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The Jancovici–Lebowitz–Manificat Law for Large Fluctuations of Random Complex Zeroes

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Abstract: Consider a Gaussian Entire Function

$$f(z) = \sum_{k=0}^{\infty} \zeta_k \frac{z^k}{\sqrt{k!}}\,,$$

where $\zeta_0, \zeta_1, \dots$ are Gaussian i.i.d. complex random variables. The zero set of this function is distribution invariant with respect to the isometries of the complex plane. Let $n(R)$ be the number of zeroes of f in the disk of radius R. It is easy to see that $\mathbb{E}n(R) = R^2$, and it is known that the variance of $n(R)$ grows linearly with R (Forrester and Honner). We prove that, for every $\alpha > 1/2$, the tail probability $\mathbb{P}\left\{ |n(R) - R^2| > R^\alpha \right\}$ behaves as $\exp\left[-R^{\varphi(\alpha)}\right]$ with some explicit piecewise linear function $\varphi(\alpha)$. For some special values of the parameter α, this law was found earlier by Sodin and Tsirelson, and by Krishnapur.

In the context of charge fluctuations of a one-component Coulomb system of particles of one sign embedded into a uniform background of another sign, a similar law was discovered some time ago by Jancovici, Lebowitz and Manificat.

1. Introduction

Consider the Fock-Bargmann space of the entire functions of one complex variable that are square integrable with respect to the measure $\frac{1}{\pi} e^{-|z|^2}\, dm(z)$, where m is the Lebesgue measure on $\mathbb{C}$. Let f be a Gaussian function associated with this space; i.e.,

$$f(z) = \sum_{k\geqslant 0} \zeta_k e_k(z),$$

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where ζ_k are independent standard complex Gaussian random variables (that is, the density of ζ_k on the complex plane $\mathbb{C}$ is $\frac{1}{\pi}e^{-|w|^2}$), and $\{e_k\}$ is an orthonormal basis in the Fock-Bargmann space. The Gaussian function f does not depend on the choice of the basis $\{e_k\}$, so usually one takes the standard basis $e_k(z) = \dfrac{z^k}{\sqrt{k!}}$, $k \in \mathbb{Z}_+$. In what follows, we call f a *Gaussian Entire Function* (G.E.F., for short). G.E.F. together with other similar models were introduced in the 90's in the works of Bogomolny, Bohigas, Lebouef [1], and Hannay [5].

A remarkable feature of the zero set $\mathcal{Z}_f = f^{-1}\{0\}$ of a G.E.F. is its distribution invariance with respect to the isometries of $\mathbb{C}$. The rotation invariance is obvious since the distribution of the function f is rotation invariant. The translation invariance follows, for instance, from the fact that the operators $(T_w g)(z) = g(w+z)e^{-z\overline{w}}e^{-|w|^2/2}$, $w \in \mathbb{C}$, are unitary operators in the Fock-Bargmann space, and therefore, if f is a G.E.F., then $T_w f$ is a G.E.F. as well (see Sect. 2.2 below). It is worth mentioning that by Calabi's rigidity [11, Sect. 3], $f(z)$ together with its scalings $f(tz)$, $t > 0$, are the only Gaussian functions analytic in $\mathbb{C}$ with the distribution of zeroes invariant with respect to the isometries of $\mathbb{C}$. See [12, Part I] for further discussion.

Let $n(R) = \text{Card}\left\{\mathcal{Z}_f \cap R\mathbb{D}\right\}$ be the number of zeroes of f in the disk of radius R. It is not hard to check that the mean number of points of $\mathcal{Z}_f$ per unit area equals $\frac{1}{\pi}$ (cf. Sect. 2.3). Therefore, $\mathbb{E}n(R) = R^2$. The asymptotics of the variance of $n(R)$ was computed by Forrester and Honner in [2]:

$$\mathbb{E}\left(n(R) - R^2\right)^2 = cR + o(R), \qquad R \to \infty,$$

with an explicitly computed positive c. In [10], Shiffman and Zelditch gave a different computation of the asymptotics of the variance valid in a more general context. The normalized random variables $\dfrac{n(R) - R^2}{\sqrt{\text{Var}\, n(R)}}$ converge in distribution to the standard Gaussian random variable. This can be proven, for instance, by a suitable modification of the argument used in [12, Part I]. In this work, we describe the probabilities of large fluctuations of the random variable $n(R) - R^2$.

Theorem 1. *For every $\alpha \geqslant \frac{1}{2}$ and every $\varepsilon > 0$,*

$$e^{-R^{\varphi(\alpha)+\varepsilon}} < \mathbb{P}\left\{|n(R) - R^2| > R^\alpha\right\} < e^{-R^{\varphi(\alpha)-\varepsilon}} \tag{1.1}$$

for all sufficiently large $R > R_0(\alpha, \varepsilon)$, where

$$\varphi(\alpha) = \begin{cases} 2\alpha - 1, & \frac{1}{2} \leqslant \alpha \leqslant 1; \\ 3\alpha - 2, & 1 \leqslant \alpha \leqslant 2; \\ 2\alpha, & \alpha \geqslant 2\,. \end{cases}$$

In a different context of charge fluctuations of a one-component Coulomb system of particles of one sign embedded into a uniform background of the opposite sign, a similar law was discovered by Jancovici, Lebowitz and Manificat in their physical paper [4]. Let us mention that it is known since Ginibre's classical paper [3] that the class of point processes considered by Jancovici, Lebowitz and Manificat contains as a special case the $N \to \infty$ limit of the eigenvalue point process of the ensemble of $N \times N$ random matrices with independent standard complex Gaussian entries. The resemblance between the zeroes of G.E.F. and the eigenvalues of Ginibre's ensemble was discussed both in the physical and the mathematical literature.

Now, let us return to the zeroes of G.E.F. In some cases, the estimate (1.1) is known. As we have already mentioned, it is known for $\alpha = \frac{1}{2}$ when it follows from the asymptotics of the variance and the asymptotic normality. In the case $\alpha = 2$ it follows from a result of Sodin and Tsirelson [12, Part III], which says that *for each* $R \geqslant 1$,

$$e^{-CR^4} \leqslant \mathbb{P}\left\{|n(R) - R^2| > R^2\right\} \leqslant e^{-cR^4}$$

with some positive numerical constants c and C. In [7], Krishnapur considered the case $\alpha > 2$ and proved that in that case

$$\mathbb{P}\left\{n(R) > R^\alpha\right\} = e^{-(\frac{\alpha}{2}-1)(1+o(1))R^{2\alpha}\log R}, \qquad R \to \infty .$$

In the same work, he also proved the lower bound in the case $1 < \alpha < 2$:

$$\mathbb{P}\left\{|n(R) - R^2| > R^\alpha\right\} \geqslant e^{-CR^{3\alpha-2}} .$$

Using a certain development of his method, we'll get the lower bound

$$\mathbb{P}\left\{|n(R) - R^2| > R^\alpha\right\} \geqslant e^{-CR^{2\alpha-1}}, \qquad \frac{1}{2} < \alpha < 1 .$$

Apparently, in the case $\frac{1}{2} < \alpha < 2$, the technique used in [12, Part III] and [7] does not allow one to treat the upper bounds in the law (1.1), which require new ideas.

Outline of the proof. Let us sketch the main ideas we use in the proof of Theorem 1.

1. We denote by $\Delta_I \arg f$ the increment of the argument of a G.E.F. f over an arc $I \subset R\mathbb{T}$ oriented counterclockwise, and set $\delta(f, I) = \Delta_I \arg f - \mathbb{E}\Delta_I \arg f$. Then by the argument principle,

$$2\pi(n(R) - R^2) = \delta(f, R\mathbb{T}) .$$

Note that the random variable $\delta(f, I)$ is set-additive and split the circumference $R\mathbb{T}$ into $N = 2\pi\dfrac{R}{r}$ disjoint arcs I_j of length r. Thus we need to estimate the probability of the event

$$\Omega_\alpha(R) = \left\{\left|\sum_{j=1}^{N} \delta(f, I_j)\right| > 2\pi R^\alpha\right\} .$$

2. Let us fix an arc I of length r and look more closely at the tails of the random variable $\delta(f, I)$. It is not hard to check that $\delta(f, I) = \delta(T_w f, I - w)$, where w is the midpoint of the arc I and $T_w f(z) = f(w + z)e^{-z\overline{w}}e^{-|w|^2/2}$. A classical complex analysis argument shows that for any analytic function g in the disk $2r\mathbb{D}$ and any "good" arc $\gamma \subset r\mathbb{D}$ of length at most r, one has

$$\left|\Delta_\gamma \arg g\right| \leqslant C \log \frac{\max_{2r\mathbb{D}} |g|}{\max_{r\mathbb{D}} |g|} ,$$

see Lemma 9. Then, estimating the probability that, for a G.E.F. $g = T_w f$, the doubling exponent $\log \dfrac{\max_{2r\mathbb{D}} |g|}{\max_{r\mathbb{D}} |g|}$ is large, we come up with the tail estimate

$$\mathbb{P}\left\{ |\delta(f, I)| > Mr^2 \right\} \leqslant \exp\left(-\frac{CM^2}{\log M} r^4\right), \qquad M \gg 1\,.$$

3. Now, let us come back to the sum $\sum_{j=1}^{N} \delta(f, I_j)$. The random variables $\delta(f, I_j)$ are not independent, however in [9, Theorem 3.2] we've introduced an "almost independence device" that allows us to think about these random variables as of independent ones, provided that the arcs I_j are well-separated from each other. Here we'll need a certain extension of that result (Lemma 5 below).

4. To see how the almost independence and the tail estimate work, first, consider the case $1 < \alpha < 2$. We split the circumference $R\mathbb{T}$ into N disjoint arcs $\{I_j\}$ of length r. In view of the tail estimate in Item 2, we need to distribute the total deviation R^α between these arcs in such a way that the "deviation per arc" R^α/N is bigger than r^2. Since $N \simeq \frac{R}{r}$, this leads to the choice of r comparable to $R^{\alpha-1}$.

Then we consider the event that for a *fixed subset* $J \subset \{1, 2, \dots, N\}$ and for every $j \in J$, one has $|\delta(f, I_j)| \geqslant m_j r^2$, where m_j are some big positive integer powers of 2 that satisfy

$$\sum_{j\in J} m_j r^2 \gtrsim R^\alpha\,. \tag{1.2}$$

Then we choose a well-separated sub-collection of arcs $J' \subset J$ that falls under the assumptions of the almost independence Lemma 5. This step weakens condition (1.2) to

$$\sum_{j\in J'} m_j^{3/2} r^2 \gtrsim R^\alpha\,,$$

which still suffices for our purposes. Then regarding the random variables $\delta(f, I_j)$, $j \in J'$, as independent ones and using the tail estimate for these variables, we see that the probability of this event does not exceed

$$\exp\left(-c\sum_{j\in J'} \frac{m_j^2}{\log m_j} r^4\right) \leqslant \exp\left(-r^2 \sum_{j\in J'} m_j^{3/2} r^2\right)$$
$$\leqslant \exp\left(-cr^2 R^\alpha\right) \leqslant \exp\left(-c_1 R^{3\alpha-2}\right)\,.$$

To get the upper bound for the probability of the event Ω_α, we need to take into account the number of possible choices of the subset J and of the numbers m_j. This factor does not exceed $2^N (\log R)^N < e^{CR\log\log R}$ which is not big enough to destroy our estimate.

5. Now, let us turn to the upper bound in the case $\frac{1}{2} < \alpha < 1$. We choose the arcs I_j of length 1. To separate them from each other, we choose from this collection $R^{1-\varepsilon}$ arcs $\{I_j\}_{j\in J}$ separated by R^ε and such that

$$\left| \sum_{j \in J} \delta(f, I_j) \right| > R^{\alpha - \varepsilon}\,.$$

For these arcs, the random variables $\delta(f, I_j)$ behave like independent ones, and since their tails have a fast decay, we can apply to them the classical Bernstein inequality (Lemma 3), which yields

$$\mathbb{P}\left\{ \left| \sum_{j \in J} \delta(f, I_j) \right| > R^{\alpha - \varepsilon} \right\} \leqslant C \exp\left(-\frac{c(R^{\alpha-\varepsilon})^2}{\text{Card}\, J} \right) = C \exp\left(-c R^{2\alpha - 1 - \varepsilon} \right).$$

6. To get the lower bound for the probability of Ω_α in the case $\frac{1}{2} < \alpha < 1$, we introduce an auxiliary Gaussian Taylor series

$$g(z) = \sum_{k=0}^{\infty} \zeta_k a_k \frac{z^k}{\sqrt{k!}},$$

where ζ_k are independent standard complex Gaussian random variables, and

$$a_k = \begin{cases} \sqrt{1 - R^{\alpha-1}}, & R^2 + R < k < R^2 + 2R\,; \\ \sqrt{1 + R^{\alpha-1}}, & R^2 - 2R < k < R^2 - R\,; \\ 1, & \text{otherwise}\,. \end{cases}$$

It is not difficult to check that for some absolute $c > 0$, the probability that the function g has at most $R^2 - cR^\alpha$ zeroes in the disk $R\mathbb{D}$ is not exponentially small (more precisely, it cannot be less than $cR^{-2+\alpha}$).

Now, let γ be the standard Gaussian measure in the space $\mathbb{C}^\infty$; i.e., the product of countably many copies of standard complex Gaussian measures on $\mathbb{C}$, and let γ_a be another Gaussian measure on $\mathbb{C}^\infty$ which is the product of complex Gaussian measures γ_{a_k} on $\mathbb{C}$ with variances a_k^2. Let $E \subset \mathbb{C}^\infty$ be the set of coefficients η_k such that the Taylor series $\sum_{k \geqslant 0} \eta_k \frac{z^k}{\sqrt{k!}}$ converges in $\mathbb{C}$ and has at most $R^2 - cR^\alpha$ zeroes in $R\mathbb{D}$. Then

$$\gamma_a(E) \geqslant cR^{-2+\alpha}\,,$$

while the quantity $\mathbb{P}\left\{ n(R) \leqslant R^2 - cR^\alpha \right\}$ we are interested in equals $\gamma(E)$. Thus, it remains to compare $\gamma(E)$ with $\gamma_a(E)$, and a more or less straightforward computation finishes the job.

