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Global Wellposed Problem for the 3-D Incompressible Anisotropic Navier-Stokes Equations in an Anisotropic Space

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Abstract: In this paper, we consider a global wellposed problem for the 3-D incompressible anisotropic Navier-Stokes equations (*ANS*). We prove the global wellposedness for *ANS* provided the initial horizontal data are sufficient small in the scaling invariant Besov-Sobolev type space $B^{0,\frac{1}{2}}$. In particular, the result implies the global wellposedness of *ANS* with large initial vertical velocity.

1. Introduction

1.1. Introduction to the anisotropic Navier-Stokes equations. In this paper, we are going to study the 3-D incompressible anisotropic Navier-Stokes equations (*ANS*), namely,

$$\begin{cases} u_t + u \cdot \nabla u - \nu_h \Delta_h u - \nu_3 \partial_{x_3}^2 u = -\nabla P, \\ \mathrm{div} u = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{1.1}$$

where $u(t, x)$ and $P(t, x)$ denote the fluid velocity and the pressure, respectively, the viscosity coefficients ν_h and ν_3 are two constants satisfying

$$\nu_h > 0, \ \nu_3 \geq 0,$$

$x = (x_h, x_3) \in \mathbb{R}^3$ and $\Delta_h = \partial_{x_1}^2 + \partial_{x_2}^2$. When $\nu_h = \nu_3 = \nu$, such a system is the classical (isotropic) Navier-Stokes system (*NS*). It appeared in geophysical fluids (see for instance [5]). In fact, instead of putting the classical viscosity $-\nu\Delta$ in *NS*, meteorologists often simulate the turbulent diffusion by putting a viscosity of the form $-\nu_h \Delta_h - \nu_3 \partial_{x_3}^2$, where ν_h and ν_3 are empiric constants, and ν_3 usually is much smaller than ν_h. We refer to the book of J. Pedlosky [14], Chap. 4 for a more complete discussion. In particular, in studying of the Ekman boundary layers for rotating fluids [5,7,8], it makes sense to consider a anisotropic viscosity of the type $-\nu_h \Delta_h - \varepsilon\beta\partial_{x_3}^2$, where ε is a very small parameter. The system (ANS) has been studied first by J.Y. Chemin, B. Desjardins,

I. Gallagher and E. Grenier in [6] and D. Iftimie in [9], where the authors proved that such a system is locally wellposed for initial data in the anisotropic Sobolev space

$$H^{0,\frac{1}{2}+\varepsilon}=\left\{u\in L^2(\mathbb{R}^3);\ \|u\|^2_{\dot H^{0,\frac{1}{2}+\varepsilon}}=\int_{\mathbb{R}^3}|\xi_3|^{1+2\varepsilon}|\hat u(\xi_h,\xi_3)|^2d\xi<+\infty\right\},$$

for some $\varepsilon>0$. Moreover, it has also been proved that if the initial data are small enough in the sense that

$$\|u_0\|^{\varepsilon}_{L^2}\|u_0\|^{1-\varepsilon}_{\dot H^{0,\frac{1}{2}+\varepsilon}}\le c\nu_h \tag{1.2}$$

for some sufficiently small constant c, then the system (1.1) is global wellposed.

Similar to the classical Navier-Stokes equations, the system (*ANS*) has a scaling invariance. Indeed, if u is a solution of *ANS* on a time interval $[0,T]$ with initial data u_0, then the vector field u_λ defined by

$$u_\lambda(t,x)=\lambda u(\lambda^2t,\lambda x)$$

is also a solution of *ANS* on the time interval $[0,\lambda^{-2}T]$ with the initial data $\lambda u_0(\lambda x)$. But the norm $\|\cdot\|_{\dot H^{0,\frac{1}{2}+\varepsilon}}$ is not scaling invariant. M. Paicu proved in [12] a similar result for the system (*ANS*) with $\nu_3=0$ in the case of the initial data $u_0\in B^{0,\frac{1}{2}}$. This space has a scaling invariant norm. Then, J.Y. Chemin and P. Zhang [4] obtained a similar result in the scaling invariant space $B_4^{-\frac{1}{2},\frac{1}{2}}$. Recently, a similar result in the scaling invariant space $B_p^{-1+\frac{2}{p},\frac{1}{2}}$ was obtained in [15]. Considering the periodic anisotropic Naiver-Stokes equations, M. Paicu obtained global wellposedness in [13].

These global results in [4,6,12,13,15] are obtained under the assumption that the initial data are sufficient small. The goal of our work is to prove that the system NS or ANS is globally wellposed when the initial horizontal data are sufficient small and the initial vertical velocity is large. Specially, we can prove that the system (NS) or (ANS) is globally wellposed for the initial data u_0^ε defined by

$$u_0^\varepsilon(x)=(\varepsilon\ln(-\ln\varepsilon)\phi^h,\ln(-\ln\varepsilon)\phi^3)(x_h,\varepsilon x_3) \tag{1.3}$$

with small enough ε, where $\phi(x)=(\phi^h,\phi^3)(x)$ is a divergence free vector field in $B^{0,\frac{1}{2}}$.

1.2. Statement of the results.. As in [4], let us begin with the definition of the spaces which we will be going to work with. It requires an anisotropic version of dyadic decomposition of the Fourier space; let us first recall the following operators of localization in Fourier space, for $(k,l)\in\mathbb{Z}^2$:

$$\Delta_k^h a=\mathcal{F}^{-1}(\varphi(2^{-k}|\xi_h|)\hat a),\ \Delta_l^v a=\mathcal{F}^{-1}(\varphi(2^{-l}|\xi_3|)\hat a),$$
$$S_k^h a=\sum_{k'\le k-1}\Delta_{k'}^h a,\ S_l^v a=\sum_{l'\le l-1}\Delta_{l'}^v a,$$

where $\mathcal{F}a$ or $\hat a$ denotes the Fourier transform of the function a, and φ is a function in $\mathcal{D}((\frac{3}{4},\frac{8}{3}))$ satisfying

$$\sum_{j\in\mathbb{Z}}\varphi(2^{-j}\tau)=1,\ \forall\,\tau>0.$$

At first, we give the definitions of $B^{0,\frac{1}{2}}$ and $B^{0,\frac{1}{2}}(T)$ as follows.

Definition 1.1 ([4,10,12]). *We denote by $B^{0,\frac{1}{2}}$ the space of distributions, which is the completion of $\mathcal{S}(\mathbb{R}^3)$ by the following norm:*

$$\|a\|_{B^{0,\frac{1}{2}}} = \sum_{l\in\mathbb{Z}} 2^{\frac{l}{2}} \|\Delta_l^v a\|_{L^2(\mathbb{R}^3)}.$$

We denote by $B^{0,\frac{1}{2}}(T)$ the space of distributions, which is the completion of $C^\infty([0,T];\mathcal{S}(\mathbb{R}^3))$ by the following norm:

$$\|a\|_{B^{0,\frac{1}{2}}(T)} = \sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}} \Big(\|\Delta_j^v a\|_{L_T^\infty(L^2(\mathbb{R}^3))} + \sqrt{\nu_h}\|\nabla_h \Delta_j^v a\|_{L_T^2(L^2(\mathbb{R}^3))} + \sqrt{\nu_3}\|\partial_3 \Delta_j^v a\|_{L_T^2(L^2(\mathbb{R}^3))}\Big).$$

Now, we present the main result of this paper.

