Commun. Math. Phys. 290, 679–717 (2009)
Digital Object Identifier (DOI) 10.1007/s00220-009-0827-z

Communications in
Mathematical Physics

On the Helicity in 3D-Periodic Navier–Stokes Equations II: The Statistical Case

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Received: 4 August 2008 / Accepted: 25 February 2009
Published online: 7 May 2009 –

Our collaborator and friend Basil Nicolaenko passed away in September of 2007, after this work was completed. Honoring his contribution and friendship, we dedicate this article to him.

Abstract: We study the asymptotic behavior of the statistical solutions to the Navier–Stokes equations using the normalization map [9]. It is then applied to the study of mean energy, mean dissipation rate of energy, and mean helicity of the spatial periodic flows driven by potential body forces. The statistical distribution of the asymptotic Beltrami flows are also investigated. We connect our mathematical analysis with the empirical theory of decaying turbulence. With appropriate mathematically defined ensemble averages, the Kolmogorov universal features are shown to be transient in time. We provide an estimate for the time interval in which those features may still be present.

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1. Introduction

This paper is the continuation of our previous work [5]. In that paper, we study the asymptotic behavior of the helicity associated with the deterministic solution of the

Navier–Stokes equations. The current paper is our study of the asymptotic properties of the statistical distributions of the solutions of the Navier–Stokes equations, including the asymptotic behavior of the statistical dynamics of the helicity.

In this paper, as in [5], we study the incompressible viscous flows which are periodic in the space variables and are driven by potential body forces. Then the velocity field $\mathbf{u}(\mathbf{x}, t)$, $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, satisfies the periodicity condition

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x} + L\mathbf{e}_j, t), \quad \mathbf{x} \in \mathbb{R}^3, \ \ j = 1, 2, 3, \tag{1.1}$$

(where $L > 0$ is the spatial period and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the standard basis of $\mathbb{R}^3$) as well as the Navier–Stokes equations

$$\frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}(\mathbf{x}, t) - \nu \Delta \mathbf{u}(\mathbf{x}, t) = -\nabla p(\mathbf{x}, t) - \nabla \varphi(\mathbf{x}, t), \tag{1.2}$$

$$\nabla \cdot \mathbf{u}(\mathbf{x}, t) = 0, \tag{1.3}$$

where ν is the viscosity of the fluid, p is the pressure and φ is the potential of the body force; here we assume that the mass density is equal to one. Using the well-known remarkable Galilean invariance of the Navier–Stokes equations we also can (by a change of the reference system) consider only the flows satisfying the following zero space average condition:

$$\int_\Omega \mathbf{u}(\mathbf{x}, t) d\mathbf{x} = 0, \quad \Omega = (-L/2, L/2)^3, \tag{1.4}$$

where $d\mathbf{x} = dx_1 dx_2 dx_3$ is the usual volume element in $\mathbb{R}^3$.

Recall that the curl of the velocity, i.e. $\nabla \times \mathbf{u}$, is usually called the vorticity of the flow and is denoted by $\boldsymbol{\omega}$. The kinetic energy/mass, the dissipation rate of energy/mass, and the helicity/mass are defined by

$$\mathcal{E}(t) = \frac{1}{2} \int_\Omega |\mathbf{u}(\mathbf{x}, t)|^2 d\mathbf{x}, \ \mathcal{F}(t) = \int_\Omega |\boldsymbol{\omega}(\mathbf{x}, t)|^2 d\mathbf{x} \tag{1.5}$$

and, respectively,

$$\mathcal{H}(t) = \int_\Omega \mathbf{u}(\mathbf{x}, t) \cdot \boldsymbol{\omega}(\mathbf{x}, t) d\mathbf{x}. \tag{1.6}$$

Above, $\mathbf{u} \cdot \boldsymbol{\omega}$ is the helicity density of the flow. For the physical importance of the helicity in fluid dynamics, see the pioneering work by Moffatt [18] and also other surveys on this topic (e.g. [19]).

In our previous paper [5], we studied mainly the asymptotic behavior of the helicity and its connections with that of the energy which had been previously determined in [8–10]. Unlike the latter behavior, the former is quite sensitive to the presence of the inertial nonlinear terms in the Navier–Stokes equations. In this paper we will present the asymptotic behavior of the statistical dynamics of all of the above three quantities $\mathcal{E}(t)$, $\mathcal{F}(t)$ and $\mathcal{H}(t)$ for our type of flows. Our main concern is to identify the asymptotic properties of $e^{2\nu(2\pi/L)^2 t}\langle \mathcal{E}(t)\rangle$, $e^{2\nu(2\pi/L)^2 t}\langle \mathcal{F}(t)\rangle$ and $e^{2\nu(2\pi/L)^2 t}\langle \mathcal{H}(t)\rangle$, where $\langle \cdot \rangle$ denotes some appropriate ensemble averages, whose rigorous mathematical definitions will be given in Sect. 8. We prove that they all have limits when $t \to \infty$ and that generically these limits are not zero.

One interesting feature in the numerical simulation of the turbulent flows is their statistical tendency to approximate Beltrami flows (see, e.g. [1,21]), i.e., the flows whose velocity and vorticity are parallel. We show that at least asymptotically this tendency is not generic.

Our rigorous results confirm the well known empirical and computational evidence that the Kolmogorov type estimates for the decaying turbulence of the spatially periodic flows lose validity for large times (even after rescaling of the physical entities in order to obtain a simulacrum of stationarity). However, as presented in Sect. 8 our methods yield some estimates of the length of the time interval in which the universal feature may still present.

In our analysis, we use the mathematically defined statistical solutions of the Navier–Stokes equations both on the phase space and the trajectory space. Since this is the first time we develop the asymptotic theory for those statistical solutions, we extend our studies to both the energy and the energy dissipation rate. Another ingredient of our method is the use of the normalization map constructed for the regular solutions to the Navier–Stokes equations in [9,10]. At this stage, we focus on the first rate of decay of the solutions, hence the first component of that normalization map is used throughout. Because the natural space to study the statistical solutions is the space of weak solutions, we extend the definition of that component of normalization map to those solutions. This newly defined map turns out to be an essential tool in describing the asymptotic behavior of the statistical solutions of the Navier–Stokes equations. Particularly, it determines the limits of the ensemble averages referred to above. We also use this map to study the flows which are asymptotically Beltrami. Moreover, the asymptotic behavior of the mean flows is connected with the nonlinear manifold $\mathcal{M}_1$ ([8–10]) of the initial data u_0 such that the corresponding solution $u(t)$ is regular for all $t \geq 0$ and decays exponentially faster than $e^{-\nu(2\pi/L)^2 t}$.

This paper is organized as follows. In Sect. 2, we present the functional settings of the Navier-Stokes equations, the asymptotic behavior of the deterministic solutions. The definitions of the statistical solutions both in the phase space and the trajectory space are recalled as well as their fundamental existence theorem. In Sect. 3, we extend the definition of the first component of the normalization map to the set of Leray-Hopf weak solutions. We prove some basic properties of that map. In Sect. 4, we study the asymptotic behavior of the mean energy, mean energy dissipation rate and mean helicity using the Vishik-Fursikov statistical solutions. For the latter two mean quantities, the moving averages in time are used to overcome the lack of regularity of the weak solutions of the Navier–Stokes equations. In Sect. 5, we construct some initial Gaussian probability measures to show that the asymptotic behavior of the above three mean quantities are not trivial. In Sect. 6, we focus on the solutions which are asymptotically Beltrami (see Definition 6.3). Different equivalent conditions for those flows are given. We prove the existence of a Vishik-Fursikov measure with Gaussian initial data which is not asymptotically Beltrami (see Definition 6.7). In Sect. 7, we first show the connections between the mean flows which decay faster than $e^{-\nu(2\pi/L)^2 t}$ with the above nonlinear manifold $\mathcal{M}_1$. We then prove the *genericity* of the mean flows with non-trivial energy or dissipation rate of energy or helicity, and of the flows which are not asymptotically Beltrami. In Sect. 8, we apply our study in the previous sections to the conventional theory of decaying turbulence. We show that the Kolmogorov type estimates are transient in time and estimate the time interval for which those estimates may still be valid. We obtain in Proposition 8.3 certain lower and upper bounds for the Kolmogorov quotient (see (8.7)). The Appendix provides various basic facts on the Navier–Stokes equations which are used in this paper.

2. Preliminaries

In this paper, we use the same notation as in [5]. We will briefly recall some of them in Subsect. 2.1. Subsection 2.2 consists of the definitions of statistical solutions — both in the phase space and the trajectory space — and their basic existence theorem.

2.1. Deterministic solutions of the Navier–Stokes equations. The initial value problem for the Navier–Stokes equations in the three-dimensional space $\mathbb{R}^3$ with a potential body force consists of Eqs. (1.2), (1.3) and the initial condition

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \tag{2.1}$$

where $\mathbf{u}_0(\mathbf{x})$ is the known initial velocity field. We consider only solutions $\mathbf{u}(\mathbf{x}, t)$ that satisfy the periodicity condition (1.1) and the zero average condition (1.4).

Let $\mathcal{V}$ be the set of all L-periodic trigonometric polynomials with values in $\mathbb{R}^3$ which are divergence-free and have zero average on Ω (see (1.1), (1.3) and (1.4)). We define $H =$ closure of $\mathcal{V}$ in $L^2(\Omega)^3$ and $V =$ closure of $\mathcal{V}$ in $H^1(\Omega)^3$.

Denote by $\langle\cdot,\cdot\rangle$ and $|\cdot|$ the inner product and norm in $L^2(\Omega)^3$. (Note that we also use $|\cdot|$ for the length of vectors in $\mathbb{R}^3$, but the context will clarify its meaning.)

On V we consider the inner product $\langle\langle\cdot,\cdot\rangle\rangle$ and the norm $\|\cdot\|$ defined by

$$\langle\langle u, v\rangle\rangle = \sum_{j,k=1}^{3} \int_\Omega \frac{\partial u_j(\mathbf{x})}{\partial x_k}\frac{\partial v_j(\mathbf{x})}{\partial x_k} d\mathbf{x} \quad \text{and} \quad \|u\| = \langle\langle u, u\rangle\rangle^{1/2},$$

for $u = \mathbf{u}(\cdot) = (u_1, u_2, u_3)$ and $v = \mathbf{v}(\cdot) = (v_1, v_2, v_3)$ in V.

Let $A = -\Delta$ be the Stokes operator on the domain $\mathcal{D}_A = V \cap H^2(\Omega)^3$.

Let P_L denote the orthogonal projector in $L^2(\Omega)^3$ onto H.

We define $B(u, v) = P_L(u \cdot \nabla v)$ for all $u, v \in \mathcal{D}_A$.

We denote by $\mathcal{R}$ the set of all initial value $u_0 \in V$ such that there is a (unique) solution $u(t), t > 0$, satisfying

$$\begin{cases} \frac{du(t)}{dt} + Au(t) + B(u(t), u(t)) = 0, \quad t > 0, \\ u(0) = u_0 \in V, \end{cases} \tag{2.2}$$

where the equation holds in H, and $u(t)$ is continuous from $[0,\infty)$ into V. Such $u(t)$ is called a regular solution of the Navier–Stokes equations.

A classical result (see, e.g., [13–15]) is that for any initial data $\mathbf{u}_0(\mathbf{x})$ in H there exists a weak solution $\mathbf{u}(\mathbf{x}, t)$ defined for all $\mathbf{x} \in \mathbb{R}^3$ and $t > 0$ which eventually becomes analytic in space and time (see also [4,7,11]), hence regular on $[t_0,\infty)$ for some $t_0 \geq 0$.

We denote by $S(t), t \geq 0$, the semigroup generated by the regular solutions of the Navier–Stokes equations, i.e., $S(t)u_0, u_0 \in \mathcal{R}$, denotes the regular solution of (2.2).

Throughout this paper, except for Sect. 8, we take $L = 2\pi$ and $\nu = 1$. The general case is easily recovered by a change of scales.

Let $\sigma(A)$ be the spectrum of the Stokes operator A. For $n \in \sigma(A)$ we denote by R_n the orthogonal projection of H onto the eigenspace of the Stokes operator A associated to n. Let $R_n = 0$ for $n \notin \sigma(A)$.

Let $\mathfrak{C} = \nabla\times$ be the curl operator mapping V into H. For each $n \in \sigma(A)$ we have

$$R_n = R_n^+ + R_n^- \quad \text{and} \quad R_nH = R_n^+H \oplus R_n^-H, \tag{2.3}$$

where R_n^+, resp. R_n^-, is the orthogonal projection of H onto the eigenspace of the curl operator $\mathfrak{C}$ associated to $\sqrt{n}$, resp. $(-\sqrt{n})$, and

$$R_n^{\pm} H = \{u \in H : \mathfrak{C}u = \pm\sqrt{n}u\}. \tag{2.4}$$

It is easy to see that

$$B(u,u) = 0 \quad \text{if} \quad u \in R_n^+ H \cup R_n^- H, \quad n \in \sigma(A). \tag{2.5}$$

Let us recall some known results on the asymptotic expansion of the regular solutions to the Navier–Stokes equations and its associated normalization map (see [8–10, 12] for more details). For any $u_0 \in \mathcal{R}$ the regular solution $u(t)$ has the asymptotic expansion

$$u(t) \sim q_1(t)e^{-t} + q_2(t)e^{-2t} + q_3(t)e^{-3t} + \cdots, \tag{2.6}$$

where $q_j(t)$ is a $\mathcal{V}$-valued polynomial in t. For any $N \in \mathbb{N}$ and $m \in \mathbb{N}$ one has

$$\|u(t) - \sum_{j=1}^{N} q_j(t)e^{-jt}\|_{H^m(\Omega)} = O\left(e^{-(N+\varepsilon)t}\right) \text{ as } \ t \to \infty \text{ for some } \varepsilon = \varepsilon_{N,m} > 0.$$

The normalization map W is defined by $W(u_0) = W_1(u_0) \oplus W_2(u_0) \oplus \cdots$, where $W_j(u_0) = R_j q_j(0)$ for $j \in \mathbb{N}$. Then W is an one-to-one analytic mapping from $\mathcal{R}$ to the Frechet space $S_A = R_1 H \oplus R_2 H \oplus \cdots$ endowed with the component-wise topology. One has $W'(0) = Id$, that is

$$W'(0)u_0 = R_1 u_0 \oplus R_2 u_0 \oplus R_3 u_0 \oplus \cdots. \tag{2.7}$$

For $u_0 \in \mathcal{R} \setminus \{0\}$, there is an eigenvalue n_0 of A such that

$$\lim_{t\to\infty} \frac{\|u(t)\|^2}{|u(t)|^2} = n_0 \quad \text{and} \quad \lim_{t\to\infty} u(t)e^{n_0 t} = w_{n_0}(u_0) \in R_{n_0} H \setminus \{0\}. \tag{2.8}$$

In this case $W_j(u_0) = 0$, $q_j = 0$ for $j < n_0$, and $q_{n_0} = w_{n_0}(u_0) = W_{n_0}(u_0) \neq 0$. In particular, for $n_0 = 1$, we have the the following limits in V:

$$W_1(u_0) = \lim_{\tau\to\infty} e^{\tau} u(\tau) = \lim_{\tau\to\infty} e^{\tau} R_1 u(\tau). \tag{2.9}$$

2.2. Statistical solutions of the Navier–Stokes equations.

Definition 2.1. *We denote by $\mathcal{T}$ the class of test functionals*

$$\Phi(u) = \phi(\langle u, g_1\rangle, \langle u, g_2\rangle, \ldots, \langle u, g_k\rangle), \quad u \in H,$$

for some $k > 0$, where ϕ is a C^1 function on $\mathbb{R}^k$ with compact support and $g_1, g_2, \ldots, g_k$ are in V.

Definition 2.2. *A family $\{\mu_t\}_{t\geq 0}$ of Borel probability measures on H is called a statistical solution of the Navier–Stokes equations with the initial data μ_0 if*

(i) *the initial kinetic energy $\int_H |u|^2 d\mu_0(u)$ is finite;*
(ii) *the function $t \mapsto \int_H \varphi(u)d\mu_t(u)$ is measurable for every bounded and continuous function φ on H;*

(iii) *the function* $t \mapsto \int_H |u|^2 d\mu_t(u)$ *belongs to* $L^\infty_{loc}([0,\infty))$;
(iv) *the function* $t \mapsto \int_H \|u\|^2 d\mu_t(u)$ *belongs to* $L^1_{loc}([0,\infty))$;
(v) μ_t *satisfies the Liouville equation*

$$\int_H \Phi(u)d\mu_t(u) = \int_H \Phi(u)d\mu_0(u) - \int_0^t \int_H \langle Au + B(u,u), \Phi'(u)\rangle d\mu_s(u)ds, \tag{2.10}$$

for all $t \geq 0$ *and* $\Phi \in \mathcal{T}$;
(vi) *the following energy inequality holds:*

$$\int_H |u|^2 d\mu_t(u) + 2\int_0^t \int_H \|u\|^2 d\mu_s(u)ds \leq \int_H |u|^2 d\mu_0(u). \tag{2.11}$$

Recall that for each $u_0 \in H$, there exists a Leray-Hopf weak solution $u(t)$ of the Navier–Stokes equations with $u(0) = u_0$ (cf. [4,17,22]). This weak solution satisfies $u \in C([0,\infty), H_{weak}) \cap L^\infty((0,\infty), H) \cap L^2((0,\infty), V)$. Additionally, let

$$\mathcal{G} = \mathcal{G}(u(\cdot)) = \{t_0 \geq 0 : \lim_{\tau \searrow 0} |u(t_0+\tau) - u(t_0)| = 0\}, \tag{2.12}$$

then $0 \in \mathcal{G}$, the Lebesgue measure of $[0,\infty)\backslash\mathcal{G}$ is zero and for any $t_0 \in \mathcal{G}$,

$$|u(t)|^2 + 2\int_{t_0}^t \|u(s)\|^2 ds \leq |u(t_0)|^2, \quad t \geq t_0. \tag{2.13}$$

Denote by Σ the set of the Leray-Hopf weak solutions of the Navier–Stokes equations on $[0,\infty)$. Hence $\Sigma \subset C([0,\infty), H_{weak})$.

