n)}} = \frac{\partial 1}{\partial u_{i-1}^{(n)}} = 0$, hence $\frac{\partial v_i^n}{\partial u_{i-1}^{(n)}} \in \mathcal{V}_{n,i-1}$. If we then replace v_i^n by $w_i^n = v_i^n - \int du_{i-1}^n \frac{\partial v_i^n}{\partial u_{i-1}^{(n)}}$, we have that $\frac{\partial w_i^n}{\partial u_i^{(n)}} = 1$ and $\frac{\partial w_i^n}{\partial u_{i-1}^{(n)}} = 0$. Proceeding by downward induction, we obtained the desired element $u_i^{(n)}$. $\square$

Clearly, the algebra R_ℓ is normal. Moreover, any extension $\mathcal{V}$ of R_ℓ can be further extended to a normal algebra, by adding missing integrals. For example, the localization of $R_1 = \mathbb{F}[u^{(n)} \,|\, n \in \mathbb{Z}_+]$ by u is not a normal algebra, since it doesn't contain $\int \frac{du}{u}$. Note that any differential algebra (A, ∂) can be viewed as a trivial algebra of differential functions with $\frac{\partial}{\partial u_i^{(n)}} = 0$. Such an algebra does not contain R_ℓ, hence it is not normal.

5.3. The complex of variational calculus. Let $\mathcal{V}$ be an algebra of differential functions. The *basic de Rham complex* $\widetilde{\Omega}^\bullet = \widetilde{\Omega}^\bullet(\mathcal{V})$ is defined as the free commutative superalgebra over $\mathcal{V}$ with odd generators $\delta u_i^{(n)}$, $i \in I, n \in \mathbb{Z}_+$. In other words $\widetilde{\Omega}^\bullet$ consists of finite sums of the form

$$\widetilde{\omega} = \sum_{i_r\in I, m_r\in\mathbb{Z}_+} f_{i_1\cdots i_k}^{m_1\cdots m_k}\, \delta u_{i_1}^{(m_1)} \wedge \cdots \wedge \delta u_{i_k}^{(m_k)}, \quad f_{i_1\cdots i_k}^{m_1\cdots m_k} \in \mathcal{V}, \tag{138}$$

and it has a (super)commutative product given by the wedge product $\wedge$. We have a natural $\mathbb{Z}_+$-grading $\widetilde{\Omega}^\bullet = \bigoplus_{k\in\mathbb{Z}_+} \widetilde{\Omega}^k$ defined by saying that elements in $\mathcal{V}$ have degree 0, while the generators $\delta u_i^{(n)}$ have degree 1. Hence $\widetilde{\Omega}^k$ is a free module over $\mathcal{V}$ with basis given by the elements $\delta u_{i_1}^{(m_1)} \wedge \cdots \wedge \delta u_{i_k}^{(m_k)}$, with $(m_1, i_1) > \cdots > (m_k, i_k)$ (with respect to the lexicographic order). In particular $\widetilde{\Omega}^0 = \mathcal{V}$ and $\widetilde{\Omega}^1 = \bigoplus_{i\in I, n\in\mathbb{Z}_+} \mathcal{V}\delta u_i^{(n)}$. Notice that there is a natural $\mathcal{V}$-linear pairing $\widetilde{\Omega}^1 \times \mathfrak{g} \to \mathcal{V}$ defined on generators by $\left(\delta u_i^{(m)}, \frac{\partial}{\partial u_j^{(n)}}\right) = \delta_{i,j}\delta_{m,n}$, and extended to $\widetilde{\Omega}^1 \times \mathfrak{g}$ by $\mathcal{V}$-bilinearity.

We let δ be an odd derivation of degree 1 of the complex $\widetilde{\Omega}^\bullet$, such that $\delta f = \sum_{i\in I,\, n\in\mathbb{Z}_+} \frac{\partial f}{\partial u_i^{(n)}} \delta u_i^{(n)}$ for $f \in \mathcal{V}$, and $\delta(\delta u_i^{(n)}) = 0$. It is immediate to check that $\delta^2 = 0$ and that, for $\widetilde{\omega} \in \widetilde{\Omega}^k$ as in (138), we have

$$\delta(\widetilde{\omega}) = \sum_{\substack{i_r\in I, m_r\in\mathbb{Z}_+\\ j\in I, n\in\mathbb{Z}_+}} \frac{\partial f_{i_1\cdots i_k}^{m_1\cdots m_k}}{\partial u_j^{(n)}} \delta u_j^{(n)} \wedge \delta u_{i_1}^{(m_1)} \wedge \cdots \wedge \delta u_{i_k}^{(m_k)}. \tag{139}$$

For $X \in \mathfrak{g}$ we define the *contraction operator* $\iota_X : \widetilde{\Omega}^\bullet \to \widetilde{\Omega}^\bullet$, as an odd derivation of $\widetilde{\Omega}^\bullet$ of degree -1, such that $\iota_X(f) = 0$ for $f \in \mathcal{V}$, and $\iota_X(\delta u_i^{(n)}) = X(u_i^{(n)})$. If $X \in \mathfrak{g}$ is as in (132) and $\widetilde{\omega} \in \widetilde{\Omega}^k$ is as in (138), we have

$$\iota_X(\widetilde{\omega}) = \sum_{i_r\in I, m_r\in\mathbb{Z}_+} \sum_{q=1}^{k} (-1)^{q+1} f_{i_1\cdots i_k}^{m_1\cdots m_k} P_{i_q,m_q} \delta u_{i_1}^{(m_1)} \wedge \stackrel{q}{\check{\cdots}} \wedge \delta u_{i_k}^{(m_k)}. \tag{140}$$

In particular, for $f \in \mathcal{V}$ we have

$$\iota_X(\delta f) = X(f). \tag{141}$$

It is easy to check that the operators $\iota_X,\ X \in \mathfrak{g}$, form an abelian (purely odd) subalgebra of the Lie superalgebra Der $\widetilde{\Omega}^\bullet$, namely

$$[\iota_X, \iota_Y] = \iota_X \circ \iota_Y + \iota_Y \circ \iota_X = 0. \tag{142}$$