Theorem 1.1. *A positive constant C_1 exists such that, if the divergence free vector field $u_0 \in B^{0,\frac{1}{2}}$ satisfies*

$$C_1 \nu_h^{-1} \|u_0^h\|_{B^{0,\frac{1}{2}}} \exp\{C_1(\nu_h^{-1}\|u_0^3\|_{B^{0,\frac{1}{2}}} + 1)^4\} \le 1, \tag{1.4}$$

then the system (1.1) with initial data u_0 has a unique global solution $u \in B^{0,\frac{1}{2}}(\infty) \cap C([0,\infty); B^{0,\frac{1}{2}})$, and $\|u\|_{B^{0,\frac{1}{2}}(\infty)}$ is independent of ν_3.

In what follows, we always use C to denote a generic positive constant independent of ν_h and ν_3.

Remark 1.1. The positive constant C_1 can be chosen in (3.9), and is independent of ν_h and ν_3.

Remark 1.2. Paicu obtained the local wellposedness for large data in [12].

Remark 1.3. Let $u_0^\varepsilon = (u_0^{h\varepsilon}, u_0^{3\varepsilon})$ be defined in (1.3). It is easy to show that $\|u_0^{h\varepsilon}\|_{B^{0,\frac{1}{2}}} \simeq C\varepsilon \ln(-\ln\varepsilon)$ and $\|u_0^{3\varepsilon}\|_{B^{0,\frac{1}{2}}} \simeq C\ln(-\ln\varepsilon)$. Thus, u_0^ε satisfies the condition (1.4) with small enough ε. From Theorem 1.1, we obtain that the system NS or ANS is globally wellposed for the initial data u_0^ε with a small enough ε. The reason may be that the initial data u_0^ε almost only depend on the horizontal variable x_h when ε is very small. From Proposition 1.1 in [3], one can obtain that

$$\|u_0^{3\varepsilon}\|_{\dot{B}^{-1}_{\infty,\infty}} \ge C\ln(-\ln\varepsilon),$$

when $\phi^3(x_h, x_3) = f(x_h)g(x_3)$, $f \in \mathcal{S}(\mathbb{R}^2)$ and $g \in \mathcal{S}(\mathbb{R})$.

We should mention that the methods introduced by Chemin-Zhang in [4] and Paicu in [12] will play a crucial role in our proof here.

Finally, we should also mention that there has been a large amount of investigation into the classical Navier-Stokes equations, see [1–3,11] and references therein. Specially, Chemin and Gallagher [3] proved that if ε is small enough, the initial data

$$(v_0^h + \varepsilon w_0^h, w_0^3)(x_h, \varepsilon x_3)$$

generates a unique global solution of NS.

2. Anisotropic Littlewood-Paley Theory

At first, we list anisotropic Berstein inequalities in the following. Please see the details in [4,12].

Lemma 2.1. *Let $\mathcal{B}_h$ (resp. $\mathcal{B}_v$) be a ball of $\mathbb{R}^2_h$ (resp. $\mathbb{R}_v$), and $\mathcal{C}_h$ (resp. $\mathcal{C}_v$) be a ring of $\mathbb{R}^2_h$ (resp. $\mathbb{R}_v$). Then, for $1 \le p_2 \le p_1 \le \infty$ and $1 \le q_2 \le q_1 \le \infty$, there holds:*

1. *If the support of $\hat{a}$ is included in $2^k\mathcal{B}_h$, then*
$$\|\partial_h^\beta a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{k(|\beta|+2(\frac{1}{p_2}-\frac{1}{p_1}))}\|a\|_{L_h^{p_2}(L_v^{q_1})},$$
where $\partial_h := \partial_{x_h}$.
2. *If the support of $\hat{a}$ is included in $2^l\mathcal{B}_v$, then*
$$\|\partial_3^N a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{l(N+\frac{1}{q_2}-\frac{1}{q_1})}\|a\|_{L_h^{p_1}(L_v^{q_2})},$$
where $\partial_3 := \partial_{x_3}$.
3. *If the support of $\hat{a}$ is included in $2^k\mathcal{C}_h$, then*
$$\|a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{-kN}\sup_{|\beta|=N}\|\partial_h^\beta a\|_{L_h^{p_1}(L_v^{q_1})}.$$
4. *If the support of $\hat{a}$ is included in $2^l\mathcal{C}_v$, then*
$$\|a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{-lN}\|\partial_3^N a\|_{L_h^{p_1}(L_v^{q_1})}.$$

Notations. In what follows, we make the convention that $(d_k)_{k\in\mathbb{Z}}$ denotes a generic element of the sphere of $l^1(\mathbb{Z})$ as in [4].

The following lemma is proved in [15] (Lemma 2.4).

Lemma 2.2. *Let w be in $B^{0,\frac{1}{2}}(T)$. Then, we have*
$$\|\Delta_j^v w\|_{L_T^{2p}(L_h^{\frac{2p}{p-1}}(L_v^2))} \lesssim d_j \nu_h^{-\frac{1}{2p}} 2^{-\frac{j}{2}}\|w\|_{B^{0,\frac{1}{2}}(T)}$$
and
$$\|w\|_{L_T^{2p}(L_h^{\frac{2p}{p-1}}(L_v^\infty))} \lesssim \|w\|_{\widetilde{L}_T^{2p}(\widetilde{L}_h^{\frac{2p}{p-1}}(B_v^{\frac{1}{2}}))} \lesssim \nu_h^{-\frac{1}{2p}}\|w\|_{B^{0,\frac{1}{2}}(T)},$$
where $\|w\|_{\widetilde{L}_T^{2p}(\widetilde{L}_h^{\frac{2p}{p-1}}(B_v^{\frac{1}{2}}))} = \sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}}\|\Delta_j^v w\|_{L_T^{2p}(L_h^{\frac{2p}{p-1}}(L_v^2))}$ *and $p \ge 2$.*

Remark 2.1. We only use the following estimate in the proof of Theorem 1.1:
$$\|\Delta_j^v w\|_{L_T^4(L_h^4(L_v^2))} \lesssim d_j \nu_h^{-\frac{1}{4}} 2^{-\frac{j}{2}}\|w\|_{B^{0,\frac{1}{2}}(T)}.$$
This estimate was proved in [4].

The following lemma is proved in [12] (Lemma 2.4).