Following [12, Parts I and II], we compare the zero point process $\mathcal{Z}_f$ with random independent perturbations of the lattice points. We fix the parameter $\nu > 0$, and consider the random point set $\{\omega + \zeta_\omega\}_{\omega \in \mathbb{Z}^2}$, where ζ_ω are independent, identical, radially distributed random variables with the tails $\mathbb{P}\{|\zeta_\omega| > t\}$ decaying as $\exp(-t^\nu)$ for $t \to \infty$. Set

$$n(R) = \text{Card}\{\omega \in \mathbb{Z}^2 \colon |\omega + \zeta_\omega| \leqslant R\}.$$

Then one can see that, for every $\alpha > \frac{1}{2}$ and every $\varepsilon > 0$,

$$e^{-R^{\varphi(\alpha,\nu)+\varepsilon}} < \mathbb{P}\left\{ |n(R) - \pi R^2| > R^\alpha \right\} < e^{-R^{\varphi(\alpha,\nu)-\varepsilon}}$$

for all sufficiently large $R > R_0(\alpha, \varepsilon)$ with

$$\varphi(\alpha, \nu) = \begin{cases} 2\alpha - 1, & \frac{1}{2} \leqslant \alpha \leqslant 1; \\ (\nu + 1)\alpha - \nu, & 1 \leqslant \alpha \leqslant 2; \\ (\nu/2 + 1)\alpha, & \alpha \geqslant 2\,. \end{cases}$$

In the range $\frac{1}{2} \leqslant \alpha \leqslant 1$, the exponent $\varphi(\alpha) = 2\alpha - 1$ seems to be determined by the asymptotic normality at the endpoint $\alpha = \frac{1}{2}$. In the range $\alpha > 1$, the Jancovici-Lebowitz-Manificat law (1.1) corresponds to the case $\nu = 2$; i.e., to the lattice perturbation with the Gaussian decay of the tails.

Convention about the constants. By c and C we denote positive numerical constants that appear in the proofs. The constants denoted by c are supposed to be small (in particular, they are always less than 1), while the constants denoted by C are supposed to be big (they are always larger than 1). Within the proof of each lemma, we start a new sequence of indices for these constants, and we never refer to these constants after the corresponding proof is completed.

Notation $A \lesssim B$ and $A \gtrsim B$ means that there exist positive numerical constants C and c such that $A \leqslant C \cdot B$ and $A \geqslant c \cdot B$ correspondingly. If $A \lesssim B$ and $A \gtrsim B$ simultaneously, then we write $A \simeq B$. Notation $A \ll B$ stands for "much less" and means that $A \leqslant c \cdot B$ with a very small positive c; similarly, $A \gg B$ stands for "much larger" and means that $A \geqslant C \cdot B$ with a very large positive C.

2. Preliminaries

2.1. A combinatorial lemma. For $j, k \in \{1, \dots, N\}$, we set

$$\begin{aligned} |j-k|_* &= \min\{|i-k| \colon i \equiv j \bmod N\} \\ &= \min\{|j-k|,\ |j-k+N|,\ |j-k-N|\}\,. \end{aligned}$$

Lemma 1. *Let m_1, ..., m_N be non-negative integers. Then, given $Q \geqslant 1$, there exists a subset $J' \subset \{1, \dots, N\}$ such that*

$$|j-k|_* \geqslant Q(\sqrt{m_j} + \sqrt{m_k})\,, \qquad j, k \in J', \quad j \neq k\,,$$

and

$$\sum_{j \in J} m_j \leqslant 5Q \sum_{j \in J'} m_j^{3/2}\,.$$

Proof of Lemma 1. We build the set J' by an inductive construction. Choose $j_1 \in \{1, \dots, N\}$ such that $m_{j_1} = \max\left\{m_j \colon j \in \{1, \dots, N\}\right\}$. Set

$$J_1' = \{j_1\}, \quad J_1'' = \left\{j \colon 0 < |j - j_1|_* < 2Q\sqrt{m_{j_1}}\right\}, \quad J_1 = J_1' \cup J_1'',$$

and note that

$$\sum_{j \in J_1} m_j \leqslant \left(4Q\sqrt{m_{j_1}} + 1\right) m_{j_1} \leqslant 5Q \sum_{j \in J_1'} m_j^{3/2}\,.$$

Now, suppose that we've made k steps of this construction. If $J_k = \{1, \dots, N\}$, then we are done with $J' = J'_k$. If $\{1, \dots, N\}\setminus J_k \neq \emptyset$, we choose $j_{k+1} \in \{1, \dots, N\}\setminus J_k$ such that

$$m_{j_{k+1}} = \max\left\{m_j : j \in \{1, \dots, N\}\setminus J_k\right\},$$

and define the sets $J'_{k+1} = J'_k \cup \{j_{k+1}\}$,

$$J''_{k+1} = J''_k \cup \left\{j \in \{1, \dots N\}\setminus J_k : 0 < |j - j_{k+1}|_* < 2Q\sqrt{m_{j_{k+1}}}\right\},$$

and $J_{k+1} = J'_{k+1} \cup J''_{k+1}$. Then, as above,

$$\sum_{j \in J_{k+1}\setminus J_k} m_j \leqslant 5Q m_{j_{k+1}}^{3/2},$$

whence

$$\sum_{j \in J_{k+1}} m_j \leqslant 5Q \sum_{j \in J'_{k+1}} m_j^{3/2}.$$

We are done. □

2.2. Probabilistic preliminaries.

Lemma 2. [9, Lemma 2.1]. *Let η_k be standard complex Gaussian random variables (not necessarily independent). Let $a_k > 0$, $S = \sum_k a_k$. Then, for every $t > 0$,*

$$\mathbb{P}\left\{\sum_k a_k|\eta_k| > t\right\} \leqslant 2e^{-\frac{1}{2}(t/S)^2}.$$

We also need the following classical Bernstein's estimate:

Lemma 3. *Let ψ_k, $k = 1, 2, \dots, n$, be independent random variables with zero mean such that, for some $K > 0$ and every $t > 0$,*

$$\mathbb{P}\{|\psi_k| > t\} \leqslant Ke^{-t}.$$

Then, for $0 < t \leqslant 5Kn$,

$$\mathbb{P}\left\{\left|\sum_k \psi_k\right| > t\right\} \leqslant 2\exp\left(-\frac{t^2}{16Kn}\right).$$

Proof. Set $S_n = \sum_{k=1}^n \psi_k$. Then $\mathbb{E}e^{\lambda S_n} = \prod_{k=1}^n \mathbb{E}e^{\lambda\psi_k}$. Note that

$$\begin{aligned}
\mathbb{E}e^{\lambda\psi_k} &= \mathbb{E}\left\{1 + \lambda\psi_k + \left(e^{\lambda\psi_k} - 1 - \lambda\psi_k\right)\right\}\\
&= 1 + \lambda\left\{\int_0^\infty \mathbb{P}\{\psi_k > t\}\left(e^{\lambda t} - 1\right)dt + \int_0^\infty \mathbb{P}\{\psi_k < -t\}\left(1 - e^{-\lambda t}\right)dt\right\}\\
&\leqslant 1 + K\lambda\left\{\int_0^\infty e^{-t}\left(e^{\lambda t} - 1\right)dt + \int_0^\infty e^{-t}\left(1 - e^{-\lambda t}\right)dt\right\}\\
&= 1 + \frac{2K\lambda^2}{1 - \lambda^2} \leqslant 1 + 4K\lambda^2 \leqslant e^{4K\lambda^2},
\end{aligned}$$

provided that $\lambda \leqslant \frac{2}{3}$. Hence, we get

$$\mathbb{P}\,\{\,S_n > t\,\} \leqslant e^{-\lambda t}\mathbb{E}e^{\lambda S_n} \leqslant e^{4Kn\lambda^2-\lambda t}\,.$$

Similarly, $\mathbb{P}\,\{\,S_n < -t\,\} \leqslant e^{4Kn\lambda^2-\lambda t}$, and therefore $\mathbb{P}\,\{\,|S_n| > t\,\} \leqslant 2e^{4Kn\lambda^2-\lambda t}$. Taking $\lambda = \dfrac{t}{8Kn}$, we get the lemma. □

2.3. Mean number of zeroes of a Gaussian Taylor series. Consider a Gaussian Taylor series

$$g(z) = \sum_{k=0}^{\infty} \zeta_k a_k z^k$$

with non-negative a_k such that $\lim\limits_{k\to\infty} \sqrt[k]{a_k} = 0$ and with independent standard complex Gaussian random variables ζ_k. Then almost surely, the series on the right-hand side has infinite radius of convergence, and hence g is an entire function. By $n_g(r)$ we denote the number of zeroes of the function g in the disk of radius r.

Lemma 4.

$$\mathbb{E}n_g(r) = \frac{1}{2}\,\frac{r\mathfrak{C}_g'(r)}{\mathfrak{C}_g(r)}\,,$$

where

$$\mathfrak{C}_g(r) = \sum_{k=0}^{\infty} a_k^2 r^{2k}\,.$$

This readily follows from the Edelman-Kostlan formula for the density of mean counting measure of zeroes of an arbitrary Gaussian analytic function, see [11, Sect. 2]. Alternatively, one can obtain this formula using the argument principle, see [6, p. 195, Exercise 5].

2.4. Operators T_w and shift invariance. For a function $g\colon \mathbb{C} \to \mathbb{C}$ and a complex number $w \in \mathbb{C}$, we define

$$T_w g(z) = g(w+z)e^{-z\overline{w}}e^{-\frac{1}{2}|w|^2}\,.$$

In what follows, we use some simple properties of these operators.

(a) T_w are unitary operators in the Fock-Bargmann space of entire functions that are square integrable with respect to the measure $\frac{1}{\pi}e^{-|z|^2}\,dm(z)$:

$$\begin{aligned}\|T_w f\|^2 &= \frac{1}{\pi}\iint_{\mathbb{C}} |f(w+z)|^2 e^{-2\mathrm{Re}\,(z\overline{w})-|w|^2-|z|^2}\,dm(z)\\ &= \frac{1}{\pi}\iint_{\mathbb{C}} |f(w+z)|^2 e^{-|w+z|^2}\,dm(z) = \|f\|^2\,.\end{aligned}$$

(b) If f is a G.E.F., then $T_w f$ is a G.E.F. as well.

In particular, the distribution of the random zero set $\mathbb{Z}_f = f^{-1}\{0\}$ is translation invariant. The property (b) also yields the distribution invariance of the function $f^*(z) = |f(z)|e^{-|z|^2/2}$ with respect to the isometries of $\mathbb{C}$. Indeed, a straightforward inspection shows that $(T_w f)^*(z) = f^*(w+z)$.

(c) By (b), if f is a G.E.F., then

$$T_w f = \sum_{k\geqslant 0} \zeta_k(w) \frac{z^k}{\sqrt{k!}} ,$$

where $\zeta_k(w)$ are independent standard complex Gaussian random variables. Recalling that $\left\{ \frac{z^k}{\sqrt{k!}} \right\}_{k\geqslant 0}$ is an orthonormal basis in the Fock-Bargmann space, and using that T_w is a unitary operator and $T_w T_{-w}$ is the identity operator in that space, we get

$$\zeta_k(w) = \left\langle T_w f, \frac{z^k}{\sqrt{k!}} \right\rangle = \left\langle f, T_{-w}\left(\frac{z^k}{\sqrt{k!}} \right) \right\rangle .$$

Note that for $w \neq w'$, the Gaussian variables $\zeta_k(w)$ and $\zeta_{k'}(w')$ are correlated and

(d)

$$\left| \mathbb{E}\left\{ \zeta_k(w)\overline{\zeta_{k'}(w')} \right\} \right| = \left| \left\langle T_{-w}\left(\frac{z^k}{\sqrt{k!}} \right), T_{-w'}\left(\frac{z^{k'}}{\sqrt{k'!}} \right) \right\rangle \right| .$$

Let $\gamma \subset \mathbb{C}$ be an oriented curve. Note that if f does not vanish on the curve γ, then

$$\Delta_{\gamma-w} \arg T_w f = \Delta_\gamma \arg f - \Delta_\gamma \mathrm{Im}(z-w)\overline{w} = \Delta_\gamma \arg f - \Delta_\gamma \mathrm{Im}(z\overline{w}) ,$$

where $\Delta_\gamma \mathrm{Im}(z\overline{w})$ is the *increment* of the function $\mathrm{Im}(z\overline{w})$ over γ, and $\gamma - w$ denotes the translation of the curve γ by $-w$.

(e) Set $\delta(f,\gamma) = \Delta_\gamma \arg f - \mathbb{E}\Delta_\gamma \arg f$. Then $\delta(T_w f, \gamma - w) = \delta(f,\gamma)$.

If $I \subset R\mathbb{T}$ is a counterclockwise oriented arc with the midpoint at w, then using rotation invariance and the argument principle, we get

$$\mathbb{E}\Delta_I \arg f = \frac{|I|}{2\pi R}\, \mathbb{E}\Delta_{R\mathbb{T}} \arg f = \frac{|I|}{2\pi R}\, \mathbb{E} 2\pi n(R) = \frac{|I|}{R}\, R^2 = |I|R$$

and

$$\delta(T_w f, I - w) = \Delta_I \arg f - |I|R .$$

2.5. Almost independence. Our approach is based on the almost independence property introduced in [9]. It says that if $\{w_j\} \subset \mathbb{C}$ is a "well-separated" set, then the G.E.F. $T_{w_j} f$ can be simultaneously approximated by independent G.E.F. The following lemma somewhat extends Theorem 3.2 from [9].

