Commun. Math. Phys. 286, 137–161 (2009)
Digital Object Identifier (DOI) 10.1007/s00220-008-0626-y

Communications in
Mathematical Physics

SNA's in the Quasi-Periodic Quadratic Family

Kristian Bjerklöv

Department of Mathematics, Royal Institute of Technology, 100 44 Stockholm, Sweden.
E-mail: bjerklov@kth.se

Received: 10 December 2007 / Accepted: 13 June 2008
Published online: 9 September 2008 –

Abstract: We rigorously show that there can exist Strange Nonchaotic Attractors (SNA) in the quasi-periodically forced quadratic (or logistic) map

$$(\theta, x) \mapsto (\theta + \omega, c(\theta)x(1 - x))$$

for certain choices of $c : \mathbb{T} \to [3/2, 4]$ and Diophantine ω.

1. Introduction

1.1. Background. Strange Nonchaotic Attractors (SNA) are certain attracting sets with a complicated geometry, but with rather simple dynamics, which have shown to appear in quasi-periodically forced maps. For the present discussion, let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the circle, and let X be $\mathbb{T}$ or a finite or infinite interval in $\mathbb{R}$. Consider a continuous mapping $\Phi : \mathbb{T} \times X \circlearrowleft$ of the form

$$(\theta, x) \mapsto (\theta + \omega, g_\theta(x)),$$

where ω is an irrational number. The graph of a non-continuous measurable function $\psi : \mathbb{T} \to X$ is called an SNA if it is invariant under Φ, i.e., $\psi(\theta + \omega) = g_\theta(\psi(\theta))$ for a.e. θ, and if the vertical Lyapunov exponent is negative on the graph of ψ. See [17] for an extensive discussion on invariant graphs.

The notion of SNA was first introduced in [8]. The phenomenon of a "strange" invariant attracting set had been observed in the projective dynamics induced by certain quasi-periodically forced $SL(2, \mathbb{R})$ cocycles [9,12,15,19] (in this case $X = \mathbb{T}$). When the cocycle is non-uniformly hyperbolic, it follows that the projectivization of the Oseledets directions must be highly discontinuous. We refer the reader to the excellent paper [13] for a detailed discussion, and also to [5–7] where we study finer properties of the SNA's appearing in the projective Schrödinger cocycle.

So far there are very few rigorous results concerning the existence of SNA's outside the class of projectivizations of linear systems. We mention [4,10,11,14]. In the papers [4,14], a class of systems introduced in [8] are considered. These examples are all so-called pinched, that is, there exists collapsed fibers in the sense that there are values of θ such that $g_\theta(x)$ is constant for all $x \in X$. In the classes considered in [10,11], the function $g_\theta(x)$ is assumed to be monotone in x for each θ.

It is therefore an interesting problem to see what happens if the system is neither pinched nor monotone (non-invertible). The model we shall consider is the quasi-periodically forced quadratic (also called the logistic) map on $\mathbb{T} \times [0, 1]$,

$$(\theta, x) \mapsto (\theta + \omega, c(\theta)x(1 - x)),$$

where $c(\theta) \in (0, 4]$. The dynamics of the one-dimensional map $x \mapsto cx(1 - x)$ $(0 < c \leq 4)$ is by now well understood (see e.g. [2,3]), so it is rather natural to consider quasi-periodic perturbations of such maps. There are several numerical papers investigating the quasi-periodically forced logistic map with fascinating results (e.g. [1,16]).

In this context we also mention the systems rigorously studied in [19]. There the base dynamics is an expanding map of the form $\theta \mapsto k\theta$ where $k > 0$ is big.

1.2. Our model. The model which we shall investigate is the following one-parameter family of a quasi-periodically forced system,

$$\Phi_\alpha : \mathbb{T} \times [0, 1] \circlearrowleft: (\theta, x) \mapsto (\theta + \omega, c_\alpha(\theta)p(x)) \quad (\mathbb{T} = \mathbb{R}/\mathbb{Z}).$$

Here p is the quadratic map

$$p(x) = x(1 - x),$$

and $c_\alpha : \mathbb{T} \to (3/2, 4]$ is defined by

$$c_\alpha(\theta) = \frac{3}{2} + \frac{5}{2}\left(\frac{1}{1 + \lambda(\cos 2\pi(\theta - \alpha/2) - \cos \pi\alpha)^2}\right), \quad \lambda > 0.$$

We shall assume that ω satisfies the Diophantine condition

$$(DC)_{\kappa,\tau} \qquad \inf_{p\in\mathbb{Z}} |q\omega - p| > \frac{\kappa}{|q|^\tau} \quad \text{for all } q \in \mathbb{Z}\backslash\{0\},$$

for some constants $\kappa > 0$ and $\tau \geq 1$. Note that if $c \in [0, 4]$ and $x \in [0, 1]$, then $cx(1 - x) \in [0, 1]$, so Φ_α indeed maps $\mathbb{T} \times [0, 1]$ into $\mathbb{T} \times [0, 1]$.

Given a point $(\theta_0, x_0) \in \mathbb{T} \times [0, 1]$, we use the notation $(\theta_n, x_n) = \Phi^n(\theta_0, x_0)$. We define the vertical (or fiber) Lyapunov exponent at (θ_0, x_0) as

$$\gamma(\theta_0, x_0) = \lim_{n\to\infty} \frac{1}{n} \log \left|\frac{\partial x_n}{\partial x_0}\right| = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log |c(\theta_k)(1 - 2x_k)|,$$

whenever the limit exists. Moreover, we define

$$\bar{\gamma}(\theta_0, x_0) = \overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{k=0}^{n-1} \log |c(\theta_k)(1 - 2x_k)|.$$

The graph of a measurable function $\psi : \mathbb{T} \to [0, 1]$ is called invariant if

$$\psi(\theta + \omega) = c(\theta) p(\psi(\theta)) \quad \text{for a.e. } \theta \in \mathbb{T}.$$

Since $p(0) = 0$ we have that the graph of $\psi_0(\theta) \equiv 0$ is invariant. By Birkhoff's ergodic theorem

$$\gamma(\theta_0, 0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log |c(\theta_k)| = \int_{\mathbb{T}} \log c(\theta) d\theta > \log(3/2) > 0 \quad \text{for all } \theta_0 \in \mathbb{T}.$$

Thus, the graph of ψ_0 is repelling. Our main theorem states that for $\lambda \gg 0$ there is a particular value of α such that there is one more invariant curve, ψ, which is highly discontinuous. This curve attracts almost all points in $\mathbb{T} \times [0, 1]$. See Fig. 2 to get an idea of what ψ can look like.

Main Theorem. *Assume that ω satisfies the Diophantine condition $(DC)_{\kappa,\tau}$ for some $\kappa > 0, \tau \geq 1$. Then for all sufficiently large $\lambda > 0$ there is a parameter value α such that the following holds for the map Φ_α.*

i) $\bar{\gamma}(\theta, x) < \frac{1}{2} \log(3/5) < 0$ *for a.e $\theta \in \mathbb{T}$ and all $x \in (0, 1)$.*
ii) $|x_n - y_n| \to 0$ *exponentially fast as $n \to \infty$ for a.e. $\theta_0 \in \mathbb{T}$ and all $x_0, y_0 \in (0, 1)$.*
iii) *For a.e $\theta_0 \in \mathbb{T}$ and all $x_0 \in (0, 1)$ there holds $x_n > 0$ for all $n \geq 0$ and* $\inf_{n\geq 0} x_n = 0$.
iv) *There exists a measurable function $\psi : \mathbb{T} \to (0, 1)$ with an invariant graph, i.e.,*

$$\psi(\theta + \omega) = \pi_x(\Phi(\theta, \psi(\theta))) \quad a.e.\ \theta \in \mathbb{T} \quad (\pi_x(\theta, x) = x).$$

Condition iii) *especially applies to $\psi(\theta)$, that is,* $\inf_{\theta\in\mathbb{T}} \psi(\theta) = 0$. *Since the line $x = 0$ is fixed, this implies*

v) *The set $\{\theta \in \mathbb{T} : \psi(\theta) < \varepsilon\}$ is dense in $\mathbb{T}$ for all $\varepsilon > 0$.*
In particular, ψ cannot be continuous. Moreover, by combining ii) *and* iv) *we get*
vi) $|x_n - \psi(\theta_n)| \to 0$ *exponentially fast as $n \to \infty$ for a.e. $\theta_0 \in \mathbb{T}$ and all $x_0 \in (0, 1)$.*
Thus the graph of ψ attracts almost all points in $\mathbb{T} \times (0, 1)$. Note that $vi)$ immediately gives
vii) $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} u(\Phi^n(\theta_0, x_0)) = \int_{\mathbb{T}} u(\theta, \psi(\theta)) d\theta$ *for all functions* $u \in C(\mathbb{T} \times [0, 1], \mathbb{R})$,
a.e. $\theta_0 \in \mathbb{T}$ and all $x_0 \in (0, 1)$.

