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Annihilation-Derivative, Creation-Derivative and Representation of Quantum Martingales*

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Abstract: On the basis of the quantum white noise theory we introduce the notion of creation- and annihilation-derivatives of Fock space operators and study the differentiability of white noise operators. We define the Hitsuda–Skorohod quantum stochastic integrals by the adjoint actions of quantum stochastic gradients and show explicit formulas for their creation- and annihilation-derivatives. As an application, we derive direct formulas for the integrands in the quantum stochastic integral representation of a regular quantum martingale.

1. Introduction

The representation theorem of regular quantum martingales, first proved by Parthasarathy–Sinha [30,31], then by Meyer [22], and later extended by Attal [2] and Ji [9] among others, says that a regular quantum martingale $\{M_t\}$ takes the form:

$$M_t = \lambda I + \int_0^t (E_s dA_s + F_s dA_s^* + G_s d\Lambda_s), \tag{1.1}$$

where the right-hand side consists of the quantum stochastic integrals of Itô type against the annihilation process $\{A_t\}$, creation process $\{A_t^*\}$ and conservation (number) process $\{\Lambda_t\}$, and the integrands $\{E_t\}$, $\{F_t\}$, $\{G_t\}$ are adapted processes uniquely determined by $\{M_t\}$, see Theorem 6.3 for the precise statement based on the recent achievement by Ji [9]. For more general discussions we refer to [14,15]. It has not been known, however, how to express those integrands directly in terms of $\{M_t\}$. In this paper we develop a new type of differential calculus for Fock space operators, in particular, for the Hitsuda–Skorohod quantum stochastic integrals and, as an application, we derive direct formulas for the integrands in (1.1).

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Our approach is based on the quantum white noise theory (e.g., [6,10,25]). Let $\Gamma(L^2(\mathbb{R}_+))$ be the Fock space over $L^2(\mathbb{R}_+)$ and equip it with the inclusion relations:

$$(E) \subset \mathcal{G} \subset \Gamma(L^2(\mathbb{R}_+)) \subset \mathcal{G}^* \subset (E)^*,$$

for details see Sect. 3. A continuous operator in $\mathcal{L}((E), (E)^*)$ is called a *white noise operator* and a white noise operator in $\mathcal{L}(\mathcal{G}, \mathcal{G}^*)$ is called *admissible*. These spaces of continuous operators enable us to treat many interesting unbounded operators in $\Gamma(L^2(\mathbb{R}_+))$ as continuous operators. The most basic white noise operators are the annihilation and creation operators at a time $t \in \mathbb{R}_+$, which are denoted by a_t and a_t^*, respectively. The pair $\{a_t\}$, $\{a_t^*\}$ is sometimes referred to as the *quantum white noise*.

As a consequence of the Fock expansion theorem for white noise operators [25], every $\Xi \in \mathcal{L}((E), (E)^*)$ is considered as a "function" of quantum white noises: $\Xi = \Xi(a_s^*, a_t\,;\, s, t \in \mathbb{R}_+)$. Then we are naturally led to a kind of functional derivatives:

$$D_t^+ \Xi = \frac{\delta \Xi}{\delta a_t^*}, \qquad D_t^- \Xi = \frac{\delta \Xi}{\delta a_t}. \tag{1.2}$$

The former is called the *pointwise creation-derivative* and the latter the *pointwise annihilation-derivative*. The heuristic notion in (1.2) will be formulated in two ways. In the previous papers [11,12] (see also Sect. 3.1), the "smeared" derivatives $D_\zeta^\pm \Xi$ are defined for any white noise operator $\Xi \in \mathcal{L}((E), (E)^*)$. In this paper we shall prove that an admissible white noise operator $\Xi \in \mathcal{L}(\mathcal{G}, \mathcal{G}^*)$ admits the pointwise derivatives $D_t^\pm \Xi$ for a.e. $t \in \mathbb{R}_+$ (Theorem 3.9). These derivatives of Fock space operators are regarded as quantum extensions of the classical stochastic derivatives widely known in the literature, see e.g., [16,20,23].

On the other hand, in [13] we introduced Hitsuda–Skorohod quantum stochastic integrals by means of the adjoint actions of quantum stochastic gradients. For a quantum stochastic process $\Xi = \{\Xi_t\} \in L^2(\mathbb{R}_+, \mathcal{L}((E), (E)^*))$ the Hitsuda–Skorohod quantum stochastic integrals $\delta^\epsilon(\Xi)$, $\epsilon \in \{+, -, 0\}$, are defined as white noise operators and their derivatives $D_\zeta^\pm \delta^\epsilon(\Xi)$ are computed explicitly (Theorem 5.2). If $\Xi = \{\Xi_t\}$ belongs to $L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}, \mathcal{G}^*))$, their Hitsuda–Skorohod quantum stochastic integrals $\delta^\epsilon(\Xi)$ admit the pointwise derivatives $D_t^\pm \delta^\epsilon(\Xi)$ for a.e. $t \in \mathbb{R}_+$. We derive formulas for these derivatives (Theorem 5.4) and, as a particular case, for an adapted process (Theorem 5.7).

Since the Hitsuda–Skorohod quantum stochastic integrals coincide with the ones of Itô type when the integrands are adapted processes, the right-hand side of (1.1) are expressible in terms of the Hitsuda–Skorohod quantum stochastic integrals. Then, by repeated application of the differential operators $D_t^\pm$ the integrands in (1.1) are obtained:

$$\begin{aligned}
E_s &= D_s^- \left[M_s - \int_0^s D_u^+ M_u \, dA_u^* \right], \\
F_s &= D_s^+ \left[M_s - \int_0^s D_u^- M_u \, dA_u \right], \\
G_s &= D_s^+ \left[\int_0^s \left\{ D_u^- \left(M_u - \int_0^u E_v \, dA_v - \int_0^u F_v \, dA_v^* \right) \right\} du \right].
\end{aligned} \tag{1.3}$$

The precise statement will be found in Theorem 6.6. The above direct formulas possess a feature quite different from the method of Parthasarathy–Sinha [30] that takes a detour through the classical Kunita–Watanabe theorem.

This paper is organized as follows. In Sect. 2 we assemble some basic notions in quantum white noise theory. In Sect. 3 we introduce the creation- and annihilation-derivatives and the quantum stochastic gradients. In Sect. 4 we define the Hitsuda–Skorohod quantum stochastic integrals by means of the adjoint actions of the quantum stochastic gradients. In Sect. 5 we show several formulas for the creation- and annihilation-derivatives of the Hitsuda–Skorohod quantum stochastic integrals. In Sect. 6 we derive formulas (1.3) for the integrands of quantum stochastic integral representation of a regular quantum martingale and discuss an example due to Parthasarathy [28] along our approach.

2. Quantum White Noise Theory

2.1. Gelfand Triple over $\mathbb{R}_+$. Let $H = L^2(\mathbb{R}_+)$ be the (complex) Hilbert space of L^2-functions on $\mathbb{R}_+ = [0, \infty)$ with respect to the Lebesgue measure dt. Here $t \in \mathbb{R}_+$ stands for a time parameter. The norm of H is denoted by $|\cdot|_0$.

Let $E = \mathcal{S}(\mathbb{R}_+)$ be the space of $\mathbb{C}$-valued continuous functions on $\mathbb{R}_+$ which are obtained by restricting rapidly decreasing functions in $\mathcal{S}(\mathbb{R})$ to $\mathbb{R}_+$. Identifying E with the quotient space $\mathcal{S}(\mathbb{R})/\mathcal{N}(\mathbb{R}_+)$, where $\mathcal{N}(\mathbb{R}_+)$ is the space of rapidly decreasing functions on $\mathbb{R}$ vanishing on $\mathbb{R}_+$, we furnish E with the natural topology. Thus E becomes a nuclear Fréchet space.

In fact, E is topologized by the Hilbertian norms $|\cdot|_p$, $p \in \mathbb{R}$, induced from the usual norms of $\mathcal{S}(\mathbb{R}) = \text{proj lim}_{p\to\infty} \mathcal{S}_p(\mathbb{R})$, see [25, Chap. 1]. Then, as in the case of $\mathcal{S}(\mathbb{R})$, the inequality

$$|\xi|_p \le \rho^q |\xi|_{p+q}, \qquad \xi \in E, \quad p \in \mathbb{R}, \quad q \ge 0,$$

holds with $\rho = 1/2$. For $p \in \mathbb{R}$ let E_p denote the Hilbert space obtained by completing E with respect to $|\cdot|_p$. Then these Hilbert spaces form a chain:

$$\cdots \subset E_p \subset \cdots \subset E_0 = H \subset \cdots \subset E_{-p} \subset \cdots, \tag{2.1}$$

where the inclusions are continuous and have dense images. We see by construction that

$$E = \mathcal{S}(\mathbb{R}_+) \cong \operatorname*{proj\,lim}_{p\to\infty} E_p$$

and its dual space (equipped with the strong dual topology) is obtained as

$$E^* \cong \operatorname*{ind\,lim}_{p\to\infty} E_{-p}.$$

Thus, we come to a complex Gelfand triple:

$$E = \mathcal{S}(\mathbb{R}_+) \subset H = L^2(\mathbb{R}_+) \subset E^* = \mathcal{S}'(\mathbb{R}_+).$$

Here the notation $\mathcal{S}'(\mathbb{R}_+)$ is reasonable, since E^* is identified with the space of tempered distributions in $\mathcal{S}'(\mathbb{R})$ with supports contained in $\mathbb{R}_+$. The canonical $\mathbb{C}$-bilinear form on $E^* \times E$ is denoted by $\langle \cdot, \cdot \rangle$. Note that $|\xi|_0^2 = \langle \bar{\xi}, \xi \rangle$.

Notation 2.1. For two locally convex spaces $\mathcal{X}, \mathcal{Y}$ we denote by $\mathcal{X} \otimes \mathcal{Y}$ the completed π-tensor product. If both $\mathcal{X}, \mathcal{Y}$ are Hilbert spaces, the Hilbert space tensor product is denoted also by $\mathcal{X} \otimes \mathcal{Y}$. The use of the same symbol will cause no confusion by contexts.

Notation 2.2. For two locally convex spaces $\mathcal{X}, \mathcal{Y}$ we denote by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the space of continuous linear maps from $\mathcal{X}$ into $\mathcal{Y}$, equipped with the bounded convergence topology. If both $\mathcal{X}, \mathcal{Y}$ are Hilbert spaces, let $\mathcal{L}_2(\mathcal{X}, \mathcal{Y})$ denote the space of Hilbert–Schmidt operators from $\mathcal{X}$ into $\mathcal{Y}$.

Notation 2.3. Let $\mathcal{H}$ be a Hilbert space. Then $L^2(\mathbb{R}_+) \otimes \mathcal{H}$ is identified with the Hilbert space of $\mathcal{H}$-valued L^2-functions on $\mathbb{R}_+$, which is denoted by $L^2(\mathbb{R}_+, \mathcal{H})$. Applying this notation in a slightly generalized context, for a locally convex space $\mathcal{X}$ we put

$$\mathcal{S}(\mathbb{R}_+, \mathcal{X}) = \mathcal{S}(\mathbb{R}_+) \otimes \mathcal{X}, \qquad \mathcal{S}'(\mathbb{R}_+, \mathcal{X}^*) = \mathcal{S}'(\mathbb{R}_+) \otimes \mathcal{X}^*,$$

which are mutually dual spaces. Incidentally, it is possible to directly define an $\mathcal{X}$-valued rapidly decreasing function [32, Chap. 44], though we do not take this approach in this paper. Furthermore, if $\mathcal{X} = \operatorname{proj\,lim}_{p\to\infty} \mathcal{X}_p$ is a countable Hilbert space, we set

$$L^2(\mathbb{R}_+, \mathcal{X}) = \operatorname*{proj\,lim}_{p\to\infty} L^2(\mathbb{R}_+, \mathcal{X}_p) \cong \operatorname*{proj\,lim}_{p\to\infty} L^2(\mathbb{R}_+) \otimes \mathcal{X}_p,$$
$$L^2(\mathbb{R}_+, \mathcal{X}^*) = \operatorname*{ind\,lim}_{p\to\infty} L^2(\mathbb{R}_+, \mathcal{X}_{-p}) \cong \operatorname*{ind\,lim}_{p\to\infty} L^2(\mathbb{R}_+) \otimes \mathcal{X}_{-p}.$$

Note that $L^2(\mathbb{R}_+, \mathcal{X}) \cong L^2(\mathbb{R}_+) \otimes \mathcal{X}$ and $L^2(\mathbb{R}_+, \mathcal{X}^*) \cong L^2(\mathbb{R}_+) \otimes \mathcal{X}^*$ do not hold in general (see Notation 2.1).