Lemma 2.3. *Let w and $\nabla_h w$ be in L^2. Then, we have*
$$\|w\|_{L_v^2(L_h^4)} \lesssim \|w\|_{L^2}^{\frac{1}{2}}\|\nabla_h w\|_{L^2}^{\frac{1}{2}}.$$

3. The Proof of an Existence Theorem

The purpose of this section is to prove the following existence theorem:

Theorem 3.1. *A constant C_1 exists such that, if the divergence free vector field $u_0 \in B^{0,\frac{1}{2}}$ satisfies*

$$C_1 \nu_h^{-1} \|u_0^h\|_{B^{0,\frac{1}{2}}} \exp\{C_1(\nu_h^{-1}\|u_0^3\|_{B^{0,\frac{1}{2}}} + 1)^4\} \leq 1,$$

then the system (1.1) with initial data u_0 has a global solution $u \in B^{0,\frac{1}{2}}(\infty) \cap C([0,\infty); B^{0,\frac{1}{2}})$, and $\|u\|_{B^{0,\frac{1}{2}}(\infty)}$ is independent of ν_3.

Proof. We shall use the classical Friedrichs' regularization method to construct the approximate solutions to (1.1). For simplicity, we just outline it here (for the details, see [4,5,12]). In order to do so, let us define the sequence of operators $(P_n)_{n\in\mathbb{N}}$ by

$$P_n a := \mathcal{F}^{-1}(1_{B(0,n)}\hat{a}),$$

and we define the following approximate system:

$$\partial_t u_n^1 + P_n(u_n \cdot \nabla u_n^1) - \nu_h \Delta_h u_n^1 - \nu_3 \partial_3^2 u_n^1 = -P_n \partial_1 (-\Delta)^{-1} \partial_j \partial_k (u_n^j u_n^k), \quad (3.1)$$
$$\partial_t u_n^2 + P_n(u_n \cdot \nabla u_n^2) - \nu_h \Delta_h u_n^2 - \nu_3 \partial_3^2 u_n^2 = -P_n \partial_2 (-\Delta)^{-1} \partial_j \partial_k (u_n^j u_n^k), \quad (3.2)$$
$$\partial_t u_n^3 + P_n(u_n \cdot \nabla u_n^3) - \nu_h \Delta_h u_n^3 - \nu_3 \partial_3^2 u_n^3 = -P_n \partial_3 (-\Delta)^{-1} \partial_j \partial_k (u_n^j u_n^k), \quad (3.3)$$
$$\mathrm{div} u_n = 0, \quad (3.4)$$
$$u_n|_{t=0} = P_n u_0, \quad (3.5)$$

where $(-\Delta)^{-1}\partial_j\partial_k$ is defined precisely by

$$(-\Delta)^{-1}\partial_j\partial_k a := -\mathcal{F}^{-1}(|\xi|^{-2}\xi_j\xi_k\hat{a}).$$

Then, the system (3.1)-(3.5) appears to be the ordinary differential equations in the space

$$L_n^2 := \{a \in L^2(\mathbb{R}^3) \big| \mathrm{div} a = 0,\ \mathrm{Supp}\hat{a} \subset B(0,n)\}.$$

Such a system is globally wellposed because

$$\frac{d}{dt}\|u_n\|_{L^2}^2 \leq 0.$$

Now, the proof of Theorem 3.1 reduces to the following two propositions, which we shall admit for the time being.

Proposition 3.1. *Let u be a divergence free vector field in $B^{0,\frac{1}{2}}(T)$ and $w \in B^{0,\frac{1}{2}}(T)$. Then, for any $j \in \mathbb{Z}$, we have*

$$F_j(T) := \int_0^T \left| \int_{\mathbb{R}^3} \Delta_j^v (u\cdot\nabla w) \Delta_j^v w dx \right| dt \lesssim d_j^2 \nu_h^{-1} 2^{-j} \|w\|_{B^{0,\frac{1}{2}}(T)}^2 \|u^h\|_{B^{0,\frac{1}{2}}(T)}.$$

Proposition 3.2. *Let u be a divergence free vector field in $B^{0,\frac{1}{2}}(T)$ and $w \in B^{0,\frac{1}{2}}(T)$. Then, for any $j \in \mathbb{Z}$, we have*

$$
\begin{aligned}
G_j(T) &:= \int_0^T \left| \sum_{k,l} \int_{\mathbb{R}^3} \Delta_j^v (-\Delta)^{-1} \partial_l \partial_k (u^l u^k) \Delta_j^v \partial_h w dx \right| dt \\
&\lesssim d_j^2 \nu_h^{-1} 2^{-j} \|u^h\|^2_{B^{0,\frac{1}{2}}(T)} \|w\|_{B^{0,\frac{1}{2}}(T)} \\
&\quad + d_j^2 2^{-j} \int_0^T \|u^h\|^{\frac{1}{2}}_{B^{0,\frac{1}{2}}} \|\nabla_h u^h\|^{\frac{1}{2}}_{B^{0,\frac{1}{2}}} \|u^3\|^{\frac{1}{2}}_{B^{0,\frac{1}{2}}} \|\nabla_h u^3\|^{\frac{1}{2}}_{B^{0,\frac{1}{2}}} \|\nabla_h w\|_{B^{0,\frac{1}{2}}} dt \\
&\quad + d_j^2 2^{-j} \int_0^T \|\nabla_h u^h\|_{B^{0,\frac{1}{2}}} \|u^3\|^{\frac{1}{2}}_{B^{0,\frac{1}{2}}} \|\nabla_h u^3\|^{\frac{1}{2}}_{B^{0,\frac{1}{2}}} \|w\|^{\frac{1}{2}}_{B^{0,\frac{1}{2}}} \|\nabla_h w\|^{\frac{1}{2}}_{B^{0,\frac{1}{2}}} dt.
\end{aligned}
$$

Conclusion of the proof of Theorem 3.1. Applying the operator Δ_j^v to (3.1) and taking the L^2 inner product of the resulting equation with $\Delta_j^v u_n^1$, we have

$$
\begin{aligned}
&\frac{d}{dt} \|\Delta_j^v u_n^1\|^2_{L^2} + 2\nu_h \|\nabla_h \Delta_j^v u_n^1\|^2_{L^2} + 2\nu_3 \|\partial_3 \Delta_j^v u_n^1\|^2_{L^2} \\
&= -2 \int_{\mathbb{R}^3} \Delta_j^v (u_n \cdot \nabla u_n^1) \Delta_j^v u_n^1 dx + 2 \sum_{k,l} \int_{\mathbb{R}^3} \Delta_j^v (-\Delta)^{-1} \partial_l \partial_k (u_n^l u_n^k) \Delta_j^v \partial_1 u_n^1 dx.
\end{aligned}
$$

