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Adiabatic Limit and the Slow Motion of Vortices in a Chern-Simons-Schrödinger System

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Abstract: We study a nonlinear system of partial differential equations in which a complex field (the Higgs field) evolves according to a nonlinear Schrödinger equation, coupled to an electromagnetic field whose time evolution is determined by a Chern-Simons term in the action. In two space dimensions, the Chern-Simons dynamics is a Galileo invariant evolution for A, which is an interesting alternative to the Lorentz invariant Maxwell evolution, and is finding increasing numbers of applications in two dimensional condensed matter field theory. The system we study, introduced by Manton, is a special case (for constant external magnetic field, and a point interaction) of the effective field theory of Zhang, Hansson and Kivelson arising in studies of the fractional quantum Hall effect. From the mathematical perspective the system is a natural gauge invariant generalization of the nonlinear Schrödinger equation, which is also Galileo invariant and admits a self-dual structure with a resulting large space of topological solitons (the moduli space of self-dual Ginzburg-Landau vortices). We prove a theorem describing the adiabatic approximation of this system by a Hamiltonian system on the moduli space. The approximation holds for values of the Higgs self-coupling constant λ close to the self-dual (Bogomolny) value of 1. The viability of the approximation scheme depends upon the fact that self-dual vortices form a symplectic submanifold of the phase space (modulo gauge invariance). The theorem provides a rigorous description of slow vortex dynamics in the near self-dual limit.

1. Introduction and Statement of Results

In this article we study vortex dynamics in a nonlinear system of evolution equations (1.5) introduced by Manton (1997). This system is in fact a special case of an effective field theory for the fractional quantum Hall effect (the Zhang-Hansson-Kivelson, or ZHK, model). In addition it is a natural gauge invariant generalization of the nonlinear Schrödinger equation, possessing important structural features (Galileo invariance and self-dual structure with existence of related moduli spaces of solitons) which make it interesting to study for mathematical reasons. After introducing the system under study, and putting it into mathematical and physical context, we explain the necessary background material in order to state our results, which appear in Sect. 1.7.

1.1. Chern-Simons vortex dynamics. We start by motivating the study of Manton's system from the mathematical perspective, before going on to show that it is equivalent to a special case of the ZHK model, and discussing its physical significance.

1.1.1. Manton's system on $\mathbb{R}^2$: mathematical context. To introduce Manton's system, we start with the nonlinear Schrödinger equation on $\mathbb{R}^2$:

$$i\frac{\partial \Phi}{\partial t} = -\Delta\Phi - \frac{\lambda}{2}(1 - |\Phi|^2)\Phi, \tag{1.1}$$

to be solved for $\Phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$; λ is a positive number. This has the following properties:

(i) it defines a globally well-posed Cauchy problem,
(ii) it admits topological soliton solutions, the Ginzburg-Landau vortices, and
(iii) it is invariant under the group of Galilean transformations.

Manton's system is a generalization of (1.1), sharing these properties, which describes the evolution of a complex field Φ, coupled to a dynamically evolving electromagnetic potential $A = A_0 dt + A_1 dx^1 + A_2 dx^2$. On $\mathbb{R}^2$ the system reads explicitly (writing $\langle a, b\rangle = \Re \bar{a} b$):

$$\begin{aligned}
\frac{\partial A_1}{\partial t} + \frac{\partial}{\partial x^1}\left(\frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2}\right) &= -\langle i\Phi, \left(\frac{\partial}{\partial x^2} - iA_2\right)\Phi\rangle,\\
\frac{\partial A_2}{\partial t} + \frac{\partial}{\partial x^2}\left(\frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2}\right) &= +\langle i\Phi, \left(\frac{\partial}{\partial x^1} - iA_1\right)\Phi\rangle,\\
i\left(\frac{\partial}{\partial t} - iA_0\right)\Phi + \sum_{j=1}^{2}\left(\frac{\partial}{\partial x^j} - iA_j\right)^2\Phi &= -\frac{\lambda}{2}(1 - |\Phi|^2)\Phi,\\
\frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2} &= +\frac{1}{2}(1 - |\Phi|^2).
\end{aligned} \tag{1.2}$$

In addition to (i)-(iii) above, this system has the following mathematical properties:

(iv) it is gauge invariant,
(v) self-dual structure and a large space of topological solitons (see Sect. 1.6).

These properties make the study of vortex dynamics in Manton's system interesting, since the self-dual structure makes a rigorous analysis possible when the vortices are arbitrarily close (see Sect. 1.6-1.7). The proof of our results makes use of special mathematical features present due to self-duality which are explained in Sect. 3; these features include complex and symplectic structures on the soliton moduli space, and a foliation of the phase space which we call the Bogomolny foliation.

1.1.2. Equivalence of Manton's system and a special case of the ZHK model. The system (1.3) can be derived from the action $S = c_0 \int s(A, \Phi) d^2x dt$, where

$$s(A, \Phi) = -\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \langle i\Phi, (\partial_t - iA_0)\Phi \rangle + A_0 + |\epsilon^{jk} \partial_j A_k|^2 \\ + |(\partial_j - iA_j)\Phi|^2 + \frac{\lambda}{4}(1 - |\Phi|^2)^2,$$

where Greek indices run over $\{0, 1, 2\}$ for space-time tensorial quantities, Roman indices run over $\{1, 2\}$, and $\epsilon^{\mu\nu\rho}$, ϵ^{jk} are the completely anti-symmetric symbols and the summation convention is understood. This action is one of a class involving the Chern-Simons term $\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$, see [17,22] for a review. It is characteristic of these theories that variation of the action with respect to A_0 gives a constraint equation involving the magnetic field, in this case the final equation of (1.3). This equation is analogous to the Gauss law in ordinary Maxwell theory, and is referred to as a constraint because the previous (dynamical) equations in (1.3) imply that its time derivative vanishes (exactly as do the dynamical Maxwell equations for the Gauss law). This constraint means that many apparently different actions give rise to the same Euler-Lagrange equations: in particular we can replace the above action density with

$$\tilde{s}(A, \Phi) = -\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \langle i\Phi, (\partial_t - iA_0)\Phi \rangle + A_0 + |(\nabla - iA)\Phi|^2 \\ + \frac{\lambda + 1}{4}(1 - |\Phi|^2)^2.$$

We now introduce the ZHK action $S_{\text{ZHK}}(a, \Phi; A^{ext}) = c \int s_{\text{ZHK}} d^2x dt$ and show that Manton's action S is in fact a special case of S_{ZHK}; essentially the same observation appears also in [22, p. 54]. The ZHK action is the action for a mean field description of the quantum Hall effect. This effect refers to the current $J_j = \sigma_{jk} E_k^{ext}$ produced in an effectively two dimensional system of electrons in a *strong* transverse magnetic field, by application of an applied electric field E_k^{ext}. In the right experimental situation the conductivity tensor σ_{jk} is found to be off-diagonal (i.e. $\sigma_{11} = 0 = \sigma_{22}$), with the non-zero entries $\sigma_{12} = -\sigma_{21} = fe^2/\hbar$, where f is an integer, or a fraction, for (respectively) the integer and fractional quantum Hall effect. This quantization of the values of σ_{12} means that as the number of charge carriers is increased there is no corresponding increase in the current - it lies on a plateau - at least until the number of carriers is sufficiently greatly increased, at which point the conductivity moves to another of the quantized values, and the current moves to another plateau. In the mean field description the field Φ interacts with an external (applied) electromagnetic potential A^{ext} and a "statistical" potential a, according to:

$$s_{\text{ZHK}} = \frac{\kappa}{2} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho + \langle i\Phi, (\partial_t - ia_0 - iA_0^{ext})\Phi \rangle + \frac{1}{2m} |(\nabla - ia - iA^{ext})\Phi|^2 \\ + \int (1 - |\Phi(x)|^2) V(x - x')(1 - |\Phi(x')|^2) d^2x',$$

(see [51], [17, Sect. 4.6], or [52, Eqs. (7)-(8)], taking note of the published erratum for the latter reference). To reduce this to $\tilde{s}$ we consider the case of a constant external magnetic field $B^{ext} = \partial_1 A_2^{ext} - \partial_2 A_1^{ext}$ with $A_0^{ext} = 0$. (The standard configuration in quantum Hall experiments involves a *strong* transverse magnetic field applied to an effectively two dimensional electron gas, with relatively small electric potentials applied along one of the planar directions.) Define $A = a + A^{ext}$. Now check that

$$\begin{aligned}
\epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho &= a_0(\partial_1 a_2 - \partial_2 a_1) - a_1(\partial_t a_2 - \partial_2 a_0) + a_2(\partial_t a_1 - \partial_1 a_0) \\
&= A_0(\partial_1 A_2 - \partial_2 A_1 - B^{ext}) - (A_1 - A_1^{ext})(\partial_t A_2 - \partial_2 A_0) \\
&\quad + (A_2 - A_2^{ext})(\partial_t A_1 - \partial_1 A_0) \\
&= \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - 2A_0 B^{ext} \\
&\quad + \partial_t(A_1^{ext} A_2 - A_2^{ext} A_1) + \partial_1(A_0 A_2^{ext}) - \partial_2(A_0 A_1^{ext}) \\
&= \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - 2A_0 B^{ext} + \epsilon^{\mu\nu\rho} \partial_\mu (A_\nu^{ext} A_\rho)
\end{aligned}$$

and deduce that $s_{\text{ZHK}} - \tilde{s}_{\text{ZHK}}$ is a derivative, where

$$\begin{aligned}
\tilde{s}_{\text{ZHK}} = \frac{\kappa}{2}\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - \kappa B^{ext} A_0 + \langle i\Phi, (\partial_t - iA_0)\Phi\rangle + \frac{1}{2m}|(\nabla - iA)\Phi|^2 \\
+ \int (1 - |\Phi(x)|^2) V(x - x')(1 - |\Phi(x')|^2) d^2x'. \qquad (1.3)
\end{aligned}$$

Now recall that derivatives in the action density do not affect the corresponding Euler-Lagrange equations (they are null Lagrangians). It follows by comparing $\tilde{s}$ and $\tilde{s}_{\text{ZHK}}$ that the equations of motion for S_{ZHK} will be identical to those of Manton if we choose $V(x) = (\lambda + 1)\delta(x)/4$, $\kappa = -2$, $B^{ext} = 1/2$ and $m = 1/2$. Therefore, we conclude that, at least as far as the classical equations of motion are concerned, the ZHK model with these values is the same as Manton's system in the case of

- a constant external magnetic field of appropriate value, and
- a point interaction $V(x) \propto \delta(x)$.

We now discuss the physical interpretation of the model in the fractional quantum Hall context. There is a microscopic model, due to Laughlin, which explains the observed phenomena in a well-accepted way in terms of a new phase of the two dimensional electron gas (for low temperature and high magnetic fields), with ground state described by the Laughlin wave function. There is an energy gap in the spectrum, so that the excitations above this new ground state have strictly positive energy - the Laughlin quasi-particles and quasi-holes, which have fractional statistics and fractional charge. It is this fractional charge, combined with the explanation of the integer quantum Hall effect, which gives rise to the fractional quantum Hall effect. The effective field theory proposed in [52], and reviewed at length in [51], gives a mean field description which is not expected to be accurate on microscopic length scales, but which does give an alternative explanation of all the main observed phenomena. In this mean field theory, the elementary excitations are described by the topological vortices, which are endowed with the same fractional charge and fractional statistics as the Laughlin quasi-particles. (The Chern-Simons term for the statistical gauge field a in the action serves to change the statistics in the well known way explained in [2,51]). It is understood that, in the mean field picture, it is the pinning of vortices which explains the observed plateaus in the Hall conductance ([34,42,51]). Thus a good understanding of vortex dynamics in the ZHK model should be useful to gain a better explanation of the phenomena within the context of the mean field approach. Needless to say there is still much work to be done to go from the results of this paper to results which would apply directly to the experimental situation: even apart from issues like the spatial domain and the real values of the coefficients in the model, it will be necessary to treat the applied electric potential which produces the Hall current flow. This means that in the above derivation we should allow for an external electric field $E_k^{ext} = -\partial_k A_0$, in addition to the static magnetic field, and investigate its effect on the vortex motion. Since the magnetic field is very strong in the experimental situation, it should be reasonable to treat the electric field perturbatively.

To conclude the discussion of the motivation for our work, the system (1.3) is one of a class of dynamical Chern-Simons vortex models whose study is mathematically interesting (due to properties (i)–(v) above), and which is physically relevant (as we have just discussed). The use of such models in condensed matter applications is phenomenological, so the precise Lagrangian and many values of the coupling constants, etc. are not precisely known. (Actually, in [51,52] the ZHK action (1.3) is derived formally from an ostensibly microscopic, second quantized, action. However, this microscopic action itself seems to have a phenomenological character, since it involves excitations which are not fundamental electrons, but rather collective excitations - see the discussion following (2.6) in [42]). In any case our main result provides a rigorous basis for understanding vortex dynamics in a prototype for a class of theories which are of interest in two dimensional condensed matter theory. The adiabatic limit system (1.22) which we derive for the vortex dynamics cannot usually be written down explicitly, but as discussed in Remark 1.7.4, the behaviour of some of its solutions can be understood in many cases, and thus information on the dynamics of vortices can be deduced within the framework of this approximation. There are reasons to hope that qualitative features of the motion in this limiting situation will have a wider validity: see Remark 1.7.4.

As a final comment on the quantum Hall effect, there is another type of soliton - a nonlocal Skyrmion - which appears in treatments of the ferromagnetic properties of quantum Hall samples (see [18,41,49 and 14] for some analytic properties of these Skyrmions in a particular case).

1.1.3. General physical context for Chern-Simons models There has been a fairly long standing interest in systems of the type (1.3) in the physics literature; we give a brief summary and refer the reader to [17,22,23] for detailed reviews. The study of Chern-Simons dynamics in 2+1 dimensional Maxwell and (non-abelian gauge) theories was started in the early 80's (see e.g. [11]) and the incorporation of vortices into this dynamics (in systems with coupling to a nonlinear Schrödinger equation) has been studied since at least the early 90's by theoretical physicists (see the review [23] for early work on Chern-Simons vortices). The reason for this interest is both because (i) the Chern-Simons models are used widely in condensed matter physics in descriptions of the quantum Hall effect and high T superconductivity, and (ii) because they provide a useful scenario in which to probe certain complex issues in field theories.

Regarding the first point, there are various time-dependent models for magnetic vortices but at very low temperatures it is argued ([3,43]) that the motion should be non-dissipative so the usual Eliashberg-Gorkov equation is not appropriate, and the Chern-Simons coupled to Schrödinger vortex dynamics is widely used instead in the condensed matter literature, both in superconductivity and the quantum Hall effect; see [34, Sect. 10.7], [32, Chap. 6] for general discussions, in addition to the references for the ZHK model in the previous section. (Relativistic invariance is broken in these condensed matter applications, so the corresponding relativistic abelian Higgs model, whose vortex dynamics are studied in [44], is not appropriate. The main application which has been suggested for the relativistic dynamics appears to be cosmic string evolution.) There have been explanations offered for the wide occurrence of Cherns-Simons types models in two dimensional condensed matter applications in terms of universality features of large scale effective actions for two dimensional interacting electronic and magnetic systems with spin ([16, Sect. 3]).

Regarding the second reason for interest in these models, it was realized in the 1980's that in two dimensions there were possible quantum statistics other than the usual fermionic and bosonic types - anyons, are two dimensional quantum particles undergoing an arbitrary phase shift on interchange. Furthermore, composite objects made up from charged particles orbiting vortices (or flux tubes) have fractional spin and statistics ([50]). In [15] the authors study the quantum theory of a Lagrangian which is closely related to (1.12), and use it to investigate the quantization of solitons, quantum statistics and anyons in a rigorous quantum field theory setting.

1.2. Organization of the article. The article is organized as follows. Our main aim is the study of vortex dynamics in the Chern-Simons-Schrödinger system with spatial domain a Riemann surface, so we start in the next section by writing down the equations in this case, and then giving necessary background including a discussion of the self-dual vortices in Sect. 1.6. We then state our main result, Theorem 1.7.2, which describes the adiabatic approximation of vortex motion in the self-dual limit. This is proved in Sect. 2 following a strategy explained in the context of a simple model problem in Sect. 1.8. The proof uses some specialized identities related to the self-dual (or Bogomolny) structure, presented in Sect. 3 (which may be read separately). Various subsidiary facts and lemmas are given in the Appendix.

1.3. The equations on a surface. The dependent variables are a complex field $\Phi(t, x)$, and an electromagnetic potential 1-form

$$A_0 dt + A_1 dx^1 + A_2 dx^2.$$

This 1-form determines a covariant derivative operator

$$D = (D_0, D_1, D_2) = \left(\frac{\partial}{\partial t} - iA_0, D_1, D_2\right) = \left(\frac{\partial}{\partial t} - iA_0, \nabla_1 - iA_1, \nabla_2 - iA_2\right), \tag{1.4}$$

which in turn determines the electric field $E = E_j dx^j$ and magnetic field $B(t, x)$ via (1.6); all these fields are defined for $(t, x) \in \mathbb{R} \times \Sigma$, where Σ is a two dimensional spatial domain, taken to be a Riemann surface with metric $g_{jk} dx^j dx^k$, area form $d\mu_g$ and complex structure $J : T^*\Sigma \to T^*\Sigma$ (where $j, k, \ldots$ take values in $\{1, 2\}$ and we use the summation convention). Introducing a covariant Laplacian operator by

$$-\Delta_A \Phi = -\frac{1}{\sqrt{\det g}} D_j \left(g^{ij} \sqrt{\det g} D_i \Phi\right)$$

(using a local frame and coordinates), the equations are

$$\begin{aligned} E_j + \frac{\partial B}{\partial x^j} &= -J_j^k \langle i\Phi, D_k \Phi\rangle, \\ i\left(\frac{\partial}{\partial t} - iA_0\right)\Phi &= -\Delta_A \Phi - \frac{\lambda}{2}(1 - |\Phi|^2)\Phi, \\ B &= \frac{1}{2}(1 - |\Phi|^2). \end{aligned} \tag{1.5}$$

The electric and magnetic field can be combined to give the space-time electromagnetic field

$$F_{\mu\nu}dx^\mu \wedge dx^\nu = E_j dt \wedge dx^j + Bd\mu_g.$$

This two form is obtained as the commutator of the space-time covariant derivative (1.4) which mediates the coupling in (1.5):

$$[D_\mu, D_\nu]\Phi = -iF_{\mu\nu}\Phi, \quad \text{where } F_{0k} = E_k, \quad \text{and} \quad \frac{1}{2}F_{jk}dx^j dx^k = Bd\mu_g. \quad (1.6)$$

(Greek indices run through 0, 1, 2 and Latin indices through 1, 2 only. Boldface is used to indicate the spatial part of a vector or one-form etc., except in Sect. 3 where time does not appear at all.)