The *Lie derivative* L_X along $X \in \mathfrak{g}$ is defined as a degree 0 derivation of the superalgebra $\widetilde{\Omega}^\bullet$, commuting with δ, and such that

$$L_X(f) = X(f) \quad \text{for} \quad f \in \widetilde{\Omega}^0. \tag{143}$$

One can easily check (on generators) Cartan's formula (cf. (45)):

$$L_X = [\delta, \iota_X] = \delta \circ \iota_X + \iota_X \circ \delta. \tag{144}$$

We next prove the following:

$$[\iota_X, L_Y] = \iota_X \circ L_Y - L_Y \circ \iota_X = \iota_{[X,Y]}. \tag{145}$$

It is clear by degree considerations that both sides of (145) act as zero on $\widetilde{\Omega}^0 = \mathcal{V}$. Moreover, it follows by (141) that $[\iota_X, L_Y](\delta f) = \iota_X\delta\iota_Y\delta f - \iota_Y\delta\iota_X\delta f = X(Y(f)) - Y(X(f)) = [X, Y](f) = \iota_{[X,Y]}(\delta f)$ for every $f \in \mathcal{V}$. Equation (145) then follows by the fact that both sides are even derivations of the wedge product in $\widetilde{\Omega}$. Finally, as immediate consequence of Eq. (145), we get that

$$[L_X, L_Y] = L_X \circ L_Y - L_Y \circ L_X = L_{[X,Y]}. \tag{146}$$

Thus, $\widetilde{\Omega}^\bullet$ is a $\mathfrak{g}$-complex, $\widehat{\mathfrak{g}}$ acting on $\widetilde{\Omega}^\bullet$ by derivations.

Note that the action of ∂ on $\mathcal{V}$ extends to a degree 0 derivation of $\widetilde{\Omega}^\bullet$, such that

$$\partial(\delta u_i^{(n)}) = \delta u_i^{(n+1)},\ i \in I,\ n \in \mathbb{Z}_+. \tag{147}$$

This derivation commutes with δ, hence we can consider the corresponding *reduced de Rham complex* $\Omega^\bullet = \Omega^\bullet(\mathcal{V})$, usually called the *complex of variational calculus*:

$$\Omega^\bullet = \bigoplus_{k\in\mathbb{Z}_+} \Omega^k, \quad \Omega^k = \widetilde{\Omega}^k/\partial\widetilde{\Omega}^k,$$

with the induced action of δ. With an abuse of notation, we denote by δ and, for $X \in \mathfrak{g}^\partial$, by ι_X, L_X, the maps induced on the quotient space Ω^k by the corresponding maps on $\widetilde{\Omega}^k$. Obviously, $\Omega^\bullet$ is a $\mathfrak{g}^\partial$-complex.

5.4. Isomorphism of the cohomology $\mathfrak{g}^\partial$-complexes $\Omega^\bullet$ and $\Gamma^\bullet$.

Proposition 14. *Let $\mathcal{V}$ be an algebra of differential functions. Consider the Lie conformal algebra $A = \oplus_{i\in I}\mathbb{F}[\partial]u_i$ with the zero λ-bracket. Then $\mathcal{V}$ is a module over the Lie conformal algebra A, with the λ-action given by*

$$u_{i\lambda} f = \sum_{n\in\mathbb{Z}_+} \lambda^n \frac{\partial f}{\partial u_i^{(n)}}. \tag{148}$$

Moreover, the λ-action of A on $\mathcal{V}$ is by derivations of the associative product in $\mathcal{V}$.

Proof. The fact that $\mathcal{V}$ is an A-module follows from the definition of an algebra of differential functions. The second statement is clear as well. □

Let $\widetilde{\Gamma}^\bullet = \widetilde{\Gamma}^\bullet(A,\mathcal{V})$ and $\Gamma^\bullet = \Gamma^\bullet(A,\mathcal{V})$ be the basic and reduced Lie conformal algebra cohomology complexes for the A-module $\mathcal{V}$, defined in Proposition 14. Thus, to every algebra of differential functions $\mathcal{V}$ we can associate two apparently unrelated types of cohomology complexes: the basic and reduced de Rham cohomology complexes, $\widetilde{\Omega}^\bullet(\mathcal{V})$ and $\Omega^\bullet(\mathcal{V})$, defined in Sect. 5.3, and the basic and reduced Lie conformal algebra cohomology complexes $\widetilde{\Gamma}^\bullet(A,\mathcal{V})$ and $\Gamma^\bullet(A,\mathcal{V})$, defined in Sect. 2.1, for the Lie conformal algebra $A = \bigoplus_{i\in I}\mathbb{F}[\partial]u_i$, with the zero λ-bracket, acting on $\mathcal{V}$, with the λ-action given by (148). We are going to prove that, in fact, these complexes are isomorphic, and all the related structures (such as exterior products, contraction operators, Lie derivatives,...) correspond via this isomorphism.

We denote, as in Sect. 4.2, by $\widetilde{\Gamma}_\bullet = \widetilde{\Gamma}_\bullet(A,\mathcal{V})$ (resp. $\Gamma_\bullet = \Gamma_\bullet(A,\mathcal{V})$) the basic (resp. reduced) space of chains of A with coefficients in $\mathcal{V}$. Recall from Sect. 4.4 that $\Pi\widetilde{\Gamma}_1$ is identified with the space $(A\otimes\mathcal{V}[[x]])\big/(\partial\otimes 1 + 1\otimes\partial_x)(A\otimes\mathcal{V}[[x]])$, and it carries a Lie algebra structure given by the Lie bracket (65), which in this case takes the form, for $i,j\in I$ and $P(x) = \sum_{m\in\mathbb{Z}_+}\frac{1}{m!}P_m x^m$, $Q(x) = \sum_{n\in\mathbb{Z}_+}\frac{1}{n!}Q_n x^n \in \mathcal{V}[[x]]$:

$$[u_i\otimes P(x), u_j\otimes Q(x)] = -u_i\otimes\sum_{n\in\mathbb{Z}_+} Q_n\frac{\partial P(x)}{\partial u_j^{(n)}} + u_j\otimes\sum_{m\in\mathbb{Z}_+} P_m\frac{\partial Q(x)}{\partial u_i^{(m)}}. \tag{149}$$

