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Twistor Actions for Self-Dual Supergravities

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Abstract: We give holomorphic Chern-Simons-like action functionals on supertwistor space for self-dual supergravity theories in four dimensions, dealing with $\mathcal{N} = 0, \ldots, 8$ supersymmetries, the cases where different parts of the R-symmetry are gauged, and with or without a cosmological constant. The gauge group is formally the group of holomorphic Poisson transformations of supertwistor space where the form of the Poisson structure determines the amount of R-symmetry gauged and the value of the cosmological constant. We give a formulation in terms of a finite deformation of an integrable $\bar{\partial}$-operator on a supertwistor space, i.e., on regions in $\mathbb{CP}^{3|8}$. For $\mathcal{N} = 0$, we also give a formulation that does not require the choice of a background.

Contents

1. Introduction

Recently it has been discovered that $\mathcal{N} = 8$ supergravity has better ultraviolet behaviour than has hitherto been anticipated, [B-Betal06, BDR07 and GRV07]. This has led some authors to speculate that it is possibly even finite. This improved behaviour relies on exact cancellations that do not follow from standard supersymmetry arguments, [St06]. One possible explanation arises from twistor string theory, [W04 and B04]. The original twistor string theories by Witten and Berkovits correspond to conformal supergravity (together with supersymmetric Yang-Mills theory), [BW04]. By gauging certain symmetries of the Berkovits twistor string, [A-ZHM08] introduced a new family of twistor string theories some of which have the appropriate field content for Einstein supergravity (including $\mathcal{N} = 4$ and $\mathcal{N} = 8$). Such a twistor string formulation of Einstein supergravity could be an explanation for the possible ultraviolet finiteness of $\mathcal{N} = 8$ supergravity if it were fully consistent in its quantum theory. However, it now appears that these twistor string theories are chiral, [N08], unlike the original twistor string theories which were parity invariant. It remains a major open question as to whether a twistor-string theory exists that gives the full content of Einstein (super)-gravity even just at tree level.

An approach to understanding what the appropriate twistor string theory might be is via a twistor action, [M05 and MS06] and [BMS07a, BMS07b]. Such actions have two terms. The first on its own gives a kinetic term for all the fields, but with only the self-dual part of the interactions. The second gives the remaining interactions of the full theory and correspond to the instanton contribution in the twistor-string theory. In the case of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, the self-dual part of the action on twistor space is a holomorphic Chern-Simons theory, [W04], see also [S95] for a closely related harmonic superspace action. [BW04] gave a twistor action for self-dual $\mathcal{N} = 4$ conformal supergravity. The purpose of this paper is to give an analogous action in the case of self-dual $\mathcal{N} = 8$ Einstein supergravity. This action is special to $\mathcal{N} = 8$ supergravity in much the same way as Witten's Chern-Simons action is special to $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. It lends general support to the idea that twistor space has something special to say about full $\mathcal{N} = 8$ supergravity and is suggestive of the existence of an underlying twistor string theory, perhaps even with explicit $\mathcal{N} = 8$ supersymmetry as opposed to those of [A-ZHM08] in which only $\mathcal{N} = 4$ supersymmetry is manifest.

Penrose's non-linear graviton construction [P76] reformulates the local data of a four-metric with self-dual Weyl tensor into the complex structure of a deformed twistor space, a three-dimensional complex manifold obtained by deforming a region in $\mathbb{CP}^3$. The space-time field equation in this case is the vanishing of the anti-self-dual part of the Weyl tensor, and in the [AHS78] approach to twistor theory, this is reformulated as the integrability of the twistor almost complex structure. [BW04] introduce a version of conformal gravity with just self-dual interactions in which the underlying conformal structure is self-dual, but in which there is also a linear anti-self dual conformal gravity field (a linearised anti-self-dual Weyl tensor B) propagating on the self dual background. This has a Lagrange multiplier action (analogous to a 'BF' action)

$$\int (B, C^-)\, \mathrm{d}\,\mathrm{vol}\,,$$

where C^- is the anti-self-dual part of the Weyl tensor, and (B, C^-) is the natural pairing. This can be extended to $\mathcal{N} = 4$ supersymmetry. [BW04] gave a corresponding (supersymmetric) twistor action of the form $\int bN$, where N is the Nijenhuis tensor of the almost complex structure and b is a Lagrange multiplier that doubles up as the Penrose transform of the field B when the field equations are satisfied. In the non-supersymmetric case, this was extended to a twistor action for full (non-self-dual) conformal gravity in [M05] with further supersymmetric extension and connections with twistor-string theory in [MS08].

For Einstein gravity we wish to encode the vanishing of the Ricci tensor. In the non-linear graviton this can be characterised by requiring that the twistor space admits a fibration over a $\mathbb{CP}^1$ together with a certain Poisson structure up the fibre. [W80] extended this to the Einstein case, with a cosmological constant; in this case, the twistor space is required to admit a holomorphic contact structure that is non-degenerate when the cosmological constant is non-zero, see [WW90 and MW96] for textbook treatments. So, for Einstein gravity, we are seeking a twistor action whose field equations not only imply the integrability of an almost complex structure, but also the existence of some compatible holomorphic geometric structure, for example the contact one-form in the case of the cosmological constant, or the fibration together with a Poisson structure up the fibres in the case of vanishing cosmological constant. The first task is to introduce suitable variables that encode the almost complex structure together with the relevant compatible geometric structure on the real six-manifold underlying the twistor space. This turns out to be a one-form with values in a line bundle, and we write down the appropriate field equations that it must satisfy and an action (depending also on a Lagrange multplier field) that gives rise to them; the Lagrange multiplier field again corresponds to an anti-self-dual linear gravitational field propagating on the self-dual background via the Penrose transform when the field equations are satisfied.

Our primary exposition will focus on the $\mathcal{N} = 8$ supersymmetric cases, and reduce them to the cases with lesser or no supersymmetry. Supersymmetric extensions of Penrose's non-linear graviton construction were first discussed by [M92a, M92b] (see also [M91, M92c]) based on work by [M88] and developed further in [A-ZHM08] and in [W07].[1] That in [W07] gives a twistor description of four-dimensional $\mathcal{N}$-extended, possibly gauged, self-dual supergravity with and without cosmological constant in terms of a deformed supertwistor space, a deformation of a region in $\mathbb{CP}^{3|\mathcal{N}}$ endowed with an even holomorphic contact structure. Here we also discuss the different gaugings in the case without a cosmological constant. It is these the integrability of the almost complex structures of these twistor spaces together with the holomorphy of the appropriate geometric structures that correspond to the field equations for our twistor actions.

There are now a number of contexts arising from conventional string theory and M-theory in which the task of finding variables and action principles whose field equations encode the integrability of complex structures compatibly with some other geometric structure. In particular Kodaira-Spencer theory, [BCOV94], leads to field equations that imply the integrability of an almost complex structure compatible with a global holomorphic volume form on a six-manifold, yielding a Calabi-Yau structure. For a compendium of such theories and relations between them, including conjectured relations to twistor-string theory, see [DGNV05]. The situations considered here are distinct from those in [DGNV05], but given that one of the form theories involved there is a self-dual form theory of four-dimensional gravity including a cosmological constant (see also [A-ZH06]) there may well be some important connections between these ideas.

The paper is structured as follows. In §2, we first review the equations of self-dual supergravity, with cosmological constant and gauged R-symmetry, and then go on to review the various twistor constructions and give a brief proof of the version of the

[1] See also [S06 and W06] and references therein for recent reviews of supertwistors and their application to supersymmetric gauge theories.

non-linear graviton construction for self-dual Einstein supergravity both with and without cosmological constant and different gaugings. In §3, we study infinitesimal deformations and show that a deformation of the contact structure determines a deformation of the almost complex structure. We develop a non-projective twistor formulation that shows that this persists in the case of a finite deformation giving a compact form for the field equations, i.e., the integrability condition for the almost complex structure. In the case of maximal supersymmetry, $\mathcal{N} = 8$, we present the twistor action and show that it gives the appropriate field equations. We give a brief discussion of its invariance properties and various reductions with lesser gauging, supersymmetry, or no cosmological constant.

A Chern-Simons action is always expressed in a given background frame and is not manifestly gauge invariant. In this gravitational context, our action is similarly not manifestly diffeomorphism invariant; we require the choice of some background, which we take to be a solution to the field equations. However, we go some way towards an invariant formulation. We give an invariant formulation of the field equations in general, but only find an explicitly diffeomorphism invariant action in the $\mathcal{N} = 0$ case with cosmological constant. We prove that on any smooth manifold of dimension $4n + 2$ equipped with a complex one-form τ up to scale (i.e., a complex line subbundle of the complexified cotangent bundle), then, if $\tau \wedge (\mathrm{d}\tau)^n = 0$, and a non-degeneracy condition is satisfied, there is a unique integrable almost complex structure for which τ is proportional to a non-degenerate holomorphic contact structure. This idea can be used to give a covariant form of the field equations in general, and a covarant action in the $\mathcal{N} = 0$ case.

In §5, we make some general concluding remarks. An action principle for $\mathcal{N} = 8$ self-dual supergravity with vanishing cosmological constant has been obtained by [KK98] in harmonic superspace for split space-time signature.[2] In that work, harmonic superspace is the spin bundle of super space-time and in Euclidean signature, it can naturally be identified with the supertwistor space. However, their action uses structures pulled back from space-time (e.g., the Laplacian) that are not locally obtainable from the complex structure and contact structure on twistor space. It is therefore not possible to regard it as a twistor action. Nevertheless, their action is closely related to ours and we show that theirs can be obtained from ours by gauge fixing in Appendix 5. In Appendix 5 we give a detailed discussion of the construction of the line bundle on a super-twistor space whose total space corresponds to a non-projective twistor space. In Appendix 5, we discuss some alternative twistor actions.

2. Twistor Constructions for Self-Dual Supergravity

We work throughout in a complex setting. This can be understood as arising from taking a real analytic metric on a real space-time, and extending it to become a holomorphic complex metric on some neighbourhood M of the real slice in complexified space-time. We can straightforwardly restrict attention to Euclidean or split signature slice by requiring invariance under appropriate anti-holomorphic involutions (for Euclidean signature, these are discussed in Appendix 5). In the Euclidean case, one needs to restrict the number of allowed supersymmetries $\mathcal{N}$ to be even.

2.1. Definitions, notation and conventions. We model our definition of chiral super space-time on the paraconformal geometries of [BE91] (see also [W07]).

[2] A similar action for $\mathcal{N} = 4$ supersymmetric Yang-Mills theory was discovered in the context of harmonic superspace by [S95].

Definition 1. *A right-chiral super space-time, $\mathscr{M}$, is a split supermanifold of super-dimension $4|2\mathcal{N}$ on which we have an identification*[3] $T\mathscr{M} \cong \mathscr{H} \otimes \widetilde{\mathscr{S}}$, *where $\widetilde{\mathscr{S}}$ is the right (dotted) spin bundle of rank $2|0$ and $\mathscr{H}$ is the sum of the left spin bundle $\mathscr{S}$ and the rank-$0|\mathcal{N}$ bundle of supersymmetry generators and so has rank $2|\mathcal{N}$. We will also assume that $\mathscr{S}$ and $\mathscr{H}$ are endowed with choices of Berezinian forms (so that $T\mathscr{M}$ does also).*

This is the superspace one would obtain from a full super space-time by eliminating the left-handed fermionic coordinates, leaving only the right-handed ones in play. Being a split supermanifold, it is locally of the form $\mathbb{C}^{4|2\mathcal{N}}$ with coordinates[4] $(x^{\mu\dot\nu}, \theta^{m\dot\nu}) := x^{M\dot\nu}$ with $x^{\mu\dot\nu}$ bosonic and $\theta^{m\dot\nu}$ fermionic where the indices range as follows: $\alpha, \ldots, \mu, \ldots = 0, 1$ for left-handed two-component spinors, $\dot\alpha, \ldots, \dot\mu, \ldots = \dot 0, \dot 1$ for right-handed spinors, $i, \ldots, m, \ldots = 1, \ldots, \mathcal{N}$ indexing the supersymmetries and $A = (\alpha, i)$, $M = (\mu, m)$; it will turn out in the following that it is natural, and simplifying in this self-dual context to group together the supersymmetry index m and the undotted spinor index μ into one index M. We use the convention that letters from the middle of the alphabets are coordinate indices whereas letters from the beginning of the alphabets are structure frame indices.

The identification $T\mathscr{M} \cong \mathscr{H} \otimes \widetilde{\mathscr{S}}$ will be specified by a choice of 'structure co-frame' given by the indexed one-forms

$$E^{A\dot\alpha} = \mathrm{d}x^{M\dot\nu} E_{M\dot\nu}{}^{A\dot\alpha}. \tag{2.1}$$

The dual vector fields will be denoted $E_{A\dot\alpha}$, $E_{A\dot\alpha} \lrcorner\, E^{B\dot\beta} = \delta_{\dot\alpha}{}^{\dot\beta}\delta_A{}^B$. When contracting a vector field V with a differential one-form α we use the notation $V \lrcorner\, \alpha$.