We now describe this set-up briefly in geometrical terms. Assume given a one dimensional complex vector bundle $L \to \Sigma$, with a real inner product h locally of the form $\langle a, b\rangle = h\Re\bar{a}b$, and corresponding norm $|a|^2 = \langle a, a\rangle$; if we employ a unitary frame over some chart then $\langle a, b\rangle = \Re\bar{a}b$. We are then solving for an S^1 connection on the bundle $\mathbb{L} \equiv \mathbb{R} \times L \to \mathbb{R} \times \Sigma$, with associated covariant derivative D, and a section Φ of $\mathbb{L}$. To be more explicit, fix a smooth connection on L determined by a covariant derivative operator ∇, so that the spatial part of D, which will be written $\mathbf{D}$, takes the form $D_j = \nabla_j - iA_j$ for a real 1-form $\mathbf{A} = A_j dx^j \in \Omega^1_{\mathbb{R}}(\Sigma)$; here ∇ is independent of time. (It is generally not possible to choose ∇ to be flat, and it will have a curvature, determined by a function b such that $[\nabla_j, \nabla_k]\Phi dx^j dx^k = -ibd\mu_g\Phi$; it is always possible to choose $b = \text{const.}$, and we will do this throughout.) In any case, with this procedure the space of connections on L can be identified with the space of real one-forms. Then at each time $t \in \mathbb{R}$ we are solving for a section $\Phi(t)$ of L, a 1-form $\mathbf{A}(t) = A_1(t)dx^1 + A_2(t)dx^2$ on Σ, and a real valued function $A_0(t)$ on Σ. The electric field is given by

$$E_j = \frac{\partial A_j}{\partial t} - \frac{\partial A_0}{\partial x^j},$$

and the magnetic field by

$$Bd\mu_g = bd\mu_g + \mathbf{dA}.$$

(Here, and elsewhere, we write $\mathbf{d}$ in boldface when it is necessary to indicate that only the spatial part is taken.) The 2-form $-iE_j dt \wedge dx^j - iBd\mu_g$ is the curvature associated to the space-time covariant derivative D, as in (1.6). For the case $\Sigma = \mathbb{R}^2$, the system was proposed by Manton (1997), who derived it as the Euler-Lagrange equation for the Lagrangian (1.12).

Notation 1.3.1. We shall always consider conformal co-ordinate systems on Σ in which the metric is of the form $g = e^{2\rho}\left((dx^1)^2 + (dx^2)^2\right)$ and the volume element is then $e^{2\rho}dx^1 \wedge dx^2$. On functions the Hodge operator acts as $*f = fd\mu_g = fe^{2\rho}dx^1 \wedge dx^2$ and $*^2 = 1$, so that $*d\omega = e^{-2\rho}(\frac{\partial\omega_2}{\partial x^1} - \frac{\partial\omega_1}{\partial x^2})$ for 1-forms ω. On 1-forms $*(\omega_1 dx^1 + \omega_2 dx^2) = \omega_1 dx^2 - \omega_2 dx^1$, which is just the negative of the complex structure J, represented in conformal co-ordinates by the anti-symmetric tensor J_i^j with $J_2^1 = -1$, $J_1^2 = +1$, the other components being zero. Correspondingly we decompose a one-form as $\omega = \omega^{(1,0)}dz + \omega^{(0,1)}d\bar{z}$; in particular for the derivative $df = \partial f dz + \bar{\partial} f d\bar{z}$, with $\bar{\partial} f = \frac{1}{2}(\frac{\partial f}{\partial x^1} + i\frac{\partial f}{\partial x^2})$, and

$$\mathbf{D}\Phi = D^{(1,0)}\Phi + D^{(0,1)}\Phi = \partial_{\mathbf{A}}\Phi dz + \bar{\partial}_{\mathbf{A}}\Phi d\bar{z},$$

with $\bar{\partial}_{\mathbf{A}}\Phi = \frac{1}{2}\left((\nabla_1 - iA_1) + i(\nabla_2 - iA_2)\right)\Phi$ etc.; see Sect. 3. For a 1-form $\mathbf{A}$ we write the co-differential $\mathbf{d}^*\mathbf{A} = -\mathrm{div}\,\mathbf{A}$, with $\mathrm{div}\,\mathbf{A} = e^{-2\rho}(\frac{\partial A_1}{\partial x^1} + \frac{\partial A_2}{\partial x^2})$, and the Laplacian on real functions is $\Delta f = e^{-2\rho}\frac{\partial^2 f}{\partial x^i \partial x^i}$, (with the summation convention), and on sections of L the covariant Laplacian is $-\Delta_{\mathbf{A}}\Phi = e^{-2\rho}(D_1^2 + D_2^2)\Phi$ when a unitary frame is used. The operators div, $*d$, Δ (resp. $\Delta_{\mathbf{A}}$) all depend on g (resp. g, h), but this is not indicated as g, h are fixed, and similarly dependence of constants in estimates on (Σ, g) and h will be suppressed throughout the article.

Notation 1.3.2. We are dealing with sections of smooth vector bundles V over Σ with an inner product $\langle\cdot,\cdot\rangle$ induced from the Riemannian metric g and the metric h on L in the standard way; since g, h are fixed throughout they will not be indicated. Thus, for example,

$$|\mathbf{D}\Phi|^2 = e^{-2\rho}\left(\langle D_1\Phi, D_1\Phi\rangle + \langle D_2\Phi, D_2\Phi\rangle\right).$$

We write $\Omega^0(V)$ for the smooth sections of V and $\Omega^p(V)$ for the smooth p-forms taking values in V. We will make use of the Sobolev spaces $H^s(V)$ of sections of V whose coefficient functions (in any frame over any open set $\Omega \subset \Sigma$) lie in the standard Sobolev space $H^s(\Omega)$; the corresponding Sobolev space of V-valued p-forms is denoted $H^s(\Omega^p(V))$. In Sect. 1 and Sect. 2 we shall generally omit explicit reference to the vector bundle, since this is usually clear, and write H^s in place of $H^s(V)$ etc. (and $\|\cdot\|_{H^s}$ for the corresponding norms). However if it is necessary to emphasize that time is fixed, and the norm is taken over Σ, we shall write $H^s(\Sigma)$.

Further notational conventions are given in the Appendix and in Sect. 3, particularly in relation to the complex structure (see also the textbook [24, Sect. 9.1] for a treatment of the background material).

1.4. Existence theory for the Cauchy problem. Inherent to the system (1.5) is the property of *gauge invariance*: let $\chi(t,x)$ be a smooth real valued function, then (A, Φ) is a smooth solution if and only if $(d\chi + A, \Phi e^{i\chi})$ is. This introduces a large degeneracy to the solution space which may be removed by a choice of gauge in various ways. We will adopt here the following gauge condition which involves the time derivatives $\dot{\mathbf{A}}$, $\dot{\Phi}$, of $\mathbf{A}$, Φ:

$$\mathrm{div}\,\dot{\mathbf{A}} - \langle i\Phi, \dot{\Phi}\rangle \equiv e^{-2\rho}(\partial_1\dot{A}_1 + \partial_2\dot{A}_2) - \langle i\Phi, \dot{\Phi}\rangle = 0. \quad (1.7)$$

We make this choice because it allows a convenient description of the complex and symplectic structures on the moduli space of vortices (see Remark 1.6.3 and Sect. 3), and also is useful in the derivation of energy estimates for the time derivatives (see Sect. 2.2 and Sect. 2.3). In this gauge global existence can be stated as follows:

Theorem 1.4.1 (Global existence in gauge (1.7)). *Consider the Cauchy problem for (1.5) with initial data $\Phi(0) \in H^2(\Sigma)$ and $\mathbf{A}(0) \in H^1(\Sigma)$. There exists a global solution satisfying (1.7) and the estimate*

$$|\Phi(t)|_{H^2(\Sigma)} \leq ce^{\alpha e^{\beta t}} \tag{1.8}$$

for some positive constants c, α, β depending only on (Σ, g), the equations, and the initial data. The solution has regularity $\Phi \in C\left([0,\infty); H^2(\Sigma)\right) \cap C^1\left([0,\infty); L^2(\Sigma)\right)$ and $\mathbf{A} \in C^1\left([0,\infty); H^1(\Sigma)\right)$. If the initial data are smooth, then the solution is also smooth.

It is explained in Appendix A.3 how to derive this theorem from the global existence result of [13], which is stated in another gauge. Bounds of the type (1.8) were derived in [10] for the cubic nonlinear Schrödinger equation on $\mathbb{R}^2$, by means of the inequality

$$|u|_{L^\infty} \leq C[1 + \sqrt{\ln(1 + \|u\|_{H^2})}], \tag{1.9}$$

valid for $u \in H^2(\mathbb{R}^2)$ and with $C = C(\|u\|_{H^1})$. The proof of global regularity for (1.5) depends on a covariant version of this inequality (given in Lemma A.11), and a careful treatment of various commutator terms $[D_\mu, D_\nu]$ which indicates that they have a comparable strength to the cubic nonlinear term.

In conclusion, Theorem 1.4.1 provides a global solution which is a continuous curve in the space $\mathcal{H}_2$, where for $s \in \mathbb{R}$ we define

$$\mathcal{H}_s \equiv \{(\mathbf{A}, \Phi) \in H^{s-1}(\Sigma) \times H^s(\Sigma)\}, \tag{1.10}$$

with the corresponding norm $\|\cdot\|_{\mathcal{H}_s}$. From now on we will consider only $(\mathbf{A}, \Phi)$ which lie (at a given time) in the space $\mathcal{H}_2$. The gauge group at fixed time is given by

$$\mathcal{G} \equiv \{g \in H^2(\Sigma; S^1)\} \tag{1.11}$$

and acts on $\mathcal{H}_2$ according to $g \cdot (\mathbf{A}, \Phi) = (\mathbf{A} + g^{-1}dg, \Phi g)$. (Restricting to the set where Φ is not identically zero the action is free and gives a principal $\mathcal{G}$−bundle structure. The gauge condition (1.7) can be then regarded as giving a connection - i.e. a family of horizontal subspaces - on this bundle.)

1.5. Variational and Hamiltonian formulation. Equations (1.5) can be derived formally as the Euler-Lagrange equations associated to the functional

$$S(A, \Phi) = \frac{1}{2}\int_{\mathbb{R}\times\Sigma} -A \wedge F + (\langle i\Phi, D_0\Phi\rangle + A_0 + 2v_\lambda(A, \Phi))\, dt d\mu_g, \tag{1.12}$$

where

$$v_\lambda(\mathbf{A}, \Phi) = \frac{1}{2}\left(B^2 + |\mathbf{D}\Phi|^2 + \frac{\lambda}{4}(1 - |\Phi|^2)^2\right) \tag{1.13}$$

is the density of the Ginzburg-Landau static energy. (The parameter λ is a positive real number.) Although S is not manifestly gauge invariant it changes by an exact form under gauge transformation, and the Euler-Lagrange equations (1.5) are gauge invariant. Vortices are critical points of the static energy

$$\mathcal{V}_\lambda(\mathbf{A}, \Phi) = \int_\Sigma v_\lambda(\mathbf{A}, \Phi) d\mu_g,$$

as will be discussed further in the next section.

To see that the system (1.5) is Hamiltonian, observe that there is a complex structure on the phase space $\mathcal{H}_2$ given by $\mathbb{J} : (\dot{\mathbf{A}}, \dot{\Phi}) = (-J\dot{\mathbf{A}}, i\dot{\Phi})$ which allows the introduction of a symplectic structure $\Omega(v, w) = \langle \mathbb{J}v, w\rangle$, where $\langle\cdot, \cdot\rangle$ is the L^2 inner product. Using this symplectic form the system (1.5), in temporal gauge $A_0 = 0$, is a Hamiltonian flow generated by the Hamiltonian functional $\mathcal{V}_\lambda(\mathbf{A}, \Phi)$, which was just defined. (A short calculation reveals that the third equation of (1.5) is preserved by the evolution, and as such is really only a condition on the initial data. It will be referred to as the constraint equation.)

1.6. Self-dual vortices and dynamics in the limit $\lambda \to 1$. The system (1.5) admits *soliton* solutions, called abelian Higgs, or Ginzburg-Landau, vortices, which are energy minimizing critical points of the static energy functional $\mathcal{V}_\lambda(\mathbf{A}, \Phi)$. We now discuss these solutions and their uses in understanding the dynamical system (1.5) via the adiabatic approximation. There is a special case, $\lambda = 1$, in which the adiabatic approximation is particularly powerful because the space of vortices is then unusually large - large enough that the motion on it can provide information on the dynamical interaction of several vortices. We call this the self-dual, or Bogomolny, case, and the corresponding solutions are called self-dual vortices. Now for such a solution, $(\mathbf{A}, \Phi)$, with a given value of the topological integer N, (the degree of L), the field Φ will have N zeros, counted with multiplicity. Each of these zeros can be thought of as the centre of a vortex. Thus the static solitons can be thought of as a nonlinear superposition of N vortices which do not interact. This was first fully understood in the case that Σ is the upper half plane with canonical metric, when the equations were solved exactly by Witten (1977) by reducing them to the Liouville equation. In general it is still possible to make a reduction to a nonlinear elliptic equation of Kazdan-Warner type, whose solutions can be completely parametrized although not explicitly given. Following this, Taubes proved an existence theorem when Σ is the Euclidean plane (Jaffe and Taubes 1982), and Bradlow (1988) did likewise for Σ a compact Riemann surface, proving the following:

Theorem 1.6.1 (Existence of vortices on a surface, [8]). *If the area of a closed Riemann surface* $|\Sigma|$ *is such that* $|\Sigma| > 4\pi N$ *the Bogomolny bound is saturated: in fact the minimum value* πN *of* $\mathcal{V}$*, where*

$$\begin{aligned} &\mathcal{V} : \mathcal{H}_2 \to \mathbb{R}, \\ &\mathcal{V}(\mathbf{A}, \Phi) \equiv \mathcal{V}_1(\mathbf{A}, \Phi) = \frac{1}{2}\int_\Sigma \left(B^2 + |\mathbf{D}\Phi|^2 + \frac{1}{4}(1 - |\Phi|^2)^2\right) d\mu_g, \end{aligned} \tag{1.14}$$

is achieved on a set $\mathcal{S}_N \subset \mathcal{H}_2$ *of pairs* $(\mathbf{A}, \Phi)$ *which solve the* **Bogomolny***, or* **self-dual vortex***, equations:*

$$\bar{\partial}_{\mathbf{A}}\Phi = 0, \qquad B - \frac{1}{2}(1 - |\Phi|^2) = 0.$$

These minimizers will be referred to as the self-dual vortices, or just vortices. The quotient of $\mathcal{S}_N$ by the gauge group $\mathcal{G}$ can be identified with $Sym^N(\Sigma)$, the symmetric N-fold product of Σ, via the mapping which takes Φ to the set of its zeros.

Remark 1.6.2 (Interaction and stability of vortices). The physical interpretation of Theorem 1.6.1 is that for $\lambda = 1$ the vortices do not interact; see [28] for a discussion of this, and some related conjectures, and [19] for some stability theorems.

Remark 1.6.3 (Bogomolny structure and Bogomolny operator). The structural feature of $\mathcal{V}$ which makes Theorem 1.6.1 possible was identified by Bogomolny in [7]. In this instance it amounts to the fact that if we introduce the Bogomolny operator $\mathcal{B}$ to be the nonlinear operator which maps $(\mathbf{A}, \Phi) \mapsto \big(B - \frac{1}{2}(1 - |\Phi|^2), \bar{\partial}_{\mathbf{A}}\Phi\big)$ then

$$\mathcal{V} = \frac{1}{2}\int |\mathcal{B}(\mathbf{A}, \Phi)|^2 d\mu_g + \pi N$$

(see Sect. 3 for more information in this regard). Also see [8] for higher dimensional versions of this decomposition, and [21] for generalizations to solutions with non-vanishing electric field.