Moreover, ∂ acts on $\widetilde{\Gamma}_1$ by (59). Its kernel $\Pi\Gamma_1$ consists of elements of the form

$$\sum_{i\in I} u_i\otimes e^{x\partial}P_i, \quad \text{where } P_i\in\mathcal{V}, \tag{150}$$

and it is a Lie subalgebra of $\Pi\widetilde{\Gamma}_1$. We also denote, as in Sect. 5.1, by $\mathfrak{g}$ the Lie algebra of all vector fields (132) acting on $\mathcal{V}$, and by $\mathfrak{g}^\partial\subset\mathfrak{g}$ the Lie subalgebra of evolutionary vector fields (133).

Proposition 15. *The map $\Phi_1 : \Pi\widetilde{\Gamma}_1 \to \mathfrak{g}$, which maps*

$$\xi = \sum_{i\in I} u_i \otimes P_i(x) = \sum_{i\in I, n\in\mathbb{Z}_+} \frac{1}{n!} u_i \otimes P_{i,n} x^n \in \widetilde{\Gamma}_1, \tag{151}$$

to

$$\Phi_1(\xi) = \sum_{i\in I,\, n\in\mathbb{Z}_+} P_{i,n} \frac{\partial}{\partial u_i^{(n)}}, \tag{152}$$

is a Lie algebra isomorphism. Moreover, the image of the space of reduced 1-chains via Φ_1 is the space of evolutionary vector fields. Hence we have the induced Lie algebra isomorphism $\Phi_1 : \Pi\Gamma_1 \xrightarrow{\sim} \mathfrak{g}^\partial$.

Proof. Clearly, Φ_1 is a bijective map, and, by (150), $\Phi_1(\Gamma_1) = \mathfrak{g}^\partial$. Hence we only need to check Φ_1 is a Lie algebra homomorphism. This is immediate from Eq. (149). $\square$

Theorem 6. *The map $\Phi^\bullet : \widetilde{\Gamma}^\bullet \to \widetilde{\Omega}^\bullet$, such that $\Phi^0 = \mathbb{I}|_\mathcal{V}$ and, for $k \geq 1$, $\Phi^k : \widetilde{\Gamma}^k \to \widetilde{\Omega}^k$ is given by*

$$\Phi^k(\widetilde{\gamma}) = \frac{1}{k!} \sum_{i_r\in I, m_r\in\mathbb{Z}_+} f_{i_1\cdots i_k}^{m_1\cdots m_k}\, \delta u_{i_1}^{(m_1)} \wedge \cdots \wedge \delta u_{i_k}^{(m_k)}, \tag{153}$$

where $f_{i_1\cdots i_k}^{m_1\cdots m_k} \in \mathcal{V}$ is the coefficient of $\lambda_1^{m_1}\cdots\lambda_k^{m_k}$ in $\widetilde{\gamma}_{\lambda_1,\dots,\lambda_k}(u_{i_1},\dots,u_{i_k})$, is an isomorphism of superalgebras, and an isomorphism of $\mathfrak{g}$-complexes, (once we identify the Lie algebras $\mathfrak{g}$ and $\Pi\widetilde{\Gamma}_1$ via Φ_1, as in Proposition 15).

Moreover, $\Phi^\bullet$ commutes with the action of ∂, hence it induces an isomorphism of the corresponding reduced $\mathfrak{g}^\partial$-complexes: $\Phi^\bullet : \Gamma^\bullet \xrightarrow{\sim} \Omega^\bullet$.

Proof. Since I is a finite index set, the RHS of (153) is a finite sum, so that $\Phi^k(\widetilde{\Gamma}^k) \subset \widetilde{\Omega}^k$. By the sesquilinearity and skew-symmetry Conditions A1 and A2 in Sect. 2.1, elements $\widetilde{\gamma} \in \widetilde{\Gamma}^k$ are uniquely determined by the collection of polynomials $\widetilde{\gamma}_{\lambda_1,\dots,\lambda_k}(u_{i_1},\dots,u_{i_k}) = \sum_{m_r\in\mathbb{Z}_+} f_{i_1\cdots i_k}^{m_1\cdots m_k} \lambda_1^{m_1}\cdots\lambda_k^{m_k}$, which are skew-symmetric with respect to simultaneous permutation of the variables λ_r and the indices i_r. We want to check that Φ^k is a bijective linear map from $\widetilde{\Gamma}^k$ to $\widetilde{\Omega}^k$. In fact, denote by $\Psi^k : \widetilde{\Omega}^k \to \widetilde{\Gamma}^k$ the linear map which to $\widetilde{\omega}$ as in (138) associates the k-cochain $\Psi^k(\widetilde{\omega})$, such that

$$\Psi^k(\widetilde{\omega})_{\lambda_1,\dots,\lambda_k}(u_{i_1},\dots,u_{i_k}) = \sum_{m_r\in\mathbb{Z}_+} \langle f\rangle_{i_1\cdots i_k}^{m_1\cdots m_k} \lambda_1^{m_1}\cdots\lambda_k^{m_k},$$

where $\langle f\rangle$ denotes the skew-symmetrization of f:

$$\langle f\rangle_{i_1\cdots i_k}^{m_1\cdots m_k} = \sum_\sigma \mathrm{sign}(\sigma) f_{i_{\sigma(1)}\cdots i_{\sigma(k)}}^{m_{\sigma(1)}\cdots m_{\sigma(k)}},$$

and $\Psi^k(\widetilde{\omega})$ is extended to $A^{\otimes k}$ by the sesquilinearity Condition A1. It is straightforward to check that $\Psi^k(\widetilde{\omega})$ is indeed a k-cochain, and that the maps Φ^k and Ψ^k are inverse to each other. This proves that $\Phi^\bullet$ is a bijective map.

