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Stochastic Porous Media Equations and Self-Organized Criticality

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Abstract: The existence and uniqueness of nonnegative strong solutions for stochastic porous media equations with noncoercive monotone diffusivity function and Wiener forcing term is proven. The finite time extinction of solutions with high probability is also proven in 1-D. The results are relevant for self-organized criticality behavior of stochastic nonlinear diffusion equations with critical states.

1. Introduction

The phenomenon of self-organized criticality is widely studied in Physics from different perspectives. (We refer to [1,2,8–10,13–19,23] for various studies). Roughly speaking it is the property of systems to have a critical point as attractor and to reach spontaneously a critical state.

In [2] Bantay and Janosi beautifully explained that the continuum limit of the sand pile model of Bak-Tang-Wiesenfeld in [1] ("BTW model"), which was based on a cellular automaton algorithm, can be interpreted as a solution of an anomalous (singular) diffusion equation of the type

$$dX(t) = \Delta(H(X(t) - x_c)dt, \tag{1.1}$$

where H is the Heaviside function and x_c is the critical value. In [13] (see also [14]) Diaz-Guilera pointed out that for this and a similar model due to Zhang [24] given by

$$dX(t) = (X(t) - x_c)\Delta(H(X(t) - x_c)dt, \tag{1.2}$$

it is more realistic to consider Eqs. (1.1) and (1.2) perturbed by (an additive) noise to model a random amount of energy put into the system varying all over the underlying domain. The resulting equations are then stochastic partial differential equations (SPDE) of evolution type, however, with very singular (non-continuous) coefficients which mathematically can only be treated as multi-valued functions.

The purpose of this paper is to analyze such type of equations within the framework of multi-valued stochastic evolution equations with (1.1) and (1.2) as the underlying motivating examples. To the best of our knowledge this is the first time this is done in the presence of a stochastic force and in such generality in a mathematically strict way. Let us introduce our framework.

Let $\mathscr{O}$ be an open bounded domain of $\mathbb{R}^d$, $d = 1, 2, 3$, with smooth boundary $\partial\mathscr{O}$. We shall study here the nonlinear stochastic diffusion equation with linear multiplicative noise,

$$\begin{cases} dX(t) - \Delta\Psi(X(t))dt \ni \sigma(X(t))dW(t), & \text{in } (0,\infty)\times\mathscr{O}, \\ \Psi(X(t)) \ni 0, \quad \text{on } (0,\infty)\times\partial\mathscr{O}, \\ X(0,x) = x \quad \text{on } \mathscr{O}, \end{cases} \tag{1.3}$$

where x is an initial datum and $\Psi : \mathbb{R} \to 2^{\mathbb{R}}$ is a maximal monotone (possibly multivalued) graph with polynomial growth and random forcing term

$$\sigma(X)dW = \sum_{k=1}^{\infty} \mu_k X d\beta_k\, e_k, \quad t \geq 0,$$

which is linear in X. Here $\{e_k\}$ is an orthonormal basis in $L^2(\mathscr{O})$, $\{\mu_k\}$ is a sequence of positive numbers and $\{\beta_k\}$ a sequence of independent standard Brownian motions on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$.

We note that the linear operator $\sigma(X)$ is defined by

$$\sigma(X)h = \sum_{k=1}^{\infty} \mu_k X\langle h, e_k\rangle_2 e_k, \quad \forall\, h \in L^2(\mathscr{O}),$$

where $\langle\cdot,\cdot\rangle_2$ is the scalar product in $L^2(\mathscr{O})$.

Apart from the self-organized criticality phenomena mentioned above, Eq. (1.3) models the dynamics of flows in porous media and more generally the phase transition (including melting and solidification processes) in the presence of a random forcing term $\sigma(X)dW$.

Existence for stochastic equations of the form (1.3) with additive and multiplicative noise was studied in [6] under the main assumption that Ψ is monotonically increasing, continuous and such that

$$\begin{cases} \Psi(0) = 0, \;\; \Psi'(r) \leq \alpha_1|r|^{m-1} + \alpha_2, \quad \forall\, r \in \mathbb{R}, \\ \displaystyle\int_0^r \Psi(s)ds \geq \alpha_3|r|^{m+1} + \alpha_4, \quad \forall\, r \in \mathbb{R}, \end{cases} \tag{1.4}$$

where $\alpha_1 \geq 0, \alpha_3 > 0, \alpha_2, \alpha_4 \geq 0$ and $m \geq 1$. (See also [7] and [22] for general growth conditions on Ψ.)

Here we shall study Eq. (1.3) under the following assumptions.

Hypothesis 1.1. (i) Ψ *is a maximal monotone multivalued function from* $\mathbb{R}$ *into* $\mathbb{R}$ *such that* $0 \in \Psi(0)$.

(ii) *There exist* $C > 0$ *and* $m \geq 1$ *such that*

$$\sup\{|\theta| : \theta \in \Psi(r)\} \leq C(1 + |r|^m), \quad \forall\, r \in \mathbb{R}.$$

(iii) *The sequence* $\{\mu_k\}$ *is such that*

$$\sum_{k=1}^{\infty} \mu_k^2 \lambda_k^2 < +\infty,$$

where λ_k *are the eigenvalues of the Laplace operator* $-\Delta$ *in* $\mathscr{O}$ *with Dirichlet boundary conditions.*

We recall that the domain of Δ is $H^2(\mathscr{O}) \cap H_0^1(\mathscr{O})$. A multivalued function $\Psi : \mathbb{R} \to 2^{\mathbb{R}}$ is said to be *maximal monotone* if it is monotone, i.e.,

$$(v_1 - v_2)(u_1 - u_2) \geq 0, \quad \forall\, v_i \in \Psi(u_i),\ u_i \in \mathbb{R},\ i = 1, 2,$$

and the range $R(I + \Psi)$ of $I + \Psi$ is all of $\mathbb{R}$.

Standard examples of maximal monotone functions (or graphs) are continuous and increasing functions, the subdifferential of the indicator function I_K of a closed interval K of the form $[a, b]$ or $(-\infty, b)$, $[0, +\infty)$, i.e.

$$I_K(r) = \begin{cases} 0, & \text{if } r \in K, \\ +\infty, & \text{if } r \notin K, \end{cases}$$

or for $-\infty = a_0 < a_1 < \cdots < a_{N+1} = \infty$ and for $0 \leq i \leq N - 1$,

$$\Psi(r) = \begin{cases} \varphi_i(r), & \text{for } a_i < r < a_{i+1}, \\ (\varphi_i(a_{i+1} - 0), \varphi_{i+1}(a_{i+1} + 0)), & \text{for } r = a_{i+1}, \end{cases}$$

where $\{\varphi_i\}_{i=1}^N$ are monotonically non-decreasing continuous functions on (a_i, a_{i+1}) and such that $\lim_{r \to a_{i+1}} \varphi_i(r) \leq \lim_{r \to a_{i+1}} \varphi_{i+1}(r)$. Of course, any linear combination of maximal monotone graphs is maximal monotone.

It should be noticed also that the subdifferential $\partial j : \mathbb{R} \to 2^{\mathbb{R}}$ of a lower semicontinuous convex function $j : \mathbb{R} \to (-\infty, +\infty]$, i.e.,

$$\partial j(r) = \{\eta \in \mathbb{R} : j(r) \leq \eta(r - \bar{r}) + j(\bar{r}),\ \forall\, \bar{r} \in \mathbb{R}\}$$

is maximal monotone and conversely every maximal monotone function Ψ is of the form ∂j, where j is a lower semicontinuous convex function on $\mathbb{R}$.

Since for $x \in H^{-1}(\mathscr{O})$,

$$|xe_k|_{-1}^2 \leq C_1 |e_k|_{H^2(\mathscr{O})}^2\, |x|_{-1}^2 \leq C_1 \lambda_k^2 |x|_{-1}^2, \tag{1.5}$$

and hence

$$\|\sigma(x)\|^2_{L_2(L^2(\mathscr{O}), H^{-1}(\mathscr{O}))} = \sum_{k=1}^{\infty} \mu_k^2 |xe_k|_{-1}^2 \leq C_1 \sum_{k=1}^{\infty} \mu_k^2 \lambda_k^2 |x|_{-1}^2, \tag{1.6}$$

it follows by (*iii*) that $\sigma(x) \in L_2(L^2(\mathscr{O}), H^{-1}(\mathscr{O}))$ (the space of all Hilbert-Schmidt operators from $L^2(\mathscr{O})$ into $H^{-1}(\mathscr{O})$) and that it is Lipschitz continuous from $H^{-1}(\mathscr{O})$ into $L_2(L^2(\mathscr{O}), H^{-1}(\mathscr{O}))$. Under these assumptions we shall prove that if $x \in L^p(\mathscr{O})$,

$p \geq \max\{2m, 4\}$, then there is a unique strong solution to Eq. (1.3) which is nonnegative if so is the initial data x.

With respect to the situation considered in [5–7], in the present case one does not assume that the range of Ψ is all of $\mathbb{R}$. This general setting, motivated by the diffusion models mentioned above, requires, however, a different treatment of existence.

It should be mentioned that several other physical problems with free boundary and with phase transition can be put into this functional setting. For instance if

$$\Psi(x) = \begin{cases} \alpha_1(x-a), & \text{for } x < a \\ [0, \rho], & \text{for } x = a \\ \alpha_2(x-a) + \rho, & \text{for } x > a, \end{cases} \tag{1.7}$$

with $a, \rho, \alpha_1, \alpha_2 \in (0, +\infty)$, then (1.3) models the phase transition in porous media or in heat conduction (Stefan problem). If $\Psi(x) = \rho \operatorname{sign} x$, where $\rho > 0$ and

$$\operatorname{sign} x = \begin{cases} \dfrac{x}{|x|}, & \text{if } x \neq 0 \\ \\ [-1, 1], & \text{if } x = 0, \end{cases} \tag{1.8}$$

then (1.3) reduces to the nonlinear singular diffusion equation

$$dX(t) - \rho \operatorname{div}\,(\delta(X(t))\nabla X(t))dt = \sigma(X(t))dW(t),$$

where δ is the Dirac measure concentrated at the origin.

