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Smooth Approximations and Exact Solutions of the 3D Steady Axisymmetric Euler Equations

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Abstract: In this paper, we prove that a class of C^1-smooth approximate solutions $\{u^\varepsilon, p^\varepsilon\}$ to the 3D steady axisymmetric Euler equations will converge strongly to 0 in $L^2_{loc}(R^3)$. The main assumptions are that the approximate solutions have uniformly finite energy and approach a constant state at far fields. We also show a Liouville type theorem that there are no non-trivial C^1-smooth exact solutions with finite energy and uniform constant state at far fields.

1. Introduction

The three-dimensional (3D) incompressible steady Euler equations in R^3 are

$$\begin{cases} (u\cdot\nabla)u+\nabla p=0, \quad x\in R^3,\\ \operatorname{div} u=0. \end{cases} \tag{1.1}$$

Here $u=(u_1(x),u_2(x),u_3(x))$ represents the velocity field and $p=p(x)$ is the pressure.

By an axisymmetric solution of (1.1), we mean that, in the cylindrical coordinate system, the unknown functions $u(x)$ and $p(x)$ do not depend on θ-variable, that is,

$$u(x)=u_r(r,z)e_r+u_\theta(r,z)e_\theta+u_z(r,z)e_z,$$
$$p(x)=p(r,z),$$

where

$$e_r=(\cos\theta,\sin\theta,0),\quad e_\theta=(-\sin\theta,\cos\theta,0),\quad e_z=(0,0,1)$$

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form the standard orthogonal bases in the cylindrical coordinate system. Furthermore, when $u_\theta \equiv 0$, which means that the axisymmetric flow has no swirls, the corresponding 3-D steady axisymetric Euler equations can be written as

$$\begin{cases} u_r \partial_r u_r + u_z \partial_z u_r + \partial_r p = 0, \\ u_r \partial_r u_z + u_z \partial_z u_z + \partial_z p = 0. \end{cases} \tag{1.2}$$

And the incompressibility condition becomes

$$\partial_r (r u_r) + \partial_z (r u_z) = 0. \tag{1.3}$$

In this case, the vorticity of the velocity is given by

$$\omega = \nabla \times u = \omega_\theta e_\theta$$

with $\omega_\theta = \partial_z u_r - \partial_r u_z$.

When the initial data is a vortex-sheets data, the 2D Euler equations have global (in time) weak solutions when the initial vorticity has a distinguished sign (see [2,7,16–18,21]) or has a changing sign with reflection symmetry (see [14,15]). However, the global existence of weak solutions for both general 2D and 3D Euler equations for general vortex-sheets initial data is still an outstanding open problem. In particular, for three-dimensional unsteady axisymmetric flows without swirls, this problem remains to be solved even in the case that the initial vorticity is of one sign. It was shown in [3] that, for the 3D unsteady axisymmetric Euler equations without swirls, a sequence of approximate solutions generated by smoothing the initial data converges either strongly in $L^2_{loc}(R^3 \times (0, \infty))$ or weakly in $L^2_{loc}(R^3 \times (0, \infty))$ to a limit which is not a classical weak solution to the Euler equations under the additional assumption that the initial vorticity has a distinguished sign. In other words, there is no concentration-cancellation occurring for one-sign axisymmetric flows without swirls which is in sharp contrast to the 2-D theory (see [5]). The authors proved in [12] that the approximate solutions, generated by smoothing the initial data, converge strongly in $L^2([0, T]; L^2_{loc}(R^3))$ provided that they have strong convergence in the region away from the symmetry axis. This means that if there would appear singularity or energy lost in the process of limit for the approximate solutions, it then must happen in the region away from the symmetry axis. It is noted that there is no restriction on the signs of initial vorticity in [12]. The convergence properties of the viscous approximations were studied in [11]. When the initial vorticity has stronger assumptions (comparing with the vortex-sheets initial data), the global existence of weak solutions was proved in [1] and the references therein.

For the two-dimensional steady Euler equations, DiPerna and Majda proved that, even though there exist approximate solutions with energy concentration, the weak limit of any approximate solutions is a weak solution, by using the shielding method (see [4]). That is, concentration-cancellation occurs in this case. The reader may refer to [6] for a more concise proof. However, for the three-dimensional steady equations, even for the axisymmetric case, it is not known whether or not there exist approximate solutions with energy concentration for the three-dimensional steady Euler equations. Recently, the authors studied some convergence properties of the approximate solutions of the 3D steady Euler equations (1.1) and the 3D steady axisymmetric Euler equations without swirls (1.2)–(1.3) (see [13]). In particular, in [13] the authors obtained a criterion for strong convergence for approximate solutions by establishing a relation between the energy distributions of the weak limit and the defect measure of the approximate solutions.

On the other hand, the existence of solutions of the 3D steady axisymmetric Euler equations without swirls (1.2)–(1.3) has been widely studied (see [8,9,19,20]). In particular, the vortex rings, which are steady, axisymmetric solutions without swirls of Eqs. (1.1), propagating with constant speed in the z-direction, has been extensively and systematically investigated, based mainly on the variational approaches (see [8,9,19] and references therein).

In this paper, we are mainly concerned with the strong convergence of C^1-smooth approximations and the existence of C^1-smooth exact solutions with finite energy and uniform constant states at the far field of the 3D steady axisymmetric Euler equations. We will prove that any C^1-approximations $\{u^\varepsilon, p^\varepsilon\}$ to the 3D steady axisymmetric Euler equations will converge strongly to 0 in $L^2_{loc}(R^3)$ under appropriate assumptions on approximate solutions and error terms (see Theorem 5.2). The main assumptions on approximate solutions are that the energy is finite and $|u^\varepsilon| \to 0$ and $p^\varepsilon \to p_0$ as $r^2 + z^2 \to \infty$, where p_0 is a constant. These kinds of approximate solutions correspond to 3D steady vortex-sheets. At the end of the paper, we obtain a Liouville type theorem that there will be no non-trivial C^1 exact solutions with finite energy to the 3D steady axisymmetric Euler equations, which satisfy that $|u| \to 0$ and $p \to p_0$ as $r^2 + z^2 \to \infty$. The Liouville theorem can be seen as a direct result of one of our main results (Theorem 5.2) and can also be proved directly. Two proofs of the Liouville theorem are presented at the end of the paper. It should be noted that contrary to the 3D steady axisymmetric Euler equations, there exist non-trivial smooth exact solutions with finite energy and there exist smooth approximate solutions with finite energy and energy concentrations in the limit process to the 2D steady Euler equations (see [4]). Also, using the spherical vortex ring given in [10], an example of approximate solutions of the 3D steady axisymmetric Euler equations which converge strongly to 0 in $L^2_{loc}(R^3)$ was constructed in [13].

Our approach is mainly based on a deliberate construction of test functions and making full use of structures of the axisymmetric Euler equations. Let $\phi_r(r, z), \phi_z(r, z) \in C_0^\infty(\bar{H})$ be two usual test functions which have compact support in $[0, \infty) \times (-\infty, \infty)$ and are divergence-free, that is, $\partial_r(r\phi_r) + \partial_z(r\phi_z) = 0$ or $r\partial_r\phi_r + \phi_r + r\partial_z\phi_z = 0$. Here $H = \{(r, z)|(r, z) \in (0, \infty) \times (-\infty, \infty)\}$ represents the $(r, z)-$plane. Then, it follows from (1.2)–(1.3) that

$$\int_H \frac{(u_r)^2}{r}\phi_r r dr dz = \int_H [(u_r)^2 - (u_z)^2]\partial_z\phi_z r dr dz + \int_H u_r u_z(\partial_r\phi_z + \partial_z\phi_r) r dr dz. \tag{1.4}$$

In particular, to study the convergence of the approximate solutions, (1.4) should be written as

$$\int_H \frac{(u_r^\varepsilon)^2}{r}\phi_r r dr dz = \int_H [(u_r^\varepsilon)^2 - (u_z^\varepsilon)^2]\partial_z\phi_z r dr dz + \int_H u_r^\varepsilon u_z^\varepsilon(\partial_r\phi_z + \partial_z\phi_r) r dr dz + h(\varepsilon), \tag{1.5}$$

where $h(\varepsilon)$ is some error term satisfying $h(\varepsilon) \to 0$ as $\varepsilon \to 0$. In the limit $\varepsilon \to 0$ (or its subsequence), there will appear more terms in the limit equation of (1.5), which corresponds to the defect measures of u^ε. Denote by u the weak limit of u^ε in $L^2(R^3)$. A key point of this paper is to prove that $\int_{\{r \geq r_0 > 0\}} \frac{u_r^2}{r^2} r dr dz = 0$ for any $r_0 > 0$, where

$\{r \geq r_0 > 0\} = \{(r, z)|(r, z) \in (0, \infty) \times (-\infty, \infty), r \geq r_0 > 0\}$ is a domain away from the symmetry axis in H. This can be obtained formally if we could choose the test functions as $\phi_r = 1$ and $\phi_z = 1$ in (1.4) or in the limit equation of (1.5). However, the test functions $\phi_r = 1$ and $\phi_z = 1$ do not satisfy the divergence-free condition and it is illegitimate to take the limit in (1.5) with $\phi_r = 1$ and $\phi_z = 1$. Thus, we should construct a new class of test functions which are divergence-free, decay at the far field and approximate the test functions $\phi_r = 1, \phi_z = 1$ in the appropriate sense such that the terms on the right-hand side of (1.4) or the limit of (1.5) will tend to zero in the approximation of the test functions. However, it is noted that the test functions denoted by φ_r, φ_z we construct in this paper do not belong to $C_0^\infty(\bar{H})$ which is required in the usual way. Especially, the test functions φ_r will have singularity $o(\frac{1}{r})$ near the symmetry axis. Due to this singularity, new difficulties will arise in our subsequent and rigorous analysis. First, in integrations by parts, there will appear the boundary term of the pressure, which is $\int_H p(0, z)\partial_z \varphi_z r dr dz$. Fortunately, by applying the special test functions we prove that the sign of this term is unchanged. Second, we need to investigate the properties of u_r near the symmetry axis more carefully. Precisely, we will obtain the estimate $\int_{R^3} \frac{1}{1+x_3^2}(\frac{u_r}{r})^2 dx \leq C$ with C an absolute constant. In the unsteady case, this estimate is naturally satisfied for the vortex-sheets initial data (see [1,11]). In steady case, however, it seems to be a nontrivial estimate. It is noted that other test functions such as those used in [12] and [13] (see also Sect. 2 of this paper) can provide us with some balance relations between the energy distributions of the velocity and the corresponding defect measures (see Theorems 2.1, 2.3 in Sect. 2) but can not yield the desired result of the vanishing of the right-hand side of (1.4).

The Liouville theorem, which says that there are no non-trivial C^1-smooth exact solutions with finite energy and uniform constant states at far fields of the 3D steady axisymmetric Euler equations, is proved at the end of the paper. It can be seen as a direct consequence of our results on the strong convergence of approximate solutions. And it can also be proved in a direct way, avoiding the technical construction of the test functions. It should be remarked that this direct method can not be applied to investigate the strong convergence of approximate solutions since one should take the limit first in the finite Radon space on both sides of (1.5) in order to study this problem. And in the process of the limit, we should use suitable test functions.

The rest of this paper is organized as follows. In Sect. 2, we review a criterion for the strong convergence of approximate solutions for the 3D steady Euler equations, which has been obtained in [13]. In Sect. 3, we construct some special test functions which will be needed later. It should be noted that these test functions do not satisfy the conditions required in the usual definition of the weak solutions but they possess some special features which are crucial in the analysis of the strong convergence of the approximate solutions. In Sect. 4, we prove the strong convergence of u_1^ε and u_2^ε in the region away from the symmetry axis. In Sect. 5, we first prove the strong convergence of u_1^ε and u_2^ε in $L^2_{loc}(R^3)$, then applying the criterion established in [13] for the strong convergence of approximate solutions (see also Sect. 2), we obtain the strong convergence of u^ε in $L^2_{loc}(R^3)$. Some appropriate conditions are imposed on the approximate solutions and error terms. In the last, we prove the Liouville theorem which says that there are no non-trivial C^1-smooth exact solutions with finite energy and uniform constant states at the far field to the 3D steady axisymmetric Euler equations. It can be seen as a direct result of the strong convergence of approximate solutions and it can also be proved in a direct way, avoiding the technical construction of the test functions.