With the capital Roman indices $A, B, \ldots$ ranging over both the bosonic $\alpha, \beta, \ldots$ and the fermionic $i, j, \ldots$ indices we use the notation $\{AB\ldots]$ for graded symmetrization and $[AB\ldots\}$ for graded skew symmetrization

$$T_{\{A_1A_2\ldots A_n]} := \tfrac{1}{n!} \sum_{\sigma\in P_n} (-)^{\bar\sigma} T_{A_{\sigma(1)}A_{\sigma(2)}\ldots A_{\sigma(n)}}, \tag{2.2a}$$

$$T_{[A_1A_2\ldots A_n\}} := \tfrac{1}{n!} \sum_{\sigma\in P_n} (-)^{\bar\sigma+|\sigma|} T_{A_{\sigma(1)}A_{\sigma(2)}\ldots A_{\sigma(n)}}, \tag{2.2b}$$

where P_n is the group of permutations of n letters, $|\sigma|$ the number of transpositions in σ and $\bar\sigma$ the number of transpositions of odd indices.

For an index such as A that ranges over indices for both odd and even coordinates, p_A will denote the Graßmann parity of the index, $p_A = 0$ for an even coordinate, and 1 for an odd one so that a graded skew form Λ_{AB} satisfies

$$\Lambda_{AB} = -(-)^{p_Ap_B}\Lambda_{BA}. \tag{2.3}$$

We introduce $\epsilon_{\dot\alpha\dot\beta} = \epsilon_{[\dot\alpha\dot\beta]}$ with $\epsilon_{\dot 0\dot 1} = -1$ and $\epsilon_{\dot\alpha\dot\gamma}\epsilon^{\dot\gamma\dot\beta} = \delta_{\dot\alpha}{}^{\dot\beta}$, and similarly for $\epsilon_{\alpha\beta}$.

In the supersymmetric setting, there is a distinction between differential and integral forms, the latter being required for integration, [M88]. Unless otherwise stated, all our forms will be differential.

[3] By $T\mathscr{M}$ we will mean $T^{(1,0)}\mathscr{M}$. There will be no role for anti-holomorphic objects on $\mathscr{M}$.

[4] The index structure on the bosonic coordinates in the curved case is not natural, but simplifies notation.

2.2. Self-dual supergravity equations. We introduce connections on $\mathscr{H}$ and $\widetilde{\mathscr{S}}$ represented by connection one-forms $\omega_A{}^B$ and $\omega_{\dot\alpha}{}^{\dot\beta}$, respectively. These determine a connection ∇ on $T\mathscr{M}$ by

$$\nabla V^{A\dot\alpha} \;=\; \mathrm{d}V^{A\dot\alpha} + V^{B\dot\alpha}\omega_B{}^A + V^{A\dot\beta}\omega_{\dot\beta}{}^{\dot\alpha} \tag{2.4}$$

so that it preserve the factorisation $T\mathscr{M} \cong \mathscr{H} \otimes \widetilde{\mathscr{S}}$. The fermionic parts of $\omega_A{}^B$ gauge the R-symmetry.

In this supersymmetric context, a choice of scale or volume form on $\mathscr{M}$ is a section of the Berezinian of $\Omega^1\mathscr{M}$. We can assume that the Berezinians of $\mathscr{H}$ and $\widetilde{\mathscr{S}}$ have been identified so that the scale is determined by a section of the Berezinian of either $\mathscr{H}^*$ or $\widetilde{\mathscr{S}}^*$. The connections can be chosen uniquely so that they preserves these sections of the Berezinians of $\mathscr{H}^*$ and $\widetilde{\mathscr{S}}^*$ and so that the connection on $T\mathscr{M}$ has torsion with vanishing supertrace.[5] We assume from hereon that such choices have been made. In the formulae that follow, we will also assume that the connection is torsion-free as that is part of the self-dual Einstein condition (the torsion will not in general vanish on the full super space-time, only on this right-chiral (or left-chiral) reduced supermanifold).

The curvature two-form $R_{A\dot\alpha}{}^{B\dot\beta}$ of ∇ decomposes into curvature two-forms for the connections on $\mathscr{H}$ and $\widetilde{\mathscr{S}}$,

$$R_{A\dot\alpha}{}^{B\dot\beta} \;=\; \delta_A{}^B R_{\dot\alpha}{}^{\dot\beta} + \delta_{\dot\alpha}{}^{\dot\beta} R_A{}^B\,. \tag{2.5}$$

Making explicit the form indices, we write the Ricci identities as

$$\begin{aligned}[\nabla_{A\dot\alpha}, \nabla_{B\dot\beta}\}V^{D\dot\delta} = {}&(-)^{p_C(p_A+p_B)}V^{C\dot\delta}R_{A\dot\alpha B\dot\beta C}{}^D \\ &+ (-)^{p_D(p_A+p_B)}V^{D\dot\gamma}R_{A\dot\alpha B\dot\beta\dot\gamma}{}^{\dot\delta},\end{aligned} \tag{2.6}$$

where $V^{A\dot\alpha}$ is a vector field on $\mathscr{M}$.

In the torsion free case, using the algebraic Bianchi identities, Prop. 2.6 of [W07] gives the decomposition of the curvature into irreducibles:

$$R_{A\dot\alpha B\dot\beta C}{}^D = -2(-)^{p_C(p_A+p_B)}R_{C[A|\dot\alpha\dot\beta|}\delta_{B\}}{}^D + \epsilon_{\dot\alpha\dot\beta}R_{ABC}{}^D, \tag{2.7a}$$

$$R_{ABC}{}^D = C_{ABC}{}^D - 2(-)^{p_C(p_A+p_B)}\Lambda_{C\{A}\delta_{B]}{}^D, \tag{2.7b}$$

$$R_{A\dot\alpha B\dot\beta\dot\gamma}{}^{\dot\delta} = C_{AB\dot\alpha\dot\beta\dot\gamma}{}^{\dot\delta} + 2\Lambda_{AB}\delta_{(\dot\alpha}{}^{\dot\delta}\epsilon_{\dot\beta)\dot\gamma} + \epsilon_{\dot\alpha\dot\beta}R_{AB\dot\gamma}{}^{\dot\delta}, \tag{2.7c}$$

where the curvature tensors satisfy the algebraic conditions

$$\begin{gathered} R_{AB\dot\alpha\dot\beta} = R_{AB\dot\alpha}{}^{\dot\gamma}\epsilon_{\dot\gamma\dot\beta} = R_{AB(\dot\alpha\dot\beta)}, \\ C_{ABC}{}^D = C_{\{ABC]}{}^D, \quad (-)^{p_C}C_{ABC}{}^C = 0, \quad \Lambda_{AB} = \Lambda_{[AB\}}. \end{gathered} \tag{2.8}$$

Here, Λ_{AB} is a natural supersymmetric extension of the scalar curvature and will be set equal to the cosmological constant when the field equations are satisfied. (See [W07] for further details of the construction and properties of the connections.)

[5] Special care needs to be taken for $\mathcal{N} = 4$, [W07].

Definition 2. *A right-chiral superspace will be said to satisfy the* $\mathcal{N}$*-extended self-dual supergravity equations if*

(i) *the unique connection that preserves the given Berezinians of* $\mathscr{H}^*$ *and* $\widetilde{\mathscr{S}}^*$ *is torsion-free and satisfies* $C_{AB\dot\alpha\dot\beta\dot\gamma}{}^{\dot\delta} = 0$*,*

(ii) $R_{AB\dot\alpha\dot\beta} = 0$*,*

(iii) *preserves some* $P^{AB} = P^{[AB\}} \in \Lambda^2\mathscr{H}$ *of rank* $2|r$ *and is flat on the odd* $(\mathcal{N}-r)$*-dimensional subspace of* $\mathscr{H}^*$ *that annihilates* P^{AB}*.*

When $\Lambda_{AB} \neq 0$ *it will be said to be Einstein, whereas if* $\Lambda_{AB} = 0$ *it will be said to be vacuum. When* $r = 0$*, the connection on* $\mathscr{H}$ *is trivial in the odd directions and the* R*-symmetry is ungauged; all supersymmetry generators are covariantly constant. For* $r > 0$*, a subgroup of the* R*-symmetry is gauged with gauge group an extension of* $SO(r, \mathbb{C})$*, the subgroup of* $SO(\mathcal{N}, \mathbb{C})$ *that preserves* P^{ij} *the odd-odd part of* P^{AB}*. For* $r = \mathcal{N}$*, the gauge group is* $SO(\mathcal{N}, \mathbb{C})$*.*

Conformal supergravity corresponds to the more general situation where condition (i) alone is satisfied, and a natural supersymmetric analogue of the hypercomplex case corresponds to conditions (i) and (ii). In this work, we shall mostly be concerned with the situation where (i)–(iii) are satisfied simultaneously.

There is only one possibility for the gauging in the Einstein case as follows:

Lemma 1. *Either* P^{AB} *and* Λ_{AB} *both have maximal rank and can be chosen to be multiples of each-other's inverse, or* $\Lambda_{AB} = 0$*.*

Proof. Condition (iii) of Def. 2 implies that $(-)^{p_C + p_C(p_D+p_E)} R_{ABC}{}^{[D} P^{E\}C} = 0$, and taking a supertrace gives the equation

$$0 \;=\; (-)^{p_C(p_A+p_E)+p_C+p_B} \Lambda_{C\{A} \delta^{[B}_{B]} P^{E\}C}\,, \tag{2.9}$$

which quickly leads to the condition that $(-)^{p_B} \Lambda_{AB} P^{BC}$ is a multiple of $\delta_A{}^C$. If this multiple is non-zero, P^{AB} and Λ_{AB} have maximal rank and are multiples of each other's inverse. If this multiple is zero, the assumption on the rank of P^{AB} implies that the rank of Λ_{AB} is less than or equal to $0|\mathcal{N}-r$. The condition that the connection is flat on the subspace of $\mathscr{H}^*$ that annihilates P^{AB} implies that $R_{ABC}{}^D e_D = 0$ for all e_D such that $P^{AB} e_B = 0$. Symmetrizing over ABC gives that $C_{ABC}{}^D e_D = 0$ so we must also have $\Lambda_{C\{A} e_{D]} = 0$. Note that for $\mathcal{N} = 2$ the multiple is always zero. □

It is a consequence of the Bianchi identities that Λ_{AB} is covariantly constant so that, when non-zero, defining P_{AB} as the inverse of P^{AB}, we can set $\Lambda_{AB} = \Lambda P_{AB}$. When Λ_{AB} is non-trivial, the curvature is non-trivial on the odd directions of $\mathscr{H}$, and so the R-symmetry is therefore necessarily gauged with gauge group $SO(\mathcal{N}, \mathbb{C})$.

We will see in §3.1 in the discussion of the deformations of twistor space how the different gaugings come about.

We also obtain $(-)^{p_C + p_C(p_D+p_E)} C_{ABC}{}^{[D} P^{E\}C} = 0$ and $\nabla_{[A\dot\alpha} C_{B\}CD}{}^E = 0$. The field equations of self-dual supergravity with zero cosmological constant lead to the Ricci identities

$$[\nabla_{A\dot\alpha}, \nabla_{B\dot\beta}\} V^{D\dot\delta} \;=\; (-)^{p_C(p_A+p_B)} V^{C\dot\delta} \epsilon_{\dot\alpha\dot\beta} C_{ABC}{}^D\,, \tag{2.10}$$

which in turn imply Ricci-flatness of $\mathscr{M}$.

The self-dual supergravity equations on chiral super space-time with vanishing cosmological constant first appeared in light-cone gauge and in their covariant formulation in the work by [S92].[6]

[6] See also [K79, K80, CDDG79, KNG92 and BS92].

2.3. Twistor constructions. Flat supertwistor space is $\mathbb{PT}'_{[\mathcal{N}]} := \mathbb{CP}^{3|\mathcal{N}} \setminus \mathbb{CP}^{1|\mathcal{N}}$ with homogeneous coordinates

$$Z^I \;:=\; (\omega^\alpha, \theta^i, \pi_{\dot\alpha}) \;=\; (\omega^A, \pi_{\dot\alpha}), \tag{2.11}$$

where ω^α and $\pi_{\dot\alpha}$ are bosonic coordinates and θ^i fermionic ones.