Definition 2.3. *A statistical solution* $\{\mu_t\}_{t\geq 0}$ *of the Navier–Stokes equations in the sense of Definition 2.2 is called a Vishik-Fursikov (VF) statistical solution if there is a Borel probability measure* $\hat{\mu}$*, called the Vishik-Fursikov (VF) measure, on the space* $C([0,\infty), H_{weak})$*, such that*

(i) $\hat{\mu}(\Sigma) = 1$;
(ii) *for each* $t \geq 0$, μ_t *is the projection measure* $Pr_t\hat{\mu}$ *on* H*, i.e.*

$$\int_H \Phi(u)d\mu_t(u) = \int_\Sigma \Phi(v(t))d\hat{\mu}(v(\cdot)), \text{ for all } \Phi \in C(H_{weak}). \tag{2.14}$$

For convenience, we also call $Pr_0\hat{\mu}$ the initial data of $\hat{\mu}$.

The existence theorems of the statistical solutions are summarized in the following ([2,3,16]).

Theorem 2.4. *Let* m *be a Borel probability measure on* H *such that* $\int_H |u|^2 dm(u)$ *is finite. Then there exists a VF statistical solution* $\{\mu_t\}_{t\geq 0}$ *with* $\mu_0 = m$.

Note that such VF statistical solution and VF measure in Theorem 2.4 are not necessarily unique.

Remark 2.5. If $u(\cdot) \in \Sigma$ then the Dirac measure $\delta_{u(\cdot)}$ is a VF measure. If $\hat{\mu}$ and $\hat{m}$ are two VF measures, so is their convex combination $(1-\theta)\hat{\mu} + \theta\hat{m}$, for any $\theta \in (0,1)$.

3. Supplementary Properties of the Normalization Map

In this section, we first extend the definition of $W_1(u_0)$, for $u_0 \in \mathcal{R}$, to $W_1(u(\cdot))$, for $u(\cdot) \in \Sigma$. This definition is more suitable for our study of the asymptotic behavior of the statistical solutions to the Navier–Stokes equations. Basic properties of $W_1(u(\cdot))$ are derived, in particular its relations with the initial value $u(0)$, hence showing the connections between the asymptotic and the initial values of the Leray-Hopf weak solutions.

First, we prove an invariant property of the first component of the normalization map which leads to the extension of that component map later.

We recall from (2.9) that for $u_0 \in \mathcal{R}$,

$$W_1(u_0) = \lim_{\tau\to\infty} e^{\tau} u(\tau) = \lim_{\tau\to\infty} e^{\tau} R_1 u(\tau),$$

where the limits are in V.

Lemma 3.1. *Let $u(\cdot) \in \Sigma$ and $t_0 \geq 0$ such that $u(t_0) \in \mathcal{R}$. Then*

$$e^t W_1(u(t)) = e^{t_0} W_1(u(t_0)), \quad t \geq t_0. \tag{3.1}$$

Proof. If $u(0) = u_0 \in \mathcal{R}$ then $S(t)u_0 \in \mathcal{R}$ for $t \geq 0$,

$$W_1(u_0) = \lim_{\tau\to\infty} e^{t+\tau} u(t+\tau) = e^t \lim_{\tau\to\infty} e^{\tau} S(\tau)S(t)u_0 = e^t W_1(S(t)u_0). \tag{3.2}$$

In general, when $u_0 \in H$, let $t_0 \geq 0$ such that $u(t_0)$ is small in V and hence belongs to $\mathcal{R}$. By (3.2), for $\tau \geq 0$ and $t = \tau + t_0 \geq t_0$,

$$W_1(u(t_0)) = e^{\tau} W_1(S(\tau)u(t_0)) = e^{-t_0} e^t W_1(u(t)), \tag{3.3}$$

thus proving (3.1). □

Remark 3.2. For the existence and estimate of the above t_0 see, e.g., Lemmas A.1 and A.2 below.

Definition 3.3. *Let $u(\cdot) \in \Sigma$. By virtue of Lemma 3.1, we define*

$$W_1(u(\cdot)) = e^{t_0} W_1(u(t_0)), \tag{3.4}$$

where $t_0 \geq 0$ such that $u(t_0) \in \mathcal{R}$.

We then have the following equivalent definition of $W_1(u(\cdot))$ which does not involve t_0 explicitly:

$$W_1(u(\cdot)) = e^{t_0} W_1(u(t_0)) = e^{t_0} \lim_{\tau\to\infty} e^{\tau} S(\tau)u(t_0) = \lim_{t\to\infty} e^t u(t), \tag{3.5}$$

where the limit is taken in V. Similarly, using the second limit in (2.9) we have

$$W_1(u(\cdot)) = \lim_{t\to\infty} e^t R_1 u(t). \tag{3.6}$$

Note that if $u_0 = u(0) \in \mathcal{R}$, then $t_0 = 0$ and $W_1(u(\cdot)) = W_1(u_0)$. Thus $W_1(u(\cdot))$ is an extension of $W_1(u_0)$, $u_0 \in \mathcal{R}$.

The following is a simple bound of $W_1(u(\cdot))$ in terms of the values $u(t)$, $t \in \mathcal{G}(u(\cdot))$, in particular, the initial value $u(0)$.

Lemma 3.4. *Let $u(\cdot) \in \Sigma$. Then*

$$|W_1(u(\cdot))| \le e^t |u(t)|, \quad t \in \mathcal{G}(u(\cdot)). \tag{3.7}$$

In particular,

$$|W_1(u(\cdot))| \le |u(0)|. \tag{3.8}$$

Proof. Let $t \in \mathcal{G}(u(\cdot))$ and $\tau \ge t$. It follows from (A.5) that

$$e^\tau |u(\tau)| \le e^\tau e^{-(\tau - t)} |u(t)| = e^t |u(t)|.$$

Hence $|W_1(u(\cdot))| = \lim_{\tau \to \infty} e^\tau |u(\tau)| \le e^t |u(t)|$. □

Remark 3.5. The estimate (3.8) gives an upper bound for $|W_1(u(\cdot))|/|u(0)|$. However, there is no positive lower bound for the quotient. Indeed, there is a sequence of solutions $u^n(\cdot)$ with nonzero $W_1(u^n(\cdot))$, such that

$$\lim_{n \to \infty} \frac{|W_1(u^n(\cdot))|}{|u^n(0)|} = 0.$$

Proof. Let $u_0^n = \xi_1 + n\xi_4$, where $\xi_j \in R_j H$, $j = 1, 4$, such that $|\xi_1| = |\xi_4| = 1$ and $B(\xi_1, \xi_1) = B(\xi_4, \xi_4) = B(\xi_1, \xi_4) = B(\xi_4, \xi_1) = 0$. For instance, we can take

$$\xi_1 = \frac{\mathbf{e}_2}{\sqrt{2(2\pi)^3}} (e^{i\mathbf{e}_1 \cdot \mathbf{x}} + e^{-i\mathbf{e}_1 \cdot \mathbf{x}}), \quad \xi_4 = \frac{\mathbf{e}_2}{\sqrt{2(2\pi)^3}} (e^{2i\mathbf{e}_1 \cdot \mathbf{x}} + e^{-2i\mathbf{e}_1 \cdot \mathbf{x}}).$$

Then $u^n(t) = \xi_1 e^{-t} + \xi_4 e^{-4t}$, $n \in \mathbb{N}$, are the corresponding regular solutions with initial data u_0^n. We thus have

$$\frac{|W_1(u^n(\cdot))|}{|u^n(0)|} = \frac{|\xi_1|}{|\xi_1 + n\xi_4|} = \frac{1}{\sqrt{1 + n^2}} \to 0, \quad n \to \infty.$$

□

In the case $|W_1(u(\cdot)|$ attains its maximum value $|u(0)|$, we have the following maximum principle.

Proposition 3.6. *Let $u(\cdot) \in \Sigma$ and $u(0) = u_0$. If $|W_1(u(\cdot))| = |u_0|$, then $u_0 \in \mathcal{R}$, $u_0 = W_1(u_0)$ and $u(t) = u_0 e^{-t}$ for all $t \ge 0$.*

Proof. By Lemma 3.4 and (A.6), we have $|u_0| = |W_1(u(\cdot))| = e^t |u(t)|$, for $t \in \mathcal{G}(u(\cdot))$. Let $I = (t_0, t_0 + s)$, $s \in (0, \infty]$ be an interval of regularity of $u(\cdot)$. Then $|u(t)|^2 = e^{-2t}|u_0|^2$ for $t \in I$, hence

$$\frac{d|u(t)|^2}{dt} + 2|u(t)|^2 = 0, \quad t \in I.$$

Comparing with the energy balance equation

$$\frac{d|u(t)|^2}{dt} + 2\|u(t)\|^2 = 0, \quad t \in I,$$

we infer that $\|u(t)\| = |u(t)|$ for all $t \in I$. Hence $u(t) \in R_1 H$ in any interval of regularity. Due to its weak continuity, $u(t) \in R_1 H$ for all $t \in [0, \infty)$. Consequently, one can check that $B(u(t), u(t)) \in R_2 H$ for all $t \in [0, \infty)$. In any interval of regularity,

$$du(t)/dt + Au(t) = -B(u(t), u(t))$$

which belongs to both $R_1 H$ and $R_2 H$. This is possible only if both sides of the equation are zero. The weak continuity of $u(\cdot)$ now implies that $u(t) = u_0 e^{-t}$ for all $t \geq 0$, hence $W_1(u(\cdot)) = u_0$. $\square$

Another consequence of Lemma 3.4 is the following.

Corollary 3.7. *Let $u(\cdot) \in \Sigma$, then for $t \geq 0$ and $T > 0$ we have*

$$|W_1(u(\cdot))|^2 \leq \frac{1}{T} \int_t^{t+T} e^{2\tau} |u(\tau)|^2 d\tau \leq |u(0)|^2, \tag{3.9}$$

and

$$\int_t^{t+T} \|u(\tau)\|^2 d\tau \leq \frac{e^{-2t}}{2} |u(0)|^2 - \frac{e^{-2(t+T)}}{2} |W_1(u(\cdot))|^2. \tag{3.10}$$

Proof. The first inequality of (3.9) comes from (3.7) and the fact that $\mathcal{G}(u(\cdot))$ is dense in $[0, \infty)$. The second inequality of (3.9) is from (A.6).

For $t_0, t_0' \in [t, t+T] \cap \mathcal{G}(u(\cdot))$ such that $t_0 < t_0'$, we have from the inequalities (2.13), (A.6) and (3.7) that

$$\int_{t_0}^{t_0'} \|u(\tau)\|^2 d\tau \leq \frac{e^{-2t_0}}{2} |u(0)|^2 - \frac{e^{-2t_0'}}{2} |W_1(u(\cdot))|^2.$$

Then (3.10) follows by taking $t_0 \searrow t$ and $t_0' \nearrow t + T$. $\square$

Note that (3.10) is a slightly better estimate than (A.7).

It follows from (2.7) that the map $W_1 : \mathcal{R} \to S_A$ is differentiable at 0 and $W_1'(0)u = R_1 u$ for all $u \in V$. Noting that $W_1(0) = 0$, we then have

$$|W_1(u) - R_1 u| = o(\|u\|), \quad \text{for } \|u\| \to 0.$$

The following lemma provides an explicit estimate for $|W_1(u) - R_1 u|$ in terms of $|u|^2$. This approximation of $W_1(u)$ by $R_1 u$ using the quadratic term $|u|^2$ when $|u| \to 0$ will be exploited in Sects. 4 and 6. But first let us note from (A.3) that

$$\begin{aligned} |\langle R_1 B(u, v), w \rangle| &= |\langle B(u, v), R_1 w \rangle| = |-\langle B(u, R_1 w), v \rangle| \\ &\leq c_3 |u|\, |v|\, |A R_1 w|^{1/2}\, |A^{3/2} R_1 w|^{1/2}. \end{aligned}$$

Since $|R_1 w| = |A R_1 w| = |A^{3/2} R_1 w|$, we obtain

$$|R_1 B(u, v)| \leq c_3 |u|\, |v|, \quad u \in V, v \in \mathcal{D}_A. \tag{3.11}$$

Lemma 3.8. *Let* $u(\cdot) \in \Sigma$, *then*

$$|R_1 u(0) - W_1(u(\cdot))| \le c_3 |u(0)|^2. \tag{3.12}$$

More generally,

$$|e^t R_1(u(t)) - W_1(u(\cdot))| \le c_3 e^t |u(t)|^2, \quad t \in \mathcal{G}(u(\cdot)). \tag{3.13}$$

Consequently, when eventually $u(t_0) \in \mathcal{R}$ *then*

$$|R_1(u(t)) - W_1(u(t))| \le c_3 |u(t)|^2, \quad t \ge t_0. \tag{3.14}$$

Proof. We have

$$\frac{dR_1 u}{dt} + R_1 u = -R_1 B(u, u),$$

whence

$$e^t R_1 u(t) = e^{t'} R_1 u(t') + \int_t^{t'} e^\tau R_1 B(u(\tau), u(\tau)) d\tau, \quad t' > t \ge 0.$$

Let $\xi_1 = W_1(u(\cdot))$ and $t \in \mathcal{G}(u(\cdot))$. Using (3.11) and (A.6), we derive

$$\begin{aligned} |e^t R_1 u(t) - \xi_1| &\le |e^{t'} R_1 u(t') - \xi_1| + \int_t^{t'} e^\tau c_3 e^{-2(\tau - t)} |u(t)|^2 d\tau \\ &\le |e^{t'} R_1 u(t') - \xi_1| + c_3 |u(t)|^2 e^{2t} (e^{-t} - e^{-t'}) \\ &\le |e^{t'} R_1 u(t') - \xi_1| + c_3 e^t |u(t)|^2. \end{aligned}$$

Letting $t' \to \infty$ gives (3.13), by (3.6). Also, when $u(t_0) \in \mathcal{R}$ and $t \ge t_0$, $\xi_1 = e^t W_1(u(t))$, hence (3.14) follows. By setting $t = 0$ in (3.13), we obtain (3.12). □

According to Remark 3.5, the quotient $|W_1(u(\cdot))|/|u(0)|$ is not bounded below by a positive constant in general. However, Lemma 3.8 immediately shows that this can be the case for $u(0)$ belonging to some "cones" in H near the origin.

Corollary 3.9. *Given* $\theta \in (0, 1)$, *there are positive numbers* α_1 *and* α_2 *such that if* $u(\cdot) \in \Sigma$ *satisfies*

$$|u(0)| \le \alpha_2 \quad \text{and} \quad |u(0) - R_1 u(0)| \le \alpha_1 |u(0)|$$

then

$$|W_1(u(\cdot))| \ge \theta |u(0)|. \tag{3.15}$$

Proof. By Lemma 3.8,

$$\begin{aligned} |W_1(u(\cdot))| &\ge |R_1 u(0)| - |R_1 u(0) - W_1(u(\cdot))| \\ &\ge |u(0)| - |u(0) - R_1 u(0)| - c_3 |u(0)|^2 \\ &\ge (1 - \alpha_1 - c_3 |u(0)|) |u(0)|. \end{aligned}$$

Then (3.15) follows with $\alpha_1 < 1 - \theta$ and $\alpha_2 = (1 - \theta - \alpha_1)/c_3$. □

The conditions in Corollary 3.9 require small $|u(0)|$. We show below that the conclusion in Corollary 3.9 still holds for u in a V-neighborhood of a special unbounded set $\mathcal{B}_1$ in H. Let

$$\mathcal{B}_1 = \{u \in R_1 H : u \neq 0,\ B(u,u) = 0\}. \tag{3.16}$$

By (2.5), the set $\mathcal{B}_1$ contains $R_1^+ H \cup R_1^- H$, hence is not empty. Also, if $u \in \mathcal{B}_1$, then $e^{-t}u$ is the regular solution with initial data u, hence $W_1(u) = u$.