In other words, the Lebesgue (*or Haar*) *measure on $\mathbb{T}$ lifted to the graph of ψ is a physical measure.*

Remark 1. It is not important that $c(\theta)$ has exactly the above form. What is needed is that c is of class C^2, has two sharp peaks, one at 0 (for simplicity) and one at α (which can be varied), and "outside" the peaks c should be close to a value $a \approx 3/2$. For such a, the map $x \mapsto ax(1 - x)$ has an attracting fixed point $x_f = (a - 1)/a$ and the repelling fixed point $x = 0$. From the peaks we need that $c(0)x_f(1 - x_f) > 1/2$ and that $c(\alpha) = 4$ (so that $c(\alpha)(1/2)(1 - 1/2) = 1$).

The proof of the main theorem is a bit technical, but the philosophy is as follows. For very large λ, the coefficient $c(\theta)$ is close to $3/2$ outside two small intervals of θ; one centered at 0 and one centered at α (see Fig. 1). α should be thought of as being very close to ω. The unperturbed one-dimensional map $x \mapsto (3/2)x(1 - x)$ has an attracting fixed point $x = 1/3$ which attracts $(0, 1)$; the fixed point $x = 0$ is repelling. The idea

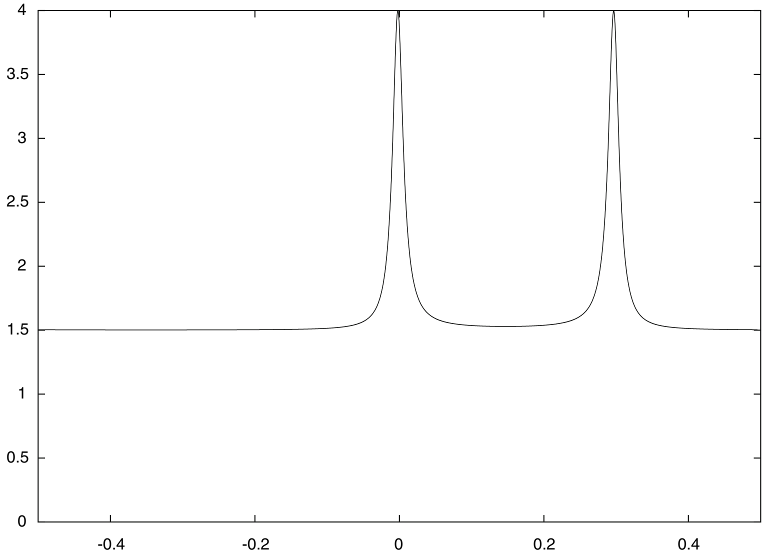

Fig. 1. A picture of $c(\theta)$; there is one peak at $\theta = 0$ and one at $\theta = \alpha$

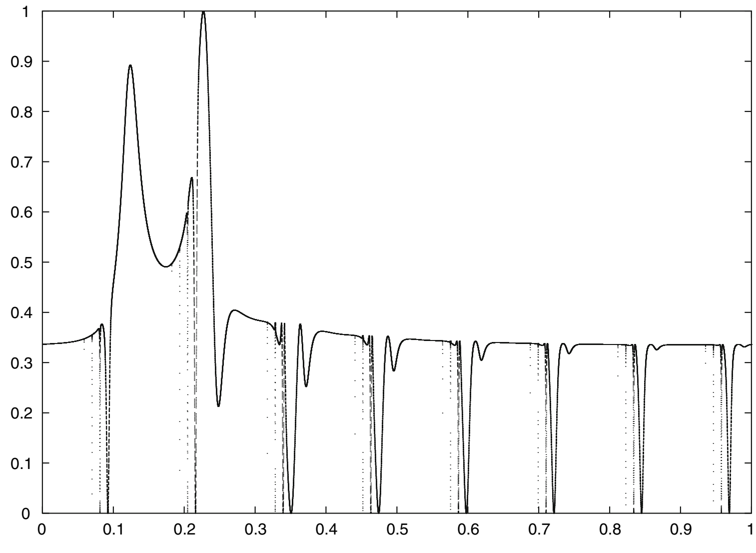

Fig. 2. An approximation of the attracting graph ψ when $\lambda = 1000, \omega = (\sqrt{5}-1)/10$ and $\alpha = \omega - 0.02047359$

is that the function $\psi = \psi(\theta)$, which we want to construct, shall be close to $1/3$ for "most" θ, but near $\theta = 0$ ($\alpha - \omega \approx 0$) there should be points on the graph of ψ which are mapped by $\pi_x(\Phi_\alpha)$ arbitrarily close to $1/2$. These points in turn should be mapped by $\pi_x(\Phi_\alpha)$ arbitrarily close to 1. In this step it is crucial that $c(\alpha) = 4$; it is only for $c = 4$ when $\max_{x \in [0,1]} cx(1-x) = 1$. Since 1 is mapped to 0, which is a repelling fixed point, we get the "strange" looking curve. This is the reason why we need two peaks on $c(\theta)$; we use the chain $1/3 \mapsto 1/2 \mapsto 1$. To get this kind of resonance phenomena we need to "fine tune" the parameter α. In Fig. 2 we can see the two peaks; one located close to ω and the second, which "touches 1", close to 2ω.

The rest of the paper is organized as follows. In Sect. 2 we derive some elementary estimates on iterations of the one-dimensional map $x \mapsto cx(1-x)$ on [0, 1]. In Sect. 3 we combine these estimates with the properties of the function $c(\theta)$ to get some general estimates on iterations of Φ_α. Section 4 contains the inductive machinery on which the construction of the proof hinges. The proof is of multi-scale type and the techniques used are similar to the ones we use in [5–7]. The methods are close in spirit to the ones used in the seminal work by Benedicks and Carleson [3]. Finally, in Sect. 5 we put everything together and derive the statements in Main Theorem.

2. Some Numerical Lemmas

This section contains certain numerical estimates for iterations of quadratic maps of the form $x \mapsto ax(1-x)$. These estimates, which are all elementary, will be used frequently in the rest of the paper.

Lemma 2.1. *Let* $P(x) = (3/2+\varepsilon)x(1-x)$. *If* $|\varepsilon|$ *is sufficiently small, then* $P(C) \subset C$, *where* C *is the interval* $[1/3 - 1/100, 1/3 + 1/100]$. *Moreover,* $0 < P'(x) < 3/5$ *holds for all* $x \in C$.

Proof. For the unperturbed map $q(x) = \frac{3}{2}x(1-x)$ we have $q(C) = [1/3 - 103/20000, 1/3 + 97/20000]$. From this the first statement follows. The second statement follows since $q'(x) = 3/2 - 3x$, and $q'(1/3 - 1/100) = 53/100$. □

Lemma 2.2. *Let* P *be as in the previous lemma. If* $1/100 \leq x \leq 99/100$, *then* $1/100 < P(x) < 2/5$, *under the condition that* $|\varepsilon|$ *is small.*

Proof. An easy computation. □

Lemma 2.3. *Assume that* $|\varepsilon_1|, |\varepsilon_2|, \ldots, |\varepsilon_{20}| < \varepsilon$. *Let* $P_i(x) = (3/2 + \varepsilon_i)x(1-x)$ $(i = 1, \ldots, 20)$. *Then* $P_{20} \circ P_{19} \circ \cdots \circ P_1(x) \in (1/3 - 1/100, 1/3 + 1/100)$ *for all* $x \in [1/100, 99/100]$, *provided that* ε *is small.*

Proof. A numerical computation shows that $q(x) = \frac{3}{2}x(1-x)$ satisfies $q^{20}([1/100, 99/100]) \subset (1/3 - 1/100, 1/3 + 1/100)$. □

Lemma 2.4. *If* $P(x) = ax(1-x)$ $(a \geq 3/2)$, *then* $P(x) \geq \frac{5}{4}x$ *for all* $x \in [0, 1/10]$.

Proof. Let $q(x) = \frac{3}{2}x(1-x)$. Then $q(x) - \frac{5}{4}x = \frac{x}{2}(\frac{1}{2} - 3x) \geq 0$. Since clearly $P(x) \geq q(x)$ for all $x \in [0, 1]$, the statement follows. □

We close this section with a return-time estimate for Diophantine rotation.

Lemma 2.5. *Assume that* $\omega \in \mathbb{T}$ *satisfies the Diophantine condition* $(DC)_{\kappa,\tau}$ *for some* $\kappa > 0$, $\tau \geq 1$. *If* $I \subset \mathbb{T}$ *is an interval of length* $\varepsilon > 0$, *then*

$$I \cap \bigcup_{0<|m|\leq N} (I + m\omega) = \emptyset$$

with $N = [(\kappa/\varepsilon)^{1/\tau}]$.