2.2. Hida–Kubo–Takenaka Space over $\mathbb{R}_+$. The (Boson) Fock space over E_p is defined by

$$\Gamma(E_p) = \left\{ \phi = (f_n)_{n=0}^{\infty} ;\ f_n \in E_p^{\widehat{\otimes} n},\ \| \phi \|_p^2 = \sum_{n=0}^{\infty} n!\, |f_n|_p^2 < \infty \right\},$$

where $E_p^{\widehat{\otimes} n}$ is the n-fold symmetric tensor power of the Hilbert space E_p. Then, (2.1) gives rise to a chain of Fock spaces:

$$\cdots \subset \Gamma(E_p) \subset \cdots \subset \Gamma(H) \subset \cdots \subset \Gamma(E_{-p}) \subset \cdots .$$

The limit spaces:

$$(E) = \operatorname*{proj\,lim}_{p\to\infty} \Gamma(E_p), \qquad (E)^* = \operatorname*{ind\,lim}_{p\to\infty} \Gamma(E_{-p}),$$

are mutually dual spaces. It is known that (E) becomes a countably Hilbert nuclear space. We thus obtain a complex Gelfand triple:

$$(E) \subset \Gamma(H) \subset (E)^*,$$

which is referred to as the *Hida–Kubo–Takenaka space* (over $\mathbb{R}_+$). By definition the topology of (E) is defined by the norms

$$\| \phi \|_p^2 = \sum_{n=0}^{\infty} n!\, |f_n|_p^2, \qquad \phi = (f_n) \in (E), \qquad p \in \mathbb{R}. \tag{2.2}$$

On the other hand, for each $\Phi \in (E)^*$ there exists $p \geq 0$ such that $\Phi \in \Gamma(E_{-p})$. In this case, we have

$$\| \Phi \|_{-p}^2 = \sum_{n=0}^{\infty} n! \, |F_n|_{-p}^2 < \infty, \qquad \Phi = (F_n).$$

The canonical $\mathbb{C}$-bilinear form on $(E)^* \times (E)$ takes the form:

$$\langle\langle \Phi, \phi \rangle\rangle = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle, \qquad \Phi = (F_n) \in (E)^*, \quad \phi = (f_n) \in (E).$$

2.3. White Noise Operators. A continuous linear operator in $\mathcal{L}((E), (E)^*)$ is called a *white noise operator*. By the nuclear kernel theorem there exists a canonical isomorphism:

$$\mathcal{K} : \mathcal{L}((E), (E)^*) \xrightarrow{\cong} (E)^* \otimes (E)^*, \tag{2.3}$$

which is defined by

$$\langle\langle \Xi \phi, \psi \rangle\rangle = \langle\langle \mathcal{K}\Xi, \psi \otimes \phi \rangle\rangle, \qquad \phi, \psi \in (E).$$

Now we recall the most fundamental white noise operators. With each $x \in \mathcal{S}'(\mathbb{R}_+)$ we associate the *annihilation operator* $a(x)$ defined by

$$a(x) : \phi = (f_n)_{n=0}^{\infty} \mapsto ((n+1) x \otimes_1 f_{n+1})_{n=0}^{\infty},$$

where $x \otimes_1 f_n$ stands for the contraction. It is known that $a(x) \in \mathcal{L}((E), (E))$. Its adjoint operator $a^*(x) \in \mathcal{L}((E)^*, (E)^*)$ is called the *creation operator* and satisfies

$$a^*(x) : \phi = (f_n)_{n=0}^{\infty} \mapsto (x \hat{\otimes} f_{n-1})_{n=0}^{\infty},$$

understanding that $f_{-1} = 0$. The following precise norm estimates are useful.

Lemma 2.1. *Let $x \in \mathcal{S}'(\mathbb{R}_+)$ and $\phi \in (E)$. For any $p \in \mathbb{R}$ and $q > 0$ we have*

$$\begin{aligned} \| a(x)\phi \|_p &\leq C_q \, | x |_{-(p+q)} \, \| \phi \|_{p+q}, \\ \left\| a^*(x)\phi \right\|_p &\leq C_q \, | x |_p \, \| \phi \|_{p+q}, \end{aligned} \tag{2.4}$$

where $C_q = \sup_{n \geq 0} \sqrt{n+1} \, \rho^{qn} < \infty$.

Lemma 2.2. *If $\zeta \in \mathcal{S}(\mathbb{R}_+)$, then $a(\zeta)$ extends to a continuous linear operator from $(E)^*$ into itself (denoted by the same symbol) and $a^*(\zeta)$ (restricted to (E)) is a continuous linear operator from (E) into itself.*

For $t \in \mathbb{R}_+$ we put

$$a_t = a(\delta_t), \qquad a_t^* = a^*(\delta_t).$$

The pair $\{a_t\}, \{a_t^*\}$ is called the *quantum white noise*.

Lemma 2.3. *The map $t \mapsto a_t$ is an $\mathcal{L}((E), (E))$-valued rapidly decreasing function, i.e., is a member of $\mathcal{S}(\mathbb{R}_+, \mathcal{L}((E), (E))) \cong \mathcal{L}((E), \mathcal{S}(\mathbb{R}_+) \otimes (E))$.*

The proofs of the above lemmas are straightforward from definition and direct computation, see also [25, Chap. 4].

Remark 2.4. The white noise operators cover a wide class of Fock space operators and provide a reasonable framework for quantum stochastic calculus. For example, if $\mathcal{X}$, $\mathcal{Y}$ are locally convex spaces admitting continuous inclusions

$$(E) \subset \mathcal{X} \subset (E)^*, \qquad (E) \subset \mathcal{Y} \subset (E)^*,$$

then the space $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ of continuous operators from $\mathcal{X}$ into $\mathcal{Y}$ is regarded as a subspace of $\mathcal{L}((E), (E)^*)$. Through the canonical isomorphism (2.3) the space of kernels corresponding to $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a subspace of $(E)^* \otimes (E)^*$. However, care must be used in expressing the space of kernels in terms of tensor product $\mathcal{Y} \otimes \mathcal{X}^*$ when lack of nuclearity [32, Chap. 50].

3. Differential Calculus for White Noise Operators

3.1. Annihilation- and Creation-Derivatives. By Lemma 2.2, for any white noise operator $\Xi \in \mathcal{L}((E), (E)^*)$ and $\zeta \in \mathcal{S}(\mathbb{R}_+)$ the commutators

$$[a(\zeta), \Xi] = a(\zeta)\Xi - \Xi a(\zeta), \qquad -[a^*(\zeta), \Xi] = \Xi a^*(\zeta) - a^*(\zeta)\Xi,$$

are well defined white noise operators, i.e., belong to $\mathcal{L}((E), (E)^*)$. We define

$$D_\zeta^+ \Xi = [a(\zeta), \Xi], \qquad D_\zeta^- \Xi = -[a^*(\zeta), \Xi].$$

We call $D_\zeta^+ \Xi$ and $D_\zeta^- \Xi$ the *creation derivative* and *annihilation derivative* of Ξ, respectively. For brevity, both together are called the *quantum white noise derivatives* or *qwn-derivatives* of Ξ.

Lemma 3.1. $\mathcal{S}(\mathbb{R}_+) \times \mathcal{L}((E), (E)^*) \ni (\zeta, \Xi) \mapsto D_\zeta^\pm \Xi \in \mathcal{L}((E), (E)^*)$ *is a continuous bilinear map.*

Lemma 3.2. *For any* $\Xi \in \mathcal{L}((E), (E)^*)$ *and* $\zeta \in \mathcal{S}(\mathbb{R}_+)$ *it holds that*

$$\mathcal{K}(D_\zeta^+ \Xi) = (a(\zeta) \otimes I)\mathcal{K}\Xi - (I \otimes a^*(\zeta))\mathcal{K}\Xi,$$
$$\mathcal{K}(D_\zeta^- \Xi) = (I \otimes a(\zeta))\mathcal{K}\Xi - (a^*(\zeta) \otimes I)\mathcal{K}\Xi.$$

Lemma 3.1 is proved by direct estimate of norms [12] and Lemma 3.2 is immediate from definition.

3.2. Admissible White Noise Operators. We shall introduce a reasonably large subspace of $\mathcal{L}((E), (E)^*)$ for differential calculus. For $p \in \mathbb{R}$ we set

$$\| \phi \|_p^2 = \sum_{n=0}^{\infty} n! e^{2pn} |f_n|_0^2, \qquad \phi = (f_n) \in \Gamma(H). \tag{3.1}$$

For $p \geq 0$ we define $\mathcal{G}_p = \{\phi = (f_n) \in \Gamma(H) ; \| \phi \|_p < \infty\}$ and $\mathcal{G}_{-p}$ to be the completion of $\Gamma(H)$ with respect to $\| \cdot \|_{-p}$. Having thus obtained a chain of Hilbert spaces:

$$\cdots \subset \mathcal{G}_p \subset \cdots \subset \mathcal{G}_0 = \Gamma(H) \subset \cdots \subset \mathcal{G}_{-p} \subset \cdots,$$

we define

$$\mathcal{G} = \operatorname*{proj\,lim}_{p\to\infty} \mathcal{G}_p, \qquad \mathcal{G}^* = \operatorname*{ind\,lim}_{p\to\infty} \mathcal{G}_{-p},$$

which are mutually dual spaces. Note that $\mathcal{G}$ is a countable Hilbert space but *not* a nuclear space.

Lemma 3.3. *Let $p \geq 0$ and $q \geq p/(-\log\rho)$. Then it holds that*

$$|\!|\!| \phi |\!|\!|_p \leq \|\phi\|_q, \qquad \phi \in (E).$$

Therefore, the canonical injection $\Gamma(E_q) \to \mathcal{G}_p$ is a contraction.