Remark 1.6.4 (Geometry of moduli space). Quotient spaces of the type arising in Theorem 1.6.1 are usually known as moduli spaces: in this case we define the moduli space $\mathcal{M}_N$ to be the space of gauge equivalence classes of self-dual vortices, so that $\mathcal{M}_N \equiv \mathrm{Sym}^N(\Sigma)$. We call the space $\mathcal{S}_N$ the vortex space and $\mathrm{proj} : \mathcal{S}_N \to \mathcal{M}_N$ the natural projection which takes $(\mathbf{A}, \Phi)$ to its gauge equivalence class $[(\mathbf{A}, \Phi)]$. The space $\mathcal{M}_N$ inherits both a metric (induced from the L^2 metric) and a symplectic structure and is a Kaehler manifold (see [9]). Explicitly, we can identify the tangent space to $\mathcal{M}_N$ with solutions $(\dot{\mathbf{A}}, \dot{\Phi})$ of the linearized Bogomolny equations which also satisfy the condition (1.7). The complex structure and symplectic structure on $\mathcal{M}_N$ are then given by restricting the formulas given in the previous section to such $(\dot{\mathbf{A}}, \dot{\Phi})$, and consequently we will use the same notation, $\mathbb{J}$ and Ω, for these objects. The existence of this complex structure on $\mathcal{M}_N$ can be seen very clearly in the formulas in Sect. 3, in which complex notation is used to combine the linearized Bogomolny equations with (1.7) into a manifestly complex linear operator $\mathcal{D}_\psi$, for $\psi = (\mathbf{A}, \Phi) \in \mathcal{S}_N$. This can all be summarized by saying that we have an identification

$$T_{[\psi]}\mathcal{M}_N \approx \mathrm{Ker}\,\mathcal{D}_\psi \equiv \{(\dot{\mathbf{A}}, \dot{\Phi}) : DB_\psi[\dot{\mathbf{A}}, \dot{\Phi}] = 0, \quad \text{and (1.7) holds}\}. \tag{1.15}$$

1.7. Statement of the adiabatic limit theorem. In order to define the adiabatic limit system, we now define a Hamiltonian function $\mathcal{M}_N \to \mathbb{R}$ by restricting the energy $\mathcal{V}_\lambda$ to the space of vortices, and observing that by gauge invariance this actually gives a smooth function on the quotient space $\mathcal{M}_N$. The corresponding Hamiltonian flow determines the slow motion of vortices for λ close to 1:

> *For $\epsilon = |\lambda - 1|$ sufficiently small, the system (1.5) can be approximated, for times of order $\frac{1}{\epsilon}$, by the Hamiltonian flow on the phase space $\mathcal{M}_N = Sym^N(\Sigma)$ associated to the Hamiltonian function $\mathcal{V}_\lambda|_{\mathcal{M}_N}$ via the symplectic form Ω.*

We now move towards a precise formulation of this in Theorem 1.7.2. Since we are interested in the regime in which $|\lambda - 1| \ll 1$, it is useful to introduce a large parameter

$$\mu = \frac{1}{|\lambda - 1|} \tag{1.16}$$

and let also, for $\lambda \neq 1$,

$$\sigma = \frac{\lambda - 1}{|\lambda - 1|} = \pm 1 \tag{1.17}$$

(also defining $\sigma = 0$ for $\lambda = 1$ where necessary). We rescale time by $\tau = \frac{t}{\mu}$, and A_0 similarly, leading to the following *rescaled equations*:

$$\begin{aligned}
\frac{\partial A_1}{\partial \tau} &= \mu\,(-\partial_1 B - \langle i\Phi, D_2\Phi\rangle) + \frac{\partial A_0}{\partial x^1},\\
\frac{\partial A_2}{\partial \tau} &= \mu\,(-\partial_2 B + \langle i\Phi, D_1\Phi\rangle) + \frac{\partial A_0}{\partial x^2},\\
i(\frac{\partial}{\partial \tau} - iA_0)\Phi &= \mu(-\Delta_A\Phi - \frac{1}{2}(1 - |\Phi|^2)\Phi) - \frac{\sigma}{2}(1 - |\Phi|^2)\Phi.
\end{aligned} \tag{1.18}$$

It is also natural to separate the energy $\mathcal{V}_\lambda$ into the (main) self-dual piece $\mathcal{V} = \mathcal{V}_1$, and a perturbation term proportional to $\lambda - 1$. Under the rescaling just introduced, the energy rescales by a factor μ, leading us to consider the Hamiltonian $H = \mu\mathcal{V} + U$, where $\mathcal{V} \equiv \mathcal{V}_1$ is as in (1.14), and the energy correction away from the self-dual, or Bogomolny, regime is given by

$$U(\Phi) = \frac{\sigma}{8}\int_\Sigma (1 - |\Phi|^2)^2\, d\mu_g. \tag{1.19}$$

The rescaled Eqs. (1.18) can be written as a Hamiltonian evolution for $\psi = (\mathbf{A}, \Phi)$ in the form

$$\mathbb{J}\frac{\partial \psi}{\partial \tau} = \mu\mathcal{V}' + U' + \mathbb{J}(dA_0, iA_0\Phi), \tag{1.20}$$

where $\mathbb{J}$ is the complex structure introduced at the end of Sect. 1.5,

$$\mathbb{J}(\dot{A}_1 dx^1 + \dot{A}_2 dx^2, \dot{\Phi}) = (-\dot{A}_2 dx^1 + \dot{A}_1 dx^2, i\dot{\Phi}) \tag{1.21}$$

with $\dot{\mathbf{A}} = \frac{\partial \mathbf{A}}{\partial \tau}$ $\dot{\Phi} = \frac{\partial \Phi}{\partial \tau}$.

Remark 1.7.1 (Explicit formulation of adiabatic limit system). We now write the equations for the adiabatic limit system in an explicit way which will be useful later. The function U is clearly gauge invariant and defines by restriction a smooth function u on $\mathcal{M}_N$. Now recall (1.15): under this identification, the gradient of the function u on $\mathcal{M}_N$ at $[\Psi_S]$ is identified with $\mathbb{P}_{\Psi_S}U'$, where $\mathbb{P}_{\Psi_S}$ is the orthogonal projector onto $\mathrm{Ker}\,\mathcal{D}_{\Psi_S}$ (see Lemma 3.3.2). The Hamiltonian differential equations for u are then equivalent to

$$\mathbb{J}\frac{\partial \Psi_S}{\partial \tau} = \mathbb{P}_{\Psi_S}U'. \tag{1.22}$$

Given an initial value $\Psi_S(0) = \psi_0 \in \mathcal{S}_N$, this equation has a unique solution $\tau \mapsto \Psi_S(\tau) \in \mathcal{S}_N$ which satisfies the gauge condition (1.7).

Main Theorem 1.7.2 (Adiabatic limit). *Let Ψ_μ be the smooth solution of* (1.20)*, satisfying the gauge condition* (1.7)*, with smooth initial data $\Psi_\mu(0)$, such that*

(i) $\lim_{\mu\to+\infty} \|\Psi_\mu(0) - \psi_0\|_{\mathcal{H}_2} = 0$, *for some smooth $\psi_0 \in \mathcal{S}_N$, and*
(ii) $\sup_{\mu\geq 1} \|\Psi_\mu(0)\|_{\mathcal{H}_2} + \|\dot{\Psi}_\mu(0)\|_{H_1} \leq K < \infty$.

Then there exists $\tau_* > 0$*, independent of* $\mu \geq 1$*, such that for* $s < 2$*,*

$$\lim_{\mu\to\infty} \sup_{[-\tau_*,\tau_*]} \left\| \Psi_\mu(\tau) - \Psi_S(\tau) \right\|_{\mathcal{H}_s} = 0, \tag{1.23}$$

where $\tau \mapsto \Psi_S(\tau) \in \mathcal{S}_N$ *is a curve in the vortex space* $\mathcal{S}_N$*, also satisfying* (1.7)*, which is the unique solution of* (1.22) *with initial data* $\Psi_S(0) = \psi_0$*. The projection onto the moduli space* $\mathcal{M}_N$*:*

$$\tau \mapsto [\Psi_S(\tau)] \in \mathcal{M}_N,$$

is the unique solution of the Hamiltonian system on $(Sym^N(\Sigma), \Omega)$ *associated to the Hamiltonian u defined in Remark* 1.7.1*, with initial value* $[\psi_0] \in \mathcal{M}_N$*.*

This theorem in proved in Sect. 2, employing a strategy which is explained in Sect. 1.8, following discussion of a very simple model problem. Some of the novel features which arise in the implementation of this strategy for (1.18) are highlighted at the beginning of Sect. 2.

Remark 1.7.3 (Related work). The approximation of the dynamical system (1.18) by a dynamical system through a space of equilibria (in this case the self-dual vortices, which are the equilibria for $\lambda = 1$) is referred to as an adiabatic limit or approximation. It was suggested in [30], following earlier conjectures of the same author on vortex and monopole dynamics in second order Lorentz invariant systems discussed in [31]. Proofs of the validity of the approximation in the case of second order dynamics were given in [44,45]; the strategy for the proof here, however, is different from that adopted in those references - see the discussion in Sect. 1.8. There has also been work on corresponding problems for σ-models, see [20,36]. A review of the analysis of adiabatic limit problems is given in [47], mostly directed towards infinite dimensional natural Lagrangian systems of the type appearing in classical field theory. (Natural Lagrangian systems are those derivable from Lagrangians of the classical “kinetic energy minus potential energy” form).

Remark 1.7.4 (Implications for Chern-Simons vortex dynamics). Although it is not generally possible to evaluate explicitly the Hamiltonian and symplectic form in the reduced system (1.22), it is possible to understand some basic features of the vortex dynamics in this model, see [27,30,31,38]. This work has been directed mostly to the case when the spatial domain is $\mathbb{R}^2$, so our Theorem 1.7.2 does not imply the validity of the approximation (1.22) in this case, see below. One general conclusion is that in the Chern-Simons model a force acting on the vortex produces motion at right angles to the direction of the force (in distinction to the behaviour in the relativistic case [31,44]). Now it is known computationally (see [28,31] and references therein), and in some special cases analytically ([46]), that the potential energy between two vortices depends on the distance between them, and is attractive for $\lambda < 1$ and repulsive for $\lambda > 1$. From this it can be deduced that two vortices will circle about one another, the direction of rotation depending upon whether $\lambda < 1$ or $\lambda > 1$. See [31, Sect. 7.13] for a discussion of these solutions in the $\mathbb{R}^2$ case. Also in the same reference it is observed that (1.22) possesses another related type of solution: a rigidly rotating p-gon, with p vortices placed at the vertices of a regular p-gon. Many of the arguments and calculations leading to the conclusions about vortex dynamics can be carried out equally well with spatial domain the standard sphere $\Sigma = S^2$ ([37]), even with explicit formulae in special limiting cases ([46]), in which case Theorem 1.7.2 implies rigorously the rotational behaviour for vortices described above. In future work results on the existence and stability of such periodic solutions for the full system (1.5) will be presented.

It is to be hoped that some of these qualitative conclusions about vortex dynamics, (which are justified for (1.5) by the Main Theorem 1.7.2) would have a wider validity for Chern-Simons models of vortex dynamics, not necessarily close to any self-dual limit. There is some numerical evidence for this in related situations, for example the scattering of vortices in the relativistic abelian Higgs model is qualitatively similar for all values of the Higgs coupling constant, even though a rigorous analysis in which the vortices actually collide is only possible in the self-dual limit; see [31,44]). On the other hand, the case of first order dynamics is in some ways numerically more problematic since it is not possible to produce any motion via choice of initial conditions (as can be done in the second order case), and it is necessary to have λ deviate from the self-dual value 1, and quite substantially so in order to get motion which is easily computationally observable. A numerical study in [27] which compares the approximation (1.22) with a computer simulation of (1.5) finds that, *in the case of spatial domain* $\Sigma = \mathbb{R}^2$, while the qualitative behaviour of two vortices is similar to that implied by (1.22) for $|\lambda - 1|$ small, there are quantitative differences between the full dynamics and the adiabatic limit, which become quite marked as λ moves away from the value 1. As the authors of [27] say, it is unclear to what extent some of these differences are genuine errors due to the neglect of radiation in the finite dimensional truncation (1.22), as compared to being a numerical artefact; certainly some of the observed behaviour is consistent with energy being transferred into radiative modes, causing the vortices to spiral in towards one another in the attractive case ([27, Fig. 6]). In any case, there is no issue with radiation *when* Σ *is a compact spatial domain*, in which case Theorem 1.7.2 does imply the validity of the approximation (1.22) for sufficiently small $|\lambda-1|$, and it seems reasonable to expect that in this case the dynamical bevaviour predicted by our analysis (relating (1.5) to (1.22) for small $|\lambda - 1|$) is at least qualitatively relevant to the applications in the theoretical physics literature.

1.8. A simple model problem and discussion of methodology. We consider here a simple two-dimensional example in order to exhibit as clearly as possible the phenomenon under study, and the strategy which will be employed in the proof of Theorem 1.7.2. (It is the basic strategy taken in [39] for finite dimensional natural Lagrangian systems, here adapted to the case of infinite dimensions and to take advantage of the Bogomolny structure.) For real numbers β and $\mu \gg 1$, we consider a linear first order Hamiltonian system for $z(\tau) = (z^1(\tau), z^2(\tau)) \in \mathbb{C}^2$:

Theorem 1.8.1. *For each* $\mu \gg 1$*, let* $\tau \mapsto Z_\mu(\tau) \in \mathbb{C}^2$ *be the solution of*

$$\begin{aligned} \dot{z}^1 &= \quad i(z^1 + \beta z^2), \\ \dot{z}^2 &= i(\beta z^1 + \mu z^2), \end{aligned} \tag{1.24}$$

with initial data satisfying $|(Z^1_\mu(0), Z^2_\mu(0)) - (\gamma, 0)| = O(\mu^{-1})$ *as* $\mu \to +\infty$*, for some fixed* $\gamma \in \mathbb{C}$*. Then*

$$\lim_{\mu\to+\infty} \max_{\tau\in\mathbb{R}} |Z_\mu(\tau) - (\gamma e^{i\tau}, 0)| = 0. \tag{1.25}$$

Remark 1.8.2. The system (1.24) is Hamiltonian with the standard symplectic structure on $\mathbb{C}^2$ and with Hamiltonian function $\mu\mathcal{V} + U$ with $\mathcal{V}(z) = \frac{1}{2}\bar{z}^2 z^2$ and

$$U(z) = \frac{1}{2}\bar{z}^1 z^1 + \beta(\bar{z}^1 z^2 + \bar{z}^2 z^1).$$

Thus $\mathcal{V}$ acts as a constraining potential for $\mu \to +\infty$, forcing the solution onto the set $\mathcal{S} = \mathbb{C} \times \{0\} \subset \mathbb{C}^2$, where $z^2 = 0$. Projecting the system to $\mathcal{S}$ gives, formally,

$$i\dot{z}^1 + z^1 = 0. \tag{1.26}$$

The theorem asserts that (1.26) indeed governs the behaviour of the limit of appropriate sequences of solutions to (1.24).

Proof. The solution with initial data $z(0) = (z^1(0), z^2(0))$ is given by:

$$\begin{aligned} z^1(\tau) = {} & \frac{\beta}{\beta(\lambda_+ - \lambda_-)} \left[\left((1-\lambda_-)e^{i\lambda_+\tau} - (1-\lambda_+)e^{i\lambda_-\tau}\right) z^1(0) \right. \\ & \left. + \beta\left(e^{i\lambda_+\tau} - e^{i\lambda_-\tau}\right) z^2(0)\right], \\ z^2(\tau) = {} & \frac{-1}{\beta(\lambda_+ - \lambda_-)} \left[(1-\lambda_+)(1-\lambda_-)(e^{i\lambda_+\tau} - e^{i\lambda_-\tau}) z^1(0)\right] \\ & + \frac{-\beta}{\beta(\lambda_+ - \lambda_-)} \left[\left((1-\lambda_+)e^{i\lambda_+\tau} - (1-\lambda_-)e^{i\lambda_-\tau}\right) z^2(0)\right]. \end{aligned}$$

Here the $\lambda_\pm$ are the characteristic values of the system:

$$\lambda_\pm = \frac{1}{2}(1+\mu)\left[1 \pm \left(1 - \frac{4(\mu - \beta^2)}{(1+\mu)^2}\right)^{\frac{1}{2}}\right],$$

which satisfy, by the binomial expansion,

$$|\lambda_+ - \mu| = O(1), \quad |\lambda_- - 1| = O(\mu^{-1})$$

as $\mu \to \infty$. From this, and the fact that $\lambda_\pm \in \mathbb{R}$ for large μ so that $|e^{i\lambda_\pm\tau}| = 1$, the behaviour in (1.25) follows for the solutions $Z_\mu(\tau)$ with initial data as described. □

Remark 1.8.3. In this example the exact solutions indicate that while $Z_\mu^2 \to 0$, the time derivatives $\dot{Z}_\mu^2$ are bounded, but cannot generally be expected to have limit zero.

In the absence of explicit formulae for $Z_\mu(\tau)$, it is still possible to prove results like Theorem 1.8.1, either

(i) by explicit perturbative construction of solutions to the full system, using solutions of the restricted system as a starting point, or
(ii) by obtaining uniform bounds for the $Z_\mu(\tau)$ which allow the extraction of convergent subsequences, and then identifying the unique limit of all such subsequences as the corresponding solution of the restricted system with Hamiltonian $U|_{\mathcal{S}}$.

In the present article we will adopt the second strategy in our proof of Theorem 1.7.2 (although it would be possible to use the first strategy, as in [44]). To make the structure of the proof transparent, it is useful to consider in some detail how to execute the second strategy to prove a variant of Theorem 1.8.1:

Theorem 1.8.4 (Weaker version of Theorem 1.8.1). *In the situation of 1.8.1,*

$$\lim_{\mu\to+\infty}\max_{a<\tau<b}|Z_\mu(\tau)-(\gamma e^{i\tau},0)|=0, \tag{1.27}$$

for every bounded interval $[a,b]\subset\mathbb{R}$.

Remark 1.8.5. Although weaker than Theorem 1.8.1, the proof of Theorem 1.8.4 that we give generalizes to the infinite dimensional problem (1.5), (1.18), in which the explicit solutions corresponding to those used in the proof of Theorem 1.8.1 are of course not available.