Remark 2. Here, and in the rest of the paper, $[\cdot]$ denotes the integer part of a real number.

Proof. The proof is standard. □

3. Definitions and Formulas

3.1. Definitions and notations. We assume from now on that $\omega \in (0, 1)$ is fixed and satisfies the Diophantine condition $(DC)_{\kappa,\tau}$ for some $\kappa > 0$, $\tau \geq 1$.

Let $\Phi_\alpha : \mathbb{T} \times [0, 1] \circlearrowleft$ be given by

$$\Phi_\alpha(\theta, x) = (\theta + \omega, c_\alpha(\theta)p(x)),$$

where

$$p(x) = x(1 - x)$$

and

$$c_\alpha(\theta) = c(\theta, \alpha) = \frac{3}{2} + \frac{5}{2}\left(\frac{1}{1 + \lambda(\cos 2\pi(\theta - \alpha/2) - \cos \pi\alpha)^2}\right).$$

Recall that c has two sharp peaks, one at $\theta = 0$ and one at $\theta = \alpha$. By taking λ large the peaks get sharper. See Fig. 1. Note that p has its maximum at $x = 1/2$.

The number α will act as a parameter and will be very close to ω. We will "fine tune" α in order to get an SNA. In the rest of the paper c and p will always be defined as above. We also stress that in this paper, λ should be thought of as being "extremely" large.

Given (θ_0, x_0) we use the notation

$$(\theta_n, x_n) = \Phi^n(\theta_0, x_0), \quad n \geq 0.$$

We define the contracting region C by

$$C = [1/3 - 1/100, 1/3 + 1/100].$$

Moreover, we let

$$\begin{aligned} I_0 &= [-\lambda^{-1/7}, \lambda^{1/7}]; \\ I_0' &= [-\lambda^{-2/5}, -\lambda^{-2/3}]; \quad \text{and} \\ \mathcal{A}_0 &= [\omega - \lambda^{-2/5}/2, \omega - 2\lambda^{-2/3}]. \end{aligned}$$

Note that $\mathcal{A}_0 \subset I_0' + \omega$. The interval I_0 contains "most" of the support of c's peak at $\theta = 0$, and on I_0' the θ-derivative of c is large. Moreover, $\mathcal{A}_0$ is our first approximation of the parameter α, that is, the α we are looking for will be in $\mathcal{A}_0$.

We define

$$M_0 = [\lambda^{1/(14\tau)}] \quad \text{and} \quad K_0 = [\lambda^{1/(28\tau)}]. \tag{3.1}$$

Note that $M_0 \approx \sqrt{N}$ and $K_0 \approx N^{1/4}$, where N is the integer in Lemma 2.5 when applied to I_0, that is, to an interval of length $2\lambda^{-1/7}$. We again stress that λ should be thought of as "extremely" large, so M_0, K_0 are big integers.

Given a set $I \subset \mathbb{T}$ and a point $\theta_0 \subset \mathbb{T}$, we denote by $N(\theta_0; I)$ the smallest non-negative integer N such that $\theta_N = \theta_0 + N\omega \in I$.

3.2. Basic lemmas. Our first lemma contains some elementary estimates on the function $c(\theta, \alpha)$. More precisely, from the definition of I_0, I_0' and $\mathcal{A}_0$ we get

Lemma 3.1. *For all large $\lambda > 0$ (depending on ω) the following holds for $\alpha \in \mathcal{A}_0$:*

a) $|c(\theta, \alpha) - 3/2|, |\partial_\theta c(\theta, \alpha)|, |\partial_\alpha c(\theta, \alpha)| < 1/\sqrt{\lambda}$ *for all* $\theta \notin I_0 \cup (I_0 + \omega)$.
b) $c(-\lambda^{-2/5}/2, \alpha) < 2$ *and* $c(-2\lambda^{-2/3}, \alpha) > 3$.
c) $\lambda^{1/6} < \partial_\theta c(\theta, \alpha) < \lambda$ *for all* $\theta \in I_0'$.
d) $|\partial_\alpha c(\theta, \alpha)| < const(\omega)$ *for all* $\theta \in I_0'$.
e) *For any* $1/2 < \delta < 1$, $\{\theta : c(\theta) \geq 4(1-\delta)\} \cap (I_0 + \omega) \subset [\alpha - \sqrt{\delta}\lambda^{-1/4}, \alpha + \sqrt{\delta}\lambda^{-1/4}]$ *holds.*

Proof. Assume that λ is sufficiently large, depending on ω. The function $g(\theta, \alpha) = \cos 2\pi(\theta - \alpha/2) - \cos \pi\alpha$ has exactly two zeroes in $[0, 1]$, namely $\theta = 0$ and $\theta = \alpha$. We have

$$\begin{aligned} g(\theta, \alpha) &= (2\pi \sin \pi\alpha)\theta + O(\theta^2) \text{ as } \theta \to 0 \text{ and} \\ g(\theta, \alpha) &= (-2\pi \sin \pi\alpha)(\theta - \alpha) + O((\theta - \alpha)^2) \text{ as } \theta \to \alpha. \end{aligned} \tag{3.2}$$

The number ω is irrational, so we must have $\sin \pi\alpha \neq 0$ and $\cos \pi\alpha \neq 0$ for all $\alpha \in \mathcal{A}_0$ provided that $\lambda \gg 1$. Since

$$c(\theta, \alpha) = \frac{3}{2} + \frac{5}{2}\left(\frac{1}{1 + \lambda g(\theta, \alpha)^2}\right),$$

this immediately gives *b*). Moreover, since also $|\alpha - \omega| < \lambda^{-2/5}$ for all $\alpha \in \mathcal{A}_0$, and $|I_0| = 2\lambda^{-1/7}$, we get

$$g^{-1}(I_0 \cup (I_0 + \omega), \alpha) \subset [-b\lambda^{-1/7}, b\lambda^{1/7}] \quad \alpha \in \mathcal{A}_0$$

for some constant $b > 0$ which only depends on ω. From this the first part of *a*) follows. Furthermore, differentiation yields

$$\begin{aligned} |\partial_\theta c(\theta, \alpha)| = \left| \frac{5\lambda g(\theta, \alpha)\partial_\theta g(\theta, \alpha)}{(1 + \lambda g(\theta, \alpha)^2)^2} \right| &< const. \frac{1}{\lambda^2 g(\theta, \alpha)^3} \\ &< \lambda^{-1/2} \quad \text{for } \theta \notin I_0 \cup (I_0 + \omega). \end{aligned}$$

Similarly for $|\partial_\alpha c|$. This gives the second part of *a*). Using (3.2) we obtain

$$c(\theta, \alpha) = \frac{3}{2} + \frac{5}{2}\left(1 - 4\pi^2 \sin^2(\pi\alpha)\lambda\theta^2 + \lambda O(\theta^4)\right) \quad \text{as } \theta \to 0.$$

Differentiating this w.r.t θ and α gives *c*) and *d*); the upper bound $\partial_\theta g < \lambda$ is trivial. Finally,

$$\begin{aligned} c(\theta) = \frac{3}{2} + \frac{5}{2} f(\theta) = \frac{3}{2} + \frac{5/2}{1 + \lambda g(\theta)^2} &= 4 - 10\pi^2\lambda \sin^2(\pi\alpha)(\theta - \alpha)^2 \\ &+ \lambda O((\theta - \alpha)^3) \end{aligned}$$

as $\theta \to \alpha$. From this *e*) follows. □

If we now combine the results in Sect. 2 with fact that $0 < c(\theta, \alpha) - 3/2 < 1/\sqrt{\lambda}$ for $\theta \notin I_0 \cup (I_0 + \omega)$ and $\alpha \in \mathcal{A}_0$, provided that λ is sufficiently large, then we get

Lemma 3.2. *For all large* $\lambda > 0$ *and* $\alpha \in \mathcal{A}_0$, *we have*

a) *If* $\theta_0 \notin I_0 \cup (I_0 + \omega)$ *and* $x_0 \in C$, *then* $x_1 \in C$, *and* $|c(\theta_0)p'(x_0)| < 3/5$.
b) *If* $\theta_0, \theta_1, \ldots, \theta_{19} \notin I_0 \cup (I_0 + \omega)$ *and* $x_0 \in [1/100, 99/100]$, *then* $x_{20} \in C$.
c) *If* $\theta_0 \notin I_0 \cup (I_0 + \omega)$ *and* $x_0 \in [1/100, 99/100]$, *then* $x_1 \in (1/100, 2/5)$.
d) *If* $x_0 \in [0, 1/10]$, *then* $x_1 \geq (5/4)x_0$ *for all* $\theta_0 \in \mathbb{T}$.

The next two lemmas will be used later to control how close to $x = 0$ the iterates x_n come.

Lemma 3.3. *If* $\theta_0 \in \mathbb{T}$, $x_0 \geq 1/100$ *and if* $x_{-1} \in (0, 1/100) \cup (99/100, 1)$, *then* $x_2 \in [1/100, 99/100]$.