Proposition 3.10. *Let $u^* \in \mathcal{B}_1$ and $\theta \in (0,1)$. There exists $\varepsilon_2 = \varepsilon_2(|u^*|, \theta) > 0$ such that if $\|v_0\| \le \varepsilon_2$ then $u_0 = u^* + v_0 \in \mathcal{R}$ and*

$$\theta |u_0| \le |W_1(u_0)| \le |u_0|. \tag{3.17}$$

Proof. Let $\varepsilon(|u^*|)$ be defined as in Lemma A.3,

$$\varepsilon_1 = \min\left\{\frac{|u^*|}{2}, \varepsilon(|u^*|)\right\} \quad \text{and} \quad \varepsilon_2 = \min\left\{\varepsilon_1, \frac{(1-\theta)(|u^*| - \varepsilon_1)}{1 + e^{c_3|u^*|}}\right\}, \tag{3.18}$$

where $c_3 > 0$ is given in the Appendix. Since $\|v_0\| \le \varepsilon_1$, we have $u_0 \in \mathcal{R}$ according to Lemma A.3. Let $u(t) = S(t)u_0$ and $v(t) = u(t) - e^{-t}u^*$. By (A.12), we obtain

$$e^t |u(t)| \ge |u^*| - e^t |v(t)| \ge |u^*| - |v_0| e^{c_3|u^*|}.$$

It follows that

$$|W_1(u_0)| = \lim_{t\to\infty} e^t |u(t)| \ge |u^*| - |v_0| e^{c_3|u^*|},$$

and we obtain

$$|W_1(u_0)| \ge |u_0| - |v_0| - |v_0| e^{c_3|u^*|} = |u_0| - |v_0|(1 + e^{c_3|u^*|}). \tag{3.19}$$

Note that $|u_0| \ge |u^*| - \varepsilon_1 > 0$, then $\|v_0\| \le \varepsilon_2$ implies

$$|v_0| \le \frac{(1-\theta)|u_0|}{1 + e^{c_3|u^*|}},$$

hence (3.19) yields $|W_1(u_0)| \ge |u_0| - (1-\theta)|u_0| = \theta |u_0|$. □

Lemma 3.11. *The function $F : u(\cdot) \in \Sigma \to W_1(u(\cdot))$ is Borel measurable. Consequently, F is $\hat{\mu}$-measurable for any VF measure $\hat{\mu}$.*

Proof. We have for each $t \ge 0$ that the function $F_t : u(\cdot) \in \Sigma \to e^t u(t) \in H$ is weakly continuous, hence H_{weak}-Borel measurable. Since the Borel sets of H_{weak} are the same as those of H, the function F_t is (strongly) Borel measurable. The fact that $W_1(u(\cdot)) = \lim_{t\to\infty} e^t u(t)$ implies that F is Borel measurable. □

4. Asymptotic Behavior of the Mean Flows

In this section, let $\hat{\mu}$ be a VF measure on the trajectory space Σ (see Definition 2.3) and let $\{\mu_t\}_{t\geq 0}$ be the family of its projections which is a statistical solution on the phase space H (see Definition 2.2). Recall that μ_0 satisfies the finite initial energy condition:

$$\int_\Sigma |u(0)|^2 d\hat{\mu}(u(\cdot)) = \int_H |u|^2 d\mu_0(u) < \infty. \tag{4.1}$$

This condition, (3.8) and the fact that $W_1(u(\cdot)) \in R_1 H$ imply

$$\int_\Sigma \|W_1(u(\cdot))\|^2 d\hat{\mu}(u(\cdot)) = \int_\Sigma |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot)) \leq \int_\Sigma |u(0)|^2 d\hat{\mu}(u(\cdot)) < \infty.$$

We first describe the asymptotic behavior of the mean energy.

Proposition 4.1. *We have*

$$\lim_{t\to\infty} e^{2t} \int_H |u|^2 d\mu_t(u) = \int_\Sigma |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot)). \tag{4.2}$$

Proof. First, $e^{2t} \int_H |u|^2 d\mu_t(u) = e^{2t} \int_\Sigma |u(t)|^2 d\hat{\mu}(u(\cdot))$. Since $e^{2t}|u(t)|^2 \leq |u(0)|^2$, by (A.5), and $\int_\Sigma |u(0)|^2 d\hat{\mu}(u(\cdot)) < \infty$, applying Lebesgue's dominated convergence theorem gives

$$\lim_{t\to\infty} e^{2t} \int_H |u|^2 d\mu_t(u) = \int_\Sigma \lim_{t\to\infty} e^{2t} |u(t)|^2 d\hat{\mu}(u(\cdot)) = \int_\Sigma |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot)).$$

□

For the mean energy dissipation rate, $\int_\Sigma \|u(t)\|^2 d\hat{\mu}(u(\cdot))$ is only defined almost everywhere on $(0,\infty)$. However, by virtue of Lemma A.2, we can study the asymptotic behavior of the mean energy dissipation rate on the set of solutions with uniformly bounded initial values in H. More precisely, we obtain:

Lemma 4.2. *For any $r > 0$, we have*

$$\lim_{t\to\infty} \int_{\{u(\cdot)\in\Sigma:|u(0)|<r\}} e^{2t} \|u(t)\|^2 d\hat{\mu}(u(\cdot)) = \int_{\{u(\cdot)\in\Sigma:|u(0)|<r\}} \|W_1(u(\cdot))\|^2 d\hat{\mu}(u(\cdot)) \tag{4.3}$$

and

$$\lim_{t\to\infty} \int_{\{u(\cdot)\in\Sigma:|u(0)|<r\}} e^{2t} \mathcal{H}(u(t)) d\hat{\mu}(u(\cdot)) = \int_{\{u(\cdot)\in\Sigma:|u(0)|<r\}} \mathcal{H}(W_1(u(\cdot))) d\hat{\mu}(u(\cdot)). \tag{4.4}$$

Proof. By virtue of Lemma A.2, there is $t(r) > 0$ such that for any $u(\cdot) \in \Sigma$ with $|u(0)| < r$ we have

$$u(t) \in \mathcal{R} \quad \text{and} \quad e^t \|u(t)\| \leq 2e|u(0)|, \quad t \geq t(r).$$

Note that the integrals on the left-hand sides of (4.4) and (4.4) are well-defined for $t \geq t(r)$. Then noting that

$$\int_{\{u(\cdot)\in\Sigma:|u(0)|<r\}} |u(0)|^2 d\hat{\mu}(u(\cdot)) \leq \int_\Sigma |u(0)|^2 d\hat{\mu}(u(\cdot)) < \infty,$$

we apply Lebesgue's dominated convergence theorem. □

Since $\int_H \|u\|^2 d\mu_t(u)$ is not known to be defined for all $t \in [t_0, \infty)$ for some $t_0 \geq 0$, we can not obtain the same result as Proposition 4.1 for the dissipation rate of energy. However, the energy inequality (2.11) suggests the consideration of the moving average in time $\frac{1}{T}\int_t^{t+T} e^{2s} \int_H \|u\|^2 d\mu_s(u)ds$ and its limit as $t \to \infty$. We also consider similar moving averages of the mean energy and helicity.

Proposition 4.3. *For any $T > 0$, we have*

$$\lim_{t\to\infty} \frac{1}{T}\int_t^{t+T} e^{2s}\int_H |u|^2 d\mu_s(u)ds = \int_\Sigma |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot)), \tag{4.5}$$

$$\lim_{t\to\infty} \frac{1}{T}\int_t^{t+T} e^{2s}\int_H \|u\|^2 d\mu_s(u)ds = \int_\Sigma \|W_1(u(\cdot))\|^2 d\hat{\mu}(u(\cdot)), \tag{4.6}$$

and

$$\lim_{t\to\infty} \frac{1}{T}\int_t^{t+T} e^{2s}\int_H \mathcal{H}(u) d\mu_s(u)ds = \int_\Sigma \mathcal{H}(W_1(u(\cdot))) d\hat{\mu}(u(\cdot)), \tag{4.7}$$

where $\mathcal{H}(u) = \langle \mathfrak{C}u, u\rangle$, for $u \in V$.

Proof. Fix $T > 0$. For the mean energy, (4.5) is a consequence of Proposition 4.1.

We prove (4.6) next. For $t \geq 0$ and $r > 0$, let

$$I(t) = \frac{1}{T}\int_t^{t+T}\int_H e^{2s}\|u\|^2 d\mu_s(u) = I_1(t,r) + I_2(t,r),$$

where

$$I_1(t,r) = \frac{1}{T}\int_t^{t+T}\int_{\{u\in H:|u|<r\}} e^{2s}\|u\|^2 d\mu_s(u),$$

$$I_2(t,r) = \frac{1}{T}\int_t^{t+T}\int_{\{u\in H:|u|\geq r\}} e^{2s}\|u\|^2 d\mu_s(u).$$

Also, let

$$J = \int_\Sigma \|W_1(u(\cdot))\|^2 d\hat{\mu}(u(\cdot)) = J_1(r) + J_2(r),$$

where

$$J_1(r) = \int_{\{u(\cdot)\in\Sigma:|u(0)|<r\}} \|W_1(u(\cdot))\|^2 d\hat{\mu}(u(\cdot)),$$

$$J_2(r) = \int_{\{u(\cdot)\in\Sigma:|u(0)|\geq r\}} \|W_1(u(\cdot))\|^2 d\hat{\mu}(u(\cdot)).$$

Then

$$|I(t) - J| \leq |I_1(t,r) - J_1(r)| + I_2(t,r) + J_2(r). \tag{4.8}$$

First, by Lemma 3.4, we have

$$J_2(r) \leq \int_{\{u(\cdot)\in\Sigma:|u(0)|\geq r\}} |u(0)|^2 d\hat{\mu}(u(\cdot)) = \int_{\{u\in H:|u|\geq r\}} |u|^2 d\mu_0(u). \tag{4.9}$$

Second, by using Fubini's theorem,

$$
\begin{aligned}
I_2(t,r) &= \frac{1}{T}\int_t^{t+T}\int_{\{u(\cdot)\in\Sigma:|u(0)|\geq r\}} e^{2s}\|u(s)\|^2 d\hat{\mu}(u(\cdot))ds \\
&= \frac{1}{T}\int_{\{u(\cdot)\in\Sigma:|u(0)|\geq r\}}\int_t^{t+T} e^{2s}\|u(s)\|^2 ds d\hat{\mu}(u(\cdot)) \\
&\leq \frac{1}{T}\int_{\{u(\cdot)\in\Sigma:|u(0)|\geq r\}} e^{2(t+T)}\int_t^{t+T}\|u(s)\|^2 ds d\hat{\mu}(u(\cdot)).
\end{aligned}
$$

Using (A.7), we continue to estimate

$$
\begin{aligned}
I_2(t,r) &\leq \frac{e^{2T}}{T}\int_{\{u(\cdot)\in\Sigma:|u(0)|\geq r\}}\frac{1}{2}|u(0)|^2 d\hat{\mu}(u(\cdot)) \\
&= \frac{e^{2T}}{2T}\int_{\{u\in H:|u|\geq r\}}|u|^2 d\mu_0(u).
\end{aligned}
$$

Given $\varepsilon > 0$. By (4.1), there is $r = r(\varepsilon) > 0$ such that

$$
\frac{e^{2T}}{2T}\int_{\{u\in H:|u|\geq r\}}|u|^2 d\mu_0(u) < \varepsilon/3,
$$

hence

$$
J_2(r) < \varepsilon/3 \quad \text{and} \quad I_2(t,r) < \varepsilon/3, \quad t \geq 0. \tag{4.10}
$$

By Lemma 4.2, there is $t_0 = t_0(r) \geq 0$ such that for all $s \geq t_0$,

$$
\left|\int_{\{u(\cdot)\in\Sigma:|u(0)|<r\}} e^{2s}\|u(s)\|^2 d\hat{\mu}(u(\cdot)) - J_1(r)\right| < \varepsilon/3.
$$

Hence

$$
\begin{aligned}
|I_1(t,r) - J_1(r)| &\leq \frac{1}{T}\int_t^{t+T}\left|\int_{\{u(\cdot)\in\Sigma:|u(0)|<r\}} e^{2s}\|u(s)\|^2 d\hat{\mu}(u(\cdot)) - J_1(r)\right| ds \\
&< \frac{1}{T}\int_t^{t+T}\varepsilon/3 ds \; = \varepsilon/3.
\end{aligned} \tag{4.11}
$$

Combining (4.8), (4.10) and (4.11), we have that for all $t \geq t_0$,

$$
|I(t) - J| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,
$$

thus proving (4.6).

For the mean helicity, the proof of (4.7) is similar. □

Motivated by the existence of the limits in Proposition 4.3 we now study the following ensemble averages of the energy, energy dissipation rate and helicity:

$$\begin{aligned}&\frac{1}{T}\int_t^{t+T}\int_H |u|^2 d\mu_s(u)ds,\ \frac{1}{T}\int_t^{t+T}\int_H \|u\|^2 d\mu_s(u)ds \text{ and}\\ &\frac{1}{T}\int_t^{t+T}\int_H \mathcal{H}(u)d\mu_s(u)ds.\end{aligned} \tag{4.12}$$

The following is a direct consequence of Proposition 4.3 and the elementary fact that if f is a measurable function on some interval (c,∞) such that

$$\lim_{t\to\infty} e^{2t} f(t) = a \in \mathbb{R},$$

then

$$\lim_{t\to\infty}\frac{e^{2t}}{T}\int_t^{t+T} f(s)ds = \frac{1-e^{-2T}}{2T}a \tag{4.13}$$

for any fixed $T > 0$.

Corollary 4.4. *We have for any $T > 0$ that*

$$\lim_{t\to\infty}\frac{e^{2t}}{T}\int_t^{t+T}\int_H |u|^2 d\mu_s(u)ds = \frac{1-e^{-2T}}{2T}\int_\Sigma |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot)), \tag{4.14}$$

$$\lim_{t\to\infty}\frac{e^{2t}}{T}\int_t^{t+T}\int_H \|u\|^2 d\mu_s(u)ds = \frac{1-e^{-2T}}{2T}\int_\Sigma |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot)), \tag{4.15}$$

and

$$\lim_{t\to\infty}\frac{e^{2t}}{T}\int_t^{t+T}\int_H \mathcal{H}(u)d\mu_s(u)ds = \frac{1-e^{-2T}}{2T}\int_\Sigma \mathcal{H}(W_1(u(\cdot)))d\hat{\mu}(u(\cdot)). \tag{4.16}$$

Remark 4.5. The limits in Corollary 4.4 yield the lower and upper bounds for the ensemble averages in (4.12) when t is large. However, we need later bounds valid for all $t \geq 0$, namely,

$$\begin{aligned}e^{-2(t+T)}\int_\Sigma |W_1 u(\cdot))|^2 d\hat{\mu}(u(\cdot)) &\leq \frac{1}{T}\int_t^{t+T}\int_H |u|^2 d\mu_\tau(u)d\tau\\ &\leq e^{-2t}\int_H |u|^2 d\mu_0(u),\end{aligned} \tag{4.17}$$

$$\begin{aligned}e^{-2(t+T)}\int_\Sigma |W_1 u(\cdot))|^2 d\hat{\mu}(u(\cdot)) &\leq \frac{1}{T}\int_t^{t+T}\int_H \|u\|^2 d\mu_\tau(u)d\tau\\ &\leq \frac{e^{-2t}}{2T}\int_H |u|^2 d\mu_0(u)\\ &\quad -\frac{e^{-2(t+T)}}{2T}\int_\Sigma |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot)),\end{aligned} \tag{4.18}$$

for $T > 0$ and $t \geq 0$. They follow readily from (3.10) and (3.9).

According to Proposition 4.1, one can understand the asymptotic behavior of the mean energy by studying $\int_\Sigma |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot))$. However, there is yet no explicit way to find $W_1(u(\cdot))$ and $\hat{\mu}$. Fortunately, $W_1(u(\cdot))$ is related to $R_1 u(0)$ by (3.12). Therefore, in some cases, we can reduce our study to $\int_H |R_1 u|^2 d\mu_0(u)$ which only involves the initial measure μ_0 and the finite rank projection R_1. Similarly, the study of the asymptotic behavior of the mean helicity can be reduced to $\int_H \mathcal{H}(R_1 u) d\mu_0(u)$.

To start, we derive some bounds for $\int_\Sigma |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot))$ using μ_0.