The supertwistor correspondence is between right-chiral complexified super space-time $\mathbb{M}_{[\mathcal{N}]} \cong \mathbb{C}^{4|2\mathcal{N}}$ with coordinates $(x^{\alpha\dot\alpha}, \theta^{i\dot\alpha}) = x^{A\dot\alpha}$, and is expressed by the incidence relation

$$\omega^A \;=\; x^{A\dot\alpha}\pi_{\dot\alpha}\,. \tag{2.12}$$

By holding $x^{A\dot\alpha}$ constant we see that points of $\mathbb{M}_{[\mathcal{N}]}$ correspond to $\mathbb{CP}^1$s in supertwistor space $\mathbb{PT}'_{[\mathcal{N}]}$ with homogeneous coordinates $\pi_{\dot\alpha}$. Alternatively, by holding Z^I constant, we see that points in $\mathbb{PT}'_{[\mathcal{N}]}$ correspond to $(2|\mathcal{N})$-dimensional isotropic superplanes. In the curved case, both sides of the correspondence are deformed, but points of super space-time still correspond to $\mathbb{CP}^1$s in supertwistor space (and points of supertwistor space to $(2|\mathcal{N})$-dimensional isotropic subsupermanifolds of $\mathscr{M}$).

Bosonic twistor space will be denoted by PT and will be a deformation of some region in $\mathbb{CP}^3$, whereas a supersymmetrically extended curved twistor space will be denoted by $\mathscr{P}\mathscr{T}$ and will be a deformation of a region in $\mathbb{CP}^{3|\mathcal{N}}$. Similarly, a bosonic space-time will be denoted by M and a supersymmetric one (which will always in this paper be right-chiral) by $\mathscr{M}$.

We recall first Ward's extension [W80] of [P76] non-linear graviton construction to the case of non-zero cosmological constant:

Theorem 1. ([P76,W80]).

(i) *There is a natural one-to-one correspondence between holomorphic conformal structures $[g]$ on some four-dimensional (complex) manifold M whose anti-self-dual Weyl curvature vanishes, and three-dimensional complex manifolds PT (the twistor space) containing a rational curve (a $\mathbb{CP}^1$) with normal bundle $N \cong \mathscr{O}(1) \oplus \mathscr{O}(1)$.*

(ii) *The existence of a conformal scale for which the trace-free Ricci tensor vanishes, but for which the scalar curvature is non-vanishing, is equivalent to PT admitting a non-degenerate contact structure.*

(iii) *The existence of a conformal scale for which the full Ricci tensor vanishes is equivalent to PT admitting a fibration $\varpi : PT \to \mathbb{CP}^1$ whose fibres admit a Poisson structure with values in the pullback of $\mathscr{O}(-2)$ from $\mathbb{CP}^1$.*

Here, $\mathscr{O}(n)$ is the complex line bundle of Chern class n on $\mathbb{CP}^1$.

The holomorphic contact structure is a rank-2 distribution $D \subset T^{(1,0)}PT$ in the holomorphic tangent bundle of PT. The quotient determines a line bundle $L := T^{(1,0)}PT/D$. It can be defined dually to be the kernel of a holomorphic $(1,0)$-form τ defined up to scale on PT, i.e. $D = \ker\tau$. If so, τ takes values in L since the map $T^{(1,0)}PT \to T^{(1,0)}PT/D := L$ is then the contraction of a vector with the $(1,0)$-form τ. The non-degeneracy condition is that for any two vector fields X and Y in D, the Frobenius form

$$\Phi \,:\, D \wedge D \;\to\; L \;:=\; T^{(1,0)}PT/D, \quad \text{with} \quad \Phi(X,Y) \;:=\; [X,Y] \bmod D \tag{2.13}$$

is non-degenerate on D. This is equivalent to $\tau \wedge \mathrm{d}\tau \neq 0$. When it is everywhere degenerate, D determines a foliation whose leaves are the fibres of the projection $\varpi : PT \to \mathbb{CP}^1$ and τ is the pullback of the one-form $\pi^{\dot\alpha}\mathrm{d}\pi_{\dot\alpha}$ from $\mathbb{CP}^1$ and L becomes the pullback of $\mathscr{O}(2)$ from $\mathbb{CP}^1$. In the non-degenerate case, we can define a Poisson structure with values in L^* to be the inverse of Φ on D. This has an analogue also in the degenerate case, now with values in $\mathscr{O}(-2)$ although its existence no longer follows from that of τ.

We can impose compatibility with, e.g., Euclidean reality conditions by requiring the existence of an anti-holomorphic involution $\rho : PT \to PT$ without fixed points sending the given Riemann sphere to itself via the antipodal map. This then induces a corresponding involution on M fixing a real slice on which the metric g is real and of Euclidean signature.

The above theorem has a supersymmetric extension as follows:

Theorem 2. (i) *There is a natural one-to-one correspondence between conformally self-dual holomorphic right-chiral space-times and complex supermanifolds $\mathscr{P}\mathscr{T}$ of dimension $3|\mathcal{N}$ with an embedded rational curve (a Riemann sphere $\mathbb{CP}^1$) with normal bundle $\mathscr{N} \cong \mathscr{O}(1)^{\oplus 2|\mathcal{N}}$.*

(ii) *Furthermore, $\mathscr{M}$ is a complex solution to the four-dimensional $\mathcal{N}$-extended self-dual supergravity equation with non-vanishing cosmological constant iff the twistor space $\mathscr{P}\mathscr{T}$ admits a non-degenerate even contact structure.*

(iii) *$\mathscr{M}$ is a complex solution to the four-dimensional $\mathcal{N}$-extended self-dual supergravity equation with vanishing cosmological constant iff the twistor space $\mathscr{P}\mathscr{T}$ admits a fibration $\varpi : \mathscr{P}\mathscr{T} \to \mathbb{CP}^{1|\mathcal{N}-r}$ and a Poisson structure of rank $2|r$ tangent ot the fibres with values in $\varpi^*\mathscr{O}(-2)$.*

Here, $\mathscr{O}(n)^{\oplus r|s} := \mathbb{C}^{r|s} \otimes \mathscr{O}(n)$. The proof breaks up into three parts; further details of the non-degenerate cosmological constant case are given in [W07].

Proof. Part (i). Let $\mathscr{F} = \mathbb{P}(\widetilde{\mathscr{S}}^*)$ be the projective co-spin bundle over $\mathscr{M}$ with holomorphic projection $p : \mathscr{F} \to \mathscr{M}$. Its fibres $p^{-1}(x)$ over $x \in \mathscr{M}$ are complex projective lines $\mathbb{CP}^1$ with homogeneous fibre coordinates $\pi_{\dot\alpha}$. We define the twistor distribution to be the rank-$2|\mathcal{N}$ distribution $\mathscr{D}_{\mathscr{F}}$ on $\mathscr{F}$ given by

$$\mathscr{D}_{\mathscr{F}} := \operatorname{span}\{\widetilde{E}_A\} := \operatorname{span}\left\{\pi^{\dot\alpha} E_{A\dot\alpha} + \pi^{\dot\alpha}\pi_{\dot\gamma}\omega_{A\dot\alpha\dot\beta}{}^{\dot\gamma}\frac{\partial}{\partial\pi_{\dot\beta}}\right\}, \tag{2.14}$$

where the $E_{A\dot\alpha}$s are the frame fields and $\omega_{\dot\alpha}{}^{\dot\beta}$ is the connection one-form on $\widetilde{\mathscr{S}}$. A few lines of algebra show that $\mathscr{D}_{\mathscr{F}}$ is integrable if and only if the connection is torsion-free and the $C_{AB(\dot\alpha\dot\beta\dot\gamma\dot\delta)}$-part of the curvature vanishes. In this case, the distribution $\mathscr{D}_{\mathscr{F}}$ defines a foliation of $\mathscr{F}$. Working locally on $\mathscr{M}$, the resulting quotient will be our supertwistor space, a $(3|\mathcal{N})$-dimensional supermanifold denoted by $\mathscr{P}\mathscr{T}$. The quotient map will be denoted by $q : \mathscr{F} \to \mathscr{P}\mathscr{T}$ so that we have the double fibration $\mathscr{P}\mathscr{T} \xleftarrow{q} \mathscr{F} \xrightarrow{p} \mathscr{M}$. We note that we can form a non-projective supertwistor space $\mathscr{T}$ by taking the quotient of $\widetilde{\mathscr{S}}^*$ by the distribution $\mathscr{D}_{\mathscr{F}}$. The integral curves of the Euler vector field $\widetilde{\Upsilon} := \pi_{\dot\alpha}\partial/\partial\pi_{\dot\alpha}$ are the fibres over $\mathbb{P}(\widetilde{\mathscr{S}}^*)$ and $\widetilde{\Upsilon}$ descends to give a vector field Υ on $\mathscr{T}$ which determines the fibration $\mathscr{T} \to \mathscr{P}\mathscr{T}$.

Since $\mathscr{F}$ is a $\mathbb{CP}^1$-bundle over $\mathscr{M}$ and the fibres are transverse to the distribution $\mathscr{D}_{\mathscr{F}}$, the submanifolds $q(p^{-1}(x)) \hookrightarrow \mathscr{P}\mathscr{T}$, for $x \in \mathscr{M}$, are $\mathbb{CP}^1$s. In the other direction, the supermanifolds $p(q^{-1}(Z)) \hookrightarrow \mathscr{M}$, for $Z \in \mathscr{P}\mathscr{T}$, are the $(2|\mathcal{N})$-dimensional isotropic subsupermanifolds of $\mathscr{M}$ given by the p projections of integral surfaces of $\mathscr{D}_{\mathscr{F}}$.

The inverse construction, i.e. starting from $\mathscr{PT}$, follows by applying a supersymmetric extension of Kodaira's deformation theory ([W86]). This allows one to reconstruct $\mathscr{M}$ as the moduli space of $\mathbb{CP}^1$s that arise as deformations of the given $\mathbb{CP}^1$ which will correspond to some $x \in \mathscr{M}$. According to Kodaira theory, $T_x\mathscr{M} \cong H^0(\mathbb{CP}^1, \mathscr{N})$, where $\mathscr{N}$ is the normal bundle to the given $\mathbb{CP}^1 \subset \mathscr{PT}$, and in order that the moduli space exist, we require the vanishing of the first cohomology of the normal bundle $\mathscr{N}$. If the given $\mathbb{CP}^1$ arises as $q(p^{-1}(x))$ for some $x \in \mathscr{M}$, then $\mathscr{N} \cong \mathscr{O}(1)^{\oplus 2|\mathcal{N}}$: this can be seen by expressing it as the quotient of the horizontal tangent vectors to $\mathscr{F}$ at $p^{-1}(x) \cong \mathbb{CP}^1$, which can be represented by $T_x\mathscr{M}$, by $\mathscr{D}_{\mathscr{F}}$,

$$0 \longrightarrow \mathscr{D}_{\mathscr{F}}|_{p^{-1}(x)} \longrightarrow T_x\mathscr{M} \longrightarrow q^*\mathscr{N} \longrightarrow 0\,. \tag{2.15}$$

Since the twistor distribution $\mathscr{D}_{\mathscr{F}}$ restricted to the fibres $p^{-1}(x)$ over $x \in \mathscr{M}$ is $\mathscr{O}(-1)^{\oplus 2|\mathcal{N}}$, and $T_x\mathscr{M} \cong \mathbb{C}^{4|2\mathcal{N}}$, $\mathscr{N}$ takes the form $\mathscr{O}(1)^{\oplus 2|\mathcal{N}}$ as stated above. Kodaira theory in turn implies that we can reconstruct $\mathscr{M}$ as the moduli space of such $\mathbb{CP}^1$s, and that the construction is stable under deformations of the complex structure on $\mathscr{PT}$. Kodaira theory identifies the tangent bundle $T_x\mathscr{M}$ with the sections of the normal bundle, $\mathscr{N} \cong \mathscr{O}(1)^{\oplus 2|\mathcal{N}}$, and these, by an extension of Liouville's theorem are linear functions of $\pi_{\dot\alpha}$, i.e., $V^{A\dot\alpha}\pi_{\dot\alpha}$ where the A index is associated to a basis of $\mathbb{C}^{2|\mathcal{N}}$. This gives the right-chiral manifold structure on $\mathscr{M}$, and it is easily seen that lines through a given point of $\mathscr{PT}$ correspond to an integrable $(2|\mathcal{N})$ manifold that will be an integral surface of the distribution $\mathscr{D}_{\mathscr{F}}$. Thus $\mathscr{D}_{\mathscr{F}}$ is integrable and the $\mathscr{M}$ is therefore conformally self-dual.