Proof. Straightforward from the definitions of norms in (2.2) and (3.1). □

From Lemma 3.3 we obtain the inclusions:

$$(E) \subset \mathcal{G} \subset \Gamma(H) \subset \mathcal{G}^* \subset (E)^*.$$

Therefore, $\mathcal{L}(\mathcal{G}, \mathcal{G}^*)$ becomes a subspace of $\mathcal{L}((E), (E)^*)$. A white noise operator in the former space is called *admissible*. Note that

$$\mathcal{L}(\mathcal{G}, \mathcal{G}^*) = \bigcup_{p,q\in\mathbb{R}} \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q) = \bigcup_{p\geq 0} \mathcal{L}(\mathcal{G}_p, \mathcal{G}_{-p}).$$

Lemma 3.4. *For any $p \geq 0$ there exists $q \geq \max\{p, p/(-\log\rho)\}$ such that*

$$\mathcal{L}(\mathcal{G}_p, \mathcal{G}_{-p}) \subset \mathcal{L}_2(\Gamma(E_q), \mathcal{G}_{-q}).$$

Proof. Given $p \geq 0$, set $r = p/(-\log\rho)$. We see from Lemma 3.3 that $\Gamma(E_r) \to \mathcal{G}_p$ is a contraction. It is known that there exists $s = s(r) > 0$ such that $\Gamma(E_{r+s}) \to \Gamma(E_r)$ is of Hilbert–Schmidt class. Take $q = \max\{r + s, p\}$. For any $\Xi \in \mathcal{L}(\mathcal{G}_p, \mathcal{G}_{-p})$ the composition

$$\Gamma(E_q) \to \Gamma(E_{r+s}) \to \Gamma(E_r) \to \mathcal{G}_p \xrightarrow{\Xi} \mathcal{G}_{-p} \to \mathcal{G}_{-q}$$

is of Hilbert–Schmidt class, which means that $\Xi \in \mathcal{L}_2(\Gamma(E_q), \mathcal{G}_{-q})$. □

Remark 3.5. The spaces $\mathcal{G}$ and $\mathcal{G}^*$ have appeared along with classical and quantum stochastic analysis, see e.g., [1,3,4,7,18,19]. The admissible white noise operators $\mathcal{L}(\mathcal{G}, \mathcal{G}^*)$ play an essential role in the recent study of quantum martingales [9], see also Sect. 6.

3.3. Classical Stochastic Gradient Acting on $\mathcal{G}^*$. First define

$$\nabla\phi(t) = a_t\phi, \qquad \phi \in (E), \quad t \in \mathbb{R}_+.$$

It follows from Lemma 2.3 that

$$\nabla : (E) \to \mathcal{S}(\mathbb{R}_+, (E)) = \mathcal{S}(\mathbb{R}_+) \otimes (E) \tag{3.2}$$

becomes a continuous linear map. We extend the domain of ∇ to $\mathcal{G}^*$, see also [1] where a slightly different proof is found.

Lemma 3.6. *Let $p \in \mathbb{R}$, $r > 0$ and set $K(p, r) = \sup_n (n+1)e^{2p-2rn} < \infty$. Then, for any $\phi \in (E)$ we have*

$$\|\nabla\phi\|^2_{L^2(\mathbb{R}_+, \mathcal{G}_{-p-r})} = \int_{\mathbb{R}_+} \| \nabla\phi(t) \|^2_{-p-r}\, dt \le K(p, r) \| \phi \|^2_{-p}. \tag{3.3}$$

Proof. Writing $\phi = (f_n)$, we have $\nabla\phi(t) = ((n+1)f_{n+1}(t, \cdot))$, where the right-hand side has a pointwise meaning since f_n is a continuous function on $\mathbb{R}^n_+$. Then

$$\begin{aligned}\int_{\mathbb{R}_+} \| \nabla\phi(t) \|^2_{-p-r}\, dt &= \sum_{n=0}^{\infty} n! e^{-2(p+r)n} \int_{\mathbb{R}_+} |(n+1)f_{n+1}(t, \cdot)|_0^2 dt \\ &= \sum_{n=0}^{\infty} (n+1)e^{2p-2rn} \times (n+1)! e^{-2p(n+1)} |f_{n+1}|_0^2 \\ &\le K(p, r) \| \phi \|^2_{-p},\end{aligned}$$

which completes the proof. □

Applying the usual approximation argument to (3.3), we obtain a continuous linear map:

$$\nabla : \mathcal{G}_{-p} \to L^2(\mathbb{R}_+, \mathcal{G}_{-p-r}) \cong L^2(\mathbb{R}_+) \otimes \mathcal{G}_{-p-r}, \tag{3.4}$$

for which the norm estimate (3.3) remains valid, where $p \in \mathbb{R}$ and $r > 0$. Finally, by taking the inductive limit, the classical stochastic gradient

$$\nabla : \mathcal{G}^* \to L^2(\mathbb{R}_+, \mathcal{G}^*)$$

is defined and becomes a continuous linear map.

We see from (3.4) that $\nabla\Phi(t)$ has a meaning as a $\mathcal{G}_{-p-r}$-valued L^2-function in $t \in \mathbb{R}_+$. Given $\zeta \in L^2(\mathbb{R}_+)$, the linear map $\mathcal{G}_{p+r} \ni \psi \mapsto \langle\langle \nabla\Phi, \zeta \otimes \psi \rangle\rangle$ is continuous. Therefore there exists a unique $\Psi \in \mathcal{G}_{-p-r}$ such that

$$\langle\langle \nabla\Phi, \zeta \otimes \psi \rangle\rangle = \langle\langle \Psi, \psi \rangle\rangle, \qquad \psi \in \mathcal{G}_{p+r}.$$

It is reasonable to write

$$\Psi = \int_{\mathbb{R}_+} \zeta(t)\nabla\Phi(t)\, dt.$$

As is easily seen, the Schwartz inequality holds:

$$\left\| \int_{\mathbb{R}_+} \zeta(t)\nabla\Phi(t)\, dt \right\|_{-p-r} \le |\zeta|_0 \| \nabla\Phi \|_{L^2(\mathbb{R}_+, \mathcal{G}_{-p-r})}. \tag{3.5}$$

Lemma 3.7. *If $\zeta \in L^2(\mathbb{R}_+)$, we have*

$$\int_{\mathbb{R}_+} \zeta(t)\nabla\Phi(t)\, dt = a(\zeta)\Phi, \qquad \Phi \in \mathcal{G}^*. \tag{3.6}$$

Proof. The left-hand side of (3.6) is denoted by $\Psi = \Psi(\Phi)$ for simplicity. Take $p \in \mathbb{R}$ and $r > 0$ arbitrarily. We see from (3.3) and (3.5) that $\Phi \mapsto \Psi(\Phi)$ is a continuous linear map from $\mathcal{G}_{-p}$ into $\mathcal{G}_{-p-r}$. As is easily verified, so is $\Phi \mapsto a(\zeta)\Phi$. Hence it is sufficient to verify (3.6) for an exponential vector $\Phi = \phi_\xi$ with ξ running over E. Since $\phi_\xi \in (E)$, the left-hand side becomes

$$\Psi(\phi_\xi) = \int_{\mathbb{R}_+} \zeta(t)\nabla\phi_\xi(t)\,dt = \int_{\mathbb{R}_+} \zeta(t)a_t\phi_\xi\,dt$$
$$= \int_{\mathbb{R}_+} \zeta(t)\xi(t)\phi_\xi\,dt = \langle\zeta, \xi\rangle\phi_\xi\,.$$

On the other hand, as is well known, ϕ_ξ is an eigenvector of $a(\zeta)$ with eigenvalue $\langle\zeta, \xi\rangle$. Hence $\Psi(\phi_\xi) = a(\zeta)\phi_\xi$, which completes the proof. □

Recall that an *exponential vector* $\phi_x \in (E)^*$ is defined by $\phi_x = (x^{\otimes n}/n!)_{n=0}^{\infty}$ for $x \in \mathcal{S}'(\mathbb{R}_+)$. The set $\{\phi_\xi\,;\,\xi \in \mathcal{S}(\mathbb{R}_+)\}$ spans a dense subspace of (E).

3.4. Pointwise QWN-Derivatives. Let $\Xi \in \mathcal{L}((E), \mathcal{G}^*)$. Noting that the kernel $\mathcal{K}\Xi$ belongs to $\mathcal{G}^* \otimes (E)^*$ on which $\nabla \otimes I$ acts, we obtain

$$(\nabla \otimes I)\mathcal{K}\Xi \in L^2(\mathbb{R}_+, \mathcal{G}^*) \otimes (E)^* \cong L^2(\mathbb{R}_+, \mathcal{G}^* \otimes (E)^*).$$

This means that $[(\nabla\otimes I)\mathcal{K}\Xi](t)$ is defined as a $\mathcal{G}^*\otimes(E)^*$-valued L^2-function in $t \in \mathbb{R}_+$. More precisely, by Lemma 3.6, for any $p \in \mathbb{R}$ and $r > 0$ we have

$$\begin{aligned}\int_{\mathbb{R}_+} \|\, [(\nabla \otimes I)\mathcal{K}\Xi](t)\,\|^2_{\mathcal{G}_{-p-r}\otimes\Gamma(E_{-p})}\,dt &= \|\,(\nabla \otimes I)\mathcal{K}\Xi\,\|^2_{L^2(\mathbb{R}_+)\otimes\mathcal{G}_{-p-r}\otimes\Gamma(E_{-p})}\\ &\le K(p, r)\,\|\,\mathcal{K}\Xi\,\|^2_{\mathcal{G}_{-p}\otimes\Gamma(E_{-p})}\\ &= K(p, r)\,\|\,\Xi\,\|^2_{\mathcal{L}_2(\Gamma(E_p),\mathcal{G}_{-p})}\,. \end{aligned} \tag{3.7}$$

On the other hand, since $a_t^* \in \mathcal{L}((E)^*, (E)^*)$, we see that $(I \otimes a_t^*)\mathcal{K}\Xi$ is well defined as a member of $\mathcal{G}^* \otimes (E)^*$ for all $t \in \mathbb{R}_+$.

Lemma 3.8. *For $\Xi \in \mathcal{L}((E), \mathcal{G}^*)$ the map $t \mapsto (I \otimes a_t^*)\mathcal{K}\Xi$ is a member of $L^2(\mathbb{R}_+, \mathcal{G}^* \otimes (E)^*)$. More precisely, for any $p \ge 1$ and $r > 0$ there exists a constant number $L = L(p, r) > 0$ such that*

$$\int_{\mathbb{R}_+} \|\,(I \otimes a_t^*)\mathcal{K}\Xi\,\|^2_{\mathcal{G}_{-p}\otimes\Gamma(E_{-p-r})}\,dt \le L(p, r)\,\|\,\Xi\,\|^2_{\mathcal{L}_2(\Gamma(E_p),\mathcal{G}_{-p})}\,. \tag{3.8}$$

Proof. In view of $\mathcal{L}((E), \mathcal{G}^*) \cong \mathcal{G}^* \otimes (E)^*$, we choose $p \ge 1$ such that $\mathcal{K}\Xi \in \mathcal{G}_{-p} \otimes \Gamma(E_{-p})$. Using the estimate

$$\|a_t^*\phi\|_{-p-r} \le C_r|\delta_t|_{-p-r}\|\phi\|_{-p}, \qquad \phi \in (E), \quad r > 0,$$

which follows from Lemma 2.1, we have

$$\int_{\mathbb{R}_+} \|(I \otimes a_t^*)\mathcal{K}\Xi\|^2_{\mathcal{G}_{-p}\otimes\Gamma(E_{-p-r})}\,dt \le C_r^2\|\mathcal{K}\Xi\|^2_{\mathcal{G}_{-p}\otimes\Gamma(E_{-p})}\int_{\mathbb{R}_+} |\delta_t|^2_{-p-r}dt.$$

Putting

$$L(p, r) = C_r^2 \int_{\mathbb{R}_+} |\delta_t|_{-p-r}^2 dt,$$

we obtain (3.8). The above integral is finite since $|\delta_t|_{-q} \le |\delta_t|_{\mathcal{S}_{-q}(\mathbb{R})}$ for all $t \in \mathbb{R}_+$ by construction of the space E and

$$\int_{\mathbb{R}_+} |\delta_t|_{-q}^2 dt \le \int_{\mathbb{R}} |\delta_t|_{\mathcal{S}_{-q}(\mathbb{R})}^2 dt < \infty, \qquad q \ge 1. \tag{3.9}$$

In fact, the right-hand side of (3.9) is the square of the Hilbert–Schmidt norm of the canonical injection $\mathcal{S}_{q+s}(\mathbb{R}) \to \mathcal{S}_s(\mathbb{R})$ (the norm is independent of s), see e.g., [25, Chap. 1]. □