From Propositions 3.1-3.2 and the Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
&2^j \left(\|\Delta_j^v u_n^1(t)\|^2_{L^2} + 2\nu_h \|\nabla_h \Delta_j^v u_n^1\|^2_{L_t^2(L^2)} + 2\nu_3 \|\partial_3 \Delta_j^v u_n^1\|^2_{L_t^2(L^2)} \right) \\
&\le 2^j \|\Delta_j^v u_n^1(0)\|^2_{L^2} + C d_j^2 \nu_h^{-1} \|u_n^h\|^3_{B^{0,\frac{1}{2}}(T)} \\
&\quad + C d_j^2 \int_0^t \|u_n^3(s)\|^{\frac{1}{2}}_{B^{0,\frac{1}{2}}} \|\nabla_h u_n^3(s)\|^{\frac{1}{2}}_{B^{0,\frac{1}{2}}} \|u_n^h(s)\|^{\frac{1}{2}}_{B^{0,\frac{1}{2}}} \|\nabla_h u_n^h(s)\|^{\frac{3}{2}}_{B^{0,\frac{1}{2}}} ds
\end{aligned}
$$

and

$$
\begin{aligned}
&\|u_n^1(t)\|_{B^{0,\frac{1}{2}}} + \sqrt{\nu_h} \|\nabla_h u_n^1\|_{\widetilde{L}_t^2(B^{0,\frac{1}{2}})} + \sqrt{\nu_3} \|\partial_3 \Delta_j^v u_n^1\|_{\widetilde{L}_t^2(B^{0,\frac{1}{2}})} \\
&\le C \|u_n^1(0)\|_{B^{0,\frac{1}{2}}} + C \nu_h^{-\frac{1}{2}} \|u_n^h\|^{\frac{3}{2}}_{B^{0,\frac{1}{2}}(T)} + \frac{\sqrt{\nu_h}}{4} \|\nabla_h u_n^h\|_{L_t^2(B^{0,\frac{1}{2}})} \\
&\quad + C \nu_h^{-\frac{3}{2}} \left(\int_0^t \|u_n^3(s)\|^2_{B^{0,\frac{1}{2}}} \|\nabla_h u_n^3(s)\|^2_{B^{0,\frac{1}{2}}} \|u_n^h(s)\|^2_{B^{0,\frac{1}{2}}} ds \right)^{\frac{1}{2}}, \qquad (3.6)
\end{aligned}
$$

where $t \in (0, T]$. Similarly, we obtain

$$
\begin{aligned}
&\|u_n^2(t)\|_{B^{0,\frac{1}{2}}} + \sqrt{\nu_h} \|\nabla_h u_n^2\|_{\widetilde{L}_t^2(B^{0,\frac{1}{2}})} + \sqrt{\nu_3} \|\partial_3 \Delta_j^v u_n^2\|_{\widetilde{L}_t^2(B^{0,\frac{1}{2}})} \\
&\le C \|u_n^2(0)\|_{B^{0,\frac{1}{2}}} + C \nu_h^{-\frac{1}{2}} \|u_n^h\|^{\frac{3}{2}}_{B^{0,\frac{1}{2}}(T)} + \frac{\sqrt{\nu_h}}{4} \|\nabla_h u_n^h\|_{L_t^2(B^{0,\frac{1}{2}})} \\
&\quad + C \nu_h^{-\frac{3}{2}} \left(\int_0^t \|u_n^3(s)\|^2_{B^{0,\frac{1}{2}}} \|\nabla_h u_n^3(s)\|^2_{B^{0,\frac{1}{2}}} \|u_n^h(s)\|^2_{B^{0,\frac{1}{2}}} ds \right)^{\frac{1}{2}} \qquad (3.7)
\end{aligned}
$$

and

$$\begin{aligned}
&\|u_n^3(t)\|_{B^{0,\frac{1}{2}}} + \sqrt{\nu_h}\|\nabla_h u_n^3\|_{\widetilde{L}_t^2(B^{0,\frac{1}{2}})} + \sqrt{\nu_3}\|\partial_3\Delta_j^v u_n^3\|_{\widetilde{L}_t^2(B^{0,\frac{1}{2}})} \\
&\le C\|u_n^3(0)\|_{B^{0,\frac{1}{2}}} + C\nu_h^{-\frac{1}{2}}\|u_n^h\|_{B^{0,\frac{1}{2}}(T)}^{\frac{1}{2}}\|u_n^3\|_{B^{0,\frac{1}{2}}(T)} + C\nu_h^{-\frac{1}{2}}\|u_n^h\|_{B^{0,\frac{1}{2}}(T)}^{\frac{3}{2}} \\
&\quad + C\left(\int_0^t \|u_n^3\|_{B^{0,\frac{1}{2}}}^{\frac{1}{2}}\|\nabla_h u_n^3\|_{B^{0,\frac{1}{2}}}^{\frac{1}{2}}\|\nabla_h u_n^h\|_{B^{0,\frac{1}{2}}}^{\frac{3}{2}}\|u_n^h\|_{B^{0,\frac{1}{2}}}^{\frac{1}{2}}\,ds\right)^{\frac{1}{2}}, \qquad (3.8)
\end{aligned}$$

where $t \in (0, T]$, since

$$\int_{\mathbb{R}^3} P_n\partial_3(-\Delta)^{-1}\partial_j\partial_k(u_n^j u_n^k)u_n^3\,dx = \int_{\mathbb{R}^3} P_n(-\Delta)^{-1}\partial_j\partial_k(u_n^j u_n^k)\mathrm{div}_h u_n^h\,dx.$$

Then, from (3.6)-(3.7) and Minkowski's inequality, we have

$$\begin{aligned}
&\|u_n^h\|_{B^{0,\frac{1}{2}}(t)}^2 \\
&\le 2C_0\|u_n^h(0)\|_{B^{0,\frac{1}{2}}}^2 + C\nu_h^{-1}\|u_n^h\|_{B^{0,\frac{1}{2}}(T)}^3 \\
&\quad + C\nu_h^{-3}\int_0^t \|u_n^3(s)\|_{B^{0,\frac{1}{2}}}^2\|\nabla_h u_n^3(s)\|_{B^{0,\frac{1}{2}}}^2\|u_n^h\|_{B^{0,\frac{1}{2}}(s)}^2\,ds,\ t\in(0,T].
\end{aligned}$$

Gronwall's inequality implies that

$$\begin{aligned}
&\|u_n^h\|_{B^{0,\frac{1}{2}}(T)}^2 \\
&\le \left(2C_0\|u_n^h(0)\|_{B^{0,\frac{1}{2}}}^2 + C\nu_h^{-1}\|u_n^h\|_{B^{0,\frac{1}{2}}(T)}^3\right)\exp\{C\nu_h^{-4}\|u_n^3\|_{B^{0,\frac{1}{2}}(T)}^4\}.
\end{aligned}$$