Part (ii). In the self-dual Einstein case with non-vanishing cosmological constant, we may introduce a one-form of homogeneity 2 on $\mathscr{F}$ by

$$\tilde\tau \;:=\; \pi^{\dot\alpha}\nabla\pi_{\dot\alpha} \;=\; \pi^{\dot\alpha}\mathrm{d}\pi_{\dot\alpha} - \omega_{\dot\alpha}{}^{\dot\beta}\pi^{\dot\alpha}\pi_{\dot\beta}, \tag{2.16}$$

where $\omega_{\dot\alpha}{}^{\dot\beta}$ is the connection one-form on $\widetilde{\mathscr{S}}$. The one-form τ automatically annihilates horizontal vectors and hence the distribution $\mathscr{D}_{\mathscr{F}}$. The form $\tilde\tau$ descends to $\mathscr{PT}$ if and only if $\mathrm{d}\tilde\tau$ is annihilated by $\mathscr{D}_{\mathscr{F}}$ also. This characterizes the self-dual Einstein equations since when $C_{AB\dot\alpha\dot\beta\dot\alpha\dot\delta} = 0$, as follows from the conformal self-duality condition,

$$\mathrm{d}\tilde\tau \;=\; \nabla\pi^{\dot\alpha}\wedge\nabla\pi_{\dot\alpha} + E^{B\dot\beta}\wedge E^{A\dot\alpha}\Lambda_{AB}\pi_{\dot\alpha}\pi_{\dot\beta} - E^{B\dot\gamma}\wedge E^{A}_{\dot\gamma}R_{AB\dot\alpha\dot\beta}\pi^{\dot\alpha}\pi^{\dot\beta}\,, \tag{2.17}$$

and this is annihilated by $\mathscr{D}_{\mathscr{F}}$ iff $R_{AB\dot\alpha\dot\beta} = 0$. Thus, $\tilde\tau$ descends to $\mathscr{PT}$, i.e., there exists a one-form τ on $\mathscr{PT}$ such that $\tilde\tau = q^*\tau$.

Non-degeneracy of the contact structure is the condition that $\mathrm{d}\tau$ is non-degenerate on the kernel $\mathscr{D}$ of τ, or equivalently, the condition that the three-form $\tau\wedge\mathrm{d}\tau$ should be non-degenerate in the sense that for any vector X, $X \lrcorner (\tau\wedge\mathrm{d}\tau) = 0 \Rightarrow X = 0$. This non-degeneracy is equivalent to the non-degeneracy of Λ_{AB} on $\mathscr{H}$. Thus, τ defines a non-degenerate holomorphic contact structure on $\mathscr{PT}$.

Part (iii). In the self-dual vacuum case, we see that the connection on $\widetilde{\mathscr{S}}$ is flat and a basis for $\widetilde{\mathscr{S}}$ can be found so that it vanishes. In this basis, $\pi_{\dot\alpha}$ are constant along the horizontal distribution on $\mathscr{F}$, and so along the distribution (2.14). They are therefore the pullback of coordinates on $\mathscr{PT}$. The condition that the connection is flat on the annihilator of P^{AB} in $\mathscr{H}^*$ means that there are $\mathcal{N} - r$ covariantly constant sections e^s_A of the odd part of $\mathscr{H}^*$, $s = r+1, \ldots, \mathcal{N}$. The forms $E^{A\dot\alpha}e^s_A$ are therefore constant and, since the connection is torsion free, these forms are exact and equal to $\mathrm{d}\theta^{s\dot\alpha}$ for some odd coordinates $\theta^{s\dot{\alpha}}$. The $\mathcal{N}-r$ functions $\theta^s = \theta^{s\dot{\alpha}}\pi_{\dot{\alpha}}$ can be seen to be constant also along the twistor distribution (2.14). The global holomorphic coordinates $(\pi_{\dot{\alpha}}, \theta^s)$ define a projection $\varpi : \mathscr{P}\mathscr{T} \to \mathbb{CP}^{1|\mathcal{N}-r}$ as promised. We now define the Poisson structure by considering a pair of local functions f, g on $\mathscr{P}\mathscr{T}$. Pulled back to $\mathscr{F}$, they satisfy

$$\pi^{\dot{\alpha}} E_{A\dot{\alpha}} f \; = \; 0, \quad \pi^{\dot{\alpha}} E_{A\dot{\alpha}} g \; = \; 0\,, \tag{2.18}$$

and this implies that

$$E_{A\dot{\alpha}} f \; = \; \pi_{\dot{\alpha}} f_A, \quad E_{A\dot{\alpha}} g \; = \; \pi_{\dot{\alpha}} g_A \tag{2.19}$$

for some f_A, g_A of weight -1 in $\pi_{\dot{\alpha}}$ (this follows from the standard fact that $\pi^{\dot{\alpha}} b_{\dot{\alpha}} = 0 \Rightarrow b_{\dot{\alpha}} = b\pi_{\dot{\alpha}}$ for some b which follows from the two-dimensionality of the spin space and the skew symmetry of $\epsilon_{\dot{\alpha}\dot{\beta}}$). We define the Poisson bracket $\{f, g\}$ of f with g to be

$$\{f, g\} \; := \; (-)^{p_A(p_f+1)} f_A P^{AB} g_B\,. \tag{2.20}$$

It is clear that this has weight -2 in $\pi_{\dot{\alpha}}$, but, as given, this expression only lives on $\mathscr{F}$. However, it is easily checked that, as a consequence of the covariant constancy of the P^{AB}, it is constant along the distribution (2.14) and descends to $\mathscr{P}\mathscr{T}$. $\square$

See Appendix 5 for more on the non-projective formulation.

3. Twistor Actions

In order to consider actions, we must allow our fields to go off-shell, and this is most straightforwardly done in the Dolbeault setting. We can take an almost complex structure that is not necessarily integrable to be the off-shell field, and regard the integrability condition to be part of the field equations. In the following we will see that if we require the almost complex structure to be compatible with a Poisson structure or complex contact structure and the almost complex structure can be encoded in a complex one-form h defined up to scale.

In the following, we will mostly work 'non-projectively' i.e., on $\mathbb{T}_{[\mathcal{N}]} = \mathbb{C}^{4|\mathcal{N}}$, or at least using homogeneous coordinates. This can also be identified as the total space of the line bundle $\mathscr{O}(-1)$ over $\mathbb{PT}$. On this space, we have the Euler homogeneity vector field Υ, and a canonically defined holomorphic volume form Ω (an integral form in this supersymmetric context) of weight $4-\mathcal{N}$, the tautological form pulled back from $\mathrm{Ber}(\mathscr{P}\mathscr{T}) \cong \mathscr{O}(\mathcal{N}-4)$ satisfying $\mathcal{L}_\Upsilon \Omega = (4-\mathcal{N})\Omega$, where $\mathcal{L}_\Upsilon$ is the Lie derivative along Υ. Similarly, τ will be a well-defined differential one-form of weight 2. See Appendix 5 for further discussion.

3.1. Deformations of twistor space. For simplicity, we take the supertwistor space $\mathscr{P}\mathscr{T}$ to be a deformation of flat twistor space $\mathbb{PT}'_{[\mathcal{N}]}$ with homogeneous coordinates as in the flat case given by[7]

$$Z^I \; = \; (\omega^\alpha, \theta^i, \pi_{\dot{\alpha}}) \; = \; (\omega^A, \pi_{\dot{\alpha}}) \; = \; (Z^a, \theta^i)\,, \tag{3.1}$$

[7] We could take a finite deformation of any curved integrable twistor space, but would then need more coordinate patches.

the latter form distinguishes between the odd, θ^i and the even, Z^a coordinates. We also assume that we are given an 'infinity twistor' a constant graded skew bi-vector

$$I^{IJ} := \mathrm{diag}(P^{AB}, \Lambda\epsilon_{\dot\alpha\dot\beta}), \tag{3.2a}$$

where

$$P^{AB} = \mathrm{diag}(\epsilon^{\alpha\beta}, P^{ij}) \quad \text{and} \quad P^{ij} = P^{(ij)}. \tag{3.2b}$$

When $\Lambda = 0$, we will take P^{ij} to be diagonal with r ones and $\mathcal{N} - r$ zeroes along the diagonal.

We also introduce the graded Poisson structure on homogeneous functions f and g by

$$[f, g\} := (-)^{p_I(p_f+1)}(\partial_I f)I^{IJ}(\partial_J g), \tag{3.3a}$$

where we introduce the notation

$$\partial_I := \frac{\partial}{\partial Z^I}, \qquad \text{and we will also use} \qquad \bar\partial_{\bar I} := \frac{\partial}{\partial \bar Z^{\bar I}}. \tag{3.3b}$$

Infinitesimally, a deformation of the almost complex structure is represented by a holomorphic tangent bundle valued (0, 1)-form j, where the deformed and undeformed anti-holomorphic exterior derivatives are related by $\bar\partial = \bar\partial_0 + j$. The first order part of the integrability condition (assuming that $\bar\partial_0^2 = 0$) is $\bar\partial_0 j = 0$. An infinitesimal diffeomorphism induced by the real part of a $(1, 0)$-vector field X gives rise to the deformation $j := -\bar\partial_0 X$, so that the infinitesimal deformations of the complex structure modulo those obtained by infinitesimal diffeomorphisms define an element of the Dolbeault cohomology group $H^1(\mathbb{PT}', T^{(1,0)}\mathbb{PT})$.

In order to impose the Einstein or vacuum conditions, we will also demand that the deformation preserves the Poisson structure $\Pi = -I^{JI}\partial_I \wedge \partial_J$ of weight -2. In this linearised context, we can ensure this by requiring that the deforming vector fields j preserve the Poisson structure $\mathcal{L}_j\Pi = 0$, where $\mathcal{L}$ is the Lie derivative. This will follow if j is Hamiltonian with respect to Π, i.e., if there exists a (0, 1)-form h of weight 2 such that

$$j = \Pi \lrcorner \mathrm{d}h = (-)^{p_I}(\partial_I h)I^{IJ}\partial_J. \tag{3.4}$$

If $h = \bar\partial\chi$ we see that j is $\bar\partial(\Pi(\chi))$ and so is pure gauge. Thus such deformations correspond to h taken to be Dolbeault representatives for elements of $H^1(\mathbb{PT}', \mathcal{O}(2))$. The Penrose transform gives the identification between elements of $H^1(\mathbb{PT}', \mathcal{O}(2))$ and linearised self-dual gravitational fields, [P68,P76] and in the supersymmetric case this will give the whole associated linearised gravitational supermultiplet.

We now consider a *finite* deformation, again determined by $h = \mathrm{d}\bar Z^{\bar a} h_{\bar a}$ which, at this stage, is an arbitrary (even) smooth function of $(Z^I, \bar Z^{\bar a})$ homogeneous of degree 2 in Z^I and 0 in $\bar Z^{\bar I}$, holomorphic in the θ^is and satisfies $\bar Z^{\bar a} h_{\bar a} = 0$; we will never allow any dependence on the complex conjugates of the fermionic cooordinates.

We then define the distribution $T^{(0,1)}\mathscr{P}\mathscr{T}$ of anti-holomorphic tangent vectors on $\mathscr{P}\mathscr{T}$ by

$$T^{(0,1)}\mathscr{P}\mathscr{T} := \mathrm{span}\{\bar D_{\bar I}\} := \mathrm{span}\left\{\bar\partial_{\bar a} + (-)^{p_I}(\partial_I h_{\bar a})I^{IJ}\partial_J, \bar\partial_{\bar i}\right\}. \tag{3.5}$$

This is to be understood as a finite perturbation of the standard complex structure on flat supertwistor space with $\bar\partial$-operator $\bar\partial_0 = \mathrm{d}\bar Z^{\bar I}\bar\partial_{\bar I}$.[8] The complex structure can be equivalently determined by specifying the space of (1,0)-forms

$$\Omega^{(1,0)}\mathscr{P}\mathscr{T} := \mathrm{span}\{DZ^I\} := \mathrm{span}\{\mathrm{d}Z^I + I^{IJ}\partial_J h\}. \tag{3.6}$$

The integrability condition for this distribution is

$$I^{IJ}\partial_J\left(\bar\partial_{\bar a}h_{\bar b} - \bar\partial_{\bar b}h_{\bar a} + [h_{\bar a}\,,h_{\bar b}\}\right) = 0 \iff I^{IJ}\partial_J\left(\bar\partial_0 h + \tfrac{1}{2}[h,h\}\right) = 0, \tag{3.7}$$

where the wedge product in the last expression is understood. When this equation is satisfied, not only is the almost complex structure integrable, but also the Poisson bracket of two holomorphic functions is again holomorphic. In the case that $\Lambda = 0$, when the Poisson structure is degenerate, the coordinates $\pi_{\dot\alpha}$ and $\theta^{r+1},\ldots,\theta^{\mathcal{N}}$ are holomorphic and define a projection to $\mathbb{CP}^{1|\mathcal{N}-r}$ as required for the characterization of a twistor space for a self-dual vacuum solution. Thus, in this case, Eq. (3.7) is the main field equation.