We already mentioned the Heavside step function

$$H(x) = \begin{cases} 0, & \text{if } x < 0 \\ [0, 1], & \text{if } x = 0 \\ 1, & \text{if } x > 0. \end{cases}$$

Furthermore, $\Psi(x) = |x|^\alpha \operatorname{sign} x$ with $0 < \alpha \leq 1$ also satisfy Hypothesis 1.1.

Typical examples considered in the literature are $\Psi(r) = (r - x_c)^\alpha$, where $\alpha < 1$ and the key result is that the density $X(t)$ of the system converges to the critical value. In the same category fall the stochastically perturbed versions of Eqs. (1.1) and (1.2), that is e.g. in the first case the highly singular diffusion equation

$$dX(t) - \Delta(H + \lambda)(X(t) - x_c)dt = \sigma(X(t) - x_c)dW(t), \tag{1.9}$$

where $\lambda \geq 0$. This is a diffusion problem with free boundary driven by a random forcing term proportional to $X(t) - x_c$, where x_c is the critical density and $X(t)$ is the density at the moment t.

Taking into account the numerical simulation in 1-D (see [2]), one might expect that the time evolution of the system displays self-organized criticality, i.e. the supercritical region $\{X(t) > x_c\}$ is absorbed asymptotically in time by the critical one $\{X(t) = x_c\}$.

A few of the previous works (see e.g. [11]) on self-organized criticality in singular diffusion equations based on numerical tests brought attention on the failure of the self-organized behavior in the presence of random fluctuations (white noise perturbation).

Here we shall prove, however, for systems of the form (1.7)-(1.9) that the self-organized criticality takes place with high probability under appropriate assumptions on the parameters and more precisely that the supercritical region "vanishes" into the critical one in finite time with high probability, at least if $\mu_k = 0$ for all $k \geq N+1$ for some $N \in \mathbb{N}$. We emphasize that this is in particular true when the noise is zero. In this case one gets an explicit bound for the time when this happens (cf. Remark 4.4 below).

The plan of this paper is the following. The main results are presented in Sect. 2 and are proven in Sect. 3. In Sect. 4 we prove a finite time extinction type result for solutions to (1.3) which displays a self-organized criticality behavior.

The following notations will be used. $L^p(\mathscr{O})$, $p \geq 1$, is the usual space of p-integrable functions with norm denoted by $|\cdot|_p$. The scalar product in $L^2(\mathscr{O})$ and the duality induced by the pivot space $L^2(\mathscr{O})$ will be denoted by $\langle\cdot,\cdot\rangle_2$. $H^k(\mathscr{O}) \subset L^2(\mathscr{O})$, $k = 1, 2$, are the standard Sobolev spaces on $\mathscr{O}$, while $H_0^1(\mathscr{O})$ is the subspace of $H^1(\mathscr{O})$ with zero trace on the boundary. For $p, q \in [1, +\infty]$ by $L_W^q((0,T); L^p(\Omega; H))$ (H a Hilbert space) we shall denote the space of all q-integrable processes $u : [0,T] \to L^p(\Omega; H)$ which are adapted to the filtration $\{\mathscr{F}_t\}_{t\geq 0}$.

By $C_W([0,T]; L^2(\Omega; H))$ we shall denote the space of all H-valued adapted processes which are mean square continuous. $L(H)$ denotes the space of bounded linear operators equipped with the usual norm.

In the following by H we shall denote the distribution space

$$H = H^{-1}(\mathscr{O}) = (H_0^1(\mathscr{O}))'$$

endowed with the scalar product and norm defined by

$$\langle u, v\rangle = \int_{\mathscr{O}} A^{-1}u(\xi)v(\xi)d\xi, \quad |u|_{-1} = \langle u, u\rangle^{1/2},$$

where $A = -\Delta$ with $D(A) = H^2(\mathscr{O}) \cap H_0^1(\mathscr{O})$.

In terms of A Eq. (1.3) can be formally rewritten as

$$\begin{cases} dX(t) + A\Psi(X(t))dt \ni \sigma(X(t))dW(t), \\ X(0,x) = x. \end{cases} \tag{1.10}$$

Its exact meaning will be precised later (see Definition 2.1 below).

It should be recalled, however, that the operator $x \to A\Psi(x)$ with the domain

$$\{x \in L^1(\mathscr{O}) \cap H^{-1}(\mathscr{O}) : \text{there is } \eta \in H_0^1(\mathscr{O}), \eta \in \Psi(x) \text{ a.e. in } \mathscr{O}\}$$

is maximal monotone in $H := H^{-1}(\mathscr{O})$ (see e.g. [3]) and so the distribution space H offers the natural functional setting for the porous media equation (1.3) or its abstract form (1.10). However, the general existence theory of infinite dimensional stochastic equations in Hilbert space with nonlinear maximal monotone operators (see [12,21]) is not applicable in the present case and so a direct approach must be used.

Finally, in this paper we use the same letter C for several different positive constants arising in chains of estimates.

2. Existence, Uniqueness and Positivity

Definition 2.1. *Let $x \in H$. An H-valued continuous $\mathscr{F}_t$-adapted process $X = X(t,x)$ is called a solution to* (1.3) (*equivalently* (1.10)) *on* $[0,T]$ *if*

$$X \in L^p(\Omega \times (0,T) \times \mathscr{O}) \cap L^2(0,T; L^2(\Omega, H)), \quad p \geq m,$$

and there exists $\eta \in L^{p/m}(\Omega \times (0,T) \times \mathscr{O})$ *such that* $\mathbb{P}$*-a.s.*

$$\langle X(t,x), e_j\rangle_2 = \langle x, e_j\rangle_2 + \int_0^t \int_{\mathscr{O}} \eta(s,\xi)\Delta e_j(\xi)d\xi ds$$

$$+ \sum_{k=1}^{\infty} \mu_k \int_0^t \langle X(s,x)e_k, e_j\rangle_2 d\beta_k(s), \quad \forall\, j \in \mathbb{N},\ t \in [0,T], \tag{2.1}$$

$$\eta \in \Psi(X) \quad \text{a.e. in } \ \Omega \times (0,T) \times \mathscr{O}. \tag{2.2}$$

Below for simplicity we often write $X(t)$ instead of $X(t,x)$.

From the stochastic point of view the solution X given by Definition 2.1 is a strong one, but from the PDE point of view it is a solution in the sense of distributions since the boundary condition $\Psi(X) \ni 0$ on $\partial\mathscr{O}$ is satisfied in a weak sense only.

Theorem 2.2 below is the main existence result.

Theorem 2.2. *Assume that* $d = 1,2,3$ *and that Hypothesis* 1.1 *holds. Then for each* $x \in L^p(\mathscr{O})$, $p \geq \max\{2m, 4\}$ *there is a unique solution* $X \in L^\infty_W(0,T; L^p(\Omega;\mathscr{O}))$ *to* (1.3)*. Moreover, if* x *is nonnegative a.e. in* $\mathscr{O}$ *then* $\mathbb{P}$*-a.s.*

$$X(t,x)(\xi) \geq 0, \quad \textit{for a.e. } (t,\xi) \in (0,\infty) \times \mathscr{O}.$$

As mentioned earlier, Theorem 2.2 was proven in [6] for a differentiable Ψ satisfying conditions (1.4) and for $p \geq \max\{m+1, 4\}$. It should be said, however, that in contrast with what happens for coercive functions Ψ arising in [6], here it seems no longer possible to extend the existence result to all $x \in H^{-1}(\mathscr{O})$, $x \geq 0$.

3. Proof of Theorem 2.2

We shall consider the approximating equation

$$\begin{cases} dX_\lambda(t) + A(\Psi_\lambda(X_\lambda(t)) + \lambda X_\lambda(t))dt = \sigma(X_\lambda(t))dW(t), \\ X_\lambda(0,x) = x, \end{cases} \tag{3.1}$$

where $\lambda > 0$ and

$$\Psi_\lambda(x) = \frac{1}{\lambda}\,(x - (1+\lambda\Psi)^{-1}(x)) \in \Psi((1+\lambda\Psi)^{-1}(x))$$

is the Yosida approximation of Ψ. We recall that Ψ_λ is Lipschitzian and monotonically increasing and so $x \to \Psi_\lambda(x) + \lambda x$ is strictly monotonically increasing and bounded by $C_1(1+|x|^m)$ and $(\Psi_\lambda(x)+\lambda x)x \geq \lambda|x|^2$ for all $x \in \mathbb{R}$. By [6, Theorem 2.2] (applied with $m = 1$), for each $x \in H^{-1}(\mathscr{O})$ Eq. (3.1) has a unique solution

$$X_\lambda \in L^2(\Omega \times (0,T) \times \mathscr{O}) \cap L^2_W(\Omega, C([0,T]; H))$$

in the sense of Definition 2.1. Here as usual $C([0,T]; H)$ is equipped with the supremum norm. Moreover, (see e.g. [21, Theorem 4.2.5]) the following Itô formula holds

$$\mathbb{E}|X_\lambda(t)|^2_{-1} \ + 2\mathbb{E}\int_0^t \int_{\mathscr{O}} (\Psi_\lambda(X_\lambda(s)) + \lambda X_\lambda(s))X_\lambda(s)d\xi\ ds$$

$$= |x|^2_{-1} + \sum_{k=1}^{\infty} \mu_k^2\, \mathbb{E}\int_0^t |X_\lambda(s)e_k|^2_{-1} ds. \tag{3.2}$$