2. A Criterion on the Strong Convergence

In this section, we give a brief review of the results in [13] on the strong convergence of approximate solutions to 3D steady Euler equations.

Similar to the unsteady case, approximate solutions for the 3D steady Euler equations (1.1) can be defined in the usual way.

Definition 2.1 (General Case). *Smooth vector-valued functions* $\{u^\varepsilon\}$ *(*$\varepsilon \in J$ *a parameter) are called* **approximate solutions** *of* (1.1) *if the following conditions are satisfied:*

(i) $u^\varepsilon(x)$ *is uniformly bounded in* $L^2(R^3)$ *and divergence free (*$\text{div}\, u^\varepsilon = 0$*);*
(ii) *For any* $\Phi(x) = (\Phi_1, \Phi_2, \Phi_3) \in C_0^\infty(R^3)$ *satisfying* $\text{div}\Phi = 0$*, it holds that*

$$\int_{R^3} u^\varepsilon \cdot (u^\varepsilon \cdot \nabla)\Phi dx = h(\varepsilon) \tag{2.1}$$

with $h(\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

In particular, when the approximate solutions are axisymmetric, one can obtain approximate solutions for the 3D steady axisymmetric Euler equations (1.2)–(1.3).

Definition 2.2 (Axisymmetric Case). *Smooth vector-valued functions* $\{u^\varepsilon\}$ *(*$\varepsilon \in J$ *a parameter) are called* **approximate solutions** *of the equations* (1.2)–(1.3) *if the following conditions are satisfied:*

(i) $u^\varepsilon(x)$ *is uniformly bounded in* $L^2(R^3)$ *and divergence free (div* $u^\varepsilon = 0$*);*
(ii) $u^\varepsilon = u_r^\varepsilon e_r + u_z^\varepsilon e_z$;
(iii) $\omega^\varepsilon = \nabla \times u^\varepsilon = \omega_\theta^\varepsilon e_\theta$;
(iv) *For* $\phi_r(r,z), \phi_z(r,z) \in C_0^\infty(\bar{H})$*, satisfying*

$$\partial_r(r\phi_r) + \partial_z(r\phi_z) = 0, \tag{2.2}$$

one has

$$\begin{aligned}&\int_H [(u_r^\varepsilon)^2 \partial_r \phi_r + (u_z^\varepsilon)^2 \partial_z \phi_z] r dr dz \\ &\quad = -\int_H u_r^\varepsilon u_z^\varepsilon (\partial_r \phi_z + \partial_z \phi_r) r dr dz + h(\varepsilon)\end{aligned} \tag{2.3}$$

with $h(\varepsilon) \to 0$ *as* $\varepsilon \to 0$. *Here* $H = \{(r,z) | (r,z) \in (0,\infty) \times (-\infty,\infty)\}$ *represents the* $(r,z)-$*plane.*

Formally, multiplying $r\phi_r$ and $r\phi_z$ on both sides of $(1.2)_1$ and $(1.2)_2$ respectively, integrating the resulting equations on $(0,\infty) \times (-\infty,\infty)$ with respect to r and z and summing over them, one obtains (2.3) with $h(\varepsilon) = 0$.

It should be noted that the assumption that the approximate solutions u^ε in Definitions 1.1–1.2 are smooth is only made for convenience and can be dispensed with.

For a sequence of approximate solutions $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ as in Definition 2.2, which is expressed by $u^\varepsilon = (u_r^\varepsilon, 0, u_z^\varepsilon)$ in the cylindrical coordinates systems, there exists a subsequence of u^ε, still denoted by itself, converging weakly in $L^2(R^3)$ and in $L^2(H; r dr dz)$. Precisely, as $\varepsilon \to 0^+$, one has

$$u_1^\varepsilon \rightharpoonup u_1, \quad u_2^\varepsilon \rightharpoonup u_2, \quad u_3^\varepsilon \rightharpoonup u_3 \tag{2.4}$$

weakly in $L^2(R^3)$, and, in the cylindrical coordinates,

$$u_r^\varepsilon \rightharpoonup u_r, \quad u_z^\varepsilon \rightharpoonup u_z \tag{2.5}$$

weakly in $L^2(H; rdrdz)$.

In what follows, a subsequence of approximate solutions will always be denoted by itself for convenience unless stated otherwise.

Since $(u^\varepsilon(x))^2$ are uniformly bounded in $L^1(R^3)$, there exists a subsequence of $(u^\varepsilon(x))^2$ which converge weakly to a Radon measure. More precisely, as $\varepsilon \to 0^+$,

$$(u_1^\varepsilon)^2 \rightharpoonup u_1^2 + \mu_1, \quad (u_2^\varepsilon)^2 \rightharpoonup u_2^2 + \mu_2, \quad (u_3^\varepsilon)^2 \rightharpoonup u_3^2 + \mu_3 \tag{2.6}$$

weakly in $M(R^3)$ which is the space of finite Radon measures. Here $\mu_i \geq 0 (i = 1, 2, 3)$ is the defect measure of $(u_i^\varepsilon)^2 (i = 1, 2, 3)$ respectively. The total variation of $\mu_i (i = 1, 2, 3)$, denoted by $|\mu_i| (i = 1, 2, 3)$, is finite.

A criterion on strong convergence of approximate solutions to the 3D steady axisymmetric Euler equations is stated as (see [13])

Theorem 2.1. *For any approximate solutions $\{u^\varepsilon\}$ defined as in Definition 2.2, there exists a subsequence of the approximate solutions satisfying (2.4)–(2.6). Moreover, it holds that*

$$\int_{R^3} u_3^2 dx - \frac{1}{2}\int_{R^3}(u_1^2 + u_2^2)dx + |\mu_3| - \frac{1}{2}(|\mu_1| + |\mu_2|) = 0. \tag{2.7}$$

Consequently, if $u^\varepsilon \to u$ strongly in $L^2_{loc}(R^3)$, then

$$\int_{R^3} u_3^2 dx - \frac{1}{2}\int_{R^3}(u_1^2 + u_2^2)dx = 0. \tag{2.8}$$

Proof. We give a sketch of proof here and refer to [13] for more details. It suffices to prove (2.7).

We choose the test functions in (2.3) as

$$\begin{aligned} \phi_r &= \frac{1}{2} r \chi_+(\frac{r}{\eta})[\chi(\frac{z - z_0}{\eta}) + \frac{z - z_0}{\eta}\chi'(\frac{z - z_0}{\eta})], \\ \phi_z &= -[\chi_+(\frac{r}{\eta}) + \frac{r}{2\eta}{\chi_+}'(\frac{r}{\eta})](z - z_0)\chi(\frac{z - z_0}{\eta}) \end{aligned} \tag{2.9}$$

for any $\eta > 0$ and any fixed $z_0 \in R$, where $\chi(s)$ and $\chi_+(s)$ are the same as (3.20) and (3.21) respectively. Then direct calculations lead to

$$\begin{aligned} \frac{\phi_r}{r} &= \frac{1}{2}\chi_+(\frac{r}{\eta})[\chi(\frac{z - z_0}{\eta}) + \frac{z - z_0}{\eta}\chi'(\frac{z - z_0}{\eta})], \\ \partial_r \phi_r &= \frac{1}{2}(\chi_+(\frac{r}{\eta}) + \frac{r}{\eta}\chi_+'(\frac{r}{\eta}))[\chi(\frac{z - z_0}{\eta}) + \frac{z - z_0}{\eta}\chi'(\frac{z - z_0}{\eta})], \\ \partial_z \phi_z &= -[\chi_+(\frac{r}{\eta}) + \frac{r}{2\eta}{\chi_+}'(\frac{r}{\eta})][\chi(\frac{z - z_0}{\eta}) + \frac{z - z_0}{\eta}\chi'(\frac{z - z_0}{\eta})], \\ \partial_z \phi_r &= \frac{1}{2} r \chi_+(\frac{r}{\eta})[\frac{2}{\eta}\chi'(\frac{z - z_0}{\eta}) + \frac{z - z_0}{\eta^2}\chi''(\frac{z - z_0}{\eta})], \\ \partial_r \phi_z &= -[\frac{3}{2\eta}\chi_+'(\frac{r}{\eta}) + \frac{r}{2\eta^2}{\chi_+}''(\frac{r}{\eta})](z - z_0)\chi(\frac{z - z_0}{\eta}). \end{aligned} \tag{2.10}$$

Letting $\varepsilon \to 0^+$ in (2.3), one can obtain

$$\begin{aligned}&\frac{1}{2\pi}\{\int_{R^3}(u_1^2+u_2^2)\partial_r\phi_r dx+\int_{R^3}u_3^2\partial_z\phi_z dx\\&\quad+\int_{R^3}\partial_r\phi_r d(\mu_1+\mu_2)+\int_{R^3}\partial_z\phi_z d\mu_3\}\\&\le\int_H(u_r^2+u_z^2)(|\partial_z\phi_r|+|\partial_r\phi_z|)rdrdz\\&\quad+\int_H(|\partial_z\phi_r|+|\partial_r\phi_z|)d(\mu_1+\mu_2+\mu_3).\end{aligned}\tag{2.11}$$

Substituting (2.10) into (2.11), and then letting $\eta \to \infty$ on both sides of (2.11), one has

$$\int_{R^3}u_3^2dx-\frac{1}{2}\int_{R^3}(u_1^2+u_2^2)dx+|\mu_3|-\frac{1}{2}(|\mu_1|+|\mu_2|)=0.$$

Equation (2.7) thus follows. The proof of the theorem is completed. □

If we choose the test functions in (2.1) as

$$\begin{aligned}\Phi_1&=\alpha_1x_1\chi_+(\frac{r}{\eta})[\chi(\frac{x_3}{\eta})+\frac{x_3}{\eta}\chi'(\frac{x_3}{\eta})],\\\Phi_2&=\alpha_2x_2\chi_+(\frac{r}{\eta})[\chi(\frac{x_3}{\eta})+\frac{x_3}{\eta}\chi'(\frac{x_3}{\eta})],\\\Phi_3&=x_3\chi(\frac{x_3}{\eta})[\alpha_3\chi_+(\frac{r}{\eta})-\frac{\alpha_1x_1^2+\alpha_2x_2^2}{\eta r}\chi_+'(\frac{r}{\eta})],\end{aligned}\tag{2.12}$$

where $\alpha_i \in R (i = 1, 2, 3)$ satisfying $\sum_{i=1}^3 \alpha_i = 0$, and $\chi(s)$ and $\chi_+(s)$ are defined as in (3.20) and (3.21) respectively, then a similar approach gives

Theorem 2.2. *For any approximate solutions $\{u^\varepsilon\}$ defined as in Definition 2.1, there exists a subsequence of the approximate solutions satisfying (2.4) and (2.6). Moreover, we have*

$$\sum_{i=1}^3\alpha_i(E_i+|\mu_i|)=0,\tag{2.13}$$

where, for $i = 1, 2,$ or 3, $E_i = \int_{R^3} u_i^2 dx$ is the energy of the i^{th} component of the limit, μ_i is same as in (2.6), and α_i is a real number satisfying $\sum_{i=1}^3 \alpha_i = 0$. Consequently, if $u^\varepsilon \to u$ strongly in $L^2_{loc}(R^3)$, then

$$E_1=E_2=E_3.\tag{2.14}$$

Theorem 2.3. *Suppose that a vector function $u = (u_1, u_2, u_3)$ is a weak solution of (1.1) in the sense that*

$$\int_{R^3}u\cdot(u\cdot\nabla)\Phi dx=0\tag{2.15}$$

for any $\Phi = \Phi(x) \in C_0^\infty(R^3)$ *satisfying* $div\Phi = 0$. *Then*

$$E_1 = E_2 = E_3, \tag{2.16}$$

where $E_i(i = 1, 2, 3)$ *are the same as in Theorem* 2.2. *Therefore, suppose that* u^ε *are exact solutions of* (1.1) *in the sense that* (2.1) *holds with* $h(\varepsilon) = 0$. *Then,*

$$E_1^\varepsilon = E_2^\varepsilon = E_3^\varepsilon, \tag{2.17}$$

where $E_i^\varepsilon = \int_{R^3}(u_i^\varepsilon)^2 dx (i = 1, 2, 3)$.