Lemma 5. *There exists a numerical constant $A > 1$ such that for every family of pairwise disjoint disks $D(w_j, r_j + A\rho_j)$ with*

$$w_j \in \mathbb{C}, \quad r_j \geqslant 1, \quad \rho_j \geqslant \max\left(1, \sqrt{\log r_j}\right),$$

one can represent the family of G.E.F. $T_{w_j} f$ as

$$T_{w_j} f = f_j + h_j,$$

where f_j are independent G.E.F. and

$$\mathbb{P}\left\{ \max_{z \in r_j \mathbb{D}} |h_j(z)| e^{-|z|^2/2} \geqslant e^{-\rho_j^2} \right\} \leqslant 2 \exp\left(-\tfrac{1}{2} e^{\rho_j^2}\right).$$

Theorem 3.2 in [9] corresponds to the case when $r_j = r \geqslant 1$ and $\rho_j = Nr$ with $N \geqslant 1$. We prove Lemma 5 in the Appendix.

2.6. Bounds for G.E.F. Our first lemma estimates the probability that the function f is very large:

Lemma 6. (cf. [9, Lemma 4.1]). *Let f be a G.E.F. Then, for each $r \geqslant 1$ and $M \geqslant 1$,*

$$\mathbb{P}\left\{ \max_{z \in r\mathbb{D}} |f(z)| e^{-|z|^2/2} \geqslant M \right\} \leqslant 18 r^2 e^{-\frac{1}{32} M^2}.$$

Proof. We cover the disk $r\mathbb{D}$ by at most $(2r+1)^2 \leqslant 9r^2$ disks $\mathcal{D}_j$ of radius 1 and show that for each j,

$$\mathbb{P}\left\{ \max_{z \in \mathcal{D}_j} |f(z)| e^{-|z|^2/2} \geqslant M \right\} \leqslant 2e^{-\frac{1}{32} M^2}.$$

By the translation invariance of the distribution of the random function $|f(z)| e^{-|z|^2/2}$ it suffices to prove this estimate in the unit disk $\mathbb{D}$. Clearly,

$$\mathbb{P}\left\{ \max_{z \in \mathbb{D}} |f(z)| e^{-|z|^2/2} \geqslant M \right\} \leqslant \mathbb{P}\left\{ \max_{z \in \mathbb{D}} |f(z)| \geqslant M \right\}$$

$$\leqslant \mathbb{P}\left\{ \sum_{k \geqslant 0} \frac{|\zeta_k|}{\sqrt{k!}} \geqslant M \right\} \overset{\text{Lemma 2}}{\leqslant} 2e^{-\frac{1}{2}(M/S)^2}$$

with $S = \sum_{k \geqslant 0} \frac{1}{\sqrt{k!}} < 4$. Hence, the lemma. $\square$

The following lemma estimates the probability that the function f is very small:

Lemma 7. (cf. Lemma 8 in [7] and Lemma 4.2 in [9]). *Let f be a G.E.F. Let $r \geqslant 1$ and $m \geqslant 3$. Then*

$$\mathbb{P}\left\{\max_{r\mathbb{D}} |f| \leqslant e^{-mr^2}\right\} \leqslant \exp\left(-\frac{m^2}{\log m} r^4\right).$$

Proof. Suppose that $|f| \leqslant e^{-mr^2}$ everywhere in $r\mathbb{D}$. Then by Cauchy's inequalities,

$$|\zeta_n| \leqslant \frac{\sqrt{n!}}{r^n} \max_{r\mathbb{D}} |f| \leqslant \frac{n^{n/2}}{r^n} e^{-mr^2}, \qquad n = 0, 1, 2, \ldots .$$

For $0 \leqslant n \leqslant \dfrac{m}{\log m} r^2$, the probabilities of these events do not exceed

$$\left(nr^{-2}\right)^n e^{-2mr^2} \leqslant \left(\frac{m}{\log m}\right)^{\frac{m}{\log m} r^2} e^{-2mr^2} < e^{-mr^2}.$$

Since these events are independent, the probability we are estimating is bounded by

$$\exp\left(-mr^2\left(\frac{m}{\log m} r^2\right)\right) = \exp\left(-\frac{m^2}{\log m} r^4\right).$$

We are done. □

The next lemma bounds the probability that a G.E.F. is small on a given curve of a given length.

Lemma 8. *Let f be a G.E.F., and let γ be a curve of length at most $r \geqslant 1$. Then, for any positive $\varepsilon \leqslant \frac{1}{4}$,*

$$\mathbb{P}\left\{\min_{z\in\gamma} |f(z)|e^{-|z|^2/2} < \varepsilon\right\} < 100 r\varepsilon\sqrt{\log\frac{1}{\varepsilon}}.$$

Proof. We split the curve γ into $\lceil r \rceil$ arcs γ_j of length at most 1, and fix the collection of disks $\mathcal{D}_j$ of radius 1 such that $\gamma_j \subset \mathcal{D}_j$. We'll show that for each j,

$$\mathbb{P}\left\{\min_{z\in\gamma_j} |f(z)|e^{-|z|^2/2} < \varepsilon\right\} < 50\varepsilon\sqrt{\log\frac{1}{\varepsilon}}.$$

Clearly, this will yield the lemma.

By the shift invariance of the distribution of the random function $|f(z)|e^{-|z|^2/2}$, we assume without loss of generality that $\mathcal{D}_j$ is the unit disk $\mathbb{D}$. Taking into account that $e^{-|z|^2/2} > \frac{1}{2}$ everywhere in the unit disk, we have

$$\mathbb{P}\left\{\min_{z\in\gamma_j} |f(z)|e^{-|z|^2/2} < \varepsilon\right\} \leqslant \mathbb{P}\left\{\min_{z\in\gamma_j} |f(z)| < 2\varepsilon\right\}.$$

We choose points $\{z_m\} \subset \gamma$ and disks $D_m = \{|z - z_m| \leqslant \kappa\varepsilon\}$ such that

$$\gamma \subset \bigcup_m D_m, \quad \text{and} \quad \mathrm{Card}\{z_m\} \leqslant \left\lceil \frac{1}{2\kappa\varepsilon} \right\rceil,$$

with the parameter κ to be specified later. Then, for $z \in D_m$,

$$|f(z)| \geqslant |f(z_m)| - |z - z_m| \max_{\mathbb{D}} |f'| \geqslant |f(z_m)| - \kappa\varepsilon \max_{\mathbb{D}} |f'| .$$

Hence, we need to estimate the probability of the events

$$\Omega_1 = \left\{ \min_m |f(z_m)| \leqslant 3\varepsilon \right\} \quad \text{and} \quad \Omega_2 = \left\{ \max_{\mathbb{D}} |f'| \geqslant \frac{1}{\kappa} \right\} .$$

If neither of these events holds, then $|f(z)| > 3\varepsilon - \varepsilon = \varepsilon$ everywhere on γ.

Recall that for any standard complex Gaussian random variable ζ and for any $t > 0$, we have $\mathbb{P}\{|\zeta| \leqslant t\} < t^2$, also recall that $f(z_m)e^{-|z_m|^2/2}$ is a standard complex Gaussian random variable. Hence, for any fixed m, we have $\mathbb{P}\{|f(z_m)| \leqslant 3\varepsilon\} \leqslant \mathbb{P}\left\{|f(z_m)|e^{-|z_m|^2/2} \leqslant 3\varepsilon\right\} < 9\varepsilon^2$. Therefore,

$$\mathbb{P}\{\Omega_1\} < \left\lceil \frac{1}{2\kappa\varepsilon} \right\rceil \cdot 9\varepsilon^2 \leqslant \frac{9}{2}\varepsilon\kappa^{-1} + 9\varepsilon^2 .$$

Next,

$$\mathbb{P}\{\Omega_2\} \leqslant \mathbb{P}\left\{ \sum_{k\geqslant 1} \frac{k}{\sqrt{k!}} |\zeta_k| \geqslant \frac{1}{\kappa} \right\} \overset{\text{Lemma 2}}{\leqslant} 2e^{-\frac{1}{2}(\kappa S)^{-2}}$$

with $S = \sum_{k\geqslant 1} \frac{k}{\sqrt{k!}} < 6$. Therefore, $\mathbb{P}\{\Omega_2\} \leqslant 2e^{-\frac{1}{72}\kappa^{-2}}$, and

$$\mathbb{P}\{\Omega_1\} + \mathbb{P}\{\Omega_2\} < \frac{9}{2}\varepsilon\kappa^{-1} + 2e^{-\frac{1}{72}\kappa^{-2}} + 9\varepsilon^2 .$$

Choosing here $\kappa^{-1} = \sqrt{72 \log \frac{1}{\varepsilon}}$, we get

$$\begin{aligned}\mathbb{P}\left\{ \min_{z\in\gamma_j} |f(z)| < 2\varepsilon \right\} &\leqslant \mathbb{P}\{\Omega_1\} + \mathbb{P}\{\Omega_2\} \\ &< 27\sqrt{2}\,\varepsilon\sqrt{\log\frac{1}{\varepsilon}} + 2\varepsilon + 9\varepsilon^2 < 50\varepsilon\sqrt{\log\frac{1}{\varepsilon}} ,\end{aligned}$$

proving the lemma. □

2.7. Upper bounds for the increment of the argument. We say that a piecewise C^1-curve $\gamma \subset r\mathbb{D}$ is *good* if its length does not exceed r and, for any $\zeta \in \mathbb{C}\setminus\{\gamma\}$, we have $\left|\Delta_\gamma \arg(z-\zeta)\right| \leqslant 2\pi$. The following lemma is classical (cf. [8, Lemma 6, Chapter VI]):

Lemma 9. *There exists a numerical constant $B > 1$ with the following property. Let g be an analytic function in the disk $2r\mathbb{D}$ such that $\sup_{2r\mathbb{D}} |g| \leqslant 1$. If $\max_{r\mathbb{D}} |g| \geqslant e^{-\beta}$, then for any good curve $\gamma \subset r\mathbb{D}$, we have*

$$\left|\Delta_\gamma \arg g\right| \leqslant B\beta .$$

Proof. By scale invariance, it suffices to prove the lemma for $r = 1$. Choose $z_0 \in r\mathbb{T}$ such that $|g(z_0)| = \max_{r\mathbb{D}} |g| \geqslant e^{-\beta}$, and denote by φ a Möbius transformation $\varphi\colon 2\mathbb{D} \to 2\mathbb{D}$ with $\varphi(0) = z_0$.

Denote by $n_g(t)$ and $n_{g\circ\varphi}(t)$ the number of zeroes of the functions g and $g \circ \varphi$ in the disk $t\mathbb{D}$, and choose $\rho < 2$ such that $\varphi^{-1}\left(\frac{3}{2}\mathbb{D}\right) \subset \rho\mathbb{D}$. Then by Jensen's formula

$$
\begin{aligned}
0 \geqslant \int_0^{2\pi} \log|(g\circ\varphi)(2e^{i\theta})| \frac{d\theta}{2\pi} &= \log|g\circ\varphi(0)| + \int_0^2 \frac{n_{g\circ\varphi}(t)}{t}\, dt \\
&\geqslant -\beta + \int_\rho^2 \frac{n_{g\circ\varphi}(t)}{t}\, dt \geqslant -\beta + n_{g\circ\varphi}(\rho)\log\tfrac{2}{\rho} \\
&\geqslant -\beta + n_g(\tfrac{3}{2})\log\tfrac{2}{\rho}\,.
\end{aligned}
$$

Thus the number of zeroes of g in the disk $\frac{3}{2}\mathbb{D}$ does not exceed $C_1\beta$. Hence, $g = pg_1$, where p is a polynomial of degree $N \leqslant C_1\beta$ with zeroes in $\frac{3}{2}\mathbb{D}$ and a unimodular leading coefficient, and g_1 does not vanish in $\frac{3}{2}\mathbb{D}$, $g_1(0) > 0$.