Next, let us prove that $\Phi^\bullet$ is an associative superalgebra homomorphism. Let $\widetilde{\alpha} \in \widetilde{\Gamma}^h$, $\widetilde{\beta} \in \widetilde{\Gamma}^k$ and let $\alpha_{i_1,\dots,i_h}^{m_1,\dots,m_h}$ be the coefficient of $\lambda_1^{m_1}\cdots\lambda_h^{m_h}$ in the polynomial $\widetilde{\alpha}_{\lambda_1,\dots,\lambda_h}(u_{i_1},\dots,u_{i_h})$, and let $\beta_{j_1,\dots,j_k}^{n_1,\dots,n_k}$ be the coefficient of $\lambda_1^{n_1}\cdots\lambda_k^{n_k}$ in $\widetilde{\beta}_{\lambda_1,\dots,\lambda_k}$

$(u_{j_1}, \dots, u_{j_k})$. By (35), the coefficient of $\lambda_1^{m_1} \cdots \lambda_{h+k}^{m_{h+k}}$ in $(\widetilde{\alpha} \wedge \widetilde{\beta})_{\lambda_1, \dots, \lambda_{h+k}}(u_{i_1}, \dots, u_{i_{h+k}})$ is

$$\sum_{\sigma \in S_{h+k}} \frac{\operatorname{sign}(\sigma)}{h!k!} \alpha_{i_{\sigma(1)}, \dots, i_{\sigma(h)}}^{m_{\sigma(1)}, \dots, m_{\sigma(h)}} \beta_{i_{\sigma(h+1)}, \cdots, i_{\sigma(h+k)}}^{m_{\sigma(h+1)}, \dots, m_{\sigma(h+k)}}.$$

The identity $\Phi^{h+k}(\widetilde{\alpha} \wedge \widetilde{\beta}) = \Phi^h(\widetilde{\alpha}) \wedge \Phi^k(\widetilde{\beta})$ follows by the definition (153) of Φ^k.

Let $\widetilde{\gamma} \in \widetilde{\Gamma}^k$, and denote by $f_{i_1 \cdots i_k}^{m_1 \cdots m_k} \in \mathcal{V}$ the coefficient of $\lambda_1^{m_1} \cdots \lambda_k^{m_k}$ in $\widetilde{\gamma}_{\lambda_1, \dots, \lambda_k}(u_{i_1}, \dots, u_{i_k})$. We want to prove that $\Phi^{k+1}(\delta \widetilde{\gamma}) = \delta \Phi^k(\widetilde{\gamma})$. By assumption, the λ-bracket on A is zero, and the λ-action of A on $\mathcal{V}$ is given by (148). Hence, recalling (10), the coefficient of $\lambda_1^{m_1} \cdots \lambda_{k+1}^{m_{k+1}}$ in the polynomial $(\delta \widetilde{\gamma})_{\lambda_1, \dots, \lambda_{k+1}}(u_{i_1}, \dots, u_{i_{k+1}})$ is

$$\sum_{r=1}^{k+1} (-1)^{r+1} \frac{\partial f_{i_1 \overset{r}{\check{\cdots}} i_{k+1}}^{m_1 \overset{r}{\check{\cdots}} m_{k+1}}}{\partial u_{i_r}^{(m_r)}}.$$

It follows that

$$\begin{aligned} \Phi^{k+1}(\delta \widetilde{\gamma}) &= \frac{1}{(k+1)!} \sum_{i_r \in I, m_r \in \mathbb{Z}_+} \sum_{q=1}^{k+1} (-1)^{q+1} \frac{\partial f_{i_1 \overset{q}{\check{\cdots}} i_{k+1}}^{m_1 \overset{q}{\check{\cdots}} m_{k+1}}}{\partial u_{i_q}^{(m_q)}} \delta u_{i_1}^{(m_1)} \wedge \cdots \wedge \delta u_{i_{k+1}}^{(m_{k+1})} \\ &= \frac{1}{k!} \sum_{i_r \in I, m_r \in \mathbb{Z}_+} \frac{\partial f_{i_1 \cdots i_k}^{m_1 \cdots m_k}}{\partial u_{i_0}^{(m_0)}} \delta u_{i_0}^{(m_0)} \wedge \cdots \wedge \delta u_{i_k}^{(m_k)} = \delta \Phi^k(\widetilde{\gamma}), \end{aligned}$$

thus proving the claim.

Similarly, the coefficient of $\lambda_1^{m_1} \cdots \lambda_k^{m_k}$ in $(\partial \widetilde{\gamma})_{\lambda_1, \dots, \lambda_k}(u_{i_1}, \dots, u_{i_k})$ is $\partial^M f_{i_1 \cdots i_k}^{m_1 \cdots m_k} + \sum_{r=1}^k f_{i_1 \cdots i_k}^{m_1 \cdots m_r - 1 \cdots m_k}$, so that

$$\begin{aligned} \Phi^k(\partial \widetilde{\gamma}) = \frac{1}{k!} \sum_{i_r \in I, m_r \in \mathbb{Z}_+} \Bigg(&\partial^M f_{i_1 \cdots i_k}^{m_1 \cdots m_k} \delta u_{i_1}^{(m_1)} \wedge \cdots \wedge \delta u_{i_k}^{(m_k)} \\ &+ f_{i_1 \cdots i_k}^{m_1 \cdots m_k} \sum_{q=1}^k \delta u_{i_1}^{(m_1)} \wedge \cdots \wedge \delta u_{i_q}^{(m_q+1)} \wedge \cdots \wedge \delta u_{i_k}^{(m_k)} \Bigg) = \partial \Phi^k(\widetilde{\gamma}). \end{aligned}$$

This proves that $\Phi^\bullet$ is compatible with the action of ∂.