Proposition 4.6. *We have*

$$\int_{R_1^+ H \cup R_1^- H} |u|^2 d\mu_0(u) \le \int_\Sigma |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot)) \le \int_H |u|^2 d\mu_0(u). \quad (4.19)$$

Proof. The second half of (4.19) follows from (3.8) and (2.14). If u_0 belongs to $R_1^+ H$ or $R_1^- H$, then $B(u_0, u_0) = 0$ and hence the corresponding solution is $u(t) = u_0 e^{-t}$, which implies $W(u(\cdot)) = u_0$. Therefore,

$$\begin{aligned}\int_\Sigma |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot)) &\ge \int_{\{u(\cdot)\in\Sigma : u(0)\in R_1^+ H \cup R_1^- H\}} |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot)) \\ &= \int_{\{u(\cdot)\in\Sigma : u(0)\in R_1^+ H \cup R_1^- H\}} |u(0)|^2 d\hat{\mu}(u(\cdot)) \\ &= \int_{R_1^+ H \cup R_1^- H} |u|^2 d\mu_0(u),\end{aligned}$$

thus yields the first half of (4.19). □

Next, we want to find some sufficient conditions in order that

$$\int_\Sigma |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot)) \ne 0 \quad \text{or} \quad \int_\Sigma \mathcal{H}(W_1(u(\cdot))) d\hat{\mu}(u(\cdot)) \ne 0.$$

Note from Proposition 4.6 that the integral $\int_\Sigma |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot))$ is positive whenever $\int_{R_1^+ H \cup R_1^- H} |u|^2 d\mu_0(u)$ is positive. However, the latter condition does not hold even when μ_0 is a Gaussian measure on $R_1 H$. Therefore we need to study other criteria which cover more classes of measures. We turn to a statistical version of (3.12) and its similar estimate for the helicity.

Lemma 4.7. *We have*

$$\int_\Sigma |R_1 u(0) - W_1(u(\cdot))| d\hat{\mu}(u(\cdot)) \le c_3 \int_H |u|^2 d\mu_0(u), \quad (4.20)$$

and for any $r > 0$,

$$\int_\Sigma |\mathcal{H}(R_1 u(0)) - \mathcal{H}(W_1(u(\cdot)))| d\hat{\mu}(u(\cdot)) \le I_r, \quad (4.21)$$

where

$$I_r = 2c_3 r \int_{\{u\in H : |u| < r\}} |u|^2 d\mu_0(u) + 4 \int_{\{u\in H : |u| \ge r\}} |u|^2 d\mu_0(u). \quad (4.22)$$

Proof. The inequality (4.20) follows directly from (3.12):

$$\int_\Sigma |R_1 u(0) - W_1(u(\cdot))| d\hat{\mu}(u(\cdot)) \le c_3 \int_\Sigma |u(0)|^2 d\hat{\mu}(u(\cdot)) = c_3 \int_H |u|^2 d\mu_0(u).$$

Note that $|\mathfrak{C} R_1 u| = \|R_1 u\| = |R_1 u|$. For the helicity,

$$\begin{aligned}
&\int_\Sigma |\mathcal{H}(R_1 u(0)) - \mathcal{H}(W_1(u(\cdot)))| d\hat{\mu}(u(\cdot)) \\
&\quad \le \int_\Sigma |\mathfrak{C} R_1 u(0) - \mathfrak{C} W_1(u(\cdot))||R_1 u(0)| d\hat{\mu}(u(\cdot)) \\
&\qquad + \int_\Sigma |\mathfrak{C} W_1(u(\cdot))||R_1 u(0) - W_1(u(\cdot))| d\hat{\mu}(u(\cdot)) \\
&\quad \le 2 \int_\Sigma |u(0)||R_1 u(0) - W_1(u(\cdot))| d\hat{\mu}(u(\cdot)) \\
&\quad \le 2 \left\{ \int_{\{u(\cdot)\in\Sigma : |u(0)|<r\}} + \int_{\{u(\cdot)\in\Sigma : |u(0)|\ge r\}} \right\} |u(0)||R_1 u(0) - W_1(u(\cdot))| d\hat{\mu}(u(\cdot)).
\end{aligned}$$

Using (3.12) for the integral on $\{u(\cdot) \in \Sigma : |u(0)| < r\}$, and using (3.8) for the integral on $\{u(\cdot) \in \Sigma : |u(0)| \ge r\}$, we obtain

$$\begin{aligned}
&\int_\Sigma |\mathcal{H}(R_1 u(0)) - \mathcal{H}(W_1(u(\cdot)))| d\hat{\mu}(u(\cdot)) \\
&\quad \le 2c_3 \int_{\{u(\cdot)\in\Sigma : |u(0)|<r\}} |u(0)|^3 d\hat{\mu}(u(\cdot)) + 4 \int_{\{u(\cdot)\in\Sigma : |u(0)|\ge r\}} |u(0)|^2 d\hat{\mu}(u(\cdot)) \\
&\quad \le 2rc_3 \int_{\{u\in H : |u|<r\}} |u|^2 d\mu_0(u) + 4 \int_{\{u\in H : |u|\ge r\}} |u|^2 d\mu_0(u) = I_r.
\end{aligned}$$

□

Using Lemma 4.7, we establish some sufficient conditions under which the integral $\int_\Sigma |W_1(u(\cdot))| d\hat{\mu}(u(\cdot))$ or $\int_\Sigma \mathcal{H}(W_1(u(\cdot))) d\hat{\mu}(u(\cdot))$ does not vanish.

Corollary 4.8. *We have the following:*

(i) *If* $\int_H |R_1 u| d\mu_0(u) > c_3 \int_H |u|^2 d\mu_0(u)$, *then* $\int_\Sigma |W_1(u(\cdot))| d\hat{\mu}(u(\cdot)) > 0$ *and subsequently,* $\int_\Sigma |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot)) > 0$.
(ii) *If* $\int_H \mathcal{H}(R_1 u) d\mu_0(u) > I_r$, *resp.* $\int_H \mathcal{H}(R_1 u) d\mu_0(u) < -I_r$, *for some* $r > 0$, *where* I_r *is defined by* (4.22), *then*

$$\int_\Sigma \mathcal{H}(W_1(u(\cdot))) d\hat{\mu}(u(\cdot)) > 0, \ \textit{resp.} \ \int_\Sigma \mathcal{H}(W_1(u(\cdot))) d\hat{\mu}(u(\cdot)) < 0.$$

Proof. By (4.20), we have

$$\begin{aligned}
\int_\Sigma |W_1(u(\cdot))| d\hat{\mu}(u(\cdot)) &\ge \int_\Sigma |R_1 u(0)| d\hat{\mu}(u(\cdot)) - \int_\Sigma |R_1 u(0) - W_1(u(\cdot))| d\hat{\mu}(u(\cdot)) \\
&\ge \int_H |R_1 u| d\mu_0(u) - c_3 \int_H |u|^2 d\mu_0(u),
\end{aligned}$$

hence obtaining (i). The proof of (ii) follows from (4.21) and the following triangular inequalities:

$$\int_\Sigma \mathcal{H}(W_1(u(\cdot)))d\hat{\mu}(u(\cdot)) \geq \int_\Sigma \mathcal{H}(R_1u(0))d\hat{\mu}(u(\cdot))$$
$$-\int_\Sigma |\mathcal{H}(R_1u(0)) - \mathcal{H}(W_1(u(\cdot)))|d\hat{\mu}(u(\cdot)),$$
$$\int_\Sigma \mathcal{H}(W_1(u(\cdot)))d\hat{\mu}(u(\cdot)) \leq \int_\Sigma \mathcal{H}(R_1u(0))d\hat{\mu}(u(\cdot))$$
$$+\int_\Sigma |\mathcal{H}(W_1(u(\cdot))) - \mathcal{H}(R_1u(0))|d\hat{\mu}(u(\cdot)).$$

□

5. Statistical Solutions with Initial Gaussian Measures

In this section, we focus on VF statistical solutions of the Navier–Stokes equations with initial Gaussian probability measures. In particular, we will construct some Gaussian measures on H to which we can apply Corollary 4.8 to obtain

$$\int_\Sigma |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot)) > 0 \quad \text{or} \quad \int_\Sigma \mathcal{H}(W_1(u(\cdot)))^2 d\hat{\mu}(u(\cdot)) \neq 0$$

for any VF measure $\hat{\mu}$ having one of those Gaussian measures as the initial data.

Example 5.1. Let $N_1(=12)$ be the dimension of R_1H and $\{w_j,\ j = 1, \ldots, N_1\}$ be an orthonormal basis in R_1H and $\{w_n,\ n > N_1\}$ be an orthonormal basis in $(I - R_1)H$. For each u of H, we write

$$u = \sum_{j=1}^{\infty} x_j w_j. \tag{5.1}$$

Let μ be a Gaussian probability measure on H such that the density of the distribution of the random variable x_j is given by $\frac{1}{\sqrt{2\pi}\sigma_j} \exp\left(-\frac{x_j^2}{2\sigma_j^2}\right)$, with $\sigma_j > 0$ for all $j \in \mathbb{N}$ and the random variable x_j, $j \in \mathbb{N}$, are independent, (see e.g. [23]). Of course, the σ_j must satisfy the condition

$$\sum_{j=1}^{\infty} \sigma_j^2 < \infty.$$

The variance of μ is

$$\sigma^2 = \int_H |u|^2 d\mu(u) = \sum_{j=1}^{\infty} \sigma_j^2. \tag{5.2}$$

This measure satisfies the following:

Lemma 5.2. *For every $r > 0$, we have*

$$\int_{\{u\in H:|u|\geq r\}} |u|^2 d\mu(u) \leq \int_{\{u\in H:|R_1 u|\geq r/2\}} |R_1 u|^2 d\mu(u) + \frac{4\underline{\sigma}^2\overline{\sigma}^2}{r^2} + \overline{\sigma}^2, \qquad (5.3)$$

where $\underline{\sigma}^2 = \sum_{j=1}^{N_1} \sigma_j^2$ *and* $\overline{\sigma}^2 = \sigma^2 - \underline{\sigma}^2$.

Proof. Note that

$$\begin{aligned}\int_{\{u\in H:|u|\geq r\}} |u|^2 d\mu(u) &\leq \int_{\{u\in H:|u|\geq r\}} |R_1 u|^2 d\mu(u) + \int_H |(I-R_1)u|^2 d\mu(u) \\ &\leq \int_{\{u\in H:|R_1 u|\geq r/2\}} |R_1 u|^2 d\mu(u) + \int_{\{u\in H:|(I-R_1)u|\geq r/2\}} |R_1 u|^2 d\mu(u) + \overline{\sigma}^2\end{aligned}$$

and

$$\begin{aligned}&\int_{\{u\in H:|(I-R_1)u|\geq r/2\}} |R_1 u|^2 d\mu(u) \\ &\quad\leq \left[\int_H |R_1 u|^2 d\mu(u)\right] \mu(\{u \in H : |(I-R_1)u| \geq r/2\}) \\ &\quad\leq \underline{\sigma}^2 \frac{\int_{\{u\in H:|(I-R_1)u|\geq r/2\}} |(I-R_1)u|^2 d\mu(u)}{(r/2)^2} \leq \frac{4\underline{\sigma}^2\overline{\sigma}^2}{r^2}.\end{aligned}$$

Thus (5.3) follows. □

Proposition 5.3. *Let $0 < \epsilon < 1/(c_3\sqrt{2\pi N_1})$ and μ be the Gaussian probability measure defined in Example 5.1 with $\sigma_j = \epsilon$, $j = 1, \ldots, N_1$, and $\sigma^2 = 2N_1\epsilon^2$. For any VF measure $\hat{\mu}$ with initial data μ, we have $\int_\Sigma |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot)) > 0$.*

Proof. From (5.2), we have

$$\int_H |u|^2 d\mu(u) = 2N_1\epsilon^2. \qquad (5.4)$$

Also,

$$\begin{aligned}\int_H |R_1 u| d\mu(u) &\geq \frac{1}{\sqrt{N_1}} \int_{\mathbb{R}^{N_1}} (|x_1| + |x_2| + \cdots + |x_{N_1}|) \\ &\qquad \times \prod_{j=1}^{N_1} \frac{1}{\sqrt{2\pi}\sigma_j} \exp\left(-\frac{x_j^2}{2\sigma_j^2}\right) dx_1 \ldots dx_{N_1} \\ &= \frac{\sqrt{2}}{\sqrt{\pi N_1}} \sum_{j=1}^{N_1} \sigma_j. \end{aligned} \qquad (5.5)$$

Hence

$$\int_H |R_1 u| d\mu(u) \geq \frac{\sqrt{2}}{\sqrt{\pi N_1}} N_1\epsilon = \frac{\sqrt{2N_1}}{\sqrt{\pi}}\epsilon. \qquad (5.6)$$

Thus, $\int_H |R_1 u| d\mu(u) > c_3 \int_H |u|^2 d\mu(u)$. Then we apply Corollary 4.8. □

If the upper bound of ϵ is slightly smaller, we obtain a lower bound for the integral $\int_\Sigma |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot))$ which is comparable with the initial kinetic energy.

Corollary 5.4. *Let μ and $\hat{\mu}$ be the measures in Proposition 5.3. If $0 < \epsilon < 1/(2c_3\sqrt{2\pi N_1})$, then*

$$\frac{1}{4\pi}\int_H |u|^2 d\mu(u) < \int_\Sigma |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot)) \le \int_H |u|^2 d\mu(u). \tag{5.7}$$

Proof. The second inequality of (5.7) is from Proposition 4.6. For the first inequality, we use (4.20), (5.6) and (5.4) to have

$$\begin{aligned}\int_\Sigma |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot)) &\ge \left[\int_\Sigma |W_1(u(\cdot))| d\hat{\mu}(u(\cdot))\right]^2\\ &\ge \left[\int_H |R_1 u| d\mu(u) - c_3 \int_H |u|^2 d\mu(u)\right]^2\\ &\ge \left[\frac{2\sqrt{N_1}}{\sqrt{2\pi}}\epsilon - 2N_1 c_3 \epsilon^2\right]^2\\ &> \epsilon^2 \frac{N_1}{2\pi} = \frac{1}{4\pi}\int_H |u|^2 d\mu(u).\end{aligned}$$

□

Remark 5.5. It is already known that

$$\int_H |u|^2 d\mu_t(u) \le e^{-2t}\int_H |u|^2 d\mu_0(u), \quad t \ge 0. \tag{5.8}$$

If $\mu_0 = \mu$ is a measure satisfying the conditions in Corollary 5.4, then Proposition 4.1 and Corollary 5.4 now imply that

$$\int_H |u|^2 d\mu_t(u) \ge \frac{1}{4\pi} e^{-2t}\int_H |u|^2 d\mu_0(u), \quad t \ge t_0, \tag{5.9}$$

for some $t_0 \ge 0$.

Example 5.6. We consider the Gaussian measure μ defined in Example 5.1. To find $\mu_0 = \mu$ that satisfies the condition in Corollary 4.8, we will be more specific in choosing the orthonormal system $w_1, w_2, \dots, w_{N_1}$ in $R_1 H$.

First we recall that R_1^+, resp. R_1^-, is the orthogonal projection of H onto the eigenspace of the curl operator $\mathfrak{C}$ corresponding to the eigenvalue 1, resp. (-1). Since $\dim R_1^+ H = \dim R_1^- H = N = N_1/2 = 6$, we choose $\{w_1, \dots, w_N\}$ to be an orthonormal basis in $R_1^+ H$ and $\{w_{N+1}, \dots, w_{2N}\}$ to be one in $R_1^- H$. Then

$$\int_H \mathcal{H}(R_1 u) d\mu(u) = \int_H |R_1^+ u|^2 d\mu(u) - \int_H |R_1^- u|^2 d\mu(u) = \sigma_+^2 - \sigma_-^2, \tag{5.10}$$

where $\sigma_+^2 = \sum_{j=1}^N \sigma_j^2$ and $\sigma_-^2 = \sum_{j=N+1}^{2N} \sigma_j^2$.

We will find a Gaussian measure that satisfies part (ii) of Corollary 4.8. For that purpose, we need to estimate the quantity I_r defined by (4.22).