We note that since

$$|X_\lambda e_k|_{-1} \le C|e_k|_{H^2(\mathscr{O})}|X_\lambda|_{-1} \le C\lambda_k|X_\lambda|_{-1},$$

(cf. (1.5)) we have by Hypothesis 1.1(iii) (cf. (1.6))

$$\sum_{k=1}^{\infty} \mu_k^2 \mathbb{E}\int_0^t |X_\lambda(s)e_k|_{-1}^2 ds \le C\mathbb{E}\int_0^t |X_\lambda(s)|_{-1}^2 ds. \tag{3.3}$$

Lemma 3.1. *There exists a constant $C > 0$ such that for all $p \ge 2$ and all $x \in L^p(\mathscr{O})$,*

$$\text{ess.sup}_{t\in[0,T]}\, \mathbb{E}|X_\lambda(t,x)|_p^p \le \exp\left(C\frac{p-1}{2}\right)\, |x|_p^p, \quad \forall\, \lambda > 0. \tag{3.4}$$

Proof. We know from [6, Lemma 3.4] (with $m = 1$) that as $\varepsilon \to 0$,

$$\begin{cases} X_\lambda^\varepsilon \to X_\lambda & \text{strongly in } L_W^\infty(0,T;L^2(\Omega;H)),\\ X_\lambda^\varepsilon \to X_\lambda & \text{in the weak* topology in } L_W^\infty(0,T;L^p(\Omega;L^p(\mathscr{O}))), \end{cases} \tag{3.5}$$

where X_λ^ε is the solution to the approximating equation

$$\begin{cases} dX_\lambda^\varepsilon(t) + (A_\lambda)_\varepsilon X_\lambda^\varepsilon(t)dt = \sigma(X_\lambda^\varepsilon(t))dW(t), \quad t \ge 0,\\ X_\lambda^\varepsilon(0) = x, \end{cases} \tag{3.6}$$

where

$$\begin{cases} A_\lambda x = A(\Psi_\lambda(x) + \lambda x) = -\Delta(\Psi_\lambda(x) + \lambda x),\\ D(A_\lambda) = \{x \in H \cap L^1(\mathscr{O}) :\ \Psi_\lambda(x) + \lambda x \in H_0^1(\mathscr{O})\}, \end{cases}$$

and $(A_\lambda)_\varepsilon$ is the Yosida approximation of A_λ,

$$(A_\lambda)_\varepsilon = \frac{1}{\varepsilon}\,(I - (I + \varepsilon A_\lambda)^{-1}), \quad \varepsilon > 0.$$

Furthermore, by [6, Lemma 3.2] we have that $X_\lambda^\varepsilon \in L^2(\Omega; C([0,T]; L^2(\mathscr{O}))$. As a matter of fact the results of [6] were proven for smooth nonlinear functions while Ψ_λ is only Lipschitz but the extension to Lipschitzian functions Ψ satisfying (1.4) is immediate. In fact, one might take a smoother approximation of Ψ, for instance the mollifier $\Psi_\lambda * \rho_\lambda$ ($\rho_\lambda(r) = \frac{1}{\lambda}\,\rho(\lambda/r)$, $\rho \in C_0^\infty(\mathbb{R})$, $\rho \ge 0$, $\int \rho dr = 1$) which still remains monotonically increasing and has all properties of Ψ_λ.

Next we apply Itô's formula (3.6) for the function $\varphi(x) = \frac{1}{p}\, |x|_p^p$. More precisely, we first apply Itô's formula to $\varphi_\gamma(x) = \frac{1}{p}\, |(1+\gamma A)^{-1}x|_p^p$, $\gamma > 0$, and then we let $\gamma \to 0$. We have (for details see the proof in [6, Lemma 3.5]),

$$\begin{aligned} &\mathbb{E}\varphi(X_\lambda^\varepsilon(t)) + \mathbb{E}\int_0^t \langle (A_\lambda)_\varepsilon X_\lambda^\varepsilon(s), |X_\lambda^\varepsilon(s)|^{p-2}X_\lambda^\varepsilon(s)\rangle_2 ds\\ &\quad = \varphi(x) + \frac{p-1}{2}\sum_{k=1}^{\infty}\mu_k^2\mathbb{E}\int_0^t\int_{\mathscr{O}} |X_\lambda^\varepsilon(s)|^{p-2}|X_\lambda^\varepsilon(s)e_k|^2 d\xi\ ds\ d\xi\\ &\quad \le \varphi(x) + \frac{p-1}{2}\,C\mathbb{E}\int_0^t\int_{\mathscr{O}} |X_\lambda^\varepsilon(s)|^p d\xi\ ds, \end{aligned} \tag{3.7}$$

since by Sobolev embedding $|e_k|_\infty \le C\lambda_k$ for all $k \in \mathbb{N}$. If Y^ε_λ is the solution to the equation

$$Y^\varepsilon_\lambda - \varepsilon\Delta(\Psi_\lambda(Y^\varepsilon_\lambda) + \lambda Y^\varepsilon_\lambda) = X^\varepsilon_\lambda, \quad \Psi_\lambda(Y^\varepsilon_\lambda) + \lambda Y^\varepsilon_\lambda \in H^1_0(\mathscr{O}),$$

then (see [6, (3.25)]) $|Y^\varepsilon_\lambda|_p \le |X^\varepsilon_\lambda|_p$ and therefore

$$\langle (A_\lambda)_\varepsilon X^\varepsilon_\lambda, |X^\varepsilon_\lambda|^{p-2}X^\varepsilon_\lambda\rangle_2 = \frac{1}{\varepsilon}\,\langle X^\varepsilon_\lambda - Y^\varepsilon_\lambda, |X^\varepsilon_\lambda|^{p-2}X^\varepsilon_\lambda\rangle_2 \ge 0.$$

Then by (3.7) it follows, via Gronwall's lemma, that

$$\mathbb{E}|X^\varepsilon_\lambda(t)|^p_p \le |x|^p_p\ \exp\left(C\frac{p-1}{2}\right),$$

where C is independent of x, λ and t. Now one obtains (3.4) by letting ε tend to 0 and taking into account (3.5). □

From now on let us assume that $p \ge \max\{4, 2m\}$ and $x \in L^p(\mathscr{O})$. From Lemma 3.1 it follows that for a subsequence $\{\lambda\} \to 0$ we have

$$\begin{cases} X_\lambda \to X \quad \text{weakly in } L^p(\Omega\times(0,T)\times\mathscr{O}),\\ \qquad\qquad \text{and weak* in } L^\infty(0,T; L^p(\Omega; L^p(\mathscr{O}))),\\ \Psi_\lambda(X_\lambda) \to \eta \quad \text{weakly in } L^{p/m}(\Omega\times(0,T)\times\mathscr{O}),\\ \qquad\qquad \text{in particular weakly in } L^2(\Omega\times(0,T)\times\mathscr{O}),\end{cases} \tag{3.8}$$

because by Hypothesis(*ii*),

$$|\Psi_\lambda(x)| \le |\Psi^0(x)| \le C(1+|x|^m), \quad \forall\, x \in \mathbb{R}.$$

(Ψ^0 is the minimal section of Ψ.) By (3.4) we have for $\lambda \to 0$,

$$\lambda X_\lambda \to 0 \quad \text{strongly in } L^p(\Omega\times(0,T)\times\mathscr{O}). \tag{3.9}$$

Clearly X and η are adapted processes. On the other hand, we have

$$\begin{aligned} &d(X_\lambda(t) - X_\mu(t)) - \Delta(\Psi_\lambda(X_\lambda(t)) - \Psi_\mu(X_\mu(t)) + \lambda X_\lambda(t) - \mu X_\mu(t))dt\\ &\quad = (\sigma(X_\lambda(t)) - \sigma(X_\mu(t)))dW(t),\end{aligned}$$

and therefore once again applying Itô's formula (cf. (3.2)) we obtain for $\alpha > 0$, $t \in [0,T]$,

$$\begin{aligned} &\frac{1}{2}\,|X_\lambda(t) - X_\mu(t))|^2_{-1}e^{-\alpha t}\\ &\quad + \int_0^t\int_{\mathscr{O}} \big[(\Psi_\lambda(X_\lambda(s)) - \Psi_\mu(X_\mu(s))\ (\lambda\Psi_\lambda(X_\lambda(s)) - \mu\Psi_\mu(X_\mu(s)))\\ &\quad + (\lambda X_\lambda(s) - \mu X_\mu(s))(X_\lambda(s) - X_\mu(s))\big]\, e^{-\alpha s} d\xi\ ds\\ &\le \left(C\sum_{k=1}^\infty \mu_k^2\lambda_k^2 - \frac{1}{2}\,\alpha\right)\int_0^t |X_\lambda(s - X_\mu(s))|^2_{-1}e^{-\alpha s}\ ds + M_{\lambda,\mu}(t), \quad \forall\,\lambda,\mu > 0,\end{aligned} \tag{3.10}$$

where

$$M_{\lambda,\mu}(t) := \int_0^t e^{-\alpha s}\langle X_\lambda(s) - X_\mu(s), \sigma(X_\lambda(s) - X_\mu(s))dW(s)\rangle_2$$

is a real local valued martingale. To derive (3.10) we used that $x = \lambda\Psi_\lambda(x) + (1+\lambda\Psi)^{-1}(x)$, and thus for all $x, y \in \mathbb{R}$,

$$\begin{aligned}(\Psi_\lambda(x) - \Psi_\mu(y))(x-y) = {} & [\Psi_\lambda(x) - \Psi_\mu(y)][(1+\lambda\Psi)^{-1}(x) - (1+\mu\Psi)^{-1}(y)] \\ & + [\Psi_\lambda(x) - \Psi_\mu(y)][\lambda\Psi_\lambda(x) - \mu\Psi_\mu(y)],\end{aligned}$$

and that the first summand on the right-hand side is nonnegative because Ψ is monotonically increasing and $\Psi_\lambda(x) \in \Psi((1+\lambda\Psi)^{-1}(x))$. Hence for $\alpha > 0$ large enough we obtain for all $\lambda, \mu \in (0,1)$ and $t \in [0,T]$,

$$\begin{aligned}&\frac{1}{2}\,|X_\lambda(t) - X_\mu(t))|_{-1}^2 e^{-\alpha t} \\ &\quad\le C\max\{\lambda,\mu\}\int_0^t\int_{\mathscr{O}}\Big(|\Psi_\lambda(X_\lambda(s))|^2 + |X_\lambda(s)|^2 + |\Psi_\mu(X_\mu(s))|^2 \\ &\qquad + |X_\mu(s)|^2\Big)\, e^{-\alpha s}d\xi\ ds + M_{\lambda,\mu}(t).\end{aligned} \tag{3.11}$$