Claim 9-1. $\displaystyle\int_0^{2\pi} \left|\log|g_1(\tfrac{3}{2}e^{i\theta})|\right| \frac{d\theta}{2\pi} \leqslant C_2\beta.$

Proof of Claim 9-1. Indeed,

$$
\begin{aligned}
\int_0^{2\pi} \left|\log|g_1(\tfrac{3}{2}e^{i\theta})|\right| \frac{d\theta}{2\pi} &\leqslant \int_0^{2\pi} \left|\log|g(\tfrac{3}{2}e^{i\theta})|\right| \frac{d\theta}{2\pi} + \int_0^{2\pi} \left|\log|p(\tfrac{3}{2}e^{i\theta})|\right| \frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} \log^-|g(\tfrac{3}{2}e^{i\theta})| \frac{d\theta}{2\pi} + \int_0^{2\pi} \left|\log|p(\tfrac{3}{2}e^{i\theta})|\right| \frac{d\theta}{2\pi}\,.
\end{aligned}
$$

To estimate the integral on the right-hand side, we note that

$$
\int_0^{2\pi} \log|g(\tfrac{3}{2}e^{i\theta})| \frac{\left(\frac{3}{2}\right)^2 - |z_0|^2}{|\frac{3}{2}e^{i\theta} - z_0|^2} \frac{d\theta}{2\pi} \geqslant \log|g(z_0)| \geqslant -\beta\,,
$$

whence

$$
\int_0^{2\pi} \log^-|g(\tfrac{3}{2}e^{i\theta})| \frac{d\theta}{2\pi} \leqslant C_3\beta\,.
$$

The estimate of the second integral on the right-hand side is also straightforward: since $p(z) = \prod_{j=1}^N (z - \lambda_j)$ with $\lambda_j \in \frac{3}{2}\mathbb{D}$, we have

$$
\int_0^{2\pi} \left|\log|p(\tfrac{3}{2}e^{i\theta})|\right| \frac{d\theta}{2\pi} \leqslant N \cdot \sup_{\lambda\in\frac{3}{2}\mathbb{D}} \int_0^{2\pi} \left|\log|\tfrac{3}{2}e^{i\theta} - \lambda|\right| \frac{d\theta}{2\pi} \overset{N\leqslant C_1\beta}{\leqslant} C_4\beta\,.
$$

Hence, the claim. □

Now, $\left|\Delta_\gamma \arg g\right| \leqslant \left|\Delta_\gamma \arg p\right| + \left|\Delta_\gamma \arg g_1\right|$, and since the curve γ is good, we have $\left|\Delta_\gamma \arg p\right| \leqslant 2\pi N \leqslant C_5\beta$. Fix the branch h of $\arg g_1$. Then

$$\left|\Delta_\gamma h\right| \leqslant 2 \max_{\overline{\mathbb{D}}} |h - h(0)| .$$

Since h harmonic in $\frac{3}{2}\mathbb{D}$, we have

$$h(z) = \int_0^{2\pi} \log|g_1(\tfrac{3}{2}e^{i\theta})| \,\mathrm{Im}\, \frac{\frac{3}{2}e^{i\theta} + z}{\frac{3}{2}e^{i\theta} - z} \frac{d\theta}{2\pi} + h(0) , \qquad |z| \leqslant 1 ,$$

and

$$|h(z) - h(0)| \leqslant C_6 \int_0^{2\pi} \left|\log|g_1(\tfrac{3}{2}e^{i\theta})|\right| \, d\theta \overset{\text{Claim 9−1}}{\leqslant} C_7\beta , \qquad |z| \leqslant 1 .$$

This proves the lemma. □

Lemma 10. *Let $r \geqslant 1$, let $\gamma \subset r\mathbb{D}$ be a good curve, let $m \geqslant 25B$, and let f be a G.E.F. Consider the event $\Omega = \left\{|\delta(f,\gamma)| \geqslant mr^2\right\}$. Then*

$$\Omega \subset \Omega' \cup \left\{\max_{r\mathbb{D}} |f| < e^{-\frac{1}{4B}mr^2}\right\} \qquad \text{with} \quad \mathbb{P}\left\{\Omega'\right\} \leqslant \exp\left(-e^{\frac{1}{6B}mr^2}\right).$$

In particular,

$$\mathbb{P}\left\{\Omega\right\} \leqslant 2\exp\left(-\tfrac{1}{16B^2}\,\tfrac{m^2r^4}{\log m}\right).$$

Proof. Introduce the events

$$\Omega_1(m) = \left\{|\Delta_\gamma \arg f| \geqslant mr^2\right\},$$

and

$$\Omega'(m) = \left\{\max_{z\in 2r\mathbb{D}} |f(z)|e^{-|z|^2/2} > e^{\frac{1}{3B}mr^2}\right\}.$$

Claim 10-1. *For $m \geqslant 12B$, $\Omega_1(m) \subset \Omega'(m) \cup \left\{\max_{r\mathbb{D}} |f| < e^{-\frac{1}{2B}mr^2}\right\}$.*

Proof of Claim 10-1. Suppose that the event $\Omega'(m)$ does not occur. Then

$$\max_{2r\mathbb{D}} |f| \leqslant e^{\frac{1}{3B}mr^2 + 2r^2} = e^{(\frac{1}{3B} + \frac{2}{m})mr^2} \overset{m\geqslant 12B}{\leqslant} e^{\frac{1}{2B}mr^2} .$$

If the event $\Omega_1(m)$ occurs, then by Lemma 9

$$mr^2 \leqslant \left|\Delta_\gamma \arg f\right| \leqslant B \log \frac{\max_{2r\mathbb{D}} |f|}{\max_{r\mathbb{D}} |f|} ,$$

whence,

$$\max_{r\mathbb{D}} |f| \leqslant e^{-\frac{1}{B}mr^2} \max_{2r\mathbb{D}} |f| \leqslant e^{-\frac{1}{2B}mr^2} ,$$

proving the claim. □

Claim 10-2. *For $m \geqslant 12B$, $\mathbb{P}\left\{\Omega'(m)\right\} \leqslant \exp\left(-e^{\frac{1}{3B}mr^2}\right)$.*

Proof of Claim 10-2. We have

$$\mathbb{P}\left\{\Omega'(m)\right\} \overset{\text{Lemma 6}}{\leqslant} 72r^2 \exp\left(-\tfrac{1}{32}e^{\frac{2}{3B}mr^2}\right) \leqslant \tfrac{6}{B}mr^2 \exp\left(-\tfrac{1}{32}e^{\frac{2}{3B}mr^2}\right).$$

It's easy to see that for $t = \frac{1}{3B}mr^2 \geqslant 4$, one has

$$\begin{aligned} 18t \exp\left(-\tfrac{1}{32}e^{2t}\right) &\leqslant 18t \exp\left(-\tfrac{e^4}{32}e^{t}\right) < 18t \exp\left(-\tfrac{3}{2}e^{t}\right) \\ &= \underbrace{18t \exp\left(-\tfrac{1}{2}e^{t}\right)}_{<1} \exp\left(-e^{t}\right) < \exp\left(-e^{t}\right). \end{aligned}$$

Hence, the claim. □

Claim 10-3. *For $m \geqslant 12B$, $\mathbb{P}\left\{\Omega_1(m)\right\} \leqslant 2\exp\left(-\frac{1}{4B^2}\frac{(mr^2)^2}{\log(m/2B)}\right)$.*

Proof of Claim 10-3. By Lemma 7,

$$\mathbb{P}\left\{\max_{r\mathbb{D}} |f| < e^{-\frac{1}{2B}mr^2}\right\} \leqslant \exp\left(-\tfrac{1}{4B^2}\tfrac{(mr^2)^2}{\log(m/2B)}\right).$$

It's easy to check that for $t = \frac{1}{B}mr^2 \geqslant 12$, one has $e^{t/3} > \frac{t^2}{4}$. Therefore,

$$e^{\frac{1}{3B}mr^2} > \tfrac{1}{4B^2}(mr^2)^2 > \tfrac{1}{4B^2}\tfrac{(mr^2)^2}{\log(m/2B)}.$$

Thus $\mathbb{P}\left\{\Omega'(m)\right\}$ also does not exceed $\exp\left(-\frac{1}{4B^2}\frac{(mr^2)^2}{\log(m/2B)}\right)$. We are done. □

Claim 10-4. $\left|\mathbb{E}\Delta_\gamma \arg f\right| \leqslant 12.5Br^2$.

Proof of Claim 10-4. By Claim 10-3, for $s \geqslant 12Br^2$, we have

$$\mathbb{P}\left\{|\Delta_\gamma \arg f| \geqslant s\right\} \leqslant 2\exp\left(-\frac{(s/2B)^2}{\log s/(2Br^2)}\right) \leqslant 2\exp\left(-\frac{(s/2B)^2}{\log s/2B}\right).$$

Therefore,

$$\begin{aligned} \left|\mathbb{E}\Delta_\gamma \arg f\right| &\leqslant 12Br^2 + 2\int_{12Br^2}^{\infty} \exp\left(-\frac{(s/2B)^2}{\log(s/2B)}\right) ds \\ &= 12Br^2 + 4B\underbrace{\int_{6r^2}^{\infty} e^{-s^2/\log s}\, ds}_{<1/8} < 12.5Br^2, \end{aligned}$$

proving the claim. □

Now, we readily finish the proof of Lemma 10. Suppose that the event Ω occurs; i.e., $|\delta(f,\gamma)| \geqslant mr^2$ with $m \geqslant 25B$. Then

$$\left|\Delta_\gamma \arg f\right| \geqslant |\delta(f,\gamma)| - \left|\mathbb{E}\Delta_\gamma \arg f\right| \geqslant (m - 12.5B)r^2 \geqslant \tfrac{1}{2}mr^2.$$

That is, $\Omega \subset \Omega_1(\frac{1}{2}m)$ and the lemma follows from Claims 10-1, 10-2 and 10-3 applied with $\frac{1}{2}m$ instead of m. □

Remark. One can get a better estimate $\left|\mathbb{E}\Delta_\gamma \arg f\right| \leqslant r^2$ than the one given in Claim 10-4 in the following way. If $\gamma\colon [0,1] \to \mathbb{C}$ is a good curve, then

$$\Delta_\gamma \arg f = \operatorname{Im} \int_0^1 \frac{f'}{f}(\gamma(t))\, \gamma'(t)\, dt.$$

Taking into account that $\mathbb{E}\frac{f'}{f}(z) = \bar{z}$, we get

$$\mathbb{E}\Delta_\gamma \arg f = \operatorname{Im} \int_0^1 \gamma'(t)\overline{\gamma(t)}\, dt,$$

whence $\left|\mathbb{E}\Delta_\gamma \arg f\right| \leqslant r \cdot \operatorname{Length}(\gamma) \leqslant r^2$.

3. The Upper Bound for $1 < \alpha < 2$

3.1. Few arcs with large increments of the argument. Given $r \geqslant 1$, we fix a collection of $N \leqslant 2\pi \frac{R}{r}$ disjoint arcs $\left\{I_j\right\}_{1\leqslant j\leqslant N}$ of length r on the circumference $R\mathbb{T}$. Then, given $\Lambda \geqslant 1$ and a positive integer L, we introduce two events. The first event $\Omega_1(r, R, \Lambda, L)$ is that the collection $\left\{I_j\right\}_{1\leqslant j\leqslant N}$ contains a sub-collection of L disjoint arcs $\{I_j\}_{j\in J}$ such that

$$\sum_{j\in J} \left|\delta(f, I_j)\right| \geqslant \Lambda\,.$$

To define the second event, we fix N independent G.E.F. f_j. Then the event $\Omega_2(r, R, \Lambda, L)$ is that the collection $\left\{I_j\right\}_{1\leqslant j\leqslant N}$ contains a sub-collection of L disjoint arcs $\{I_j\}_{j\in J}$ such that,

$$\sum_{j\in J} \left|\delta(f_j, \widetilde{I}_j)\right| \geqslant \Lambda\,.$$

Here, $\widetilde{I}_j = I_j - w_j$, where w_j are the centers of the arcs I_j.

Lemma 11. *Suppose that R is sufficiently big. Suppose also that*

$$R^{1/2} \leqslant \Lambda \leqslant R^2 \qquad \text{and} \qquad 1 < L < \frac{b\Lambda}{r^2 + \log R}$$

with a sufficiently small positive numerical constant b. Then the probabilities of the events Ω_i, $i = 1, 2$, do not exceed $e^{-b_1 r^2 \Lambda}$ with a positive numerical constant b_1.

Proof of Lemma 11. First, we estimate the probability of the event Ω_2; this is a simpler part of the job. Suppose that the event $\Omega_2(r, R, \Lambda, L)$ occurs. We choose $M_j \leqslant r^{-2}\left|\delta(f_j, \widetilde{I}_j)\right|$ such that $\sum_{j\in J} M_j r^2 = \Lambda$. Let B be the constant from Lemma 9. Note that the arcs $\widetilde{I}_j$ with $M_j < 50B$ can contribute at most $50BL < 50Bb\Lambda < \frac{1}{2}\Lambda$ to the total sum, provided that $b < \frac{1}{100B}$. We discard the arcs $\widetilde{I}_j$ with $M_j < 50B$ and denote by J the collection of remaining arcs.

Now, let m_j be the largest positive integer power of 2 such that $m_j \leqslant M_j$, $j \in J$. Then

$$\frac{1}{4}\Lambda \leqslant \sum_{j\in J} m_j r^2 \leqslant \Lambda \qquad \text{and} \qquad m_j \geqslant 25B\,, \tag{3.1.1}$$

and

$$\mathbb{P}\,\{\,\Omega_2\,\} \leqslant \sum_J \sum_{\{m_j\}} \mathbb{P}\left\{\bigcap_{j\in J}\left\{|\delta(f_j,\widetilde{I}_j)| \geqslant m_j r^2\right\}\right\},$$

where the first sum is taken over all subsets $J \subset \{1, ..., N\}$ of cardinality at most L, and the second sum is taken over all possible choices of m_j, $j \in J$, that are positive integer powers of 2 satisfying restrictions (3.1.1). Since f_j are independent, we have

$$\mathbb{P}\left\{\bigcap_{j\in J}\left\{|\delta(f_j,\widetilde{I}_j)| \geqslant m_j r^2\right\}\right\} = \prod_{j\in J} \mathbb{P}\left\{|\delta(f_j,\widetilde{I}_j)| \geqslant m_j r^2\right\}.$$

The probabilities of the events on the right-hand side were estimated in Lemma 10:

$$\mathbb{P}\left\{|\delta(f_j,\widetilde{I}_j)| \geqslant m_j r^2\right\} \leqslant 2\exp\left(-\frac{1}{16B^2}\,\frac{m_j^2 r^4}{\log m_j}\right).$$

Therefore,

$$\begin{aligned}\mathbb{P}\left\{\bigcap_{j\in J}\left\{|\delta(f_j,\widetilde{I}_j)| \geqslant m_j r^2\right\}\right\} &\leqslant 2^L \exp\left(-\frac{1}{16B^2}\,r^2 \sum_{j\in J}\frac{m_j^2 r^2}{\log m_j}\right)\\ &< 2^L \exp\left(-\frac{1}{16B^2}\,r^2 \sum_{j\in J} m_j r^2\right)\\ &\leqslant 2^L \exp\left(-\frac{1}{64B^2}\,r^2\Lambda\right),\end{aligned}$$

and

$$\mathbb{P}\,\{\,\Omega_2\,\} < 2^L \exp\left(-\frac{1}{64B^2}\,r^2\Lambda\right)\sum_J \sum_{\{m_j\}} 1\,.$$

To get rid of the sums on the right-hand side, we need to estimate the number of different ways to choose the "data" J, $\{m_j\}_{j\in J}$. Since m_j is an integer power of 2 and $m_j \leqslant \Lambda$, for each $j \in J$, there are at most $2\log\Lambda$ ways to choose the integer m_j. Hence, given a set J of cardinality at most L, we have at most $(2\log\Lambda)^L$ ways to choose the collection $\{m_j\}_{j\in J}$. Also there are at most

$$\sum_{0\leqslant \ell\leqslant L}\binom{N}{\ell} < (N+1)^L \overset{N\leqslant 2\pi R}{<} e^{CL\log R}$$

ways to choose the subset $J \subset \{1, 2, \dots, N\}$ of cardinality at most L. Therefore,

$$2^L \sum_J \sum_{\{m_j\}} 1 \leqslant (4 \log \Lambda)^L e^{CL \log R}$$
$$< e^{C' L(\log R + \log\log \Lambda)} < e^{C'' L \log R} < e^{C'' b\Lambda},$$

which is a negligible factor with respect to $\exp\left(-\frac{1}{64B^2} r^2 \Lambda\right)$, provided that $b \ll B^{-2}$. This completes the estimate of $\mathbb{P}\{\Omega_2\}$. □

The estimate of the probability of the event Ω_1 follows a similar pattern. Now, the events $\left\{|\delta(f, I_j)| \geqslant m_j r^2\right\}$, $j \in J$, are not independent. To get around this obstacle, we'll use the almost independence lemma, which brings in some awkward technicalities. We split the proof into several steps.