Proof. This is an easy verification. Recall that p is growing on $[0, 1/2)$ and that $p(1 - x) = p(x)$ (so $p((0, 1/100)) = p((99/100, 1))$):

$$\begin{aligned} x_0 = c(\theta_{-1})p(x_{-1}) &< 4p(1/100) < 1/25,\\ 1/100 < \frac{3}{2}p(1/100) < x_1 = c(\theta_0)p(x_0) &< 4p(1/25) < 4/25,\\ 1/100 < x_2 = c(\theta_1)p(x_1) &< 4p(4/25) < 16/25 < 99/100. \end{aligned}$$

□

Lemma 3.4. *For all large* $\lambda > 0$ *we have the following. Fix* $\alpha \in \mathcal{A}_0$, *let* $M > 100$ *be any integer, and let*

$$J = \{\theta : c(\theta) \geq 4(1 - (4/5)^M)\} \cap (I_0 + \omega).$$

If $\theta_0 \in (I_0 - \omega)\setminus(J - 2\omega)$ *and* $x_0 \in [1/100, 99/100]$, *then there is a* k, $3 \leq k \leq M - 7$, *such that* $x_k \in [1/100, 99/100]$. *Moreover, if* $\theta_0 \in (I_0 + \omega)\setminus J$ *and* $x_0 \in [1/100, 99/100]$, *then there is a* k, $1 \leq k \leq M - 7$, *such that* $x_k \in [1/100, 99/100]$.

Proof. Assume first that $\theta_0 \in (I_0 - \omega)\setminus(J - 2\omega)$ and $x_0 \in [1/100, 99/100]$. For large $\lambda > 0$ we have that $(I_0 - \omega) \subset \mathbb{T}\setminus(I_0 \cup (I_0 + \omega))$. Thus, by Lemma 3.2, $1/100 < x_1 < 2/5$, and therefore $1/100 < x_2 < 4p(2/5) = 24/25 < 99/100$. Since $\theta_2 \in (I_0 + \omega)\setminus J$, we have $c(\theta_2)p(1/2) < 1 - (4/5)^M$. Consequently $1/100 < x_3 = c(\theta_2)p(x_2) < 1 - (4/5)^M$. If $x_3 \leq 99/100$ we are done. Assume now that $99/100 < x_3 < 1 - (4/5)^M$. Since p has the property that $p(x) = p(1 - x)$ for all x, we get the same orbit $x_4, x_5, x_6, \ldots$ if we use $y_3 = 1 - x_3$ instead of x_3. Note that $(4/5)^M < y_3 < 1/100$. If $x_4, x_5, \ldots, x_{M-8} < 1/100$, it follows by repeated use of Lemma 2.4 that $x_{M-7} \geq (5/4)^{M-10}(4/5)^M = (4/5)^{10} > 1/100$. To get the upper bound, we do as in the proof of the previous lemma, i.e., we use the fact that if $x_k < 1/100$ and $x_{k+1} > 1/100$, then $x_{k+1} < 1/25$.

The proof of the second statement is included in the one above. □

3.3. Formulas. We shall now derive some formulas which will be needed to control the geometry in the inductive construction in the next section.

We begin with an easy formula. Assume that (a_k) and (b_k) are sequences of real numbers and that (x_n) is defined inductively by $x_{n+1} = a_n + b_n x_n$. Given x_0, we get

$$x_{n+1} = a_n + \sum_{k=1}^{n} a_{k-1} b_n \cdots b_k + b_n \cdots b_0 x_0, \quad n \geq 0.$$

Now, assume that $x_0 = x_0(\theta, \alpha)$ is given, and that x_n is defined by $(\theta + n\omega, x_n) = \Phi^n(\theta, x_0)$. Then $x_{n+1} = c(\theta + n\omega, \alpha)p(x_n)$. Differentiating this with respect to θ and α, respectively, yields

$$\partial_\theta x_{n+1} = (\partial_\theta c_n)p(x_n) + c_n p'(x_n)\partial_\theta x_n;$$
$$\partial_\alpha x_{n+1} = (\partial_\alpha c_n)p(x_n) + c_n p'(x_n)\partial_\alpha x_n.$$

Here we use the notation $c_n = c(\theta + n\omega, \alpha)$. Applying the above formula now gives us

$$\partial_\theta x_{n+1} = (\partial_\theta c_n)p(x_n) + \partial_\theta x_0 \prod_{j=0}^{n} c_j p'(x_j) + \sum_{k=1}^{n} \left((\partial_\theta c_{k-1})p(x_{k-1}) \prod_{j=k}^{n} c_j p'(x_j) \right) \tag{3.3}$$

and

$$\partial_\alpha x_{n+1} = (\partial_\alpha c_n)p(x_n) + \partial_\alpha x_0 \prod_{j=0}^{n} c_j p'(x_j) + \sum_{k=1}^{n} \left((\partial_\alpha c_{k-1})p(x_{k-1}) \prod_{j=k}^{n} c_j p'(x_j) \right). \tag{3.4}$$

Lemma 3.5. *Assume that $x_0 \in [0, 1]$, $\partial_\alpha x_0 = \partial_\theta x_0 = 0$ and $\prod_{j=k}^{T} |c_j p'(x_j)| < (3/5)^{(T-k+1)/2}$ for all $k \in [0, T]$, where $T > 10 \log \lambda$ is an integer. Assume moreover that $|\partial_\alpha c_k|, |\partial_\theta c_k| < 1/\sqrt{\lambda}$ for $k \in [T - 10 \log \lambda, T]$. Then $|\partial_\alpha x_{T+1}|, |\partial_\theta x_{T+1}| < \lambda^{-1/4}$, provided that λ is bigger than a numerical constant.*

Proof. Using the given estimates, together with the fact that $|\partial_\theta c(\theta, \alpha)|, |\partial_\alpha c(\theta, \alpha)| < \lambda$ (by an easy estimate) and $0 \le p(x_i) \le p(1/2) = 1/4$, the above formulas give

$$|\partial_\alpha x_{T+1}|, |\partial_\theta x_{T+1}| < \frac{1}{4\sqrt{\lambda}} + \frac{1}{4\sqrt{\lambda}} \sum_{k=T-10\log\lambda}^{T} (3/5)^{(T-k+1)/2} + \frac{\lambda}{4} \sum_{k=1}^{T-10\log\lambda} (3/5)^{(n-k+1)/2}.$$

Since, $(3/5)^4 < 1/e$, we have

$$\sum_{k=1}^{T-10\log\lambda} (3/5)^{(T-k+1)/2} < \sum_{j=10\log\lambda}^{\infty} (\sqrt{3/5})^j = \frac{\sqrt{3/5}^{10\log\lambda}}{1 - \sqrt{3/5}} < 1/\lambda^2.$$

From this the statement of the lemma follows. □

4. The Induction

In this section we present the inductive construction on which the proof of Theorem 1 hinges. Inductively we will obtain fine estimates on longer and longer orbits of "many" initial points.

4.1. Basic step. We begin with the basic step. It follows more or less straightforwardly from the definitions and results in the previous two sections. We recall that $I_0' \subset I_0$ and $\mathcal{A}_0 \subset I_0' + \omega$; on I_0' the θ-derivative of $c(\theta)$ is $\gg 1$.

Lemma 4.1. *There is a $\lambda_1 > 0$ such that for all $\lambda > \lambda_1$, the following holds:*

(i) *If $\alpha \in \mathcal{A}_0$, $\theta_0 \notin I_0 \cup (I_0 + \omega)$ and $x_0, y_0 \in C$, then, letting $N = N(\theta_0; I_0)$,*

$$\prod_{i=k}^{N-1} |c(\theta_i)p'(x_i)| < (3/5)^{N-k} \quad \text{for all } k \in [0, N-1]; \tag{4.1}$$

$$x_k \in C \quad \text{for all } k \in [0, N]; \quad \text{and} \tag{4.2}$$

$$|x_k - y_k| \leq (3/5)^k |x_0 - y_0|, \quad \text{for all } k \in [1, N]. \tag{4.3}$$

(ii)$_0$ *If Γ is a horizontal line segment $\Gamma = (I_0 - M_0\omega) \times \{x\}$, where $x \in C$, then*

$$\Phi_\alpha^{M_0+1}(\Gamma) = \{(\theta, \varphi(\theta, \alpha)) : \theta \in I_0 + \omega\} \quad (\alpha \in \mathcal{A}_0),$$

where the function $\varphi : (I_0 + \omega) \times \mathcal{A}_0 \to \mathbb{R}$ satisfies

$$\begin{aligned} &3/10 < \varphi^{\pm}(\theta, \alpha) < 99/100; \\ &\quad |\partial_\alpha \varphi(\theta, \alpha)| < const(\omega) \quad \text{and} \quad \lambda^{1/7} < \partial_\theta \varphi(\theta, \alpha) < \lambda \\ &\quad\quad \text{for all } \theta \in I_0' + \omega, \alpha \in \mathcal{A}_0. \end{aligned}$$