Hence by the Burkholder-Davis-Gundy inequality (for $p = 1$) we get for all $\lambda, \mu \in (0,1)$, $r \in [0,T]$,

$$\begin{aligned}&\frac{1}{2}\,\mathbb{E}\sup_{t\in[0,r]}|X_\lambda(t) - X_\mu(t))|_{-1}^2 e^{-\alpha t} \\ &\quad\le C\max\{\lambda,\mu\}\mathbb{E}\int_0^r\int_{\mathscr{O}}\Big(|\Psi_\lambda(X_\lambda(s))|^2 + |X_\lambda(s)|^2 + |\Psi_\mu(X_\mu(s))|^2 \\ &\qquad + |X_\mu(s)|^2\Big)\, e^{-\alpha s}d\xi\ ds + C\mathbb{E}\left(\int_0^r |X_\lambda(s) - X_\mu(s)|_{-1}^4 e^{-2\alpha s}ds\right)^{1/2}.\end{aligned} \tag{3.12}$$

But

$$\begin{aligned}&\mathbb{E}\left(\int_0^r |X_\lambda(s) - X_\mu(s)|_{-1}^4 e^{-2\alpha s}ds\right)^{1/2} \\ &\quad\le \mathbb{E}\sup_{s\in[0,r]}|X_\lambda(s) - X_\mu(s))|_{-1}e^{-\frac{\alpha}{2}s}\left(\int_0^r |X_\lambda(s) - X_\mu(s)|_{-1}^2 e^{-\alpha s}ds\right)^{1/2} \\ &\quad\le \frac{1}{4}\,\mathbb{E}\sup_{s\in[0,r]}|X_\lambda(s) - X_\mu(s))|_{-1}^2 e^{-\alpha s} + C\mathbb{E}\int_0^r |X_\lambda(s) - X_\mu(s)|_{-1}^2 e^{-\alpha s}ds.\end{aligned} \tag{3.13}$$

Taking into account that by Hypothesis 1.1(ii),

$$|\Psi_\lambda(X_\lambda)| \le C(1 + |X_\lambda|^m), \quad \forall\,\lambda > 0,$$

and that by (3.4) $\{X_\lambda\}$ is bounded in $L^p(\Omega\times(0,T)\times\mathscr{O})$ for $p\geq\max\{4,2m\}$, we infer by (3.12), (3.13) and Gronwall's lemma that $\{X_\lambda\}$ is a Cauchy net in $L^2(\Omega;C([0,T];H))$ Hence for $\lambda\to 0$,

$$X_\lambda \to X \quad \text{in } L^2(\Omega; C([0,T];H)). \tag{3.14}$$

In order to complete the proof of the existence part of Theorem 2.2 it suffices to show that

$$\eta(\omega,t,\xi)\in\Psi(X(\omega,t,\xi)) \quad \text{a.e in } \ \Omega\times(0,T)\times\mathscr{O}. \tag{3.15}$$

Since the operator

$$L^p(\Omega\times(0,T)\times\mathscr{O})\to L^{\frac{p}{m}}(\Omega\times(0,T)\times\mathscr{O})\subset L^{\frac{p}{p-1}}(\Omega\times(0,T)\times\mathscr{O}),\quad X\to\Psi(X),$$

in the duality pair

$$\left(L^p(\Omega\times(0,T)\times\mathscr{O}), L^p(\Omega\times(0,T)\times\mathscr{O})'=L^{\frac{p}{p-1}}(\Omega\times(0,T)\times\mathscr{O})\right),$$

is maximal monotone, it suffices to show that (see e.g. [3])

$$\liminf_{\lambda\to 0}\mathbb{E}\int_0^T\int_{\mathscr{O}}\Psi_\lambda(X_\lambda)X_\lambda d\xi dt \leq \mathbb{E}\int_0^T\int_{\mathscr{O}}\eta X d\xi dt. \tag{3.16}$$

To prove (3.16) we first note that by (3.2) we have

$$\begin{aligned}\liminf_{\lambda\to 0}\mathbb{E}&\int_0^T\int_{\mathscr{O}}\Psi_\lambda(X_\lambda)X_\lambda d\xi dt+\frac{1}{2}\,\mathbb{E}|X(t)|_{-1}^2\\ &=\frac{1}{2}\,|x|_{-1}^2+\frac{1}{2}\,\sum_{k=1}^\infty\mu_k^2\,\mathbb{E}\int_0^t|X(s)e_k|_{-1}^2ds,\end{aligned} \tag{3.17}$$

because by (1.5), $|(X_\lambda - X)e_k|_{-1}\leq C\lambda_k|X_\lambda - X|_{-1}$ and so by Hypothesis 1.1(iii),

$$\lim_{\lambda\to 0}\sum_{k=1}^\infty\mu_k^2\mathbb{E}\int_0^t|X_\lambda(s)e_k|_{-1}^2ds=\sum_{k=1}^\infty\mu_k^2\mathbb{E}\int_0^t|X(s)e_k|_{-1}^2ds.$$

Next letting λ tend to zero in (3.1) and using (3.8) we see that $\mathbb{P}$-a.s., for all $t\in[0,T]$,

$$\langle X(t),e_j\rangle_2=\langle x,e_j\rangle_2+\int_0^t\langle\eta(s),\Delta e_j\rangle_2 ds+\sum_{k=1}^\infty\mu_k\int_0^t\langle X(s)e_k,e_j\rangle_2 d\beta_k(s). \tag{3.18}$$

Note that by continuity the $\mathbb{P}$-zero set does not depend on $t\in[0,T]$, since

$$\sum_{k=1}^\infty\mu_k\int_0^t\langle X(s)e_k,e_j\rangle_2 d\beta_k(s)=\int_0^t\langle e_j,\sigma(X(s))dW(s)\rangle_2.$$

In order to get (3.18) we have used the fact that by (3.14) we have

$$\mathbb{E}\left|\int_0^t \langle X_\lambda(s)e_k, e_j\rangle_2 d\beta_k(s)ds - \int_0^t \langle X(s)e_k, e_j\rangle_2 d\beta_k(s)ds\right|^2$$
$$= \mathbb{E}\int_0^t \langle (X_\lambda(s) - X(s))e_k, e_j\rangle_2^2 ds \le C\lambda_j^2\lambda_k^2 T|X_\lambda - X|^2_{L^2(\Omega,C([0,T];H))},$$

and therefore

$$\lim_{\lambda\to 0}\sum_{k=1}^\infty \mu_k \int_0^t \langle X_\lambda(s)e_k, e_j\rangle_2 d\beta_k ds = \sum_{k=1}^\infty \mu_k \int_0^t \langle X(s)e_k, e_j\rangle_2 d\beta_k ds.$$

Therefore (3.18) follows and this yields, via Itô's formula (applied to $\langle X(t), e_j\rangle_2^2$, $t \in [0, T]$) and summation over j that

$$\frac{1}{2}\,\mathbb{E}|X(t)|_{-1}^2 + \mathbb{E}\int_0^t\int_{\mathscr{O}} \eta X d\xi\, ds$$
$$= \frac{1}{2}\,\mathbb{E}|x|_{-1}^2 + \frac{1}{2}\sum_{k=1}^\infty \mu_k^2\,\mathbb{E}\int_0^t |X(s)e_k|_{-1}^2 ds,\ \forall\, t \in [0, T]. \tag{3.19}$$

Comparing (3.17) and (3.19) we get (3.16). Hence X is a solution to (1.3) as claimed.

To prove uniqueness we take two solutions $X^{(1)}$ and $X^{(2)}$ with corresponding $\eta^{(1)}$ and $\eta^{(2)}$. Repeating the argument above we obtain

$$\frac{1}{2}\,\mathbb{E}|X^{(1)}(t) - X^{(2)}(t)|_{-1}^2$$
$$+\mathbb{E}\int_0^t\int_{\mathscr{O}} (\eta^{(1)}(s) - \eta^{(2)}(s))(X^{(1)}(s) - X^{(2)}(s))d\xi ds$$
$$= \frac{1}{2}\sum_{k=1}^\infty \mu_k^2\,\mathbb{E}\int_0^t |(X^{(1)}(s) - X^{(2)}(s))e_k|_{-1}^2\, ds, \quad \forall\, t \in [0, T].$$

Since, because Ψ is monotone, the second term on the left is positive, by (1.5), Hypothesis 1.1(iii) this implies $X^{(1)} = X^{(2)}$ by Gronwall's lemma.