Lemma 5.7. *Let $r > 0$, $\delta \in (0, 1)$ and μ be the measure constructed in Example 5.1 with $0 < \sigma_j \leq r/(2N_1 M_\delta)$, $j = 1, 2, \ldots, N_1$, where $M_\delta > 0$ satisfies*

$$\frac{1}{\sqrt{2\pi}} \int_{\{t \in \mathbb{R} : |t| \geq M_\delta\}} t^2 e^{-t^2/2} dt = \delta. \tag{5.11}$$

Let I_r be defined by (4.22) with $\mu_0 = \mu$. Then

$$I_r \leq \underline{\sigma}^2 \left(2c_3 r + 4\delta + \frac{16\overline{\sigma}^2}{r^2} \right) + (2c_3 r + 4)\overline{\sigma}^2. \tag{5.12}$$

Proof. Recall that, for $r > 0$,

$$I_r = 2c_3 r \int_{\{u \in H : |u| < r\}} |u|^2 d\mu(u) + 4 \int_{\{u \in H : |u| \geq r\}} |u|^2 d\mu(u).$$

By the change of variable $t = x_j/\sigma_j$, we have

$$\begin{aligned} &\int_{\{x_j \in \mathbb{R} : |x_j| \geq r/(2N_1)\}} |x_j|^2 \frac{1}{\sqrt{2\pi}\sigma_j} \exp\left(-\frac{x_j^2}{2\sigma_j^2} \right) dx_j \\ &= \frac{1}{\sqrt{2\pi}} \int_{\{|t| \geq r/(2N_1 \sigma_j)\}} \sigma_j^2 |t|^2 e^{-|t|^2/2} dt \leq \frac{1}{\sqrt{2\pi}} \int_{\{|t| \geq M_\delta\}} \sigma_j^2 |t|^2 e^{-|t|^2/2} dt \leq \delta \sigma_j^2. \end{aligned}$$

Hence

$$\begin{aligned} \int_{\{u \in H : |R_1 u| \geq r/2\}} |R_1 u|^2 d\mu(u) &= \int_{\{x \in \mathbb{R}^{N_1} : |x| \geq r/2\}} |x|^2 \prod_{j=1}^{N_1} \frac{1}{\sqrt{2\pi}\sigma_j} \exp\left(-\frac{x_j^2}{2\sigma_j^2} \right) dx \\ &\leq \sum_{j=1}^{N_1} \int_{\{x_j \in \mathbb{R} : |x_j| \geq r/(2N_1)\}} |x_j|^2 \frac{1}{\sqrt{2\pi}\sigma_j} \exp\left(-\frac{x_j^2}{2\sigma_j^2} \right) dx_j \\ &\leq \delta \underline{\sigma}^2. \end{aligned}$$

Applying Lemma 5.2, we obtain

$$\int_{\{u \in H : |u| \geq r\}} |u|^2 d\mu(u) \leq \delta \underline{\sigma}^2 + \frac{4\underline{\sigma}^2 \overline{\sigma}^2}{r^2} + \overline{\sigma}^2.$$

Since $2c_3 r \int_{\{u \in H : |u| < r\}} |u|^2 d\mu(u) \leq 2c_3 r \sigma^2$, we have

$$I_r \leq 2c_3 r \left(\underline{\sigma}^2 + \overline{\sigma}^2 \right) + 4 \left(\frac{4\underline{\sigma}^2 \overline{\sigma}^2}{r^2} + \overline{\sigma}^2 + \delta \underline{\sigma}^2 \right),$$

and (5.12) follows. □

Now, the condition in the second statement of Corollary 4.8 can be fulfilled by some explicit Gaussian measures.

Proposition 5.8. *There exists a Gaussian probability measure $\mu_{0,+}$, resp. $\mu_{0,-}$, as defined in Example 5.6 such that any VF measure $\hat{\mu}_+$, resp. $\hat{\mu}_-$, with initial data $\mu_{0,+}$, resp. $\mu_{0,-}$, satisfies*

$$\int_\Sigma \mathcal{H}(W_1(u(\cdot)))d\hat{\mu}_+(u(\cdot)) > 0, \tag{5.13}$$

resp.

$$\int_\Sigma \mathcal{H}(W_1(u(\cdot)))d\hat{\mu}_-(u(\cdot)) < 0. \tag{5.14}$$

Proof. Let $\sigma_j = \epsilon_+$ for $j = 1, \dots, N$, and $\sigma_j = \epsilon_-$ for $j = N+1, \dots, 2N$.

For (5.13), we take $\epsilon_+ = \sqrt{2}\epsilon$ and $\epsilon_- = \epsilon$, for some $\epsilon > 0$. By (5.10), we have

$$\int_H \mathcal{H}(R_1 u)d\mu(u) = N(\epsilon_+^2 - \epsilon_-^2) = N\epsilon^2. \tag{5.15}$$

Note that $\underline{\sigma}^2 = 3N\epsilon^2$. In addition, if

$$2c_3 r < \frac{1}{18},\ 0 < 4\delta < \frac{1}{18},\ 0 < \epsilon < \frac{r}{4\sqrt{2}N_1 M_\delta},\ \frac{16\overline{\sigma}^2}{r^2} < \frac{1}{18} \text{ and } (\frac{1}{18} + 4)\overline{\sigma}^2 < \frac{\epsilon^2 N}{2},$$

where M_δ is defined in (5.11), then it follows from Lemma 5.7 that

$$\begin{aligned} I_r &\le 3N\epsilon^2(2c_3 r + 4\delta + \frac{16\overline{\sigma}^2}{r^2}) + (2c_3 r + 4)\overline{\sigma}^2 \\ &< 3N\epsilon^2(\frac{1}{18} + \frac{1}{18} + \frac{1}{18}) + \frac{\epsilon^2 N}{2} \\ &= N\epsilon^2 = \int_H \mathcal{H}(R_1 u)d\mu(u). \end{aligned}$$

Then applying Corollary 4.8 (ii), we obtain (5.13).

For (5.14), we choose $\epsilon_- = \sqrt{2}\epsilon$ and $\epsilon_+ = \epsilon$, then the sum $\epsilon_+^2 + \epsilon_-^2$ is still $3N\epsilon^2$, hence I_r remains less than $N\epsilon^2$. However, $\int_H \mathcal{H}(R_1 u)d\mu(u) = -N\epsilon^2$ and therefore $\int_H \mathcal{H}(R_1 u)d\mu(u) < -I_r$. Again, (5.14) follows from Corollary 4.8 (ii). □

Next, we construct Gaussian measures μ's such that for the corresponding VF measures $\hat{\mu}$'s the integrals $\int_\Sigma |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot))$ are arbitrarily large.

Proposition 5.9. *For any $M > 0$, there exists a VF measure $\hat{\mu}$ such that its initial data is a Gaussian probability measure and*

$$\int_\Sigma |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot)) \ge M. \tag{5.16}$$

Proof. Let μ be a Gaussian measure as in Example 5.6 and let $\hat{\mu}$ be a VF measure with initial data μ. Fix $M > 0$ and $\theta \in (0, 1)$. Take $\sigma_+ > 0$ such that $\theta^5 \sigma_+^2 \ge M$. Since $\lim_{K\to\infty} \int_{\{u\in H: 1/K \le |R_1^+ u| \le K\}} |R_1^+ u|^2 d\mu(u) = \sigma_+^2$, we can choose K sufficiently large so that

$$\int_{\{u\in H: 1/K \le |R_1^+ u| \le K\}} |R_1^+ u|^2 d\mu(u) \ge \theta\sigma_+^2. \tag{5.17}$$

Let $\mathcal{B}_1(\theta) = \{u + v : u \in \mathcal{B}_1,\ \|v\| < \varepsilon_2(|u|, \theta)\}$, where $\mathcal{B}_1$ is defined by (3.16) and $\varepsilon_2(|u|, \theta)$ is in (3.18). By virtue of Proposition 3.10, (3.18) and (A.14), there is $\varepsilon > 0$ depending on K and θ such that

$$\mathcal{B}_1^{\pm}(K, \theta) \stackrel{\text{def}}{=} \{u + v : u \in R_1^{\pm} H,\ 1/K \le |u| \le K,\ \|v\| \le 2\varepsilon\} \subset \mathcal{B}_1(\theta).$$

According to Proposition 3.10, $|W_1(u)| \ge \theta |u|$ for $u \in \mathcal{B}_1^{\pm}(K, \theta)$. Thus we have

$$\begin{aligned}
\int_\Sigma |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot)) &\ge \int_{\{u(\cdot)\in\Sigma : u(0)\in\mathcal{B}_1(\theta)\}} \theta^2 |u(0)|^2 d\hat{\mu}(u(\cdot)) \\
&= \theta^2 \int_{\{u\in H : u\in\mathcal{B}_1(\theta)\}} |u|^2 d\mu(u) \\
&\ge \theta^2 \int_{\mathcal{B}_1^+(K,\theta)} |u|^2 d\mu(u) \\
&\ge \theta^2 \int_{\left\{\substack{u\in H : 1/K\le|R_1^+u|\le K, |R_1^- u|\le\varepsilon,\\ \|(I-R_1)u\|\le\varepsilon}\right\}} |R_1^+ u|^2 d\mu(u) \\
&= \theta^2 \left\{ \int_{\{u\in H : 1/K\le|R_1^+u|\le K\}} |R_1^+ u|^2 d\mu(u) \right\} \mu(\{u \in H : |R_1^- u| \le \varepsilon\}) \\
&\quad \times \mu(\{u \in H : \|(I - R_1)u\| \le \varepsilon\}).
\end{aligned} \tag{5.18}$$

Assume for the moment that there are σ_j, for $j > N$, such that

$$\mu(\{u \in H : |R_1^- u| \le \varepsilon\}) \ge \theta, \tag{5.19}$$

$$\mu(\{u \in H : \|(I - R_1)u\| \le \varepsilon\}) \ge \theta. \tag{5.20}$$

Then combining (5.17), (5.18), (5.19) and (5.20) we obtain

$$\int_\Sigma |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot)) \ge \theta^2 (\theta\sigma_+^2)\theta^2 = \theta^5 \sigma_+^2 \ge M, \tag{5.21}$$

hence (5.16). It remains to verify (5.19) and (5.20).
Verification of (5.19). We have

$$\begin{aligned}
\mu(\{u \in H : |R_1^- u| \le \varepsilon\}) &= \int_{\{x\in\mathbb{R}^N : |x|\le\varepsilon\}} \prod_{j=1}^N \frac{1}{\sqrt{2\pi}\,\sigma_{N+j}} e^{-\frac{x_j^2}{2\sigma_{N+j}^2}} dx \\
&\ge \prod_{j=1}^N \int_{\{|x_j|\le\varepsilon/N\}} \frac{1}{\sqrt{2\pi}\,\sigma_{N+j}} e^{-\frac{x_j^2}{2\sigma_{N+j}^2}} dx_j \\
&= \prod_{j=1}^N \int_{-\varepsilon/(N\sigma_{N+j})}^{\varepsilon/(N\sigma_{N+j})} \frac{1}{\sqrt{2\pi}} e^{-\frac{y_j^2}{2}} dy_j.
\end{aligned}$$

For $\delta \in (0, 1)$, let $m(\delta)$ be the positive number such that

$$\int_{-m(\delta)}^{m_(\delta)} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = \delta.$$

Let $\sigma_{N+j} = \epsilon_- \le \varepsilon/(Nm(\theta^{1/N}))$ for $j = 1, \dots, N$. We then obtain

$$\mu(\{u \in H : |R_1^- u| \le \varepsilon\}) \ge \left(\int_{-m(\theta^{1/N})}^{m(\theta^{1/N})} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \right)^N = \theta.$$

Verification of (5.20). We will determine σ_j for $j > N_1$ such that (5.20) holds. Suppose $Aw_j = \lambda_j w_j$, where $(\lambda_j)_{j=1}^\infty$ is the increasing sequence of eigenvalues of the Stokes operator. For $w = \sum_{j=N_1+1}^\infty x_j w_j \in (I - R_1)H$, we have $\|w\|^2 = \sum_{j=N_1+1}^\infty \lambda_j |x_j|^2$. Note that

$$\{ \sum_{j=N_1+1}^\infty x_j w_j : |x_j| \le \frac{2^N \varepsilon}{2^{j/2} \lambda_j^{1/2}} \} \subset \{ w \in (I - R_1)H : \|w\| \le \varepsilon \}.$$

For $j > N_1$, let $\sigma_j > 0$ be sufficiently small such that

$$\frac{2^N \varepsilon}{2^{j/2} \lambda_j^{1/2} \sigma_j} \ge m(\theta^{2^{N_1}/2^j}).$$

We obtain

$$\begin{aligned} \mu(\{u \in H : \|(I - R_1)u\| \le \varepsilon\}) &\ge \prod_{j=N_1+1}^\infty \int_{\{|x_j| \le \frac{\varepsilon}{2^{j/2}\lambda_j^{1/2}}\}} \frac{1}{\sqrt{2\pi}\sigma_j} e^{-\frac{x_j^2}{2\sigma_j^2}} dx_j \\ &\ge \prod_{j=N_1+1}^\infty \int_{-m(\theta^{2^{N_1}/2^j})}^{m(\theta^{2^{N_1}/2^j})} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \\ &= \prod_{j=N_1+1}^\infty \theta^{2^{N_1}/2^j} = \theta, \end{aligned}$$

hence (5.20) is satisfied. The proof is complete. □

Remark 5.10. In the above proof, if $\sigma_-^2, \overline{\sigma}^2 \le 1$ and $\sigma_+^2 \ge 2\alpha$, where $\alpha = \theta/(1-\theta)$, then $\sigma_+^2 \ge \alpha(\sigma_-^2 + \overline{\sigma}^2)$ and

$$\sigma_+^2 \ge \alpha\sigma^2/(1+\alpha) = \theta\sigma^2. \tag{5.22}$$

From (5.22) and (5.21), we have

$$\int_\Sigma |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot)) \ge \theta^6 \int_H |u|^2 d\mu(u).$$

Hence we have proved that for any given $M > 0$ and $\theta \in (0, 1)$, there exists a VF measure $\hat{\mu}$ with Gaussian initial data μ such that

$$\int_H |u|^2 d\mu(u) \ge M \tag{5.23}$$

and

$$\theta \int_H |u|^2 d\mu(u) \le \int_\Sigma |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot)) \le \int_H |u|^2 d\mu(u). \tag{5.24}$$

6. Asymptotic Beltrami Flows

A C^1 vector field $\mathbf{u}(\mathbf{x})$ in $\mathbb{R}^3$ is a said to be Beltrami if

$$\nabla \times \mathbf{u}(\mathbf{x}) = \alpha(\mathbf{x})\mathbf{u}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3, \text{ some } \alpha(\mathbf{x}) \in \mathbb{R}. \tag{6.1}$$

Note that if $u = \mathbf{u}(\cdot)$ is an eigenfunction of the curl operator $\mathfrak{C}$, then (6.1) holds with $\alpha \equiv \pm\sqrt{n}$, for some $n \in \sigma(A)$. The converse is considered in the following:

Lemma 6.1. ([1]) *Let $u = \mathbf{u}(\cdot) \in R_n H \backslash \{0\}$, where $n \in \sigma(A)$. If $\nabla \times \mathbf{u}(\mathbf{x}) = \alpha(\mathbf{x})\mathbf{u}(\mathbf{x})$ a.e., for some $\alpha(\cdot) \in \mathbb{R}$, then u is an eigenfunction of the curl operator, i.e., $\mathfrak{C}u = \sqrt{n}u$ or $\mathfrak{C}u = -\sqrt{n}u$.*

Corollary 6.2. *Let $u \in R_n H \backslash \{0\}$, where $n \in \sigma(A)$. Then u is Beltrami if and only if u is an eigenfunction of the curl operator, i.e., $\mathfrak{C}u = \sqrt{n}u$ or $\mathfrak{C}u = -\sqrt{n}u$.*

Let $u(\cdot) \in \Sigma$ be such that there is $t_0 \geq 0$, $u(t_0) \in \mathcal{R} \setminus \{0\}$. Then

$$n = \lim_{\tau\to\infty} \frac{\|u(t_0+\tau)\|^2}{|u(t_0+\tau)|^2} = \lim_{t\to\infty} \frac{\|u(t)\|^2}{|u(t)|^2}$$

is an eigenvalue of the Stokes operator A.

Note that the eigenvalue n above depends on the asymptotic behavior of the solution but not on the value of t_0. Therefore we define

$$n_*(u(\cdot)) = \lim_{t\to\infty} \frac{\|u(t)\|^2}{|u(t)|^2}. \tag{6.2}$$

Denote $n_* = n_*(u(\cdot))$. Define

$$W_*(u(\cdot)) = \lim_{t\to\infty} e^{n_* t} u(t), \tag{6.3}$$

and

$$\overline{W}_*(u(\cdot)) = \frac{W_*(u(\cdot))}{|W_*(u(\cdot))|} = \lim_{t\to\infty} \frac{u(t)}{|u(t)|}, \tag{6.4}$$

where the limits in both (6.3) and (6.4) are taken in either H or V. Recall that both $W_*(u(\cdot))$ and $\overline{W}_*(u(\cdot))$ belong to $R_{n_*} H \setminus \{0\}$.

Since $u(t_0) \in \mathcal{R}$, we have

$$\lim_{t\to\infty} e^{n_* t} u(t) = e^{n_* t_0} \lim_{\tau\to\infty} e^{n_* \tau} u(t_0+\tau) = e^{n_* t_0} W_{n_*}(u(t_0)),$$

hence

$$W_*(u(\cdot)) = e^{n_* t_0} W_{n_*}(u(t_0)). \tag{6.5}$$

In particular, if $t_0 = 0$, i.e., $u_0 = u(0) \in \mathcal{R}$, then $W_*(u(\cdot)) = W_{n_*}(u_0)$.