Moreover, there is an $\alpha \in \mathcal{A}_0$ such that

$$\varphi(\alpha, \alpha) = 1/2.$$

(iii)$_0$ *If $\alpha \in \mathcal{A}_0$, $1/100 \leq x_0 \leq 99/100$ and $\theta_0 \notin I_0 \cup (I_0 + \omega)$, then*

$$1/100 \leq x_k \leq 99/100 \quad \text{for all } k \in [0, N].$$

Before we prove the above lemma, we comment a bit on the statement. Conditions $(i)_0$ and $(iii)_0$ just state that we have good control on iteration outside $I_0 \cup (I_0 + \omega)$ for $\alpha \in \mathcal{A}_0$. They follow directly from Lemma 3.2. Condition $(ii)_0$ gives a first approximation of the function Ψ which we want to construct. We iterate the line segment Γ under the mapping Φ_α ($\alpha \in \mathcal{A}_0$). When it comes over $I_0 + \omega$ we have a good control on how it looks, see Fig. 3. The interesting part is the one over $I_0' + \omega$. The last statement says that there is a parameter value $\alpha \in \mathcal{A}_0 \subset I_0' + \omega$ such that $\Phi_\alpha^{M_0+1}(\Gamma)$ contains the point $(\alpha, 1/2)$. This point will be mapped to $(\alpha + \omega, 1)$ by Φ_α, and then to $(\alpha + 2\omega, 0)$. Recall that $x = 0$ is fixed. Inductively we will later (Proposition 4.2 below) get better and better approximations of Ψ, and make sure that we have this "collision" between Ψ and the point $(\alpha, 1/2)$ for a certain value of α.

Proof. We assume that λ is sufficiently large. First we verify $(i)_0$. Assume that $\theta_0 \notin I_0 \cup (I_0 + \omega)$ and $x_0, y_0 \in C$. Let $N = N(\theta_0; I_0) > 0$. Since $\theta_0, \theta_1, \ldots, \theta_{N-1} \notin I_0 \cup (I_0 + \omega)$, it follows directly by a repeated application of Lemma 3.2 that

$$x_k \in C \quad \text{for all } k \in [0, N], \quad \text{and} \quad |c(\theta_k)p'(x_k)| < 3/5 \quad \text{for all } k \in [0, N-1].$$

In particular this gives

$$\prod_{i=k}^{N-1} |c(\theta_i)p'(x_i)| < (3/5)^{N-k} \; (k \in [0, N-1]).$$

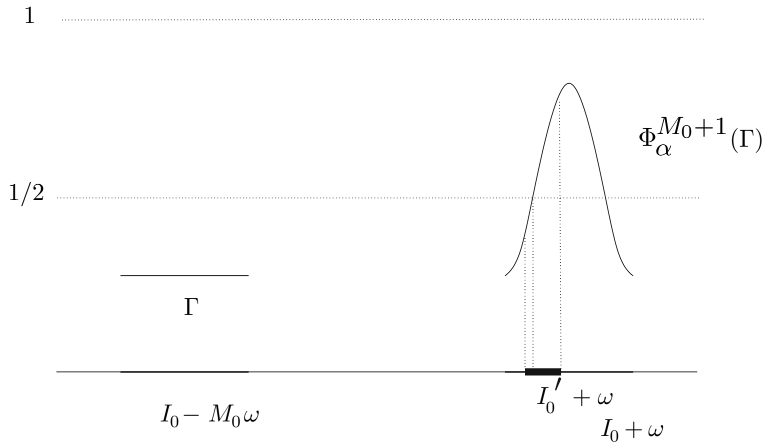

Fig. 3. A picture of condition $(ii)_0$

Furthermore, by the mean value theorem

$$|x_k - y_k| = \prod_{i=0}^{k-1} |c(\theta_i)p'(\xi_i)||x_0 - y_0| \ (k = 1, 2, \ldots, N),$$

where ξ_i is between x_i and y_i. Since $x_i, y_i \in C$ and $\theta_i \notin I_0 \cup (I_0+\omega)$ $(i = 0, 1, \ldots, N-1)$, it follows from Lemma 3.2 that $|c(\theta_i)p'(\xi_i)| < 3/5$. Thus,

$$|x_k - y_k| \leq (3/5)^k |x_0 - y_0| \quad \text{for all } k \in [1, N].$$

To obtain $(ii)_0$, take any $x \in C$ and let $x_0 = x_0(\theta, \alpha) = x$ for $\theta \in I_0 - M_0\omega$ and $\alpha \in \mathcal{A}_0$. From the definition of M_0 in (3.1), we have that $[I_0 \cup (I_0 + \omega)] \cap \bigcup_{m=1}^{M_0} (I_0 - m\omega) = \emptyset$ (recall the discussion below the definition of M_0). Thus it follows from $(4.2)_0$ in $(i)_0$ that $x_k \in C$ for $k = 0, 1, \ldots, M_0$. Furthermore, by Lemma 3.1 we have $|\partial_\theta c(\theta_k, \alpha)|, |\partial_\alpha c(\theta_k, \alpha)| < 1/\sqrt{\lambda}$ for $k = 0, 1, \ldots, M_0 - 1$ if $\theta_0 \in I_0 - M_0\omega$. Thus, using this fact and $(4.1)_0$ in $(i)_0$, and noticing that $M_0 \gg 10 \log \lambda$, it follows from Lemma 3.5 that

$$|\partial_\alpha x_{M_0}|, |\partial_\theta x_{M_0}| < \lambda^{-1/4}. \tag{4.4}$$

From the fact that $x_{M_0} \in C$ and $3/2 < c(\theta) \leq 4$ for all θ, we get

$$3/10 < \frac{3}{2} p(1/3 - 1/100) \leq x_{M_0+1} \leq 4p(1/3 + 1/100) < 99/100.$$

To continue, let

$$\psi(\theta, \alpha) = x_{M_0}(\theta - M_0\omega, \alpha), \quad \theta \in I_0, \alpha \in \mathcal{A}_0; \quad \text{and}$$
$$\varphi(\theta, \alpha) = x_{M_0+1}(\theta - (M_0 + 1)\omega, \alpha), \quad \theta \in I_0 + \omega, \alpha \in \mathcal{A}_0.$$

Then, by definition, ψ and φ are related like

$$\varphi(\theta, \alpha) = c(\theta - \omega, \alpha)p(\psi(\theta - \omega, \alpha)).$$

Differentiating this, we obtain, using the estimates in Lemma 3.1, together with (4.4),

$$|\partial_\alpha \varphi(\theta,\alpha)| < \text{const} \quad \text{and} \quad \lambda^{1/7} < \partial_\theta \varphi(\theta,\alpha) < \lambda, \quad \theta \in I_0' + \omega, \alpha \in \mathcal{A}_0.$$

Moreover, from Lemma 3.1 we have $c(-\lambda^{-2/5}/2, \alpha) < 2$ and $c(-2\lambda^{-2/3}, \alpha) > 3$ for all $\alpha \in \mathcal{A}_0$. Since $x_{M_0} \in C$, it therefore follows that

$$\varphi(-\lambda^{-2/5}/2 + \omega, \alpha) < 1/2 - 1/10 \quad \text{and} \quad \varphi(-2\lambda^{-2/3} + \omega, \alpha) > 1/2 + 1/10 \\ \text{for all } \alpha \in \mathcal{A}_0.$$

Thus, for each $\alpha \in \mathcal{A}_0$ there must be a $\theta = \theta(\alpha) \in [-\lambda^{-2/5}/2 + \omega, -2\lambda^{-2/3} + \omega] = \mathcal{A}_0$ (recall the definition of $\mathcal{A}_0$) such that $\varphi(\theta(\alpha), \alpha) = 1/2$. Since the mapping $\mathcal{A}_0 \ni \alpha \mapsto \theta(\alpha) \in \mathcal{A}_0$ clearly is continuous, there must be a fixed point, that is, an α such that $\theta(\alpha) = \alpha$. Hence, we have $\varphi(\alpha, \alpha) = 1/2$ for some $\alpha \in \mathcal{A}_0$. This finishes the proof.