Finally, if $x \ge 0$ a.e. in $\mathscr{O}$ we know by [6, Theorem 2.2] that $X_\lambda \ge 0$ $\mathbb{P}$-a.s. and so by (3.14) it follows that $X \ge 0$, a.e in $\Omega \times (0, T) \times \mathscr{O}$ as desired. This completes the proof of Theorem 2.2. □

Remark 3.2. Theorem 2.2 extends to any dimension $d \ge 1$ if one modifies condition (*iii*) in Hypothesis 1.1 as in [6, Condition 4.1], i.e., one assumes

$$\sum_{k=1}^\infty \mu_k^2(|e_k|_\infty + \lambda_k|e_k|_{L^{\frac{4d}{d+6}}(\mathscr{O})})^2 < +\infty.$$

Remark 3.3. The existence part of Theorem 2.2 remains true for stochastic porous media equations with additive noise, i.e.

$$dX - \Delta\Psi(X)dt = \sqrt{Q}\, dW(t),$$

where Ψ satisfies Hypothesis 1.1 and

$$\sqrt{Q}\, dW(t) = \sum_{k=1}^{\infty} \mu_k e_k d\beta_k(t)$$

with

$$\sum_{k=1}^{\infty} \lambda_k^{-1} \mu_k^2 < +\infty.$$

The proof is exactly the same and so, it will be omitted.

Proposition 3.4. *Let X_λ, $\lambda \in (0,1)$, be as above, $x \in L^4(\mathscr{O})$. Assume that Ψ satisfies Hypothesis 1.1 with $m = 1$ and for some $\delta > 0$,*

$$(\tilde{x} - \tilde{y})(x - y) \geq \delta(x - y)^2, \quad \forall\, (x, \tilde{x}), (y, \tilde{y}) \in \Psi. \tag{3.20}$$

Then $X_\lambda, X \in L^2_W(0, T; L^2(\Omega, H^1_0(\mathscr{O})))$ and

$$\lim_{\lambda \to 0} \mathbb{E}|X_\lambda - X|^2_{L^2(0,T;L^2(\mathscr{O}))} = 0. \tag{3.21}$$

Proof. A simple calculation reveals that

$$(\Psi_\lambda(x) - \Psi_\lambda(y))(x - y) \geq \frac{\delta}{2}\, |x - y|^2, \quad \forall\, x, y \in \mathbb{R}$$

for λ sufficiently small. Then $\tilde{\Psi}_\lambda$ defined by $\tilde{\Psi}_\lambda(r) := \Psi_\lambda(r) - \frac{\delta}{2}\, r,\ r \in \mathbb{R}$, is increasing and so by Itô's formula we have

$$\mathbb{E}|X_\lambda(t)|_2^2 + \frac{\delta}{2}\, \mathbb{E} \int_0^t |X_\lambda(s)|^2_{H^1_0(\mathscr{O})} ds \leq C. \tag{3.22}$$

As a matter of fact, we shall apply Itô's formula not directly to Eq. (3.1) but to Eq. (3.6) (cf. the proof of Lemma 3.1 to obtain (3.7)). Thus we get

$$\frac{1}{2}\, \mathbb{E}|X^\varepsilon_\lambda(t)|_2^2 + \mathbb{E} \int_0^t \langle (A_\lambda)_\varepsilon X^\varepsilon_\lambda(s), X^\varepsilon_\lambda(s)\rangle_2 ds \leq \frac{1}{2}\, |x|_2^2 + C\mathbb{E} \int_0^t |X^\varepsilon_\lambda(s)|_2^2 ds.$$

Next we have

$$\langle (A_\lambda)_\varepsilon X^\varepsilon_\lambda, X^\varepsilon_\lambda\rangle_2 = \langle A_\lambda(1 + \varepsilon A_\lambda)^{-1} X^\varepsilon_\lambda, (1 + \varepsilon A_\lambda)^{-1} X^\varepsilon_\lambda\rangle_2 + \varepsilon|(A_\lambda)_\varepsilon X^\varepsilon_\lambda|_2^2.$$

Taking into account that $A_\lambda = \Delta(\Psi_\lambda + \lambda I)$ and that $r \to \Psi_\lambda(r) - \delta r/2$ is monotonically increasing we get

$$\langle (A_\lambda)_\varepsilon X^\varepsilon_\lambda, X^\varepsilon_\lambda\rangle_2 \geq \frac{\delta}{2} \int_{\mathscr{O}} |\nabla(1 + \varepsilon A_\lambda)^{-1} X^\varepsilon_\lambda|^2 d\xi + \varepsilon|(A_\lambda)_\varepsilon X^\varepsilon_\lambda|_2^2.$$

Hence

$$\mathbb{E}\int_0^t |(1+\varepsilon A_\lambda)^{-1} X_\lambda^\varepsilon(s)|^2_{H_0^1(\mathscr{O})} ds \le C$$

and letting $\varepsilon \to 0$ we get (3.22) and the first assertion (taking also into account (3.5)).

To prove the second part we note that

$$d(X_\lambda - X_\mu) - \Delta[\tilde{\Psi}_\lambda(X_\lambda) - \tilde{\Psi}_\mu(X_\mu) + \lambda X_\lambda - \mu X_\mu + \frac{1}{2}\,\delta\,(X_\lambda - X_\mu)]dt$$
$$= (\sigma(X_\lambda) - \sigma(X_\mu))dW.$$

Hence exactly the same arguments to derive (3.11) lead to

$$\frac{1}{2}\,|X_\lambda(t) - X_\mu(t)|^2_{-1} e^{-\alpha t} + \frac{\delta}{2}\int_0^t |X_\lambda(s) - X_\mu(s)|_2^2 e^{-\alpha s} ds$$
$$\le C\max\{\lambda,\mu\}\int_0^t \Big(|\Psi_\lambda(X_\lambda(s))|_2^2 + |\Psi_\mu(X_\mu(s))|_2^2$$
$$+ |X_\lambda(s)|_2^2 + |X_\mu(s)|_2^2\Big)\, e^{-\alpha s} ds + M_{\lambda,\mu}(t),$$

for α large enough and $\lambda, \mu \in (0,1)$, $t \in [0,T]$. Since $m = 1$, we have $|\Psi_\lambda(x)| \le C(1+|x|)$ for all $x \in \mathbb{R}$, $\lambda \in (0,1)$, hence taking the expectation we get

$$\frac{\delta}{2}\,\mathbb{E}\int_0^t |X_\lambda(s) - X_\mu(s)|_2^2 ds \le C\max\{\lambda,\mu\}\mathbb{E}\int_0^t (|X_\lambda(s)|^2 + |X_\mu(s)|^2) ds.$$

By Lemma 3.1 with $p = 2$ and (3.8) this implies (3.21). □

Besides Hypothesis 1.1, we shall now assume the following:

(iv) $\Psi(r) = \rho$ sign $r + \widetilde{\Psi}(r)$, *for* $r \in \mathbb{R}$, *where* $\rho \ge 0$, $\widetilde{\Psi} : \mathbb{R} \to \mathbb{R}$ *is Lipschitz,* $\widetilde{\Psi} \in C^1(\mathbb{R}\setminus\{0\})$ *and for some* $\delta > 0$ *it satisfies* $\widetilde{\Psi}'(r) \ge \delta$ *for all* $r \in \mathbb{R}\setminus\{0\}$.

Here the signum is defined by (1.8).

Below we shall use an approximation to Ψ which is slightly different from Ψ_λ defined before. Namely, below we consider

$$\Psi_\lambda(r) := \rho\ (\mathrm{sign})_\lambda(r) + \widetilde{\Psi}(r) + \lambda r, \quad r \in \mathbb{R},$$

where $(\mathrm{sign})_\lambda$ is the Yosida approximation of the sign, i.e.

$$(\mathrm{sign})_\lambda(r) := \begin{cases} 1 & \text{if } r > \lambda \\ \frac{r}{\lambda} & \text{if } r \in [-\lambda,\lambda] \\ -1 & \text{if } r < -\lambda. \end{cases}$$

We shall use the symbol Ψ_λ also for this approximation and denote also by X_λ the corresponding solution of (3.1). This approximation in the special case of condition (iv) is much more convenient. We emphasize that all previous results remain true for this modified approximation. The proofs are the same and some parts even simplify. We therefore shall use all previous results for Ψ_λ and X_λ as above without further notice.

The following technical result will be used in Sect. 4 (cf. Lemma 4.1) in a crucial way.

Proposition 3.5. *The solutions* X_λ *to* (3.1) *and* X *to* (1.3) *satisfy all conditions of Proposition* 3.4 *and in addition*

$$\mathbb{E}\int_0^T\int_{\mathscr{O}} |\nabla(\text{sign})_\lambda(X_\lambda)|^2 d\xi dt \le C, \quad \forall\, \lambda > 0,$$

and consequently $\eta \in L^2_W(0,T;L^2(\Omega;H^1_0(\mathscr{O}))$.

Proof. We set

$$g_\lambda(r) := \int_0^r (\text{sign})_\lambda(s)ds, \quad r \in \mathbb{R},$$

and choose $\varphi_\lambda \in C^2(\mathbb{R})$ such that

(i) $\varphi_\lambda(0) = 0$.
(ii) $\varphi'_\lambda(r) = \frac{r}{\lambda}$ for $|r| \le \lambda$, $\varphi'_\lambda(r) = 1+\lambda$ for $r \ge 2\lambda$, $\varphi'_\lambda(r) = -1-\lambda$ for $r \le -2\lambda$.
(iii) $0 \le \varphi''_\lambda(r) \le \frac{C}{\lambda}$ for all $r \in \mathbb{R}$.