(i) Suppose that the event $\Omega_1(r, R, \Lambda, L)$ occurs. As above, we choose $M_j \leqslant r^{-2} \left|\delta(f, I_j)\right|$ such that $\sum_{j\in J} M_j r^2 = \Lambda$. Then we fix a sufficiently large positive numerical constant $C_1 \geqslant 25B$ and note that the arcs I_j with $M_j < 2C_1(1 + r^{-2}\log\Lambda)$ can contribute to the total deviation Λ at most

$$2C_1 L(r^2 + \log \Lambda) < 2C_1 \frac{b\Lambda(r^2 + 2\log R)}{r^2 + \log R} \leqslant 4bC_1\Lambda,$$

which is much smaller than Λ provided that the constant b is sufficiently small. We choose $b < \frac{1}{8C_1}$ and conclude that *at least half of the deviation* Λ *must come from the arcs* I_j *with sufficiently large* M_j. From now on, we discard the arcs I_j with $M_j < 2C_1(1+r^{-2}\log\Lambda)$ and denote by J the set of the remaining arcs.

Now, let m_j be the largest positive integer power of 2 such that $m_j \leqslant M_j$, $j \in J$. Then

$$\frac{1}{4}\Lambda \leqslant \sum_{j\in J} m_j r^2 \leqslant \Lambda, \qquad \text{and} \qquad m_j r^2 \geqslant C_1\left(r^2 + \log\Lambda\right) \tag{3.1.2}$$

and

$$\mathbb{P}\{\Omega_1\} \leqslant \sum_J \sum_{\{m_j\}} \mathbb{P}\left\{\bigcap_{j\in J}\left\{|\delta(f, I_j)| \geqslant m_j r^2\right\}\right\}, \tag{3.1.3}$$

where the first sum is taken over all subsets $J \subset \{1, \dots, N\}$ of cardinality at most L, and the second sum is taken over all possible choices of m_j, $j \in J$, that are positive integer powers of 2 satisfying restrictions (3.1.2).

As in the previous case, it suffices to show that, for a *fixed* subset $J \subset \{1, 2, \dots, N\}$ with $\operatorname{Card} J \leqslant L$, and for *fixed* m_j, $j \in J$, that are integer powers of 2 and satisfy conditions (3.1.2), one has

$$\mathbb{P}\left\{\bigcap_{j\in J}\left\{|\delta(f, I_j)| \geqslant m_j r^2\right\}\right\} \leqslant e^{-cr^2\Lambda}. \tag{3.1.4}$$

Since we have at most $e^{C'' L \log R} < e^{C'' b\Lambda}$ possible combinations of the "data" J and $\{m_j\}_{j\in J}$, the two sums on the right-hand side of (3.1.3) contribute by a negligible factor with respect to $e^{-cr^2\Lambda}$, provided that $b < \frac{c}{2C''}$.

(ii) From now on, we fix a set J of cardinality at most L, and m_j, $j\in J$, that are integer powers of 2 and satisfy conditions (3.1.2). Let w_j be the centers of the arcs I_j, $\widetilde{I}_j = I_j - w_j$, and let

$$\Omega_j \stackrel{\text{def}}{=} \left\{|\delta(f, I_j)| \geqslant m_j r^2\right\} = \left\{|\delta(T_{w_j} f, \widetilde{I}_j)| \geqslant m_j r^2\right\}.$$

By Lemma 10 applied to the G.E.F. $T_{w_j} f$ with $\gamma = \widetilde{I}_j$ and $m = m_j$, we have

$$\Omega_j \subset \Omega'_j \cup \left\{\max_{r\mathbb{D}} |T_{w_j} f| < e^{-\frac{1}{4B} m_j r^2}\right\}$$

with $\mathbb{P}\left\{\Omega'_j\right\} < \exp\left(-e^{-\frac{1}{6B} m_j r^2}\right)$, whence,

$$\bigcap_{j\in J}\left\{\left|\delta(f, I_j)\right| \geqslant m_j r^2\right\} \subset \left(\bigcup_{j\in J}\Omega'_j\right) \cup \bigcap_{j\in J}\left\{\max_{r\mathbb{D}} |T_{w_j} f| < e^{-\frac{1}{4B} m_j r^2}\right\}$$

with

$$\mathbb{P}\left\{\bigcup_{j\in J}\Omega'_j\right\} \overset{m_j r^2 \geqslant C_1 \log \Lambda}{\leqslant} L \exp\left(-e^{\frac{1}{6B} C_1 \log \Lambda}\right) \overset{C_1 \geqslant 25B}{\leqslant} L e^{-\Lambda^4} \overset{L<\Lambda}{<} \Lambda e^{-\Lambda^4}.$$

Since $r^2 < \Lambda$, this is much less than $e^{-r^2\Lambda}$ when $R \gg 1$.

Discarding the event $\bigcup_{j\in J}\Omega'_j$, we need to estimate the probability of the event

$$\bigcap_{j\in J}\left\{\max_{r\mathbb{D}} |T_{w_j} f| \leqslant \exp\left(-\tfrac{1}{4B} m_j r^2\right)\right\}.$$

(iii) Combinatorial Lemma 1 applied with $m_j = 0$ for $j \notin J$ and with the constant $Q = \frac{\pi}{2}(A+1)$, gives us a subset $J' \subset J$ such that

$$|j-k|_* \geqslant Q(\sqrt{m_j} + \sqrt{m_k}), \qquad j, k \in J', \quad j \neq k,$$

and

$$\sum_{j\in J'} m_j^{3/2} \geqslant \frac{1}{5Q}\sum_{j\in J} m_j. \tag{3.1.5}$$

Hence, the centers w_j of the arcs from J' are well-separated:

$$|w_j - w_k| = 2R \sin\frac{|j-k|_* r}{2R} \geqslant \frac{2}{\pi}|j-k|_* r \geqslant (A+1)\left(\sqrt{m_j}\, r + \sqrt{m_k}\, r\right)$$

for $j, k \in J'$, $j \neq k$. By the almost independence Lemma 5 applied with $r_j = \rho_j = \sqrt{m_j} r$, we have $T_{w_j} f = f_j + h_j$, $j \in J'$, where f_j are *independent* G.E.F., and

$$\mathbb{P}\left\{\max_{z\in r\mathbb{D}} |h_j(z)| e^{-|z|^2/2} \geqslant e^{-m_j r^2}\right\} \leqslant 2\exp\left(-\tfrac{1}{2} e^{m_j r^2}\right)$$
$$\overset{m_j r^2 \geqslant C_1 \log \Lambda}{\leqslant} 2\exp\left(-\tfrac{1}{2}\Lambda^{C_1}\right).$$

Introduce the event

$$\mathcal{F} = \bigcup_{j\in J'} \left\{ \max_{r\mathbb{D}} |h_j| > \exp\left(-\tfrac{1}{2}m_j r^2\right) \right\}.$$

Claim 11 -1. *For* $R \gg 1$, $\mathbb{P}\{\mathcal{F}\} \leqslant e^{-r^2\Lambda}$.

Proof of Claim 11 -1. If for some $j \in J'$,

$$\max_{r\mathbb{D}} |h_j| > \exp\left(-\tfrac{1}{2}m_j r^2\right),$$

then

$$\max_{z\in r\mathbb{D}} |h_j(z)|e^{-|z|^2/2} > \exp\left(-\tfrac{1}{2}(m_j+1)r^2\right) \overset{m_j \geqslant 25}{>} e^{-m_j r^2}.$$

Therefore,

$$\mathbb{P}\{\mathcal{F}\} \leqslant \sum_{j\in J'} \mathbb{P}\left\{ \max_{z\in r\mathbb{D}} |h_j(z)|e^{-|z|^2/2} > \exp\left(-\tfrac{1}{2}m_j r^2\right) \right\}$$

$$\leqslant L\cdot 2\exp\left(-\tfrac{1}{2}\Lambda^{C_1}\right) \overset{L\leqslant\Lambda}{\leqslant} 2\Lambda \exp\left(-\tfrac{1}{2}\Lambda^{C_1}\right).$$

Since $r^2 < \Lambda$, this is much less than $e^{-r^2\Lambda}$, provided that $R \gg 1$. □

If the event $\mathcal{F}$ does not occur, then for each $j \in J'$,

$$\max_{r\mathbb{D}} |f_j| \leqslant \max_{r\mathbb{D}} |T_{w_j} f| + \max_{r_j\mathbb{D}} |h_j|$$

$$\leqslant e^{-\frac{1}{4B}m_j r^2} + e^{-\frac{1}{2}m_j r^2} \overset{B>1}{<} 2e^{-\frac{1}{4B}m_j r^2} \overset{m_j r^2 \geqslant 25B}{<} e^{-\frac{1}{6B}m_j r^2}.$$

We conclude that *if R is sufficiently big, then outside of an event of probability less than* $\exp(-r^2\Lambda)$, *we have*

$$\max_{r\mathbb{D}} |f_j| < \exp\left(-\tfrac{1}{6B}m_j r^2\right)$$

for each $j \in J'$.

(iv) Our problem boils down to the estimate of the probability that the *independent* events

$$\left\{ \max_{r\mathbb{D}} |f_j| < e^{-\frac{1}{6B}m_j r^2} \right\}, \qquad j \in J',$$

occur. By Lemma 7 the logarithm of the probability of each of these events doesn't exceed $-\frac{c_2 m_j^2}{\log m_j} r^4$ with $c_2 = \frac{1}{36B^2}$. Therefore, the logarithm of the probability that all these events happen doesn't exceed

$$-c_2 \sum_{j\in J'} \frac{m_j^2}{\log m_j} r^4 < -c_2 r^2 \sum_{j\in J'} m_j^{3/2} r^2$$

$$\overset{(3.1.5)}{<} -\frac{c_2}{5\frac{\pi}{2}(A+1)} r^2 \sum_{j\in J} m_j r^2 \overset{(3.1.2)}{\leqslant} -cr^2\Lambda$$

with

$$c = \frac{1}{4}\,\frac{2c_2}{5\pi(A+1)} = \frac{1}{360\pi B^2(A+1)}\,.$$

This completes the proof of (3.1.4) and, thereby, of the lemma. □

3.2. Proof of Theorem 1: the upper bound for $1 < \alpha < 2$. We need to estimate the probability of the event $\Omega_\alpha = \left\{|n(R) - R^2| > R^\alpha\right\}$. Let b be the constant from the previous lemma. We fix a small positive $\delta \in (\frac{1}{4}b, \frac{1}{2}b)$ such that the number $N = \frac{2\pi}{\delta}R^{2-\alpha}$ is an integer, take $r = \delta R^{\alpha-1}$, and split the circumference $R\mathbb{T}$ into N disjoint arcs $\{I_j\}$ of length r. By the argument principle, $\Omega_\alpha = \left\{\left|\sum_j \delta(f, I_j)\right| > 2\pi R^\alpha\right\}$. In the case $1 < \alpha < 2$, the cancelations between different random variables $\delta(f, I_j)$ are not important, so we are after the upper bound for the probability of the bigger event $\Omega'_\alpha = \left\{\sum_j \left|\delta(f, I_j)\right| > 2\pi R^\alpha\right\}$.

We take $\Lambda = 2\pi R^\alpha$, and check that Lemma 11 can be applied to the whole collection of arcs $\{I_j\}$; i.e., with $L = N$. If R is big enough then $\log R \ll r^2$, and

$$1 \leqslant L = \frac{2\pi}{\delta}R^{2-\alpha} < \frac{1}{2}\,\frac{b\cdot 2\pi R^\alpha}{\delta^2 R^{2\alpha-2}} = \frac{1}{2}\,\frac{b\Lambda}{r^2} < \frac{b\Lambda}{r^2+\log R}\,.$$

Therefore, the assumptions of Lemma 11 are fulfilled, and we get

$$\mathbb{P}\left\{\Omega'_\alpha\right\} \leqslant e^{-2\pi b_1 r^2 R^\alpha} < e^{-cR^{3\alpha-2}}\,.$$

Done! □

4. The Upper Bound for $\frac{1}{2} < \alpha < 1$

4.1. Approximating the total increment of arg f *by the sum of increments of arguments of independent G.E.F.*

Lemma 12. *Suppose that R is sufficiently big, that $1 \leqslant r \leqslant 2$, and that $3R^{1/2} \leqslant \Lambda \leqslant R$. Then, given a collection of disjoint arcs $\left\{I_j\right\}$ of length r of the circumference $R\mathbb{T}$ that are separated by arcs of length at least* $\log R$*, there exists a collection of independent G.E.F. $\{f_j\}$ such that*

$$\mathbb{P}\left\{\left|\sum_j \delta(f, I_j) - \sum_j \delta(f_j, \widetilde{I}_j)\right| \geqslant \Lambda\right\} \leqslant e^{-b_2\Lambda},$$

where $\widetilde{I}_j = I_j - w_j$ and b_2 is a positive numerical constant.