Finally, we prove that $\Phi^\bullet$ is compatible with the contraction operators. Let $\widetilde{\gamma} \in \widetilde{\Gamma}^k$ be as in the statement of the theorem, and let $\xi \in \widetilde{\Gamma}_1$ be as in (151). By Eq. (63), we have the following formula for the contraction operator ι_ξ,

$$(\iota_\xi \widetilde{\gamma})_{\lambda_2, \dots, \lambda_k}(u_{i_2}, \dots, u_{i_k}) = \sum_{i_1 \in I} \big\langle P_{i_1}(x_1), \widetilde{\gamma}_{\lambda_1, \lambda_2, \dots, \lambda_k}(u_{i_1}, u_{i_2}, \dots, u_{i_k}) \big\rangle,$$

where $\langle \, , \, \rangle$ denotes the contraction of x_1 with λ_1 defined in (62). Hence, the coefficient of $\lambda_2^{m_2} \cdots \lambda_k^{m_k}$ in $(\iota_\xi \widetilde{\gamma})_{\lambda_2, \dots, \lambda_k}(u_{i_2}, \dots, u_{i_k})$ is

$$\sum_{i_1 \in I, m_1 \in \mathbb{Z}_+} P_{i_1, m_1} f_{i_1 i_2 \cdots i_k}^{m_1 m_2 \cdots m_k}.$$

It follows that

$$\Phi^{k-1}(\iota_\xi(\widetilde{\gamma})) = \frac{1}{(k-1)!} \sum_{i_r \in I, m_r \in \mathbb{Z}_+} P_{i_1,m_1} f_{i_1 i_2 \cdots i_k}^{m_1 m_2 \cdots m_k} \delta u_{i_2}^{(m_2)} \wedge \cdots \wedge \delta u_{i_k}^{(m_k)},$$

which, recalling (140) and (152), is the same as $\iota_{\Phi_1(\xi)}(\Phi^k(\widetilde{\gamma}))$. This completes the proof of the theorem. □

5.5. An explicit construction of the $\mathfrak{g}^\partial$*-complex of variational calculus.* Let $\mathcal{V}$ be an algebra of differential functions in the variables $\{u_i\}_{i\in I}$, let $A = \bigoplus_{i\in I} \mathbb{F}[\partial]u_i$ be the free $\mathbb{F}[\partial]$-module of rank ℓ, considered as a Lie conformal algebra with the zero λ-bracket, and consider the A-module structure on $\mathcal{V}$, with the λ-action given by (148). By Theorem 6, the $\mathfrak{g}^\partial$-complex of variational calculus $\Omega^\bullet(\mathcal{V})$ is isomorphic to the $\Pi\Gamma_1$-complex $\Gamma^\bullet(A,\mathcal{V})$. Furthermore, due to Theorems 2 and 4, the $\Pi\Gamma_1$-complex $\Gamma^\bullet(A,\mathcal{V})$ is isomorphic to the ΠC_1-complex $C^\bullet(A,\mathcal{V}) = \bigoplus_{k\in\mathbb{Z}_+} C^k$, which is explicitly described in Sects. 2.3 and 4.6.

In this section we use this isomorphism to describe explicitly the $\Pi C_1 \simeq \mathfrak{g}^\partial$-complex of variational calculus $C^\bullet(A,\mathcal{V}) \simeq \Omega^\bullet(\mathcal{V})$, both in terms of "poly-symbols", and in terms of skew-symmetric "poly-differential operators". We shall identify these two complexes via this isomorphism.

We start by describing all vector spaces Ω^k and the maps $d: \Omega^k \to \Omega^{k+1}$, $k \in \mathbb{Z}_+$. First, we have

$$\Omega^0 = \mathcal{V}/\partial\mathcal{V}. \tag{154}$$

Next, $\Omega^1 = \mathrm{Hom}_{\mathbb{F}[\partial]}(A,\mathcal{V})$, hence we have a canonical identification

$$\Omega^1 = \mathcal{V}^{\oplus\ell}. \tag{155}$$

Comparing (12) and (148), we see that $d: \Omega^0 \to \Omega^1$ is given by the variational derivative:

$$d{\textstyle\int} f = \frac{\delta f}{\delta u}. \tag{156}$$

For arbitrary $k \geq 1$, the space Ω^k can be identified with the space of k-*symbols* in u_i, $i \in I$. By definition, a k-symbol is a collection of expressions of the form

$$\left\{u_{i_1}{}_{\lambda_1} u_{i_2}{}_{\lambda_2} \cdots u_{i_{k-1}}{}_{\lambda_{k-1}} u_{i_k}\right\} \in \mathbb{F}[\lambda_1,\dots,\lambda_{k-1}] \otimes \mathcal{V}, \tag{157}$$

where $i_1,\dots,i_k \in I$, satisfying the following skew-symmetry property:

$$\left\{u_{i_1}{}_{\lambda_1} u_{i_2}{}_{\lambda_2} \cdots u_{i_{k-1}}{}_{\lambda_{k-1}} u_{i_k}\right\} = \mathrm{sign}(\sigma) \left\{u_{i_{\sigma(1)}}{}_{\lambda_{\sigma(1)}} \cdots u_{i_{\sigma(k-1)}}{}_{\lambda_{\sigma(k-1)}} u_{i_{\sigma(k)}}\right\}, \tag{158}$$

for every permutation $\sigma \in S_k$, where λ_k is replaced, if it occurs in the RHS, by $\lambda_k^\dagger = -\sum_{j=1}^{k-1}\lambda_j - \partial$, with ∂ acting from the left. Clearly, by sesquilinearity, for $k \geq 1$, the space $\Omega^k = C^k$ of k-λ-brackets is one-to-one correspondence with the space of k-symbols.

For example, the space of 1-symbols is the same as $\mathcal{V}^{\oplus\ell}$. A 2-symbol is a collection of elements $\{u_{i\,\lambda}u_j\} \in \mathbb{F}[\lambda]\otimes\mathcal{V}$, for $i, j \in I$, such that

$$\{u_{i\,\lambda}u_j\} = -\{u_{j\,-\lambda-\partial}u_i\}.$$

A 3-symbol is a collection of elements $\left\{u_{i\,\lambda}u_{j\,\mu}u_k\right\} \in \mathbb{F}[\lambda,\mu]\otimes\mathcal{V}$, for $i, j, k \in I$, such that

$$\left\{u_{i\,\lambda}u_{j\,\mu}u_k\right\} = -\left\{u_{j\,\mu}u_{i\,\lambda}u_k\right\} = -\left\{u_{i\,\lambda}u_{k\,-\lambda-\mu-\partial}u_j\right\},$$

and similarly for $k > 3$.