It remains to verify $(iii)_0$. Since $\theta_k \notin I_0 \cup (I_0 + \omega)$ $(k = 0, \ldots, N-1)$, $(iii)_0$ follows directly by repeated use of Lemma 3.2. □

4.2. The Inductive Step. Before we state the inductive lemma, we introduce the following notation. Given intervals $I_0, I_1, \ldots, I_{n-1}$ and integers $M_0, M_1, \ldots, M_{n-1}$ and $K_0, K_1, \ldots, K_{n-1}$, we define

$$\Theta_{n-1} = \mathbb{T} \setminus \bigcup_{i=0}^{n-1} \bigcup_{m=-M_i}^{M_i} (I_i + m\omega), \quad \Theta_{-1} = \mathbb{T} \setminus (I_0 \cup (I_0 + \omega)); \quad \text{and}$$

$$G_{n-1} = \bigcup_{i=0}^{n-1} \bigcup_{m=0}^{3K_i} (I_i + m\omega), \quad G_{-1} = \emptyset.$$

Proposition 4.2. *There is a $\lambda_2 > 0$ such that the following hold for all $\lambda > \lambda_2$: Assume that for some $n \geq 0$, closed intervals $I_0' \supset I_1 \supset \cdots \supset I_n$ have been constructed, and integers $M_0 < M_1 < \cdots < M_n$ and $K_0 < K_1 < \cdots < K_n$ have been chosen, satisfying*

$$|I_k| = (4/5)^{K_{k-1}}, \quad K_k \in [(5/4)^{K_{k-1}/(4\tau)}, 2(5/4)^{K_{k-1}/(4\tau)}] \\ \text{for } k = 1, 2, \ldots, n; \tag{4.5}$$

$$M_k \in [(5/4)^{K_{k-1}/(2\tau)}, 2(5/4)^{K_{k-1}/(2\tau)}] \ \text{for } k = 1, 2, \ldots, n; \quad \text{and} \tag{4.6}$$

$$\bigcup_{k=0}^{20} (I_n + (2K_n + k)\omega) \subset \Theta_{n-1}, \quad I_n - M_n\omega \subset \Theta_{n-1}. \tag{4.7}$$

Assume further that a non-empty interval $\mathcal{A}_n = [\alpha_n^-, \alpha_n^+] \subset I_n + \omega$ ($\subset I_0' + \omega$ if $n = 0$) has been constructed such that, writing $I_n + \omega = [a_n, b_n]$ ($I_0' + \omega = [a_0, b_0]$ if $n = 0$), there holds

$$\alpha_n^- - a_n > (4/5)^{K_n} \quad \text{and} \quad b_n - \alpha_n^+ > (4/5)^{K_n}, \tag{4.8}$$

and the following holds:

$(i)_n$ *If* $\alpha \in \mathcal{A}_n$, $\theta_0 \in \Theta_{n-1}$ *and* $x_0, y_0 \in C$, *then, letting* $N = N(\theta_0; I_n)$,

$$\prod_{i=k}^{N-1} |c(\theta_i)(1-2x_i)| < (3/5)^{(1/2+1/2^{n+1})(N-k+1)} \quad \text{for all } k \in [0, N-1]; \tag{4.9}$$

$$x_k \notin C \text{ and } k \in [0, N] \Rightarrow \theta_k \in G_{n-1}; \text{ and} \tag{4.10}$$

$$|x_k - y_k| \le (3/5)^{(1/2+1/2^{n+1})k} |x_0 - y_0| \quad \text{for all } k \in [1, N]. \tag{4.11}$$

$(ii)_n$ *If* Γ *is a horizontal line segment* $\Gamma = (I_n - M_n\omega) \times \{x\}$, *where* $x \in C$, *then*

$$\Phi_\alpha^{M_n+1}(\Gamma) = \{(\theta, \varphi(\theta, \alpha)) : \theta \in I_n + \omega\} \quad (\alpha \in \mathcal{A}_n),$$

where the function $\varphi : (I_n + \omega) \times \mathcal{A}_n \to \mathbb{R}$ *satisfies*

$$3/10 < \varphi(\theta, \alpha) < 99/100, \tag{4.12}$$

$$\lambda^{1/7} < \partial_\theta \varphi(\theta, \alpha) < \lambda,$$

$$|\partial_\alpha \varphi| < const(\omega) \quad \text{for all} \begin{cases} \theta \in I_0' + \omega, \alpha \in \mathcal{A}_0 & \text{if } n = 0 \\ \theta \in I_n + \omega, \alpha \in \mathcal{A}_n & \text{if } n > 0 \end{cases}. \tag{4.13}$$

Moreover, there is an $\alpha \in \mathcal{A}_n$ *such that*

$$\varphi(\alpha, \alpha) = 1/2. \tag{4.14}$$

$(iii)_n$ *If* $\alpha \in \mathcal{A}_n$, $1/100 \le x_0 < 99/100$ *and* $\theta_0 \notin I_0 \cup (I_0 + \omega)$, *then*

$$x_k \notin [1/100, 99/100] \text{ and } k \in [0, N(\theta_0; I_n)] \Rightarrow \theta_k \in G_{n-1}.$$

Then there are non-degenerate closed intervals $I_{n+1} \subset I_n$ ($I_1 \subset I_0'$ *if* $n = 0$) *and* $\mathcal{A}_{n+1} \subset (I_{n+1} + \omega) \cap \mathcal{A}_n$, *and integers* M_{n+1}, K_{n+1} *such that* $(4.5\text{-}4.8)_{n+1}$ *and* $(i - iii)_{n+1}$ *hold.*

Proof. Along the proof, which consists of several parts, we assume that $n \ge 0$ is given and that λ is sufficiently large. We stress that λ does not depend on n.

By using Lemma 2.5, the length estimates on the I_k in (4.5) imply the minimal return time to I_k is >

$$\begin{cases} [(\kappa(5/4)^{K_{k-1}})^{1/\tau}] := N_k & k \ge 1 \\ \left[(\kappa\lambda^{1/7}/2)^{1/\tau}\right] := N_0 & k = 0. \end{cases} \tag{4.15}$$

Thus, the M_k and K_k have been chosen to be approximately $\sqrt{N_k}$ and $N_k^{1/4}$, respectively. This implies, in particular, that

$$I_k \cap \bigcup_{0<|m|\le 10M_k} (I_k + m\omega) = \emptyset \quad \text{for all } k = 0, 1, \ldots, n. \tag{4.16}$$

By condition $(4.7)_n$, combined with (4.16), we have that

$$(I_n - M_n\omega) \cap G_n = \emptyset. \tag{4.17}$$

Moreover, since $I_n \subset I_k$ $(k = 0, 1, \ldots, n-1)$ and $(I_k - \omega) \cap \bigcup_{m=0}^{3K_k} (I_k + m\omega) = \emptyset$ $(k = 0, 1, \ldots, n)$, we have

$$(I_n - \omega) \cap G_n = \emptyset. \tag{4.18}$$

Defining $\mathcal{A}_{n+1}$ *and* I_{n+1}. Take $x, y \in C$ and let φ_1, φ_2 be the corresponding functions given by $(ii)_n$, that is, let $\Gamma_1 = (I_n - M_n\omega) \times \{x\}$ and $\Gamma_2 = (I_n - M_n\omega) \times \{y\}$, and let φ_i be defined by $\Phi^{M_n+1}(\Gamma_i) = (\theta, \varphi_i(\theta, \alpha))$. Then they satisfy $(4.12\text{-}4.13)_n$. Moreover,

$$|\varphi_1 - \varphi_2| < \frac{2}{25}(3/5)^{M_n/2}. \tag{4.19}$$

Indeed, by $(4.7)_n$ we have $I_n - M_n\omega \subset \Theta_{n-1}$, and from (4.16) it follows that if $\theta_0 \in I_n - M_n\omega$, then $N(\theta_0; I_n) = M_n$. Thus, taking $\theta_0 \in I_n - M_n\omega$ and $x_0, y_0 \in C$, we obtain from $(i)_n$ that

$$|x_{M_n} - y_{M_n}| \leq (3/5)^{M_n/2}|x_0 - y_0| \leq \frac{1}{50}(3/5)^{M_n/2}.$$

Since $x_{M_n+1} = c(\theta_n)p(x_{M_n})$ and $y_{M_n+1} = c(\theta_n)p(y_{M_n})$, $3/2 < c(\theta_{M_n}) \leq 4$, and since $|p'(x)| \leq 1$ in $[0, 1]$, we have that

$$|x_{M_n+1} - y_{M_n+1}| \leq \frac{2}{25}(3/5)^{M_n/2}.$$

By recalling the definition of $\varphi_{1,2}$, this gives (4.19).

Next, by $(4.14)_n$ there are $\alpha_1, \alpha_2 \in \mathcal{A}_n$ such that $\varphi_i(\alpha_i, \alpha_i) = 1/2$ $(i = 1, 2)$. By letting $g_i(\theta) = \varphi_i(\theta, \theta)$, for example, an easy computation, using the above estimates on φ_i, shows that $|\alpha_1 - \alpha_2| < (3/5)^{M_n/2}$.

Summing up, since $x, y \in C$ were arbitrarily chosen, the above calculation shows that there exists a closed interval $\mathcal{A}_{n+1} \subset \mathcal{A}_n$, of length $< (3/5)^{M_n/2}$, with the following properties: for any $x \in C$, with corresponding function φ (we use $(ii)_n$), there is an $\alpha \in \mathcal{A}_{n+1}$ such that $\varphi(\alpha, \alpha) = 1/2$.