Proof of Lemma 12. Set $\rho = \sqrt{C_1 \log R}$ with $C_1 \gg 1$. Let A be the constant from the almost independence lemma. If R is big enough, then by our assumptions, the disks $D(w_j, r + A\rho)$ are disjoint. So the almost independence Lemma 5 yields a decomposition $T_{w_j} f = f_j + h_j$ with independent G.E.F. $\{f_j\}$ and

$$\mathbb{P}\left\{\max_{r\mathbb{D}} |h_j(z)|e^{-|z|^2/2} \geqslant R^{-C_1}\right\} \leqslant 2\exp\left(-\tfrac{1}{2}R^{C_1}\right).$$

In what follows, we assume that

$$\max_j \max_{r\mathbb{D}} |h_j(z)|e^{-|z|^2/2} \leqslant R^{-C_1}\,.$$

For this, we throw away an event of probability at most

$$2\pi R \cdot 2e^{-\frac{1}{2}R^{C_1}} \ll e^{-\Lambda}\,.$$

Since $\delta(f, I_j) = \delta(T_{w_j} f, \widetilde{I}_j)$, we need to estimate the probability of the event

$$\left\{ \left| \sum_j \left[\delta(T_{w_j} f, \widetilde{I}_j) - \delta(f_j, \widetilde{I}_j)\right] \right| \geqslant \Lambda \right\},$$

introduce the events

$$\Omega_j = \left\{ \min_{z\in\widetilde{I}_j} |f_j(z)|e^{-|z|^2/2} \leqslant R^{-C_1/2} \right\},$$

and note that if Ω_j does not occur, then

$$\begin{aligned}\left|\delta(T_{w_j} f, \widetilde{I}_j) - \delta(f_j, \widetilde{I}_j)\right| &= \left|\Delta_{\widetilde{I}_j} \arg T_{w_j} f - \Delta_{\widetilde{I}_j} \arg f_j\right| \\ &= \left|\Delta_{\widetilde{I}_j} \arg\left(1 + \frac{h_j}{f_j}\right)\right| \lesssim R^{-C_1/2}\end{aligned}$$

(we have used that $\mathbb{E}\Delta_{\widetilde{I}_j} \arg T_{w_j} f = \mathbb{E}\Delta_{\widetilde{I}_j} \arg f_j$), whence

$$\sum_{j:\,\Omega_j \text{ doesn't occur}} \left|\delta(T_{w_j} f, \widetilde{I}_j) - \delta(f_j, \widetilde{I}_j)\right| \lesssim 2\pi R \cdot R^{-C_1/2} \ll 1\,.$$

Therefore, we conclude that

$$\begin{aligned}&\left|\sum_j \left(\delta(T_{w_j} f, \widetilde{I}_j) - \delta(f_j, \widetilde{I}_j)\right)\right| \\ &\quad\leqslant \sum_{j:\,\Omega_j \text{ occurs}} \left|\delta(T_{w_j} f, \widetilde{I}_j)\right| + \sum_{j:\,\Omega_j \text{ occurs}} \left|\delta(f_j, \widetilde{I}_j)\right| + 1 \\ &\quad= \sum_{j:\,\Omega_j \text{ occurs}} \left|\delta(f, I_j)\right| + \sum_{j:\,\Omega_j \text{ occurs}} \left|\delta(f_j, \widetilde{I}_j)\right| + 1.\end{aligned}$$

To estimate the size of the two sums on the right-hand side, we introduce the (random) counter $L = \text{Card}\{j:\ \Omega_j \text{ occurs}\}$. Lemma 11 (applied to $\frac{1}{3}\Lambda$ instead of Λ) handles the case

$$L \leqslant \frac{b}{6}\,\frac{\Lambda}{\log R} \quad \left(< \frac{b\Lambda}{3(r^2 + \log R)} \quad \text{for} \quad R \gg 1\right).$$

It yields that outside of some event Ω' of probability at most $2e^{-b_1 r^2 \Lambda}$, each of these two sums does not exceed $\frac{1}{3}\Lambda$.

Now, consider the second case when $L > \frac{b}{6}\frac{\Lambda}{\log R}$. Denote by Q the integer part of $\dfrac{b}{6}\dfrac{\Lambda}{\log R}$. Then at least Q *independent events* $\Omega_{j_1}, \dots, \Omega_{j_Q}$ must occur. By Lemma 8 applied with $\gamma = \widetilde{I}_j$ and $\varepsilon = R^{-C_1/2}$, we have

$$\mathbb{P}\left\{\Omega_j\right\} \leqslant 100 r R^{-C_1/2}\sqrt{\tfrac{1}{2}C_1 \log R} \leqslant R^{-C_1/3},$$

provided that R is sufficiently big. Therefore,

$$\begin{aligned}\mathbb{P}\left\{L \geqslant \tfrac{1}{6}\tfrac{b\Lambda}{\log R}\right\} &\leqslant \underbrace{\binom{\mathrm{Card}\{I_j\}}{Q}}_{\leqslant (2\pi R)^Q}\left(R^{-C_1/3}\right)^Q \\ &< e^{-\frac{1}{4}C_1 Q\log R} \leqslant e^{-c_2\Lambda}.\end{aligned}$$

Thereby,

$$\begin{aligned}&\mathbb{P}\left\{\left|\sum_j \delta(f, I_j) - \sum_j \delta(f_j, \widetilde{I}_j)\right| \geqslant \Lambda\right\} \\ &\leqslant \mathbb{P}\left\{\Omega'\right\} + \mathbb{P}\left\{L \geqslant \tfrac{1}{6}\tfrac{b\Lambda}{\log R}\right\} < 2e^{-b_1 r^2\Lambda} + e^{-c_2\Lambda} < e^{-c_3\Lambda},\end{aligned}$$

and we are done. □

4.2. Proof of Theorem 1: the upper bound in the case $\frac{1}{2} < \alpha < 1$. We split the circumference $R\mathbb{T}$ into $N = \lfloor 2\pi R\rfloor$ disjoint arcs $\{I_j\}$ of equal length r, $1 \leqslant r \leqslant 2$. We fix a positive $\varepsilon < \frac{1-\alpha}{4}$ and suppose that

$$\left|\sum_{j=1}^{N} \delta(f, I_j)\right| > 2\pi R^\alpha.$$

Then we split the set $\{1, \dots, N\}$ into $n = \lfloor 2R^\varepsilon\rfloor$ disjoint arithmetic progressions $J_1, \dots, J_n$. If R is sufficiently big, then the cardinality of each of these arithmetic progressions cannot be less than

$$\frac{N}{n} - 1 \geqslant \frac{2\pi R - 1}{2R^\varepsilon} - 1 > 2R^{1-\varepsilon},$$

and cannot be larger than

$$\frac{N}{n} + 1 \leqslant \frac{2\pi R}{2R^\varepsilon - 1} + 1 < 4R^{1-\varepsilon}.$$

For at least one of these progressions, say for J_l, we have

$$\left|\sum_{j\in J_l} \delta(f, I_j)\right| > 2\pi\,\frac{R^\alpha}{n} > 2R^{\alpha-\varepsilon}.$$

Given a collection $\{I_j\}_{j\in J}$ with $2R^{1-\varepsilon} < \operatorname{Card} J < 4R^{1-\varepsilon}$ of R^ε-separated arcs of length r, we show that

$$\mathbb{P}\left\{\left|\sum_{j\in J}\delta(f, I_j)\right| > 2R^{\alpha-\varepsilon}\right\} \leqslant C_1 e^{-c_2 R^{2\alpha-1-\varepsilon}}.$$

Since we have $n \ll R$ such collections $\{I_j\}$, this will prove the upper bound in the case $\frac{1}{2} < \alpha < 1$.

Now, suppose that $\left|\sum_{j\in J}\delta(f, I_j)\right| > 2R^{\alpha-\varepsilon}$. By Lemma 12 applied with $\Lambda = R^{\alpha-\varepsilon}$, we see that there is a collection of independent G.E.F. $\{f_j\}$ such that throwing away an event of probability at most

$$e^{-b_2\Lambda} = e^{-b_2 R^{\alpha-\varepsilon}} \overset{\varepsilon<1-\alpha}{\ll} e^{-R^{2\alpha-1}},$$

we have

$$\left|\sum_{j\in J}\delta(f_j, \widetilde{I}_j)\right| > 2R^{\alpha-\varepsilon} - \Lambda = R^{\alpha-\varepsilon}.$$

To estimate the probability of the event $\mathbb{P}\left\{\left|\sum_{j\in J}\delta(f_j, \widetilde{I}_j)\right| > R^{\alpha-\varepsilon}\right\}$, we apply Bernstein's estimate (Lemma 3) to the independent identically distributed random variables $\psi_j = \delta(f_j, \widetilde{I}_j)$. By Lemma 10, the tails of these random variables decay superexponentially:

$$\mathbb{P}\left\{|\psi_j| \geqslant t\right\} \leqslant \exp\left(-\frac{c_3 t^2}{\log t}\right)$$

for $t \gg 1$. The number of the random variables ψ_j is bigger than $2R^{1-\varepsilon}$. Hence, the Bernstein estimate can be applied with $t = R^{\alpha-\varepsilon}$. We see that the probability we are interested in does not exceed

$$2\exp\left(-c_4 t^2/\operatorname{Card} J\right) < \exp\left(-c_5 R^{2\alpha-1-\varepsilon}\right),$$

completing the argument. □

5. Proof of Theorem 1: The Lower Bound for $\frac{1}{2} < \alpha < 1$

We fix $\alpha \in (\frac{1}{2}, 1)$ and show that, for some positive numerical constant c_0 and for each $R > R_0(\alpha)$, one has

$$\mathbb{P}\left\{n(R) \leqslant R^2 - c_0 R^\alpha\right\} \geqslant e^{-3R^{2\alpha-1}}.$$

Everywhere below, we assume that $R > 2$. Let $N = \lfloor R \rfloor$. Let $\mathcal{J}_-$ be a set consisting of N integers between $R^2 - 2R$ and $R^2 - R$, and let $\mathcal{J}_+$ be a set consisting of N integers between $R^2 + R$ and $R^2 + 2R$. Let

$$a_k = \begin{cases} \sqrt{1 - R^{\alpha-1}}, & k \in \mathcal{J}_+; \\ \sqrt{1 + R^{\alpha-1}}, & k \in \mathcal{J}_-; \\ 1, & k \notin \mathcal{J}_+ \cup \mathcal{J}_-. \end{cases}$$

Consider the Gaussian Taylor series

$$g(z) = \sum_{k=0}^{\infty} \zeta_k a_k \frac{z^k}{\sqrt{k!}},$$

and denote by $n_g(R)$ the number of its zeroes in the disk $R\mathbb{D}$.

Claim 5.1. *For* $R \geqslant 1$, *we have* $\mathbb{E} n_g(R) \leqslant R^2 - c_1 R^\alpha$.

Proof of Claim 5.1. By Lemma 4,

$$\mathbb{E} n_g(R) = \frac{1}{2} \frac{R \sum_{k\geqslant 0} a_k^2 \cdot 2k \cdot \frac{R^{2k-1}}{k!}}{\sum_{k\geqslant 0} a_k^2 \cdot \frac{R^{2k}}{k!}} = \frac{\sum_{k\geqslant 0} a_k^2 \cdot k \cdot \frac{R^{2k}}{k!}}{\sum_{k\geqslant 0} a_k^2 \cdot \frac{R^{2k}}{k!}}.$$

The ratio on the right-hand side can be written as

$$R^2 + \frac{\sum_{k\geqslant 0} a_k^2 \cdot (k - R^2) \cdot \frac{R^{2k}}{k!}}{\sum_{k\geqslant 0} a_k^2 \cdot \frac{R^{2k}}{k!}}.$$

Note that

$$\sum_{k\geqslant 0} (k - R^2) \cdot \frac{R^{2k}}{k!} = 0,$$

so the numerator in the second term equals

$$\sum_{k\in\mathcal{J}_-} R^{\alpha-1} \cdot (k - R^2) \cdot \frac{R^{2k}}{k!} + \sum_{k\in\mathcal{J}_+} (-R^{\alpha-1}) \cdot (k - R^2) \cdot \frac{R^{2k}}{k!}$$
$$\leqslant -R^\alpha \sum_{k\in\mathcal{J}_-\cup\mathcal{J}_+} \frac{R^{2k}}{k!}.$$

Since $R \geqslant 1$, we have $a_k^2 \leqslant 2$, and the denominator cannot be bigger than $2e^{R^2}$. Hence,

$$\mathbb{E} n_g(R) \leqslant R^2 - \frac{1}{2} R^\alpha e^{-R^2} \sum_{k\in\mathcal{J}_-\cup\mathcal{J}_+} \frac{R^{2k}}{k!}.$$

Now, observe that

$$\sum_{k\in\mathcal{J}_-\cup\mathcal{J}_+} \frac{R^{2k}}{k!} \geqslant c e^{R^2}$$

with some absolute $c > 0$. To see this, note that the function $k \mapsto \frac{R^{2k}}{k!}$ decreases for $k \in \mathcal{J}_+$ and increases for $k \in \mathcal{J}_-$. We set $K = \lceil R^2 + 2R \rceil$. Applying Stirling's formula, we get

$$\frac{R^{2k}}{k!} \geqslant \frac{R^{2K}}{K!} \gtrsim \frac{1}{\sqrt{K}} \left(\frac{eR^2}{K} \right)^K$$
$$\gtrsim \frac{e^{R^2+2R-1}}{R\left(1+\frac{2}{R}\right)^{R^2+2R}} \gtrsim \frac{e^{R^2+2R}}{Re^{2R+4}} \gtrsim \frac{e^{R^2}}{R}$$

for $k \in \mathcal{J}_+$. A similar estimate holds for $k \in \mathcal{J}_-$. Therefore,

$$\mathbb{E}n_g(R) \leqslant R^2 - \frac{1}{2}R^\alpha e^{-R^2} \sum_{k\in\mathcal{J}_-\cup\mathcal{J}_+} \frac{R^{2k}}{k!} \leqslant R^2 - \frac{1}{2}R^\alpha e^{-R^2} \cdot ce^{R^2},$$

proving the claim. □

Claim 5.2. *For $R \geqslant 1$, we have*

$$\mathbb{P}\left\{n_g(R) \leqslant R^2 - \frac{c_1}{2}R^\alpha\right\} \geqslant \frac{c_1}{2}R^{-2+\alpha}.$$