Proof.
- Differentiation of Eqs. (1.24) in time gives the identical system $\zeta=\dot z$. Use the energy identity:

$$\mu\mathcal{V}(\zeta(\tau))+U(\zeta(\tau))=\mu\mathcal{V}(\zeta(0))+U(\zeta(0)),$$

  together with the identical estimate for $z(\tau)$, to deduce (using Cauchy-Schwarz) that the solutions Z_μ of Theorem 1.8.1 satisfy $|Z_\mu(\tau)|+|\dot Z_\mu(\tau)|\le C$, with C independent of $\mu\gg1$.
- By the previous item, deduce that the family of functions $\tau\mapsto Z_\mu(\tau)$ is uniformly (in $\mu\gg1$) bounded and equicontinuous, and so the Arzela-Ascoli theorem implies *subsequential* convergence $Z_{\mu_j}\to Z$ in $C(I)$ for any bounded interval $I\subset\mathbb{R}$.
- The energy estimate implies that, for large μ there exists $C>0$, independent of μ, such that $\mu\bar Z^2Z^2\le C$. It follows that $Z_\mu^2\to0$ along any convergent subsequence. Now consider the integrated form of the first equation of (1.24) (i.e. project the system onto $\mathcal{S}=\mathbb{C}\times\{0\}\subset\mathbb{C}^2$, where $z^2=0$). Taking the limit $\mu_j\to\infty$, it follows that the limit $Z=(Z^1,Z^2)$ of any convergent subsequence satisfies $Z^1(\tau)=i\int_0^\tau Z^1(\tau')d\tau'$ and $Z^1(0)=\gamma$. This integral equation has unique solution $Z^1(\tau)=\gamma e^{i\tau}$, and hence the C_{loc} limit of any convergent subsequence is $(\gamma e^{i\tau},0)$. It follows that Z_μ converges to this limit in C_{loc} without restriction to subsequences. This proves Theorem 1.8.4. (In view of Remark 1.8.3 we should not expect this convergence to be in C^1_{loc}.) □

The general situation to which Theorem (1.8.4), and its proof, potentially generalize is the following: on a phase space $\mathcal{H}$ we consider the integral curves $Z_\mu(\tau)$ for a Hamiltonian $\mu\mathcal{V}+U$ for large μ ("the full system"). Under the assumption that $\mathcal{S}=\{z\in\mathcal{H}:\min\mathcal{V}=\mathcal{V}(z)\}$ is a symplectic submanifold of $\mathcal{H}$, we can consider the "restricted system" on $\mathcal{S}$ determined by the Hamiltonian $U|_{\mathcal{S}}$, and try to prove that this Hamiltonian system can be used to describe the limiting behaviour of $Z_\mu(\tau)$ as $\mu\to+\infty$. An infinite dimensional example of this situation is provided by the Chern-Simons-Schrödinger system (1.18): in the next section we will provide a proof of Theorem 1.7.2 employing the same strategy to that used in the proof of Theorem 1.8.4 just given.

2. Uniform Bounds and Proof of the Main Theorem

In this section we prove our main result, Theorem 1.7.2, along the lines suggested by the discussion of the simple model problem in the last section. The crucial stage is the proof of the main estimate, Theorem 2.3.1, which asserts the existence of a time interval, *independent of* μ, on which the solution $\psi=(\mathbf{A},\Phi)$ is uniformly bounded in $\mathcal{H}_2$, and its time derivative is uniformly bounded in H^1 as $\mu\to+\infty$. Given this bound, Theorem

1.7.2 can be deduced using a variant of the Lions-Aubin lemma, and a careful analysis of the $\mu \to +\infty$ limit of (1.18). Before obtaining the uniform bound, we collect some identities used in the proof. Some more specialized identities related to the self-dual structure are collected separately in Sect. 3, and referred to as needed. Specifically, we draw the reader's attention to the following two uses made of these more specialized identities:

(i) Differentiation in time gives rise to Eq. (2.31) for $\zeta = \dot{\psi}$ in which the dominant term (as $\mu \to +\infty$) involves $\overline{L}_\psi$, the Hessian of $\mathcal{V}$ defined in (2.40). It is shown in Sect. 3 that this operator takes the special form

$$\overline{L}_\psi = \mathcal{D}_\psi^* \mathcal{D}_\psi + O(|\mathcal{B}|), \tag{2.28}$$

with $\mathcal{D}_\psi$ complex linear (see (3.62)), and $\mathcal{B}$ as in Remark 1.6.3. Observing that the L^2 norm is exactly preserved for equations of the form $\mathbb{J}\dot{\zeta} = \mathcal{D}_\psi^* \mathcal{D}_\psi \zeta$, it is easy to believe that the stated structure of $\overline{L}_\psi$ is useful in the derivation of μ-independent bounds for (2.31), (for initial data as in the theorem); this indeed turns out to be the case - see the proof of Theorem 2.3.1.

(ii) After obtaining a convergent subsequence of solutions of (1.20) it is necessary to take the limit of the equation itself along the subsequence $\mu = \mu_j \to +\infty$. For this purpose it is very convenient to be able to eradicate the term $\mu\mathcal{V}'$ on the right hand side, since this is clearly hard to control for large μ: this can be done by applying a projection operator $\mathbb{P}_\mu$ whose existence close to the set of self-dual vortices is assured by the Bogomolny structure: see Lemmas 3.3.1 and 3.3.2. (In geometrical terms there is a foliation of the phase space $\mathcal{H}_2$, and the range of $\mathbb{P}_\mu$ is the tangent space to the leaves of this foliation, after dividing out by the action of the gauge group using (1.7).)

Although our final conclusions are in terms of the standard Sobolev norms based on the fixed connection ∇, it will be convenient to obtain bounds for the corresponding Sobolev norms defined at each fixed time with respect to the connection $\mathbf{D} = \nabla - i\mathbf{A}$, see (A.2). These can be related to the standard norms by (A.3)–(A.5).

2.1. The evolution equations and associated identities. In addition to the rescaled Eq. (1.20) for $\psi = (\mathbf{A}, \Phi)$:

$$\mathbb{J}\frac{\partial \psi}{\partial \tau} = \mu\mathcal{V}' + U' + \mathbb{J}(dA_0, iA_0\Phi),$$

we will use the differentiated equation for $\zeta = \dot{\psi} \equiv \frac{\partial \psi}{\partial \tau}$. To write this down we need the linearization of the operator $\mathcal{V}'(\psi)$, i.e. the second order linear differential operator L_ψ obtained by differentiation of the map $\psi \mapsto \mathcal{V}'(\psi)$:

$$L_\psi = D\mathcal{V}'(\psi),$$

or equivalently, $\langle \zeta, L_\psi \zeta \rangle_{L^2} = \frac{d^2}{ds^2}\mathcal{V}(\psi + s\zeta)|_{s=0}$. Explicitly, with $\zeta = (\dot{\mathbf{A}}, \dot{\Phi})$, we have

$$\langle \zeta, L_\psi \zeta \rangle_{L^2} = \int \Big(|d\dot{\mathbf{A}}|^2 + |D\dot{\Phi}|^2 + |\Phi|^2|\dot{\mathbf{A}}|^2 - 2\langle \mathbf{D}\Phi, i\dot{\mathbf{A}}\dot{\Phi}\rangle - 2\langle \mathbf{D}\dot{\Phi}, i\dot{\mathbf{A}}\Phi\rangle \tag{2.29}$$
$$+ \langle \Phi, \dot{\Phi}\rangle^2 - \frac{1}{2}(1 - |\Phi|^2)|\dot{\Phi}|^2 \Big) \, d\mu_g.$$

Remark 2.1.1. There is a slightly simpler version of this formula, given in (2.40) below, when ζ is restricted by the gauge condition (1.7). Furthermore in Sect. 3 it is shown that the self-dual structure provides a useful way of rewriting this formula as in (2.28), in terms of the complex structure defined in (1.21), and using the complex one-form $\dot{\alpha}dz$, where $\dot{\alpha} = \frac{\dot{A}_1 - i\dot{A}_2}{2}$, in place of the real one-form $\dot{A}_1 dx^1 + \dot{A}_2 dx^2$, see (3.60). Since this is used only at one point in the proof - in Lemma 2.3.8 - this formulation is presented separately in Sect. 3, and referred to only as needed.

The linearization of U' is the linear operator $K_\psi = DU'(\psi)$, given by

$$K_\psi = (\dot{\mathbf{A}}, \dot{\Phi}) \mapsto \left(0, \frac{\sigma}{2}(1 - |\Phi|^2)\dot{\Phi} + \sigma\langle\Phi, \dot{\Phi}\rangle\Phi\right), \tag{2.30}$$

with σ defined in (1.17). Given these definitions, the chain rule implies that, if ψ is a smooth solution of (1.20), then $\zeta(\tau) = \dot{\psi}(\tau)$ solves

$$\mathbb{J}\frac{\partial\zeta}{\partial\tau} = \mu L_\psi\zeta + K_\psi\zeta + \mathbb{J}\frac{\partial}{\partial\tau}(dA_0, iA_0\Phi). \tag{2.31}$$

We also need identities for the evolution of the Bogomolny operator $\mathcal{B}$ defined in Remark 1.6.3 and discussed in more detail in Sect. 3. The first component is preserved

$$\frac{\partial}{\partial\tau}\left((B - \frac{1}{2}(1 - |\Phi|^2)\right) = e^{-2\rho}(\partial_1\dot{A}_2 - \partial_2\dot{A}_1) + \langle\Phi, \dot{\Phi}\rangle = 0, \tag{2.32}$$

as a consequence of (1.18). We will require that the initial data are such that $B - \frac{1}{2}(1 - |\Phi|^2) = 0$ initially, and hence for all times. The second component of the Bogomolny operator $\mathcal{B}$ will be denoted

$$\eta = \bar{\partial}_{\mathbf{A}}\Phi = \frac{1}{2}(D_1 + iD_2)\Phi, \tag{2.33}$$

(see Sect. 3), and we have the following identity:

$$i(\partial_\tau - iA_0)\eta = \mu(-4\bar{\partial}_{\mathbf{A}}(e^{-2\rho}\partial_{\mathbf{A}}\eta) + |\Phi|^2\eta) - \frac{\sigma}{2}\bar{\partial}_{\mathbf{A}}\left((1 - |\Phi|^2)\Phi\right). \tag{2.34}$$

(To verify this identity: substitute $\Delta_{\mathbf{A}}\Phi = 4e^{-2\rho}\partial_{\mathbf{A}}\bar{\partial}_{\mathbf{A}}\Phi - B\Phi$ into the third line of (1.18) and then apply $\bar{\partial}_{\mathbf{A}}$ to the resulting equation and use the identity $(E_1 + iE_2)\Phi = -2\mu|\Phi|^2\bar{\partial}_{\mathbf{A}}\Phi$ which follows from the first two lines of (1.18).)

Of course, the energy

$$\mathcal{E}(\tau) = \mu\mathcal{V}(\psi(\tau)) + U(\psi(\tau)) = \mathcal{E}_0 > 0 \tag{2.35}$$

is independent of time τ for regular solutions, as is the L^2 norm,

$$\|\Phi(\tau)\|_{L^2} = L > 0. \tag{2.36}$$

2.2. Choice of gauge condition and related estimates. The divergence of E can be calculated to be:

$$\begin{aligned}\operatorname{div} E &= e^{-2\rho}(\partial_1 E_1 + \partial_2 E_2)\\ &= \mu\left((-\Delta B - e^{-2\rho}\partial_1\langle i\Phi, D_2\Phi\rangle + e^{-2\rho}\partial_2\langle i\Phi, D_1\Phi\rangle\right)\\ &= \mu(4e^{-2\rho}|\eta|^2) + \langle i\Phi, (\frac{\partial}{\partial t} - iA_0)\Phi\rangle - \sigma B|\Phi|^2.\end{aligned}$$

In the last line we have used $B = \frac{1}{2}(1-|\Phi|^2)$, so that $\Delta B = -\langle\Phi, \Delta_{\mathbf{A}}\Phi\rangle - e^{-2\rho}(|D_1\Phi|^2 + |D_2\Phi|^2)$, the equation for Φ and the definition of η in (2.33). Under the gauge condition (1.7) we get the following equation for A_0:

$$(-\Delta + |\Phi|^2)A_0 = 4\mu e^{-2\rho}|\eta|^2 - \sigma B|\Phi|^2. \tag{2.37}$$

Lemma 2.2.1 (Estimates for A_0). *Assume $\tau \mapsto \psi(\tau) = (\mathbf{A}(\tau), \Phi(\tau))$, is a smooth solution, of* (1.20) *which satisfies the gauge condition* (1.7)*,* (2.35) *and* (2.36)*. Then for all $r < \infty$, there exists $c_0(\mathcal{E}_0, L, r) > 0$ such that,*

$$\|A_0(\tau)\|_{L^r} \leq c_0(\mathcal{E}_0, L, r) \tag{2.38}$$

and there exists $c_0(\mathcal{E}_0, L) > 0$ such that

$$\|A_0(\tau)\|_{H^2} \leq c_0(\mathcal{E}_0, L)(1 + \mu\|\bar{\partial}_{\mathbf{A}}\Phi(\tau)\|_{L^\infty}). \tag{2.39}$$

Remark 2.2.2. This shows that in the original system (before rescaling) the time component of the potential A_0 is $O(|\lambda - 1|)$ in the gauge defined by (1.7).

Proof. The crucial point here is the μ independence of the bounds. The second inequality follows from standard elliptic theory once the first is established. By (2.37) it is possible to write $A_0 = A_0^+ + \hat{A}_0$, where $(-\Delta + |\Phi|^2)A_0^+ = 4\mu e^{-2\rho}|\eta|^2$, so that $A_0^+ \geq 0$ by the maximum principle, and $(-\Delta + |\Phi|^2)\hat{A}_0 = -\sigma B|\Phi|^2$. The bounds stated in the lemma will follow by the triangle inequality once they are proved for A_0^+, since they are immediate for $\hat{A}_0$. Now integrating the equation for A_0^+ implies that $\||\Phi|^2 A_0^+\|_{L^1} = \int_\Sigma |\Phi|^2 A_0^+ d\mu_g \leq C(\mathcal{E}_0, L)$ since $A_0^+ \geq 0$; this bound is independent of $\mu \gg 1$ on account of (2.35). The standard elliptic theory for $-\Delta u = f \in L^1$ now gives the L^r estimates for A_0^+ and hence the lemma. □

Lemma 2.2.3 (Estimates for $\dot{\mathbf{A}}$). *Let $\zeta = (\dot{\mathbf{A}}, \dot{\Phi})$ satisfy the gauge condition* (1.7)*, as well as the linearized constraint equation* (2.32)*. Then there exists a constant $c_1 > 0$ such that $\|\dot{\mathbf{A}}\|_{H^1} \leq c_1\|\Phi\dot{\Phi}\|_{L^2}$, and more generally, for any $1 < p < \infty$, there exists a constant $c_1(p) > 0$ such that $\|\dot{\mathbf{A}}\|_{W^{1,p}} \leq c_1\|\Phi\dot{\Phi}\|_{L^p}$. In particular these estimates hold for a smooth solution, $\tau \mapsto \psi(\tau) = (\mathbf{A}(\tau), \Phi(\tau))$, of* (1.20) *which satisfies the gauge condition* (1.7)*.*

Proof. These are the standard estimates for the Hodge system, proved by using the Hodge decomposition to reduce to the Calderon-Zygmund estimate for the Laplacian. □

On the subspace of $\zeta = (\dot{\mathbf{A}}, \dot{\Phi})$ satisfying the gauge condition (1.7), the operator L_ψ has a simpler form: $L_\psi \zeta = \overline{L}_\psi \zeta$, where $\overline{L}_\psi$ is the operator defined by

$$\langle \zeta, \overline{L}_\psi \zeta \rangle_{L^2} = \int \Big(|\mathbf{d}\dot{\mathbf{A}}|^2 + |\mathrm{div}\, \dot{\mathbf{A}}|^2 + |\mathbf{D}\dot{\Phi}|^2 + |\Phi|^2 (|\dot{\mathbf{A}}|^2 + |\dot{\Phi}|^2) - 4\langle \mathbf{D}\Phi, i\dot{\mathbf{A}}\dot{\Phi}\rangle - \frac{1}{2}(1 - |\Phi|^2)|\dot{\Phi}|^2 \Big) \, d\mu_g. \tag{2.40}$$

Lemma 2.2.4 (The Hessian). *Let $\psi = (\mathbf{A}, \Phi)$ be smooth. Then the second order differential operator $\overline{L}_\psi$ is a self-adjoint operator with domain H^2, and there exist numbers c_2, c_3 such that*

$$\langle \zeta, \overline{L}_\psi \zeta \rangle_{L^2} \geq c_2 \|\zeta\|^2_{H^1_A} - c_3 \|\zeta\|^2_{L^2}.$$

The numbers c_2, c_3 depend only on the numbers L and $\mathcal{E}_0$, defined as in (2.35), (2.36).

Proof. First of all, observe that

$$\int \Big(|\mathbf{d}\dot{\mathbf{A}}|^2 + |\mathrm{div}\, \dot{\mathbf{A}}|^2 + |\mathbf{D}\dot{\Phi}|^2 + |\Phi|^2 (|\dot{\mathbf{A}}|^2 + |\dot{\Phi}|^2) \Big) \, d\mu_g \geq c(\mathcal{E}_0, L) \|(\dot{\mathbf{A}}, \dot{\Phi})\|^2_{H^1_A}.$$

This can be proved by a straightforward contradiction argument that is very similar to the proof of Lemma 3.2.2 given below, so the details will be omitted. Next, to deduce the stated result, just bound the final two terms in (2.40) using the Holder inequality with $1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{4}$, the interpolation inequality in Lemma A.9 and Cauchy-Schwarz. □

Corollary 2.2.5. *Assume given a smooth solution, $\tau \mapsto \psi(\tau) = (\mathbf{A}(\tau), \Phi(\tau))$, of (1.20) which satisfies the gauge condition (1.7), (2.35) and (2.36). Then the quantity*

$$\mathcal{E}_1(\tau) = \frac{1}{2} \langle \zeta(\tau), (L_\psi + \mu^{-1} K_\psi) \zeta(\tau) \rangle_{L^2}, \tag{2.41}$$

where $\psi = \psi(\tau)$, satisfies for $\mu \geq 1$,

$$\mathcal{E}_1 \ \geq \ c_4 \|\zeta\|^2_{H^1_A} - c_5 \|\zeta\|^2_{L^2}$$

with c_4, c_5 depending only on $\mathcal{E}_0, L$.