Comparing (13) and (148) we see that, if $F \in \mathcal{V}^{\oplus\ell}$, its differential dF corresponds to the following 2-symbol:

$$\{u_{i\,\lambda}u_j\} = \sum_{n\in\mathbb{Z}_+}\left(\lambda^n\frac{\partial F_j}{\partial u_i^{(n)}} - (-\lambda-\partial)^n\frac{\partial F_i}{\partial u_j^{(n)}}\right) = (D_F)_{ji}(\lambda) - (D_F^*)_{ji}(\lambda), \tag{159}$$

where D_F is the Frechet derivative defined by (9). More generally, the differential of a k-symbol for $k \geq 1$ is given by the following formula:

$$\begin{aligned} d&\left(\{u_{i_1\,\lambda_1}\cdots u_{i_{k-1}\,\lambda_{k-1}}u_{i_k}\}\right)_{i_1,\dots,i_k\in I} \\ &= \left(\sum_{n\in\mathbb{Z}_+}\sum_{s=1}^{k}(-1)^{s+1}\lambda_s^n\frac{\partial}{\partial u_{i_s}^{(n)}}\left\{u_{i_1\,\lambda_1}\overset{s}{\check{\cdots}}\,u_{i_k\,\lambda_k}u_{i_{k+1}}\right\}\right. \\ &\quad\left.+(-1)^k\sum_{n\in\mathbb{Z}_+}\left(-\sum_{j=1}^{k}\lambda_j-\partial\right)^n\frac{\partial}{\partial u_{i_{k+1}}^{(n)}}\left\{u_{i_1\,\lambda_1}\cdots u_{i_{k-1}\,\lambda_{k-1}}u_{i_k}\right\}\right)_{i_1,\dots,i_{k+1}\in I}. \end{aligned} \tag{160}$$

Provided that $\mathcal{V}$ is an algebra of differential functions extension of R_ℓ, an equivalent language is that of skew-symmetric poly-differential operators. By definition, a *k-differential operator* is an $\mathbb{F}$-linear map $S : (\mathcal{V}^\ell)^k \to \mathcal{V}/\partial\mathcal{V}$, of the form

$$S(P^1,\dots,P^k) = \int\sum_{\substack{n_1,\dots,n_k\in\mathbb{Z}_+\\ i_1,\dots,i_k\in I}} f_{i_1,\dots,i_k}^{n_1,\dots,n_k}(\partial^{n_1}P_{i_1}^1)\cdots(\partial^{n_k}P_{i_k}^k). \tag{161}$$

The operator S is called skew-symmetric if

$$S(P^1,\dots,P^k) = \operatorname{sign}(\sigma)S(P^{\sigma(1)},\dots,P^{\sigma(k)}),$$

for every $P^1,\dots,P^k \in \mathcal{V}^\ell$ and every permutation $\sigma \in S_k$. Given a k-symbol

$$\left\{u_{i_1\,\lambda_1}\cdots u_{i_{k-1}\,\lambda_{k-1}}u_{i_k}\right\} = \sum_{n_1,\dots,n_{k-1}\in\mathbb{Z}_+} f_{i_1,\dots,i_{k-1},i_k}^{n_1,\dots,n_{k-1}}\lambda_1^{n_1}\cdots\lambda_{k-1}^{n_{k-1}}, \quad i_1,\dots,i_k\in I, \tag{162}$$

where $f_{i_1,\dots,i_k}^{n_1,\dots,n_{k-1}} \in \mathcal{V}$, we associate to it the following poly-differential operator: $S : (\mathcal{V}^\ell)^k \to \mathcal{V}/\partial\mathcal{V}$, is

$$S(P^1,\dots,P^k) = \int\sum_{\substack{n_1,\dots,n_{k-1}\in\mathbb{Z}_+\\ i_1,\dots,i_k\in I}} f_{i_1,\dots,i_{k-1},i_k}^{n_1,\dots,n_{k-1}}(\partial^{n_1}P_{i_1}^1)\cdots(\partial^{n_{k-1}}P_{i_{k-1}}^{k-1})P_{i_k}^k. \tag{163}$$

Clearly, the skew-symmetry property of the k-symbol is translated to the skew-symmetry of the poly-differential operator. Conversely, integrating by parts, any k-differential operator can be written in the form (163). Thus we have a surjective map Ξ from the space of k-symbols to the space of skew-symmetric k-differential operators. Provided that $\mathcal{V}$ is an algebra of differential functions extension of R_ℓ, by Lemma 10(c), the k-differential operator S can be written uniquely in the form (163). Hence, the map Ξ is an isomorphism.

Note that the space of 1-differential operators $S : \mathcal{V}^\ell \to \mathcal{V}/\partial\mathcal{V}$ can be canonically identified with the space $\Omega^1 = \mathcal{V}^{\oplus\ell}$. Explicitly, to the 1-differential operator $S(P) = \int \sum_{i\in I, n\in\mathbb{Z}_+} f_i^n \partial^n P_i$, we associate:

$$\left(\sum_{n\in\mathbb{Z}_+} (-\partial)^n f_i^n\right)_{i\in I} \in \mathcal{V}^{\oplus\ell}. \tag{164}$$