It is easily seen that such a function exists and can be constructed simply by smoothing the function $(\text{sign})_\lambda$. Let us denote the resulting function by f_λ. Then define

$$\varphi_\lambda(r) := \int_0^r f_\lambda(s)ds, \quad r \in \mathbb{R}.$$

As mentioned above the arguments of the previous proofs extends to the present situation in order to prove that $\{X_\lambda\}$ is convergent to the solution X to (1.3).

Now we shall apply Itô's formula to Eq. (3.1) (or, more exactly, to (3.6) and then let $\varepsilon \to 0$ as in the proof of Proposition 3.4) with Ψ_λ defined as above and to the function $\int_{\mathscr{O}} \varphi_\lambda(X_\lambda)d\xi$.

Arguing as in the proof of Lemma 3.1 to obtain (3.7), we get (recall that $X_\lambda(t) \in H^1_0(\mathscr{O})$),

$$\begin{aligned}
&\mathbb{E}\int_{\mathscr{O}} \varphi_\lambda(X_\lambda(t))d\xi - \mathbb{E}\int_0^t \langle \Delta(\text{sign})_\lambda(X_\lambda(s)) + \Delta\widetilde{\Psi}(X_\lambda(s)), \varphi'_\lambda(X_\lambda(s))\rangle_2\, ds \\
&\quad\le \int_{\mathscr{O}} \varphi_\lambda(x)d\xi + C\sum_{k=1}^\infty \mu_k^2 \mathbb{E}\int_0^t\int_{\mathscr{O}} \varphi''_\lambda(X_\lambda(s))|X_\lambda(s)e_k|^2 d\xi ds \\
&\quad\le \int_{\mathscr{O}} \varphi_\lambda(x)d\xi + 4\lambda C\sum_{k=1}^\infty \mu_k^2\lambda_k^2 \mathbb{E}\int_0^t\int_{\mathscr{O}} 1_\lambda(s,\xi)|e_k|^2 d\xi ds,
\end{aligned}$$

where 1_λ is the characteristic function of the set $\{(s,\xi) : 0 \le |X_\lambda(s,\xi)| \le 2\lambda\}$.

Concerning the first line we note that, since φ'_λ and $\tilde{\Psi}$ are monotonically increasing while as seen earlier $X_\lambda(t) \in H^1_0(\mathscr{O})$, we have by the Green formula that

$$\langle \Delta\widetilde{\Psi}(X_\lambda), \varphi'_\lambda(X_\lambda)\rangle_2 = -\int_{\mathscr{O}} \widetilde{\Psi}'(X_\lambda)\varphi''_\lambda(X_\lambda)|\nabla X_\lambda|^2 d\xi \le 0.$$

This yields

$$\mathbb{E}\int_0^T\int_{\mathscr{O}} \langle \nabla(\text{sign})_\lambda(X_\lambda), \nabla\varphi'_\lambda(X_\lambda)\rangle_2 \, d\xi ds \le C, \quad \forall\, \lambda \in (0,1).$$

Taking into account that

$$-\langle \Delta(\text{sign})_\lambda(X_\lambda), \varphi'_\lambda(X_\lambda)\rangle_2 = \langle \nabla(\text{sign})_\lambda(X_\lambda), \nabla\varphi'_\lambda(X_\lambda)\rangle_2 \ge 0, \quad \text{a.e.}$$

and that $\nabla\varphi'_\lambda(X_\lambda) = \frac{1}{\lambda}\,\nabla X_\lambda$ on $\{(s,\xi):\ |X_\lambda(s,\xi)| < \lambda\}$, we get

$$\mathbb{E}\int_0^T\int_{\mathscr{O}} |\nabla(\text{sign})_\lambda(X_\lambda)|^2 d\xi ds \le C, \quad \forall\, \lambda \in (0,1),$$

because $\nabla(\text{sign})_\lambda(X_\lambda) = \frac{1}{\lambda}\nabla(X_\lambda)$ if $|X_\lambda)| < \lambda$ and $\nabla(\text{sign})_\lambda(X_\lambda) = 0$ if $|X_\lambda)| \ge \lambda$.

Then we get the desired estimate and since also by (3.22),

$$\mathbb{E}\int_0^T\int_{\mathscr{O}} |\nabla\widetilde{\Psi}(X_\lambda)|^2 d\xi ds \le C, \quad \forall\, \lambda \in (0,1)$$

and $(\text{sign})_\lambda(X_\lambda) + \widetilde{\Psi}(X_\lambda) \to \eta$ weakly in $L^2(\Omega \times (0,T) \times \mathscr{O})$ as $\lambda \to 0$, we infer that $\eta \in L^2_W(0,T; L^2(\Omega; H^1_0(\mathscr{O}))$ as claimed. □

4. Extinction in Finite Time and Self-Organized Criticality

In this section we shall prove a finite extinction property for solutions of (1.3) in 1-D for a special density dependent diffusion coefficient function Ψ. However, Lemma 4.1 below can be proved without restriction on dimension. So, for the moment we remain in our general framework.

For simplicity we choose the Wiener process

$$W(t) = \sum_{k=1}^N \mu_k e_k \beta_k(t), \quad t \ge 0, \tag{4.1}$$

where $N \in \mathbb{N}$.

Besides Hypothesis 1.1, we shall assume Hypothesis (iv) (following the proof of Prop. 3.4), i.e.

(iv) $\Psi(r) = \rho$ sign $r + \widetilde{\Psi}(r)$, *for* $r \in \mathbb{R}$, *where* $\rho > 0$, $\widetilde{\Psi}: \mathbb{R} \to \mathbb{R}$ *is Lipschitzian,* $\widetilde{\Psi} \in C^1(\mathbb{R} \setminus \{0\})$ *and for some* $\delta > 0$ *it satisfies* $\widetilde{\Psi}'(r) \ge \delta$ *for all* $r \in \mathbb{R} \setminus \{0\}$.

Here the signum is defined by (1.8).

Now let τ be the stopping time

$$\tau = \inf\{t \ge 0:\ |X(t,x)|_{-1} = 0\},$$

where $X(t,x), t \ge 0$, is the solution to (1.3) given by Theorem 2.2 for $x \in L^p(\mathscr{O})$, $p \ge \max\{4, 2m\}$.

Lemma 4.1. *Under assumptions* (i)−(iv) *we have*

$$X(t,x)=0, \quad \text{for } t \geq \tau, \quad \mathbb{P}\text{-a.s.}$$

Proof. Set $A=-\Delta$, $D(A)=H^2(\mathscr{O})\cap H_0^1(\mathscr{O})$. Define $\mu:[0,T]\times\Omega\to C_b^2(\mathscr{O};\mathbb{R})$ by

$$\mu(t):=-\sum_{k=1}^N \mu_k e_k \beta_k(t), \quad t\in[0,T],$$

and $\tilde{\mu}:[0,T]\to C_b^2(\mathscr{O};\mathbb{R})$ by

$$\tilde{\mu}:=\sum_{k=1}^N \mu_k^2 e_k^2.$$

Define

$$Y(t)=e^{\mu(t)}X(t), \quad t\geq 0.$$

Let $D(A)$ be equipped with the graph norm of A and let $D(A)'$ be its dual space, hence

$$D(A)\subset H_0^1(\mathscr{O})\subset L^2(\mathscr{O})\subset H^{-1}(\mathscr{O})\subset D(A)'. \tag{4.2}$$

It is easy to see that for all $\omega\in\Omega$, $t\in[0,T]$ the function $e^{\mu(t,\omega)}$ is a multiplier both in $D(A)$ and in H, hence $e^{\mu(t,\omega)}\Delta z\in D(A)'$ is well defined for all $z\in L^2(\mathscr{O})$ and $Y(t)\in H$.

Claim. We have

$$Y(t)=x+\int_0^t e^{\mu(s)}\Delta\eta(s)ds-\frac{1}{2}\int_0^t \tilde{\mu}Y(s)ds, \quad t\in[0,T], \tag{4.3}$$

where the fist integral on the right-hand side is a Bochner integral in $D(A)'$, the second by (3.8) is one in $L^p(\mathscr{O})\subset L^2(\mathscr{O})$. In particular a posteriori the first integal is in H, continuous in H as a function of $t\in[0,T]$, $\mathbb{P}$-a.s.