We have thus seen that $t \mapsto [(\nabla \otimes I)\mathcal{K}\Xi](t) - (I \otimes a_t^*)\mathcal{K}\Xi$ is defined as a $\mathcal{G}^* \otimes (E)^*$-valued L^2-function in $t \in \mathbb{R}_+$. We define $D_t^+\Xi$ by

$$\mathcal{K}(D_t^+\Xi) = [(\nabla \otimes I)\mathcal{K}\Xi](t) - (I \otimes a_t^*)\mathcal{K}\Xi. \tag{3.10}$$

Then $D_t^+\Xi$ becomes an $\mathcal{L}((E), \mathcal{G}^*)$-valued L^2-function in $t \in \mathbb{R}_+$. We call $D_t^+\Xi$ the *pointwise creation-derivative*. Combining (3.7) and (3.8), we see that for any $p \ge 1$ and $r > 0$ there exists a constant number $C = C(p, r) > 0$ such that

$$\int_{\mathbb{R}_+} \| D_t^+\Xi \|_{\mathcal{L}_2(\Gamma(E_{p+r}), \mathcal{G}_{-p-r})}^2 dt \le C(p, r) \| \Xi \|_{\mathcal{L}_2(\Gamma(E_p), \mathcal{G}_{-p})}^2. \tag{3.11}$$

By a parallel argument as above, for $\Xi \in \mathcal{L}(\mathcal{G}, (E)^*) \cong (E)^* \otimes \mathcal{G}^*$ we can define $D_t^-\Xi$ by

$$\mathcal{K}(D_t^-\Xi) = [(I \otimes \nabla)\mathcal{K}\Xi](t) - (a_t^* \otimes I)\mathcal{K}\Xi.$$

Then $D_t^-\Xi$ is an $\mathcal{L}(\mathcal{G}, (E)^*)$-valued L^2-function in $t \in \mathbb{R}_+$. We call $D_t^-\Xi$ the *pointwise annihilation-derivative*. Moreover, for any $p \ge 1$ and $r > 0$ we have

$$\int_{\mathbb{R}_+} \| D_t^-\Xi \|_{\mathcal{L}_2(\mathcal{G}_{p+r}, \Gamma(E_{-p-r}))}^2 dt \le C(p, r) \| \Xi \|_{\mathcal{L}_2(\mathcal{G}_p, \Gamma(E_{-p}))}^2. \tag{3.12}$$

In conclusion,

Theorem 3.9. *Every admissible white noise operator $\Xi \in \mathcal{L}(\mathcal{G}, \mathcal{G}^*)$ is pointwisely qwn-differentiable in the sense that $D_t^{\pm}\Xi \in \mathcal{L}((E), (E)^*)$ is determined for a.e. $t \in \mathbb{R}_+$. The norm estimates are given in* (3.11) *and* (3.12)*.*

Example 3.10. For $\zeta \in L^2(\mathbb{R}_+)$, the annihilation and creation operators $a^{\pm}(\zeta)$ belong to $\mathcal{L}(\mathcal{G}, \mathcal{G}^*)$. Their derivatives are given by

$$D_t^{\pm}(a^{\pm}(\zeta)) = D_t^{\pm} \int_{\mathbb{R}_+} \zeta(s) a_s^{\pm}\, ds = \zeta(t) I,$$

$$D_t^{\pm}(a^{\mp}(\zeta)) = D_t^{\pm} \int_{\mathbb{R}_+} \zeta(s) a_s^{\mp}\, ds = 0.$$

For the number operator we have

$$D_t^+ \int_{\mathbb{R}_+} a_s^* a_s \, ds = a_t, \qquad D_t^- \int_{\mathbb{R}_+} a_s^* a_s \, ds = a_t^*.$$

Here the formal integral representations of white noise operators (the so-called integral kernel operators [25]) give us a good intuition.

Proposition 3.11. *The bilinear map in Lemma 3.1 yields the continuous bilinear maps:*

$$L^2(\mathbb{R}_+) \times \mathcal{L}((E), \mathcal{G}^*) \ni (\zeta, \Xi) \mapsto D_\zeta^+ \Xi \in \mathcal{L}((E), \mathcal{G}^*),$$
$$L^2(\mathbb{R}_+) \times \mathcal{L}(\mathcal{G}, (E)^*) \ni (\zeta, \Xi) \mapsto D_\zeta^- \Xi \in \mathcal{L}(\mathcal{G}, (E)^*).$$

Moreover, for $\zeta \in L^2(\mathbb{R}_+)$ we have

$$\int_{\mathbb{R}_+} \zeta(t) D_t^\pm \Xi \, dt = D_\zeta^\pm \Xi.$$

Proof. The continuity follows from direct norm estimates, of which argument is similar to the case of $D_t^\pm \Xi$. The integral formula is straightforward. □

4. Quantum Stochastic Integrals

4.1. White Noise Integrals. As a general rule, a one-parameter family $\{\Xi_t\} \subset \mathcal{L}((E), (E)^*)$ is called a *quantum stochastic process*, where t runs over an interval of $\mathbb{R}_+$. Slightly generalizing this notation, we shall deal with an element $\Xi \in L^2(\mathbb{R}_+, \mathcal{L}((E), (E)^*))$ also as a quantum stochastic process. For such Ξ we may choose $p \geq 0$ such that $\Xi \in L^2(\mathbb{R}_+, \mathcal{L}_2(\Gamma(E_p), \Gamma(E_{-p})))$, which means that $\Xi_t \in \mathcal{L}_2(\Gamma(E_p), \Gamma(E_{-p}))$ makes sense only for a.e. $t \in \mathbb{R}_+$. Along this line an element of $\mathcal{S}'(\mathbb{R}_+, \mathcal{L}((E), (E)^*))$ is called a *generalized quantum stochastic process* [26,27].

Let $\{\Xi_t\}$ be a quantum stochastic process, where t runs over a (finite or infinite) interval $T \subset \mathbb{R}_+$. If $t \mapsto \langle\langle \Xi_t \phi, \psi \rangle\rangle$ is integrable on T for any $\phi, \psi \in (E)$ and if the bilinear form on $(E) \times (E)$ defined by

$$(\phi, \psi) \mapsto \int_T \langle\langle \Xi_t \phi, \psi \rangle\rangle \, dt$$

is continuous, then there exists a white noise operator $\Xi_T \in \mathcal{L}((E), (E)^*)$ such that

$$\langle\langle \Xi_T \phi, \psi \rangle\rangle = \int_T \langle\langle \Xi_t \phi, \psi \rangle\rangle \, dt, \qquad \phi, \psi \in (E).$$

In this case, we say that $\{\Xi_t\}$ is *white noise integrable* on T and write

$$\Xi_T = \int_T \Xi_t \, dt.$$

The white noise integrability can be checked with the famous characterization theorem for operator symbols [5,24,25]. It is proved that the white noise integrals:

$$A_t = \int_0^t a_s ds, \quad A_t^* = \int_0^t a_s^* ds, \quad \Lambda_t = \int_0^t a_s^* a_s ds$$

are defined. These are called respectively the *annihilation process*, the *creation process* and the *conservation process*, which play an essential role in quantum stochastic calculus [8,21,29].

As for $\Xi = \{\Xi_t\} \in L^2(\mathbb{R}_+, \mathcal{L}((E), (E)^*))$ we only mention the following

Proposition 4.1. *For any $\Xi \in L^2(\mathbb{R}_+, \mathcal{L}((E), (E)^*))$ and $\zeta \in L^2(\mathbb{R}_+)$ the quantum stochastic process $\zeta\Xi = \{\zeta(t)\Xi_t\}$ is white noise integrable on $\mathbb{R}_+$. In particular, every $\Xi \in L^2(\mathbb{R}_+, \mathcal{L}((E), (E)^*))$ is white noise integrable on any finite interval.*

4.2. Classical Hitsuda–Skorohod Integrals. Let δ denote the adjoint map of ∇ in (3.2). Then

$$\delta = \nabla^* : \mathcal{S}'(\mathbb{R}_+, (E)^*) \to (E)^*$$

becomes a continuous linear map. We call $\delta(\Psi) \in (E)^*$ the *(classical) Hitsuda–Skorohod integral* of $\Psi \in \mathcal{S}'(\mathbb{R}_+, (E)^*)$, though $\delta(\Psi)$ is understood only through duality.

Proposition 4.2. *If $\Psi \in L^2(\mathbb{R}_+, (E)^*)$, we have*

$$\langle\langle \delta(\Psi), \phi \rangle\rangle = \int_{\mathbb{R}_+} \langle\langle \Psi(t), \nabla\phi(t) \rangle\rangle dt, \qquad \phi \in (E).$$

Proof. It is sufficient to show that $t \mapsto \langle\langle \Psi(t), \nabla\phi(t) \rangle\rangle$ is integrable on $\mathbb{R}_+$. This is in fact immediate from (2.4) and (3.9) with the Schwartz inequality. □

4.3. Quantum Hitsuda–Skorohod Integrals. The quantum Hitsuda–Skorohod integrals are defined in the same spirit as the classical one, where the quantum stochastic gradients are employed.

4.3.1. Creation Integrals The *creation gradient* ∇^+ is by definition the composition of linear maps:

$$\begin{aligned} \nabla^+ : \mathcal{L}((E)^*, (E)) &\xrightarrow{\cong} (E) \otimes (E) \xrightarrow{\nabla \otimes I} (\mathcal{S}(\mathbb{R}_+) \otimes (E)) \otimes (E) \\ &\xrightarrow{\cong} \mathcal{S}(\mathbb{R}_+) \otimes ((E) \otimes (E)) \xrightarrow{\cong} \mathcal{S}(\mathbb{R}_+, \mathcal{L}((E)^*, (E))). \end{aligned} \tag{4.1}$$

The *creation integral* δ^+ is defined to be its adjoint:

$$\delta^+ = (\nabla^+)^* : \mathcal{S}'(\mathbb{R}_+, \mathcal{L}((E), (E)^*)) \longrightarrow \mathcal{L}((E), (E)^*).$$

By definition one can check easily [13] that

$$\langle\langle \delta^+(\Xi)\phi, \psi \rangle\rangle = \langle\langle \Xi\phi, \nabla\psi \rangle\rangle, \quad \Xi \in \mathcal{S}'(\mathbb{R}_+, \mathcal{L}((E), (E)^*)), \quad \phi, \psi \in (E).$$

If $\Xi \in L^2(\mathbb{R}_+, \mathcal{L}((E), (E)^*))$, the above identity becomes

$$\langle\langle \delta^+(\Xi)\phi, \psi \rangle\rangle = \int_{\mathbb{R}_+} \langle\langle \Xi_t\phi, \nabla\psi(t) \rangle\rangle \, dt. \tag{4.2}$$

Put $(\Xi\phi)(t) = \Xi_t\phi$. Then, by Proposition 4.2, (4.2) becomes

$$= \int_{\mathbb{R}_+} \langle\langle (\Xi\phi)(t), \nabla\psi(t) \rangle\rangle \, dt = \langle\langle \Xi\phi, \nabla\psi \rangle\rangle = \langle\langle \delta(\Xi\phi), \psi \rangle\rangle .$$

Thus, we come to the relation between the creation integral and the classical Hitsuda–Skorohod integral:

$$\delta^+(\Xi)\phi = \delta(\Xi\phi), \qquad \Xi \in L^2(\mathbb{R}_+, \mathcal{L}((E), (E)^*)), \quad \phi \in (E). \tag{4.3}$$

4.3.2. Annihilation Integrals The *annihilation gradient* ∇^- is defined in a manner similar to (4.1) as follows:

$$\begin{aligned}\nabla^- : \mathcal{L}((E)^*, (E)) &\xrightarrow{\cong} (E) \otimes (E) \xrightarrow{I\otimes\nabla} (E) \otimes (\mathcal{S}(\mathbb{R}_+) \otimes (E)) \\ &\xrightarrow{\cong} \mathcal{S}(\mathbb{R}_+) \otimes ((E) \otimes (E)) \xrightarrow{\cong} \mathcal{S}(\mathbb{R}_+, \mathcal{L}((E)^*, (E))).\end{aligned}$$