In the case $u(t_0) = 0$ for some $t_0 \geq 0$, we let

$$n_*(u(\cdot)) = \infty \quad \text{and} \quad W_*(u(\cdot)) = 0. \tag{6.6}$$

Denote $n_* = n_*(u(\cdot))$ and $\xi_{n_*} = W_*(u(\cdot))$, $\overline{\xi}_{n_*} = \overline{W}_*(u(\cdot))$. Recall that

$$\frac{u(t)}{|u(t)|} \to \overline{\xi}_{n_*} \text{ in } H \quad \text{and} \quad V, \quad t \to \infty, \tag{6.7}$$

and hence

$$\frac{\mathfrak{C}u(t)}{|u(t)|} \to \mathfrak{C}\overline{\xi}_{n_*} \text{ in } H, \quad t \to \infty. \tag{6.8}$$

It is known (e.g. [12]) that $e^{n_* t}u(t) \to \xi_{n_*}$, for $t \to \infty$, in any $H^m(\Omega)$ norm, $m \in \mathbb{N}$, consequently, in sup norm. Hence for $\mathbf{x} \in \mathbb{R}^3$,

$$\lim_{t\to\infty} e^{n_* t}\mathbf{u}(t, \mathbf{x}) = \xi_{n_*}(\mathbf{x}) \quad \text{and} \quad \lim_{t\to\infty} e^{n_* t}\nabla \times \mathbf{u}(t, \mathbf{x}) = \nabla \times \xi_{n_*}(\mathbf{x}). \tag{6.9}$$

Since $\xi_{n_*}(\mathbf{x})$ is analytic, we have

$$\lim_{t\to\infty} e^{n_* t}|\mathbf{u}(t, \mathbf{x})| = |\xi_{n_*}(\mathbf{x})| \neq 0, \quad \text{a.e.} \tag{6.10}$$

Definition 6.3. *We say that a time dependent vector field $\mathbf{u}(\mathbf{x}, t)$ is asymptotically Beltrami if there are $\alpha(\mathbf{x}, t) \in \mathbb{R}$ such that*

$$\lim_{t\to\infty} \frac{\nabla \times \mathbf{u}(\mathbf{x}, t) - \alpha(\mathbf{x}, t)\mathbf{u}(\mathbf{x}, t)}{|\mathbf{u}(\mathbf{x}, t)|} = 0, \ \textit{a.e. on } \mathbb{R}^3. \tag{6.11}$$

Remark 6.4. The limit in (6.11) requires that a.e. on $\mathbb{R}^3$, $u(\mathbf{x}, t) \neq 0$, for all $t \geq t_0(\mathbf{x})$.

We obtain the following equivalent conditions for a Leray-Hopf solution to be asymptotically Beltrami.

Theorem 6.5. *Let $u(\cdot) \in \Sigma$ such that $u(t_0) \in \mathcal{R}\setminus\{0\}$, for some $t_0 > 0$. The following are equivalent:*

(i) *$u(\cdot)$ is asymptotically Beltrami.*
(ii) *There is a subsequence $t_k \nearrow \infty$ and $\alpha(\mathbf{x}, t_k) \in \mathbb{R}$ such that*

$$\lim_{k\to\infty} \frac{\nabla \times \mathbf{u}(\mathbf{x}, t_k) - \alpha(\mathbf{x}, t_k)\mathbf{u}(\mathbf{x}, t_k)}{|\mathbf{u}(\mathbf{x}, t_k)|} = 0, \ \textit{a.e. on } \mathbb{R}^3. \tag{6.12}$$

(iii) *$W_*(u(\cdot))$ is a Beltrami vector field.*
(iv) *For $n_* = n_*(u(\cdot))$,*

$$\lim_{t\to\infty} \frac{|\mathfrak{C}u(t) - \varepsilon\sqrt{n_*}u(t)|}{|u(t)|} = 0, \tag{6.13}$$

where $\varepsilon = 1$ or -1.

Proof. Assume (i). Of course, (ii) follows.

Assume (ii). From (6.7) and (6.8) we can assume, without loss of generality, that

$$\lim_{k\to\infty} \mathbf{u}(\mathbf{x}, t_k)/|u(t_k)| = \overline{\xi}_{n_*}(\mathbf{x}) \neq 0, \ \text{a.e.}, \tag{6.14}$$

$$\lim_{k\to\infty} \nabla \times \mathbf{u}(\mathbf{x}, t_k)/|u(t_k)| = \mathfrak{C}\overline{\xi}_{n_*}(\mathbf{x}), \ \text{a.e.} \tag{6.15}$$

Thus from (6.14),

$$\lim_{k\to\infty} |\mathbf{u}(\mathbf{x}, t_k)|/|u(t_k)| = |\overline{\xi}_{n_*}(\mathbf{x})| \neq 0, \ \text{a.e.}, \tag{6.16}$$

and together with (6.12),

$$\lim_{k\to\infty} \frac{\nabla \times \mathbf{u}(\mathbf{x}, t_k) - \alpha(\mathbf{x}, t_k)\mathbf{u}(\mathbf{x}, t_k)}{|u(t_k)|} = 0, \text{ a.e.} \tag{6.17}$$

From (6.15) and (6.17), it follows that

$$\lim_{k\to\infty} \alpha(\mathbf{x}, t_k)\frac{\mathbf{u}(\mathbf{x}, t_k)}{|u(t_k)|} = \mathfrak{C}\overline{\xi}_{n_*}(\mathbf{x}), \text{ a.e.,} \tag{6.18}$$

and hence $\lim_{k\to\infty} \alpha(\mathbf{x}, t_k) = \alpha(\mathbf{x})$ exists a.e. on $\mathbb{R}^3$ and $\mathfrak{C}\overline{\xi}_{n_*}(\mathbf{x}) = \alpha(\mathbf{x})\overline{\xi}_{n_*}(\mathbf{x})$ a.e. on $\mathbb{R}^3$. By Lemma 6.2, $\overline{\xi}_{n_*}$ is an eigenfunction of the curl operator $\mathfrak{C}$. Hence $\overline{\xi}_{n_*}$ is Beltrami, so is $W_*(u(\cdot))$, and we have (iii).

Assume (iii). Then $\mathfrak{C}\overline{\xi}_{n_*} = \varepsilon\sqrt{n_*}\overline{\xi}_{n_*}$, where $\varepsilon = 1$ or -1. By (6.7) and (6.8),

$$\lim_{t\to\infty} \left[\frac{\mathfrak{C}u(t)}{|u(t)|} - \varepsilon\sqrt{n_*}\frac{u(t)}{|u(t)|}\right] = \mathfrak{C}\overline{\xi}_{n_*} - \varepsilon\sqrt{n_*}\overline{\xi}_{n_*} = 0,$$

where the limit is taken in H, thus proving (iv).

Assume (iv). The limit in (iv) is $|\mathfrak{C}\overline{\xi}_{n_*} - \varepsilon\sqrt{n_*}\overline{\xi}_{n_*}|$, hence $\mathfrak{C}\overline{\xi}_{n_*} = \varepsilon\sqrt{n_*}\overline{\xi}_{n_*}$ and

$$\nabla \times \xi_{n_*}(\mathbf{x}) = \varepsilon\sqrt{n_*}\xi_{n_*}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3. \tag{6.19}$$

By (6.9) and (6.10),

$$\lim_{t\to\infty} \frac{\nabla \times \mathbf{u}(\mathbf{x}, t) - \varepsilon\sqrt{n_*}\mathbf{u}(\mathbf{x}, t)}{|\mathbf{u}(\mathbf{x}, t)|} = \frac{\nabla \times \xi(\mathbf{x}) - \varepsilon\sqrt{n_*}\xi_{n_*}(\mathbf{x})}{|\xi_{n_*}(\mathbf{x})|}, \text{ a.e..}$$

This limit is zero by (6.19), hence (6.11) holds with $\alpha(\mathbf{x}, t) \equiv \varepsilon\sqrt{n_*}$. □

Corollary 6.6. *Let $u(\cdot) \in \Sigma$ be not identically zero in (t_0, ∞), for some $t_0 > 0$. If $u(t)$ is asymptotically Beltrami then $\mathfrak{C}W_*(u(\cdot)) = \varepsilon\sqrt{n_*(u(\cdot))}W_*(u(\cdot))$, with $\varepsilon = 1$ or $\varepsilon = -1$, and (6.11) holds with $\alpha = \varepsilon\sqrt{n_*(u(\cdot))}$.*

We now turn to the statistical study of the asymptotically Beltrami flows using the statistical solutions of the Navier–Stokes equations.

Definition 6.7. *Let $\hat{\mu}$ be a VF measure on Σ as in Definition 2.3. We say that the $\hat{\mu}$ is asymptotically Beltrami if almost surely every solution $u(\cdot)$ in Σ is asymptotic Beltrami; more precisely,*

$$\hat{\mu}(\{u(\cdot) \in \Sigma : u(\cdot) \textit{ is asymptotically Beltrami}\}) = 1. \tag{6.20}$$

We infer from Theorem 6.5 and Corollary 6.2 that if a Leray-Hopf solution $u(\cdot)$ is asymptotically Beltrami then

$$\mathfrak{C}W_1(u(\cdot)) = W_1(u(\cdot)) \text{ or } \mathfrak{C}W_1(u(\cdot)) = -W_1(u(\cdot))$$

(this trivially holds if $W_1(u(\cdot)) = 0$), or equivalently,

$$R_1^- W_1(u(\cdot)) = 0 \text{ or } R_1^+ W_1(u(\cdot)) = 0.$$

Therefore, the necessary condition for $\hat{\mu}$ to be asymptotically Beltrami is that

$$\hat{\mu}(\{u(\cdot) \in \Sigma : |R_1^+ W_1(u(\cdot))|\,|R_1^- W_1(u(\cdot))| = 0\}) = 1, \tag{6.21}$$

or equivalently,

$$\int_{\Sigma} |R_1^+ W_1(u(\cdot))|\, |R_1^- W_1(u(\cdot))| d\hat{\mu}(u(\cdot)) = 0. \tag{6.22}$$

Another alternative interpretation of (6.22) is the following. Since $R_1^+ u$ and $R_1^- u$ are orthogonal, we have that

$$|R_1^+ u||R_1^- u| = 0 \text{ if and only if } |R_1^+ u| + |R_1^- u| - |R_1 u| = 0.$$

Therefore, the necessary condition (6.22) for $\hat{\mu}$ to be asymptotically Beltrami is equivalent to

$$\int_{\Sigma} \left[|R_1^+ W_1(u(\cdot))| + |R_1^- W_1(u(\cdot))| - |W_1(u(\cdot))| \right] d\hat{\mu}(u(\cdot)) = 0. \tag{6.23}$$

Proposition 6.8. *If $\hat{\mu}$ is a VF measure with initial data μ satisfying*

$$\int_{H} \left[|R_1^+ u| + |R_1^- u| - |R_1 u| \right] d\mu(u) > 3c_3 \int_{H} |u|^2 d\mu(u), \tag{6.24}$$

then $\hat{\mu}$ is not asymptotically Beltrami.

Proof. Suppose (6.24) holds. Using (4.20) with $\mu_0 = \mu$, one can show that (6.23) does not hold. □

Theorem 6.9. *There exists a VF measure $\hat{\mu}$ with initial Gaussian probability measure such that $\hat{\mu}$ is not asymptotically Beltrami.*

Proof. Let μ be a Gaussian measure as in Example 5.6 and $\hat{\mu}$ be a VF measure with initial data μ. Let $\sigma_j = \epsilon > 0$ for $j = 1, \ldots, 2N$. Let ω_n be the area of the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$, $n \geq 2$. We have

$$\begin{aligned} \int_{\mathbb{R}^n} |z| \frac{e^{-|z|^2/(2\epsilon^2)}}{(2\pi)^{n/2}\epsilon^n} dz &= \frac{1}{(2\pi)^{n/2}\epsilon^n} \int_0^\infty r e^{-r^2/(2\epsilon^2)} \omega_n r^{n-1} dr, \quad y = \frac{r}{\sqrt{2}\epsilon} \\ &= \frac{\omega_n \sqrt{2}\epsilon}{\pi^{n/2}} \int_0^\infty y^n e^{-y^2} dy \\ &= \alpha_n \epsilon. \end{aligned}$$

Condition (6.24) is now equivalent to

$$(2\alpha_N - \alpha_{2N})\epsilon > 3c_3(N\epsilon^2 + N\epsilon^2 + \overline{\sigma}^2). \tag{6.25}$$

Since $2\alpha_N - \alpha_{2N} > 0$, condition (6.25) is satisfied with

$$\overline{\sigma}^2 = N\epsilon^2 \text{ and } \epsilon < (2\alpha_N - \alpha_{2N})/(9Nc_3).$$

□

Remark 6.10. In Proposition 6.8, we can use (6.22) instead of (6.23) to replace (6.24) with the following condition:

$$\int_H |R_1^+ u|\,|R_1^- u| d\mu(u) > I_r, \tag{6.26}$$

for some $r > 0$, where I_r is defined by (4.22) with $\mu_0 = \mu$. Also one can adjust the construction of μ in the proof of Theorem 6.9 such that μ should satisfy (6.26), hence

$$\int_\Sigma |R_1^+ W_1(u(\cdot))|\,|R_1^- W_1(u(\cdot))| d\hat{\mu}(u(\cdot)) > 0$$

and $\hat{\mu}$ is not asymptotically Beltrami.

7. Some Generic Properties of VF Measures

First we will show that

$$\int_\Sigma |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot)) > 0 \tag{7.1}$$

is a generic property for a VF measure $\hat{\mu}$. For this we will give a useful characterization of a VF measure with that property.

Let

$$\Sigma_1 = \{u(\cdot) \in \Sigma : W_1(u(\cdot)) = 0\}. \tag{7.2}$$

It is easy to see that

$$\Sigma_1 = \{u(\cdot) \in \Sigma : u(t) \in \mathcal{M}_1, \text{ for all } t \geq t_0 = t_0(u(\cdot))\}, \tag{7.3}$$

where $\mathcal{M}_1 = \{u \in \mathcal{R} : W_1(u) = 0\}$ (see [8]). It is worth mentioning that $\mathcal{M}_1$ is a manifold in V. For our convenience, we will also define

$$\Sigma_{1,t} = \{u(\cdot) \in \Sigma : u(t) \in \mathcal{M}_1\}. \tag{7.4}$$

Then $\Sigma_{1,t} \subset \Sigma_{1,t'}$ for $t \leq t'$ and $\Sigma_1 = \cup_{t \geq 0} \Sigma_{1,t}$.

Proposition 7.1. *Relation (7.1) holds if and only if $\hat{\mu}(\Sigma_1) < 1$.*

Proof. Suppose (7.1) does not hold. Let $r > 0$. According to Lemma A.2, there is $t_1 = t_1(r) > 0$ such that $u(t_1) \in \mathcal{R}$ whenever $|u(0)| < r$. Then

$$\begin{aligned} 0 &= \int_{\{u(\cdot)\in\Sigma:|u(0)|<r\}} |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot)) \\ &= \int_{\{u(\cdot)\in\Sigma:|u(0)|<r\}} e^{t_1} |W_1(u(t_1))|^2 d\hat{\mu}(u(\cdot)). \end{aligned}$$

Hence $W_1(u(t_1)) = 0$ $\hat{\mu}$-a.e. on $\{u(\cdot) \in \Sigma : |u(0)| < r\}$. Thus

$$\begin{aligned} 1 \geq \hat{\mu}(\Sigma_1) &\geq \hat{\mu}(\{u(\cdot) \in \Sigma : |u(0)| < r,\ u(t_1(r)) \in \mathcal{M}_1\}) \\ &= \hat{\mu}(\{u(\cdot) \in \Sigma : |u(0)| < r,\ W_1(u(t_1(r))) = 0\}) \\ &= \hat{\mu}(\{u(\cdot) \in \Sigma : |u(0)| < r\}). \end{aligned}$$

Letting $r \to \infty$, we obtain $\hat{\mu}(\Sigma_1) = 1$.

We now assume that $\hat{\mu}(\Sigma_1) = 1$. Since $W_1(\Sigma_1) = \{0\}$, we have $W_1(u(\cdot)) = 0$ $\hat{\mu}$-a.e. on Σ, thus (7.1) fails. □

For the initial data μ_0 of $\hat{\mu}$ we obtain the following.

Corollary 7.2. *Let*

$$\mathcal{N}_1 = \{u_0 \in H : \exists u(\cdot) \in \Sigma,\ u(0) = u_0,\ \textit{and}\ u(t) \in \mathcal{M}_1,\ t \geq t_0 = t_0(u(\cdot))\}.$$

If $\mu_0(\mathcal{N}_1) < 1$ *then* (7.1) *holds.*

Proof. Since $\mathcal{N}_1 = Pr_0\Sigma_1$, we have

$$1 > \mu_0(\mathcal{N}_1) = \hat{\mu}(Pr_0^{-1}\mathcal{N}_1) \geq \hat{\mu}(\Sigma_1).$$

Hence (7.1) holds, by virtue of Proposition 7.1. □

Remark 7.3. We do not know if $\mu_0(\mathcal{N}_1) = 1$ implies $\hat{\mu}(\Sigma_1) = 1$.