□

Proof of the Claim. Let $\varphi\in D(A)$. As before we shall use $\langle\cdot,\cdot\rangle_2$ also for the extended dualizations with pivot space $L^2(\mathscr{O})$ as the ones in (4.2).Then for $t\in[0,T]$,

$$\langle\varphi, e^{\mu(t)}X(t)\rangle_2=\sum_{j=1}^\infty \langle e_j, e^{\mu(t)}\varphi\rangle_2\ \langle e_j, X(t)\rangle_2.$$

Furthermore, we have by Itô's formula for all $\xi\in\mathscr{O}$,

$$e^{\mu(t,\xi)}=1+\int_0^t e^{\mu(s,\xi)}d\mu(s,\xi)+\frac{1}{2}\int_0^t e^{\mu(s,\xi)}\tilde{\mu}(\xi)ds.$$

Now fix $j \in \mathbb{N}$. Then by the stochastic Fubini Theorem

$$\langle e_j, e^{\mu(t)}\varphi\rangle_2 = \langle e_j, \varphi\rangle_2 - \sum_{k=1}^{N} \mu_k \int_0^t \langle e_j, e_k e^{\mu(s)}\varphi\rangle_2 d\beta_k(s) + \frac{1}{2}\int_0^t \langle e_j, \tilde{\mu} e^{\mu(s)}\varphi\rangle_2 ds, \quad t \in [0, T].$$

By Itô's product rule and (3.18) we hence obtain

$$\begin{aligned}
&\langle e_j, e^{\mu(t)}\varphi\rangle_2 \ \langle e_j, X(t)\rangle_2 \\
&= \langle e_j, \varphi\rangle_2 \ \langle e_j, x\rangle_2 \quad + \int_0^t \langle e_j, e^{\mu(s)}\varphi\rangle_2 \ \langle \Delta e_j, \eta(s)\rangle_2 \ ds \\
&+ \sum_{k=1}^{N} \mu_k \int_0^t \langle e_j, e^{\mu(s)}\varphi\rangle_2 \ \langle e_j, X(s)e_k\rangle_2 \ d\beta_k(s) \\
&+ \frac{1}{2}\int_0^t \langle e_j, X(s)\rangle_2 \ \langle e_j, \tilde{\mu}\ e^{\mu(s)}\varphi\rangle_2 \ ds \\
&- \sum_{k=1}^{N} \mu_k \int_0^t \langle e_j, X(s)\rangle_2 \ \langle e_j, e_k e^{\mu(s)}\varphi\rangle_2 \ d\beta_k(s) \\
&- \sum_{k=1}^{N} \mu_k^2 \int_0^t \langle e_j, e_k e^{\mu(s)}\varphi\rangle_2 \ \langle e_j, X(s)e_k\rangle_2 \ d\beta_k(s).
\end{aligned}$$

After summing over $j \in \mathbb{N}$ the two stochastic terms cancel and the claim follows since $\varphi \in D(A)$ was arbitrary.

Below we work for $\mathbb{P}$-a.s. $\omega \in \Omega$, ω fixed. Hence all constants C appearing below may depend on ω.

Consider the solution $X_\lambda \in L^2_W(0, T; L^2(\Omega, H_0^1(\mathscr{O})))$ to Eq. (3.1). By Proposition 3.4 we have

$$\lim_{\lambda \to 0} \mathbb{E}|X_\lambda - X|^2_{L^2(0,T;L^2(\mathscr{O}))} = 0$$

and $\Psi_\lambda(X_\lambda) \in L^2_W(0, T; L^2(\Omega, H_0^1(\mathscr{O})))$ because Ψ_λ is Lipschitz.

On the other hand,we have as in (4.3) for $Y_\lambda = e^{\mu} X_\lambda$,

$$\frac{dY_\lambda(t)}{dt} = e^{\mu(t)}\Delta\eta_\lambda(t) - \frac{1}{2}\ \tilde{\mu}(t)Y_\lambda(t), \quad \forall\, t \geq 0, \tag{4.4}$$

where

$$\eta_\lambda(t) = \Psi_\lambda(X_\lambda(t)) \in H_0^1(\mathscr{O}).$$

It follows by (3.21) that

$$\lim_{\lambda\to 0} \mathbb{E}|Y_\lambda - Y|^2_{L^2(0,T;L^2(\mathscr{O}))} = 0, \tag{4.5}$$

and therefore for some sequence $\lambda_n \to 0$,

$$\lim_{n\to\infty} |Y_{\lambda_n} - Y|_{L^2(0,T;L^2(\mathscr{O}))} = 0 \quad \text{a.e. on } \Omega. \tag{4.6}$$

Below we simply write λ instead of λ_n. Next we have by (4.4) that

$$\left\langle \frac{dY_\lambda(t)}{dt}, Y_\lambda(t)\right\rangle_2 = \left\langle \eta_\lambda(t), \Delta(e^{\mu(t)}Y_\lambda(t))\right\rangle_2 - \frac{1}{2}\,\langle\tilde{\mu}(t)Y_\lambda(t), Y_\lambda(t)\rangle_2 \quad \text{a.e. } t\in[0,T]. \tag{4.7}$$

Also we have (for simplicity we take $\rho = 1$)

$$\begin{aligned}
&\langle \eta_\lambda(t), \Delta(e^{\mu(t)}Y_\lambda(t))\rangle_2\\
&\quad= \langle (\mathrm{sign})_\lambda\ (e^{-\mu(t)}Y_\lambda(t)) + \widetilde{\Psi}(e^{-\mu(t)}Y_\lambda(t)), \Delta(e^{\mu(t)}Y_\lambda(t))\rangle_2\\
&\quad= -\int_{\mathscr{O}} (\nabla(\mathrm{sign})_\lambda\ (e^{-\mu(t)}Y_\lambda(t)), \nabla(e^{\mu(t)}Y_\lambda(t)))d\xi\\
&\qquad -\int_{\mathscr{O}} \widetilde{\Psi}'(e^{-\mu(t)}Y_\lambda(t))(\nabla(e^{-\mu(t)}Y_\lambda(t)), \nabla(e^{\mu(t)}Y_\lambda(t)))d\xi\\
&\quad= -\frac{1}{\lambda}\int_{\mathscr{O}} (|\nabla Y_\lambda(t)|^2 - |Y_\lambda(t)|^2\ |\nabla\mu(t)|^2)1_\lambda(t,\xi)d\xi\\
&\qquad -\int_{\mathscr{O}} \widetilde{\Psi}'(e^{-\mu(t)}Y_\lambda(t))(|\nabla Y_\lambda(t)|^2 - |Y_\lambda(t)|^2\ |\nabla\mu(t)|^2)d\xi,
\end{aligned}$$

because for $y \in H^1_0(\mathscr{O})$,

$$\nabla\ (\mathrm{sign})_\lambda\ (y) = \begin{cases} 0, & \text{on } \{y \notin (-\lambda,\lambda)\},\\ \frac{1}{\lambda}\ \nabla y, & \text{on } \{y\in(-\lambda,\lambda)\}.\end{cases}$$

(Here 1_λ is the characteristic function of $\{(\xi,t)\in\mathscr{O}\times[0,T]:\ |e^{-\mu(t,\xi)}Y_\lambda(t,\xi))| < \lambda\}$ and $(\cdot,\cdot)$ is the euclidean scalar product in $\mathbb{R}^n$.) Since $\widetilde{\Psi}' \geq \delta$ and $\widetilde{\Psi}' \in L^\infty(\mathbb{R})$, $\mu \in C([0,T]\times\mathscr{O})$ this yields

$$\langle \eta_\lambda(t), \Delta(e^{\mu(t)}Y_\lambda(t))\rangle_2 \leq C\left(|Y_\lambda(t)|^2_2 + \lambda\right). \tag{4.8}$$

Hence (4.7) and Gronwall's lemma imply

$$|Y_\lambda(t)|^2_2 \leq e^{C(t-s)}\left(|Y_\lambda(s)|^2_2 + C\lambda T\right) \quad \text{a.e. } t > s.$$

Now taking into account (4.6) and letting $\lambda\to 0$ we get

$$|Y(t)|^2_2 \leq e^{C(t-s)}|Y(s)|^2_2 \quad \text{a.e. } t > s. \tag{4.9}$$

If $Y(\cdot)$ is $L^2(\mathscr{O})$-continuous then (4.9) holds for all $s,t\in[0,T]$, $t\geq s$. Taking in (4.9) $s = \tau\wedge T$ we get $Y(t) = 0$ for all $t \geq \tau\wedge T$ and since $T > 0$ was arbitrary for all

$t \geq \tau$ as claimed. So, we have to prove that Y is $L^2(\mathscr{O})$-continuous on $[0, T]$. For this we recall that by Proposition 3.5 we have

$$e^{\mu}\, \eta \in L^2(0, T; H_0^1(\mathscr{O})), \quad \mathbb{P}\text{-a.s.} \tag{4.10}$$

Then by Eq. (4.3) we have $\frac{dY}{dt} \in L^2(0, T; H^{-1}(\mathscr{O}))$ and so, since $Y \in L^2(0, T; H_0^1(\mathscr{O}))$ $\mathbb{P}$-a.s. by Proposition 3.4, by a well known interpolation result (see e.g. [3]), we conclude that $Y \in C([0, T]; L^2(\mathscr{O}))$. This concludes the proof of Lemma 4.1. □

For proving our extinction result we need $\mathscr{O} \subset \mathbb{R}$, i.e. $d = 1$. To be more specific let $\mathscr{O} = (0, \pi)$. Then $e_k(\xi) = \sqrt{\frac{2}{\pi}}\ \sin k\xi, \quad \xi \in [0, \pi]$, $\lambda_k = k^2$ and $L^1(0, \pi) \subset H$ continuously, so

$$\gamma = \inf \left\{ \frac{|x|_{L^1}}{|x|_{-1}} : \ x \in L^1(0, \pi) \right\} > 0. \tag{4.11}$$