The detail of the proofs of Theorem 2.2 and Theorem 2.3 is referred to [13] and omitted here. It should be remarked that Theorem 2.2 and Theorem 2.3 hold for any n-dimensional ($n \geq 2$) steady Euler equations.

3. A Special Class of Test Functions and Estimates

Suppose that the approximate solutions $u^\varepsilon, p^\varepsilon \in C^1(R^3)$ satisfy

$$\begin{cases} u_r^\varepsilon \partial_r u_r^\varepsilon + u_z^\varepsilon \partial_z u_r^\varepsilon + \partial_r p^\varepsilon = h_r^\varepsilon(r, z), \\ u_r^\varepsilon \partial_r u_z^\varepsilon + u_z^\varepsilon \partial_z u_z^\varepsilon + \partial_z p^\varepsilon = h_z^\varepsilon(r, z), \end{cases} \tag{3.18}$$

and

$$\partial_r(r u_r^\varepsilon) + \partial_z(r u_z^\varepsilon) = 0, \tag{3.19}$$

where $h_r^\varepsilon(r, z)$ and $h_z^\varepsilon(r, z)$ are some error terms.

To study the structures and properties of approximate solutions satisfying (3.18) and (3.19), we need to construct a special class of test functions.

Let $\chi = \chi(s)$ be a nonnegative smooth function satisfying

$$\begin{cases} \chi(s) = 1, & |s| \leq 1, \\ \chi(s) = 0, & |s| > 2. \end{cases} \tag{3.20}$$

Denote by $\chi_+(s) = \chi(s)|_{s \geq 0}$ the restriction of $\chi(s)$ on $\{s \geq 0\}$. Then

$$\begin{cases} \chi_+(s) = 1, & 0 \leq s \leq 1, \\ \chi(s) = 0, & s > 2. \end{cases} \tag{3.21}$$

For any $\eta > 1$, we define

$$\psi(r, z) = z\chi_+(\frac{r}{\eta}) f_\eta(z), \ \ (r, z) \in H,$$

with

$$f_\eta(z) = \begin{cases} 1, & |z| \leq \eta, \\ a_1\eta^{\alpha_1}|z|^{-\alpha_1} + a_2\eta^{\alpha_2}|z|^{-\alpha_2} + a_3\eta^{\alpha_3}|z|^{-\alpha_3}, & |z| \geq \eta. \end{cases} \tag{3.22}$$

Here $1 \leq \alpha_1 < \alpha_2 < \alpha_3$ and a_1, a_2, a_3 are constants to be determined such that $f_\eta(z)$ is a C^2-smooth function satisfying

$$f_\eta(z) + z f_\eta'(z) \geq 0, \ \ z \in R, \tag{3.23}$$

and

$$|z||f_\eta'(z)| + z^2|f_\eta''(z)| \leq C, \ \ z \in R \tag{3.24}$$

with C an absolute constant. To be more precise, we consider the case $z \geq 0$ and the case $z \leq 0$ can be treated similarly. Note that when $z \geq \eta > 1$ we have

$$\begin{aligned} f_\eta(z) &= a_1\eta^{\alpha_1}z^{-\alpha_1} + a_2\eta^{\alpha_2}z^{-\alpha_2} + a_3\eta^{\alpha_3}z^{-\alpha_3}, \\ f_\eta'(z) &= -\alpha_1 a_1\eta^{\alpha_1}z^{-\alpha_1-1} - \alpha_2 a_2\eta^{\alpha_2}z^{-\alpha_2-1} - \alpha_3 a_3\eta^{\alpha_3}z^{-\alpha_3-1}, \\ f_\eta''(z) &= \alpha_1(\alpha_1+1)a_1\eta^{\alpha_1}z^{-\alpha_1-2} + \alpha_2(\alpha_2+1)a_2\eta^{\alpha_2}z^{-\alpha_2-2} \\ &\quad + \alpha_3(\alpha_3+1)a_3\eta^{\alpha_3}z^{-\alpha_3-2}. \end{aligned}$$

To guarantee that $f_\eta(z) \in C^2(R)$, one requires that

$$\begin{cases} a_1 + a_2 + a_3 = 1, \\ \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 = 0, \\ \alpha_1(\alpha_1+1)a_1 + \alpha_2(\alpha_2+1)a_2 + \alpha_3(\alpha_3+1)a_3 = 0. \end{cases} \tag{3.25}$$

Solving (3.25), one has

$$\begin{cases} a_1 = \frac{\alpha_2\alpha_3(\alpha_3-\alpha_2)}{\alpha_2\alpha_3(\alpha_3-\alpha_2)+\alpha_1\alpha_3(\alpha_1-\alpha_3)+\alpha_1\alpha_2(\alpha_2-\alpha_1)}, \\ a_2 = \frac{\alpha_1\alpha_3(\alpha_1-\alpha_3)}{\alpha_2\alpha_3(\alpha_3-\alpha_2)+\alpha_1\alpha_3(\alpha_1-\alpha_3)+\alpha_1\alpha_2(\alpha_2-\alpha_1)}, \\ a_3 = \frac{\alpha_1\alpha_2(\alpha_2-\alpha_1)}{\alpha_2\alpha_3(\alpha_3-\alpha_2)+\alpha_1\alpha_3(\alpha_1-\alpha_3)+\alpha_1\alpha_2(\alpha_2-\alpha_1)}. \end{cases} \tag{3.26}$$

We note that (3.23) is clearly satisfied when $z \leq \eta$. To guarantee that (3.23) is satisfied for all $z \in R$, we choose some particular $1 \leq \alpha_1 < \alpha_2 < \alpha_3$, for example, $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 10$. Then for any $z = a\eta$ with $a \geq 1$, direct calculations show that

$$\begin{aligned} & f_\eta(z) + zf_\eta'(z) \\ &= a_1\eta^{\alpha_1}z^{-\alpha_1}(1-\alpha_1) + a_2\eta^{\alpha_2}z^{-\alpha_2}(1-\alpha_2) + a_3\eta^{\alpha_3}z^{-\alpha_3}(1-\alpha_3) \\ &= \frac{\alpha_2\alpha_3(\alpha_3-\alpha_2)(1-\alpha_1)a^{-\alpha_1} + \alpha_1\alpha_3(\alpha_1-\alpha_3)(1-\alpha_2)a^{-\alpha_2}}{\alpha_2\alpha_3(\alpha_3-\alpha_2)+\alpha_1\alpha_3(\alpha_1-\alpha_3)+\alpha_1\alpha_2(\alpha_2-\alpha_1)} \\ &\quad + \frac{\alpha_1\alpha_2(\alpha_2-\alpha_1)(1-\alpha_3)a^{-\alpha_3}}{\alpha_2\alpha_3(\alpha_3-\alpha_2)+\alpha_1\alpha_3(\alpha_1-\alpha_3)+\alpha_1\alpha_2(\alpha_2-\alpha_1)}, \end{aligned}$$

and

$$\begin{aligned} & \alpha_2\alpha_3(\alpha_3-\alpha_2)+\alpha_1\alpha_3(\alpha_1-\alpha_3)+\alpha_1\alpha_2(\alpha_2-\alpha_1) = 72, \\ & \alpha_2\alpha_3(\alpha_3-\alpha_2)(1-\alpha_1)a^{-\alpha_1} = 0, \\ & \alpha_1\alpha_3(\alpha_1-\alpha_3)(1-\alpha_2)a^{-\alpha_2} = 90a^{-2}, \\ & \alpha_1\alpha_2(\alpha_2-\alpha_1)(1-\alpha_3)a^{-\alpha_3} = -18a^{-10}. \end{aligned}$$

Therefore

$$f_\eta(z) + zf_\eta'(z) = \frac{5a^{-2} - a^{-10}}{4} > 0$$

for all $z = a\eta$ with $a \geq 1$ and (3.23) is satisfied for all $z \in R$. Moreover, (3.24) is clearly satisfied.

Now we choose the test functions as follows:

$$r\varphi_z = -\partial_r\psi = -\frac{z}{\eta}\chi_+'(\frac{r}{\eta})f_\eta(z), \tag{3.27}$$

$$r\varphi_r = \partial_z\psi = \chi_+(\frac{r}{\eta})f_\eta(z) + z\chi_+(\frac{r}{\eta})f_\eta'(z). \tag{3.28}$$

In view of (3.23), one has $r\varphi_r \geq 0$. Note that the test functions defined in (3.27) and (3.28) do not satisfy the conditions required in Definition 2.2. Especially, the test functions φ_r has singularity $o(\frac{1}{r})$ near the symmetry axis. But for these test functions, we have

Theorem 3.1. *Suppose that the approximate solutions $u^\varepsilon, p^\varepsilon \in C^1(R^3)$ satisfy (3.18)–(3.19) and the following conditions:*

$$\|u^\varepsilon\|_{L^2(R^3)} \leq C, \tag{3.29}$$

$$\int_{R^3} \frac{1}{1+x_3^2}(\frac{u_r^\varepsilon}{r})^2 dx \leq C, \tag{3.30}$$

$$|u^\varepsilon| \to 0, \quad p^\varepsilon \to p_0 \text{ as } r^2+z^2 \to \infty, \tag{3.31}$$

where $C(>0)$ and p_0 are some absolute constants. Suppose further that

$$\int_H (|h_z^\varepsilon| + \frac{|h_r^\varepsilon|}{r}) r dr dz \leq C \text{ or } \int_H (\frac{|h_z^\varepsilon|}{r} + \frac{|h_r^\varepsilon|}{r}) r dr dz \leq C, \tag{3.32}$$

$$\int_{-\infty}^{z} h_z^\varepsilon(0,z)dz \leq 0 \tag{3.33}$$

for all $z \in R$. Then for the test functions defined as in (3.27)–(3.28), it holds that

$$\begin{aligned}
&\int_H (u_r^\varepsilon)^2 \varphi_r dr dz \\
&\leq \int_H |[(u_r^\varepsilon)^2 - (u_z^\varepsilon)^2][\frac{1}{\eta}\chi_+'(\frac{r}{\eta})f_\eta(z) + \frac{z}{\eta}\chi_+'(\frac{r}{\eta})f_\eta'(z)]| dr dz \\
&\quad + \int_H |u_r^\varepsilon u_z^\varepsilon[-\varphi_z - \frac{z}{\eta^2}\chi_+''(\frac{r}{\eta})f_\eta(z)]| dr dz \\
&\quad + \int_H |u_r^\varepsilon u_z^\varepsilon[2\chi_+(\frac{r}{\eta})f_\eta'(z) + z\chi_+(\frac{r}{\eta})f_\eta''(z)]| dr dz + h(\varepsilon),
\end{aligned} \tag{3.34}$$

where $h(\varepsilon) = \int_H |[h_r^\varepsilon(r,z)\varphi_r + h_z^\varepsilon(r,z)\varphi_z]| r dr dz$.

Proof. Without loss of generality, we assume that

$$p^\varepsilon \to 0 \text{ as } r^2+z^2 \to \infty. \tag{3.35}$$

Otherwise, one may replace p^ε by $\tilde{p}^\varepsilon = p^\varepsilon - p_0$ in (3.18).