The *annihilation integral* δ^- is by definition the adjoint map of the annihilation gradient:

$$\delta^- = (\nabla^-)^* : \mathcal{S}'(\mathbb{R}_+, \mathcal{L}((E), (E)^*)) \to \mathcal{L}((E), (E)^*).$$

For $\varXi \in L^2(\mathbb{R}_+, \mathcal{L}((E), (E)^*))$ we have

$$\langle\langle \delta^-(\varXi)\phi, \psi \rangle\rangle = \int_{\mathbb{R}_+} \langle\langle \varXi_t(\nabla\phi(t)), \psi \rangle\rangle \, dt, \qquad \phi, \psi \in (E), \tag{4.4}$$

by definition. Hence,

$$\delta^-(\varXi)\phi = \int_{\mathbb{R}_+} \varXi_t(\nabla\phi(t))\, dt, \quad \varXi \in L^2(\mathbb{R}_+, \mathcal{L}((E), (E)^*)), \quad \phi \in (E). \tag{4.5}$$

The creation and annihilation integrals are related directly. Comparing (4.2) and (4.4), we obtain the simple formula:

$$(\delta^-(\varXi))^* = \delta^+(\varXi^*), \qquad \varXi \in L^2(\mathbb{R}_+, \mathcal{L}((E), (E)^*)). \tag{4.6}$$

4.3.3. Conservation Integrals

Lemma 4.3. *For $\varPhi, \varPsi \in \mathcal{S}(\mathbb{R}_+, (E))$ we define $\varOmega = \varOmega(\varPhi, \varPsi) \in \mathcal{S}(\mathbb{R}_+, (E) \otimes (E))$ by $\varOmega(t) = \varPhi(t) \otimes \varPsi(t)$. Then, $(\varPhi, \varPsi) \mapsto \varOmega(\varPhi, \varPsi)$ is a continuous bilinear map.*

Proof. Consider first $\varPhi = \xi \otimes \phi$ and $\varPsi = \eta \otimes \psi$, where $\xi, \eta \in \mathcal{S}(\mathbb{R}_+)$ and $\phi, \psi \in (E)$. Then, $\varOmega(\varPhi, \varPsi) = (\xi\eta) \otimes \phi \otimes \psi$ and for any $p \geq 0$ we have

$$\|\varOmega(\xi \otimes \phi, \eta \otimes \psi)\|_{E_p \otimes \varGamma(E_p) \otimes \varGamma(E_p)} = |\xi\eta|_p \|\phi\|_p \|\psi\|_p. \tag{4.7}$$

Since the pointwise multiplication of $\mathcal{S}(\mathbb{R}_+)$ yields a continuous bilinear map, there exist $q > 0$ and $C = C(p, q) > 0$ such that $|\xi\eta|_p \leq C|\xi|_{p+q}|\eta|_{p+q}$ for all $\xi, \eta \in \mathcal{S}(\mathbb{R}_+)$. Hence (4.7) becomes

$$\begin{aligned}&\|\varOmega(\xi \otimes \phi, \eta \otimes \psi)\|_{E_p \otimes \varGamma(E_p) \otimes \varGamma(E_p)} \\ &\quad \leq C|\xi|_{p+q}|\eta|_{p+q}\|\phi\|_p\|\psi\|_p \\ &\quad \leq C\|\xi \otimes \phi\|_{E_{p+q} \otimes \varGamma(E_{p+q})}\|\eta \otimes \psi\|_{E_{p+q} \otimes \varGamma(E_{p+q})}.\end{aligned}$$

Then, by definition of the π-tensor product, for $\varPhi, \varPsi \in \mathcal{S}(\mathbb{R}_+, (E))$ we have

$$\|\varOmega(\varPhi, \varPsi)\|_{E_p \otimes \varGamma(E_p) \otimes \varGamma(E_p)} \leq C\|\varPhi\|_{E_{p+q} \otimes_\pi \varGamma(E_{p+q})}\|\varPsi\|_{E_{p+q} \otimes_\pi \varGamma(E_{p+q})}. \tag{4.8}$$

Note that

$$\mathcal{S}(\mathbb{R}_+) \otimes (E) \cong \operatorname*{proj\,lim}_{p\to\infty} E_p \otimes_\pi \varGamma(E_p) \cong \operatorname*{proj\,lim}_{p\to\infty} E_p \otimes \varGamma(E_p),$$

which follows from the nuclearity of $\mathcal{S}(\mathbb{R}_+)$ (or (E)). Hence the assertion follows from (4.8). □

We need the "diagonalized" tensor product $\nabla \oslash \nabla$ of the stochastic gradients. For each $\phi, \psi \in (E)$ we define

$$[(\nabla \oslash \nabla)(\phi \otimes \psi)](t) = \nabla\phi(t) \otimes \nabla\psi(t), \qquad t \in \mathbb{R}_+.$$

Noting that $\nabla\phi, \nabla\psi \in \mathcal{S}(\mathbb{R}_+, (E))$, we have $(\nabla \oslash \nabla)(\phi \otimes \psi) = \Omega(\nabla\phi, \nabla\psi)$ by Lemma 4.3. Therefore,

$$\nabla \oslash \nabla : (E) \otimes (E) \to \mathcal{S}(\mathbb{R}_+, (E) \otimes (E))$$

is a continuous linear map.

The *conservation gradient* is now defined by compositions of continuous linear maps:

$$\begin{aligned} \nabla^0 : \mathcal{L}((E)^*, (E)) &\xrightarrow{\cong} (E) \otimes (E) \xrightarrow{\nabla\oslash\nabla} \mathcal{S}(\mathbb{R}_+) \otimes (E) \otimes (E) \\ &\xrightarrow{\cong} \mathcal{S}(\mathbb{R}_+, (E) \otimes (E)) \xrightarrow{\cong} \mathcal{S}(\mathbb{R}_+, \mathcal{L}((E)^*, (E))). \end{aligned} \tag{4.9}$$

The *conservation integral* δ^0 is by definition the adjoint map of the creation gradient ∇^0. Taking the adjoint map of (4.9), we have

$$\delta^0 = (\nabla^0)^* : \mathcal{S}'(\mathbb{R}_+, \mathcal{L}((E), (E)^*)) \to \mathcal{L}((E), (E)^*).$$

For $\Xi \in L^2(\mathbb{R}_+, \mathcal{L}((E), (E)^*))$ we have

$$\langle\langle \delta^0(\Xi)\phi, \psi \rangle\rangle = \int_{\mathbb{R}_+} \langle\langle \Xi_t(\nabla\phi(t)), \nabla\psi(t) \rangle\rangle \, dt, \qquad \phi, \psi \in (E).$$

Therefore,

$$\delta^0(\Xi)\phi = \delta(\Xi\nabla\phi), \quad \Xi \in L^2(\mathbb{R}_+, \mathcal{L}((E), (E)^*)), \quad \phi \in (E), \tag{4.10}$$

where $\Xi\nabla\phi$ is a classical stochastic process defined by $[\Xi\nabla\phi](t) = \Xi_t(\nabla\phi(t))$.

Remark 4.4. During the above discussion the domain of δ^ϵ is taken as large as possible in the sense that $\delta^\epsilon(\Xi)$ is defined as a white noise operator. This was achieved by taking the smallest possible domain of ∇^ϵ. From this aspect some regularity properties of the quantum stochastic integrals $\delta^\epsilon(\Xi)$ are studied systematically in terms of extendability of ∇^ϵ, see [13] for details.

Remark 4.5. We see from (4.3), (4.5) and (4.10) that our definitions of the Hitsuda–Skorohod quantum stochastic integrals coincide with the ones introduced by Belavkin [3] and Lindsay [17] for a common integrand. In fact, their definition starts with the right-hand sides of (4.3), (4.5) and (4.10) for suitably chosen Ξ and ϕ. Our definition is more direct thanks to the quantum stochastic gradients acting on white noise operators.

5. Differential Calculus for Quantum Stochastic Integrals

5.1. QWN-Derivatives of Quantum Hitsuda–Skorohod Integrals. For each

$$\Xi \in L^2(\mathbb{R}_+, \mathcal{L}((E), (E)^*)) \cong L^2(\mathbb{R}_+, (E)^* \otimes (E)^*)$$

we may choose $p \geq 0$ such that

$$\Xi \in L^2(\mathbb{R}_+, \mathcal{L}_2(\Gamma(E_p), \Gamma(E_{-p}))) \cong L^2(\mathbb{R}_+) \otimes \mathcal{L}_2(\Gamma(E_p), \Gamma(E_{-p})).$$

In view of this identification, we write $D_\zeta^\pm \Xi = (I \otimes D_\zeta^\pm)\Xi$ for simplicity. Then $D_\zeta^\pm \Xi \in L^2(\mathbb{R}_+, \mathcal{L}((E), (E)^*))$ for all $\zeta \in \mathcal{S}(\mathbb{R}_+)$.

Lemma 5.1. *It holds that*

$$\nabla[a^*(\zeta)\phi](t) = a^*(\zeta)[\nabla\phi(t)] + \zeta(t)\phi, \qquad \phi \in (E), \quad \zeta \in \mathcal{S}(\mathbb{R}_+).$$

Proof. This is nothing else but the canonical commutation relation $[a_t, a^*(\zeta)] = \zeta(t)I$. Note that both $a_t, a^*(\zeta)$ are members of $\mathcal{L}((E), (E))$. □

Theorem 5.2. *Let $\zeta \in \mathcal{S}(\mathbb{R}_+)$ and $\Xi \in L^2(\mathbb{R}_+, \mathcal{L}((E), (E)^*))$. It holds that*

$$D_\zeta^+(\delta^+(\Xi)) = \delta^+(D_\zeta^+\Xi) + \int_{\mathbb{R}_+} \zeta(t)\Xi_t\, dt, \tag{5.1}$$

$$D_\zeta^-(\delta^+(\Xi)) = \delta^+(D_\zeta^-\Xi), \tag{5.2}$$

$$D_\zeta^+(\delta^-(\Xi)) = \delta^-(D_\zeta^+\Xi), \tag{5.3}$$

$$D_\zeta^-(\delta^-(\Xi)) = \delta^-(D_\zeta^-\Xi) + \int_{\mathbb{R}_+} \zeta(t)\Xi_t\, dt. \tag{5.4}$$

$$D_\zeta^+(\delta^0(\Xi)) = \delta^0(D_\zeta^+\Xi) + \delta^-(\zeta\Xi), \tag{5.5}$$

$$D_\zeta^-(\delta^0(\Xi)) = \delta^0(D_\zeta^-\Xi) + \delta^+(\zeta\Xi), \tag{5.6}$$

where $\zeta\Xi \in L^2(\mathbb{R}_+, \mathcal{L}((E), (E)^*)$ is defined by $(\zeta\Xi)(t) = \zeta(t)\Xi_t$.