2.3. The main estimate. We say that a smooth solution, $\tau \mapsto \psi(\tau) = (A(\tau), \Phi(\tau))$, of (1.20) satisfies conditions (AE) and (AI), if the following conditions hold:

- (AE) There exist positive numbers $\mathcal{E}_0, L$ such that $\|\Phi(\tau)\|_{L^2} = L$ and $\mathcal{E}(\tau) = \mathcal{E}_0$, for all times $\tau \in \mathbb{R}$, where $\mathcal{E}(\tau)$ is the energy (2.35). (Recall that both these quantities are independent of τ.)
- (AI) The initial data are such that $\|\psi(0)\|_{\mathcal{H}_2} + \|\dot{\psi}(0)\|_{H^1} \leq K < \infty$. (Recall the definition of the norms in (1.10)).

Theorem 2.3.1. *For $\mu \geq 1$ let $\tau \mapsto \psi(\tau)$ be a smooth solution of (1.20) satisfying conditions (AE) and (AI), for some fixed numbers $K, L, \mathcal{E}_0$. There exist numbers $\tau_* > 0$ and $M_* > 0$, independent of μ, such that*

$$\max_{|\tau| \leq \tau_*} \left| \left(\psi(\tau), \frac{\partial}{\partial \tau} \psi(\tau) \right) \right|_{\mathcal{H}_2 \times H^1} \ \leq \ M_*. \tag{2.42}$$

Beginning of proof of Theorem 2.3.1. By time reversal invariance it is sufficient to prove the bound for $0 \le \tau \le \tau_*$, for some $\tau_* > 0$ independent of μ. Let

$$\zeta(\tau) = \frac{\partial}{\partial \tau}\psi(\tau) = \dot{\psi}(\tau).$$

For any $M > \|\zeta(0)\|_{L^2}$ there exists a time $T(M, \mu) > 0$ such that

$$\sup_{0 \le \tau \le T(M,\mu)} \|\zeta(\tau)\|_{L^2} \le M. \tag{2.43}$$

We will prove that there exist positive numbers M_*, τ_*, independent of μ, such that $T(M_*, \mu) \ge \tau_*$, and hence $\sup_{0\le\tau\le\tau_*} \|\zeta(\tau)\|_{L^2} \le M_*$. The proof proceeds by obtaining a series of μ-independent bounds, predicated upon (2.43), which imply boundedness of $\big(\psi(\tau), \dot{\psi}(\tau)\big)$ in the Hilbert space $\mathcal{H}_2$ defined in (1.10) for $0 \le \tau \le \tau_*$. These bounds are now stated in a sequence of lemmas, all of which refer to a smooth solution of (1.20),(1.7) which verifies (AE), (AI) and (2.43) for all τ under consideration.

Lemma 2.3.2 (Estimate for Φ in H^2). *There exists $C_1 = C_1(\mathcal{E}_0, L) > 0$, independent of μ, such that*

$$\|\Phi(\tau)\|_{H_A^2} \le C_1(1 + \|\zeta(\tau)\|_{L^2}) \le C_1(1 + M).$$

Proof. Using the third equation of (1.18) for Φ, we bound

$$\|\Delta_{\mathbf{A}}\Phi\|_{L^2} \le \|\dot{\Phi}\|_{L^2} + \|A_0\Phi\|_{L^2} + \frac{1}{2}\|\Phi(1 - |\Phi|^2)\|_{L^2}.$$

Now, by Lemma A.2.2, we can bound $\|\nabla_{\mathbf{A}}\nabla_{\mathbf{A}}\Phi\|_{L^2} \le \|\Delta_{\mathbf{A}}\Phi\|_{L^2} + c(\mathcal{E}_0)\|\nabla_{\mathbf{A}}\Phi\|_{L^4}$, and hence, by Lemma A.9 and Cauchy-Schwarz: $\|\nabla_{\mathbf{A}}\nabla_{\mathbf{A}}\Phi\|_{L^2} \le 2\|\Delta_{\mathbf{A}}\Phi\|_{L^2} + c(\mathcal{E}_0, L)$. Therefore, using also Lemma 2.2.1, we deduce the bound $\|\Phi(t)\|_{H^2} \le c(1 + \|\zeta(\tau)\|_{L^2}) \le c(1 + M)$, for some $c = c(\mathcal{E}_0, L) > 0$, and the result follows. □

Corollary 2.3.3. $\exists C_2 = C_2(\mathcal{E}_0, L) > 0$ *such that,* $\|\Phi(\tau)\|_{L^\infty} \le C_2\left(1 + \sqrt{\ln(1 + M)}\right)$.

Proof. This follows from Lemma A.11 and the previous lemma. □

Lemma 2.3.4 (Energy estimate for $\zeta = \dot{\psi}$). *There is a constant $C_3(\mathcal{E}_0, L) > 0$ such that,*

$$\left|\frac{d\mathcal{E}_1}{d\tau}\right| \le C_3(1 + \|\Phi\|_{L^\infty}^2)\|\zeta\|_{H_A^1}^2 + C_3\|\zeta\|_{L^2}^6 + C_3\|\zeta\|_{L^2}^4. \tag{2.44}$$

where $\mathcal{E}_1$ is the quantity defined in (2.41).

Proof. Compute $\frac{d}{dt}\mathcal{E}_1$, substitute from (2.31), and use the observation that

$$\langle \mathbb{J}\dot{\zeta}, (d\dot{A}_0, i\Phi\dot{A}_0)\rangle_{L^2} = 0, \tag{2.45}$$

by the constraint equation $B = \frac{1}{2}(1 - |\Phi|^2)$ in (1.5), to obtain

$$\frac{d\mathcal{E}_1}{d\tau} = \langle i\dot{\Phi}, iA_0\dot{\Phi}\rangle_{L^2} + \frac{1}{2}\langle\zeta, [\frac{\partial}{\partial\tau}, L_\psi + \mu^{-1}K_\psi]\zeta\rangle_{L^2}.$$

To handle the second term, we make use of the following bounds (written schematically, i.e. suppressing indices and inner products which play no role):

$$\begin{aligned}\|\Phi\zeta^3\|_{L^1} &\le \|\Phi\|_{L^\infty}\|\zeta\|_{L^2}\|\zeta\|^2_{L^4} \le c\|\Phi\|_{L^\infty}\|\zeta\|^2_{L^2}\|\zeta\|_{H^1_A}\\ \|\dot\Phi\dot{\mathbf{A}}\nabla_{\mathbf{A}}\dot\Phi\|_{L^1} &\le \|\nabla_{\mathbf{A}}\dot\Phi\|_{L^2}\|\dot{\mathbf{A}}\|_{L^4}\|\dot\Phi\|_{L^4} \le c\|\Phi\|_{L^\infty}\|\zeta\|^{3/2}_{H^1_A}\|\zeta\|^{3/2}_{L^2}\\ \|\dot\Phi^2\nabla\dot{\mathbf{A}}\|_{L^1} &\le \|\zeta\|^2_{L^4}\|\nabla\dot{\mathbf{A}}\|_{L^2} \le c\|\Phi\|_{L^\infty}\|\zeta\|^2_{L^2}\|\zeta\|_{H^1_A}.\end{aligned}$$

All of these bounds follow directly from Holder's inequality, the interpolation inequality in Lemma A.2.1, Lemma 2.2.3 and the bound

$$\|\dot{\mathbf{A}}\|_{L^4} + \|\dot{\mathbf{A}}\|_{H^1} \le c\|\Phi\|_{L^\infty}\|\dot\Phi\|_{L^2}.$$

It then follows, by inspection of the formulae for L_ψ, K_ψ in (2.29) and (2.30), that the second term in $\frac{d\mathcal{E}_1}{d\tau}$ can be bounded by a sum of terms of this type, and hence:

$$\left|\left\langle \zeta,\ [\frac{\partial}{\partial\tau}\ ,\ L_\psi + \mu^{-1}K_\psi]\zeta\right\rangle_{L^2}\right| \le c(1+\|\Phi\|^2_{L^\infty})\|\zeta\|^2_{H^1_A} + c\|\zeta\|^6_{L^2} + c\|\zeta\|^4_{L^2}.$$

Also, we can bound

$$|\langle i\dot\Phi, iA_0\dot\Phi\rangle_{L^2}| \le c\|A_0\|_{L^r}\|\dot\Phi\|^2_{L^{2r'}} \le c\|A_0\|_{L^r}\|\dot\Phi\|^2_{H^1_A},$$

where $r > 1$ and $1/r + 1/r' = 1$. Combining these with Lemma 2.2.1, we obtain (2.44), completing the proof of the lemma. □

Corollary 2.3.5. *There is a constant* $C_4 = C_4(\mathcal{E}_0, K, L, M) > 0$ *such that,* $\|\zeta(\tau)\|_{H^1_A} \le C_4(1+\tau)$, *for all times* $\tau \in [0, T(M,\mu)]$.

Lemma 2.3.6 (Estimate for $\eta = \bar\partial_{\mathbf{A}}\Phi$). *There exists* $C_5 = C_5(\mathcal{E}_0) > 0$ *such that, at each time* τ,

$$\mu\|\eta\|_{H^2_A} \le C\left(\|\dot\Phi\|_{H^1_A} + \|\dot{\mathbf{A}}\|^2_{L^2} + \|\Phi\|^2_{L^\infty}\right). \tag{2.46}$$

Proof. From Eq. (2.34) for η, and using the interpolation inequality in Lemma A.9, the elliptic term

$$\mathcal{L}_{(\mathbf{A},\Phi)}\eta \equiv (-4\bar\partial_{\mathbf{A}}(e^{-2\rho}\partial_{\mathbf{A}}\eta) + |\Phi|^2\eta)$$

satisfies, for some $c = c(\mathcal{E}_0) > 0$,

$$\mu\|\mathcal{L}_{(\mathbf{A},\Phi)}\eta\|_{L^2} \le \|\dot\Phi\|_{H^1_A} + \|\Phi\|_{L^\infty}\|\dot{\mathbf{A}}\|_{L^2} + c\|A_0\|_{L^4}(1+\|\eta\|^{1/2}_{H^1}) + c\|\Phi\|^2_{L^\infty}. \tag{2.47}$$

We next see that (2.46) follows from the usual elliptic regularity estimate. Firstly, observe that associated to the operator $\mathcal{L}_{(\mathbf{A},\Phi)}$ is the quadratic form

$$Q_{(\mathbf{A},\Phi)}(\eta) = \langle \eta, \mathcal{L}_{(\mathbf{A},\Phi)}\eta\rangle_{L^2(\Sigma)} = \int_\Sigma \left(4|\partial_{\mathbf{A}}\eta|^2 e^{-4\rho} + |\Phi|^2|\eta|^2 e^{-2\rho}\right) d\mu_g,$$

which is bounded below by $c\|\eta\|^2_{H^1_A}$, where $c = c(\mathcal{E}_0, L) > 0$ by Lemma 3.2.2. It follows that $\|\eta\|_{H^1_A} \le c\|\mathcal{L}_{(\mathbf{A},\Phi)}\eta\|_{L^2}$, a result which can be strengthened by the following:

Claim. $\|\nabla_{\mathbf{A}}\nabla_{\mathbf{A}}\eta\|_{L^2} \leq c\|\mathcal{L}_{(\mathbf{A},\Phi)}\eta\|_{L^2}$ where $c = c(\mathcal{E}_0, L) > 0$.
By the Garding inequality,

$$\|\nabla_{\mathbf{A}}\nabla_{\mathbf{A}}\eta\|_{L^2} \leq \|\mathcal{L}_{(\mathbf{A},\Phi)}\eta\|_{L^2} + c(\mathcal{E}_0, L)(\|\nabla_{\mathbf{A}}\eta\|_{L^4} + \|\eta\|_{H^1_A}).$$

Finally, using the interpolation inequality (A.9) and the Cauchy-Schwarz inequality, we deduce the inequality claimed. □

Corollary 2.3.7. *There is a constant* $C_6 = C_6(\mathcal{E}_0, K, L, M) > 0$ *such that,* $\mu\|\bar{\partial}_{\mathbf{A}}\Phi(\tau)\|_{L^\infty} \leq C_6(1+\tau)$.

Lemma 2.3.8 (Closing the argument: estimate for ζ in L^2). *There is a constant* $C_7(\mathcal{E}_0, L, M)$ *such that* $\zeta = \frac{\partial\psi}{\partial\tau}$ *satisfies*

$$\|\zeta(\tau)\|^2_{L^2} \leq \|\zeta(0)\|^2_{L^2} e^{C_7\int_0^\tau(\|\mu\bar{\partial}_{\mathbf{A}}\Phi(s)\|_{L^\infty}+\|\Phi(s)\|^2_{L^\infty})ds}.$$

Proof. Compute, using (2.31), that

$$\frac{d}{d\tau}\|\zeta(\tau)\|^2_{L^2} = 2\langle\mathbb{J}\zeta, (\mu L_\psi + K_\psi)\zeta\rangle$$

since (by the gauge condition) $\langle\zeta, (d\dot{A}_0, i\Phi\dot{A}_0)\rangle_{L^2} = 0$, and $\langle\zeta, (0, iA_0\dot{\Phi})\rangle_{L^2} = 0$ (using $\langle i\dot{\Phi}, \dot{\Phi}\rangle = 0$ pointwise). By Corollary 3.2.1 and the formula for K_ψ, there exists $C_7 = C_7(\mathcal{E}_0, L) > 0$ such that

$$\left|\frac{d}{d\tau}\|\zeta(\tau)\|^2_{L^2}\right| \leq C_7(\mu\|\bar{\partial}_A\Phi(\tau)\|_{L^\infty} + \|\Phi(\tau)\|^2_{L^\infty})\|\zeta(\tau)\|^2_{L^2}$$

and so the stated inequality follows by the Gronwall lemma. □

Completion of proof of Theorem 2.3.1. The previous lemma allows us to validate the claim that (2.43), and thus all the bounds in Lemmas 2.3.2-2.3.8, in fact hold on a μ-independent interval $[0, \tau_*]$, thus closing the argument. Indeed, by Corollaries 2.3.3 and 2.3.7 we have $\mu\|\bar{\partial}_A\Phi(\tau)\|_{L^\infty}+\|\Phi(\tau)\|^2_{L^\infty} \leq C_8(1+\tau)$ for some $C_8 = C_8(\mathcal{E}_0, L, M)$. Now let τ_*, M_* be such that

$$\|\zeta(0)\|^2_{L^2} e^{C_7C_8(\tau_*+\tau_*^2/2)} \leq M_*^2.$$

(This is always possible for $M_* > \|\zeta(0)\|_{L^2}$ and τ_* small.) Then it follows that (2.43) holds with $T(M_*, \mu) \geq \tau_*$, and that the bounds given in Lemma 2.3.2 through Corollary 2.3.7 hold on the interval $[0, \tau_*]$. To conclude, we explain how to derive the bounds in (2.42). For $\zeta = \dot{\psi}$ we have boundedness of $\|\zeta(\tau)\|_{H^1_A}$ by Corollary 2.3.5. Integrating in τ gives the bound for $\|\mathbf{A}\|_{H^1}$ in (2.42). Also the Kato and Sobolev inequalities ([28]) give a bound for $\dot{\Phi}$ in L^p for $2 \leq p < \infty$. Together with the boundedness of $\|\Phi\|_{L^\infty}$ this implies boundedness of $\|\dot{\mathbf{A}}\|_{W^{1,p}}$ by Lemma 2.2.3. Hence, integrating in τ and applying Sobolev's inequality we deduce boundedness of $\|\mathbf{A}\|_{L^\infty}$. Putting all this information into (A.3),(A.4) we can deduce, from Lemma 2.3.2 and Corollary 2.3.5, that $\big(\Phi(\tau), \dot{\Phi}(\tau)\big)$ is bounded in the (τ-independent) norm $H^2 \times H^1$ as claimed in (2.42). □

2.4. Proof of Theorem 1.7.2. There are three stages to the proof:

- Deduce, from the uniform bounds of Theorem 2.3.1 and the compactness Lemma 2.4.1, that for any sequence $\mu_j \to +\infty$, there exists a subsequence along which the Ψ_{μ_j} converge.
- Identify the limit of these convergent subsequences.
- Deduce, from the uniqueness of the limit just identified, that the Ψ_μ do in fact converge as $\mu \to +\infty$ (without restriction to subsequences).