In the Einstein case, we must produce a holomorphic contact structure. On the flat twistor space, introduce the contact structure

$$\tau_0 = \mathrm{d}Z^I Z^J I_{JI}, \tag{3.8a}$$

where

$$(-)^{p_K} I_{IK}I^{KJ} = \Lambda\delta_I{}^J \quad\text{and}\quad I_{IJ} = \mathrm{diag}(\Lambda P_{AB}, \epsilon^{\dot\alpha\dot\beta}). \tag{3.8b}$$

For the Einstein case, from Thm. 2, we need to know that we have a holomorphic contact structure on the deformed space. The deformed one can be taken to be

$$\tau := DZ^I Z^J I_{JI} = \mathrm{d}Z^I Z^J\omega_{JI} + Z^J\underbrace{(-)^{p_I}I_{JI}I^{IK}}_{=\Lambda\delta_J{}^K}\partial_K h = \tau_0 + 2\Lambda h, \tag{3.9}$$

where the last equation follows from the homogeneity relation $Z^I\partial_I h = 2h$. The condition that $\bar\partial\tau = 0 \Leftrightarrow \bar D_{\bar I} \lrcorner\, \mathrm{d}\tau = 0$ is

$$F^{(0,2)} := \bar\partial_0 h + \tfrac{1}{2}[h,h\} = 0. \tag{3.10}$$

Thus, integrability of the complex structure follows from the holomorphy of the contact structure when $\Lambda \neq 0$. (When $\Lambda = 0$, τ_0 remains holomorphic trivially.) Thus, not only is (3.10) our main equation in the Einstein case, it also implies (3.7) in the other cases, and so we will focus on this as the main equation in what follows.

The choice of the Poisson structure reduces the diffeomorphism freedom to (infinitesimal) Hamiltonian coordinate transformations of the form

$$\delta Z^I = [Z^I,\chi\} \rightsquigarrow h \mapsto h + \delta h, \quad\text{with}\quad \delta h = \bar\partial_0\chi + [h,\chi\}, \tag{3.11}$$

where χ is some smooth function of weight 2. Under this transformation, the 'curvature' $F^{(0,2)}$ behaves as $F^{(0,2)} \mapsto F^{(0,2)} + \delta F^{(0,2)}$ with $\delta F^{(0,2)} = [F^{(0,2)},\chi\}$. Thus, the field equation (3.10) is invariant under these transformations.

[8] As in the linearised context, we eventually want to impose the Einstein condition on the space-time manifold. Therefore, we are only interested in a subclass of (finite) deformations $\bar\partial_0 \mapsto \bar\partial_0 + j$ with j given by $j = \mathrm{d}\bar Z^{\bar a} j_{\bar a}{}^I\partial_I = \mathrm{d}\bar Z^{\bar a}(-)^{p_I}\partial_I h_{\bar a} I^{IJ}\partial_J$.

We can see that, at least in linear theory, h encodes a supergravity multiplet as follows. The form h may be expanded in the odd coordinates as

$$h = h_0 + \sum_{r=1}^{\mathcal{N}} \tfrac{1}{r!}\,\theta^{i_1}\cdots\theta^{i_r} h_{i_1\cdots i_r}\,. \tag{3.12}$$

If we further linearise (3.10) around the trivial solution $h = 0$, it tells us that $\bar\partial_0 h = 0$, or equivalently, $\bar\partial_0 h_0 = 0 = \bar\partial_0 h_{i_1\cdots i_r}$. Because of the gauge invariance (3.11), which at the linearised level reduces to $\delta h = \bar\partial_0\chi$, we see that $h_0 \in H^1(PT, \mathcal{O}(2))$ and $h_{i_1\cdots i_r} \in H^1(PT, \mathcal{O}(2-r))$, where PT represents the body of the supermanifold $\mathscr{P}\mathscr{T}$ (so that PT is a finite deformation of $\mathbb{PT}'_{[0]}$). By virtue of the Penrose transform, [P68], h_0 corresponds on space-time to a helicity $s = 2$ field while $h_{i_1\cdots i_r}$ to a helicity $s = (4-r)/2$ field. Hence, for maximal $\mathcal{N} = 8$ supersymmetry, we find $(s_m) = (-2_1, -\frac{3}{2}{}_8, -1_{28}, -\frac{1}{2}{}_{56}, 0_{70}, \frac{1}{2}{}_{56}, 1_{28}, \frac{3}{2}{}_8, 2_1)$ which is precisely the (on-shell) spectrum of $\mathcal{N} = 8$ Einstein supergravity; the subscript 'm' refers to the respective multiplicity. Altogether, we see that a single element $h \in H^1(\mathbb{PT}, \mathcal{O}(2))$ encodes the full particle content of maximally supersymmetric linearised Einstein gravity in four dimensions.

In this linearised context, it is straightforward to see how the gauging works. The bundle of R-symmetry generators on twistor space is the tangent bundle to the odd directions spanned by $\partial/\partial\theta^i$. The linearised variation in the $\bar\partial$-operator on this bundle is $P^{ik}\partial^2 h/\partial\theta^j\partial\theta^k$ because the part of $\bar\partial f^i\partial/\partial\theta^i$ tangent to the odd directions is $(\bar\partial f^i + P^{ik}\partial^2 h/\partial\theta^j\partial\theta^k f_j)\partial/\partial\theta^i$. Because θ^i anti-commute, $\partial^2 h/\partial\theta^i\partial\theta^j$ is skew symmetric in ij. Thus, in the case of non-degenerate P^{ij}, this gives an element of the Lie algebra of $SO(\mathcal{N}, \mathbb{C})$, and so corresponds to the maximal gauging of the R-symmetry, with gauge group $SO(\mathcal{N}, \mathbb{C})$. When P^{ij} has rank r, for $r < \mathcal{N}$, the gauging of the R-symmetry will be reduced to the subgroup of $SO(\mathcal{N}, \mathbb{C})$ that preserves P^{ij}.

In Appendix 5, where we compare our approach with that of [KK98], we also make some comments on the space-time fields in the non-linearised setting for zero cosmological constant.

3.2. Action functionals. We will be interested in integrating Lagrangian densities over twistor space for which we will need the holomorphic volume *integral* form

$$\Omega_{\mathcal{N}} = D(DZ^I) = \tfrac{1}{4!}\epsilon_{abcd} Z^a DZ^b \wedge DZ^c \wedge DZ^d \otimes \prod_{i=1}^{\mathcal{N}} D\theta^i\,, \tag{3.13}$$

which has weight $4 - \mathcal{N}$ on account of the Berezinian integration rule $\int \mathrm{d}\theta^i\theta^j = \delta^{ij}$ implying $\mathrm{d}(\lambda\theta^i) = \lambda^{-1}\mathrm{d}\theta^i$ for $\lambda \in \mathbb{C}^*$. Here, we use Manin's notation [M88] to denote integral forms associated with a given basis of differential one-forms. We will not integrate over any complex conjugated odd coordinates.

For maximal supersymmetry, $\mathcal{N} = 8$, we can write down an action functional reproducing the field equations (3.10) and hence also (3.7),

$$\begin{aligned} S[h] &= \int \Omega_8 \wedge \big(h \wedge \bar\partial_0 h + \tfrac{1}{3} h \wedge [h, h\}\big) \\ &= \int \Omega_8^{(0)} \wedge \big(h \wedge \bar\partial_0 h + \tfrac{1}{3} h \wedge [h, h\}\big), \end{aligned} \tag{3.14}$$

where the integral form

$$\Omega_8^{(0)} = D(\mathrm{d}Z^I) = \tfrac{1}{4!}\epsilon_{abcd}Z^a\mathrm{d}Z^b \wedge \mathrm{d}Z^c \wedge \mathrm{d}Z^d \otimes \prod_{i=1}^{8}\mathrm{d}\theta^i. \tag{3.15}$$

It can be seen that the weights balance as h has weight 2, $[\cdot,\cdot\}$ weight -2 and Ω_8 (respectively, $\Omega_8^{(0)}$) has weight -4. This is the only value of $\mathcal{N}$ for which there is such a balance.

The action (3.14) is invariant under (3.11). This follows from the Bianchi identity for $F^{(0,2)}$,

$$\bar{\partial}_0 F^{(0,2)} + [h, F^{(0,2)}\} = 0, \tag{3.16}$$

implied by the (graded) Jacobi identity for the Poisson structure.

It is clear that the almost complex structure, integrability conditions and action formulation (the latter for $\mathcal{N} = 8$) only depend on the Poisson structure I^{IJ} and not on I_{IJ} directly. It is also clear that if I^{IJ} is degenerate, the above field equations and action (the latter for $\mathcal{N} = 8$) all make good sense, although the action most directly yields (3.10) rather than the superficially weaker Eq. (3.7), that is sufficient to determine the relevant structures on the deformed twistor space.

The action (3.14) can be compared with the Kodaira-Spencer actions introduced in [BCOV94], the compendium of topological M-theory related actions in [DGNV05] and the Lagrange multiplier-type action involving the Nijenhuis tensor given in [BW04] in the $\mathcal{N} = 4$ case. Our action is local in contra-distinction with the non-local Kodaira-Spencer action. Our action is given for a non-Calabi Yau space (due the isomorphism (B.5), the holomorphic Berezinian is only trivial when $\mathcal{N} = 4$). Ours is most closely related to that in Berkovits & Witten, although our basic variable, the one-form h which is a "potential" for the deformation j, considered in deformation theory (i.e. j is a holomorphic derivative of h) and is most naturally expressed for $\mathcal{N} = 8$ rather than $\mathcal{N} = 4$.

We close this subsection by discussing the cases with $\mathcal{N} < 8$ supersymmetries. We start from the action (3.14) with $\mathcal{N} = 8$ but restrict the dependence of h on θ^i by requiring invariance under an $SO(8-\mathcal{N}, \mathbb{C})$ subgroup of the R-symmetry. Thus, we set

$$h = f + \theta^{\mathcal{N}+1}\cdots\theta^8\, b, \tag{3.17}$$

where f and b are now one forms depending on the bosonic twistor coordinates and $\theta^1, \ldots, \theta^{\mathcal{N}}$, f has weight 2, and b has weight $\mathcal{N} - 6$. We can now integrate out the anti-commuting variables $\theta^{\mathcal{N}+1}, \ldots, \theta^8$ and integrate by parts to obtain the action

$$S[b, f] = \int \Omega_r \wedge b \wedge \left(\bar{\partial}_0 f + \tfrac{1}{2}[f, f\}\right). \tag{3.18}$$

This action is now of 'BF' form where b acts as a Lagrange multiplier for the field equation

$$\bar{\partial}_0 f + \tfrac{1}{2}[f, f\} = 0. \tag{3.19}$$

which, as we have seen, implies that integrability of the complex structure is compatible with a holomorphic Poisson structure. Varying f yields the equation

$$\bar{\partial}_f b = 0 \tag{3.20}$$

and, together with the gauge freedom $b \mapsto b + \bar{\partial}_f \chi$, this implies that b defines an element of the cohomology group $H^1(\mathscr{P}\mathscr{T}, \mathscr{O}(\mathcal{N}-6))$ and so is the Penrose transform of a superfield of helicity $-2 + \mathcal{N}/2$.

4. Covariant Approach, Covariant Action for $\mathcal{N}=0$ and Special Geometry

The above actions are non-covariant in the sense that they explicitly depend on the chosen background one has started with so that diffeomorphism invariance is broken. This is normal in the context of Chern-Simons actions for which a frame of the Yang-Mills bundle must be chosen. Nevertheless, we will see that at least for τ non-degenerate and $\mathcal{N}=0$ we can give a covariant version.

The geometric structure we are concerned with here is closely related to a (real) six-dimensional special geometry introduced by [CE03]. In their geometry, a real rank-4 distribution (subbundle of the tangent bundle) D is introduced and, if suitably non-degenerate and satisfying a positivity condition, it is shown that there is a canonically defined almost complex structure J for which the distribution is an almost complex contact distribution. Furthermore, the obstruction to the integrability of J is identified. Our situation is somewhat different in that the primary structure on a smooth manifold, P, is a complex one-form τ defined up to complex rescalings (or more abstractly, a complex line bundle $L^* \subset \mathbb{C}T^*P := \mathbb{C} \otimes T^*P$). This is more information in the sense that D is defined directly as the kernel of τ, but τ is only defined by D up to $\tau \mapsto a\tau + b\bar{\tau}$, where a, b are complex valued functions on P. Given D, there is a unique choice of τ that is compatible with the Cap-Eastwood almost complex structure but a priori, one does not know if that is the τ that has been chosen. Our analogue of the Cap-Eastwood theorem works in higher dimensions also and we state it in greater generality than we need.

Theorem 3. *Suppose that on a* (*smooth*) *manifold P of dimension $4n+2$ we are given a complex line subbundle $L^* \subset \mathbb{C}T^*P$, represented by a complex one-form τ defined up to complex rescalings. Suppose further that*

$$\tau \wedge (\mathrm{d}\tau)^{n+1} = 0 \quad \text{and} \quad \tau \wedge (\mathrm{d}\tau)^n \wedge \bar{\tau} \wedge (\mathrm{d}\bar{\tau})^n \neq 0,$$

then there is a unique integrable almost complex structure for which τ is proportional to a non-degenerate holomorphic contact structure. Here, $(\mathrm{d}\tau)^n := \mathrm{d}\tau \wedge \cdots \wedge \mathrm{d}\tau$ (*n-times*).