Proof. We first prove (5.1). By applying Lemma 3.2 we have

$$\mathcal{K}(D_\zeta^+(\delta^+(\Xi))) = (a(\zeta) \otimes I)\mathcal{K}(\delta^+(\Xi)) - (I \otimes a^*(\zeta))\mathcal{K}(\delta^+(\Xi)). \tag{5.7}$$

Let $\phi, \psi \in (E)$. As for the first term in the right-hand side of (5.7), we have

$$\begin{aligned}\langle\langle (a(\zeta) \otimes I)\mathcal{K}(\delta^+(\Xi)),\, \psi \otimes \phi \rangle\rangle &= \langle\langle \mathcal{K}(\delta^+(\Xi)),\, a^*(\zeta)\psi \otimes \phi \rangle\rangle \\ &= \langle\langle \delta^+(\Xi)\phi,\, a^*(\zeta)\psi \rangle\rangle \\ &= \int_{\mathbb{R}_+} \langle\langle \Xi_t\phi,\, [\nabla(a^*(\zeta)\psi)](t) \rangle\rangle dt,\end{aligned}$$

where the last equality is due to (4.2). By virtue of Lemma 5.1, the last integral becomes

$$\begin{aligned}&= \int_{\mathbb{R}_+} \langle\langle \Xi_t\phi,\, a^*(\zeta)[\nabla\psi(t)] \rangle\rangle dt + \int_{\mathbb{R}_+} \langle\langle \Xi_t\phi,\, \zeta(t)\psi \rangle\rangle dt \\ &= \langle\langle \delta^+(a(\zeta)\Xi)\phi,\, \psi \rangle\rangle + \int_{\mathbb{R}_+} \zeta(t)\, \langle\langle \Xi_t\phi,\, \psi \rangle\rangle\, dt.\end{aligned} \tag{5.8}$$

Similarly, for the second term in the right-hand side of (5.7) we have

$$\langle\langle (I \otimes a^*(\zeta))\mathcal{K}(\delta^+(\Xi)),\ \psi \otimes \phi \rangle\rangle = \langle\langle \delta^+(\Xi a(\zeta))\phi,\ \psi \rangle\rangle. \tag{5.9}$$

Inserting (5.8) and (5.9) into (5.7), we have

$$\begin{aligned}\langle\langle D_\zeta^+(\delta^+(\Xi))\phi,\ \psi\rangle\rangle &= \langle\langle \delta^+(a(\zeta)\Xi - \Xi a(\zeta))\phi,\ \psi\rangle\rangle + \int_{\mathbb{R}_+} \zeta(t)\, \langle\langle \Xi_t\phi,\ \psi\rangle\rangle\, dt \\ &= \langle\langle \delta^+(D_\zeta^+\Xi)\phi,\ \psi\rangle\rangle + \int_{\mathbb{R}_+} \zeta(t)\, \langle\langle \Xi_t\phi,\ \psi\rangle\rangle\, dt,\end{aligned}$$

which proves (5.1).

We next prove (5.5) by mimicking the above argument. In fact, we have

$$\mathcal{K}(D_\zeta^+(\delta^0(\Xi))) = (a(\zeta)\otimes I)\mathcal{K}(\delta^0(\Xi)) - (I \otimes a^*(\zeta))\mathcal{K}(\delta^0(\Xi)). \tag{5.10}$$

For any $\phi, \psi \in (E)$ we have

$$\begin{aligned}\langle\langle (a(\zeta)\otimes I)\mathcal{K}(\delta^0(\Xi)),\ \psi\otimes\phi\rangle\rangle &= \langle\langle \mathcal{K}(\delta^0(\Xi)),\ a^*(\zeta)\psi\otimes\phi\rangle\rangle \\ &= \langle\langle \delta^0(\Xi)\phi,\ a^*(\zeta)\psi\rangle\rangle \\ &= \int_{\mathbb{R}_+} \langle\langle \Xi_t(\nabla\phi(t)),\ [\nabla a^*(\zeta)\psi](t)\rangle\rangle dt.\end{aligned}$$

By Lemma 5.1 the last expression becomes

$$\begin{aligned}&= \int_{\mathbb{R}_+} \langle\langle a(\zeta)\Xi_t(\nabla\phi(t)),\ (\nabla\psi)(t)\rangle\rangle\, dt + \int_{\mathbb{R}_+} \zeta(t)\, \langle\langle \Xi_t(\nabla\phi(t)),\ \psi\rangle\rangle\, dt \\ &= \langle\langle \delta^0(a(\zeta)\Xi)\phi,\ \psi\rangle\rangle + \langle\langle \delta^-(\zeta\,\Xi)\phi,\ \psi\rangle\rangle.\end{aligned} \tag{5.11}$$

On the other hand, one can see easily that

$$\langle\langle (I\otimes a^*(\zeta))\mathcal{K}(\delta^0(\Xi)),\ \psi\otimes\phi\rangle\rangle = \langle\langle \delta^0(\Xi a(\zeta))\phi,\ \psi\rangle\rangle. \tag{5.12}$$

Inserting (5.11) and (5.12) into (5.10), we obtain

$$\langle\langle D_\zeta^+(\delta^0(\Xi))\phi,\ \psi\rangle\rangle = \langle\langle \delta^0(D_\zeta^+\Xi)\phi,\ \psi\rangle\rangle + \langle\langle \delta^-(\zeta\,\Xi)\phi,\ \psi\rangle\rangle,$$

which shows (5.5). The rest is verified in a similar manner. □

5.2. Pointwise QWN-Derivatives of Quantum Hitsuda–Skorohod Integrals. The formulas for pointwise qwn-derivatives (Theorem 5.4 below) formally follow from (5.1)–(5.6) by setting $\zeta = \delta_t$. For mathematical rigor we repeat the argument in Sect. 3.4 at a level of quantum stochastic processes.

First we set

$$L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}, \mathcal{G}^*)) = \bigcup_{p,q\in\mathbb{R}} L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q)) = \bigcup_{p\ge 0} L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p, \mathcal{G}_{-p})).$$

For $\Xi = \{\Xi_s\} \in L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}, \mathcal{G}^*))$ we shall define $D_t^\pm \Xi$.

Lemma 5.3. *For any $p \geq 0$ there exists $q \geq \max\{p, p/(-\log\rho)\}$ such that*

$$L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p, \mathcal{G}_{-p})) \subset L^2(\mathbb{R}_+, \mathcal{L}_2(\Gamma(E_q), \mathcal{G}_{-q})).$$

Proof. Let $\Xi \in L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p, \mathcal{G}_{-p}))$. Then, $\Xi_s \in \mathcal{L}(\mathcal{G}_p, \mathcal{G}_{-p})$ for a.e. $s \in \mathbb{R}_+$. From the proof of Lemma 3.4 we see that

$$\|\Xi_s\|_{\mathcal{L}_2(\Gamma(E_q),\mathcal{G}_{-q})} \leq L(p,q)\|\Xi_s\|_{\mathcal{L}(\mathcal{G}_p,\mathcal{G}_{-p})}, \tag{5.13}$$

where $L(p,q) > 0$ is the Hilbert–Schmidt norm of $\Gamma(E_q) \to \Gamma(E_r)$, where $r = p/(-\log\rho)$. Then the assertion follows by integrating (5.13). □

Now let $\Xi = \{\Xi_s\} \in L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}, \mathcal{G}^*))$. With the help of Lemma 5.3 we may choose $p \geq 1$ satisfying $\Xi \in L^2(\mathbb{R}_+, \mathcal{L}_2(\Gamma(E_p), \mathcal{G}_{-p}))$. In particular, $\Xi_s \in \mathcal{L}_2(\Gamma(E_p), \mathcal{G}_{-p})$ for a.e. $s \in \mathbb{R}_+$. Then, by virtue of Theorem 3.9, for any $r > 0$ it holds that

$$\int_{\mathbb{R}_+} \|D_t^+ \Xi_s\|^2_{\mathcal{L}_2(\Gamma(E_{p+r}),\mathcal{G}_{-p-r})} dt \leq C(p,r) \parallel \Xi_s \parallel^2_{\mathcal{L}_2(\Gamma(E_p),\mathcal{G}_{-p})}.$$

Integrating both sides with respect to s over $\mathbb{R}_+$, we obtain

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \|D_t^+ \Xi_s\|^2_{\mathcal{L}_2(\Gamma(E_{p+r}),\mathcal{G}_{-p-r})} dtds \leq C(p,r) \parallel \Xi \parallel^2_{L^2(\mathbb{R}_+,\mathcal{L}_2(\Gamma(E_p),\mathcal{G}_{-p}))}.$$

By the Fubini theorem we see that for a.e. $t \in \mathbb{R}_+$, $s \mapsto D_t^+ \Xi_s$ is an L^2-function in $s \in \mathbb{R}_+$ with values in $\mathcal{L}_2(\Gamma(E_{p+r}), \mathcal{G}_{-p-r}) \subset \mathcal{L}((E), \mathcal{G}^*)$. Thus the pointwise annihilation-derivative $D_t^+ \Xi \in L^2(\mathbb{R}_+, \mathcal{L}((E), \mathcal{G}^*))$ is defined for a.e. $t \in \mathbb{R}_+$.

In a similar manner, noting that $\mathcal{L}(\mathcal{G}, \mathcal{G}^*) \subset \mathcal{L}(\mathcal{G}, (E)^*)$, we define the pointwise annihilation derivative $D_t^- \Xi \in L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}, (E)^*)$ for a.e. $t \in \mathbb{R}_+$.

Next, mimicking the argument in Sect. 4.3, we define the quantum stochastic gradients as continuous maps:

$$\nabla^\epsilon : \begin{array}{l} \mathcal{L}(\mathcal{G}^*, (E)) \to L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}^*, (E))), \\ \mathcal{L}((E)^*, \mathcal{G}) \to L^2(\mathbb{R}_+, \mathcal{L}((E)^*, \mathcal{G})), \end{array}$$

and by their adjoint actions the quantum Hitsuda–Skorohod integrals:

$$\delta^\epsilon : \begin{array}{l} L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}, (E)^*)) \to \mathcal{L}(\mathcal{G}, (E)^*), \\ L^2(\mathbb{R}_+, \mathcal{L}((E), \mathcal{G}^*)) \to \mathcal{L}((E), \mathcal{G}^*), \end{array} \tag{5.14}$$

where $\epsilon \in \{+, -, 0\}$, for more details see [13].

Theorem 5.4. *Let $\Xi \in L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}, \mathcal{G}^*))$. Then for a.e. $t \in \mathbb{R}_+$ we have*

$$D_t^+(\delta^+(\Xi)) = \delta^+(D_t^+\Xi) + \Xi_t, \tag{5.15}$$

$$D_t^-(\delta^+(\Xi)) = \delta^+(D_t^-\Xi), \tag{5.16}$$

$$D_t^+(\delta^-(\Xi)) = \delta^-(D_t^+\Xi), \tag{5.17}$$

$$D_t^-(\delta^-(\Xi)) = \delta^-(D_t^-\Xi) + \Xi_t, \tag{5.18}$$

$$D_t^+(\delta^0(\Xi)) = \delta^0(D_t^+\Xi) + \Xi_t a_t, \tag{5.19}$$

$$D_t^-(\delta^0(\Xi)) = \delta^0(D_t^-\Xi) + a_t^*\Xi_t. \tag{5.20}$$

Proof. We shall prove (5.15). Since $\mathcal{L}(\mathcal{G},\mathcal{G}^*) \subset \mathcal{L}((E),\mathcal{G}^*)$, we see from (5.14) that $\delta^+(\Xi) \in \mathcal{L}((E),\mathcal{G}^*)$. Applying the creation derivative (see Sect. 3.4), we have $D_t^+(\delta^+(\Xi))$ as an $\mathcal{L}((E),\mathcal{G}^*)$-valued L^2-function in t. On the other hand, we see from the above argument with (5.14) that $\delta^+(D_t^+\Xi)$ is $\mathcal{L}((E),\mathcal{G}^*)$-valued L^2-function in t. Thus, both sides of (5.15) are $\mathcal{L}((E),\mathcal{G}^*)$-valued L^2-functions in t. It is then sufficient to show their inner products with an arbitrary $\zeta \in L^2(\mathbb{R}_+)$ coincide, which is immediate from Theorem 5.2.