The first stage of the proof depends upon the following version of the Lions-Aubin Compactness Lemma (see [29, Lemma 10.4]), which is proved by a modification of the standard proof of the usual Ascoli-Arzela theorem:

Lemma 2.4.1. *Assume that (V, h) is a smooth vector bundle with inner product, over a compact Riemannian manifold (Σ, g), which is endowed with a smooth unitary connection ∇ and corresponding Sobolev norms $\|\cdot\|_{H^s}$ on the space of sections defined as in [33]. Assume that l, s are positive numbers with $l < s$. Assume $f_n(\tau)$ is a sequence of smooth time-dependent sections of V which satisfy*

$$\max_{|\tau|\le\tau_*} \left(\|f_n(\tau)\|_{H^s} + \|\dot{f}_n(\tau)\|_{H^l}\right) \le C.$$

Then there exists a subsequence $\{f_{n_j}\}_{j=1}^\infty$ which converges to a limiting time-dependent section $f \in C([-\tau_*, \tau_*]; H^s(V))$, in the sense that, $\max_{|\tau|\le\tau_*} \|(f_n(\tau, \cdot) - f(\tau, \cdot))\|_{H^r} \to 0$, *for every $r < s$.*

Applying this we infer immediately the existence of a subsequence $\mu_j \to +\infty$ along which the solutions $\Psi_{\mu_j} = (\mathbf{A}^{\mu_j}, \Phi^{\mu_j})$ converge to a limit $\Psi_S(\tau)$ in the sense that

$$\lim_{\mu_j\to\infty} \sup_{[-\tau_*,\tau_*]} \left\|\Psi_{\mu_j}(\tau) - \Psi_S(\tau)\right\|_{\mathcal{H}_r} = 0, \tag{2.48}$$

for $r < 2$. It follows from Corollary (2.3.7), that

$$\lim_{\mu\to+\infty} \sup_{[-\tau_*,\tau_*]} \|\bar{\partial}_{\mathbf{A}^\mu} \Phi^\mu\|_{L^\infty} = 0,$$

and since the other Bogomolny equation $B = \frac{1}{2}(1 - |\Phi|^2)$ is satisfied as a constraint, we deduce by Theorem 1.6.1, that $\Psi_S(\tau) \in \mathcal{S}_N$, i.e. the limit $\Psi_S(\tau)$ is a self-dual vortex for each $\tau \in [-\tau_*, \tau_*]$. In addition, by (2.42) we have

$$\|\Psi_\mu(\tau_1) - \Psi_\mu(\tau_2)\|_{H^1} \le M_*|\tau_1 - \tau_2|$$

so that, by (2.48), the limit Ψ_S will also satisfy

$$\|\Psi_S(\tau_1) - \Psi_S(\tau_2)\|_{H^{r'}} \le c|\tau_1 - \tau_2|$$

for $r' < 1$, i.e. the limit is Lipschitz, and in particular lies in $W^{1,\infty}([-\tau_*, \tau_*]; L^2)$.

For the second stage, we need to identify the limiting curve $\tau \mapsto \Psi_S(\tau) \in \mathcal{S}_N$ as that described in Remark 1.7.1. It is clear, from the conditions on the initial data in the statement of Theorem 1.7.2, that $\Psi_S(0) = \psi_0 \in \mathcal{S}_N$, and so it remains to deduce the ordinary differential equation (1.22) which then determines the curve completely. To do this it is necessary to take the limit of (1.20):

$$\mathbb{J}\frac{\partial\Psi_\mu}{\partial\tau} = \mu\mathcal{V}' + U' + \mathbb{J}(dA_0^\mu, iA_0^\mu\Phi^\mu) \tag{2.49}$$

as $\mu \to \infty$. The first term on the right hand side is the most evidently problematic. However, since the limiting motion is constrained to the vortex space $\mathcal{S}_N$, it is only necessary to take a limit projected onto the tangent space $T_{\Psi_S}\mathcal{S}_N$. To this end, it is actually most convenient to introduce $\mathbb{P}_\mu(\tau) = \mathbb{P}_{\Psi_\mu(\tau)}$ the spectral projection operator onto $\operatorname{Ker}\mathcal{D}_{\Psi_\mu(\tau)} = \operatorname{Ker}\mathcal{D}^*_{\Psi_\mu(\tau)}\mathcal{D}_{\Psi_\mu(\tau)}$, discussed in Lemma 3.3.2. By the final statement of Lemma 3.3.2, and the convergence of Ψ_{μ_j} in (2.48), we know that $\mathbb{P}_\mu(\tau)$ converge, in the $L^2 \to L^2$ operator norm, to the operator $\mathbb{P}_{\Psi_S(\tau)}$, which is the spectral projection operator onto $\operatorname{Ker}\mathcal{D}_{\Psi_S(\tau)} = \operatorname{Ker}\mathcal{D}^*_{\Psi_S(\tau)}\mathcal{D}_{\Psi_S(\tau)}$. (This latter operator is also the orthogonal L^2 projector onto the tangent space $T_{\Psi_S}\mathcal{S}_N$ (subject to the gauge condition (1.7).) Apply the operator $\mathbb{P}_\mu(\tau)$ to Eq. (1.20), to obtain:

$$\mathbb{P}_\mu(\tau)\mathbb{J}\frac{\partial \Psi_\mu}{\partial \tau} = \mathbb{P}_\mu(\tau)U'(\Psi_\mu(\tau)), \tag{2.50}$$

since $\mathbb{J}(dA_0, iA_0\Phi_\mu)$ and $\mathcal{V}'(\Psi_\mu)$ are both in the kernel of $\mathbb{P}_\mu$ by Lemma 3.3.2. We can now identify the limit of the right hand side as $\mathbb{P}_{\Psi_S(\tau)}U'(\Psi_S(\tau))$ at each τ, and the convergence is strong in $L^2(\Sigma)$, by (2.48) and the above mentioned convergence of $\mathbb{P}_\mu(\tau)$. For the left hand side it is necessary to consider the limit of the derivatives $\frac{\partial\Psi_\mu}{\partial\tau}$. Noting that these are bounded in e.g. $L^2([-\tau_*, \tau_*]; L^2(\Sigma))$, we may assume (by restricting to a further subsequence if necessary), the weak in L^2 subsequential convergence to a limit which is the weak time derivative of Ψ_S:

$$\langle \tilde{f}, \frac{\partial \Psi_{\mu_j}}{\partial \tau}\rangle_{L^2([-\tau_*,\tau_*];L^2(\Sigma))} \to \langle \tilde{f}, \frac{\partial \Psi_S}{\partial \tau}\rangle_{L^2([-\tau_*,\tau_*];L^2(\Sigma))},$$

for every $\tilde{f} \in L^2([-\tau_*, \tau_*]; L^2(\Sigma))$. Now to identify the limit along a convergent subsequence $\mu_j \to +\infty$, consider the projection operator $\mathbb{P}_{\Psi_S(\tau)}$. Choosing $\tilde{f}(\tau, \cdot) = \mathbb{P}_{\Psi_S(\tau)}(f(\tau, \cdot))$, and using the symmetry of $\mathbb{P}_{\Psi_S(\tau)}$ this implies that

$$\begin{aligned}\int_{-\tau_*}^{+\tau_*} \langle f, \ \mathbb{P}_{\Psi_S(\tau)}\mathbb{J}\frac{\partial \Psi_{\mu_j}}{\partial \tau}\rangle_{L^2(\Sigma)} d\tau &= \int_{-\tau_*}^{+\tau_*} \langle \mathbb{P}_{\Psi_S(\tau)} f, \ \mathbb{J}\frac{\partial \Psi_{\mu_j}}{\partial \tau}\rangle_{L^2(\Sigma)} d\tau \\ \to \int_{-\tau_*}^{+\tau_*} \langle \mathbb{P}_{\Psi_S(\tau)} f, \ \mathbb{J}\frac{\partial \Psi_S}{\partial \tau}\rangle_{L^2(\Sigma)} d\tau &= \int_{-\tau_*}^{+\tau_*} \langle f, \ \mathbb{P}_{\Psi_S(\tau)}\mathbb{J}\frac{\partial \Psi_S}{\partial \tau}\rangle_{L^2(\Sigma)} d\tau,\end{aligned}$$

for any $f \in L^2([-\tau_*, \tau_*]; L^2(\Sigma))$. On the other hand, by the above mentioned convergence of $\mathbb{P}_\mu(\tau)$ to $\mathbb{P}_{\Psi_S(\tau)}$ and the bounded convergence theorem we have

$$\int_{-\tau_*}^{+\tau_*} \left[\langle \mathbb{P}_{\mu_j}(\tau) f, \mathbb{J}\frac{\partial \Psi_{\mu_j}}{\partial \tau}\rangle_{L^2(\Sigma)} d\tau - \langle \mathbb{P}_{\Psi_S(\tau)} f, \mathbb{J}\frac{\partial \Psi_{\mu_j}}{\partial \tau}\rangle_{L^2(\Sigma)} \right] d\tau \to 0,$$

on account of the bound (2.42). Therefore, we have in the limit:

$$\int_{-\tau_*}^{+\tau_*} \langle f, \mathbb{P}_{\Psi_S(\tau)}\mathbb{J}\frac{\partial \Psi_S}{\partial \tau}\rangle_{L^2(\Sigma)} d\tau = \int_{-\tau_*}^{+\tau_*} \langle f, \mathbb{P}_{\Psi_S(\tau)} U'(\Psi_S(\tau))\rangle_{L^2(\Sigma)} d\tau, \tag{2.51}$$

for any $f \in L^2([-\tau_*, \tau_*]; L^2(\Sigma))$. But since the limit is known by the above to be in $W^{1,\infty}([-\tau_*, \tau_*]; L^2)$, it is differentiable (with respect to τ, as a map into L^2) almost everywhere (the standard result extends to Hilbert space-valued functions, see, e.g., [4, Prop. 6.41]); the derivative lies in the tangent space $T_{\Psi_S}\mathcal{S}_N$, which is the range of the projector $\mathbb{P}_{\Psi_S(\tau)}$. Consequently (2.51) implies that $\tau \mapsto \Psi_S(\tau)$ is a solution of (1.22), with equality holding in L^2 for almost every τ. But this in turn implies that $\tau \mapsto \Psi_S(\tau)$ is actually continuously differentiable into L^2, and we have a classical solution of (1.22).

Finally for the third stage: we have now identified the limit as a solution of the limiting Hamiltonian system specified using Remark 1.7.1. Choosing smooth co-ordinates on $\mathcal{M}_N$ as in [46] we see that this is a smooth finite dimensional Hamiltonian system, and as such its solutions (for given initial data) are unique. Therefore all subsequences have the same limit, and so we can assert full convergence without resorting to subsequences.

3. Equations and Identities Related to the Self-Dual Structure

Notation change: *In this section time does not appear at all, and so the boldface* **A** *for the spatial component is not used: i.e. in this section only, A refers to the spatial part of the connection, $A = A_1 dx^1 + A_2 dx^2$.*

Ginzburg-Landau vortices are critical points of the static Ginzburg Landau energy functional $\mathcal{V}_\lambda = \int_\Sigma v_\lambda(A, \Phi) d\mu_g$ introduced following (1.13). The *coupling* constant $\lambda > 0$ is central to the theory of critical points of the Ginzburg-Landau functional and the value $\lambda = 1$ is special as in this case the functional admits the *Bogomolny decomposition* introduced in Remark 1.6.3. This allows for a detailed understanding of the critical points not available for general values of λ, and the theory of critical points for such general values is incomplete. (There is, however, a substantial literature on the asymptotic behaviour of critical points in the $\lambda \to +\infty$ limit, starting with [6]; see [40] and references therein.) This decomposition of $\mathcal{V} \equiv \mathcal{V}_1$ has proved to be very useful not only for the analysis of critical points, but also for the associated time-dependent equations of vortex motion. For our purposes we need in particular to derive a special form for the operator L_ψ associated to the Hessian of $\mathcal{V}$, see (3.61).

3.1. Complex structure. To discuss the Bogomolny structure in detail it is useful to use a complex formulation, so we introduce the complex co-ordinate $z = x^1 + ix^2$ for the complex structure J on Σ. Using this, there is a decomposition of the complex 1-forms $\Omega^1_{\mathbb{C}} = \Omega^{1,0} \oplus \Omega^{0,1}$ into the $\pm i$ eigenspaces of J, see Notation 1.3.1. Let $\Omega^p(L)$ be the space of p-forms taking values in the bundle L: then for $p = 1$ there is a similar decomposition,

$$\Omega^1(L) = \Omega^{1,0}(L) \oplus \Omega^{0,1}(L).$$

Applying this decomposition to $D\Phi \in \Omega^1(L)$ we are led to introduce the operator $D^{0,1}$ given by

$$D^{0,1}\Phi = \frac{1}{2}\left((\nabla_1 - iA_1) + i(\nabla_2 - iA_2)\right)\Phi d\bar{z} \;=\; \bar{\partial}_A \Phi d\bar{z}.$$

For *real* 1-forms $A_1 dx^1 + A_2 dx^2 \in \Omega^1_{\mathbb{R}}$ this decomposition reads

$$A_1 dx^1 + A_2 dx^2 = \alpha dz + \bar{\alpha} d\bar{z},$$

where $\alpha = \frac{A_1 - iA_2}{2}$, and the map $A \mapsto \alpha$ (resp. $A \mapsto \bar{\alpha}$) is an $\mathbb{R}$-linear isomorphism from $\Omega^1_{\mathbb{R}}$ to $\Omega^{1,0}$ (resp. $\Omega^{0,1}$), and $\|A\|^2_{L^2} = 4\int \bar{\alpha}\alpha e^{-2\rho} d\mu_g$. With this α notation we can write

$$\bar{\partial}_A \Phi = \frac{\partial \Phi}{\partial \bar{z}} - i\bar{\alpha}\Phi.$$

3.2. The Hessian. The Bogomolny decomposition amounts to the observation that, with $\lambda = 1$,

$$\mathcal{V}(A,\Phi) \equiv \mathcal{V}_1(A,\Phi) = \frac{1}{2}\int_\Sigma \left(4|\bar\partial_A\Phi|^2 e^{-2\rho} + (B - \frac{1}{2}\left(1-|\Phi|^2\right)\right)^2\right) d\mu_g + \pi N,$$

where $N = deg L$. If the following first order equations, called the Bogomolny equations,

$$\begin{aligned} \bar\partial_A\Phi &= 0, \\ B - \frac{1}{2}(1-|\Phi|^2) &= 0, \end{aligned} \tag{3.52}$$

have solutions in a given class, they will automatically minimize $\mathcal{V}$ within that class.

We introduce the nonlinear Bogomolny operator associated to this decomposition,

$$\begin{aligned} \mathcal{B} \,:\, \Omega^1_{\mathbb{R}} \oplus \Omega^0(L) &\longrightarrow \Omega^0_{\mathbb{R}} \oplus \Omega^{0,1}(L), \\ (A,\Phi) &\mapsto \left(B - \frac{1}{2}(1-|\Phi|^2)\,,\ \bar\partial_A\phi\right). \end{aligned}$$

Using the norm $\|(\beta,\eta)\|^2_{L^2} = \int(|\beta|^2 + 4e^{-2\rho}|\eta|^2)d\mu_g$ induced from the metric on the target space, we see that $\mathcal{V}(A,\Phi) = \frac{1}{2}\|\mathcal{B}(A,\Phi)\|^2_{L^2} + \pi N$ as in Remark 1.6.3; see [8]. The derivative of $\mathcal{B}$ at $\psi = (A,\Phi)$ is the map $D\mathcal{B}_\psi : \Omega^1_{\mathbb{R}} \oplus \Omega^0(L) \longrightarrow \Omega^0_{\mathbb{R}} \oplus \Omega^{0,1}(L)$ given by

$$(\dot A, \dot\Phi) \mapsto (*d\dot A + \langle\Phi,\dot\Phi\rangle\,,\ \bar\partial_A\dot\Phi - i\dot{\bar\alpha}\Phi), \tag{3.53}$$

where $\alpha = \frac{A_1 - iA_2}{2}$ and $\dot\alpha = \frac{\dot A_1 - i\dot A_2}{2}$. Using this complex notation allows a simple unified formulation, which takes account the *gauge condition* (1.7): this condition is the real part of

$$4e^{-2\rho}\bar\partial\dot\alpha - i\Phi\dot{\bar\Phi} = 0, \tag{3.54}$$

while the imaginary part of this expression is just the condition $*d\dot A + (\Phi,\dot\Phi) = 0$, appearing in the linearized Bogomolny equations. This suggests the introduction of the operators

$$\begin{aligned} \mathcal{D}_\psi &: \left(\Omega^{1,0} \oplus \Omega^0(L)\right) \longrightarrow \left(\Omega^0_{\mathbb{C}} \oplus \Omega^{0,1}(L)\right), \\ \mathcal{D}^*_\psi &: \left(\Omega^0_{\mathbb{C}} \oplus \Omega^{0,1}(L)\right) \longrightarrow \left(\Omega^{1,0} \oplus \Omega^0(L)\right), \end{aligned} \tag{3.55}$$

given by

$$\begin{aligned} \mathcal{D}_\psi(\dot\alpha,\dot\Phi) &= (4e^{-2\rho}\bar\partial\dot\alpha - i\Phi\dot{\bar\Phi},\ \bar\partial_A\dot\Phi - i\dot{\bar\alpha}\Phi), \\ \mathcal{D}^*_\psi(\beta,\eta) &= (-\partial\beta - i\Phi\bar\eta,\ -4e^{-2\rho}\partial_A\eta - i\Phi\bar\beta). \end{aligned} \tag{3.56}$$

We use the real inner product associated to the L^2 norms induced from the metric as above, i.e.:

$$\left\langle(\dot\alpha,\dot\Phi),(\alpha',\Phi')\right\rangle_{L^2} = \int\left(4e^{-2\rho}\Re\dot{\bar\alpha}\alpha' + \Re\dot{\bar\Phi}\Phi'\right)d\mu_g \qquad \text{on } \Omega^{1,0}\oplus\Omega^0(L),$$

$$\left\langle(\beta,\eta),(\beta',\eta')\right\rangle_{L^2} = \int\left(\Re\bar\beta\beta' + 4e^{-2\rho}\Re\bar\eta\eta'\right)d\mu_g \qquad \text{on } \Omega^0_{\mathbb{C}}\oplus\Omega^{0,1}(L)\,.$$

Integrating by parts we deduce that

$$\left\langle \mathcal{D}_\psi(\dot\alpha,\dot\Phi),(\beta,\eta)\right\rangle_{L^2} = \left\langle (\dot\alpha,\dot\Phi),\mathcal{D}^*_\psi(\beta,\eta)\right\rangle_{L^2},$$

so that $\mathcal{D}^*_\psi$ is the L^2 adjoint of $\mathcal{D}_\psi$ and

$$\begin{aligned}\mathcal{D}^*_\psi\mathcal{D}_\psi(\dot\alpha,\dot\Phi) &= \Big(-\partial(4e^{-2\rho}\bar\partial\dot\alpha - i\Phi\bar{\dot\Phi}) - i\Phi\,(\overline{\bar\partial_A\dot\Phi} + i\dot\alpha\bar\Phi)\,,\\ &\quad -4e^{-2\rho}\partial_A(\bar\partial_A\dot\Phi - i\bar{\dot\alpha}\Phi) - i\Phi(4e^{-2\rho}\partial\bar{\dot\alpha} + i\bar\Phi\dot\Phi)\Big)\\ &= \Big(-\partial(4e^{-2\rho}\bar\partial\dot\alpha) + i(\partial_A\Phi)\bar{\dot\Phi} + |\Phi|^2\dot\alpha\,,\\ &\quad -4e^{-2\rho}\partial_A\bar\partial_A\dot\Phi + |\Phi|^2\dot\Phi + i4e^{-2\rho}\bar{\dot\alpha}\partial_A\Phi\Big)\,. \end{aligned} \tag{3.57}$$