We now define I_{n+1} by

$$I_{n+1} + \omega = [\alpha_{n+1}^- - (4/5)^{K_n}/2, \alpha_{n+1}^- + (4/5)^{K_n}/2],$$

where we use the notation $\mathcal{A}_{n+1} = [\alpha_{n+1}^-, \alpha_{n+1}^+]$. Since $\mathcal{A}_{n+1} \subset \mathcal{A}_n$, we have that

$$\alpha_n^- \leq \alpha_{n+1}^- < \alpha_{n+1}^+ \leq \alpha_n^+,$$

where $\mathcal{A}_n = [\alpha_n^-, \alpha_n^+]$. Denoting the endpoints in $I_{n+1} + \omega$ as $a_{n+1} = \alpha_{n+1}^- - (4/5)^{K_n}/2$ and $b_{n+1} = \alpha_{n+1}^- + (4/5)^{K_n}/2$, and $I_n + \omega = [a_n, b_n]$ ($I_0' + \omega = [a_0, b_0]$ if $n = 0$), it follows from $(4.8)_n$ that

$$a_{n+1} - a_n = \alpha_{n+1}^- - a_n - (4/5)^{K_n}/2 \geq \alpha_n^- - a_n - (4/5)^{K_n}/2 > 0$$

and

$$b_n - b_{n+1} = b_n - (\alpha_{n+1}^- + (4/5)^{K_n}/2) \geq b_n - \alpha_n^+ - (4/5)^{K_n}/2 > 0.$$

Thus

$$I_{n+1} \subset I_n \quad (I_1 \subset I_0' \text{ if } n = 0).$$

Below we will choose the integer K_{n+1} to be of the size $(5/4)^{K_n/(4\tau)}$, that is $K_{n+1} \gg K_n$. Using this fact, the above definitions yields

$$\alpha_{n+1}^- - a_{n+1} = (4/5)^{K_n}/2 > (4/5)^{K_{n+1}}$$

and, since $\alpha_{n+1}^+ - \alpha_{n+1}^- < (3/5)^{M_n/2}$,

$$b_{n+1} - \alpha_{n+1}^+ = \alpha_{n+1}^- + (4/5)^{K_n}/2 - \alpha_{n+1}^+ > (4/5)^{K_n}/2 - (3/5)^{M_n/2} > (4/5)^{K_{n+1}}.$$

This shows that $(4.8)_{n+1}$ holds, once we have defined K_{n+1} of the above mentioned size.

To continue, we note that since the length of $\mathcal{A}_{n+1}$ is $< (3/5)^{M_n/2}$, we have

$$I_{n+1} + \omega \supset [\alpha - (4/5)^{K_n}/\lambda^{1/4}, \alpha + (4/5)^{K_n}/\lambda^{1/4}] \quad \text{for all } \alpha \in \mathcal{A}_{n+1}.$$

By Lemma 3.1 we therefore have

$$c(\theta,\alpha) < 4(1-(4/5)^{2K_n}) \quad \text{for all } \theta \in (I_0+\omega)\backslash(I_{n+1}+\omega), \alpha \in \mathcal{A}_{n+1}. \tag{4.20}$$

Choosing M_{n+1} and K_{n+1}. For each $j \in [0,n]$, let N'_j be the positive integer given by Lemma 2.5 when it is applied to $I = 3I_j$. Here $3I_j$ is the interval with the same center as I_j, but three times longer. By the estimates in $(4.16)_n$ we have

$$N'_j = \left[(\kappa(5/4)^{K_{j-1}}/3)^{1/\tau}\right], \quad j \in [1,n], \quad N'_0 = \left[(\kappa\lambda^{1/7}/6)^{1/\tau}\right].$$

By this choice we have

$$(3I_j) \cap \bigcup_{0<|m|\le N'_j} (3I_j + m\omega) = \emptyset, \quad j \in [0,n].$$

Since M_j is of size $\sqrt{N'_j}$ one easily checks that the following holds for each $j \in [0,n]$: Given any integer $t \in \mathbb{Z}$, there are at most $4M_j + 1$ integers p in $[t, t+N'_j]$ such that

$$(I_j + p\omega) \cap \bigcup_{m=-2M_j}^{2M_j} (I_j + m\omega) \neq \emptyset.$$

Thus, in the interval $[t, t+N'_n]$ there are at most

$$s = (4M_n+1) + (4M_{n-1}+1)(\left[N'_n/N'_{n-1}\right]+1) + \cdots (4M_0+1)(\left[N'_n/N'_0\right]+1)$$

integers p such that

$$(I_n + p\omega) \cap \bigcup_{j=0}^{n}\bigcup_{m=-2M_j}^{2M_j} (I_j + m\omega) \neq \emptyset.$$

Since

$$s < 100\left(M_n + M_{n-1}\frac{N'_n}{N'_{n-1}} + \cdots + M_0\frac{N'_n}{N'_0}\right) = 100N_n\left(\frac{M_n}{N'_n} + \cdots \frac{M_0}{N'_0}\right) \ll N'_n$$

by the estimates on M_k and N_k, there must be a $p \in [t, t+N'_n]$ such that

$$(I_n + p\omega) \cap \bigcup_{j=0}^{n}\bigcup_{m=-2M_j}^{2M_j} (I_j + m\omega) = \emptyset,$$

and thus

$$\bigcup_{k=0}^{20}(I_n+(p+k)\omega)\cap\bigcup_{j=0}^{n}\bigcup_{m=-M_j}^{M_j}(I_j+m\omega)=\emptyset.$$

Consequently, letting $t=(5/4)^{K_n/(4\tau)}$ and noticing that $(5/4)^{K_n/(4\tau)}\gg N'_n$, we can find an integer K_{n+1} in the interval $[(5/4)^{K_n/(4\tau)},2(5/4)^{K_n/(4\tau)}]$ such that the first condition in (4.7)$_n$ holds. Similarly we can find an integer M_{n+1} in $[(5/4)^{M_n/(2\tau)},2(5/4)^{M_n/(2\tau)}]$ such that the second condition in (4.7)$_n$ holds.

Verifying $(i)_{n+1}$. Assume that $\alpha\in\mathcal{A}_{n+1}\subset\mathcal{A}_n$, $\theta_0\in\Theta_n\subset\Theta_{n-1}$ and $x_0,y_0\in C$, and let $N=N(\theta_0;I_{n+1})$. We need to prove that

$$\prod_{i=k}^{N-1}|c(\theta_i)(1-2x_i)|<(3/5)^{(1/2+1/2^{n+2})(N-k)}\quad\text{for all } k\in[0,N-1];\qquad(4.21)$$

$$x_k\notin C\text{ and }k\in[0,N]\Rightarrow\theta_k\in G_n;\quad\text{and}\qquad(4.22)$$

$$|x_k-y_k|\le(3/5)^{(1/2+1/2^{n+2})k}|x_0-y_0|\quad\text{for all }k\in[1,N].\qquad(4.23)$$

We denote by (4.21)$[T]$–(4.23)$[T]$ the three above conditions when N is replaced by an integer $T>0$, respectively. We shall inductively prove that (4.21)[N]–(4.23)[N] hold.

From the assumption that $\theta_0\in\Theta_n$ it immediately follows from (4.16) that $N>M_n$. Let $0<s_1<s_2<\cdots<s_r=N$ be the times $k\in[0,N]$ when $\theta_k\in I_n$. Note that we could have $r=1$. We have the estimates (recall (4.15))

$$s_1>M_n;\quad\text{and}$$
$$s_j-s_{j-1}>N_n\quad\text{for } j=2,3,\dots,r.\qquad(4.24)$$

From $(i)_n$, which holds by assumption, we automatically get that the weaker conditions (4.21)$[s_1]$–(4.23)$[s_1]$ hold. If $r=1$ we are done. If not, assume that we have shown that (4.21)$[s_l]$–(4.23)$[s_l]$ hold for some l, $1\le l<r$. Since $s_l>M_n$ and since (4.17) holds, it follows from (4.22)$[s_l]$ that

$$x_{s_l-M_n}\in C.\qquad(4.25)$$

Recall that $\theta_{s_l-M_n}\in I_n-M_n\omega$. Thus, from $(ii)_n$ we get that $3/10<x_{s_l+1}<99/100$. Since (4.20) holds, it now follows from Lemma 3.4 that there is a t, $2\le t\le 2K_n-7$, such that $x_{s_l+t}\in[1/100,99/100]$. We now prove that $x_{s_l+2K_n}\in[1/100,99/100]$.