Proof. We claim that, with the assumptions above, the $(2n+1)$-form $\tau \wedge (\mathrm{d}\tau)^n$ is simple, i.e., that the space of vectors $X \in \Gamma(P, \mathbb{C}TP)$ such that $X \lrcorner (\tau \wedge \mathrm{d}\tau) = 0$ is $(2n+1)$-dimensional. This follows because the kernel of τ is $(4n+1)$-dimensional, whereas $\mathrm{d}\tau$ defines a skew form on this kernel and so must have even rank. However, its rank is less than $2n+2$ by $\tau \wedge (\mathrm{d}\tau)^{n+1} = 0$ but greater than or equal to $2n$ because $\tau \wedge (\mathrm{d}\tau)^n \neq 0$. Hence, the kernel of $\tau \wedge (\mathrm{d}\tau)^n$ is $(2n+1)$-dimensional and we will take this kernel to be the space of anti-holomorphic tangent vectors spanning $T^{(0,1)}P$. The condition that $T^{(0,1)}P$ should contain no real vectors follows from the second assumption of the theorem.

We have that $X \lrcorner (\tau \wedge (\mathrm{d}\tau)^n) = 0 \Leftrightarrow X \lrcorner (\tau \wedge \mathrm{d}\tau) = 0$ and we will use this latter characterisation of $T^{(0,1)}P$ in the following.

We now consider the integrability of the distribution. Let X and Y satisfy

$$X \lrcorner (\tau \wedge \mathrm{d}\tau) = 0 = Y \lrcorner (\tau \wedge \mathrm{d}\tau). \tag{4.1}$$

Then clearly $X \lrcorner \tau = 0 = Y \lrcorner \tau$ and

$$\tau \wedge (X \lrcorner \mathrm{d}\tau) = 0\,, \tag{4.2}$$

so that $X \lrcorner \mathrm{d}\tau \propto \tau$ and $\mathcal{L}_X \tau \propto \tau$, and similarly for Y. Here, $\mathcal{L}_X$ denotes the Lie derivative along X. Thus,

$$[X, Y] \lrcorner \tau = X(Y \lrcorner \tau) - Y(X \lrcorner \tau) - X \lrcorner (Y \lrcorner \mathrm{d}\tau) = 0\,, \tag{4.3}$$

since $X \lrcorner \tau = 0 = Y \lrcorner \tau$ by assumption and so $X \lrcorner (Y \lrcorner \mathrm{d}\tau) = 0$ from above. Furthermore,

$$\begin{aligned} [X, Y] \lrcorner (\tau \wedge \mathrm{d}\tau) &= -\tau \wedge ([X, Y] \lrcorner \mathrm{d}\tau) \\ &= -\tau \wedge ([X, Y] \lrcorner \mathrm{d}\tau + \mathrm{d}([X, Y] \lrcorner \tau) \\ &= -\tau \wedge (\mathcal{L}_{[X,Y]}\tau) = -\tau \wedge (\mathcal{L}_X \mathcal{L}_Y \tau - \mathcal{L}_Y \mathcal{L}_X \tau) = 0\,, \end{aligned} \tag{4.4}$$

since $\mathcal{L}_X \tau = X \lrcorner \mathrm{d}\tau \propto \tau$, so $\mathcal{L}_X \mathcal{L}_Y \tau \propto \tau$.

Thus, the almost complex structure is integrable. □

In the twistor context, we will take P to be a six-dimensional manifold with topology $U \times S^2$ with $U \subset \mathbb{R}^4$ and, as before, we shall denote it by PT. With this theorem, then, our data is simply a complex line subbundle $L^* \subset \mathbb{C}T^*PT$ represented by a differential one-form τ with values in L subject to the open condition $\tau \wedge \mathrm{d}\tau \wedge \bar{\tau} \wedge \mathrm{d}\bar{\tau} \neq 0$. We will also require that the line bundle L has Chern class 2. The field equation is $\tau \wedge (\mathrm{d}\tau)^2 = 0$. The $\mathcal{N} = 0$ action above is simply

$$S[b, \tau] = \int b \wedge \tau \wedge (\mathrm{d}\tau)^2\,, \tag{4.5}$$

where $b \in \Omega^1 PT \otimes (L^*)^3$ is a Lagrange multiplier. Clearly, the field equation obtained by varying b is $\tau \wedge (\mathrm{d}\tau)^2 = 0$, as desired. The action is clearly diffeomorphism invariant, and enjoys a gauge invariance given by $\tau \mapsto \chi\tau$ and $b \mapsto \chi^{-3} b$, where χ is a non-vanishing complex-valued function on PT. This gauge freedom corresponds to the fact that τ takes values in a line bundle L which we shall also denote by $\mathcal{O}(2)$ since it becomes that on-shell, and hence b is a differential one-form with values in $\mathcal{O}(-6)$.

The action is also invariant under $b \mapsto b+\gamma$, where $\gamma \wedge \tau \wedge (\mathrm{d}\tau)^2 = 0$, and the space of such γ is two-dimensional when the field equations are not satisfied, but three-dimensional when they are. (When they are satisfied, this freedom can be used to ensure that b is a (0, 1)-form.) There is also a gauge freedom in b obtained as follows. We can define a partial connection $\bar{\partial}$ on $\mathcal{O}(n)$ by defining for χ, now assumed to be a section of $\mathcal{O}(-6)$, $\bar{\partial}\chi$ to be the differential one-form modulo the kernel of $\bar{\partial}\chi \mapsto \bar{\partial}\chi \wedge \tau \wedge (\mathrm{d}\tau)^2$ defined by $\bar{\partial}\chi \wedge \tau \wedge (\mathrm{d}\tau)^2 := \mathrm{d}(\chi\tau \wedge (\mathrm{d}\tau)^2)$. It is clear from this definition that the integrand of the action evaluated on such a $b = \bar{\partial}\chi$ is a boundary integral and so this represents a gauge freedom. On-shell, the above definition becomes trivial, and $\bar{\partial}\chi$ needs to be defined a little differently by $\bar{\partial}\chi^{2/3} \wedge (\tau \wedge \mathrm{d}\tau) := \mathrm{d}(\chi^{2/3}\tau \wedge \mathrm{d}\tau)$, and in this case it leads to an honest $\bar{\partial}$-operator on the line bundles $\mathcal{O}(n)$.

The field equation for b is

$$\mathrm{d}b \wedge \tau \wedge \mathrm{d}\tau - \tfrac{3}{2} b \wedge (\mathrm{d}\tau)^2 = 0 \tag{4.6}$$

and when the field equation for τ is satisfied, this is the $\bar\partial$-closure condition for sections of $\Omega^{(0,1)}PT \otimes \mathscr{O}(-6)$. Taking into account the gauge freedom $b \mapsto b + \bar\partial\chi$ with χ a section of $\mathscr{O}(-6)$, b will correspond to an element of $H^1(PT, \mathscr{O}(-6))$.

Thus, solutions to the field equations correspond to a complex three-dimensional manifold PT with holomorphic contact structure τ, and the condition on the Chern class of L implies that it satisfies the topological assumption of Ward's theorem, so that, if it contains a holomorphic rational curve of degree one in the S^2-factor, then it corresponds to a space-time M with self-dual Einstein metric. The field $b \in H^1(PT, \mathscr{O}(-6))$ then corresponds via the Penrose transform to a right-handed linearised gravitational field propagating on that self-dual background. Thus, we have the self-dual sector of non-supersymmetric Einstein gravity.

4.1. The supersymmetric case. In the supersymmetric situation, we will assume that $\mathscr{PT}$ is a smooth supermanifold with six real bosonic dimensions and $\mathcal{N}$ complex fermionic dimensions. Without loss of generality, we can always assume that the supermanifold is split in the smooth category [B79], and that locally the odd coordinates are θ^i, $i = 1, \dots, \mathcal{N}$, and that we will only ever have holomorphic dependence on θ^i, their complex conjugates will not enter the formalism, so, in particular, the transition functions for the supermanifold will be holomorphic in θ^i.[9] We can still encode the structure of a supersymmetric non-linear graviton into a complex contact form τ as follows. We will assume that τ is a complex differential one-form on the supermanifold $\mathscr{PT}$, again with only holomorphic dependence on the θ^i, i.e., $\tau = \mathrm{d}x^a\tau_a + \mathrm{d}\theta^i\tau_i$, where the x^as are the real bosonic coordinates on $\mathscr{PT}$, $a = 1, \dots, 6$, and τ_a and τ_i are holomorphic in θ^i with τ_i odd and τ_a even functions on $\mathscr{PT}$. On the body of the supermanifold, $\theta^i = 0$, we can assume that we have the equations $\tau \wedge (\mathrm{d}\tau)^2 = 0$ as before, but these will not hold when $\theta^i \neq 0$, even for standard flat supertwistor space as, in general, $(\mathrm{d}\theta)^n \neq 0\,\forall\, n$ for an odd variable θ. Thus, we cannot express the conditions we need quite so simply in the supersymmetric case.

Nevertheless, much of Thm. 3 works in the supersymmetric case also. We will require firstly, as a genericity assumption, that the complexified kernel $\mathbb{C}\mathscr{D}$ of τ has dimension $5|2\mathcal{N}$ (here we are taking $\partial/\partial\theta^i$ and $\partial/\partial\bar\theta^{\bar i}$ to be independent). Secondly, we require that on this complexified kernel of τ, the two form $\mathrm{d}\tau$ has rank $2|\mathcal{N}$ so that the kernel of $\tau \wedge \mathrm{d}\tau$ is $3|\mathcal{N}$-dimensional and further, that $\ker(\tau \wedge \mathrm{d}\tau)$ has no real vectors, i.e.

$$\ker(\tau \wedge \mathrm{d}\tau) \cap \overline{\ker(\tau \wedge \mathrm{d}\tau)} \;=\; \{0\}. \tag{4.7}$$

The fact that we have required that τ depends only on θ^i and not $\bar\theta^{\bar i}$ means that $\mathrm{d}\tau$ annihilates $\partial/\partial\bar\theta^{\bar i}$, for $i = 1, \dots, \mathcal{N}$ and so the rank of $\mathrm{d}\tau$ is at most $5|\mathcal{N}$ in any case. With these assumptions, the proof of Thm. 3 follows without modification to show that $\ker(\tau \wedge \mathrm{d}\tau)$ is integrable and that τ is a holomorphic complex contact structure so that

$$T^{(0,1)}\mathscr{PT} \;:=\; \ker(\tau \wedge \mathrm{d}\tau). \tag{4.8}$$

The main field equation is therefore the condition that $\tau \wedge \mathrm{d}\tau$ annihilates a complex distribution of dimension $3|\mathcal{N}$. In the supersymmetric context, we do not yet have an equation on τ analogous to the bosonic equation $\tau \wedge (\mathrm{d}\tau)^{n+1} = 0$ for higher dimensional complex contact structures nor an action that produces this condition as its Euler-Lagrange equation. As a consequence, we have so far been unable to find a covariant supersymmetric action functional.

[9] In a Dolbeault context, this assumption is, in effect a gauge choice.

5. Conclusions

Given that these actions are 'Chern-Simons-like' one is led to ask the extent to which they can be interpreted coherently as holomorphic Chern-Simons theories. Clearly, in some sense, the gauge group should be taken to be the diffeomorphisms of the supertwistor space that preserve the holomorphic Poisson structure. This is most easily made sense of in a complexified context so that the holomorphic twistor variables are freed up and become independent from the conjugate twistor variables. Then the theory becomes a complexified Chern-Simons theory with gauge group the holomorphic contact transformations of the holomorphic supertwistor space, a region in $\mathbb{CP}^{3|8}$, on the conjugate supertwistor space (which is just $\mathbb{CP}^3$ as we have no anti-holomorphic fermionic coordinates). A similar connection between the self-dual vacuum equations and a gauge theory with a diffeomorphism group gauge group was given on space-time in [MN89] (here the gauge theory was the self-dual Yang-Mills equations); see also [W07] for a supersymmetric extension thereof.

The fact that Thm. 3 works in $4n+2$ dimensions is suggestive of applications of this framework to the twistor theory for quaternionic Kähler manifolds with non-zero scalar curvature in $4n$ dimensions. It is straightforward to write down a Lagrange multiplier action $\int b \wedge \tau \wedge (\mathrm{d}\tau)^{n+1}$ analogous to our $\mathcal{N}=0$ action, but with b a $(2n-1)$-form, although in this context the interpretation of b is less clear.

An attractive feature is that we have a fully supersymmetrically invariant and Lorentz invariant off-shell formulation of the theory. However, we have so far been unable to find an action functional of $\mathcal{N}=8$ self-dual supergravity that does not depend on a given integrable background. Such an action functional would, however, be desirable as one would hope for an explicitly diffeomorphism invariant action principle for $\mathcal{N}=8$ self-dual supergravity. In particular, if one wishes to be able to extend the ideas to the full theory along the lines of [M05] for conformal supergravity,[10] then it would seem awkward to have to identify a Minkowski background.