We compare this expression with the operator defined in (2.40):

$$\overline{L}_\psi\ :\ \left(\Omega^{1,0}\oplus\Omega^0(L)\right) \longrightarrow \left(\Omega^{1,0}\oplus\Omega^0(L)\right), \tag{3.58}$$

which defines the Hessian of $\mathcal{V}$ on the subspace on which the gauge condition (1.7) is satisfied, i.e.,

$$\langle\dot\psi,\overline{L}_\psi\dot\psi\rangle_{L^2} = D^2\mathcal{V}_\psi(\dot\psi,\dot\psi) = \frac{d^2}{d\epsilon^2}|_{\epsilon=0}\mathcal{V}(\psi+\epsilon\dot\psi), \tag{3.59}$$

for $\dot\psi = (\dot A,\dot\Phi)$ satisfying (1.7). Using mixed real/complex notation for A/α, (2.40) implies the following formula:

$$\begin{aligned}\overline{L}_\psi = &\Big(-4\partial(e^{-2\rho}\bar\partial\dot\alpha) + |\Phi|^2\dot\alpha - (i\dot\Phi, D_1\Phi) + i(i\dot\Phi, D_2\Phi)\ ,\\ &-\Delta_A\dot\Phi - \frac{1}{2}(1-3|\Phi|^2)\dot\Phi + 2ie^{-2\rho}\dot A\cdot D\Phi\Big).\end{aligned} \tag{3.60}$$

Calculate $\dot A\cdot D\Phi = 2\dot\alpha\bar\partial_A\Phi + 2\bar{\dot\alpha}\partial_A\Phi$ and $-(i\dot\Phi, D_1\Phi) + i(i\dot\Phi, D_2\Phi) = i\bar{\dot\Phi}\partial_A\Phi - i\dot\Phi\overline{\bar\partial_A\Phi}$, from which it follows that

$$(\overline{L}_\psi - \mathcal{D}^*_\psi\mathcal{D}_\psi)\dot\psi = \begin{pmatrix} -i\dot\Phi\overline{\bar\partial_A\Phi} \\ \left(B - \frac{1}{2}(1-|\Phi|^2)\right)\dot\Phi + 4ie^{-2\rho}\dot\alpha\bar\partial_A\Phi \end{pmatrix}. \tag{3.61}$$

(Incidentally, observing that

$$\mathcal{B}(A+\dot A,\Phi+\dot\Phi) = \mathcal{B}(A,\Phi) + \mathcal{D}_\psi\dot\psi + \left(\frac{1}{2}|\dot\Phi|^2, -\bar{\dot{i}}\alpha\dot\Phi\right),$$

with $\dot\psi = (\dot A,\dot\Phi)$ satisfying (1.7), the identity (3.61) can also be read off from the quadratic part of the Taylor expansion for $\mathcal{V}(A+\dot A,\Phi+\dot\Phi)$:

$$\begin{aligned}\frac{1}{2}\langle\dot\psi,\overline{L}_\psi\dot\psi\rangle_{L^2} &= \frac{1}{2}|\mathcal{D}_\psi\dot\psi|^2_{L^2} + \left\langle \mathcal{B}(\psi)\,,\left(\frac{1}{2}|\dot\Phi|^2, -i\bar{\dot\alpha}\dot\Phi\right)\right\rangle\\ &= \frac{1}{2}|\mathcal{D}_\psi\dot\psi|^2_{L^2} + \int_\Sigma \left(\frac{1}{2}(B - \frac{1}{2}(1-|\Phi|^2))|\dot\Phi|^2\right.\\ &\quad \left. +4e^{-2\rho}\langle\bar\partial_A\Phi, -i\bar{\dot\alpha}\dot\Phi\rangle\right)\,d\mu_g,\end{aligned}$$

using the inner product on $\Omega^{1,0}\oplus\Omega^0(L)$ defined above.)

Corollary 3.2.1. *Let* $\mathbb{J}$ *denote the complex structure defined in* (1.21). *There exists a number* $c > 0$, *independent of* $\psi = (\alpha, \Phi)$ *and* $\zeta = \dot{\psi} = (\dot{\alpha}, \dot{\Phi}) \in \Omega^{1,0} \oplus \Omega^0(L)$, *such that*

$$|\langle \mathbb{J}\zeta, L_\psi \zeta \rangle_{L^2}| \leq c\, |\mathcal{B}(\psi)|_{L^\infty} |\zeta|^2_{L^2}.$$

Proof. By (3.61) $|\langle \mathbb{J}\zeta, L_\psi \zeta\rangle_{L^2} - \langle \mathcal{D}_\psi \mathbb{J}\zeta, \mathcal{D}_\psi \zeta\rangle_{L^2}| \leq |\mathcal{B}(\psi)|_{L^\infty}|\zeta|^2_{L^2}$. Now the complex structure $\mathbb{J}$ written in complex notation, i.e. acting on $\Omega^{1,0} \oplus \Omega^0_(L)$, is given by $\mathbb{J}(\dot{\alpha}, \dot{\Phi}) = (-i\dot{\alpha}, i\dot{\Phi})$. Correspondingly, on $\Omega^0_{\mathbb{C}} \oplus \Omega^{0,1}(L)$ we introduce the complex structure $\mathbb{J}'(\beta, \eta) = (i\beta, -i\eta)$. Then, by observation

$$\mathcal{D}_\psi \mathbb{J}\zeta = -\mathbb{J}'\mathcal{D}_\psi \zeta. \tag{3.62}$$

Therefore, writing $w = \mathcal{D}_\psi \zeta$, we have $\langle \mathcal{D}_\psi \mathbb{J}\zeta, \mathcal{D}_\psi \zeta\rangle_{L^2} = \langle -\mathbb{J}'w, w\rangle_{L^2} = 0$ by skew-symmetry, and the result follows. □

Lemma 3.2.2. *Assume there are positive numbers* $L, \mathcal{E}_0$ *such that* $|\Phi|_{L^2} = L$, *and* $\mathcal{V}_\lambda(A, \Phi) = \mathcal{E}_0$ *and* $\lambda > 0$. *Then the quadratic forms*

$$\tilde{Q}_\Phi(\beta) = \int_\Sigma 4|\partial\beta|^2 e^{-2\rho} + |\Phi|^2|\beta|^2 d\mu_g \quad on \quad \oplus\, \Omega^0_{\mathbb{C}} \quad and$$
$$Q_{(A,\Phi)}(\eta) = \int_\Sigma 4e^{-4\rho}|\partial_A \eta|^2 + e^{-2\rho}|\Phi|^2|\eta|^2 d\mu_g \quad on \quad \Omega^{0,1}(L)$$

are strictly positive, and in fact bounded below by (respectively) $C\|\beta\|^2_{H^1}$ *and* $C\|\eta\|^2_{H^1_A}$, *where* C *is a positive number depending only upon the numbers* $L, \mathcal{E}_0$.

Proof. We will present the proof for the quadratic form $Q_{(A,\Phi)}(\eta)$ as the other is similar but easier. Clearly $Q_{(A,\Phi)}(\eta) \geq 0$ and in fact $Q_{(A,\Phi)}(\eta) = 0$ if and only if $\eta \equiv 0$ on Σ (because if $\partial_A\eta \equiv 0$ then η has isolated zeros (as in [28], Sect. 3.5); if $\Phi\eta \equiv 0$ then $\eta \equiv 0$, since $\Phi = 0$ $a.e.$ contradicts $\int_\Sigma |\Phi|^2 = L > 0$. Furthermore, we show that $Q_{(A,\Phi)}(\eta) \geq c|\eta|^2_{L^2}$ for a constant c; to be precise there exists $c = c(L, \mathcal{E}_0)$ such that

$$Q_{(A,\Phi)}(\eta) \geq c, \text{ for all } \eta \text{ such that } \|\eta\|_{L^2} = 1. \tag{3.63}$$

We will prove this by contradiction. First we obtain some bounds. By gauge invariance we are free to assume that the Coulomb gauge condition $\operatorname{div} A = 0$ holds. With this gauge condition, we have the bound $\|A\|_{H^1} \leq c(\mathcal{E}_0)$, and so A is bounded in every L^p space. Now use $\|\partial\eta\|_{L^p} \leq \|\partial_A\eta\|_{L^p} + \|A\eta\|_{L^p}$ to deduce that

$$\|\partial\eta\|^2_{L^p} \leq C(1 + Q_{(A,\Phi)}(\eta))$$

for every $p < 2$, by Holder's inequality. This in turn implies, by the L^p estimate for the inhomogeneous Cauchy-Riemann system, that η is bounded similarly in L^4, and so since A is also we can bound $\partial\eta$ in L^2 and hence η in H^1. Finally, since A and η are bounded similarly in L^4, this imples that $\|\eta\|^2_{H^1_A} \leq C(1 + Q_{(A,\Phi)}(\eta))$, with C depending only upon $\mathcal{E}_0$, L. To conclude, in Coulomb gauge the A, Φ, η are all bounded in H^1 in terms of L, $\mathcal{E}_0$, $Q_{(A,\Phi)}(\eta)$.

The contradiction argument now starts: assume (3.63) fails. Then, by the bounds just obtained and the Banach-Alaoglu and Rellich theorems, there is a sequence $(A_\nu, \Phi_\nu, \eta_\nu)$ with

$$\|A_\nu\|_{H^1} + \|\nabla\Phi_\nu\|_{L^2} \le K(\mathcal{E}_0, L),$$

$\|\Phi_\nu\|_{L^2} = L$ and $\|\eta_\nu\|_{L^2} = 1$, such that

$$\begin{aligned} &Q_{(A_\nu,\Phi_\nu)}(\eta_\nu) \longrightarrow 0,\\ &A_\nu \longrightarrow A \ \text{ weakly in } H^1,\\ &\Phi_\nu \longrightarrow \Phi \ \text{ weakly in } H^1 \text{ and strongly in } L^p \text{ for any } p < \infty,\\ &\eta_\nu \longrightarrow \eta \ \text{ weakly in } H^1 \text{ and strongly in } L^p. \end{aligned}$$

This implies that $|\Phi|_{L^2} = L > 0$, $Q_{(A,\Phi)}(\eta) = 0$ which implies as above that $\Phi = 0$ $a.e.$ and contradicts as above that $|\Phi|_{L^2}$ is constant. This leads to

$$Q_{(A,\Phi)}(\eta) \ge c_1|\eta|^2_{L^2} \text{ where } c_1 = c_1(L, \mathcal{E}_0).$$

Finally just apply the bound above for $\|D\eta\|_{L^2}$ to improve this up to the H^1_A lower bound claimed. □

3.3. The Bogomolny foliation. We introduce a foliation associated to the Bogomolny operator, which we regard as a map between the following Hilbert spaces:

$$\begin{aligned} \mathcal{B} \ : \ H^1\left(\Omega^1_{\mathbb{R}} \oplus \Omega^0(L)\right) &\longrightarrow L^2\left(\Omega^0_{\mathbb{R}} \oplus \Omega^{0,1}(L)\right),\\ (A,\Phi) &\mapsto \left(B - \frac{1}{2}(1-|\Phi|^2)\ ,\ \bar{\partial}_A\phi\right). \end{aligned}$$

With this choice of norms $\mathcal{B}$ is a smooth function. The next result shows that it is a submersion if the energy is close to the minimum value:

Lemma 3.3.1. *There exists $\theta_* > 0$ such that $\|\bar{\partial}_A\Phi\|_{L^2} < \theta_*$ implies that $Ker\mathcal{D}^*_\Psi = \{0\}$, and $Ker\mathcal{D}_\Psi$ is $2N$ dimensional (where $N = degL$).*

Proof. $\mathcal{D}^*(\beta,\eta) = 0$ is equivalent to

$$\begin{aligned} -\partial\beta - i\Phi\bar{\eta} &= 0,\\ -4e^{-2\rho}\partial_A\eta - i\Phi\bar{\beta} &= 0. \end{aligned}$$

Apply the operations $4\bar{\partial}$ to the first and $4\bar{\partial}_A$ to the second of these equations to deduce that

$$\begin{aligned} -4e^{-2\rho}\bar{\partial}\partial\beta + |\Phi|^2\beta - 4ie^{-2\rho}\bar{\partial}_A\Phi\bar{\eta} &= 0,\\ -4\bar{\partial}_A(e^{-2\rho}\bar{\partial}_A\eta) + |\Phi|^2\eta - i(\bar{\partial}_A\Phi)\beta &= 0. \end{aligned}$$

The first two terms of these two equations are respectively the Euler- Lagrange operators associated to the quadratic forms $\tilde{Q}_\Phi(\beta)$ and $Q_{A,\Phi}(\eta)$ studied in the previous lemma. Then we get the estimates

$$\begin{aligned} \tilde{Q}_\Phi(\beta) &\le c|\bar{\partial}_A\Phi|_{L^2}|\beta|_{L^4}|\eta|_{L^4},\\ Q_{A,\Phi}(\eta) &\le c|\bar{\partial}_A\Phi|_{L^2}|\beta|_{L^4}|\eta|_{L^4}, \end{aligned}$$

which implies the result, since $\tilde{Q}_\Phi(\beta) \ \ge\ c|\beta|^2_{H^1}$ and $Q_{A,\Phi}(\eta) \ \ge\ c|\eta|^2_{H^1_A}$. □

The natural geometrical context for the results of this section will now be explained. Define $\mathcal{O}_* \equiv \{(A, \Phi) \in H^1(\Omega^1_{\mathbb{R}} \oplus \Omega^0(L)) : \|\bar\partial_A \Phi\|_{L^2} < \theta_*\}$ which is an open set containing $\{\psi = (A, \Phi) : \mathcal{B}(\psi) = 0\} \subset H^1(\Omega^1_{\mathbb{R}} \oplus \Omega^0(L))$. Furthermore, the previous lemma implies that $\mathcal{D}_\psi : (\Omega^{1,0} \oplus \Omega^0(L)) \longrightarrow (\Omega^0_{\mathbb{C}} \oplus \Omega^{0,1}(L))$ is surjective for $\psi \in \mathcal{O}_*$. By the discussion in the paragraph preceding (3.55), this implies that $D\mathcal{B}_\psi : \Omega^1_{\mathbb{R}} \oplus \Omega^0(L) \longrightarrow \Omega^0_{\mathbb{R}} \oplus \Omega^{0,1}(L)$ is also surjective for $\psi \in \mathcal{O}_*$, and hence the level sets of $\mathcal{B}$ form a foliation of $\mathcal{O}_*$ whose leaves have tangent space equal to $\mathrm{Ker}\, D\mathcal{B}_\psi$ by [1, Sect. 3.5 and Sect. 4.4]. The intersection of this tangent space with $\mathcal{S}L_\psi = \{(\dot A, \dot\Phi) : (\dot A, \dot\Phi)$ satisfies (1.7)$\}$ is $\mathrm{Ker}\,\mathcal{D}_\Psi$.

Lemma 3.3.2. *Assume $\psi \in (\Omega^{1,0} \oplus \Omega^0_{\mathbb{C}}(L)) \cap \mathcal{O}_*$. The operators $\mathcal{D}^*_\psi \mathcal{D}_\psi$ defined in (3.57) are self-adjoint operators on L^2, with domain H^2, with $2N$-dimensional kernel equal to $\mathrm{Ker}\,\mathcal{D}_\psi$, and*

$$\|\mathcal{D}^*_\psi \mathcal{D}_\psi \zeta\|_{L^2} + \|\zeta\|_{L^2} \geq c\|\zeta\|_{H^2}. \tag{3.64}$$

Let $\mathbb{P}_\psi$ be the orthogonal spectral projector onto $\mathrm{Ker}\,\mathcal{D}^*_\psi \mathcal{D}_\psi = \mathrm{Ker}\,\mathcal{D}_\psi$. Then $\mathbb{P}_\psi(\mathcal{V}'(\psi)) = 0$ and $\mathbb{P}_\psi(\mathbb{J}(d\chi, i\chi\Phi_\mu)) = 0$ for any smooth real valued function χ. Finally, if also $\psi^{(j)} \in (\Omega^{1,0} \oplus \Omega^0_{\mathbb{C}}(L)) \cap \mathcal{O}_*$, and $\sup_j \|\psi^{(j)}\|_{\mathcal{H}_2} < \infty$ and $\lim_{j\to+\infty} \|\psi^{(j)} - \psi\|_{\mathcal{H}_r} = 0$, for all $r < 2$, the corresponding projectors $\mathbb{P}_{\psi^{(j)}}$ converge to $\mathbb{P}_\psi$ in $L^2 \to L^2$ operator norm.

Proof. The first assertion and the bound (3.64) follow from Lemma 3.3.1 and standard elliptic theory. The next statement follows by noting that if $n \in \mathrm{Ker}\,\mathcal{D}_\psi$, then differentiation of $\mathcal{V}(\psi) = \frac{1}{2}\int |\mathcal{B}(\psi)|^2 d\mu_g + \pi N$ yields

$$\langle n, \mathcal{V}'(\psi)\rangle_{L^2} = \frac{d}{ds}\bigg|_{s=0} \mathcal{V}(\psi + sn) = \langle \mathcal{B}(\psi), D\mathcal{B}_\psi(n)\rangle_{L^2} = 0,$$

since $\mathrm{Ker}\,\mathcal{D}_\psi \subset \mathrm{Ker}\, D\mathcal{B}_\psi$ by the discussion preceding (3.55). Next, $n \in \mathrm{Ker}\,\mathcal{D}_\psi$ implies that $\mathbb{P}_\psi(\mathbb{J}(d\chi, i\chi\Phi_\mu)) = 0$ since integration by parts reduces this to the fact that n solves the first component of $D\mathcal{B}_\psi n = 0$ in (3.53).