Let $\bar{p}^\varepsilon = p^\varepsilon - p^\varepsilon(0,z)$. Then

$$\begin{cases} u_r^\varepsilon \partial_r u_r^\varepsilon + u_z^\varepsilon \partial_z u_r^\varepsilon + \partial_r \bar{p}^\varepsilon = h_r^\varepsilon(r,z), \\ u_r^\varepsilon \partial_r u_z^\varepsilon + u_z^\varepsilon \partial_z u_z^\varepsilon + \partial_z \bar{p}^\varepsilon + \partial_z p^\varepsilon(0,z) = h_z^\varepsilon(r,z). \end{cases} \tag{3.36}$$

For the test functions $r\varphi_r$ and $r\varphi_z$ defined in (3.27) and (3.28), multiplying $r\varphi_r$ and $r\varphi_z$ on both sides of $(3.36)_1$ and $(3.36)_2$ respectively and integrating on H, we have

$$\int_H [u_r^\varepsilon \partial_r u_r^\varepsilon + u_z^\varepsilon \partial_z u_r^\varepsilon + \partial_r \bar{p}^\varepsilon]\varphi_r r dr dz = \int_H h_r^\varepsilon(r,z)\varphi_r r dr dz, \tag{3.37}$$

$$\int_H [u_r^\varepsilon \partial_r u_z^\varepsilon + u_z^\varepsilon \partial_z u_z^\varepsilon + \partial_z \bar{p}^\varepsilon + \partial_z p^\varepsilon(0,z)]\varphi_z r dr dz = \int_H h_z^\varepsilon(r,z)\varphi_z r dr dz. \tag{3.38}$$

Since $u^\varepsilon \in C^1(R^3)$ and $u^\varepsilon = u_r^\varepsilon e_r + u_z^\varepsilon e_z$, so $u_r^\varepsilon|_{r=0} = 0$. Formally, it follows from (3.37) and (3.38) through integrating by parts that

$$\begin{aligned}\int_H [(u_r^\varepsilon)^2 \partial_r \varphi_r + (u_z^\varepsilon)^2 \partial_z \varphi_z] r dr dz + \int_H p^\varepsilon(0,z)\partial_z\varphi_z r dr dz \\ = -\int_H u_r^\varepsilon u_z^\varepsilon(\partial_r\varphi_z + \partial_z\varphi_r) r dr dz + \bar{h}(\varepsilon),\end{aligned} \tag{3.39}$$

where $\bar{h}(\varepsilon) = \int_H [h_r^\varepsilon(r,z)\varphi_r + h_z^\varepsilon(r,z)\varphi_z] r dr dz.$

It follows from (3.27) that

$$r\partial_r\varphi_z = -\varphi_z - \frac{z}{\eta^2}\chi_+''(\frac{r}{\eta}) f_\eta(z), \tag{3.40}$$

with

$$\varphi_z = \begin{cases} 0, & 0 \le r \le \eta, \\ -\frac{z}{r\eta}\chi_+'(\frac{r}{\eta}) f_\eta(z), & \eta \le r \le 2\eta, \\ 0, & r \ge 2\eta, \end{cases} \tag{3.41}$$

and

$$r\partial_z\varphi_z = -\frac{1}{\eta}\chi_+'(\frac{r}{\eta}) f_\eta(z) - \frac{z}{\eta}\chi_+'(\frac{r}{\eta}) f_\eta'(z). \tag{3.42}$$

While (3.28) yields

$$r\partial_r\varphi_r = -\varphi_r + \frac{1}{\eta}\chi_+'(\frac{r}{\eta}) f_\eta(z) + \frac{z}{\eta}\chi_+'(\frac{r}{\eta}) f_\eta'(z), \tag{3.43}$$

and

$$r\partial_z\varphi_r = 2\chi_+(\frac{r}{\eta}) f_\eta'(z) + z\chi_+(\frac{r}{\eta}) f_\eta''(z). \tag{3.44}$$

Substitute (3.40)–(3.44) into (3.39) to obtain

$$\begin{aligned}\int_H (u_r^\varepsilon)^2 \varphi_r dr dz &= \int_H p^\varepsilon(0,z)\partial_z\varphi_z r dr dz \\ &+ \int_H [(u_r^\varepsilon)^2 - (u_z^\varepsilon)^2][\frac{1}{\eta}\chi_+'(\frac{r}{\eta}) f_\eta(z) + \frac{z}{\eta}\chi_+'(\frac{r}{\eta}) f_\eta'(z)] dr dz \\ &+ \int_H u_r^\varepsilon u_z^\varepsilon[-\varphi_z - \frac{z}{\eta^2}\chi_+''(\frac{r}{\eta}) f_\eta(z)] dr dz \\ &+ \int_H u_r^\varepsilon u_z^\varepsilon[2\chi_+(\frac{r}{\eta}) f_\eta'(z) + z\chi_+(\frac{r}{\eta}) f_\eta''(z)] dr dz + \bar{h}(\varepsilon).\end{aligned} \tag{3.45}$$

In view of (3.36)$_2$, one has

$$\partial_z p^\varepsilon(0,z) = -u_z^\varepsilon(0,z)\partial_z u_z^\varepsilon(0,z) + h_z^\varepsilon(0,z). \tag{3.46}$$

Thus

$$p^\varepsilon(0,z) = -\frac{1}{2}(u_z^\varepsilon(0,z))^2 + \int_{-\infty}^{z} h_z^\varepsilon(0,z)dz \le 0, \tag{3.47}$$

where the assumptions (3.31) and (3.33) have been used.

Thanks to (3.23), (3.42), we have

$$r\partial_z\varphi_z = -\frac{1}{\eta}\chi_+'(\frac{r}{\eta})f_\eta(z) - \frac{z}{\eta}\chi_+'(\frac{r}{\eta})f_\eta'(z) \ge 0, \tag{3.48}$$

since $\chi_+'(s) \le 0$ for $s \ge 0$. Thus, combining (3.47), (3.48) with (3.45) shows

$$\begin{aligned}
&\int_H (u_r^\varepsilon)^2\varphi_r drdz\\
&\le \int_H |[(u_r^\varepsilon)^2 - (u_z^\varepsilon)^2][\frac{1}{\eta}\chi_+'(\frac{r}{\eta})f_\eta(z) + \frac{z}{\eta}\chi_+'(\frac{r}{\eta})f_\eta'(z)]|drdz\\
&\quad + \int_H |u_r^\varepsilon u_z^\varepsilon[-\varphi_z - \frac{z}{\eta^2}\chi_+''(\frac{r}{\eta})f_\eta(z)]|drdz\\
&\quad + \int_H |u_r^\varepsilon u_z^\varepsilon[2\chi_+(\frac{r}{\eta})f_\eta'(z) + z\chi_+(\frac{r}{\eta})f_\eta''(z)]|drdz + h(\varepsilon)\\
&\equiv I,
\end{aligned} \tag{3.49}$$

where $h(\varepsilon) = \int_H |[h_r^\varepsilon(r,z)\varphi_r + h_z^\varepsilon(r,z)\varphi_z]|rdrdz$.

Each term on the right-hand side of (3.49) is well-defined. In fact, there exists a constant $C = C(\eta)$ such that

$$\int_H |[(u_r^\varepsilon)^2 - (u_z^\varepsilon)^2][\frac{1}{\eta}\chi_+'(\frac{r}{\eta})f_\eta(z) + \frac{z}{\eta}\chi_+'(\frac{r}{\eta})f_\eta'(z)]|drdz \le C(\eta)\|u^\varepsilon\|_{L^2(R^3)}^2;$$

$$\int_H |u_r^\varepsilon u_z^\varepsilon[-\varphi_z - \frac{z}{\eta^2}\chi_+''(\frac{r}{\eta})f_\eta(z)]|drdz \le C(\eta)\|u^\varepsilon\|_{L^2(R^3)}^2.$$

Moreover, by (3.24), one has

$$|(1+z^2)^{\frac{1}{2}}[2\chi_+(\frac{r}{\eta})f_\eta'(z) + z\chi_+(\frac{r}{\eta})f_\eta''(z)]| \le C, \tag{3.50}$$

and hence

$$\begin{aligned}
&|\int_H u_r^\varepsilon u_z^\varepsilon[2\chi_+(\frac{r}{\eta})f_\eta'(z) + z\chi_+(\frac{r}{\eta})f_\eta''(z)]drdz|\\
&\quad \le C(\int_H \frac{1}{1+z^2}(\frac{u_r^\varepsilon}{r})^2 rdrdz)^{\frac{1}{2}}(\int_H (u_z^\varepsilon)^2 rdrdz)^{\frac{1}{2}}.
\end{aligned}$$

Due to (3.32), one has $h(\varepsilon) \le C$. Consequently, using (3.29), (3.30), one has

$$|I| \le C, \tag{3.51}$$

with C an absolute constant.

To obtain (3.49) rigorously, we should prove that the left-hand side of (3.49) is well-defined. To this end, we denote $H_M = (0, \infty) \times [-M, M]$ for any $M > 0$. Multiplying $r\varphi_r$ and $r\varphi_z$ on both sides of $(3.36)_1$ and $(3.36)_2$ respectively and integrating on H_M with respect to (r, z), we have

$$\int_{H_M} [u_r^\varepsilon \partial_r u_r^\varepsilon + u_z^\varepsilon \partial_z u_r^\varepsilon + \partial_r \bar{p}^\varepsilon]\varphi_r r dr dz = \int_{H_M} h_r^\varepsilon(r, z)\varphi_r r dr dz, \tag{3.52}$$

$$\begin{aligned} &\int_{H_M} [u_r^\varepsilon \partial_r u_z^\varepsilon + u_z^\varepsilon \partial_z u_z^\varepsilon + \partial_z \bar{p}^\varepsilon + \partial_z p^\varepsilon(0, z)]\varphi_z r dr dz \\ &= \int_{H_M} h_z^\varepsilon(r, z)\varphi_z r dr dz. \end{aligned} \tag{3.53}$$

Integrating by parts in (3.52) and (3.53) and then adding the resulting equations show that

$$\begin{aligned} \int_{H_M} (u_r^\varepsilon)^2 \varphi_r dr dz &= \int_{H_M} p^\varepsilon(0, z)\partial_z \varphi_z r dr dz \\ &+ \int_{H_M} [(u_r^\varepsilon)^2 - (u_z^\varepsilon)^2][\frac{1}{\eta}\chi_+'(\frac{r}{\eta}) f_\eta(z) + \frac{z}{\eta}\chi_+'(\frac{r}{\eta}) f_\eta'(z)] dr dz \\ &+ \int_{H_M} u_r^\varepsilon u_z^\varepsilon[-\varphi_z - \frac{z}{\eta^2}\chi_+''(\frac{r}{\eta}) f_\eta(z)] dr dz \\ &+ \int_{H_M} u_r^\varepsilon u_z^\varepsilon[2\chi_+(\frac{r}{\eta}) f_\eta'(z) + z\chi_+(\frac{r}{\eta}) f_\eta''(z)] dr dz \\ &+ h^M(\varepsilon) + S_b^M, \end{aligned} \tag{3.54}$$

where $h^M(\varepsilon) = \int_{H_M} [h_r^\varepsilon(r, z)\varphi_r + h_z^\varepsilon(r, z)\varphi_z] r dr dz$ and

$$\begin{aligned} S_b^M = &- \int_0^\infty [u_z^\varepsilon u_r^\varepsilon \partial_z \varphi_r + (u_z^\varepsilon)^2 \partial_z \varphi_z]|_{z=-M}^M r dr \\ &- \int_0^\infty [(\bar{p}^\varepsilon + p^\varepsilon(0, z))\partial_z \varphi_z]|_{z=-M}^M r dr \end{aligned}$$

which is the boundary term. It follows from (3.47) and (3.48) that

$$\begin{aligned} &\int_{H_M} (u_r^\varepsilon)^2 \varphi_r dr dz \\ &\le \int_{H_M} [(u_r^\varepsilon)^2 - (u_z^\varepsilon)^2][\frac{1}{\eta}\chi_+'(\frac{r}{\eta}) f_\eta(z) + \frac{z}{\eta}\chi_+'(\frac{r}{\eta}) f_\eta'(z)] dr dz \\ &\quad + \int_{H_M} u_r^\varepsilon u_z^\varepsilon[-\varphi_z - \frac{z}{\eta^2}\chi_+''(\frac{r}{\eta}) f_\eta(z)] dr dz \\ &\quad + \int_{H_M} u_r^\varepsilon u_z^\varepsilon[2\chi_+(\frac{r}{\eta}) f_\eta'(z) + z\chi_+(\frac{r}{\eta}) f_\eta''(z)] dr dz \\ &\quad + h^M(\varepsilon) + S_b^M. \end{aligned} \tag{3.55}$$

Since

$$|S_b^M| \le C \max(|u^\varepsilon|^2 + |p^\varepsilon|)|[\int_0^\infty (\partial_z \varphi_r + \partial_z \varphi_z) r dr]|_{z=-M}^M|,$$

it is clear to deduce that

$$|S_b^M| \to 0$$

for any fixed $\varepsilon > 0$ and $\eta > 1$ as $M \to \infty$. Combining this with (3.51) and noting that $|h^M(\varepsilon)| \le C$ by (3.33), we obtain that the term on the left-hand side of (3.55) is uniformly bounded with respect to M. Therefore, taking the limit $M \to \infty$ on both sides of (3.55), we obtain (3.49). The proof of the theorem is finished. □

4. Strong Convergence in Region Away From the Symmetry Axis

For any $r_0 > 0$, we define $\Omega_{r_0} = \{x | x \in R^3, x_1^2 + x_2^2 > r_0^2\}$. Then we have

Theorem 4.1. *Suppose that the assumptions of Theorem 3.1 hold and $h(\varepsilon) \to 0$ as $\varepsilon \to 0$, where $h(\varepsilon)$ is same as in (3.34). Then*

$$u_1^\varepsilon \to 0, \quad u_2^\varepsilon \to 0 \tag{4.1}$$

strongly in $L^2_{loc}(\Omega_{r_0})$ for any $r_0 > 0$ as $\varepsilon \to 0$.