A task for the future is to start with the superfield expansions (in the non-linear setting) of τ and h and reproduce the covariant form of the field equations and of the action functional of $\mathcal{N}=8$ self-dual supergravity in four dimensions as given in [S92].[11] In the zero cosmological constant case, our twistor action and field equations must correspond via the Penrose transform to Siegel's results.

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Appendix A. Prepotential Formulation

The subject of this appendix is the comparison of [KK98] approach with ours. Their formulation is based on an anti-holomorphic involution which picks a real slice in complexified space-time being of split signature. Pretty much the same holds true, however,

[10] See also [A-ZH06] for a space-time action for expanding about the self-dual sector in the case of Einstein gravity.

[11] Similar expansions for certain supersymmetric gauge theories were performed in [PW04,PS05], Sämann (2005), [PSW05 and LS06].

for Euclidean signature and it is this latter case we are interested in here. As already indicated, this works only for an even number of supersymmetries. In the following, we shall use conventions from [W06].

A.1. Real structures on $\mathbb{PT}'_{[\mathcal{N}]}$ *and* $\mathbb{M}_{[\mathcal{N}]}$. Let us first consider the supertwistor space $\mathbb{PT}'_{[\mathcal{N}]} = \mathbb{CP}^{3|\mathcal{N}} \setminus \mathbb{CP}^{1|\mathcal{N}}$ with (homogeneous) coordinates $(\omega^A, \pi_{\dot\alpha})$ for flat super spacetime $\mathbb{M}_{[\mathcal{N}]} \cong \mathbb{C}^{4|2\mathcal{N}}$. An Euclidean signature real slice follows from the anti-holomorphic involution without fixed points $\rho : \mathbb{PT}'_{[\mathcal{N}]} \to \mathbb{PT}'_{[\mathcal{N}]}$ given by

$$(\hat\omega^A, \hat\pi_{\dot\alpha}) := \rho(\omega^A, \pi_{\dot\alpha}) := (\bar\omega^B C_B{}^A, C_{\dot\alpha}{}^{\dot\beta}\bar\pi_{\dot\beta}), \tag{A.1}$$

where bar denotes complex conjugation and $(C_A{}^B) = \mathrm{diag}((C_\alpha{}^\beta), (C_i{}^j))$, with

$$(C_\alpha{}^\beta) = \epsilon, \quad (C_i{}^j) = \mathrm{diag}(\underbrace{\epsilon, \ldots, \epsilon}_{\frac{\mathcal{N}}{2}-\text{times}}), \quad (C_{\dot\alpha}{}^{\dot\beta}) = -\epsilon, \quad \epsilon := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A.2}$$

We can extend ρ to a map from a holomorphic function f on $\mathbb{PT}'_{[\mathcal{N}]}$ another holomorphic function by

$$\rho(f(\cdots)) := \overline{f(\rho(\cdots))}. \tag{A.3}$$

By virtue of the incidence relation, $\omega^A = x^{A\dot\alpha}\pi_{\dot\alpha}$, we obtain an induced involution on $\mathbb{M}_{[\mathcal{N}]}$ explicitly given by

$$\rho(x^{A\dot\alpha}) = -\bar x^{B\dot\beta} C_B{}^A C_{\dot\beta}{}^{\dot\alpha}. \tag{A.4}$$

We shall use the same notation ρ for the anti-holomorphic involution induced on the different (super)manifolds in the twistor correspondence. The fixed point set of this involution, that is, $\rho(x) = x$ for $x \in \mathbb{M}_{[\mathcal{N}]}$, defines Euclidean right-chiral superspace $\mathbb{M}^\rho_{[\mathcal{N}]} \cong \mathbb{R}^{4|2\mathcal{N}}$ inside $\mathbb{M}_{[\mathcal{N}]}$.

Following [AHS78], the supertwistor space $\mathbb{PT}'_{[\mathcal{N}]}$ can be identified with

$$\mathcal{O}(1)^{\oplus 2|\mathcal{N}} \to \mathbb{CP}^1 \tag{A.5}$$

and so it can be covered by two (acyclic) coordinate patches $\mathscr{U}_\pm$ and coordinatised by $(\omega^A_\pm, \pi_\pm)$, where $\omega^A_\pm$ are local fibre coordinates with $\omega^A_+ := \omega^A/\pi_{\dot 0}$, $\omega^A_- := \omega^A/\pi_{\dot 1}$ and $\pi_+ := \pi_{\dot 1}/\pi_{\dot 0}$, $\pi_- := \pi_{\dot 0}/\pi_{\dot 1}$ are the standard local holomorphic coordinates on $\mathbb{CP}^1$, with $\pi_+ = \pi_-^{-1}$ on $\mathscr{U}_+ \cap \mathscr{U}_+ \subset \mathbb{PT}'_{[\mathcal{N}]}$. On the other hand, since $\mathbb{PT}'_{[\mathcal{N}]}$ is diffeomorphic to $\mathbb{M}^\rho_{[\mathcal{N}]} \times S^2 \cong \mathbb{R}^{4|2\mathcal{N}} \times S^2$, one may equivalently coordinatise it by using $(x^{A\dot\alpha}, \lambda_\pm)$, where $\lambda_\pm$ are the standard local holomorphic coordinates on $S^2 \cong \mathbb{CP}^1$. Note that $(\omega^A_\pm, \pi_\pm) = (x^{A\dot\alpha}\lambda^\pm_{\dot\alpha}, \lambda_\pm)$, where

$$(\lambda^{\dot\alpha}_+) := \begin{pmatrix} \lambda_+ \\ -1 \end{pmatrix} \quad \text{and} \quad (\lambda^{\dot\alpha}_-) := \begin{pmatrix} 1 \\ -\lambda_- \end{pmatrix}. \tag{A.6}$$

The explicit inverse transformation laws are simply

$$x^{A\dot{\alpha}} = \frac{\omega^A_\pm \hat{\pi}^{\dot{\alpha}}_\pm - \hat{\omega}^A_\pm \pi^{\dot{\alpha}}_\pm}{\hat{\pi}^{\dot{\beta}}_\pm \pi_{\pm\dot{\beta}}}, \tag{A.7}$$

where $\pi^{\dot{\alpha}}_\pm$ are similarly defined as in (A.6). Altogether, we have obtained a non-holomorphic fibration

$$\pi : \mathbb{PT}'_{[\mathcal{N}]} \to \mathbb{M}^\rho_{[\mathcal{N}]}. \tag{A.8}$$

Introduce

$$(\hat{\lambda}^{\dot{\alpha}}_+) := \begin{pmatrix} 1 \\ \bar{\lambda}_+ \end{pmatrix}, \quad (\hat{\lambda}^{\dot{\alpha}}_-) := \begin{pmatrix} \bar{\lambda}_- \\ 1 \end{pmatrix}, \quad \gamma^{-1}_\pm := \hat{\lambda}^{\dot{\alpha}}_\pm \lambda^\pm_{\dot{\alpha}} = 1 + \lambda_\pm \bar{\lambda}_\pm, \tag{A.9}$$

like for $\hat{\pi}^{\dot{\alpha}}_\pm = \rho(\pi^{\dot{\alpha}}_\pm)$. Then, due to the above diffeomorphism, we have the following transformation laws between the coordinate vector fields:

$$\frac{\partial}{\partial \omega^A_\pm} = \gamma_\pm \hat{\lambda}^{\dot{\alpha}}_\pm \frac{\partial}{\partial x^{A\dot{\alpha}}}, \tag{A.10a}$$

$$\frac{\partial}{\partial \pi_+} = \frac{\partial}{\partial \lambda_+} - \gamma_+ x^{A\dot{1}} \hat{\lambda}^{\dot{\alpha}}_+ \frac{\partial}{\partial x^{A\dot{\alpha}}}, \tag{A.10b}$$

$$\frac{\partial}{\partial \pi_-} = \frac{\partial}{\partial \lambda_-} - \gamma_- x^{A\dot{0}} \hat{\lambda}^{\dot{\alpha}}_- \frac{\partial}{\partial x^{A\dot{\alpha}}} \tag{A.10c}$$

for the holomorphic tangent vector fields and

$$\frac{\partial}{\partial \bar{\omega}^{\bar{A}}_\pm} = -\gamma_\pm C_A{}^B \lambda^{\dot{\alpha}}_\pm \frac{\partial}{\partial x^{B\dot{\alpha}}}, \tag{A.10d}$$

$$\frac{\partial}{\partial \bar{\pi}_+} = \frac{\partial}{\partial \bar{\lambda}_+} - \gamma_+ x^{A\dot{0}} \lambda^{\dot{\alpha}}_+ \frac{\partial}{\partial x^{A\dot{\alpha}}}, \tag{A.10e}$$

$$\frac{\partial}{\partial \bar{\pi}_-} = \frac{\partial}{\partial \bar{\lambda}_-} + \gamma_- x^{A\dot{1}} \lambda^{\dot{\alpha}}_- \frac{\partial}{\partial x^{A\dot{\alpha}}} \tag{A.10f}$$

for the anti-holomorphic ones.

A.2. Comparison of the two approaches. In what follows, we shall restrict our discussion to the $\mathscr{U}_+$-patch only and for notational simplicity suppress the patch index. Of course, a similar discussion carries over to the $\mathscr{U}_-$-patch.

To begin with, let us write down the field Eqs. (3.10) more explicitly. If we let the deformation be $h = \mathrm{d}\bar{\omega}^{\bar{\alpha}} h_{\bar{\alpha}} + \mathrm{d}\bar{\pi} h_{\bar{\pi}}$, they read as

$$\frac{\partial}{\partial \bar{\omega}^{\bar{\alpha}}} h_{\bar{\beta}} - \frac{\partial}{\partial \bar{\omega}^{\bar{\beta}}} h_{\bar{\alpha}} + [h_{\bar{\alpha}}, h_{\bar{\beta}}\} = 0, \tag{A.11a}$$

$$\frac{\partial}{\partial \bar{\pi}} h_{\bar{\alpha}} - \frac{\partial}{\partial \bar{\omega}^{\bar{\alpha}}} h_{\bar{\pi}} + [h_{\bar{\pi}}, h_{\bar{\alpha}}\} = 0. \tag{A.11b}$$

Using the incidence relation $\omega^A = x^{A\dot{\alpha}}\pi_{\dot{\alpha}}$ and the involutions introduced in the preceding subsection, h can also be expressed in the coordinates $(x^{A\dot{\alpha}}, \lambda)$ as

$$h = -\gamma\hat{\lambda}_{\dot{\beta}}\mathrm{d}x^{\alpha\dot{\beta}}\,\Phi_\alpha + \mathrm{d}\bar{\lambda}\,\Phi_{\bar{\lambda}}, \tag{A.12}$$

where $\Phi_\alpha := -\gamma^{-1}C_\alpha{}^\beta h_{\bar{\beta}}$ and $\Phi_{\bar{\lambda}} := h_{\bar{\pi}} + \gamma x^{\alpha\dot{0}}\Phi_\alpha$.

In order to compare our approach with those by [KK98], we notice that their formulation deals with the 'vacuum case', i.e. with the case of vanishing cosmological constant. Upon also recalling point (iii) of Thm. 2, we must therefore ensure that the fibration of the supertwistor space is preserved, and so (i) h is of the form

$$h = -\gamma\hat{\lambda}_{\dot{\beta}}\mathrm{d}x^{\alpha\dot{\beta}}\,\Phi_\alpha, \tag{A.13}$$

i.e. $\Phi_{\bar{\lambda}} = 0 \Leftrightarrow h_{\bar{\pi}} = -\gamma x^{\alpha\dot{0}}\Phi_\alpha$ and (ii) the relative symplectic structure needs to be preserved which amounts to requiring a degeneracy of the Poisson structure $\omega = (I^{IJ})$ introduced in Sect. 3.1 according to $\omega = (I^{AB})$. Notice further that Φ_α must be of weight 3 in order for h to be of weight 2.

Some algebra then reveals that in the 'vacuum case' the above equations for $h_{\bar{\alpha}}$ and $h_{\bar{\pi}}$ translate into the following set:

$$\epsilon^{\alpha\beta}\bar{\partial}_\alpha\Phi_\beta + \tfrac{1}{2}\epsilon^{\alpha\beta}[\Phi_\alpha, \Phi_\beta\} = 0, \tag{A.14a}$$

$$\partial_{\bar{\lambda}}\Phi_\alpha + \gamma^{-2}\epsilon^{\beta\gamma}(\partial_\beta\Phi_\alpha)\Phi_\gamma = 0, \tag{A.14b}$$

where $\bar{\partial}_A := \lambda^{\dot{\alpha}}\partial/\partial x^{A\dot{\alpha}}$ and $\partial_A := \gamma\hat{\lambda}^{\dot{\alpha}}\partial/\partial x^{A\dot{\alpha}}$.