The final statement follows by [25, Sect. IV.3], if it can be established that $T_j \equiv \mathcal{D}^*_{\psi^{(j)}} \mathcal{D}_{\psi^{(j)}}$ converges to $T \equiv \mathcal{D}^*_\psi \mathcal{D}_\psi$ in the generalized sense of Kato (see [25, Sect. IV.2.6]), or equivalently in the norm resolvent sense:

$$\lim_{j\to\infty} \|(i + T)^{-1} - (i + T_j)^{-1}\|_{L^2 \to L^2} = 0. \tag{3.65}$$

To verify this convergence, it is convenient first of all to verify it in Coulomb gauge. So let $\tilde\psi^{(j)} = (\tilde A^{(j)}, \tilde\Phi^{(j)}) = e^{i\chi_j} \cdot \psi^{(j)}$ and $\tilde\psi = (\tilde A, \tilde\Phi) = e^{i\chi} \cdot \psi$ be gauge transforms (as defined following (1.11)), such that $\mathrm{div}\, \tilde A^{(j)} = 0 = \mathrm{div}\, \tilde A$. The assumed properties of $\psi^{(j)}$ ensure that $\sup \|\chi_j\|_{H^2} < \infty$ and that $\lim \|\chi_j - \chi\|_{H^r} = 0$, $\forall r < 2$, so that also $\tilde\psi^{(j)} \to \tilde\psi$ in $\mathcal{H}_r$ for $r < 2$. Now observe that *in Coulomb gauge* the formula (3.57) does not involve any derivatives of the connection one-form A at all. From this it is then immediate by inspection that (writing $\tilde T_j \equiv \mathcal{D}^*_{\tilde\psi^{(j)}} \mathcal{D}_{\tilde\psi^{(j)}}$, and $\tilde T \equiv \mathcal{D}^*_{\tilde\psi} \mathcal{D}_{\tilde\psi}$,)

$$\|(\tilde T - \tilde T_j)\zeta\|_{L^2} \leq \delta_j \|\zeta\|_{H^2} \leq c\delta_j(\|\zeta\|_{L^2} + \|\tilde T\zeta\|_{L^2}), \tag{3.66}$$

where $\delta_j \to 0$ as $j \to +\infty$. But this last fact implies (by [25, Theorems IV.2.24-25]) that $\tilde{T}_j$ converges to $\tilde{T}$ in the generalized sense, and hence in the resolvent sense:

$$\lim_{j\to\infty} \|(i+\tilde{T})^{-1} - (i+\tilde{T}_j)^{-1}\|_{L^2\to L^2} = 0. \tag{3.67}$$

This would establish the convergence of the corresponding spectral projectors in Coulomb gauge. To go back to the original ψ_j it is just necessary to make use of the following gauge invariance property: on $\zeta = (\dot{\alpha}, \dot{\Phi})$ the induced action of the gauge group is $g \bullet (\dot{\alpha}, \dot{\Phi}) = (\dot{\alpha}, g\dot{\Phi})$ for any S^1 valued function g, and

$$\tilde{T}\left(e^{i\chi} \bullet \zeta\right) = e^{i\chi} \bullet (T\zeta),$$

and similarly with T_j, χ_j replaced by T, χ. This gauge invariance property implies that $(i+T_j)^{-1} = e^{-i\chi_j} \circ (i+\tilde{T}_j)^{-1} \circ e^{i\chi_j}$ and $(i+T)^{-1} = e^{-i\chi} \circ (i+\tilde{T})^{-1} \circ e^{i\chi}$, where by $\circ$ we mean operator composition, and $e^{i\chi}$ is shorthand for the operator $e^{i\chi}\bullet$ etc. Finally, using $\lim \|\chi_j - \chi\|_{H^r} = 0$, $\forall r < 2$, we see that (3.66) and (3.67) imply (3.65), completing the proof. □

Appendix

A.1. Operators. To describe in detail the Laplacian operators which appear in the text, we assume Σ to be covered by an atlas of charts U_α on each of which is a local trivialisation of L determined by a choice of a local unitary frame. (A smooth section Φ of L then corresponds to a family of smooth functions $\Phi_\alpha : U_\alpha \to \mathbb{C}$ so that on $U_\alpha \cap U_\beta$ we have $\Phi_\alpha = e^{i\theta_{\alpha\beta}}\Phi_\beta$ with $e^{i\theta_{\alpha\beta}} : U_\alpha \cap U_\beta \to S^1$ smooth.) We assume given a smooth connection $\mathbf{D} = \nabla - i\mathbf{A}$ on L acting as a covariant derivative operator on sections of L. Working in such a chart, and suppressing the index α, the Laplacian on sections Φ of L is given by

$$-\Delta_A\Phi = -\frac{1}{\sqrt{g}} D_j\left(g^{ij}\sqrt{g} D_i\Phi\right) = -e^{-2\rho}\,(D_i D_i \Phi). \tag{A.1}$$

This satisfies $\langle -\Delta_A\Phi, \Phi'\rangle_{L^2} = \frac{d}{d\epsilon}\frac{1}{2}|\mathbf{D}(\Phi+\epsilon\Phi')|^2_{L^2}|_{\epsilon=0}$.

Next we need the Laplacian on one-forms. Starting with $\mathbf{A} = A_1dx^1 + A_2dx^2 \in \Omega^1_{\mathbb{R}}$, the negative Laplacian is the Euler-Lagrange operator associated to the Dirichlet form $\frac{1}{2}\int(|\mathrm{div}\,\mathbf{A}|^2 + |\mathbf{dA}|^2)d\mu_g$ (with the norms inside the integral determined by g in the standard way). Transferring to complex form $\alpha = \frac{1}{2}(A_1 - iA_2) \in \Omega^{1,0}$, this Dirichlet form is just $I(\alpha) = 8\int e^{-4\rho}\overline{\bar\partial\alpha}\bar\partial\alpha\, d\mu_g$. The corresponding negative Laplacian $-\Delta^{1,0}$ is then defined by $\langle -\Delta^{1,0}\alpha, \beta\rangle_{L^2} = \frac{d}{d\epsilon}I(\alpha+\epsilon\beta)|_{\epsilon=0}$, where we use the induced inner product $\Omega^{1,0}$ as in Sect. 3. This leads to the following formula for the negative Laplacian $-\Delta^{1,0}$ on $\alpha \in \Omega^{1,0}$:

$$-\Delta^{1,0}\alpha = -4\partial(e^{-2\rho}\bar\partial\alpha),$$

which is precisely the operator appearing in Sect. 3. Similarly, on $\Omega^{0,1}(L)$ the negative Laplacian is

$$-\Delta^{0,1}_A\eta = -4\bar\partial_A(e^{-2\rho}\partial_A\eta),$$

which is the operator in (2.34).

A.2. Norms and inequalities. We define the Sobolev norms defined with the covariant derivative $\mathbf{D} = \nabla_{\mathbf{A}} = \nabla - i\mathbf{A}$. (We write $\nabla_{\mathbf{A}}$ in place of $\mathbf{D}$ for emphasis here.) The first Sobolev norm is defined by

$$|\Phi|^2_{H^1_{\mathbf{A}}} = \int_\Sigma \left(|\Phi|^2 + |\nabla_A \Phi|^2\right) d\mu_g. \tag{A.2}$$

In the above integral the inner products are the standard ones induced from h and g. The higher norms $H^2_{\mathbf{A}}, \ldots$ are defined similarly, as are the $W^{k,p}_{\mathbf{A}}$ norms for integral k and any $p \in [1, \infty]$. The L^p norms of the higher covariant derivatives arising from the connections $\nabla_{\mathbf{A}}$ and ∇ are related as expressed schematically in the following:

$$\|\nabla\Phi\|_{L^p} \le \|\nabla_{\mathbf{A}}\Phi\|_{L^p} + c\|\mathbf{A}\|_{L^\infty}\|\Phi\|_{L^p}, \tag{A.3}$$

$$\begin{aligned}\|\nabla\nabla\Phi\|_{L^p} \le\ & \|\nabla_{\mathbf{A}}\nabla_{\mathbf{A}}\Phi\|_{L^p} + c\|\mathbf{A}\|_{L^\infty}\|\nabla_{\mathbf{A}}\Phi\|_{L^p} \\ & + c(1 + \|\nabla\mathbf{A}\ \Phi\|_{L^p} + \|\mathbf{A}\|^2_{L^\infty}\|\Phi\|_{L^p}),\end{aligned} \tag{A.4}$$

$$\begin{aligned}\|\nabla\nabla\nabla\Phi\|_{L^p} \le\ & \|\nabla_{\mathbf{A}}\nabla_{\mathbf{A}}\nabla_{\mathbf{A}}\Phi\|_{L^p} + c\|\mathbf{A}\|_{L^\infty}\|\nabla_{\mathbf{A}}\nabla_{\mathbf{A}}\Phi\|_{L^p} \\ & + c(1 + \|\nabla\mathbf{A}\|_{L^\infty} + \|\mathbf{A}\|^2_{L^\infty})\|\nabla_{\mathbf{A}}\Phi\|_{L^p} \\ & + c\left(1 + \|\nabla^2\mathbf{A}\|_{L^q}\|\Phi\|_{L^r} + \|\mathbf{A}\|^3_{L^\infty}\|\Phi\|_{L^p}\right),\end{aligned} \tag{A.5}$$

where $q^{-1} + r^{-1} = p^{-1}$.

We now collect together some inequalities from [13].

The system of equations

$$B = f \qquad \operatorname{div}\mathbf{A} = g \tag{A.6}$$

(where as above div : $\Omega^1 \to \Omega^0$ is minus the adjoint of d) is a first order elliptic system which can be solved for $\mathbf{A}$ subject to the condition on $\int f d\mu_g$ dictated by an integer N, the degree of L. It can be rewritten

$$\mathbf{dA} = (f - b)d\mu_g \qquad \operatorname{div}\mathbf{A} = g \tag{A.7}$$

and solved via Hodge decomposition as long as the right hand sides have zero integral. There is a solution unique up to addition of harmonic 1-forms which satisfies $\|\mathbf{A}\|_{W^{1,p}} \le c_p(1 + \|f\|_{L^p} + \|g\|_{L^p})$ for $p < \infty$.

Lemma A.2.1 (Covariant Sobolev and Gagliardo-Nirenberg inequalities). *For* (Σ, g) *as above and for* $(\mathbf{A}, \Phi) \in (H^1 \times H^2_{\mathbf{A}})(\Sigma)$, *then* $\nabla_{\mathbf{A}}\Phi \in L^4(\Sigma)$ *and*

$$\|\nabla_{\mathbf{A}}\Phi\|_{L^4} \le c\|\nabla_{\mathbf{A}}\Phi\|_{H^1_{\mathbf{A}}} \tag{A.8}$$

and also for all $1 \le p < \infty$, $H^2_{\mathbf{A}} \hookrightarrow W^{1,p}_{\mathbf{A}} \hookrightarrow L^\infty$ *continuously on* Σ. *Also*

$$\|\nabla_{\mathbf{A}}\Phi\|_{L^4} \le c\|\nabla_{\mathbf{A}}\Phi\|^{1/2}_{L^2}\left(\|\nabla_{\mathbf{A}}\Phi\|^{1/2}_{L^2} + \|\nabla_{\mathbf{A}}\nabla_{\mathbf{A}}\Phi\|^{1/2}_{L^2}\right), \tag{A.9}$$

where c depends only on (Σ, g).

Lemma A.2.2 (Covariant version of the Garding inequality). *For* $\Psi = (\mathbf{A}, \Phi)$ *such that the norms on* Σ *appearing below are finite we have*

$$\begin{aligned}\|\nabla_{\mathbf{A}}\nabla_{\mathbf{A}}\Phi\|_{L^2} \le\ & \|\Delta_{\mathbf{A}}\Phi\|_{L^2} + c\|B\|^{1/2}_{L^\infty}\|\nabla_{\mathbf{A}}\Phi\|_{L^2} \\ & + c\|\Phi\|^{1/2}_{L^\infty}\|\nabla_{\mathbf{A}}\Phi\|^{1/2}_{L^2}\|\nabla B\|^{1/2}_{L^2},\end{aligned} \tag{A.10}$$

where c is a number depending only on (Σ, g).

Lemma A.2.3 (Covariant version of the Brezis-Gallouet inequality). *If* $\mathbf{A} \in H^1(\Sigma)$ *and* $\Phi \in H^2_{\mathbf{A}}(\Sigma)$ *then*

$$\|\Phi\|_{L^\infty(\Sigma)} \leq c\left(1 + \|\Phi\|_{H^1_{\mathbf{A}}}\sqrt{\ln(1 + \|\Phi\|_{H^2_{\mathbf{A}}})}\right), \tag{A.11}$$

where c depends only on (Σ, g).

A.3. Global existence results and different choices of gauge. In this section we will summarize the existence theory for (1.5) from [5] and [13], and explain how Theorem 1.4.1 can be deduced from it. Existence theory can be worked out using various gauge conditions, and a choice of gauge is usually made to facilitate the calculations. The simplest condition for the statement of the theorem, which also is convenient if we wish to make the Hamiltonian structure manifest - see Sect. 1.5, is the temporal gauge condition $A_0 = 0$; however, the regularity is stronger in Coulomb gauge $\operatorname{div}\mathbf{A} = 0$. We have the following statements.

Theorem A.3.1 (Global existence in temporal gauge). *Given data* $\Phi(0) \in H^2(\Sigma)$ *and* $\mathbf{A}(0) \in H^1(\Sigma)$*, there exists a global solution for the Cauchy problem for* (1.5) *satisfying* $A_0 = 0$*, with regularity* $\Phi \in C\left([0,\infty); H^2(\Sigma)\right) \cap C^1\left([0,\infty); L^2(\Sigma)\right)$ *and* $\mathbf{A} \in C^1\left([0,\infty); H^1(\Sigma)\right)$. *Furthermore, it is the unique such solution satisfying* $A_0 = 0$ *and satisfies the estimate*

$$\|\Phi(t)\|_{H^2(\Sigma)} \leq ce^{\alpha e^{\beta t}},$$

for some positive constants c, α, β *depending only on* (Σ, g)*, the equations, and the initial data.*

This can be derived from Theorem 1.1 in [13], by applying a gauge transformation to put the solution obtained there into temporal gauge. To be precise the cited result gives a global solution $(a_0, \mathbf{a}, \phi)$ of the system (1.5) satisfying the parabolic gauge condition $a_0 = \operatorname{div}\mathbf{a}$, and the gauge invariant growth estimate

$$\|\phi\|_{H^2_a(\Sigma)}(t) \leq ce^{\alpha e^{\beta t}}. \tag{A.12}$$

The solution satisfies $\phi \in C\left([0,\infty); H^2(\Sigma)\right) \cap C^1\left([0,\infty); L^2(\Sigma)\right)$, $\mathbf{a} \in C\left([0,\infty); H^1(\Sigma)\right)$ and $a_0 \in C\left([0,\infty); L^2(\Sigma)\right)$. Now define $\chi \in C^1\left([0,\infty); L^2(\Sigma)\right)$ by $\partial_t\chi + a_0 = 0$ and $\chi(0) = 0$. Define $(\Phi, A) = (\phi e^{it\chi}, a + d\chi)$: this gives a solution to (1.5) satisfying the properties asserted in Theorem A.3.1. (Most of this can be read off immediately, except perhaps to verify that $\mathbf{A} \in C^1\left([0,\infty); H^1(\Sigma)\right)$, but this follows from the first equation in (1.5), using the fact that $A_0 = 0$ and the right hand side is continuous into L^2.)

An alternative approach to local existence is given in [5], where it is shown that, in Coulomb gauge, systems of the type (1.5) can be put in the form of an abstract evolution equation to which Kato's theory ([26]) applies. This yields the existence of a *local* solution denoted (A', Φ') with Φ' continuous into H^2 on a time interval of length determined by the H^2 norm of the initial data. But the estimate (A.12) above is gauge invariant, and allows continuation of the local solution to provide a global solution in Coulomb gauge with regularity $\Phi' \in C\left([0,\infty); H^2(\Sigma)\right) \cap C^1\left([0,\infty); L^2(\Sigma)\right)$

and $\mathbf{A}' \in C\left([0,\infty); H^3(\Sigma)\right) \cap C^1\left([0,\infty); H^1(\Sigma)\right)$ satisfying the Coulomb gauge condition $\operatorname{div} A' = 0$.

Finally, we explain how to obtain Theorem 1.4.1 from these results. Given a solution $\mathbf{A}'$, Φ' in Coulomb gauge, as just described, define $\chi(t,x)$ to be the solution of

$$(-\Delta + |\Phi'|^2)\dot{\chi} = \operatorname{div}\dot{\mathbf{A}}' - \langle i\Phi', \dot{\Phi}'\rangle = -\langle i\Phi', \dot{\Phi}'\rangle,$$

with $\chi(0,x) = 0$. Then it is easy to verify that $\mathbf{A} = \mathbf{A}' + \mathbf{d}\chi$, $\Phi = \Phi' e^{i\chi}$ satisfies (1.7). Under the condition $\|\Phi(t)\|^2_{L^2(\Sigma)} = L > 0$ the solution exists and is unique at time t; this condition is natural because $\|\Phi(t)\|_{L^2(\Sigma)}$ is independent of time for solutions of (1.5). Now by the above mentioned Coulomb gauge regularity and the basic estimates for the Laplacian we deduce that $\chi \in C^1([0,\infty); H^2)$. This gives the global existence theorem in the gauge stated in Theorem 1.4.1.

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