Proof. Due to (3.23), for any $r > 0$, we have

$$\varphi_r = \frac{1}{r}\chi_+(\frac{r}{\eta}) f_\eta(z) + \frac{z}{r}\chi_+(\frac{r}{\eta}) f'_\eta(z) \ge 0. \tag{4.2}$$

For any $r = r_n = \frac{1}{n} > 0 (n = 1, 2, \cdots)$, it follows from (4.2) and (3.34) that

$$\begin{aligned}
&|\int_{\{r \ge r_n\}} (u_r^\varepsilon)^2 \frac{1}{r^2}\chi_+(\frac{r}{\eta}) f_\eta(z) r dr dz| \\
&\le \frac{1}{r_n^2}\int_H |(u_r^\varepsilon)^2 z \chi_+(\frac{r}{\eta}) f'_\eta(z)| r dr dz \\
&\quad + \int_H |[(u_r^\varepsilon)^2 - (u_z^\varepsilon)^2][\frac{1}{r\eta}\chi'_+(\frac{r}{\eta}) f_\eta(z) + \frac{z}{r\eta}\chi'_+(\frac{r}{\eta}) f'_\eta(z)]| r dr dz \\
&\quad + \frac{1}{2}\int_H |[(u_r^\varepsilon)^2 + (u_z^\varepsilon)^2][\frac{\varphi_z}{r} + \frac{z}{r\eta^2}\chi''_+(\frac{r}{\eta}) f_\eta(z)]| r dr dz \\
&\quad + \int_H |u_r^\varepsilon u_z^\varepsilon [2\chi_+(\frac{r}{\eta}) f'_\eta(z) + z\chi_+(\frac{r}{\eta}) f''_\eta(z)]| dr dz + h(\varepsilon) \\
&\equiv I_1 + I_2 + I_3 + I_4 + h(\varepsilon).
\end{aligned} \tag{4.3}$$

Note that

$$\begin{aligned}
|I_4| &\le \frac{1}{2}\int_H \frac{1}{1+z^2}(\frac{u_r^\varepsilon}{r})^2 (1+z^2)^{\frac{1}{2}} |[2\chi_+(\frac{r}{\eta}) f'_\eta(z) + z\chi_+(\frac{r}{\eta}) f''_\eta(z)]| r dr dz \\
&\quad + \frac{1}{2}\int_H (u_z^\varepsilon)^2 (1+z^2)^{\frac{1}{2}} |[2\chi_+(\frac{r}{\eta}) f'_\eta(z) + z\chi_+(\frac{r}{\eta}) f''_\eta(z)]| r dr dz \\
&= \frac{1}{2}\int_{\{|z| \ge \eta\}} \frac{1}{1+z^2}(\frac{u_r^\varepsilon}{r})^2 (1+z^2)^{\frac{1}{2}} |[2\chi_+(\frac{r}{\eta}) f'_\eta(z) + z\chi_+(\frac{r}{\eta}) f''_\eta(z)]| r dr dz \\
&\quad + \frac{1}{2}\int_{\{|z| \ge \eta\}} (u_z^\varepsilon)^2 (1+z^2)^{\frac{1}{2}} |[2\chi_+(\frac{r}{\eta}) f'_\eta(z) + z\chi_+(\frac{r}{\eta}) f''_\eta(z)]| r dr dz.
\end{aligned}$$

Equation (4.3) becomes

$$
\begin{aligned}
&|\int_{\{r\ge r_n\}}(u_r^\varepsilon)^2\frac{1}{r^2}\chi_+(\frac{r}{\eta})f_\eta(z)rdrdz|\\
&\le \frac{1}{r_n^2}\int_H |(u_r^\varepsilon)^2 z\chi_+(\frac{r}{\eta})f'_\eta(z)|rdrdz\\
&\quad+\int_H |[(u_r^\varepsilon)^2-(u_z^\varepsilon)^2][\frac{1}{r\eta}\chi'_+(\frac{r}{\eta})f_\eta(z)+\frac{z}{r\eta}\chi'_+(\frac{r}{\eta})f'_\eta(z)]|rdrdz\\
&\quad+\frac{1}{2}\int_H |[(u_r^\varepsilon)^2+(u_z^\varepsilon)^2][\frac{\varphi_z}{r}+\frac{z}{r\eta^2}\chi''_+(\frac{r}{\eta})f_\eta(z)]|rdrdz\\
&\quad+\frac{1}{2}\int_{\{|z|\ge\eta\}}\frac{1}{1+z^2}(\frac{u_r^\varepsilon}{r})^2(1+z^2)^{\frac{1}{2}}|[2\chi_+(\frac{r}{\eta})f'_\eta(z)+z\chi_+(\frac{r}{\eta})f''_\eta(z)]|rdrdz\\
&\quad+\frac{1}{2}\int_{\{|z|\ge\eta\}}(u_z^\varepsilon)^2(1+z^2)^{\frac{1}{2}}|[2\chi_+(\frac{r}{\eta})f'_\eta(z)+z\chi_+(\frac{r}{\eta})f''_\eta(z)]|rdrdz+h(\varepsilon)\\
&\equiv I_1+I_2+I_3+I_5+I_6+h(\varepsilon).
\end{aligned}
\tag{4.4}
$$

Applying a diagonal procedure, taking the limit $\varepsilon\to 0$, one can get

$$
\begin{aligned}
&\int_{\{r\ge r_n\}}(u_r^\varepsilon)^2\frac{1}{r^2}\chi_+(\frac{r}{\eta})f_\eta(z)rdrdz\\
&=\frac{1}{2\pi}\int_{R^3\backslash\{r\le r_n\}}[(u_1^\varepsilon)^2+(u_2^\varepsilon)^2]\frac{1}{r^2}\chi_+(\frac{r}{\eta})f_\eta(z)dx\to I_0\\
&\equiv\frac{1}{2\pi}\int_{R^3\backslash\{r\le r_n\}}[(u_1)^2+(u_2)^2]\frac{1}{r^2}\chi_+(\frac{r}{\eta})f_\eta(z)dx\\
&\quad+\frac{1}{2\pi}\int_{R^3\backslash\{r\le r_n\}}\frac{1}{r^2}\chi_+(\frac{r}{\eta})f_\eta(z)d(\mu_1+\mu_2)
\end{aligned}
\tag{4.5}
$$

for any $r_n=\frac{1}{n}>0(n=1,2,\cdots)$ and $\eta>0$. Then we obtain

$$
I_0\to\frac{1}{2\pi}\int_{R^3\backslash\{r\le r_n\}}[(u_1)^2+(u_2)^2]\frac{1}{r^2}dx+\frac{1}{2\pi}\int_{R^3\backslash\{r\le r_n\}}\frac{1}{r^2}d(\mu_1+\mu_2)\tag{4.6}
$$

as $\eta\to\infty$.

I_1, I_2 and I_3 can be treated in a similar way (see also the proof of Theorem 2.1). Taking the limit $\varepsilon\to 0$ first for any $\eta>1$ and then taking the limit $\eta\to\infty$ in I_1, I_2 and I_3, we can obtain

$$
I_1+I_2+I_3\to 0.\tag{4.7}
$$

Now we consider the convergence of I_5 and I_6. Due to (3.30), we have

$$\frac{1}{1+z^2}(\frac{u_r^\varepsilon}{r})^2 r dr dz \rightharpoonup g + \mu_w \tag{4.8}$$

weakly in M as $\varepsilon \to 0$, where $g \in L^1(H)$ and μ_w is a Radon measure. Note that for any fixed $\eta > 1$,

$$|(1+z^2)^{\frac{1}{2}}[2\chi_+(\frac{r}{\eta})f_\eta'(z) + z\chi_+(\frac{r}{\eta})f_\eta''(z)]| = O(\frac{\eta^{\alpha_1}}{|z|^{\alpha_1}})$$

as $|z| \to \infty$. Then, taking the limit $\varepsilon \to 0$ in I_5 shows that

$$\begin{aligned} I_5 \to \tilde{I}_5 \equiv \frac{1}{2}\int_{\{|z|\geq \eta-1\}} &|g(1+z^2)^{\frac{1}{2}}[2\chi_+(\frac{r}{\eta})f_\eta'(z) + z\chi_+(\frac{r}{\eta})f_\eta''(z)]|r dr dz \\ &+\frac{1}{2}\int_{\{|z|\geq \eta-1\}} |(1+z^2)^{\frac{1}{2}}[2\chi_+(\frac{r}{\eta})f_\eta'(z) + z\chi_+(\frac{r}{\eta})f_\eta''(z)]|d\mu_w \end{aligned} \tag{4.9}$$

for any $\eta > 1$. Furthermore, thanks to (3.24), one has

$$|(1+z^2)^{\frac{1}{2}}[2\chi_+(\frac{r}{\eta})f_\eta'(z) + z\chi_+(\frac{r}{\eta})f_\eta''(z)]| \leq C$$

with C an absolute constant, which yields

$$\tilde{I}_5 \to 0 \tag{4.10}$$

as $\eta \to \infty$. Similarly, taking the limit $\varepsilon \to 0$ first for any $\eta > 1$ and then taking the limit $\eta \to \infty$ in I_6, we obtain

$$I_6 \to 0. \tag{4.11}$$

Combining (4.5)–(4.7) and (4.9)–(4.11), taking the limit (up to a subsequence) $\varepsilon \to 0$ first for any $\eta > 1$ and then taking the limit $\eta \to \infty$ in (4.4) show

$$\frac{1}{2\pi}\int_{R^3\backslash\{r\leq r_n\}} [(u_1)^2 + (u_2)^2]\frac{1}{r^2}dx + \frac{1}{2\pi}\int_{R^3\backslash\{r\leq r_n\}} \frac{1}{r^2}d(\mu_1+\mu_2) = 0 \tag{4.12}$$

for any $r_n = \frac{1}{n}(n = 1, 2, \cdots)$. Therefore, for any $r_0 > 0$, in the region $\Omega_{r_0} = \{x|x \in R^3, x_1^2 + x_2^2 > r_0^2\}$,

$$u_1 = u_2 = 0, x \in \Omega_{r_0},$$

and

$$\mu_1(\Omega_{r_0}) = \mu_2(\Omega_{r_0}) = 0.$$

Consequently,

$$u_1^\varepsilon \to 0, \quad u_2^\varepsilon \to 0 \tag{4.13}$$

strongly in $L^2_{loc}(\Omega_{r_0})$ as $\varepsilon \to 0$. The proof of the theorem is finished. □

5. Strong Convergence in R^3

Theorem 5.1. *Under the assumptions of Theorem 4.1, it holds that*

$$u^\varepsilon \to 0 \tag{5.1}$$

strongly in $L^2_{loc}(R^3)$ *as* $\varepsilon \to 0$.