From (3.8) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\|u_n^3\|_{B^{0,\frac{1}{2}}(T)} \\
&\le 2C_0\|u_n^3(0)\|_{B^{0,\frac{1}{2}}} + C\nu_h^{-\frac{1}{2}}\|u_n^h\|_{B^{0,\frac{1}{2}}(T)}^{\frac{1}{2}}\|u_n^3\|_{B^{0,\frac{1}{2}}(T)} + C\nu_h^{-\frac{1}{2}}\|u_n^h\|_{B^{0,\frac{1}{2}}(T)}^{\frac{3}{2}} \\
&\quad + C\nu_h^{-\frac{1}{2}}\|u_n^h\|_{B^{0,\frac{1}{2}}(T)}\|u_n^3\|_{B^{0,\frac{1}{2}}(T)}^{\frac{1}{2}} \\
&\le 2C_0\|u_n^3(0)\|_{B^{0,\frac{1}{2}}} + C\nu_h^{-\frac{1}{2}}\|u_n^h\|_{B^{0,\frac{1}{2}}(T)}^{\frac{1}{2}}\|u_n^3\|_{B^{0,\frac{1}{2}}(T)} + C\nu_h^{-\frac{1}{2}}\|u_n^h\|_{B^{0,\frac{1}{2}}(T)}^{\frac{3}{2}}.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
\|u_n^h\|_{B^{0,\frac{1}{2}}(T)}^2 \le e^{C\nu_h^{-4}(4C_0\|u_0^3\|_{B^{0,\frac{1}{2}}}+\nu_h)^4}\Big(&2C_0\|u_0^h\|_{B^{0,\frac{1}{2}}}^2 \\
&+ C\nu_h^{-1}(4C_0\|u_0^h\|_{B^{0,\frac{1}{2}}}^2)^{\frac{3}{2}}e^{\frac{3}{2}C\nu_h^{-4}(4C_0\|u_0^3\|_{B^{0,\frac{1}{2}}}+\nu_h)^4}\Big)
\end{aligned}$$

and

$$\begin{aligned}\|u_n^3\|_{B^{0,\frac{1}{2}}(T)} &\le 2C_0\|u_0^3\|_{B^{0,\frac{1}{2}}} + C\nu_h^{-\frac{1}{2}}(4C_0\|u_0^h\|^2_{B^{0,\frac{1}{2}}})^{\frac{3}{4}} e^{\frac{3}{4}C\nu_h^{-4}(4C_0\|u_0^3\|_{B^{0,\frac{1}{2}}}+\nu_h)^4} \\ &+C\nu_h^{-\frac{1}{2}}(4C_0\|u_0^h\|^2_{B^{0,\frac{1}{2}}})^{\frac{1}{4}} e^{\frac{1}{4}C\nu_h^{-4}(4C_0\|u_0^3\|_{B^{0,\frac{1}{2}}}+\nu_h)^4}\Big(4C_0\|u_0^3\|_{B^{0,\frac{1}{2}}}+\nu_h\Big),\end{aligned}$$

for all $T < T_n$, where

$$\begin{aligned}T_n := \sup\{t > 0;\ \|u_n^h\|^2_{B^{0,\frac{1}{2}}(t)} &\le 4C_0\|u_0^h\|^2_{B^{0,\frac{1}{2}}} e^{C\nu_h^{-4}(4C_0\|u_0^3\|_{B^{0,\frac{1}{2}}}+\nu_h)^4}, \\ \|u_n^3\|_{B^{0,\frac{1}{2}}(t)} &\le 4C_0\|u_0^3\|_{B^{0,\frac{1}{2}}} + \nu_h\}.\end{aligned}$$

Then, if u_0 satisfies

$$C_1\nu_h^{-1}\|u_0^h\|_{B^{0,\frac{1}{2}}} \exp\{C_1(\nu_h^{-1}\|u_0^3\|_{B^{0,\frac{1}{2}}} + 1)^4\} \le 1,$$

where

$$C_1 = 2^9 C^2 C_0^4, \tag{3.9}$$

we get that for any n and for any $T < T_n$,

$$\|u_n^h\|_{B^{0,\frac{1}{2}}(T)} \le \frac{5}{2}C_0\|u_0^h\|^2_{B^{0,\frac{1}{2}}} e^{C\nu_h^{-4}(4C_0\|u_0^3\|_{B^{0,\frac{1}{2}}}+\nu_h)^4}$$

and

$$\|u_n^3\|_{B^{0,\frac{1}{2}}(t)} \le \frac{5}{2}C_0\|u_0^3\|_{B^{0,\frac{1}{2}}} + \frac{1}{2}\nu_h.$$

Thus, $T_n = +\infty$. Then, the existence follows from the classical compactness method, the details of which are omitted (see [5,12]).

Then, we prove the continuity of the solution u as follows. From (1.1), we have

$$\Delta_j^v u_t = \nu_h\Delta_j^v\Delta_h u + \nu_3\Delta_j^v\partial_3^2 u - \Delta_j^v(u\cdot\nabla u) - \Delta_j^v\nabla P.$$

We can easily obtain that for any $T > 0$ and $j \in \mathbb{Z}$,

$$\nu_3\Delta_j^v\partial_3^2 u \in L^\infty([0,T];L^2),\ \nu_h\Delta_j^v\Delta_h u \in L^2(0,T;L_v^2(\dot H_h^{-1}))$$

and

$$(\nu_h\Delta_j^v\Delta_h u + \nu_3\Delta_j^v\partial_3^2 u|\Delta_j^v u)_{L^2} \in L^1([0,T]).$$

From Proposition 3.1, we have

$$(\Delta_j^v(u\cdot\nabla u)|\Delta_j^v u)_{L^2} \in L^1([0,T]).$$

Thus, we have that $\frac{d}{dt}\|\Delta_j^v u(t)\|^2_{L^2} \in L^1([0,T])$, for any $T > 0$ and $j \in \mathbb{Z}$. Combining it with $u \in B^{0,\frac{1}{2}}(\infty)$, we can easily get that $u \in C([0,\infty); B^{0,\frac{1}{2}})$. Then Theorem 3.1 is proved provided of course that we have proved Propositions 3.1–3.2. □

To prove Propositions 3.1–3.2, we need the following lemma.