Definition 7.4. *Let* $\hat{\mu}$ *and* $\tilde{\mu}$ *be two Borel measures on* Σ. *We define* $d_1(\hat{\mu}, \tilde{\mu})$ *by the total variation of the measure* $\hat{\mu} - \tilde{\mu}$, *that is,*

$$d_1(\hat{\mu}, \tilde{\mu}) = \sup\left\{\sum_{j=1}^{N} |\hat{\mu}(E_j) - \tilde{\mu}(E_j)|\right\}, \tag{7.5}$$

where the supremum is taken over all Borel partitions $\{E_1, E_2, \ldots, E_N\}$, $N \in \mathbb{N}$, *of* Σ. *It is known that the space of finite Borel measures on* Σ *with metric* d_1 *is complete.*

For our study, it is more suitable to let $\mathfrak{M}$ be the set of all VF measures and define the following metric for $\hat{\mu}$ and $\tilde{\mu}$ in $\mathfrak{M}$:

$$d(\hat{\mu}, \tilde{\mu}) = d_1(\hat{\mu}, \tilde{\mu}) + \int_{\Sigma} |u(0)|^2 d|\hat{\mu} - \tilde{\mu}|(u(\cdot)), \tag{7.6}$$

where $|\hat{\mu} - \tilde{\mu}|$ is the total variation measure of the signed measure $(\hat{\mu} - \tilde{\mu})$.

We have:

Proposition 7.5. *The metric space* $(\mathfrak{M}, d)$ *is complete.*

Proof. Let $(\hat{\mu}^n)_{n=1}^{\infty}$ be a Cauchy sequence in $(\mathfrak{M}, d)$. Then $(\hat{\mu}^n)_{n=1}^{\infty}$ is a Cauchy sequence with respect to d_1. Therefore there is a Borel measure $\hat{\mu}$ on Σ such that $\lim_{n\to\infty} d_1(\hat{\mu}^n, \hat{\mu}) = 0$. Obviously, $\hat{\mu}$ is a probability measure on Σ. For $r > 0$, let

$$B_\Sigma(r; 0) = \{u(\cdot) \in \Sigma : |u(0)| < r\}.$$

We have the function $u(\cdot) \in B_\Sigma(r; 0) \to P_k u(0)$ is continuous for $r > 0$, $k \in \mathbb{N}$. Given $\varepsilon > 0$, there is $N > 0$ such that for $n' > n > N$, we have

$$\int_{B_\Sigma(r;0)} |P_k u(0)|^2 d|\hat{\mu}^n - \hat{\mu}^{n'}|(u(\cdot)) < \varepsilon,$$

for any $r > 0$ and $k \in \mathbb{N}$. Letting $n' \to \infty$ and then $r \to \infty$, $k \to \infty$, we obtain

$$\int_{\Sigma} |u(0)|^2 d|\hat{\mu}^n - \hat{\mu}|(u(\cdot)) \leq \varepsilon.$$

Thus $\lim_{n\to\infty} d(\hat{\mu}^n, \hat{\mu}) = 0$. Since $\int_\Sigma |u(0)|^2 d\hat{\mu}^n(u(\cdot))$ is finite for each n, it follows that $\int_\Sigma |u(0)|^2 d\hat{\mu}(u(\cdot))$ is finite. Hence $\hat{\mu}$ is a VF measure. Therefore $(\mathfrak{M}, d)$ is complete. □

In what follows, $\mathfrak{M}$ is considered as a metric space with metric d. A property $P(\hat{\mu})$ of a VF measure $\hat{\mu}$ is called *generic* if the set of all VF measures $\hat{\mu}$ enjoying the property $P(\hat{\mu})$ contains an intersection of dense open sets in $\mathfrak{M}$.

Lemma 7.6. *Let $\hat{\mu}, \hat{m} \in \mathfrak{M}$, $\varepsilon \in (0, 1)$ and $\tilde{\mu} = (1-\varepsilon)\hat{\mu} + \varepsilon\hat{m}$. Then $\tilde{\mu} \in \mathfrak{M}$ and*

$$d(\tilde{\mu}, \hat{\mu}) \leq 2\varepsilon + \varepsilon \left\{ \int_\Sigma |u(0)|^2 d\hat{\mu}(u(\cdot)) + \int_\Sigma |u(0)|^2 d\hat{m}(u(\cdot)) \right\}. \tag{7.7}$$

Proof. The fact that $\tilde{\mu} \in \mathfrak{M}$ follows from Remark 2.5. For any Borel partition $\{E_j, j = 1, \ldots, N\}$, some $N \in \mathbb{N}$, of Σ, we have

$$\sum_{j=1}^N |\tilde{\mu}(E_j) - \hat{\mu}(E_j)| = \varepsilon \sum_{j=1}^N \left\{ \hat{\mu}(E_j) + \hat{m}(E_j) \right\} \leq 2\varepsilon,$$

thus yielding $d_1(\hat{\mu}, \tilde{\mu}) \leq 2\varepsilon$. Moreover,

$$\begin{aligned} \int_\Sigma |u(0)|^2 d|\tilde{\mu} - \hat{\mu}|(u(\cdot)) &= \int_\Sigma |u(0)|^2 d|\varepsilon\hat{m} - \varepsilon\hat{\mu}|(u(\cdot)) \\ &\leq \varepsilon \int_\Sigma |u(0)|^2 d\hat{\mu}(u(\cdot)) + \varepsilon \int_\Sigma |u(0)|^2 d\hat{m}(u(\cdot)). \end{aligned}$$

Hence (7.7) follows. □

Theorem 7.7. *The set $\mathfrak{M}_E$ of all $\hat{\mu} \in \mathfrak{M}$ such that (7.1) holds is open and dense in $\mathfrak{M}$. Subsequently, (7.1) is generic.*

Proof. For the density, suppose $\hat{\mu} \in \mathfrak{M} \backslash \mathfrak{M}_E$ and $\varepsilon \in (0, 1)$. Denote $M = \int_\Sigma |u(0)|^2 d \hat{\mu}(u(\cdot))$. Let $u_0 \in R_1 H \backslash \{0\}$ such that $\mathfrak{C} u_0 = u_0$ and $|u_0| = 1$. Then $S(t)u_0 = u_0(t) = e^{-t}u_0$, for all $t \geq 0$. Clearly, $u_0 \in \mathcal{R}$, $W_1(u_0) = u_0$ and $W_1(u_0(\cdot)) = u_0$, by Definition 3.3. Set $\tilde{\mu} = (1-\varepsilon)\hat{\mu} + \varepsilon\delta_{u_0(\cdot)}$. Then $\tilde{\mu} \in \mathfrak{M}$ and

$$\begin{aligned} \int_\Sigma |W_1(u(\cdot))|^2 d\tilde{\mu}(u(\cdot)) &= (1-\varepsilon) \int_\Sigma |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot)) + \varepsilon |W_1(u_0(\cdot))|^2 \\ &= 0 + \varepsilon |u_0|^2 \neq 0, \end{aligned}$$

hence $\tilde{\mu} \in \mathfrak{M}_E$. By Lemma 7.6, we have $d(\hat{\mu}, \tilde{\mu}) \leq \varepsilon(M + 3)$. Therefore $\mathfrak{M}_E$ is dense in $\mathfrak{M}$.

Now suppose $\hat{\mu} \in \mathfrak{M}_E$. By Proposition 7.1, we have $\hat{\mu}(\Sigma_1) < 1$, hence $\delta = \hat{\mu}(\Sigma \setminus \Sigma_1) > 0$. Assume $\tilde{\mu} \in \mathfrak{M}$ satisfies $d(\tilde{\mu}, \hat{\mu}) < \delta$. We have

$$\tilde{\mu}(\Sigma_1) \leq \hat{\mu}(\Sigma_1) + d_1(\tilde{\mu}, \hat{\mu}) < \hat{\mu}(\Sigma_1) + \delta = 1,$$

thus $\tilde{\mu} \in \mathfrak{M}_E$ thanks to Proposition 7.1 again. Thus $\mathfrak{M}_E$ is open. □

We now study the genericity of the following property:

$$\int_\Sigma \mathcal{H}(W_1(u(\cdot))) d\hat{\mu}(u(\cdot)) \neq 0. \tag{7.8}$$

For that purpose, we denote by $\mathfrak{M}_H$ the set of all $\hat{\mu} \in \mathfrak{M}$ such that (7.8), holds. Note that $\mathfrak{M}_H = \mathfrak{M}_H^+ \cup \mathfrak{M}_H^-$, where

$$\mathfrak{M}_H^+ = \Big\{\hat{\mu} \in \mathfrak{M} : \int_\Sigma \mathcal{H}(W_1(u(\cdot)))d\hat{\mu}(u(\cdot)) > 0\Big\}, \tag{7.9}$$

$$\mathfrak{M}_H^- = \Big\{\hat{\mu} \in \mathfrak{M} : \int_\Sigma \mathcal{H}(W_1(u(\cdot)))d\hat{\mu}(u(\cdot)) < 0\Big\}. \tag{7.10}$$

Theorem 7.8. *The set $\mathfrak{M}_H$ is open and dense in $\mathfrak{M}$. Subsequently, (7.8) is generic.*

Proof. First, let $\hat{\mu} \in \mathfrak{M}\backslash\mathfrak{M}_H$ and $\varepsilon \in (0, 1)$. Let $u_0(\cdot)$, $\tilde{\mu}$ and M be as in Theorem 7.7. Above we proved $d(\tilde{\mu}, \mu) \le \varepsilon(3 + M)$. Also

$$\int_\Sigma \mathcal{H}(W_1(u(\cdot)))d\tilde{\mu}(u(\cdot)) = (1-\varepsilon)\int_\Sigma \mathcal{H}(W_1(u(\cdot)))d\hat{\mu}(u(\cdot)) + \varepsilon\mathcal{H}(W_1(u_0(\cdot)))$$
$$= \varepsilon\langle \mathfrak{C}u_0, u_0\rangle = \varepsilon|u_0|^2 > 0,$$

hence $\tilde{\mu} \in \mathfrak{M}_H$. Thus $\mathfrak{M}_H$ is dense in $\mathfrak{M}$.

Second, let $\hat{\mu} \in \mathfrak{M}_H^+$ such that

$$\int_\Sigma \mathcal{H}(W_1(u(\cdot)))d\hat{\mu}(u(\cdot)) = \delta > 0.$$

Suppose $\tilde{\mu} \in \mathfrak{M}$ satisfies $d(\tilde{\mu}, \hat{\mu}) < \delta$. Then we have

$$\left|\int_\Sigma \mathcal{H}(W_1(u(\cdot)))d\tilde{\mu}(u(\cdot)) - \int_\Sigma \mathcal{H}(W_1(u(\cdot)))d\hat{\mu}(u(\cdot))\right|$$
$$\le \int_\Sigma |\mathcal{H}(W_1(u(\cdot)))|d|\tilde{\mu} - \hat{\mu}|(u(\cdot)) \le \int_\Sigma |u(0)|^2 d|\tilde{\mu} - \hat{\mu}|(u(\cdot)) < \delta.$$

Thus $\int_\Sigma \mathcal{H}(W_1(u(\cdot)))d\tilde{\mu}(u(\cdot)) > 0$ or $\tilde{\mu} \in \mathfrak{M}_H^+$. Therefore $\mathfrak{M}_H^+$ is open. Similarly, $\mathfrak{M}_H^-$ is open and hence so is $\mathfrak{M}_H$. The proof is complete. □

We now discuss the genericity of the VF measures which are asymptotically Beltrami (see Definition 6.7). We let

$$\mathfrak{M}_B = \{\hat{\mu} \in \mathfrak{M} : \hat{\mu} \text{ is asymptotically Beltrami}\}. \tag{7.11}$$

Proposition 7.9. *$\mathfrak{M}\backslash\mathfrak{M}_B$ contains an open and dense subset of $\mathfrak{M}$. Consequently, the property "$\hat{\mu}$ is not asymptotically Beltrami" for a VF measure $\hat{\mu}$ is generic.*

Proof. Let

$$\mathfrak{N}_B = \left\{\hat{\mu} \in \mathfrak{M} : \int_\Sigma |R_1^+ W_1(u(\cdot))|\,|R_1^- W_1(u(\cdot))|d\hat{\mu}(u(\cdot)) > 0\right\}.$$

We know from the necessary condition (6.22) that $\mathfrak{N}_B$ is a subset of $\mathfrak{M}\backslash\mathfrak{M}_B$. Similar to Theorem 7.8, one can easily prove that $\mathfrak{N}_B$ is open. It suffices to show that $\mathfrak{N}_B$ is dense.

Suppose $\hat{\mu} \in \mathfrak{M}\backslash\mathfrak{N}_B$. Let $\hat{m}$ be in $\mathfrak{M}$ having initial data μ_0 as in Remark 6.10. We have $\int_\Sigma |R_1^+ W_1(u(\cdot))|\,|R_1^- W_1(u(\cdot))|d\hat{m}(u(\cdot)) > 0$. Given $\varepsilon \in (0, 1)$, let $\tilde{\mu} = (1-\varepsilon)\hat{\mu} + \varepsilon\hat{m}$. Then

$$\int_\Sigma |R_1^+ W_1(u(\cdot))|\,|R_1^- W_1(u(\cdot))|d\tilde{\mu}(u(\cdot)) = \varepsilon\int_\Sigma |R_1^+ W_1(u(\cdot))|\,|R_1^- W_1(u(\cdot))|d\hat{m}(u(\cdot)),$$

which is positive, hence $\tilde{\mu} \in \mathfrak{N}_B$. Also, it follows from Lemma 7.6 that

$$d(\tilde{\mu}, \hat{\mu}) < \varepsilon(M+2) \text{ where } M = \int_\Sigma |u(0)|^2 d\hat{\mu}(u(\cdot)) + \int_\Sigma |u(0)|^2 d\hat{m}(u(\cdot)).$$

Therefore $\mathfrak{N}_B$ is dense. The proof is complete. □

8. A Connection to the Empirical Theory of Turbulence

In this section, we connect our analytic study of the statistical solutions of the Navier–Stokes equations to the empirical theory of decaying turbulence. However, unlike the preceding sections which are based on rigorous mathematical arguments, our following discussion involves also heuristic inferences. Here x, L, t, ν are the *dimensional* spatial variable, period, time and viscosity. To apply the results established in the previous sections, we use the following change of scales:

$$x = \frac{x'}{\sqrt{\lambda_1}}, \quad t = \frac{1}{\lambda_1 \nu} t',$$

where $\lambda_1 = (2\pi/L)^2$ denotes the first eigenvalue of the Stokes operator. Then x' and t' play the roles of corresponding *adimensional* variables of the previous sections.

Let us recall the basic features of Kolmogorov's empirical theory of turbulence. In that theory, the following quantities are essential:

$$U^2 = \frac{1}{L^3}\langle \int_{[0,L]^3} |\mathbf{u}(\mathbf{x},t)|^2 dx\rangle \text{ and } \epsilon = \frac{\nu}{L^3}\langle \int_{[0,L]^3} |\nabla \times \mathbf{u}(\mathbf{x},t)|^2 dx\rangle,$$

where $\langle \cdot \rangle$ denotes an "adequate" ensemble average. Note that U^2 is twice the mean energy/mass and ϵ is the mean energy dissipation rate/mass. These two quantities are connected by

$$U^2 \sim \int_{k_i}^{k_d} S(k)dk, \quad \epsilon \sim \nu \int_{k_i}^{k_d} k^2 S(k)dk,$$

where $S(k)$ is the energy spectrum and $[k_i, k_d]$ is called the "inertial range" of the turbulent flows. Assume $k_i \sim k_0 = \sqrt{\lambda_1} = 2\pi/L$, $k_d \sim (\epsilon/\nu^3)^{1/4}$ and $S(k) \sim \epsilon^{2/3}k^{-5/3}$ (based on the dimensional analysis), we obtain

$$U^2 \sim \epsilon^{2/3} \int_{k_i}^{k_d} k^{-5/3} dk \sim \epsilon^{2/3} k_i^{-2/3} \sim (L\epsilon)^{2/3}. \tag{8.1}$$

In the empirical theory of turbulence, both quantities U^2 and ϵ are often considered time-independent. However, in our study, the body force is potential hence they decay exponentially. We propose the following seemingly suitable candidates for these quantities based on our mathematical studies in the previous sections.

Let $(\mu_t)_{t\geq 0}$ be a VF statistical solution to the Navier–Stokes equations with the VF measure $\hat{\mu}$ and $T > 0$. We define for $t \geq 0$,

$$U_t^2 = \lambda_1^{3/2} \frac{1}{T} \int_t^{t+T} \int_H |u|^2 d\mu_\tau(u) d\tau, \tag{8.2}$$

and

$$\epsilon_t = \nu\lambda_1^{3/2}\frac{1}{T}\int_t^{t+T}\int_H \|u\|^2 d\mu_\tau(u)d\tau, \tag{8.3}$$

where we recall that $|u|$ denotes the L^2-norm on $\Omega = (-L/2, L/2)^3$ and $\|u\| = |\nabla u| = |\nabla \times u|$. For our asymptotic study, the first component of the normalization map is defined now by

$$W_1(u(\cdot)) = \lim_{t\to\infty} e^{\nu\lambda_1 t}u(t), \tag{8.4}$$

where the limit is taken in any Sobolev norms. We also let

$$\alpha_0^2 = \lambda_1^{3/2}\int_H |u|^2 d\mu_0(u) \text{ and } \alpha_1^2 = \lambda_1^{3/2}\int_\Sigma |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot)).$$

For the long time dynamics of U_t^2 and ϵ_t, we have the following dimensional version of the related results in Corollary 4.4.