Proof. For any $X_3 >> 1$ large enough and $r_0 > 0$, we have

$$\begin{aligned}
&\int_{\{|x_3|\le X_3, r\ge 0\}} (u_r^\varepsilon)^2 r dr dz \\
&\quad \le \int_{\{|x_3|\le X_3, r> r_0\}} (u_r^\varepsilon)^2 r dr dz + \int_{\{|x_3|\le X_3, 0\le r\le r_0\}} (u_r^\varepsilon)^2 r dr dz \\
&\quad \le \int_{\{|x_3|\le X_3, r> r_0\}} (u_r^\varepsilon)^2 r dr dz + (1+X_3^2)\int_{\{|x_3|\le X_3, 0\le r\le r_0\}} \frac{(u_r^\varepsilon)^2}{1+x_3^2} r dr dz \\
&\quad \le \int_{\{|x_3|\le X_3, r> r_0\}} (u_r^\varepsilon)^2 r dr dz + r_0^2(1+X_3^2)\int_H \frac{1}{1+x_3^2}(\frac{u_r^\varepsilon}{r})^2 r dr dz \\
&\quad \le \int_{\{|x_3|\le X_3, r> r_0\}} (u_r^\varepsilon)^2 r dr dz + r_0^2(1+X_3^2) C,
\end{aligned} \tag{5.2}$$

where (3.30) has been used. For any $\delta_0 > 0$ and $X_3 >> 1$, we choose $r_0 > 0$ small enough such that $r_0^2(1+X_3^2)C \le \delta_0$. Using (4.13) and taking the limit $\varepsilon \to 0$ in (5.2) yield

$$\int_{\{|x_3|\le X_3, r\ge 0\}} (u_r)^2 r dr dz + \int_{\{|x_3|\le X_3, r> 0\}} d\mu_r \le \delta_0. \tag{5.3}$$

Since δ_0 is arbitrary, (5.3) shows that $u_r = 0$ and $\mu_r = 0$. Consequently,

$$u_1^\varepsilon \to 0, \quad u_2^\varepsilon \to 0 \tag{5.4}$$

strongly in $L^2_{loc}(R^3)$ as $\varepsilon \to 0$. This, together with (2.7), shows that

$$\int_{R^3} u_3^2 dx + |\mu_3| = 0,$$

which implies

$$u_3 = \mu_3 = 0. \tag{5.5}$$

Consequently, combining (5.4) with (5.5) shows that

$$u^\varepsilon \to 0 \tag{5.6}$$

strongly in $L^2_{loc}(R^3)$ as $\varepsilon \to 0$. The proof of the theorem is finished. □

Now we investigate the validity of the condition (3.30).

Lemma 5.1. *Suppose that the approximate solutions* u^ε, $p^\varepsilon \in C^2(R^3)$ *satisfy* (3.18) *and* (3.19) *with* h_r^ε, h_z^ε *some error terms satisfying* $\partial_z h_r^\varepsilon, \partial_r h_z^\varepsilon \in C(H)$. *Moreover, suppose that*

$$\|u^\varepsilon\|_{L^2(R^3)} \le C, \tag{5.7}$$

$$|\omega_\theta^\varepsilon| \le C(\varepsilon), \quad (r, z) \in \bar{H} = [0, \infty) \times (0, \infty), \tag{5.8}$$

$$\int_H \Big|\frac{\partial_z h_r^\varepsilon - \partial_r h_z^\varepsilon}{r}\Big| r dr dz \le C, \tag{5.9}$$

$$|u^\varepsilon| \to 0, \quad \text{as } r^2 + z^2 \to \infty, \tag{5.10}$$

where C *is an absolute constant and* $C(\varepsilon)$ *is a constant which may depend on* ε. *Then* (3.30) *holds.*

Proof. It follows from (3.18) and (3.19) that

$$u_r^\varepsilon \partial_r\Big(\frac{\omega_\theta^\varepsilon}{r}\Big) + u_z^\varepsilon \partial_z\Big(\frac{\omega_\theta^\varepsilon}{r}\Big) = \frac{\partial_z h_r^\varepsilon - \partial_r h_z^\varepsilon}{r}. \tag{5.11}$$

Set $\rho(x_3) = \int_{-\infty}^{x_3} \frac{1}{1+\tau^2} d\tau$. For any $\eta > 0$, we define $\varphi(r, z) = \chi_+(\frac{r}{\eta})\rho(z)$ with χ_+ the same as in (3.21).

In the following, we will multiply the test functions $r\varphi(r, z)$ on both sides of (5.11) and make the integration on H with respect to r and z. Similar as in the proof of Theorem 3.1, especially as the rigorous derivation of (3.49), the proof can be completed rigorously by integrating on $H_M = (0, \infty) \times [-M, M]$ instead of H and we will omit the details for conciseness.

Multiplying $r\varphi(r, z)$ on both sides of (5.11), integrating the resulting identity with respect to (r, z) over $(0, \infty) \times (-\infty, \infty)$, and using (3.19) and (5.8), we obtain

$$\int_H u_r^\varepsilon \omega_\theta^\varepsilon \partial_r \varphi dr dz + \int_H u_z^\varepsilon \omega_\theta^\varepsilon \partial_z \varphi dr dz = -\int_H \frac{\partial_z h_r^\varepsilon - \partial_r h_z^\varepsilon}{r} \varphi r dr dz. \tag{5.12}$$

That is

$$\begin{aligned} &\int_H u_z^\varepsilon \omega_\theta^\varepsilon \chi_+\Big(\frac{r}{\eta}\Big)\rho'(z) dr dz \\ &= -\int_H u_r^\varepsilon \omega_\theta^\varepsilon \frac{1}{\eta} \chi_+'\Big(\frac{r}{\eta}\Big)\rho(z) dr dz \\ &\quad - \int_H \frac{\partial_z h_r^\varepsilon - \partial_r h_z^\varepsilon}{r} \chi_+\Big(\frac{r}{\eta}\Big)\rho(z) r dr dz. \end{aligned} \tag{5.13}$$

Note that

$$\begin{aligned} \int_H u_z^\varepsilon \omega_\theta^\varepsilon \chi_+\Big(\frac{r}{\eta}\Big)\rho'(z) dr dz &= \int_H \rho' u_z^\varepsilon (\partial_z u_r^\varepsilon - \partial_r u_z^\varepsilon)\chi_+\Big(\frac{r}{\eta}\Big) dr dz \\ = \frac{1}{2}\int_{-\infty}^{\infty} \rho' (u_z^\varepsilon)^2(0, z) dz &+ \frac{1}{2}\int_H \rho' (u_z^\varepsilon)^2 \frac{1}{\eta}\chi_+'\Big(\frac{r}{\eta}\Big) dr dz \\ - \int_H (\rho'' u_z^\varepsilon u_r^\varepsilon &+ \rho' u_r^\varepsilon \partial_z u_z^\varepsilon)\chi_+\Big(\frac{r}{\eta}\Big) dr dz. \end{aligned} \tag{5.14}$$

Therefore, one has

$$\begin{aligned}
&\int_H u_z^\varepsilon \omega_\theta^\varepsilon \chi_+(\frac{r}{\eta})\rho'(z)drdz \\
&\geq \frac{1}{2}\int_H \rho'(u_z^\varepsilon)^2\frac{1}{\eta}\chi_+'(\frac{r}{\eta})drdz \\
&\quad -\int_H (\rho'' u_z^\varepsilon u_r^\varepsilon + \rho' u_r^\varepsilon(-\frac{u_r^\varepsilon}{r} - \partial_r u_r^\varepsilon))\chi_+(\frac{r}{\eta})drdz \\
&= \int_H \rho'\frac{(u_r^\varepsilon)^2}{r}\chi_+(\frac{r}{\eta})drdz - \int_H \rho'' u_z^\varepsilon u_r^\varepsilon \chi_+(\frac{r}{\eta})drdz \\
&\quad -\frac{1}{2}\int_H \rho'(u_r^\varepsilon)^2\frac{1}{\eta}\chi_+'(\frac{r}{\eta})drdz + \frac{1}{2}\int_H \rho'(u_z^\varepsilon)^2\frac{1}{\eta}\chi_+'(\frac{r}{\eta})drdz. \qquad (5.15)
\end{aligned}$$

It follows from (5.13) and (5.15) that

$$\begin{aligned}
&\int_H \rho'\frac{(u_r^\varepsilon)^2}{r}\chi_+(\frac{r}{\eta})drdz - \int_H \rho'' u_z^\varepsilon u_r^\varepsilon \chi_+(\frac{r}{\eta})drdz \\
&\leq \frac{1}{2}\int_H \rho'(u_r^\varepsilon)^2\frac{1}{\eta}\chi_+'(\frac{r}{\eta})drdz - \frac{1}{2}\int_H \rho'(u_z^\varepsilon)^2\frac{1}{\eta}\chi_+'(\frac{r}{\eta})drdz \\
&\quad -\int_H u_r^\varepsilon \omega_\theta^\varepsilon \frac{1}{\eta}\chi_+'(\frac{r}{\eta})\rho(z)drdz - \int_H \frac{\partial_z h_r^\varepsilon - \partial_r h_z^\varepsilon}{r}\chi_+(\frac{r}{\eta})\rho(z)rdrdz. \qquad (5.16)
\end{aligned}$$

For any $N > 1$, we choose $\eta > N$ large enough such that

$$\begin{aligned}
&|\int_H \rho'' u_z^\varepsilon u_r^\varepsilon \chi_+(\frac{r}{\eta})drdz| \\
&\leq |\int_{-N}^N\int_0^N \rho'' u_z^\varepsilon u_r^\varepsilon drdz| + \int_{H\setminus(-N,N)\times(0,N)} |\rho'' u_z^\varepsilon u_r^\varepsilon|drdz \\
&= |\int_{-N}^N\int_0^N \rho'' u_z^\varepsilon u_r^\varepsilon drdz| + [\int_{-\infty}^{-N}\int_0^N + \int_N^\infty\int_0^N + \int_{-\infty}^{-N}\int_N^\infty \\
&\quad + \int_N^\infty\int_N^\infty + \int_{-N}^N\int_N^\infty]|\frac{2z}{(1+z^2)^2}u_z^\varepsilon u_r^\varepsilon|drdz \\
&\equiv |\int_{-N}^N\int_0^N \rho'' u_z^\varepsilon u_r^\varepsilon drdz| + \sum_{i=1}^5 I_i. \qquad (5.17)
\end{aligned}$$

The following estimates are direct:

$$
\begin{aligned}
I_1 &= \int_{-\infty}^{-N}\int_0^N |\frac{2z}{(1+z^2)^2}u_z^\varepsilon u_r^\varepsilon|drdz \le C\max|u^\varepsilon|^2\frac{1}{N};\\
I_2 &= \int_N^{\infty}\int_0^N |\frac{2z}{(1+z^2)^2}u_z^\varepsilon u_r^\varepsilon|drdz \le C\max|u^\varepsilon|^2\frac{1}{N};\\
I_3 &= \int_{-\infty}^{-N}\int_N^{\infty} |\frac{2z}{(1+z^2)^2}u_z^\varepsilon u_r^\varepsilon|drdz\\
&\le C\frac{1}{N^4}\int_{-\infty}^{-N}\int_N^{\infty}|u_z^\varepsilon u_r^\varepsilon|rdrdz \le C\frac{1}{N^4}\|u^\varepsilon\|_{L^2(R^3)}^2;\\
I_4 &= \int_N^{\infty}\int_N^{\infty} |\frac{2z}{(1+z^2)^2}u_z^\varepsilon u_r^\varepsilon|drdz \le C\frac{1}{N^4}\|u^\varepsilon\|_{L^2(R^3)}^2;\\
I_5 &= \int_{-N}^{N}\int_N^{\infty} |\frac{2z}{(1+z^2)^2}u_z^\varepsilon u_r^\varepsilon|drdz \le C\frac{1}{N^4}\|u^\varepsilon\|_{L^2(R^3)}^2.
\end{aligned}
$$