Lemma 3.1. *Let w and u be in $B^{0,\frac{1}{2}}(T)$. We have*

$$\|\Delta_j^v(u\partial_h w)\|_{L_T^{\frac{4}{3}}(L_h^{\frac{4}{3}}(L_v^2))} \lesssim d_j \nu_h^{-\frac{3}{4}} 2^{-\frac{j}{2}} \|w\|_{B^{0,\frac{1}{2}}(T)} \|u\|_{B^{0,\frac{1}{2}}(T)}.$$

Proof. Using Bony's decomposition in the vertical variable, we obtain

$$\Delta_j^v(u\partial_h w) = \sum_{|j-j'|\le 5} \Delta_j^v(S_{j'-1}^v u \partial_h \Delta_{j'}^v w) + \sum_{j'\ge j-N_0} \Delta_j^v(\Delta_{j'}^v u \partial_h S_{j'+2}^v w).$$

Using Hölder's inequality and Lemma 2.2, we get

$$\begin{aligned}\|\Delta_j^v(S_{j'-1}^v u\partial_h \Delta_{j'}^v w)\|_{L_T^{\frac{4}{3}}(L_h^{\frac{4}{3}}(L_v^2))} &\lesssim \|S_{j'-1}^v u\|_{L_T^4(L_h^4(L_v^\infty))} \|\Delta_{j'}^v \partial_h w\|_{L_T^2(L^2(\mathbb{R}^3))} \\ &\lesssim d_{j'} \nu_h^{-\frac{1}{2}} 2^{-\frac{j'}{2}} \|w\|_{B^{0,\frac{1}{2}}(T)} \nu_h^{-\frac{1}{4}} \|u\|_{B^{0,\frac{1}{2}}(T)}\end{aligned}$$

and

$$\begin{aligned}\|\Delta_j^v(\Delta_{j'}^v u\partial_h S_{j'+2}^v w)\|_{L_T^{\frac{4}{3}}(L_h^{\frac{4}{3}}(L_v^2))} &\lesssim \|S_{j'+2}^v(\partial_h w)\|_{L_T^2(L_h^2(L_v^\infty))} \|\Delta_{j'}^v u\|_{L_T^4(L_h^4(L_v^2))} \\ &\lesssim d_{j'} \nu_h^{-\frac{1}{2}} 2^{-\frac{j'}{2}} \|w\|_{B^{0,\frac{1}{2}}(T)} \nu_h^{-\frac{1}{4}} \|u\|_{B^{0,\frac{1}{2}}(T)}.\end{aligned}$$

Then, we can immediately finish the proof. □

Similarly, we can obtain the following lemma.

Lemma 3.2. *Let $\nabla_h w$, u and $\nabla_h u$ be in $B^{0,\frac{1}{2}}$. We have*

$$\|\Delta_j^v(u\partial_h w)\|_{L_h^{\frac{4}{3}}(L_v^2)} \lesssim d_j 2^{-\frac{j}{2}} \|\nabla_h w\|_{B^{0,\frac{1}{2}}} \|u\|_{B^{0,\frac{1}{2}}}^{\frac{1}{2}} \|\nabla_h u\|_{B^{0,\frac{1}{2}}}^{\frac{1}{2}}.$$

Lemma 3.3. *Let u, $\nabla_h u$, w and $\nabla_h w$ be in $B^{0,\frac{1}{2}}$. We have*

$$\|\Delta_j^v(uw)\|_{L^2(\mathbb{R}^3)} \lesssim d_j 2^{-\frac{j}{2}} \|u\|_{B^{0,\frac{1}{2}}}^{\frac{1}{2}} \|\nabla_h u\|_{B^{0,\frac{1}{2}}}^{\frac{1}{2}} \|w\|_{B^{0,\frac{1}{2}}}^{\frac{1}{2}} \|\nabla_h w\|_{B^{0,\frac{1}{2}}}^{\frac{1}{2}}.$$

Proof. Using Bony's decomposition in the vertical variable, we obtain

$$\Delta_j^v(uw) = \sum_{|j-j'|\le 5} \Delta_j^v(S_{j'-1}^v u \Delta_{j'}^v w) + \sum_{j'\ge j-N_0} \Delta_j^v(S_{j'+2}^v w \Delta_{j'}^v u).$$

Using Hölder's inequality and Lemma 2.3, we get

$$\begin{aligned}\|\Delta_j^v(S_{j'-1}^v u\Delta_{j'}^v w)\|_{L^2(\mathbb{R}^3)} &\lesssim \|S_{j'-1}^v u\|_{L_v^\infty(L_h^4)} \|\Delta_{j'}^v w\|_{L_v^2(L_h^4)} \\ &\lesssim d_{j'} 2^{-\frac{j'}{2}} \|u\|_{B^{0,\frac{1}{2}}}^{\frac{1}{2}} \|\nabla_h u\|_{B^{0,\frac{1}{2}}}^{\frac{1}{2}} \|w\|_{B^{0,\frac{1}{2}}}^{\frac{1}{2}} \|\nabla_h w\|_{B^{0,\frac{1}{2}}}^{\frac{1}{2}}\end{aligned}$$

and

$$\|\Delta_j^v(S_{j'+2}^v w\Delta_{j'}^v u)\|_{L_T^2(L^2(\mathbb{R}^3))} \lesssim d_{j'} 2^{-\frac{j'}{2}} \|u\|_{B^{0,\frac{1}{2}}}^{\frac{1}{2}} \|\nabla_h u\|_{B^{0,\frac{1}{2}}}^{\frac{1}{2}} \|w\|_{B^{0,\frac{1}{2}}}^{\frac{1}{2}} \|\nabla_h w\|_{B^{0,\frac{1}{2}}}^{\frac{1}{2}}.$$

Then, we can immediately finish the proof. □

Proof of Proposition 3.1. We distinguish the terms with horizontal derivatives from the terms with vertical ones, writing

$$F_j(T) \le F_j^h(T) + F_j^v(T),$$

where

$$F_j^h(T) := \int_0^T \left| \int_{\mathbb{R}^3} \Delta_j^v (u^h \cdot \nabla_h w) \Delta_j^v w dx \right| dt$$

and

$$F_j^v(T) := \int_0^T \left| \int_{\mathbb{R}^3} \Delta_j^v (u^3 \partial_3 w) \Delta_j^v w dx \right| dt.$$

Using Hölder's inequality, Lemmas 2.2 and 3.1, we obtain

$$\begin{aligned} F_j^h(T) &\le \|\Delta_j^v(u^h \cdot \nabla_h w)\|_{L_T^{\frac{4}{3}}(L_h^{\frac{4}{3}}(L_v^2))} \|\Delta_j^v w\|_{L_T^4(L_h^4(L_v^2))} \\ &\lesssim d_j^2 \nu_h^{-1} 2^{-j} \|w\|^2_{B^{0,\frac{1}{2}}(T)} \|u^h\|_{B^{0,\frac{1}{2}}(T)}. \end{aligned}$$

Applying the trick from [4,12], using paradifferential decomposition in the vertical variable to $\Delta_j^v(u^3\partial_3 w)$ first, then by a commutator process, one gets

$$\begin{aligned} \Delta_j^v(u^3\partial_3 w) &= S_{j-1}^v u^3 \partial_3 \Delta_j^v w + \sum_{|j-l|\le 5} [\Delta_j^v; S_{l-1}^v u^3] \partial_3 \Delta_l^v w \\ &\quad + \sum_{|j-l|\le 5} (S_{l-1}^v u^3 - S_{j-1}^v u^3) \partial_3 \Delta_j^v \Delta_l^v w + \sum_{l \ge j - N_0} \Delta_j^v (\Delta_l^v u^3 \partial_3 S_{l+2}^v w). \end{aligned}$$