Proof of Claim 5.2. We have

$$c_1R^\alpha \leqslant \mathbb{E}(R^2 - n_g(R)) \leqslant \frac{c_1}{2}R^\alpha + R^2\,\mathbb{P}\left\{n_g(R) \leqslant R^2 - \frac{c_1}{2}R^\alpha\right\}$$

whence

$$\mathbb{P}\left\{n_g(R) \leqslant R^2 - \frac{c_1}{2}R^\alpha\right\} \geqslant R^{-2}\cdot\frac{c_1}{2}R^\alpha = \frac{c_1}{2}R^{-2+\alpha}.$$

□

Claim 5.3. *Let $0 \leqslant t \leqslant N$. Then*

$$\mathbb{P}\left\{\sum_{k\in\mathcal{J}_-}|\zeta_k|^2 - \sum_{k\in\mathcal{J}_+}|\zeta_k|^2 \geqslant t\right\} \leqslant 2\exp\left(-\frac{t^2}{16(e+1)N}\right).$$

Proof of Claim 5.3. Note first of all that $\mathbb{P}\left\{|\zeta_k|^2 \geqslant t\right\} = e^{-t}$ and $\mathbb{E}|\zeta_k|^2 = 1$, whence, for $t > 0$,

$$\mathbb{P}\left\{|\zeta_k|^2 - 1 > t\right\} < e^{-t}$$

and

$$\mathbb{P}\left\{|\zeta_k|^2 - 1 < -t\right\} = \max\left\{1 - e^{t-1}, 0\right\} < e^{1-t}.$$

Thus we can apply Bernstein's Lemma 3 with $K = e + 1$ to the random variables $\pm(|\zeta_k|^2 - 1)$, which yields the desired conclusion. □

In particular,

$$\mathbb{P}\left\{\sum_{k\in\mathcal{J}_-}|\zeta_k|^2 - \sum_{k\in\mathcal{J}_+}|\zeta_k|^2 \geqslant R^{1/2}\log R\right\} \leqslant 2\exp\left(-c_2\log^2 R\right) \leqslant \frac{c_1}{4}R^{-2+\alpha},$$

provided that $R > R_0(\alpha)$.

Now everything is ready to make the final estimate. Let γ be the standard Gaussian measure on the space $\mathbb{C}^\infty$; i.e., the product of countably many copies of the measures $\frac{1}{\pi}e^{-|\eta_k|^2}\,dm(\eta_k)$, and let γ_a be another Gaussian measure on $\mathbb{C}^\infty$ that is the product of the Gaussian measures $\frac{1}{\pi a_k^2}e^{-|\eta_k|^2/a_k^2}\,dm(\eta_k)$. Let $E \subset \mathbb{C}^\infty$ be the set of coefficients

η_k such that the Taylor series $\sum_{k\geqslant 0}\eta_k\frac{z^k}{\sqrt{k!}}$ converges in $\mathbb{C}$ and has at most $R^2-\frac{c_1}{2}R^\alpha$ zeroes in $R\mathbb{D}$. Then Claim 5.2 can be rewritten as

$$\gamma_a(E)\geqslant\frac{c_1}{2}R^{-2+\alpha}\,,$$

while the quantity $\mathbb{P}\left\{n(R)\leqslant R^2-\frac{c_1}{2}R^\alpha\right\}$ we are interested in equals $\gamma(E)$. Thus, it remains to compare $\gamma(E)$ with $\gamma_a(E)$.

Let

$$U=\left\{\sum_{k\in\mathcal{J}_-\cup\mathcal{J}_+}|\eta_k|^2\geqslant\sum_{k\in\mathcal{J}_-\cup\mathcal{J}_+}\frac{|\eta_k|^2}{a_k^2}+R^{\alpha-\frac12}\log R\right\},$$

and

$$\widetilde{U}=\left\{\sum_{k\in\mathcal{J}_-\cup\mathcal{J}_+}a_k^2|\eta_k|^2\geqslant\sum_{k\in\mathcal{J}_-\cup\mathcal{J}_+}|\eta_k|^2+R^{\alpha-\frac12}\log R\right\}.$$

Note that

$$\gamma_a(U)=\gamma(\widetilde{U})=\mathbb{P}\left\{\sum_{k\in\mathcal{J}_-}|\zeta_k|^2-\sum_{k\in\mathcal{J}_+}|\zeta_k|^2\geqslant R^{1/2}\log R\right\}\leqslant\frac{c_1}{4}R^{-2+\alpha}\,.$$

Hence,

$$\gamma_a(E\backslash U)\geqslant\frac{c_1}{4}R^{-2+\alpha}\,.$$

But on $E\backslash U$, we can bound the density of γ_a with respect to γ:

$$\frac{d\gamma_a}{d\gamma}\leqslant e^{R^{\alpha-\frac12}\log R}(1-R^{2\alpha-2})^{-N}<e^{2R^{2\alpha-1}}$$

for $R>R_0(\alpha)$. The rest is obvious:

$$\begin{aligned}\gamma(E)&\geqslant\gamma(E\backslash U)\geqslant e^{-2R^{2\alpha-1}}\gamma_a(E\backslash U)\\&\geqslant\frac{c_1}{4}R^{-2+\alpha}e^{-2R^{2\alpha-1}}\geqslant e^{-3R^{2\alpha-1}},\end{aligned}$$

provided that $R>R_0(\alpha)$. This proves the lower bound in Theorem 1. □

Appendix: Asymptotic Almost Independence. Proof of Lemma 5

A-1. Elementary inequalities.

Claim A-1.1. *For all positive k and t,*

$$k\log t-t\leqslant k\log k-k-(\sqrt{t}-\sqrt{k})^2\,.$$

Proof. The function $\varphi(\tau) = k\log(\tau^2) - \tau^2$ attains its maximum at $\tau = \sqrt{k}$, and

$$\varphi''(\tau) = -\frac{2k}{\tau^2} - 2 \leqslant -2 \qquad \text{for all } \tau > 0\,.$$

Hence,

$$\varphi(\tau) \leqslant \varphi(\sqrt{k}) - (\tau - \sqrt{k})^2 \qquad \text{for all } \tau > 0\,.$$

Replacing τ^2 by t, we get the claim. □

Claim A-1.2. *Let k be a positive integer and $u \geqslant k$. Then*

$$\int_u^\infty \frac{t^k e^{-t}}{k!}\,dt \leqslant e^{-(\sqrt{u}-\sqrt{k})^2}\,.$$

Proof.

$$\begin{aligned}
\int_u^\infty \frac{t^k e^{-t}}{k!}\,dt &= \int_k^\infty \frac{[t+(u-k)]^k e^{-t-(u-k)}}{k!}\,dt \\
&= \int_k^\infty \frac{t^k e^{-t}}{k!}\left[1+\frac{u-k}{t}\right]^k e^{-(u-k)}\,dt \\
&\leqslant \left[1+\frac{u-k}{k}\right]^k e^{-(u-k)} \underbrace{\int_k^\infty \frac{t^k e^{-t}}{k!}\,dt}_{\leqslant 1} \\
&\leqslant \exp\{[k\log u - u] - [k\log k - k]\} \overset{\text{Claim A-1.1}}{\leqslant} e^{-(\sqrt{u}-\sqrt{k})^2}\,,
\end{aligned}$$

proving the claim. □

Corollary A-1.3.

$$\frac{1}{\pi}\int_{|z|\geqslant\sqrt{k}+d} \frac{|z|^{2k}}{k!} e^{-|z|^2}\,dm_2(z) = \int_{(\sqrt{k}+d)^2}^\infty \frac{t^k}{k!} e^{-t}\,dt \leqslant e^{-d^2}\,.$$

Claim A-1.4. *Let w', w'' be points in $\mathbb{C}$ and let k', k'' be non-negative integers. Then*

$$\left|\mathbb{E}\left\{\xi_{k'}(w')\overline{\xi_{k''}(w'')}\right\}\right| \leqslant 2e^{-\frac{d^2}{8}}\,,$$

provided that $|w'-w''| \geqslant \sqrt{k'} + \sqrt{k''} + d$, $d > 0$.

Proof. By Sect. 2.4(d),

$$\begin{aligned}
&\left|\mathbb{E}\left\{\xi_{k'}(w')\overline{\xi_{k''}(w'')}\right\}\right| \\
&= \left|\left\langle T_{-w'}\left(\frac{z^{k'}}{\sqrt{k'!}}\right), T_{-w''}\left(\frac{z^{k''}}{\sqrt{k''!}}\right)\right\rangle\right| \\
&= \left|\frac{1}{\pi}\int_{\mathbb{C}} \frac{(z-w')^{k'}}{\sqrt{k'!}}\,\frac{(\overline{z-w''})^{k''}}{\sqrt{k''!}}\, e^{-z\overline{w}'-\frac{1}{2}|w'|^2} e^{-\bar{z}w''-\frac{1}{2}|w''|^2} e^{-|z|^2}\,dm_2(z)\right|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left|\mathbb{E}\left\{\xi_{k'}(w')\overline{\xi_{k''}(w'')}\right\}\right| \\
&\leqslant \frac{1}{\pi}\int_{\mathbb{C}} \frac{|z-w'|^{k'}}{\sqrt{k'!}}\, e^{-\frac{1}{2}|z-w'|^2}\, \frac{|z-w''|^{k''}}{\sqrt{k''!}}\, e^{-\frac{1}{2}|z-w''|^2}\, dm_2(z) \\
&\leqslant \frac{1}{\pi}\int_{\left\{|z-w'|\geqslant\sqrt{k'}+\frac{d}{2}\right\}} + \frac{1}{\pi}\int_{\left\{|z-w''|\geqslant\sqrt{k''}+\frac{d}{2}\right\}} = I' + I''\,.
\end{aligned}$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}
I' &\leqslant \left\{\frac{1}{\pi}\int_{\left\{|z-w'|\geqslant\sqrt{k'}+\frac{d}{2}\right\}} \frac{|z-w'|^{2k'}}{k'!}\, e^{-|z-w'|^2}\, dm_2(z)\right\}^{1/2} \\
&\quad\times\left\{\frac{1}{\pi}\int_{\mathbb{C}} \frac{|z-w''|^{2k''}}{k''!}\, e^{-|z-w''|^2}\, dm_2(z)\right\}^{1/2} \\
&\overset{\text{Claim A-1.3}}{\leqslant} e^{-\frac{d^2}{8}}\cdot 1 = e^{-\frac{d^2}{8}}\,.
\end{aligned}$$

Similarly, $I''\leqslant e^{-\frac{d^2}{8}}$. Hence, $I'+I''\leqslant 2e^{-\frac{d^2}{8}}$, and we are done. □

Claim A-1.5. *Assume that the disks $D(w_j, R_j+8\sigma_j)$ are pairwise disjoint and $R_j\geqslant 1$, $\sigma_j\geqslant\max\left(1,\sqrt{\log R_j}\right)$. Let $D_{ij}=|w_i-w_j|-R_i-R_j$ be the distance between the disks $D(w_i,R_i)$ and $D(w_j,R_j)$. Then, for each i,*

$$2\sum_{j:\ j\neq i}(1+R_j^2)e^{-\frac{1}{8}D_{ij}^2}\leqslant e^{-2\sigma_i^2}\,.$$

Proof. Indeed, since $D_{ij}\geqslant 8\sigma_j$, we have

$$\frac{1}{16}D_{ij}^2\geqslant 4\sigma_j^2\geqslant 2\sigma_j^2+2\geqslant\log(e^2R_j^2)\geqslant\log(4R_j^2)\geqslant\log\left[2(1+R_j^2)\right]\,.$$

Thus, it suffices to estimate the sum $\sum_{j:\ j\neq i} e^{-\frac{1}{16}D_{ij}^2}$. For each $j\neq i$, consider the disk $\mathcal{D}_j\subset D(w_j,R_j+8\sigma_j)$ of radius 4 closest to w_i.