We can write down the expression of the differential $d : \Omega^k \to \Omega^{k+1}$ in terms of poly-differential operators. First, if $F \in \Omega^1 = \mathcal{V}^{\oplus\ell}$, the 2-differential operator corresponding to $dF \in \Omega^2$ is obtained by looking at Eq. (159):

$$\begin{aligned}(dF)(P,Q) &= \textstyle\int \sum_{i\in I} \big(Q_i X_P(F_i) - P_i X_Q(F_i)\big) \\ &= \textstyle\int \sum_{i,j\in I} \big(Q_i D_F(\partial)_{ij} P_j - P_i D_F(\partial)_{ij} Q_j\big),\end{aligned} \tag{165}$$

where X_P denotes the evolutionary vector field associated to $P \in \mathcal{V}^\ell$, defined in (133), and $D_F(\partial)$ is the Frechet derivative (9). Next, if $S : (\mathcal{V}^\ell)^k \to \mathcal{V}/\partial\mathcal{V}$ is a skew-symmetric k-differential operator, its differential dS, obtained by looking at (160), is the following $k+1$-differential operator:

$$(dS)(P^1,\dots,P^{k+1}) = \sum_{s=1}^{k+1} (-1)^{s+1} (X_{P^s} S)(P^1, \overset{s}{\check{\cdots}}, P^{k+1}). \tag{166}$$

In the above formula, if S is as in (161), $X_P S$ denotes the k-differential operator obtained from S by replacing the coefficients $f_{i_1,\dots,i_k}^{n_1,\dots,n_k}$ by $X_P(f_{i_1,\dots,i_k}^{n_1,\dots,n_k})$.

Remark 8. For $k \geq 2$, a k-differential operator can also be understood as a map $S : (\mathcal{V}^\ell)^{k-1} \to \mathcal{V}^{\oplus\ell}$ of the following form:

$$S(P^1,\dots,P^{k-1})_{i_k} = \sum_{\substack{n_1,\dots,n_{k-1}\in\mathbb{Z}_+\\ i_1,\dots,i_{k-1}\in I}} f_{i_1,\dots,i_{k-1},i_k}^{n_1,\dots,n_{k-1}} (\partial^{n_1} P_{i_1}^1)\cdots(\partial^{n_{k-1}} P_{i_{k-1}}^{k-1}). \tag{167}$$

This corresponds to the k-symbol (162) in the obvious way. With this notation, the differential dS is the following map $(\mathcal{V}^\ell)^k \to \mathcal{V}^{\oplus\ell}$:

$$\begin{aligned}(dS)(P^1,\dots,P^k)_i &= \sum_{s=1}^{k} (-1)^{s+1} (X_{P^s} S)(P^1, \overset{s}{\check{\cdots}}, P^k)_i \\ &\quad + (-1)^k \sum_{j\in I, n\in\mathbb{Z}_+} (-\partial)^n \left(P_j^k \frac{\partial S}{\partial u_i^{(n)}}(P^1,\dots,P^{k-1})_j\right).\end{aligned} \tag{168}$$

Recall that the Lie algebra $\mathfrak{g}^\partial \simeq \Pi C_1$ is identified with the space $\mathcal{V}^\ell$ via the map $P \mapsto X_P$, defined in (133). Given $P \in \mathcal{V}^\ell$, we want to describe explicitly the action of the corresponding contraction operator ι_P and the Lie derivative $L_P = [d, \iota_P]$. First, for $F \in \mathcal{V}^{\oplus\ell} = \Omega^1$, we have (cf. (90)):

$$\iota_P(F) = \int \sum_{i \in I} P_i F_i \in \mathcal{V}/\partial\mathcal{V} = \Omega^0. \tag{169}$$

Next, the contraction of a k-symbol for $k \geq 2$ is given by the following formula (cf. (91)):

$$\begin{aligned}&\iota_P \left(\left\{u_{i_1\,\lambda_1} \cdots u_{i_{k-1}\,\lambda_{k-1}} u_{i_k}\right\}\right)_{i_1,\dots,i_k \in I} \\ &= \left(\sum_{i_1 \in I} \left\{u_{i_1\,\partial} u_{i_2\,\lambda_2} \cdots u_{i_{k-1}\,\lambda_{k-1}} u_{i_k}\right\}_{\to} P_{i_1}\right)_{i_2,\dots,i_k \in I},\end{aligned} \tag{170}$$

where, as usual, the arrow in the RHS means that ∂ is moved to the right. For $k = 2$, the above formula becomes

$$\iota_P \left(\left\{u_{i\,\lambda} u_j\right\}\right)_{i,j \in I} = \left(\sum_{j \in I} \left\{u_{j\,\partial} u_i\right\}_{\to} P_j\right)_{i \in I} \in \mathcal{V}^{\oplus\ell} = \Omega^1. \tag{171}$$

We can write the above formulas in the language of poly-differential operators. For a k-differential operator S, we have

$$(\iota_{P^1} S)(P^2, \dots, P^k) = S(P^1, P^2, \dots, P^k). \tag{172}$$

For $k = 2$ $\iota_{P^1} S$ is a 1-differential operator which, by (164), is the same as an element of $\mathcal{V}^{\oplus\ell} = \Omega^1$.

Remark 9. In the interpretation (167) of a k-differential operator, the action of the contraction operator is given by

$$(\iota_{P^1} S)(P^2, \dots, P^{k-1})_{i_k} = S(P^1, P^2, \dots, P^{k-1})_{i_k}.$$

Next, we write the formula for the Lie derivative $L_Q : \Omega^k \to \Omega^k$, associated to $Q \in \mathcal{V}^\ell \simeq \mathfrak{g}^\partial$, using Cartan's formula $L_Q = [\iota_Q, d]$. Recalling (156) and (169), after integration by parts we obtain, for $\int f \in \Omega^0 = \mathcal{V}/\partial\mathcal{V}$:

$$L_Q \left(\int f\right) = \int X_Q(f), \tag{173}$$

where X_Q is the evolutionary vector field corresponding to Q (cf. (133)). Similarly, recalling (159) and (171), we obtain, for $F \in \Omega^1 = \mathcal{V}^{\oplus\ell}$:

$$\begin{aligned} d\iota_Q(F) &= D_F(\partial)^* Q + D_Q(\partial)^* F, \\ \iota_Q d(F) &= D_F(\partial) Q - D_F(\partial)^* Q, \end{aligned}$$

where $D_F(\partial)$ denotes the Frechet derivative (9), and $D_F(\partial)^*$ is the adjoint differential operator. Putting the above formulas together, we get:

$$L_Q F = D_F(\partial) Q + D_Q(\partial)^* F. \tag{174}$$

For $k \geq 2$, L_Q acts on a k-symbol in Ω^k by the following formula, which can be derived from (160) and (170):

$$
\begin{aligned}
&L_Q\{u_{i_1 \lambda_1} \cdots u_{i_{k-1} \lambda_{k-1}} u_{i_k}\} = X_Q\{u_{i_1 \lambda_1} \cdots u_{i_{k-1} \lambda_{k-1}} u_{i_k}\} \\
&+ \sum_{s=1}^{k-1} (-1)^{s+1} \sum_{j \in I} \{u_{j_{\lambda_s+\partial}} u_{i_1 \lambda_1} \overset{s}{\check{\cdots}} u_{i_{k-1} \lambda_{k-1}} u_{i_k}\}_{\to} D_Q(\lambda_s)_{j i_s} \\
&+ (-1)^{k+1} \sum_{j \in I} \{u_{j_{\lambda_k^\dagger+\partial}} u_{i_1 \lambda_1} \cdots u_{i_{k-2} \lambda_{k-2}} u_{i_{k-1}}\}_{\to} D_Q(\lambda_k^\dagger)_{j i_k}.
\end{aligned}
$$

In the RHS the evolutionary vector field X_Q is applied to the coefficients of the k-symbol, in the last two terms the arrow means, as usual, that we move ∂ to the right, $D_Q(\lambda)$ denotes the Frechet derivative (9) considered as a polynomial in λ, and, in the last term, $\lambda_k^\dagger = -\lambda_1 - \cdots - \lambda_{k-1} - \partial$, where ∂ is moved to the left. This formula takes a much nicer form in the language of k-differential operators. Namely we have:

$$
(L_Q S)(P^1, \ldots, P^k) = (X_Q S)(P^1, \ldots, P^k) + \sum_{s=1}^{k} S(P^1, \ldots, X_Q P^s, \ldots, P^k). \tag{175}
$$

Here $X_Q S$ has the same meaning as in Eq. (166). This formula can be obtained from the previous one by integration by parts.

5.6. An application to the classification of symplectic differential operators. Recall that $\mathcal{C} \subset \mathcal{V}$ denotes the subspace (127) of constant functions. In [BDK] we prove the following:

Theorem 7. *If $\mathcal{V}$ is normal, then $H^k(\Omega^\bullet, d) = \delta_{k,0}\mathcal{C}/(\mathcal{C} \cap \partial\mathcal{V})$.*

Recall that a *symplectic differential operator* (cf. [D] and [BDK]) is a skew-adjoint differential operator $S(\partial) = \big(S_{i,j}(\partial)\big)_{i,j \in I} : \mathcal{V}^\ell \to \mathcal{V}^{\oplus\ell}$, which is closed, namely the following condition holds (cf. (168)):

$$
u_{i\,\lambda} S_{kj}(\mu) - u_{j\,\mu} S_{ki}(\lambda) - u_{k_{-\lambda-\mu-\partial}} S_{ji}(\lambda) = 0, \tag{176}
$$

where the λ-action of u_i on $\mathcal{V}$ is defined by (148). We have the following corollary of Theorem 7.

Corollary 1. *If $\mathcal{V}$ is a normal algebra of differential functions, then any symplectic differential operator is of the form: $S_F(\partial) = D_F(\partial) - D_F(\partial)^*$, for some $F \in \mathcal{V}^{\oplus\ell}$. Moreover, $S_F = S_G$ if and only if $F - G = \frac{\delta f}{\delta u}$ for some $f \in \mathcal{V}$.*

A skew-symmetric k-differential operator $S : (\mathcal{V}^\ell)^k \to \mathcal{V}/\partial\mathcal{V}$ is called *symplectic* if it is closed, i.e.

$$
\sum_{s=1}^{k+1} (-1)^{s+1} (X_{P^s} S)(P^1, \overset{s}{\check{\cdots}}, P^{k+1}) = 0.
$$

The following corollary of Theorem 7 is a generalization of Corollary 1 and uses Proposition 13

Corollary 2. *If $\mathcal{V}$ is a normal algebra of differential functions, then any symplectic k-differential operator, for $k \geq 1$, is of the form:*

$$S(P^1, \ldots, P^k) = \sum_{s=1}^{k} (-1)^{s+1} (X_{P^s} T)(P^1, \overset{s}{\check{\ldots}}, P^k),$$

for some skew-symmetric $k-1$-differential operator T. Moreover, T is defined up to adding a symplectic $k-1$-differential operator.

Remark 10. It follows from the proof of Theorem 7 that, Corollaries 1 and 2 hold in any algebra of differential functions $\mathcal{V}$, provided that we are allowed to take F and T respectively in an extension of $\mathcal{V}$, obtained by adding finitely many integrals of elements of $\mathcal{V}$ (an integral of an element $f \in \mathcal{V}_{n,i}$ is a preimage $\int du_i^{(n)} f$ of $\frac{\partial}{\partial u_i^{(n)}}$ independent on $u_j^{(m)}$ with $(m, j) > (n, i)$).

Remark 11. The map Ξ defined in Sect. 5.5 may have a non-zero kernel if $\mathcal{V}$ is not an extension of the algebra R_ℓ, but, of course, for any $\mathcal{V}$ the image of Ξ is a $\mathfrak{g}^\partial$-complex. The 0^{th} term of this complex is $\mathcal{V}/\partial\mathcal{V}$ and the k^{th} term, for $k \geq 1$, is the space of skew-symmetric k-differential operators $S : (\mathcal{V}^\ell)^k \to \mathcal{V}/\partial\mathcal{V}$.

Remark 12. Throughout this section we assumed that the number ℓ of variables u_i is finite, but this assumption is not essential, and our arguments go through with minor modifications. This is the reason for distinguishing $\mathcal{V}^\ell$ from $\mathcal{V}^{\oplus\ell}$, in order to accommodate the case $\ell = \infty$.

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Communicated by Y. Kawahigashi