Theorem 4.2. *Let $x \in L^p(0, \pi)$, $p \geq \max\{2m, 4\}$, be such that*

$$|x|_{-1} < C_N^{-1} \rho \gamma,$$

where

$$C_N := \frac{\pi}{4} \sum_{k=1}^{N} (1+k)^2 \mu_k^2. \tag{4.12}$$

Then, for each $n \in \mathbb{N}$,

$$\mathbb{P}(\tau \leq n) \geq 1 - \frac{|x|_{-1}}{\rho\gamma} \left(\int_0^n e^{-C_N s} ds \right)^{-1}, \tag{4.13}$$

where by Lemma 4.1 we have

$$\tau(\omega) = \sup\{t \geq 0 : \ |X(t, x)|_{-1} > 0\}.$$

Proof. By condition (*iv*) we see that

$$r\Psi(r) \geq \rho |r|, \quad \forall\, r \in \mathbb{R}. \tag{4.14}$$

Consider the solution $X_\lambda \in L^2_W(0, T; L^2(\Omega; H_0^1(0, \pi)))$ to Eq. (3.1). Then by first applying Krylov-Rozovskii's Itô formula (cf. [20, Theorem I.3.1] or e.g. [21, Theorem 4.2.5]) and then the classical Itô formula to the real valued semi-martingale $|X_\lambda(t)|^2_{-1}$, $t \in [0, T]$, and the function

$$\varphi_\varepsilon(r) = (r + \varepsilon^2)^{1/2}, \quad r \in \mathbb{R},$$

we find

$$
\begin{aligned}
&d\varphi_\varepsilon(|X_\lambda(t)|_{-1}^2) + (|X_\lambda(t)|_{-1}^2 + \varepsilon^2)^{-1/2}\langle X_\lambda(t), \Psi_\lambda(X_\lambda(t))\rangle_2 dt\\
&\quad = \frac{1}{2}\sum_{k=1}^N \mu_k^2 \frac{|X_\lambda(t)e_k|_{-1}^2(|X_\lambda(t)|_{-1}^2+\varepsilon^2) - |\langle X_\lambda(t)e_k, X_\lambda(t)\rangle_{-1}|^2)}{(|X_\lambda(t)|_{-1}^2+\varepsilon^2)^{3/2}}\, dt\\
&\qquad + \langle \sigma(X_\lambda(t))dW(t), \varphi'_\varepsilon(|X_\lambda(t)|_{-1}^2)X_\lambda(t)\rangle_{-1}\\
&\quad \le \frac{1}{2}\sum_{k=1}^N \mu_k^2 \frac{|X_\lambda(t)e_k|_{-1}^2}{(|X_\lambda(t)|_{-1}^2+\varepsilon^2)^{1/2}} dt + \langle \sigma(X_\lambda(t))dW(t), \varphi'_\varepsilon(|X_\lambda(t)|_{-1}^2)X_\lambda(t)\rangle_{-1}\\
&\quad \le C_N \frac{|X_\lambda(t)|_{-1}^2}{(|X_\lambda(t)|_{-1}^2+\varepsilon^2)^{1/2}}\, dt + 2\langle \sigma(X_\lambda(t))dW(t), \varphi'_\varepsilon(|X_\lambda(t)|_{-1}^2)X_\lambda(t)\rangle_{-1}. \qquad (4.15)
\end{aligned}
$$

Here C_N is given by (4.12) and

$$
\sigma(X_\lambda(t))dW(t) = \sum_{k=1}^N \mu_k X_\lambda(t) e_k d\beta_k(t).
$$

Integrating over t and letting $\lambda \to 0$ we see that the right-hand side of (4.15) converges to the right-hand side of (4.16) below. But by (3.8), (3.12), (3.13) and by Proposition 3.4 the same is true for the left-hand side with limit

$$
\varphi_\varepsilon(|X(t)|_{-1}^2) - \varphi_\varepsilon(|x|_{-1}^2) + \int_0^t \int_{\mathscr{O}} \frac{X(s)}{(|X(s)|_{-1}^2+\varepsilon)^{1/2}}\, \eta(s) d\xi ds.
$$

Taking into account (2.2) and (4.14) we altogether obtain

$$
\begin{aligned}
&d\varphi_\varepsilon(|X(t)|_{-1}^2) + \rho \frac{|X(t)|_{L^1(0,\pi)}}{(|X(t)|_{-1}^2+\varepsilon^2)^{1/2}} dt\\
&\quad \le C_N \frac{|X(t)|_{-1}^2}{(|X(t)|_{-1}^2+\varepsilon^2)^{1/2}}\, dt + 2\langle \sigma(X(t))dW(t), \varphi'_\varepsilon(|X(t)|_{-1}^2)X(t)\rangle.
\end{aligned}
$$

Consequently by Lemma 4.1 for all $t \ge 0$,

$$
\begin{aligned}
&\varphi_\varepsilon(|X(t)|_{-1}^2) + \gamma\rho \int_0^{t\wedge\tau} \frac{|X(s)|_{-1}}{(|X(s)|_{-1}^2+\varepsilon^2)^{1/2}} ds\\
&\quad \le \varphi_\varepsilon(|x|_{-1}^2) + C_N \int_0^{t\wedge\tau} \frac{|X(s)|_{-1}^2}{(|X(s)|_{-1}^2+\varepsilon^2)^{1/2}} ds\\
&\qquad + 2\int_0^{t\wedge\tau} \langle \sigma(X(s))dW(s), \varphi'_\varepsilon(|X(s)|_{-1}^2)X(s)\rangle, \quad \mathbb{P}\text{-a.s.}, \qquad (4.16)
\end{aligned}
$$

where γ is defined by (4.4).

Clearly, we have

$$\lim_{\varepsilon\to 0}\int\limits_0^{t\wedge\tau} \frac{|X(s)|_{-1}}{(|X(s)|_{-1}^2+\varepsilon^2)^{1/2}}\,ds = t\wedge\tau, \qquad \mathbb{P}\text{-a.s.}$$

Now, letting ε tend to zero we get

$$\begin{aligned}&|X(t)|_{-1}+\gamma\rho(t\wedge\tau)\\ &\leq |x|_{-1}+C_N\int\limits_0^t |X(s)|_{-1}ds+\int\limits_0^t 1_{[0,\tau]}(s)\langle\sigma(X(s))dW(s),X(s)|X(s)|_{-1}^{-1}\rangle \quad \mathbb{P}\text{-a.s.}\end{aligned} \tag{4.17}$$

Hence by a standard comparison result

$$\begin{aligned}&|X(t)|_{-1}+\rho\gamma\int\limits_0^t e^{C_N(t-s)}1_{[0,\tau]}(s)ds\\ &\quad\leq e^{C_Nt}|x|_{-1}+\int\limits_0^t e^{C_N(t-s)}1_{[0,\tau]}(s)\langle\sigma(X(s))dW(s),X(s)|X(s)|_{-1}^{-1}\rangle.\end{aligned}$$

Taking the expectation and multiplying by $(\rho\gamma)^{-1}e^{-C_Nt}$, we obtain

$$\int\limits_0^t e^{-C_Ns}\mathbb{P}(\tau>s)ds \leq \frac{|x|_{-1}}{\rho\gamma}.$$

Writing $\mathbb{P}(\tau>s)=1-\mathbb{P}(\tau\leq s)$ we deduce that

$$\mathbb{P}(\tau\leq t)\geq 1-\frac{|x|_{-1}}{\rho\gamma}\left(\int\limits_0^t e^{-C_Ns}ds\right)^{-1}$$

and (4.13) follows. □

In particular Theorem 4.2 applies to self-organized criticality stochastic models (1.9),

$$\begin{cases} dX(t)-\Delta(\rho\text{ sign }(X(t)-x_c)+\widetilde{\Psi}(X(t)-x_c))dt\\ \qquad\qquad \ni \sigma(X(t)-x_c)\sum_{k=1}^N \mu_k e_k d\beta_k, \quad t\geq 0,\\ \rho\text{ sign }(X(t)-x_c)+\widetilde{\Psi}(X(t)-x_c)\ni 0, \quad \text{on } \partial[0,\pi],\\ X(0,x)=x.\end{cases} \tag{4.18}$$

Here the function $\widetilde{\Psi}$ is as in assumption (iv) and $x_c\in\mathbb{R}$.

Corollary 4.3. *Assume that*

$$|x - x_c|_{-1} < \rho\gamma C_N^{-1},$$

where C_N *is as in* (4.12) *and* γ *as in* (4.11). *Then for each* $n \in \mathbb{N}$,

$$\mathbb{P}(\tau_c \le n) \ge 1 - \frac{|x - x_c|_{-1}}{\rho\gamma}\left(\int_0^n e^{-C_N s} ds\right)^{-1}, \tag{4.19}$$

where

$$\tau_c = \inf\{t \ge 0 : \ |X(t) - x_c|_{-1} = 0\} = \sup\{t \ge 0 : \ |X(t) - x_c|_{-1} > 0\},$$

and $X = X(t, x)$ *is the solution to* (4.18) *in the sense of Definition* 2.1.

We note that Eq. (1.9) reduces to (4.18) by shifting the Heavside function with x_c.

Remark 4.4. One must notice that if $x > x_c$, i.e. if the initial state is in the supercritical region then by the positivity result in Theorem 2.2 we have $X(t) \ge x_c$, $\mathbb{P}$-a.s. for all $t \ge 0$. This means that the state remains in the supercritical-critical region for all time. However, by (4.19) if $\frac{C_N |x|_{-1}}{\rho\gamma}$ is small, it reaches the critical state x_c with high probability in a finite time, i.e. the supercritical-critical region is completely absorbed by the critical one in a finite time. In contrast, if $\frac{C_N |x|_{-1}}{\rho\gamma}$ is not small, i.e., if the magnitude of the random fluctuations induced by the noise is large compared with the initial state x then the above conclusion might fail because the random perturbations can push the density $X(t)$ over the singularity x_c.

So, in general we cannot expect $\tau_c < \infty$, $\mathbb{P}$-a.s. However, by (4.19) we see that

$$\mathbb{P}(\tau_c < \infty) = \lim_{n\to\infty} \mathbb{P}(\tau_c \le n) \ge 1 - \frac{|x - x_c|_{-1}}{\rho\gamma C_N}.$$

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Note added in proof. Employing a supermartingale argument it is possible to prove Lemma 4.1 without the assumption that N in (4.1) is finite. Then also Theorem 4.2 holds for $N = \infty$. In addition, Lemma 4.1 also holds without assuming in (iv) that $\delta > 0$, but rather only that $\tilde{\Psi}'(r) \ge 0$ for all $r \in \mathbb{R}\backslash\{0\}$. Details on this will be included in a forthcoming paper.

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