Consequently, one has from (5.17) that

$$
\begin{aligned}
&|\int_H \rho'' u_z^\varepsilon u_r^\varepsilon \chi_+(\frac{r}{\eta})drdz|\\
&\quad\le |\int_{-N}^{N}\int_0^N \rho'' u_z^\varepsilon u_r^\varepsilon drdz| + C\frac{1}{N}(\max|u^\varepsilon|^2 + \|u^\varepsilon\|_{L^2(R^3)}^2)
\end{aligned}
\tag{5.18}
$$

for any $N > 1$ and $\eta > N$. Combining (5.16) with (5.18), one has

$$
\begin{aligned}
&\int_{-N}^{N}\int_0^N \rho'\frac{(u_r^\varepsilon)^2}{r}drdz\\
&\quad\le |\int_{-N}^{N}\int_0^N \rho'' u_z^\varepsilon u_r^\varepsilon drdz| + C\frac{1}{N}(\max|u^\varepsilon|^2 + \|u^\varepsilon\|_{L^2(R^3)}^2)\\
&\qquad + C\int_H |\frac{\partial_z h_r^\varepsilon - \partial_r h_z^\varepsilon}{r}|rdrdz + |J|,
\end{aligned}
\tag{5.19}
$$

where

$$
\begin{aligned}
J \equiv{}& \frac{1}{2}\int_H \rho'(u_r^\varepsilon)^2\frac{1}{\eta}\chi_+'(\frac{r}{\eta})drdz - \frac{1}{2}\int_H \rho'(u_z^\varepsilon)^2\frac{1}{\eta}\chi_+'(\frac{r}{\eta})drdz\\
&- \int_H u_r^\varepsilon\omega_\theta^\varepsilon\frac{1}{\eta}\chi_+'(\frac{r}{\eta})\rho(z)drdz.
\end{aligned}
\tag{5.20}
$$

The last term on the right-hand side of (5.20) can be rewritten as

$$\begin{aligned}
&\int_H u_r^\varepsilon \omega_\theta^\varepsilon \frac{1}{\eta}\chi_+'(\frac{r}{\eta})\rho(z)drdz \\
&= \int_H u_r^\varepsilon(\partial_z u_r^\varepsilon - \partial_r u_z^\varepsilon)\frac{1}{\eta}\chi_+'(\frac{r}{\eta})\rho(z)drdz \\
&= -\frac{1}{2}\int_H (u_r^\varepsilon)^2 \frac{1}{\eta}\chi_+'(\frac{r}{\eta})\rho'(z)drdz + \int_H \partial_r u_r^\varepsilon u_z^\varepsilon \frac{1}{\eta}\chi_+'(\frac{r}{\eta})\rho(z)drdz \\
&\quad + \int_H u_r^\varepsilon u_z^\varepsilon \frac{1}{\eta^2}\chi_+''(\frac{r}{\eta})\rho(z)drdz \\
&= -\frac{1}{2}\int_H (u_r^\varepsilon)^2 \frac{1}{\eta}\chi_+'(\frac{r}{\eta})\rho'(z)drdz - \int_H (\frac{u_r^\varepsilon}{r} + \partial_z u_z^\varepsilon) u_z^\varepsilon \frac{1}{\eta}\chi_+'(\frac{r}{\eta})\rho(z)drdz \\
&\quad + \int_H u_r^\varepsilon u_z^\varepsilon \frac{1}{\eta^2}\chi_+''(\frac{r}{\eta})\rho(z)drdz \\
&= -\frac{1}{2}\int_H (u_r^\varepsilon)^2 \frac{1}{\eta}\chi_+'(\frac{r}{\eta})\rho'(z)drdz - \int_H \frac{u_r^\varepsilon}{r} u_z^\varepsilon \frac{1}{\eta}\chi_+'(\frac{r}{\eta})\rho(z)drdz \\
&\quad + \frac{1}{2}\int_H (u_z^\varepsilon)^2 \frac{1}{\eta}\chi_+'(\frac{r}{\eta})\rho'(z)drdz + \int_H u_r^\varepsilon u_z^\varepsilon \frac{1}{\eta^2}\chi_+''(\frac{r}{\eta})\rho(z)drdz. \qquad (5.21)
\end{aligned}$$

It follows from (5.20) and (5.21) that

$$|J| \le C\frac{1}{\eta^2}\|u^\varepsilon\|_{L^2(R^3)}^2 \to 0, \qquad (5.22)$$

as $\eta \to \infty$.

Taking the limit $\eta \to \infty$ on both sides of (5.19) yields

$$\begin{aligned}
&\int_{-N}^N \int_0^N \rho' \frac{(u_r^\varepsilon)^2}{r} drdz \le |\int_{-N}^N \int_0^N \rho'' u_z^\varepsilon u_r^\varepsilon drdz| \\
&+ C\frac{1}{N}(\max |u^\varepsilon|^2 + \|u^\varepsilon\|_{L^2(R^3)}^2) + C\int_H |\frac{\partial_z h_r^\varepsilon - \partial_r h_z^\varepsilon}{r}| rdrdz \qquad (5.23)
\end{aligned}$$

for any $N > 1$.

Since $\rho'(x_3) > 0$ for all $x_3 \in R$, it follows from (5.23) and (5.9) that

$$\begin{aligned}
\int_{-N}^N \int_0^N \rho' \frac{(u_r^\varepsilon)^2}{r} drdz &\le (\int_{-N}^N \int_0^N \rho' \frac{(u_r^\varepsilon)^2}{r} drdz)^{\frac{1}{2}} (\int_{-N}^N \int_0^N (u_z^\varepsilon)^2 \frac{(\rho'')^2}{\rho'} rdrdz)^{\frac{1}{2}} \\
&\quad + C\frac{1}{N}(\max |u^\varepsilon|^2 + \|u^\varepsilon\|_{L^2(R^3)}^2) + C,
\end{aligned}$$

where C is an absolute constant independent of ε and N. By the Cauchy-Schwartz inequality, we obtain

$$\int_{-N}^N \int_0^N \rho' \frac{(u_r^\varepsilon)^2}{r} drdz \le C\frac{1}{N}(\max |u^\varepsilon|^2 + \|u^\varepsilon\|_{L^2(R^3)}^2) + C, \qquad (5.24)$$

where C is an absolute constant independent of ε and N. Letting $N \to \infty$ on both sides of (5.24) yields (3.30) and the proof of the theorem is finished. □

Lemma 5.2. *Suppose that the approximate solutions u^ε, $p^\varepsilon \in C^1(R^3)$ satisfy (3.18) and (3.19) with some error terms h_r^ε and h_z^ε satisfying $h_r^\varepsilon, h_z^\varepsilon \in C^1(H)$ and $h_z^\varepsilon|_{r=0} = 0$. Suppose further that (5.7),(5.9) and (5.10) are satisfied and $p^\varepsilon \to p_0$ as $r^2 + z^2 \to \infty$, where p_0 is a constant. Then (3.30) holds.*

Proof. Without loss of generality, we assume that

$$p^\varepsilon \to 0 \text{ as } r^2 + z^2 \to \infty.$$

For any $\eta > 0$, we let $\varphi(r, z) = \chi_+(\frac{r}{\eta})\rho(z)$ be the same as in the proof of Lemma 5.1. Similar to the proof of Lemma 5.1, it is assumed that the following integrations make sense and the rigorous proof by integration on H_M instead of H will be omitted for conciseness.

Multiplying $\partial_z\varphi$ and $\partial_r\varphi$ on both sides of $(3.36)_1$ and $(3.36)_2$ respectively and integrating on H, one may get

$$\int_H [u_r^\varepsilon \partial_r u_r^\varepsilon + u_z^\varepsilon \partial_z u_r^\varepsilon + \partial_r \bar{p}^\varepsilon]\partial_z\varphi drdz = \int_H h_r^\varepsilon(r, z)\partial_z\varphi drdz, \tag{5.25}$$

$$\int_H [u_r^\varepsilon \partial_r u_z^\varepsilon + u_z^\varepsilon \partial_z u_z^\varepsilon + \partial_z \bar{p}^\varepsilon + \partial_z p^\varepsilon(0, z)]\partial_r\varphi drdz$$
$$= \int_H h_z^\varepsilon(r, z)\partial_r\varphi drdz, \tag{5.26}$$

where $\bar{p}^\varepsilon = p^\varepsilon(r, z) - p^\varepsilon(0, z)$.

Since

$$\int_H [u_r^\varepsilon \partial_r u_r^\varepsilon + u_z^\varepsilon \partial_z u_r^\varepsilon]\partial_z\varphi drdz$$
$$= \int_H u_r^\varepsilon \partial_z u_r^\varepsilon \partial_r\varphi drdz + \int_H u_z^\varepsilon \partial_z u_r^\varepsilon \partial_z\varphi drdz, \tag{5.27}$$

and

$$\int_H [u_r^\varepsilon \partial_r u_z^\varepsilon + u_z^\varepsilon \partial_z u_z^\varepsilon]\partial_r\varphi drdz$$
$$= \int_H u_r^\varepsilon \partial_r u_z^\varepsilon \partial_r\varphi drdz + \int_H u_z^\varepsilon \partial_r u_z^\varepsilon \partial_z\varphi drdz$$
$$+ \frac{1}{2}\int_{-\infty}^{\infty} (u_z^\varepsilon)^2(0, z)\partial_z\varphi(0, z)dz, \tag{5.28}$$

subtracting (5.26) from (5.25) and then integrating by parts, with help of (5.27) and (5.28), one has

$$\int_H u_r^\varepsilon \omega_\theta^\varepsilon \partial_r\varphi drdz + \int_H u_z^\varepsilon \omega_\theta^\varepsilon \partial_z\varphi drdz - \frac{1}{2}\int_{-\infty}^{\infty} (u_z^\varepsilon)^2(0, z)\rho'(z)dz$$
$$+ \int_H p^\varepsilon(0, z)\partial_r\partial_z\varphi drdz = -\int_H \frac{\partial_z h_r^\varepsilon - \partial_r h_z^\varepsilon}{r}\varphi rdrdz. \tag{5.29}$$

Moreover, since $\chi_+'(s) \le 0$ $(s \in R)$, $\rho' > 0$ and $p^\varepsilon(0, z) \le 0$ due to (3.46), (3.47) and the assumption that $h_z^\varepsilon(0, z) = 0$, it holds that

$$\int_H p^\varepsilon(0, z)\partial_r\partial_z\varphi = \int_H p^\varepsilon(0, z)\frac{1}{\eta}\chi_+'(\frac{r}{\eta})\rho' drdz \ge 0. \tag{5.30}$$

It follows from (5.29), (5.30) and (5.14) that

$$
\begin{aligned}
&-\int_H (\rho'' u_z^\varepsilon u_r^\varepsilon + \rho' u_r^\varepsilon \partial_z u_z^\varepsilon)\chi_+(\frac{r}{\eta})drdz \\
&\le -\int_H u_r^\varepsilon \omega_\theta^\varepsilon \partial_r \varphi drdz - \frac{1}{2}\int_H \rho'(u_z^\varepsilon)^2 \frac{1}{\eta}\chi_+'(\frac{r}{\eta})drdz - \int_H \frac{\partial_z h_r^\varepsilon - \partial_r h_z^\varepsilon}{r}\varphi r drdz.
\end{aligned}
\tag{5.31}
$$