The proof of the rest is similar. For (5.19) and (5.20) we employ the following formulas:

$$\langle\langle \delta^-(\zeta\Xi)\phi,\ \psi\rangle\rangle = \int_{\mathbb{R}_+} \zeta(t)\, \langle\langle \Xi_t a_t\phi,\ \psi\rangle\rangle\, dt,$$

$$\langle\langle \delta^+(\zeta\Xi)\phi,\ \psi\rangle\rangle = \int_{\mathbb{R}_+} \zeta(t)\, \langle\langle \Xi_t\phi,\ a_t\psi\rangle\rangle\, dt = \int_{\mathbb{R}_+} \zeta(t)\langle\langle a_t^*\Xi_t\phi,\ \psi\rangle\rangle dt,$$

for $\phi, \psi \in (E)$. □

5.3. QWN-Derivatives of Adapted Integrals. First we recall that for all $t \in \mathbb{R}_+$, the space $\mathcal{G}_p$ admits a factorization

$$\mathcal{G}_p = \mathcal{G}_p([0,t]) \otimes \mathcal{G}_p([t,\infty)), \tag{5.21}$$

which is derived from $L^2(\mathbb{R}_+) = L^2([0,t]) \oplus L^2([t,\infty))$. A quantum stochastic process $\{\Xi_t\}_{t\ge 0} \subset \mathcal{L}(\mathcal{G}_p,\mathcal{G}_q)$ is said to be *adapted* if for all $t \in \mathbb{R}_+$, Ξ_t admits a factorization

$$\Xi_t = \Xi_{[0,t]} \otimes I_{[t}\,,$$

according to (5.21), where $I_{[t}$ is the identity operator on $\mathcal{G}_p([t,\infty))$.

Proposition 5.5. *Let $\{\Xi_t\} \in \mathcal{L}(\mathcal{G}_p,\mathcal{G}_q)$ be an adapted process. Then, for any $\zeta \in L^2(\mathbb{R}_+)$, $\{D_\zeta^\pm \Xi_t\}$ is an adapted process. In fact, for any $t \in \mathbb{R}_+$ we have*

$$D_\zeta^+\Xi_t = \left(D^+_{\zeta_{[0,t]}}\Xi_{[0,t]}\right) \otimes I_{[t}, \quad D_\zeta^-\Xi_t = \left(D^-_{\zeta_{[0,t]}}\Xi_{[0,t]}\right) \otimes I_{[t}, \tag{5.22}$$

where $\Xi_t = \Xi_{[0,t]} \otimes I_{[t}$ and $\zeta_{[0,t]} = \zeta\mathbf{1}_{[0,t]}$.

Proof. By using the fact that for any $\zeta, \xi \in \mathcal{S}(\mathbb{R}_+)$,

$$a(\zeta)\phi_\xi = \left(a(\zeta_{[0,t]})\phi_{\xi_{[0,t]}}\right) \otimes \phi_{\xi_{[t}} + \phi_{\xi_{[0,t]}} \otimes \left(a(\zeta_{[t})\phi_{\xi_{[t}}\right),$$

where $\xi_{[t} = \xi\mathbf{1}_{[t,\infty)}$, we can easily see that for any $\xi \in \mathcal{S}(\mathbb{R}_+)$,

$$D_\zeta^+\Xi_t\phi_\xi = \left(\left(D^+_{\zeta_{[0,t]}}\Xi_{[0,t]}\right) \otimes I_{[t}\right)\phi_\xi.$$

Since $\{\phi_\xi\,;\ \xi \in \mathcal{S}(\mathbb{R}_+)\}$ spans a dense subspace of $\mathcal{G}_p$, the first relation in (5.22) follows by continuity. The second relation is verified in a similar fashion. □

Proposition 5.6. *Let* $\{\Xi_t\} \subset \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q)$ *be an adapted process. Then for any* $s \geq 0$ *and* $\zeta \in L^2([s, \infty))$ *we have*

$$D_\zeta^\pm \Xi_s = 0.$$

Therefore, for any $s \geq 0$ *it holds that*

$$D_t^\pm \Xi_s = 0 \quad \textit{for a.e. } t \geq s.$$

Proof. Since $a(\zeta)\phi = 0$ for all $\phi \in \mathcal{G}_p([0, t])$, $D^\pm_{\zeta_{[0,t]}} \Xi_{[0,t]} = 0$ on $\mathcal{G}_p([0, t])$. Hence the proof is obvious from (5.22). □

Combining Theorem 5.4 and Proposition 5.6, we come to the following

Theorem 5.7. *Let* $\Xi \in L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q))$ *be an adapted process. Then for a.e.* $t \in \mathbb{R}_+$ *we have*

$$\begin{aligned}
D_t^+(\delta^+(\Xi)) &= \delta^+(\mathbf{1}_{[t,\infty)} D_t^+ \Xi) + \Xi_t, \\
D_t^-(\delta^+(\Xi)) &= \delta^+(\mathbf{1}_{[t,\infty)} D_t^- \Xi), \\
D_t^+(\delta^-(\Xi)) &= \delta^-(\mathbf{1}_{[t,\infty)} D_t^+ \Xi), \\
D_t^-(\delta^-(\Xi)) &= \delta^-(\mathbf{1}_{[t,\infty)} D_t^- \Xi) + \Xi_t, \\
D_t^+(\delta^0(\Xi)) &= \delta^0(\mathbf{1}_{[t,\infty)} D_t^+ \Xi) + \Xi_t a_t, \\
D_t^-(\delta^0(\Xi)) &= \delta^0(\mathbf{1}_{[t,\infty)} D_t^- \Xi) + a_t^* \Xi_t.
\end{aligned}$$

Remark 5.8. Let $\Xi \in L^2(\mathbb{R}_+, \mathcal{L}((E), (E)^*))$. Then $\{a_t^* \Xi\}$, $\{\Xi_t a_t\}$ and $\{a_t^* \Xi_t a_t\}$ are white noise integrable on a finite interval. Moreover, it is easily checked that

$$\delta^+(\mathbf{1}_{[0,t]}\Xi) = \int_0^t a_s^* \Xi_s \, ds, \qquad \delta^-(\mathbf{1}_{[0,t]}\Xi) = \int_0^t \Xi_s a_s \, ds,$$
$$\delta^0(\mathbf{1}_{[0,t]}\Xi) = \int_0^t a_s^* \Xi_s a_s \, ds.$$

If $\Xi \in L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}, \mathcal{G}^*))$ is adapted, we have

$$\delta^+(\mathbf{1}_{[0,t]}\Xi) = \int_0^t \Xi_s dA_s^*, \qquad \delta^-(\mathbf{1}_{[0,t]}\Xi) = \int_0^t \Xi_s dA_s,$$
$$\delta^0(\mathbf{1}_{[0,t]}\Xi) = \int_0^t \Xi_s d\Lambda_s,$$

where the right-hand sides are quantum stochastic integrals of Itô type [9].

6. Application to Quantum Martingales

6.1. Regular Quantum Martingales. An adapted process $\{M_t\}_{t\geq 0} \subset \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q)$ is called a *quantum martingale* if

$$\langle\langle M_t \phi_{\xi_{s]}}, \phi_{\eta_{s]}} \rangle\rangle = \langle\langle M_s \phi_{\xi_{s]}}, \phi_{\eta_{s]}} \rangle\rangle, \quad \xi, \eta \in L^2(\mathbb{R}_+), \quad 0 \leq s \leq t.$$

The above condition is equivalent to

$$\langle\langle \mathbf{E}_s M_t \mathbf{E}_s \phi_\xi ,\, \phi_\eta \rangle\rangle = \langle\langle \mathbf{E}_s M_s \mathbf{E}_s \phi_\xi ,\, \phi_\eta \rangle\rangle, \qquad \xi, \eta \in H, \quad 0 \le s \le t,$$

where $\mathbf{E}_t$ is the conditional expectation defined by

$$\mathbf{E}_t \Phi = \Gamma(\mathbf{1}_{[0,t]})\Phi = (\mathbf{1}_{[0,t]}^{\otimes n} F_n), \qquad \Phi = (F_n) \in \mathcal{G}^*.$$

After the recent work [9], a quantum martingale $\{M_t\} \subset \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q)$ is said to be *regular with respect to a Radon measure* $\mathfrak{m}$ on $\mathbb{R}_+$, or simply *regular* if

$$\begin{aligned} &|\!|\!| (M_t - M_s)\phi |\!|\!|_q^2 \le |\!|\!| \phi |\!|\!|_p^2 \ \mathfrak{m}([s,t]), \\ &\Big|\!\Big|\!\Big| (M_t^* - M_s^*)\psi \Big|\!\Big|\!\Big|_{-p}^2 \le |\!|\!| \psi |\!|\!|_{-q}^2 \ \mathfrak{m}([s,t]), \end{aligned}$$

for all $\phi \in \mathcal{G}_p([0,s])$, $\psi \in \mathcal{G}_{-q}([0,s])$ and $0 \le s < t$.

Example 6.1. Let $l, m \ge 0$ be integers. As is easily checked, for any $p \in \mathbb{R}$ and $q > 0$ there exists a constant $C \ge 0$ such that

$$|\!|\!|((A_t^*)^l A_t^m - (A_s^*)^l A_s^m)\phi|\!|\!|_p^2 \le C \,|\!|\!| \phi |\!|\!|_{p+q}^2 \,(t^l - s^l) s^m$$

for all $\phi \in \mathcal{G}_{p+q}([0,s])$ and $0 \le s < t$. Hence $\{(A_t^*)^l A_t^m\}_{t \ge 0}$ is a regular quantum martingale in $\mathcal{L}(\mathcal{G}_{p+q}, \mathcal{G}_p)$. In particular, so are the annihilation process $\{A_t\}$ and the creation process $\{A_t^*\}$.

Example 6.2. The conservation process $\{\Lambda_t\}_{t \ge 0}$ is a regular quantum martingale in $\mathcal{L}(\mathcal{G}_{p+q}, \mathcal{G}_p)$ for any $p \in \mathbb{R}$ and $q > 0$. In fact,

$$|\!|\!| (\Lambda_t - \Lambda_s)\phi |\!|\!|_p^2 = 0$$

for all $\phi \in \mathcal{G}_{p+q}([0,s])$ and $0 \le s < t$.

We now recall the fundamental result due to Ji [9].

Theorem 6.3. *Let $\{M_t\}_{t \ge 0} \subset \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q)$ be a quantum martingale, regular with respect to a Radon measure $\mathfrak{m}$ on $\mathbb{R}_+$. Then there exist adapted processes $\{E_t\}$, $\{F_t\}$, $\{G_t\}$ in $\mathcal{L}(\mathcal{G}_p, \mathcal{G}_q)$ and $\lambda \in \mathbb{C}$ such that*

$$M_t = \lambda I + \int_0^t (E_s dA_s + F_s dA_s^* + G_s d\Lambda_s) \tag{6.1}$$

as operators in $\mathcal{L}((E), \mathcal{G}^*)$, and $s \mapsto \|G_s\|_{\mathcal{L}(\mathcal{G}_p, \mathcal{G}_q)}$ is locally bounded and

$$\max\{\|E_s\|_{\mathcal{L}(\mathcal{G}_p,\mathcal{G}_q)}^2, \|F_s\|_{\mathcal{L}(\mathcal{G}_p,\mathcal{G}_q)}^2\} \le \mathfrak{m}'_{\mathrm{ac}}(s) \quad \textit{for all } s \ge 0,$$

where $\mathfrak{m}'_{\mathrm{ac}}$ denotes the density of the absolutely continuous part of $\mathfrak{m}$. Such a triple $(\{E_t\}, \{F_t\}, \{G_t\})$ is unique. Conversely, if $\{M_t\} \subset \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q)$ admits the integral representation (6.1) with adapted processes $\{E_t\}, \{F_t\}, \{G_t\}$ in $\mathcal{L}(\mathcal{G}_p, \mathcal{G}_q)$ such that $\|E_s\|_{\mathcal{L}(\mathcal{G}_p,\mathcal{G}_q)}$ and $\|F_s\|_{\mathcal{L}(\mathcal{G}_p,\mathcal{G}_q)}$ are locally square integrable in $s \in \mathbb{R}_+$, then $\{M_t\}$ is a regular quantum martingale.