Correspondingly, we decompose $F_j^v(T)$ as

$$F_j^v(T) := F_j^{1,v}(T) + F_j^{2,v}(T) + F_j^{3,v}(T) + F_j^{4,v}(T).$$

Using integration by parts and the fact that div$u$=0, we have

$$\begin{aligned} F_j^{1,v}(T) &= \frac{1}{2} \int_0^T \left| \int_{\mathbb{R}^3} S_{j-1}^v \mathrm{div}_h u^h |\Delta_j^v w|^2 dx \right| dt \\ &= \int_0^T \left| \int_{\mathbb{R}^3} S_{j-1}^v u^h \cdot \nabla_h \Delta_j^v w \Delta_j^v w dx \right| dt. \end{aligned}$$

From Lemma 2.2 and Hölder's inequality, we get

$$\begin{aligned} F_j^{1,v}(T) &\le \|S_{j-1}^v u^h\|_{L_T^4(L_h^4(L_v^\infty))} \|\Delta_j^v w\|_{L_T^4(L_h^4(L_v^2))} \|\nabla_h \Delta_j^v w\|_{L_T^2(L^2(\mathbb{R}^3))} \\ &\lesssim d_j^2 \nu_h^{-1} 2^{-j} \|u^h\|_{B^{0,\frac{1}{2}}(T)} \|w\|^2_{B^{0,\frac{1}{2}}(T)}. \end{aligned}$$

To deal with the commutator in $F_j^{2,v}$, we first use the Taylor formula to get

$$\begin{aligned}
&F_j^{2,v}(T)\\
&=\sum_{|j-l|\leq 5}\int_0^T\Big|\int_{\mathbb{R}^3}2^j\int_{\mathbb{R}}h(2^j(x_3-y_3))\int_0^1 S_{l-1}^v\partial_3 u^3(x_h,\tau y_3+(1-\tau)x_3)d\tau\\
&\quad\times(y_3-x_3)\partial_3\Delta_l^v w(x_h,y_3)dy_3\Delta_j^v w(x)dx\Big|dt,
\end{aligned}$$

where $h=\mathcal{F}^{-1}\varphi$. Using div$u=0$ and integration by parts, we have

$$\begin{aligned}
F_j^{2,v}(T)=&\sum_{|j-l|\leq 5}\int_0^T\Big|\int_{\mathbb{R}^3}\int_{\mathbb{R}}\bar{h}(2^j(x_3-y_3))\int_0^1 S_{l-1}^v u^h(x_h,\tau y_3+(1-\tau)x_3)d\tau\\
&\cdot\nabla_h\partial_3\Delta_l^v w(x_h,y_3)dy_3\Delta_j^v w(x)dx\Big|dt\\
&+\sum_{|j-l|\leq 5}\int_0^T\Big|\int_{\mathbb{R}^3}\int_{\mathbb{R}}\bar{h}(2^j(x_3-y_3))\int_0^1 S_{l-1}^v u^h(x_h,\tau y_3+(1-\tau)x_3)d\tau\\
&\times\partial_3\Delta_l^v w(x_h,y_3)dy_3\cdot\nabla_h\Delta_j^v w(x)dx\Big|dt,
\end{aligned}$$

where $\bar{h}(x_3)=x_3h(x_3)$. Using Hölder's inequality, Young's inequality and Lemma 2.2, we obtain

$$\begin{aligned}
&F_j^{2,v}(T)\\
&\lesssim\sum_{|j-l|\leq 5}2^{l-j}\|S_{l-1}^v u^h\|_{L_T^4(L_h^4(L_v^\infty))}\|\nabla_h\Delta_l^v w\|_{L_T^2(L^2(\mathbb{R}^3))}\|\Delta_j^v w\|_{L_T^4(L_h^4(L_v^2))}\\
&\quad+\sum_{|j-l|\leq 5}2^{l-j}\|S_{l-1}^v u^h\|_{L_T^4(L_h^4(L_v^\infty))}\|\nabla_h\Delta_j^v w\|_{L_T^2(L^2(\mathbb{R}^3))}\|\Delta_l^v w\|_{L_T^4(L_h^4(L_v^2))}\\
&\lesssim d_j^2\nu_h^{-1}2^{-j}\|u^h\|_{B^{0,\frac12}(T)}\|w\|^2_{B^{0,\frac12}(T)}.
\end{aligned}$$

It is easy to see that

$$F_j^{3,v}(T)\leq\sum_{\substack{|j-l'|\leq 5\\|j-l|\leq 5}}\int_0^T\Big|\int_{\mathbb{R}^3}\Delta_{l'}^v u^3\partial_3\Delta_j^v\Delta_l^v w\Delta_j^v w dx\Big|dt.$$

We can rewrite $\Delta_{l'}^v u^3$ as follows:

$$\Delta_{l'}^v u^3=\int_{\mathbb{R}}g^v(2^{l'}(x_3-y_3))\partial_3\Delta_{l'}^v u^3(x_h,y_3)dy_3,\tag{3.10}$$

where $g^v\in\mathcal{S}(\mathbb{R})$ satisfying $\mathcal{F}(g^v)(\xi_3)=\frac{\tilde{\varphi}(|\xi_3|)}{i\xi_3}$, and $\tilde{\varphi}$ is a function in $\mathcal{D}((\frac12,3))$ satisfying $\tilde{\varphi}(\tau)=1$ with $\tau\in(\frac34,\frac83)$. Using div$u=0$, integration by parts, Young's inequality and Lemma 2.2, we get