Proposition 8.1. *We have for each $T > 0$ that*

$$\lim_{t\to\infty} e^{2\nu\lambda_1 t}U_t^2 = \frac{1-e^{-2T}}{2T}\alpha_1^2, \tag{8.5}$$

$$\lim_{t\to\infty} e^{2\nu\lambda_1 t}\epsilon_t = \frac{1-e^{-2T}}{2T}\alpha_1^2. \tag{8.6}$$

If (8.1) applies to U_t^2 and ϵ_t then there are absolute positive constants c_K and C_K such that

$$c_K \le \frac{U_t^2}{(L/2\pi)^{2/3}\epsilon_t^{2/3}} = \frac{\lambda_1^{1/3}U_t^2}{\epsilon_t^{2/3}} \le C_K. \tag{8.7}$$

By virtue of Proposition 8.1, relation (8.7) will not hold when t is sufficiently large and $\alpha_1^2 > 0$. (The case $\alpha_1^2 > 0$ is, in fact, *generic* according to our study in Sect. 7.) We will estimate the time interval when (8.7) may still be valid, hence the universal features of the turbulent flows may only be observed on that interval of time. Furthermore, we find rigorous lower and upper bounds for the quotient $\lambda_1^{1/3}U_t^2/\epsilon_t^{2/3}$.

To start, we restate the inequalities in Remark 4.5 in their dimensional forms.

Lemma 8.2. *We have for $T > 0$ and $t \ge 0$ that*

$$e^{-2\nu\lambda_1(t+T)}\alpha_1^2 \le U_t^2 \le e^{-2\nu\lambda_1 t}\alpha_0^2, \tag{8.8}$$

$$\nu\lambda_1 U_t^2 \le \epsilon_t \le \frac{e^{-2\nu\lambda_1 t}}{2T}(\alpha_0^2 - e^{-2\nu\lambda_1 T}\alpha_1^2). \tag{8.9}$$

Proposition 8.3. *Let $Q = \alpha_1^2/\alpha_0^2$. We have for $t \ge 0$ that*

$$Q\left\{\frac{2\nu\lambda_1 T e^{-2\nu\lambda_1 T}}{1-e^{-2\nu\lambda_1 T}Q}\right\}^{2/3}\left\{\frac{e^{-2\nu\lambda_1 t}\alpha_0^2}{\lambda_1\nu^2}\right\}^{1/3} \le \frac{\lambda_1^{1/3}U_t^2}{\epsilon_t^{2/3}} \le \left\{\frac{e^{-2\nu\lambda_1 t}\alpha_0^2}{\lambda_1\nu^2}\right\}^{1/3}, \tag{8.10}$$

or, equivalently,

$$\left\{\frac{2\nu\lambda_1 T}{Q^{-1}e^{2\nu\lambda_1 T}-1}\right\}^{2/3}\left\{\frac{e^{-2\nu\lambda_1 t}\alpha_1^2}{\lambda_1\nu^2}\right\}^{1/3} \le \frac{\lambda_1^{1/3}U_t^2}{\epsilon_t^{2/3}} \le \left\{\frac{e^{-2\nu\lambda_1 t}\alpha_0^2}{\lambda_1\nu^2}\right\}^{1/3}. \tag{8.11}$$

Proof. For the upper bound of $\lambda_1^{1/3}U_t^2/\epsilon_t^{2/3}$, we have from Lemma 8.2,

$$\frac{\lambda_1^{1/3}U_t^2}{\epsilon_t^{2/3}} \le \frac{\lambda_1^{1/3}U_t^2}{(\lambda_1\nu U_t^2)^{2/3}} = \frac{U_t^{2/3}}{\lambda_1^{1/3}\nu^{2/3}} \le \frac{\left\{e^{-2\nu\lambda_1 t}\alpha_0^2\right\}^{1/3}}{\lambda_1^{1/3}\nu^{2/3}}.$$

For the lower bound:

$$\begin{aligned}\frac{\lambda_1^{1/3}U_t^2}{\epsilon_t^{2/3}} &\ge \frac{\lambda_1^{1/3}e^{-2\nu\lambda_1(t+T)}Q\alpha_0^2}{\left\{\frac{e^{-2\nu\lambda_1 t}}{2T}(\alpha_0^2 - e^{-2\nu\lambda_1 T}\alpha_1^2)\right\}^{2/3}} \\ &= Q\left\{\frac{e^{-2\nu\lambda_1 t}\alpha_0^2}{\lambda_1\nu^2}\right\}^{1/3}\left\{\frac{2\nu\lambda_1 T e^{-2\nu\lambda_1 T}}{1 - e^{-2\nu\lambda_1 T}Q}\right\}^{2/3}.\end{aligned}$$

Hence we obtain (8.10). The estimates in (8.11) follow immediately. □

Corollary 8.4. *The relation* (8.1) *may only be valid on the time interval* $[t_K, T_K]$ *where*

$$t_K = \frac{1}{2\nu\lambda_1}\left(\log\frac{\alpha_1^2}{\lambda_1\nu^2} - 3\log C_K - 2\log\frac{Q^{-1}e^{2\nu\lambda_1 T} - 1}{2\nu\lambda_1 T}\right), \tag{8.12}$$

$$T_K = \frac{1}{2\nu\lambda_1}\left(\log\frac{\alpha_0^2}{\lambda_1\nu^2} - 3\log c_K\right). \tag{8.13}$$

Proof. For $t \ge 0$ such that (8.1) holds, it follows from (8.8) that

$$c_K \le \frac{\lambda_1^{1/3}U_t^2}{\epsilon_t^{2/3}} \le \frac{\left\{e^{-2\nu\lambda_1 t}\alpha_0^2\right\}^{1/3}}{\lambda_1^{1/3}\nu^{2/3}},$$

thus yielding $t \le T_K$. Similarly, using (8.9), we have

$$\left\{\frac{2\nu\lambda_1 T}{Q^{-1}e^{2\nu\lambda_1 T} - 1}\right\}^{2/3}\left\{\frac{e^{-2\nu\lambda_1 t}\alpha_1^2}{\lambda_1\nu^2}\right\}^{1/3} \le C_K,$$

hence we obtain $t \ge t_K$. □

Example 8.5. Let $L = 2\pi$ ($\lambda_1 = 1$), $\nu = 1$ and $\hat{\mu}$ be the VF measure in Corollary 5.4. We have $(4\pi)^{-1} \le Q \le 1$. It follows from Proposition 8.3 that

$$\left(\frac{1}{4\pi}\right)^{1/3}\left(\frac{2T}{4\pi e^{2T} - 1}\right)^{2/3}(e^{-2t}\alpha_0^2)^{1/3} \le \frac{\lambda_1^{1/3}U_t^2}{\epsilon_t^{2/3}} \le (e^{-2t}\alpha_0^2)^{1/3},$$

for all $t \ge 0$. Also, by Corollary 8.4, we derive

$$\begin{aligned}T_K &= \frac{1}{2}\{\log(\alpha_0^2) - 3\log c_K\}, \\ t_K &\ge \frac{1}{2}\left\{\log\frac{\alpha_0^2}{4\pi} - 3\log C_K - 2\log\frac{4\pi e^{2T} - 1}{2T}\right\}.\end{aligned}$$

Now, if we let $M > 0, \theta \in (0, 1)$ and $\hat{\mu}$ be a VF measure satisfying (5.23) and (5.24), then $\alpha_0^2 \ge M$ and $\theta \le Q \le 1$ and t_K in (8.12) can be bounded below by

$$t_K \ge \frac{1}{2}\left(\log M - 3\log C_K - 2\log\frac{\theta^{-1}e^{2T} - 1}{2T}\right).$$

Appendix A

In this paper we need several well-known estimates for the non-linear term $B(u, u)$ in the Navier–Stokes equations (2.2). For the convenience of the reader, we list them below. There are positive constants $c_j,\ j = 1, 2, 3$, such that

$$|\langle B(u, v), w\rangle| \le c_1\|u\|\|v\|^{1/2}|Av|^{1/2}\,|w|, \tag{A.1}$$

$$|\langle B(u, v), w\rangle| \le c_2\|u\|^{1/2}|Au|^{1/2}\|v\|\,|w|, \tag{A.2}$$

$$|\langle B(u, v), w\rangle| \le c_3|u||Av|^{1/2}|A^{3/2}v|^{1/2}\,|w|. \tag{A.3}$$

The numbering of the constants is done in order to indicate the estimate in which the constant c_j appears. Thus

$$|B(u, v)| \le \min\{c_1\|u\|\|v\|^{1/2}|Av|^{1/2},\ c_2\|u\|^{1/2}|Au|^{1/2}\|v\|,\ c_3|u||Av|^{1/2}|A^{3/2}v|^{1/2}\}. \tag{A.4}$$

Let $u(\cdot)$ be a Leray-Hopf solution on $[0, \infty)$ and $\mathcal{G} = \mathcal{G}(u(\cdot))$ be defined by (2.12). It is known that for $t_0 \in \mathcal{G}$, we have

$$|u(t)| \le e^{-(t-t_0)}|u(t_0)|, \quad t \ge t_0. \tag{A.5}$$

In particular, $0 \in \mathcal{G}$ and

$$|u(t)| \le e^{-t}|u(0)|, \quad t \ge 0. \tag{A.6}$$

For $t' > t \ge 0$, let $t_0 \in [t, t') \cap \mathcal{G}$, then by (2.13)

$$2\int_{t_0}^{t'} \|u(s)\|ds \le |u(t_0)|^2 \le e^{-2t_0}|u(0)|^2.$$

Letting $t_0 \to t$, we obtain

$$\int_t^{t'} \|u(s)\|^2 ds \le \frac{e^{-2t}}{2}|u(0)|^2, \quad t' > t \ge 0. \tag{A.7}$$

Lemma A.1. *There is $\varepsilon_0 > 0$ such that if $\|u_0\| \le \varepsilon_0$ then $u_0 \in \mathcal{R}$ and*

$$\|u(t)\| \le 2e^{-t}\|u_0\|, \quad t > 0. \tag{A.8}$$

Proof. Though this is a consequence of the convergence of the asymptotic expansion of the regular solution when the initial data is small (cf. [6]), we present below an elementary proof to make our paper self-contained. The calculations are formal but can be made rigorous using the Galerkin approximations.

Let $C_0 = \min\{c_1, c_2\}$. It follows from (2.2) and (A.4) that

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + |Au|^2 \le |\langle B(u, u), Au\rangle| \le C_0|Au|^{3/2}\|u\|^{3/2} \le \frac{1}{2}|Au|^2 + 2C_0^4\|u\|^6.$$

Let $C_1 = 1/(2C_0\sqrt[4]{2})$ and $\|u_0\| < C_1$. By the standard small initial data argument, we have $u_0 \in \mathcal{R}$ and $\|u(t)\| \le e^{-t/2}\|u_0\|$. Now, using interpolating inequality $\|u\|^2 \le |u||Au|$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + |Au|^2 \le C_0|Au|^{3/2}\|u\|^{3/2} \le C_0|Au|^2|u|^{1/2}\|u\|^{1/2},$$

hence

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + (1 - C_0|u|^{1/2}\|u\|^{1/2})|Au|^2 \leq 0.$$

Using $|u| \leq \|u\| \leq |Au|$ and $\|u(t)\| \leq \|u_0\| \leq C_1$, we derive

$$\begin{aligned}\|u(t)\|^2 &\leq e^{-2\int_0^t (1-C_0|u(\tau)|^{1/2}\|u(\tau)\|^{1/2})d\tau}\|u_0\|^2 \leq e^{2C_0\int_0^\infty \|u(\tau)\|^2 d\tau} e^{-2t}\|u_0\|^2 \\ &\leq e^{2C_0\int_0^\infty e^{-\tau/2}\|u_0\|d\tau} e^{-2t}\|u_0\|^2 \leq e^{4C_0\|u_0\|}e^{-2t}\|u_0\|^2.\end{aligned}$$

Thus (A.8) holds for

$$\varepsilon_0 = \min\left\{C_1, \frac{\log 2}{2C_0}\right\} > 0. \tag{A.9}$$

□

We give an estimate of t_0 for which $u(t_0) \in \mathcal{R}$ in terms of $|u_0|$ and ε_0 defined in (A.9).

Lemma A.2. *Let $u(\cdot) \in \Sigma$, then there is $t_0 \in [0, \log^+(|u(0)|/\varepsilon_0)+1)$ such that $u(t_0) \in \mathcal{R}$ and*

$$\|u(t)\| \leq 2e|u(0)|e^{-t}, \quad t \geq t_0. \tag{A.10}$$

(Above $\log^+ \alpha = \log(\max\{1, \alpha\})$, for $\alpha \in \mathbb{R}$.)

Proof. Let $u_0 = u(0)$. Take $t_* = \log^+(|u_0|/\varepsilon_0)$. By (A.7),

$$2\int_{t_*}^{t_*+1} \|u(s)\|^2 ds \leq e^{-2t_*}|u_0|^2. \tag{A.11}$$

This implies that the Lebesgue measure of $\{s : \|u(s)\|^2 \leq e^{-2t_*}|u_0|^2\}$ is greater or equal to 1/2. Hence there is $t_0 \in (t_*, t_* + 1)$ such that $\|u(t_0)\| \leq e^{-t_*}|u_0| \leq \varepsilon_0$. Applying Lemma A.1 to $u(t_0)$ gives

$$\|u(t)\| \leq 2e^{-(t-t_0)}\|u(t_0)\| \leq 2e^{-t}e^{t_*+1}e^{-t_*}|u_0| = 2e^{-t+1}|u_0|, \quad t \geq t_0,$$

thus proving (A.10). □

Concerning the perturbation problem for the Navier–Stokes equations when the initial data u_0 is in a neighborhood of a fixed $u_0^* \in \mathcal{R}$, we have the following result which is similar to but much simpler than that in [20]. For our purpose, we focus on the case u_0^* belonging to the set $\mathcal{B}_1$ consisting of $u \in R_1 H\setminus\{0\}$ such that $B(u, u) = 0$.

Lemma A.3. *Let $u_0^* \in \mathcal{B}_1$, there is $\varepsilon = \varepsilon(|u_0^*|)$ such that if $\|v_0\| \leq \varepsilon$ then $u_0 = u_0^* + v_0 \in \mathcal{R}$ and*

$$|S(t)u_0 - e^{-t}u_0^*| \leq |v_0|e^{c_3|u_0^*|}e^{-t}, \quad t > 0. \tag{A.12}$$

Proof. Let $u^*(t) = S(t)u_0^* = e^{-t}u_0^*$ and $v(t) = S(t)u_0 - u^*(t)$. The equation for $v(t)$ is

$$\frac{dv}{dt} + Av + B(v, v) + B(u^*, v) + B(v, u^*) = 0. \tag{A.13}$$

Using (A.4) and the fact that $u^* \in R_1 H$, we have

$$\frac{1}{2}\frac{d|v|^2}{dt} + \|v\|^2 \le |\langle B(v, u^*), v\rangle| \le c_3|v|^2|u^*| \le c_3|v|^2|u_0^*|e^{-t}.$$

Hence

$$|v(t)|^2 \le |v_0|^2 e^{-2t} e^{2c_3|u_0^*|\int_0^t e^{-\tau}d\tau} \le |v_0|^2 e^{2c_3|u_0^*|} e^{-2t},$$

thus yielding (A.12). We also have

$$\begin{aligned}\frac{1}{2}\frac{d\|v\|^2}{dt} + |Av|^2 &\le C_0|Av|^{3/2}\|v\|^{3/2} + c_2|Av|\|v\||u^*| + c_3|Av||v||u^*| \\ &\le C_0|Av|^{3/2}\|v\|^{3/2} + c_2|Av|^{3/2}|v|^{1/2}|u^*| + c_3|Av||v||u^*| \\ &\le \frac{1}{2}|Av|^2 + C_2\|v\|^6 + C_3|v|^2|u^*|^2(1 + |u^*|^2) \\ &\le \frac{1}{2}|Av|^2 + C_2\|v\|^6 + C_3|v_0|^2 e^{2c_3|u_0^*|}|u_0^*|^2(1 + |u_0^*|^2)e^{-2t},\end{aligned}$$

where $C_2, C_3 > 0$. Take $\varepsilon > 0$ satisfying

$$C_2\varepsilon^4 + C_3\varepsilon^2 e^{2c_3|u_0^*|}|u_0^*|^2(1 + |u_0^*|^2) < \frac{1}{4}. \tag{A.14}$$

The argument becomes standard now and we omit the details. □

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Communicated by P. Constantin