For each $z\in\mathcal{D}_j$, we have $|z-w_i|\leqslant D_{ij}+R_i$. Also, the disks $\mathcal{D}_j$ are disjoint and $\bigcup_j\mathcal{D}_j\subset\mathbb{C}\backslash D(w_i,R_i+8\sigma_i)$. Hence,

$$\begin{aligned}
\sum_{j:\ j\neq i} e^{-\frac{1}{16}D_{ij}^2} &\leqslant \frac{1}{16\pi}\int_{\{|z-w_i|\geqslant R_i+8\sigma_i\}} e^{-\frac{1}{16}(|z-w_i|-R_i)^2}\, dm_2(z) \\
&= \frac{1}{16\pi}\int_{\{|z|\geqslant R_i+8\sigma_i\}} e^{-\frac{1}{16}(|z|-R_i)^2}\, dm_2(z) = \frac{1}{8}\int_{8\sigma_i}^{\infty}(R_i+t)e^{-\frac{1}{16}t^2}\, dt \\
&\leqslant (1+\tfrac{1}{8}R_i)\int_{8\sigma_i}\frac{t}{8}e^{-\frac{1}{16}t^2}\, dt = (1+\tfrac{1}{8}R_i)e^{-4\sigma_i^2} \\
&\leqslant \frac{1+\frac{1}{8}R_i}{e^{\sigma_i^2+1}}e^{-2\sigma_i^2}\leqslant\frac{8+R_i}{8eR_i}e^{-2\sigma_i^2}\leqslant\frac{9}{8e}e^{-2\sigma_i^2}<e^{-2\sigma_i^2}
\end{aligned}$$

proving the claim. □

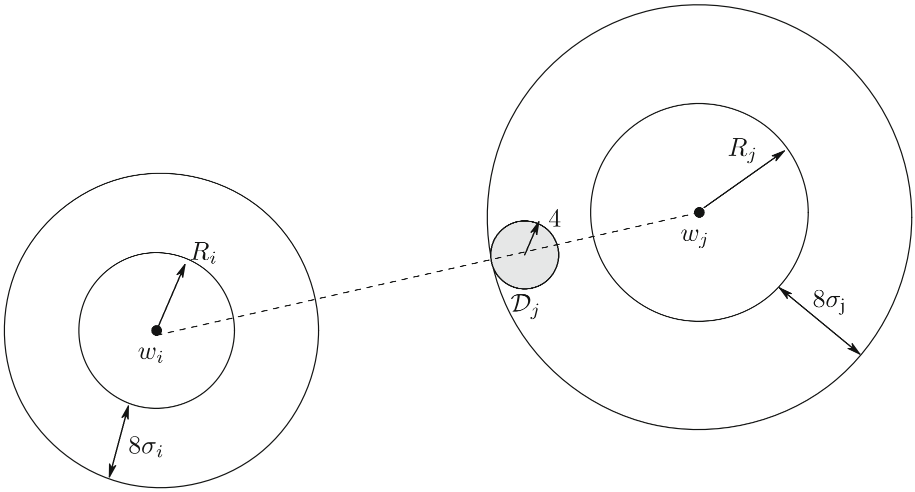

Fig. 1. The disks $D(w_i, R_i)$, $D(w_j, R_j)$ and $\mathcal{D}_j$

A-2. Almost orthogonal standard Gaussian random variables are almost independent.

Claim A-2.1. *Let ξ_j be standard complex Gaussian random variables such that their covariance matrix $\Gamma_{ij} = \mathbb{E}\left\{\xi_i \overline{\xi_j}\right\}$ satisfies*

$$\sum_{j:\ j\neq i} |\Gamma_{ij}| \leqslant \delta_i \leqslant \frac{1}{3}\,.$$

Then $\xi_j = \zeta_j + s_j\eta_j$, where ζ_j are independent standard complex Gaussian random variables, η_j are standard complex Gaussian random variables, and $s_j \in [0, \delta_j]$.

Proof. Let $\Gamma = I - \Delta$, where I is the identity matrix. Put

$$\zeta_i = \sum_j (\Gamma^{-1/2})_{ij}\xi_j\,.$$

Then ζ_i are independent standard complex Gaussian random variables. We set $\widetilde{\Delta} = I - \Gamma^{-1/2}$ and $s_i\eta_i = \sum_j \widetilde{\Delta}_{ij}\xi_j$, and estimate the sum $\sum_j |\widetilde{\Delta}_{ij}|$.

We have

$$\Gamma^{-1/2} = I + \frac{1}{2}\Delta + \sum_{k\geqslant 2} \alpha_k \Delta^k$$

with $|\alpha_k| \leqslant 1$ for all $k \geqslant 2$. Then

$$|\widetilde{\Delta}_{ij}| \leqslant \frac{1}{2}|\Delta_{ij}| + \sum_{k\geqslant 2} |(\Delta^k)_{ij}|,$$

whence

$$\sum_j |\widetilde{\Delta}_{ij}| \leqslant \frac{1}{2}\sum_j |\Delta_{ij}| + \sum_{k\geqslant 2}\sum_j |(\Delta^k)_{ij}| \leqslant \frac{\delta_i}{2} + \sum_{k\geqslant 2}\sum_j |(\Delta^k)_{ij}|\,.$$

To estimate the sum on the right-hand side, we note that for any two square matrices A and B of the same size, we have

$$\sum_j |(AB)_{ij}| \leqslant \sum_{j,\ell} |A_{i\ell}|\,|B_{\ell j}| = \sum_\ell \left[|A_{i\ell}| \cdot \sum_j |B_{\ell j}| \right] \leqslant \left(\sum_j |A_{ij}| \right) \cdot \sup_\ell \sum_j |B_{\ell j}|\,.$$

Applying this observation to the matrices $\Delta^k = \Delta \cdot \Delta^{k-1}$ (with $k \geqslant 1$), we conclude by induction that

$$\sum_j |(\Delta^k)_{ij}| \leqslant \left(\sum_j |\Delta_{ij}| \right) 3^{-(k-1)} \leqslant \frac{\delta_i}{3^{k-1}}\,.$$

Thus

$$\sum_j |\widetilde{\Delta}_{ij}| \leqslant \frac{\delta_i}{2} + \sum_{k\geqslant 2} \frac{\delta_i}{3^{k-1}} = \delta_i\,,$$

and we are done. □

A-3. Proof of the lemma. We fix two big constants $A \gg a \gg 1$. Let $R_j = r_j + a\rho_j$, $\sigma_j = \frac{A-a}{8}\rho_j$. Clearly, $R_j \geqslant 1$, $\sigma_j \geqslant 1$. Also,

$$\begin{aligned} \sigma_j &= 2\rho_j + \left(\frac{A-a}{8} - 2 \right) \rho_j \\ &\geqslant 2\sqrt{\log r_j} + \frac{A-a-16}{8a} \log(1 + a\rho_j) \\ &\geqslant 2\sqrt{\log r_j} + 2\sqrt{\log(1 + a\rho_j)} \\ &\geqslant 2\sqrt{\log r_j(1 + a\rho_j)} \geqslant 2\sqrt{\log R_j}\,, \end{aligned}$$

provided that $a \geqslant 2$ and $A \geqslant 17a + 16$.

We consider now the family of standard Gaussian random variables $\zeta_k(w_j)$, $k \leqslant R_j^2$. Applying to this family Claim A-1.4, we get

$$\left| \mathbb{E}\left\{ \zeta_k(w_i)\overline{\zeta_\ell(w_j)} \right\} \right| \leqslant 2e^{-\frac{1}{8}D_{ij}^2}$$

where, as before, $D_{ij} = |w_i - w_j| - R_i - R_j$ is the distance between the disks $D(w_i, R_i)$ and $D(w_j, R_j)$. Now, Claim A-1.5 implies that the sum of absolute values of the covariances of $\zeta_k(w_i)$ with all other $\zeta_l(w_j)$ in our family does not exceed $e^{-2\sigma_i^2} \leqslant e^{-2} < \dfrac{1}{3}$. Claim A-2.1 then allows us to write

$$\zeta_k(w_i) = \zeta_{ik} + s_{ik}\eta_{ik}\,, \qquad k \leqslant R_i^2,$$

where ζ_{ik} are independent standard Gaussian complex random variables, η_{ik} are standard Gaussian complex random variables, and $s_{ik} \in [0, e^{-2\sigma_i^2}]$.

Next, we choose ζ_{ik}, $k > R_i^2$, in such a way that the whole family ζ_{ik} of standard Gaussian complex random variables is independent and put

$$f_i = \sum_k \zeta_{ik} \frac{z^k}{\sqrt{k!}},$$

$$h_i = \sum_{k \leqslant R_i^2} s_{ik} \eta_{ik} \frac{z^k}{\sqrt{k!}} + \sum_{k > R_i^2} [\zeta_k(w_i) - \zeta_{ik}] \frac{z^k}{\sqrt{k!}}.$$

By construction, $T_{w_i} f = f_i + h_i$.

To estimate the probability

$$\mathbb{P}\left\{ \max_{z \in r_i \mathbb{D}} |h_i(z)| e^{-\frac{1}{2}|z|^2} > e^{-\rho_j^2} \right\},$$

it suffices to estimate the expression

$$\sum_{k \leqslant R_i^2} s_{ik} \max_{z \in r_i \mathbb{D}} \frac{|z|^k}{\sqrt{k!}} e^{-\frac{1}{2}|z|^2} + 2 \sum_{k > R_i^2} \max_{z \in r_i \mathbb{D}} \frac{|z|^k}{\sqrt{k!}} e^{-\frac{1}{2}|z|^2}.$$

If this expression is less than $e^{-2\rho_j^2}$, then by Lemma 2, we get what Lemma 5 asserts:

$$\mathbb{P}\left\{ \max_{z \in r_j \mathbb{D}} |h_j(z)| e^{-|z|^2/2} \geqslant e^{-\rho_j^2} \right\} \leqslant 2 \exp\left(-\frac{1}{2} e^{2\rho_j^2} \right).$$

For every $k \geqslant 1$, we have $\frac{|z|^k}{\sqrt{k!}} e^{-|z|^2/2} \leqslant 1$ and thereby,

$$\begin{aligned} \sum_{k \leqslant R_i^2} s_{ik} \max_{z \in r_i \mathbb{D}} \frac{|z|^k}{\sqrt{k!}} e^{-|z|^2/2} &\leqslant \sum_{k \leqslant R_i^2} s_{ik} \\ &\leqslant (1 + R_i^2) e^{-2\sigma_i^2} \leqslant \frac{1 + R_i^2}{e^{\sigma_i^2}} e^{-\sigma_i^2} \\ &\leqslant \frac{1 + R_i^2}{R_i^4} e^{-\sigma_i^2} \leqslant 2 e^{-\frac{(A-a)^2}{64}\rho_i^2} \leqslant \frac{1}{2} e^{-2\rho_i^2}, \end{aligned}$$

provided that $A > a + 16$.

For $k > R_i^2$, Claim A-1.1 implies that

$$\frac{|z|^{2k}}{k!} e^{-|z|^2} \leqslant \frac{k^k}{k!} e^{-k} e^{-(\sqrt{k} - |z|)^2} \leqslant e^{-(\sqrt{k} - r_i)^2}$$

for all $z \in r_i \mathbb{D}$. Hence,

$$\max_{z \in r_i \mathbb{D}} \frac{|z|^k}{\sqrt{k!}} e^{-|z|^2/2} \leqslant e^{-\frac{1}{2}(\sqrt{k} - r_i)^2}, \qquad k > R_i^2,$$

and it suffices to show that

$$2 \sum_{k > R_i^2} e^{-\frac{1}{2}(\sqrt{k} - r_i)^2} \leqslant \frac{1}{2} e^{-2\rho_i^2}.$$

Now,

$$\sum_{k > R_i^2} = \sum_{R_i^2 < k \leqslant 4r_i^2} + \sum_{k > \max(R_i^2, 4r_i^2)}$$

with the usual convention that the sum taken over the empty set equals zero. The first sum does not exceed

$$(1 + 4r_i^2) e^{-\frac{1}{2}a^2\rho_i^2} \leqslant \frac{5r_i^2}{e^{4+2\rho_i^2}} e^{(-\frac{1}{2}a^2 - 6)\rho_i^2} \overset{\rho_i^2 \geqslant \log r_i}{\leqslant} \frac{5}{e^4} e^{-2\rho_i^2} < \frac{1}{8} e^{-2\rho_i^2},$$

provided that $a \geqslant 4$. At last, the remaining sum does not exceed

$$\sum_{k \geqslant a\rho_i^2} e^{-\frac{1}{8}k} \leqslant \frac{1}{1 - e^{-1/8}} e^{-\frac{1}{8}a^2\rho_i^2}$$

$$\leqslant 9e^{-\frac{1}{8}a^2\rho_i^2} \leqslant \frac{9}{e^6} e^{-(\frac{1}{8}a^2 - 6)\rho_i^2} \leqslant \frac{1}{8} e^{-2\rho_i^2},$$

provided that $a \geqslant 8$. This finishes off the proof of Lemma 5. □

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References

1. Bogomolny, E., Bohigas, O., Leboeuf, P.: Distribution of roots of random polynomials. Phys. Rev. Lett. **68**, 2726–2729 (1992); Quantum chaotic dynamics and random polynomials. J. Stat. Phys. **85**, 639–679 (1996)
2. Forrester, P.J., Honner, G.: Exact statistical properties of the zeros of complex random polynomials. J. Phys. A **32**, 2961–2981 (1999)
3. Ginibre, J.: Statistical Ensembles of Complex, Quaternion, and Real Matrices. J. Math. Phys. **6**, 440–449 (1965)
4. Jancovici, B., Lebowitz, J.L., Manificat, G.: Large charge fluctuations in classical Coulomb systems. J. Statist. Phys. **72**, 773–787 (1993)
5. Hannay, J.H.: Chaotic analytic zero points: exact statistics for those of a random spin state. J. Phys. A **29**, L101–L105 (1996); The chaotic analytic function. J. Phys. A **31**, L755–L761 (1998)
6. Kahane, J.-P.: *Some random series of functions*. 2nd edition. Cambridge: Cambridge University Press, 1985
7. Krishnapur, M.: Overcrowding estimates for zeroes of Planar and Hyperbolic Gaussian analytic functions. J. Statist. Phys. **124**, 1399–1423 (2006)
8. Levin, B.Ya.: *Distribution of zeros of entire functions. Revised edition*. Translations of Mathematical Monographs, **5**. Providence, R.I.: Amer. Math. Soc., 1980
9. Nazarov, F., Sodin, M., Volberg, A.: Transportation to random zeroes by the gradient flow. Geom. and Funct. Anal. **17**, 887–935 (2007)
10. Shiffman, B., Zelditch, S.: *Number variance of random zeros on complex manifold*. Geom. and Funct. Anal., to appear. available at http://arXiv.org/list/math.CV/0608743, 2006
11. Sodin, M.: Zeros of Gaussian analytic functions. Math. Res. Lett. **7**, 371–381 (2000)
12. Sodin M., Tsirelson, B.: Random complex zeroes. I. Asympotic normality. Israel J. Math. **144**, 125–149 (2004); II. Perturbed Lattice, ibid **152**, 105-124 (2006); III. Decay of the hole probability, ibid **147**, 371–379 (2005)

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