Before going any further, let us say a few words about gauge symmetries. The original equations for h transformed covariantly under gauge transformations of the form $h \mapsto h + \delta h$, with $\delta h = \bar{\partial}_0\chi + [h, \chi\}$ for some function χ of weight 2. However, the above equations will no longer transform covariantly under generic gauge transformations, since we have incorporated the constraint $\Phi_{\bar{\lambda}} = 0$. Nevertheless, some residual gauge symmetry remains, and which is determined as follows. In order to preserve the constraint $\Phi_{\bar{\lambda}} = 0$, we must have $\delta h_{\bar{\pi}} = -\gamma x^{\alpha\dot{0}}\delta\Phi_\alpha$, where $\delta\Phi_\alpha = -\gamma^{-1}C_\alpha{}^\beta\delta h_{\bar{\beta}}$, i.e. transformations of $h_{\bar{\pi}}$ are determined by those of $h_{\bar{\alpha}}$. It is not difficult to verify that the remaining gauge symmetry is given by the following transformation laws:

$$\delta\Phi_\alpha = -(\bar{\partial}_\alpha\chi + [\Phi_\alpha, \chi\}), \qquad \text{with} \qquad \partial_{\bar{\lambda}}\chi + \gamma^{-2}\epsilon^{\beta\gamma}(\partial_\beta\chi)\Phi_\gamma = 0. \tag{A.15}$$

In particular, the last of these equations shows that the 2nd equation for Φ_α from above does not constrain Φ_α any further, so that the only remaining field equation we are left with is

$$\epsilon^{\alpha\beta}\bar{\partial}_\alpha\Phi_\beta + \tfrac{1}{2}\epsilon^{\alpha\beta}[\Phi_\alpha, \Phi_\beta\} = 0. \tag{A.16}$$

Since in particular $\Phi_\alpha = \partial_\alpha\Phi$ (see also [W85]), where Φ is some function of weight 4 (recall that Φ_α is of weight 3) and $\omega = (I^{AB})$, we end up with

$$\Box\Phi + \tfrac{1}{2}\epsilon^{\alpha\beta}(-)^{p_A}\partial_A\partial_\alpha\Phi\, I^{AB}\,\partial_B\partial_\beta\Phi = 0 \qquad \text{and} \qquad \Box := \epsilon^{\alpha\beta}\bar{\partial}_\alpha\partial_\beta, \tag{A.17}$$

which is [KK98] result.

As before, in the case of maximal supersymmetry, $\mathcal{N} = 8$, the field Eqs. (A.17) can be derived from an action principle,

$$S[\Phi] = \int \mathrm{d}\,\mathrm{vol}\left\{\Phi\Box\Phi + \tfrac{1}{3!}\epsilon^{\alpha\beta}(-)^{p_A}\Phi\partial_A\partial_\alpha\Phi\, I^{AB}\,\partial_B\partial_\beta\Phi\right\}, \tag{A.18a}$$

where the measure d vol is given by

$$\mathrm{d}\,\mathrm{vol} = \mathrm{d}^4x\,\gamma^2\mathrm{d}\lambda\mathrm{d}\bar{\lambda}\,\mathrm{d}\theta^1\cdots\mathrm{d}\theta^8. \tag{A.18b}$$

It remains to give the superfield expansion of Φ. For brevity, let us only discuss the $\mathcal{N} = 8$ case. We find

$$\begin{aligned}\Phi = g &+ \theta^i\psi_i + \theta^{i_1i_2}A_{[i_1i_2]} + \theta^{i_1i_2i_3}\chi_{i_1i_2i_3} + \theta^{i_1i_2i_3i_4}\phi_{i_1i_2i_3i_4}\\ &+ \theta_{i_1i_2i_3}\tilde{\chi}^{i_1i_2i_3} + \theta_{i_1i_2}\tilde{A}^{i_1i_2} + \theta_i\tilde{\psi}^i + \theta\tilde{g},\end{aligned} \tag{A.19}$$

where

$$\theta^{i_1\cdots i_r} := \tfrac{1}{r!}\theta^{i_1}\cdots\theta^{i_r}, \quad \text{for} \quad r = 1,\ldots,4, \tag{A.20a}$$

$$\theta_{i_1\cdots i_{8-r}} := \tfrac{1}{r!}\epsilon_{i_1\cdots i_{8-r}i_{9-r}\cdots i_8}\theta^{i_{9-r}}\cdots\theta^{i_8}, \quad \text{for} \quad r = 5,\ldots,8. \tag{A.20b}$$

Here, $\epsilon_{i_1\cdots i_8} = \epsilon_{[i_1\cdots i_8]}$ and $\epsilon_{1\ldots 8} = 1$. Keeping in mind (A.13), we find the following space-time fields:

Table 1. Space-time fields and their helicities and multiplicities

Field	g	ψ	A	χ	ϕ	$\tilde{\chi}$	$\tilde{A}$	$\tilde{\phi}$	$\tilde{g}$
Helicity	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-\frac{3}{2}$	-2
Multiplicity	1	8	28	56	70	56	28	8	1

Appendix B. Holomorphic Volume Forms and Non-Projective Twistor Space

It is often convenient to work on the non-projective twistor space $\mathbb{T}$ as many of the geometric structures can be formulated globally there and sections of the line bundles $\mathcal{O}(n)$ become ordinary functions of weight n under the action of the Euler vector field $\Upsilon = Z^I\partial/\partial Z^I$. In the curved case, as in the proof of Theorem 2, the non-projective space can be defined as the quotient of the non-projective co-spin bundle $\mathscr{S}^*$ by $\mathscr{D}_{\mathscr{F}}$. We can also define it intrinsically as follows.

In the bosonic case, given a contact structure defined by a one-form τ with values in a line bundle L, we can see that $\tau\wedge\mathrm{d}\tau$ defines a (non-vanishing) section of $\Omega^{(3,0)}PT\otimes L^2$. Thus, we must have $L^{-2}\cong\Omega^{(3,0)}PT$. In the flat case, non-projective twistor space $\mathbb{T}_{[0]}\cong\mathbb{C}^4$ is the total space of the (tautological) line bundle $\mathcal{O}(-1)$ over the projective twistor space $\mathbb{PT}'_{[0]}$, and $\Omega^{(3,0)}\mathbb{PT}'_{[0]}\cong\mathcal{O}(-4)$. In the general (non-supersymmetric) case, we can define the non-projective twistor space T to be the total space of the line bundle $\mathcal{O}(-1)$ now defined to be the 4$^{\text{th}}$ root of $\Omega^{(3,0)}PT$. If so, we see that $L\cong\mathcal{O}(2)$. The non-projective space has an Euler vector field Υ that generates the $\mathbb{C}^*$ action on the fibres of $\mathcal{O}(-1)$. The weights of functions and forms pulled back from PT are translated into the weights along Υ on the non-projective space. In this context,

τ defines a 1-form of weight 2 on the non-projective space, and the non-degeneracy of the contact structure translates into the condition that the two-form $\mathrm{d}\tau$ is non-degenerate as a two-form on T and being closed defines a holomorphic symplectic structure. Its inverse Π therefore defines a non-degenerate holomorphic Poisson structure on T of weight -2. This descends to give a Poisson structure on PT with values in $\mathscr{O}(-2)$.

We can extend this reasoning to the supersymmetric case as follows. We again consider a holomorphic differential one-form τ with values in a complex line bundle $\mathscr{L}$. It defines as its kernel the contact distribution $\mathscr{D}$, which now is of rank $2|\mathcal{N}$, leading to a short exact sequence as follows:

$$0 \longrightarrow \mathscr{D} \longrightarrow T^{(1,0)}\mathscr{P}\mathscr{T} \longrightarrow \mathscr{L} \longrightarrow 0\,. \tag{B.1}$$

Since we assume that τ defines a non-degenerate holomorphic contact structure, $\mathrm{d}\tau$ provides a non-degenerate skew form on $\mathscr{D}$. Taking its Berezinian, we get an element

$$\mathrm{Ber}(\mathrm{d}\tau|_{\mathscr{D}}) \in \mathscr{L}^{2-\mathcal{N}} \otimes (\mathrm{Ber}\,\mathscr{D})^{-2}. \tag{B.2}$$

(This follows from the fact that in the definition of the Berezinian, the odd-odd part of the matrix is inverted before its determinant is taken leading to inverse weights associated to the odd directions relative to their bosonic counterparts.) When $\mathscr{L}$ has a square root, we can take its square root to get an isomorphism

$$\sqrt{(\mathrm{Ber}(\mathrm{d}\tau|_{\mathscr{D}}))} : \mathrm{Ber}\,\mathscr{D} \to \mathscr{L}^{1-\mathcal{N}/2}\,. \tag{B.3}$$

The above exact sequence then gives an identification

$$\mathrm{Ber}\,T^{(1,0)}\mathscr{P}\mathscr{T} \cong \mathrm{Ber}\,\mathscr{D} \otimes \mathscr{L} \cong \mathscr{L}^{2-\mathcal{N}/2}\,, \tag{B.4}$$

and so finally we obtain the isomorphism

$$\mathrm{Ber}(\mathscr{P}\mathscr{T}) := \mathrm{Ber}\,\Omega^{(1,0)}\mathscr{P}\mathscr{T} \cong \mathscr{L}^{\mathcal{N}/2-2}. \tag{B.5}$$

We will take the body of the supertwistor space to have topology $U \times S^2$, where U is an open subset of $\mathbb{R}^4$ (or more generally the total space of the projective co-spin bundle of a real smooth spin four-manifold M). The assumption on the normal bundle of a rational curve in supertwistor space implies that the holomorphic Berezinian bundle $\mathrm{Ber}(\mathscr{P}\mathscr{T})$ has Chern class $\mathcal{N}-4$, and with the topological assumptions we have made, this will have an $|\mathcal{N}-4|$-th root and we may introduce the (consistent) notation $\mathscr{O}(n) := (\mathrm{Ber}(\mathscr{P}\mathscr{T}))^{n/(\mathcal{N}-4)}$. Thus, $\mathscr{L} \cong \mathscr{O}(2)$ and $\mathrm{Ber}(\mathscr{P}\mathscr{T}) \cong \mathscr{O}(\mathcal{N}-4)$.

Appendix C. Supersymmetric BF-Type Theory

In this appendix we wish to present an alternative interpretion of the holomorphic Chern-Simons-type theory (3.14). We shall see that this theory can be viewed as a certain supersymmetric holomorphic BF-type theory. In what follows, we will borrow ideas of [W89].

To begin with, consider some (0|2)-dimensional space $\mathscr{T}$ with odd coordinates ψ^1 and ψ^2, which we collectively denote by ψ^α. On $\mathscr{P}\mathscr{T} \times \mathscr{T}$, we may introduce a (0, 1)-form H of weight 2 according to

$$H = h + \psi^\alpha \chi_\alpha + \psi^1 \psi^2 b. \tag{C.1}$$

Here, h and b are even and χ_α are odd $(0,1)$-forms of weight 2 on $\mathscr{P}\mathscr{T}$. As before, we assume that these fields have no dependence on the $\bar{\theta}^{\bar{i}}$ coordinates.

In analogy to (3.14), we may consider the action functional

$$S[b,h,\chi_\alpha] \;=\; \int \mathrm{d}\psi^1 \mathrm{d}\psi^2 \int \Omega_8^{(0)} \wedge \left(H \wedge \bar{\partial}_0 H + \tfrac{1}{3} H \wedge [H,H\}\right). \tag{C.2}$$

A short calculation reveals that this action reduces after integration over the ψ^α coordinates to

$$S[h,b,\chi_\alpha] \;=\; \int \Omega_8^{(0)} \wedge \left\{ b \wedge F^{(0,2)} - \tfrac{1}{2}\epsilon^{\alpha\beta}\chi_\alpha \wedge (\bar{\partial}_0 \chi_\beta + [h,\chi_\beta\}) \right\}. \tag{C.3}$$

The equations of motion that follow from this action are

$$F^{(0,2)} = 0, \tag{C.4a}$$

$$\bar{\partial}_0 b + [h,b\} = \tfrac{1}{2}\epsilon^{\alpha\beta}[\chi_\alpha,\chi_\beta\}, \tag{C.4b}$$

$$\bar{\partial}_0 \chi_\alpha + [h,\chi_\alpha\} = 0. \tag{C.4c}$$

The first equation is the field equation (3.10). Note that for $\chi_\alpha = 0$ we get (3.18).

The supersymmetry transformations are straightforwardly worked out as they follow from infinitesimal translations in the odd coordinates ψ^α. We find

$$\delta_\alpha h \;=\; \chi_\alpha, \qquad \delta_\alpha \chi_\beta \;=\; \epsilon_{\alpha\beta} b \qquad \text{and} \qquad \delta_\alpha b \;=0, \tag{C.5}$$

with $\{\delta_\alpha,\delta_\beta\} = 0$. Therefore, the supersymmetric holomorphic BF-type action (C.3) can also be written as

$$S[h,b,\chi_\alpha] \;=\; -\tfrac{1}{2}\delta_1\delta_2 S[h], \tag{C.6}$$

where $S[h]$ is the action (3.14).

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