Remark 6.4. Recall that $\{A_t\}, \{A_t^*\}, \{\Lambda_t\}$ are excluded from the class of regular quantum martingales in the sense of Parthasarathy–Sinha [30] due to their unboundedness in the Fock space $\Gamma(L^2(\mathbb{R}_+))$. The choice of Fock chain $\{\mathcal{G}_p\}$ has the advantage of including a wider class of regular quantum martingales possibly unbounded in $\Gamma(L^2(\mathbb{R}_+))$.

6.2. Calculating the Integrands. We are now in a position to discuss how the integrands in (6.1) are obtained from $\{M_t\}$. We start with the following

Lemma 6.5. *Let $\{\Xi_t\} \subset \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q)$ be an adapted quantum stochastic process satisfying*

$$\int_0^t \|\Xi_s\|^2_{\mathcal{L}(\mathcal{G}_p,\mathcal{G}_q)} ds < \infty \quad \text{for all } t \geq 0.$$

Then for a.e. $t \in \mathbb{R}_+$ we have

$$\begin{aligned} D_t^+(\delta^+(\Xi \mathbf{1}_{[0,t]})) &= \Xi_t, & D_t^-(\delta^+(\Xi \mathbf{1}_{[0,t]})) &= 0, \\ D_t^+(\delta^-(\Xi \mathbf{1}_{[0,t]})) &= 0, & D_t^-(\delta^-(\Xi \mathbf{1}_{[0,t]})) &= \Xi_t, \\ D_t^+(\delta^0(\Xi \mathbf{1}_{[0,t]})) &= \Xi_t a_t, & D_t^-(\delta^0(\Xi \mathbf{1}_{[0,t]})) &= a_t^* \Xi_t. \end{aligned}$$

Proof. Straightforward from Theorem 5.7. □

Theorem 6.6. *Let $\{M_t\}_{t\geq 0}$ be a regular quantum martingale in $\mathcal{L}(\mathcal{G}_p, \mathcal{G}_q)$ with the integral representation:*

$$M_t = \lambda I + \int_0^t E_s dA_s + \int_0^t F_s dA_s^* + \int_0^t G_s d\Lambda_s, \qquad t \geq 0, \tag{6.2}$$

as described in Theorem 6.3. Then the integrands in (6.2) satisfy the following relations:

$$E_s = D_s^- \left[M_s - \int_0^s D_u^+ M_u dA_u^* \right], \tag{6.3}$$

$$F_s = D_s^+ \left[M_s - \int_0^s D_u^- M_u dA_u \right], \tag{6.4}$$

$$G_s = D_s^+ \left[\int_0^s \left\{ D_u^- \left(M_u - \int_0^u E_v dA_v - \int_0^u F_v dA_v^* \right) \right\} du \right]. \tag{6.5}$$

Proof. First note that (6.2) is written in the form:

$$M_t = \lambda I + \delta^-(\mathbf{1}_{[0,t]} E) + \delta^+(\mathbf{1}_{[0,t]} F) + \delta^0(\mathbf{1}_{[0,t]} G).$$

Then, applying the formulas in Lemma 6.5, we have

$$D_t^+ M_t = F_t + G_t a_t, \qquad D_t^- M_t = E_t + a_t^* G_t,$$

and hence,

$$\begin{aligned} M_t - \int_0^t D_s^- M_s dA_s &= \lambda I + \int_0^t F_s dA_s^* = \lambda I + \delta^+(\mathbf{1}_{[0,t]} F), \\ M_t - \int_0^t D_s^+ M_s dA_s^* &= \lambda I + \int_0^t E_s dA_s = \lambda I + \delta^-(\mathbf{1}_{[0,t]} E). \end{aligned}$$

Applying the formulas in Lemma 6.5 again, we obtain

$$E_t = D_t^- \left[M_t - \int_0^t D_s^+ M_s dA_s^* \right], \quad F_t = D_t^+ \left[M_t - \int_0^t D_s^- M_s dA_s \right],$$

which proves (6.3) and (6.4). On the other hand, it follows from (6.2) that

$$\int_0^t G_s d\Lambda_s = \delta^0(\mathbf{1}_{[0,t]}G) = M_t - \lambda I - \int_0^t E_s dA_s - \int_0^t F_s dA_s^*.$$

Applying D_t^- leads

$$a_t^* G_t = D_t^- \left[M_t - \int_0^t E_u dA_u - \int_0^t F_u dA_u^* \right].$$

Integrating both sides with respect to t, we come to

$$\int_0^t G_s dA_s^* = \int_0^t \left\{ D_s^- \left(M_s - \int_0^s E_u dA_u - \int_0^s F_u dA_u^* \right) \right\} ds.$$

Finally, applying D_t^+ we have

$$G_t = D_t^+ \left[\int_0^t \left\{ D_s^- \left(M_s - \int_0^s E_u dA_u - \int_0^s F_u dA_u^* \right) \right\} ds \right],$$

which proves (6.5). □

6.3. An Example. We shall discuss an instructive example due to Parthasarathy [28] along our approach.

Consider an operator K of Hilbert–Schmidt class on $L^2(\mathbb{R}_+)$ with the corresponding integral kernel $\kappa \in L^2(\mathbb{R}_+ \times \mathbb{R}_+)$, i.e.,

$$K\xi(u) = \int_0^\infty \kappa(u, v)\xi(v)dv, \qquad \xi \in L^2(\mathbb{R}_+).$$

In the following we fix $p \in \mathbb{R}$ and $q \geq \max\{0, \log \|K\|_{\mathrm{op}}\}$ arbitrarily, where $\|K\|_{\mathrm{op}}$ is the operator norm of K. Then, the second quantization $\Gamma(K)$ is a member of $\mathcal{L}(\mathcal{G}_{p+q}, \mathcal{G}_p)$, as is seen from the obvious inequalities:

$$|\!|\!| \Gamma(K)\phi |\!|\!|_p^2 \leq \sum_{n=0}^\infty n! e^{2pn} \|K\|_{\mathrm{op}}^{2n} |f_n|_0^2 \leq |\!|\!| \phi |\!|\!|_{p+q}^2.$$

Define a quantum stochastic process $\{M_t\}$ by

$$M_t = \mathbf{E}_t \Gamma(K) \mathbf{E}_t, \qquad t \geq 0.$$

We shall see that for any $p \in \mathbb{R}$ there exists $q \geq 0$ such that $\{M_t\}$ is a regular quantum martingale in $\mathcal{L}(\mathcal{G}_{p+q}, \mathcal{G}_p)$. In fact, as is easily verified, $\{M_t\}$ is a quantum martingale with the property that $\|M_t\|_{\mathcal{L}(\mathcal{G}_{p+q}, \mathcal{G}_p)}$ is locally bounded in $t \in \mathbb{R}_+$. We need to check that $\{M_t\}$ is regular. Note that for any $0 \leq s < t$ and $\phi = (f_n) \in \mathcal{G}_p([0, s])$ we have

$$\begin{aligned} |\!|\!| (M_t - M_s)\phi |\!|\!|_p^2 &= \sum_{n=0}^\infty n! e^{2pn} \left| \sum_{i=1}^n \left(\mathbf{1}_{[0,t]}^{\otimes(n-i)} \otimes \mathbf{1}_{[s,t]} \otimes \mathbf{1}_{[0,s]}^{\otimes(i-1)} \right) K^{\otimes n} f_n \right|_0^2 \\ &\leq \mathfrak{m}([s, t]) \sum_{n=0}^\infty n! e^{2pn} n \|K\|_{\mathrm{op}}^{2(n-1)} |f_n|_0^2, \end{aligned}$$

where $\mathfrak{m}$ is a Radon measure on $\mathbb{R}_+$ defined by

$$\mathfrak{m}([s,t]) = \int_s^t \int_0^\infty |\kappa(u,v)|^2 dv du, \qquad 0 \le s < t.$$

Replacing q with a larger one satisfying $n\|K\|_{\mathrm{op}}^{2(n-1)} \le e^{2qn}$ for all $n \ge 1$ if necessary, we obtain

$$\| (M_t - M_s)\phi \|_p^2 \le \| \phi \|_{p+q}^2 \, \mathfrak{m}([s,t]),$$

as desired. The second half of the regularity condition is verified similarly.

From Theorem 6.6 we see that M_t admits a unique integral representation as in (6.2). In fact, for any $\zeta \in L^2(\mathbb{R}_+)$ we have

$$\begin{aligned} a(\zeta)M_t &= M_t a\left(\mathbf{1}_{[0,t]} \int_0^t \kappa(u,\cdot)\zeta(u)du\right), \\ M_t a^*(\zeta) &= a^*\left(\mathbf{1}_{[0,t]} \int_0^t \kappa(\cdot,v)\zeta(v)du\right) M_t, \end{aligned}$$

which implies that for a.e. $u \in \mathbb{R}_+$,

$$\begin{aligned} D_u^+ M_u &= M_u \left(a(\mathbf{1}_{[0,u]}\kappa(u,\cdot)) - a_u\right), \\ D_u^- M_u &= \left(a^*(\mathbf{1}_{[0,u]}\kappa(\cdot,u)) - a_u^*\right) M_u. \end{aligned} \tag{6.6}$$

Noting that $\|M_u a(\mathbf{1}_{[0,u]}\kappa(u,\cdot))\|_{\mathcal{L}(\mathcal{G}_p,\mathcal{G}_q)}$ is locally square integrable in $u \in \mathbb{R}_+$ for some $p, q \in \mathbb{R}$, we obtain

$$\delta^+(\mathbf{1}_{[0,s]}(u) D_u^+ M_u) = \delta^+(\mathbf{1}_{[0,s]}(u) M_u a(\mathbf{1}_{[0,u]}\kappa(u,\cdot)) - \delta^0(\mathbf{1}_{[0,s]}(u) M_u),$$

where the integrals are taken with respect to u.

Now applying the formulas in (6.3) and in Lemma 6.5, we have

$$\begin{aligned} E_s &= D_s^- \left(M_s - \delta^+(\mathbf{1}_{[0,s]}(u) D_u^+ M_u)\right) \\ &= D_s^- \left(M_s - \delta^+(\mathbf{1}_{[0,s]}(u) M_u a(\mathbf{1}_{[0,u]}\kappa(u,\cdot)) + \delta^0(\mathbf{1}_{[0,s]}(u) M_u)\right) \\ &= a^*(\mathbf{1}_{[0,s]}\kappa(\cdot,s)) M_s. \end{aligned}$$

Similarly, we obtain

$$F_s = M_s a(\mathbf{1}_{[0,s]}\kappa(s,\cdot)).$$

On the other hand, we see from (6.6) and Lemma 6.5 that

$$D_s^- \left(M_s - \int_0^s E_u dA_u - \int_0^s F_u dA_u^*\right) = -a_s^* M_s.$$

Applying the formulas in (6.5) and Lemma 6.5, we come to

$$\begin{aligned} G_t &= D_t^+ \left[\int_0^t \left\{D_s^- \left(M_s - \int_0^s E_u dA_u - \int_0^s F_u dA_u^*\right)\right\} ds\right] \\ &= -D_t^+ \left[\int_0^t M_s dA_s^*\right] \\ &= -M_s. \end{aligned}$$

Consequently, the stochastic integral representation of $\{M_t\}$ is given by

$$M_t = I + \int_0^t a^*(\mathbf{1}_{[0,s]}\kappa(\cdot, s))M_s dA_s + \int_0^t M_s a(\mathbf{1}_{[0,s]}\kappa(s, \cdot))dA_s^* - \int_0^t M_s d\Lambda_s.$$

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