$$
\begin{aligned}
&F_j^{3,v}(T)\\
&\quad\le \sum_{\substack{|j-l'|\le 5\\ |j-l|\le 5}} \int_0^T \Big(\Big| \int_{\mathbb{R}^3}\int_{\mathbb{R}} g^v(2^{l'}(x_3-y_3))\Delta_{l'}^v u^h(x_h,y_3)dy_3\cdot\nabla_h\partial_3\Delta_j^v\Delta_l^v w\Delta_j^v w dx\Big|\\
&\qquad + \Big|\int_{\mathbb{R}^3}\int_{\mathbb{R}} g^v(2^{l'}(x_3-y_3))\Delta_{l'}^v u^h(x_h,y_3)dy_3\cdot\nabla_h\Delta_j^v w\partial_3\Delta_j^v\Delta_l^v w dx\Big|\Big)dt\\
&\quad\lesssim \sum_{\substack{|j-l'|\le 5\\ |j-l|\le 5}} 2^{l-l'}\|\Delta_{l'}^v u^h\|_{L_T^4(L_h^4(L_v^\infty))}\|\nabla_h\Delta_j^v w\|_{L_T^2(L^2(\mathbb{R}^3))}\|\Delta_j^v w\|_{L_T^4(L_h^4(L_v^2))}\\
&\quad\lesssim d_j^2\nu_h^{-1}2^{-j}\|u^h\|_{B^{0,\frac12}(T)}\|w\|^2_{B^{0,\frac12}(T)}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&F_j^{4,v}(T)\\
&\quad\le \sum_{l\ge j-N_0}\int_0^T\Big(\Big|\int_{\mathbb{R}^3}\Delta_j^v\Big(\int_{\mathbb{R}} g^v(2^l(x_3-y_3))\Delta_l^v u^h(x_h,y_3)dy_3\cdot\nabla_h\partial_3 S_{l+2}^v w\Big)\Delta_j^v w dx\Big|\\
&\qquad+\Big|\int_{\mathbb{R}^3}\Delta_j^v\Big(\int_{\mathbb{R}} g^v(2^l(x_3-y_3))\Delta_l^v u^h(x_h,y_3)dy_3\partial_3 S_{l+2}^v w\Big)\cdot\nabla_h\Delta_j^v w dx\Big|\Big)dt\\
&\quad\lesssim \sum_{l\ge j-N_0}\|\Delta_l^v u^h\|_{L_T^4(L_h^4(L_v^2))}\|\nabla_h S_{l+2}^v w\|_{L_T^2(L_h^2(L_v^\infty))}\|\Delta_j^v w\|_{L_T^4(L_h^4(L_v^2))}\\
&\qquad+\sum_{l\ge j-N_0}\|\Delta_l^v u^h\|_{L_T^4(L_h^4(L_v^2))}\|S_{l+2}^v w\|_{L_T^4(L_h^4(L_v^\infty))}\|\nabla_h\Delta_j^v w\|_{L_T^2(L_h^2(L_v^2))}\\
&\quad\lesssim d_j^2\nu_h^{-1}2^{-j}\|u^h\|_{B^{0,\frac12}(T)}\|w\|^2_{B^{0,\frac12}(T)}.
\end{aligned}
$$

This completes the proof of Proposition 3.1. □

Proof of Proposition 3.2. We distinguish the terms with horizontal derivatives from the terms with vertical ones, writing

$$G_j(T)\le G_j^h(T)+2G_j^{v1}(T)+G_j^{v2}(T),$$

where

$$
\begin{aligned}
G_j^h(T) &:= \sum_{l=1}^2\sum_{k=1}^2\int_0^T\Big|\int_{\mathbb{R}^3}\Delta_j^v(-\Delta)^{-1}\partial_l\partial_k(u^l u^k)\Delta_j^v\partial_h w dx\Big|dt,\\
G_j^{v1}(T) &:= \sum_{k=1}^2\int_0^T\Big|\int_{\mathbb{R}^3}\Delta_j^v(-\Delta)^{-1}\partial_3\partial_k(u^3u^k)\Delta_j^v\partial_h w dx\Big|dt,
\end{aligned}
$$

and

$$G_j^{v2}(T):=\int_0^T\Big|\int_{\mathbb{R}^3}\Delta_j^v(-\Delta)^{-1}\partial_3(2u^3\partial_3u^3)\Delta_j^v\partial_h w dx\Big|dt.$$

Using Hölder's inequality and Lemma 3.3, we get

$$\begin{aligned}
G_j^h(T) &\lesssim \sum_{l=1}^{2}\sum_{k=1}^{2}\int_0^T \|\Delta_j^v(u^l u^k)\|_{L^2}\|\Delta_j^v\partial_h w\|_{L^2}dt \\
&\lesssim \int_0^T d_j 2^{-\frac{j}{2}}\|u^h\|_{B^{0,\frac12}}\|\nabla_h u^h\|_{B^{0,\frac12}}\|\Delta_j^v\partial_h w\|_{L^2}dt \\
&\lesssim d_j^2\nu_h^{-1}2^{-j}\|u^h\|^2_{B^{0,\frac12}(T)}\|w\|_{B^{0,\frac12}(T)}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
G_j^{v1}(T) &\lesssim \int_0^T d_j 2^{-\frac{j}{2}}\|u^h\|^{\frac12}_{B^{0,\frac12}}\|\nabla_h u^h\|^{\frac12}_{B^{0,\frac12}}\|u^3\|^{\frac12}_{B^{0,\frac12}}\|\nabla_h u^3\|^{\frac12}_{B^{0,\frac12}}\|\Delta_j^v\partial_h w\|_{L^2}dt \\
&\lesssim d_j^2 2^{-j}\int_0^T \|u^h\|^{\frac12}_{B^{0,\frac12}}\|\nabla_h u^h\|^{\frac12}_{B^{0,\frac12}}\|u^3\|^{\frac12}_{B^{0,\frac12}}\|\nabla_h u^3\|^{\frac12}_{B^{0,\frac12}}\|\nabla_h w\|_{B^{0,\frac12}}dt.
\end{aligned}$$

Since $\mathrm{div} u = 0$, we obtain

$$\begin{aligned}
G_j^{v2}(T) &= \int_0^T \left|\int_{\mathbb{R}^3}\Delta_j^v(-\Delta)^{-1}\partial_3(2u^3\mathrm{div}_h u^h)\Delta_j^v\partial_h w dx\right|dt \\
&= \int_0^T \left|\int_{\mathbb{R}^3}\Delta_j^v(2u^3\mathrm{div}_h u^h)\Delta_j^v(-\Delta)^{-1}\partial_3\partial_h w dx\right|dt.
\end{aligned}$$

Then, using Hölder's inequality, Minkowski's inequality, Lemmas 2.3 and 3.2, we get

$$\begin{aligned}
G_j^{v2}(T) &\lesssim \int_0^T \|\Delta_j^v(u^3\mathrm{div}_h u^h)\|_{L_h^{\frac43}(L_v^2)}\|\Delta_j^v(-\Delta)^{-1}\partial_h\partial_3 w\|_{L_h^4(L_v^2)}dt \\
&\lesssim d_j 2^{-\frac{j}{2}}\int_0^T \|\nabla_h u^h\|_{B^{0,\frac12}}\|u^3\|^{\frac12}_{B^{0,\frac12}}\|\nabla_h u^3\|^{\frac12}_{B^{0,\frac12}} \\
&\quad\times\|\Delta_j^v(-\Delta)^{-1}\partial_h\partial_3 w\|^{\frac12}_{L^2}\|\Delta_j^v\nabla_h(-\Delta)^{-1}\partial_h\partial_3 w\|^{\frac12}_{L^2}dt \\
&\lesssim d_j^2 2^{-j}\int_0^T \|\nabla_h u^h\|_{B^{0,\frac12}}\|u^3\|^{\frac12}_{B^{0,\frac12}}\|\nabla_h u^3\|^{\frac12}_{B^{0,\frac12}}\|w\|^{\frac12}_{B^{0,\frac12}}\|\nabla_h w\|^{\frac12}_{B^{0,\frac12}}dt.
\end{aligned}$$

This completes the proof of Proposition 3.2. □

Using a similar argument as that in [12], one can easily obtain the uniqueness of the solution u. Thus, we finish the proof of Theorem 1.1.

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