Noting that the left-hand side of (5.31) is

$$
\begin{aligned}
&-\int_H (\rho'' u_z^\varepsilon u_r^\varepsilon + \rho' u_r^\varepsilon(-\frac{u_r^\varepsilon}{r} - \partial_r u_r^\varepsilon))\chi_+(\frac{r}{\eta})drdz \\
&= \int_H \rho' \frac{(u_r^\varepsilon)^2}{r}\chi_+(\frac{r}{\eta})drdz - \int_H \rho'' u_z^\varepsilon u_r^\varepsilon \chi_+(\frac{r}{\eta})drdz - \frac{1}{2}\int_H \rho'(u_r^\varepsilon)^2\frac{1}{\eta}\chi_+'(\frac{r}{\eta})drdz,
\end{aligned}
\tag{5.32}
$$

one has

$$
\begin{aligned}
&\int_H \rho' \frac{(u_r^\varepsilon)^2}{r}\chi_+(\frac{r}{\eta})drdz - \int_H \rho'' u_z^\varepsilon u_r^\varepsilon \chi_+(\frac{r}{\eta})drdz \\
&\le \frac{1}{2}\int_H \rho'(u_r^\varepsilon)^2\frac{1}{\eta}\chi_+'(\frac{r}{\eta})drdz - \frac{1}{2}\int_H \rho'(u_z^\varepsilon)^2\frac{1}{\eta}\chi_+'(\frac{r}{\eta})drdz \\
&\quad - \int_H u_r^\varepsilon \omega_\theta^\varepsilon \frac{1}{\eta}\chi_+'(\frac{r}{\eta})\rho(z)drdz - \int_H \frac{\partial_z h_r^\varepsilon - \partial_r h_z^\varepsilon}{r}\chi_+(\frac{r}{\eta})\rho(z) r drdz \\
&\equiv J - \int_H \frac{\partial_z h_r^\varepsilon - \partial_r h_z^\varepsilon}{r}\chi_+(\frac{r}{\eta})\rho(z) r drdz,
\end{aligned}
\tag{5.33}
$$

where J is same as in (5.20). Using similar arguments as (5.17)–(5.22), we obtain (5.23) from (5.33) and hence (5.24) by the Cauchy-Schwartz inequality. Letting $N \to \infty$ on both sides of (5.24) yields (3.30) and the proof of the theorem is finished. □

Remark 5.1. For unsteady 3D axisymmetric Euler equations with vortex-sheets initial data, Chae and Imanuvilov proved in [1] that the smooth approximate solutions constructed through regularizing the initial data satisfy

$$
\int_0^T \int_{R^3} \frac{1}{1+x_3^2}(\frac{u_r^\varepsilon}{r})^2 dx \le C,
$$

where C is a constant depending on initial energy and total variation of initial vorticity. Corresponding viscous approximations can be found in [11]. Lemma 5.1 and Lemma 5.2 above concern the steady approximations with error terms and in particular in Lemma 5.2 we only need that approximate solutions are C^1-smooth.

Based on Theorem 5.1, Lemma 5.1 and Lemma 5.2, we have

Theorem 5.2. i) *Suppose that the approximate solutions u^ε, $p^\varepsilon \in C^2(R^3)$ satisfy* (3.18) *and* (3.19) *with error terms h_r^ε and h_z^ε satisfying $\partial_z h_r^\varepsilon, \partial_r h_z^\varepsilon \in C(H)$. Moreover, suppose that*

$$\|u^\varepsilon\|_{L^2(R^3)} \le C, \tag{5.34}$$

$$|\omega_\theta^\varepsilon| \le C(\varepsilon), \quad (r,z) \in \bar{H} = [0,\infty) \times (0,\infty), \tag{5.35}$$

$$\int_H (|h_z^\varepsilon| + \frac{|h_r^\varepsilon|}{r}) r dr dz \le C \ \text{ or } \ \int_H (\frac{|h_z^\varepsilon|}{r} + \frac{|h_r^\varepsilon|}{r}) r dr dz \le C, \tag{5.36}$$

$$\int_H |\frac{\partial_z h_r^\varepsilon - \partial_r h_z^\varepsilon}{r}| r dr dz \le C, \tag{5.37}$$

$$|u^\varepsilon| \to 0, \ p^\varepsilon \to p_0, \ \text{as } r^2 + z^2 \to \infty, \tag{5.38}$$

where C, p_0 are some constants and $C(\varepsilon)$ is a constant which may depend on ε. Then $u^\varepsilon \to 0$ strongly in $L^2_{loc}(R^3)$.

ii) *Suppose that the approximate solutions u^ε, $p^\varepsilon \in C^1(R^3)$ satisfy* (3.18) *and* (3.19) *with error terms h_r^ε and h_z^ε satisfying $h_r^\varepsilon, h_z^\varepsilon \in C^1(H)$ and $h_z^\varepsilon|_{r=0} = 0$. Assume further that* (5.34) *and* (5.36)–(5.38) *are satisfied. Then $u^\varepsilon \to 0$ strongly in $L^2_{loc}(R^3)$.*

Remark 5.2. Contrary to the 3D steady axisymmetric Euler equations, there exist non-trivial smooth exact solutions with finite energy and there exist smooth approximate solutions with finite energy appearing energy concentrations in the limit process to the 2D steady Euler equations (see [4]). More precisely, in 2D steady case, choose a velocity field,

$$u(x) = r^{-2} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \int_0^r s\omega(s) ds,$$

satisfying supp$\omega \subset \{|x| \le 1\}$ and $\int_0^1 s\omega(s)ds = 0$. Set $u^\varepsilon(x) = \epsilon^{-1} u(x/\epsilon)$. Then u^ϵ are the exact solutions of the two-dimensional steady Euler equations. Moreover,

$$\int_{R^2} |u^\epsilon|^2 dx + \int_{R^2} |\nabla u^\epsilon| dx \le C,$$

and

$$u^\epsilon \rightharpoonup 0$$

weakly in $L^2(R^2)$. However,

$$u^\epsilon \otimes u^\epsilon \rightharpoonup C_1 \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_0 \end{pmatrix}$$

weakly in $M(\Omega)$, the finite Radon space, where $u^\epsilon \otimes u^\epsilon = (u_i^\varepsilon u_j^\varepsilon)$ is a 2×2 matrix, δ_0 is a Dirac measure supported at the origin and C_1 is a positive constant.

Remark 5.3. Using the spherical vortex rings given in [10], an example of the approximate solutions of the 3D steady axisymmetric Euler equations which converge strongly to 0 in $L^2_{loc}(R^3)$ was constructed in [13].

Based on Theorem 5.2 ii), we obtain a Liouville type theorem which reads:

Theorem 5.3. *Suppose that* $u, p \in C^1(R^3)$ *are exact solutions of 3D steady axisymmetric Euler equations* (1.2)-(1.3) *satisfying*

$$\|u\|_{L^2(R^3)} \leq C,$$
$$|u| \to 0, \quad p \to p_0 \ \text{ as } \ r^2+z^2 \to \infty,$$

where C *and* p_0 *are some constants. Then* $u \equiv 0$ *and* $p \equiv p_0$.

Proof (I). Taking $u^\varepsilon = u$, $p^\varepsilon = p$, $h_r^\varepsilon, h_z^\varepsilon = 0$ in Theorem 5.2 ii), we obtain that $u \equiv 0$ directly. While (1.1) and the fact that $p \to p_0$ as $r^2+z^2 \to \infty$ shows that $p \equiv p_0$. The proof of the theorem is complete. □

The following is a direct proof of Theorem 5.3. The merit of this proof is that we do not need the technical test functions above.

Proof (II). Without loss of generality, we assume that

$$p \to 0 \text{ as } r^2+z^2 \to \infty. \tag{5.39}$$

Otherwise, one may replace p by $\tilde{p} = p - p_0$ in (1.2)–(1.3).

Let $\bar{p} = p - p(0,z)$. Then it follows from (1.2)–(1.3) that

$$\begin{cases} u_r\partial_r u_r + u_z\partial_z u_r + \partial_r \bar{p} = 0, \\ u_r\partial_r u_z + u_z\partial_z u_z + \partial_z \bar{p} + \partial_z p(0,z) = 0. \end{cases} \tag{5.40}$$

Note that $(5.40)_1$ can be rewritten as

$$(ru_r)\partial_r \frac{u_r}{r} + (ru_z)\partial_z \frac{u_r}{r} + \frac{u_r^2}{r} = -\partial_r \bar{p}. \tag{5.41}$$

Integrating (5.41) with respect to r over $[0, R]$, and then with respect to z over $[-Z, Z]$, using (1.3) and the fact that $u_r(0,z) = 0$, we have

$$\int_{-Z}^{Z} u_r^2(R,z)dz + \int_0^R u_r(r,z)u_z(r,z)|_{z=-Z}^{Z} dr + \int_{-Z}^{Z}\int_0^R \frac{u_r^2}{r} dr dz$$
$$= -\int_{-Z}^{Z} \bar{p}(R,z)dz. \tag{5.42}$$

Letting $R \to \infty$ on both sides of (5.42), and using the fact that $|u| \to 0$, $p \to 0$ as $r^2+z^2 \to \infty$, one can obtain

$$\int_0^\infty u_r(r,z)u_z(r,z)|_{z=-Z}^{Z} dr + \int_{-Z}^{Z}\int_0^\infty \frac{u_r^2}{r} dr dz = \int_{-Z}^{Z} p(0,z)dz. \tag{5.43}$$

Taking $r = 0$ on both sides of $(5.40)_2$, one has

$$\partial_z p(0,z) = -u_z(0,z)\partial_z u_z(0,z), \quad z \in (-\infty, \infty).$$

Thus

$$p(0,z) = -\frac{1}{2}(u_z(0,z))^2 \leq 0, \ \ z \in (-\infty, \infty). \tag{5.44}$$

Substitute (5.44) into (5.43) to obtain

$$\int_{-Z}^{Z}\int_{0}^{\infty}\frac{u_r^2}{r}drdz \le \int_0^{\infty}|u_r(r,Z)u_z(r,Z)|dr + \int_0^{\infty}|u_r(r,-Z)u_z(r,-Z)|dr. \tag{5.45}$$

Since $u \in L^2(R^3)$, we have

$$\int_{-\infty}^{\infty}\int_0^{\infty}|u_r(r,z)u_z(r,z)|rdrdz < \infty.$$

Consequently,

$$\int_{-\infty}^{\infty}\int_1^{\infty}|u_r(r,z)u_z(r,z)|drdz \le \int_{-\infty}^{\infty}\int_1^{\infty}|u_r(r,z)u_z(r,z)|rdrdz < \infty.$$

Thus there exists a sequence of number $Z_i > 0 (i = 1, 2, \cdots)$ satisfying $Z_i \to \infty$ as $i \to \infty$ such that

$$\int_1^{\infty}|u_r(r,Z_i)u_z(r,Z_i)|dr + \int_1^{\infty}|u_r(r,-Z_i)u_z(r,-Z_i)|dr \to 0 \tag{5.46}$$

as $i \to \infty$. Note that

$$\int_0^{\infty}|u_r(r,Z_i)u_z(r,Z_i)|dr = (\int_0^1 + \int_1^{\infty})|u_r(r,Z_i)u_z(r,Z_i)|dr. \tag{5.47}$$

Since $u \in C^1(R^3)$ and $|u| \to 0$, we obtain that

$$\int_0^1|u_r(r,Z_i)u_z(r,Z_i)|dr \to 0 \tag{5.48}$$

as $i \to \infty$. It follows from (5.46)–(5.48) that

$$\int_0^{\infty}|u_r(r,Z_i)u_z(r,Z_i)|dr \to 0 \tag{5.49}$$

as $i \to \infty$. Similarly, one has

$$\int_0^{\infty}|u_r(r,-Z_i)u_z(r,-Z_i)|dr \to 0 \tag{5.50}$$

as $i \to \infty$. Replacing Z by Z_i in (5.45) and taking the limit $i \to \infty$ on both sides of (5.45), we obtain

$$\int_{-\infty}^{\infty}\int_0^{\infty}\frac{u_r^2}{r}drdz = 0$$

and $u_r = 0$. This, combined with (1.3), implies that $\partial_z u_z = 0$, from which we have $u_r(r,z) = u_z(r,z) = 0$ for all $(r,z) \in R_+ \times R$. Equations (1.2) and the fact that $p \to p_0$ as $r^2 + z^2 \to \infty$ show that $p \equiv p_0$. The proof of the theorem is complete. □

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