If $t=2$ or $t=3$, then, since $\theta_{s_l+t}\in I_n+t\omega$ and $(I_n+t\omega)\cap(I_0\cup(I_0+\omega))=\emptyset$ $(t=2,3)$, we can use $(iii)_n$ to get

$$x_{s_l+k}\notin[1/100,99/100]\quad\text{and}\quad k\in[t,s_{l+1}-s_l]\Rightarrow\theta_{s_l+k}\in G_{n-1}.$$

Since $(I_n+2K_n\omega)\cap G_{n-1}=\emptyset$ by (4.7)$_n$ we must have that $x_{s_l+2K_n}\in[1/100,99/100]$.

If $t>3$, assume that t was chosen as small as possible, i.e., assume that $x_{s_l+k}\notin[1/100,99/100]$ for $k=2,3,\dots,t-1$. Then we must have $x_{s_l+k}<1/100$ for $k=3,4,\dots,t-1$. If $\theta_{s_l+t}\notin I_0\cup(I_0+\omega)$ we can use $(iii)_n$, as above, and obtain $x_{s_l+2K_n}\in[1/100,99/100]$. If $\theta_{s_l+t}\in I_0\cup(I_0+\omega)$, then we use Lemma 3.3 to get $x_{s_l+t+2}\in[1/100,99/100]$. Since $\theta_{s_l+t+2}\notin I_0\cup(I_0+\omega)$, we can proceed as above, i.e., apply $(iii)_n$ to the point $(\theta_{s_l+t+2},x_{s_l+t+2})$.

Thus, we know that $x_{s_l+2K_n} \in [1/100, 99/100]$. Since $(4.7)_n$ holds, so in particular we have $\theta_{s_l+2K_n}, \dots, \theta_{s_l+2K_n+20} \notin I_0 \cup (I_0+\omega)$, it follows from Lemma 3.2 that $x_{s_l+2K_n+20} \in C$. Now we know, again using $(4.7)_n$, that $\theta_{s_l+2K_n+20} \in \Theta_{n-1}$. Therefore we can apply $(i)_n$ to the point $(\theta_{s_l+2K_n+20}, x_{s_l+2K_n+20})$ and deduce (recall the definition of s_{l+1})

$$\prod_{i=k}^{s_{l+1}-1} |c(\theta_i)(1-2x_i)| < (3/5)^{(1/2+1/2^{n+1})(s_{l+1}-k)}$$

$$\text{for all } k \in [s_l + 2K_n + 20, s_{l+1} - 1]; \tag{4.26}$$

$$x_k \notin C \text{ and } k \in [s_l + 2K_n + 20, s_{l+1}] \Rightarrow \theta_k \in G_{n-1}; \text{ and} \tag{4.27}$$

$$|x_k - y_k| \le (3/5)^{(1/2+1/2^{n+1})(k-s_l-2K_n-20)} |x_{s_l+2K_n+20} - y_{s_l+2K_n+20}|$$

$$\text{for all } k \in [s_l + 2K_n + 20 + 1, s_{l+1}]. \tag{4.28}$$

During the passage from $k = s_l + 1$ to $k = s_l + 2K_n + 20$ we could have had $x_k \notin C$, but for $k \in [s_l + 1, s_l + 2K_n + 20]$ we have $\theta_k \in \bigcup_{m=0}^{2K_n+20}(I_n + m\omega) \subset G_n$. Combining this with $(4.22)[s_l]$ and (4.27) gives $(4.22)[s_{l+1}]$.

To continue, we notice that $|c(\theta)| \le 4$ ($\theta \in \mathbb{T}$) and $|1-2x| \le 1$ for $x \in [0, 1]$. Thus we always have the trivial estimates

$$\begin{aligned} \prod_{i=k}^{s_l+2K_n+19} |c(\theta_i)(1-2x_i)| &\le 4^{s_l+2K_n+20-k}, \quad k \le s_l + 2K_n + 19; \quad \text{and} \\ |x_{s_l+k} - y_{s_l+k}| &\le 4^k |x_{s_l} - y_{s_l}|, \quad k > 0. \end{aligned} \tag{4.29}$$

To show that $(4.21)[s_{l+1}]$ holds, it follows from $(4.21)[s_l]$ and (4.26) that it is enough to prove that

$$\prod_{i=k}^{s_{l+1}-1} |c(\theta_i)(1-2x_i)| < (3/5)^{(1/2+1/2^{n+2})(s_{l+1}-k)} \quad \text{for all } k \in [s_l, s_l + 2K_n + 19].$$

Take $k \in [s_l, s_l + 2K_n + 19]$. By the above estimates we have

$$\begin{aligned} \prod_{i=k}^{s_{l+1}-1} |c(\theta_i)(1-2x_i)| &= \prod_{i=k}^{s_l+2K_n+19} |c(\theta_i)(1-2x_i)| \prod_{i=s_l+2K_n+20}^{s_{l+1}-1} |c(\theta_i)(1-2x_i)| \\ &< 4^{s_l+2K_n+20-k} (3/5)^{(1/2+1/2^{n+1})(s_{l+1}-s_l-2K_n-20)} \\ &< (3/5)^{(1/2+1/2^{n+1})(s_{l+1}-s_l-2K_n-20)-3(s_l+2K_n+20-k)}. \end{aligned}$$

In the last inequality we used the fact that $(5/3)^3 > 4$. Denoting the above exponent by $z(k)$, we have to show that $z(k) > w(k) := (1/2+1/2^{n+2})(s_{l+1} - k)$. Subtracting, using $2K_n + 20 < 3K_n$ and the fact that the worst case is when $k = s_l$, we obtain

$$z(k) - w(k) > (s_{l+1} - s_l)/(2^{n+2}) - 12K_n.$$

By (4.24), and the estimates on N_n and K_n in (4.15), it is clear that this is positive.

It remains to verify $(4.23)[s_{l+1}]$. Since $\theta_{s_l-M_n} \in I_n - M_n\omega \subset \Theta_{n-1}$, and since (4.25) holds, we can apply $(i)_n$ to the point $(\theta_{s_l-M_n}, x_{s_l-M_n})$ and obtain the estimate

$$|x_{s_l} - y_{s_l}| \le (3/5)^{(1/2+1/2^{n+1})M_n} |x_{s_l-M_n} - y_{s_l-M_n}|. \tag{4.30}$$

Applying the estimate (4.29) for $k \in [1, 2K_n + 20]$, we have, again using the fact that $(5/3)^3 > 4$,

$$|x_{s_l+k} - y_{s_l+k}| \leq (3/5)^{(1/2+1/2^{n+1})M_n - 3k} |x_{s_l - M_n} - y_{s_l - M_n}|.$$

Now we note that

$$\begin{aligned} &((1/2 + 1/2^{n+1})M_n - 3k) - (1/2 + 1/2^{n+2})(M_n + k) \\ &= M_n/2^{n+2} - 4k > M_n/2^{n+2} - 12K_n > 0 \end{aligned}$$

by the estimates on K_n and M_n. Thus,

$$\begin{aligned} |x_{s_l+k} - y_{s_l+k}| &\leq (3/5)^{(1/2+1/2^{n+2})(M_n+k)} |x_{s_l - M_n} - y_{s_l - M_n}|, \\ &k \in [1, 2K_n + 20]. \end{aligned} \tag{4.31}$$

Combining this estimate with (4.23)$[s_l]$ and (4.28) now yields (4.23)$[s_{l+1}]$.

Verifying $(ii)_{n+1}$. Take $x \in C$ and let $\Gamma = (I_{n+1} - M_{n+1}\omega) \times \{x\}$. Let $\varphi : (I_{n+1} + \omega) \times \mathcal{A}_{n+1} \to \mathbb{R}$ be defined such that $\Phi^{M_{n+1}+1}(\Gamma) = \{(\theta, \varphi(\theta, \alpha)) : \theta \in I_{n+1} + \omega\}$ for fixed $\alpha \in \mathcal{A}_{n+1}$.

Fix $\alpha \in \mathcal{A}_{n+1}$, and let (θ_0, x_0) be a point on Γ, that is, $\theta_0 \in I_{n+1} - M_{n+1}\omega$ and $x_0 = x \in C$. We note that $N(\theta_0; I_{n+1}) = M_{n+1}$. Since $I_{n+1} - M_{n+1}\omega \subset \Theta_n$, by $(4.7)_{n+1}$, we can apply $(i)_{n+1}$ and get

$$\prod_{i=k}^{M_{n+1}-1} |c(\theta_i)(1 - 2x_i)| < (3/5)^{(N-k+1)/2}$$
$$\text{for all } k \in [0, M_{n+1} - 1]; \quad \text{and} \tag{4.32}$$
$$x_k \notin C \text{ and } k \in [0, M_{n+1}] \Rightarrow \theta_k \in G_n. \tag{4.33}$$

The latter, together with (4.17), implies that $x_{M_{n+1}-M_n} \in C$, because $\theta_{M_{n+1}-M_n} \in I_{n+1} - M_n\omega \subset I_n - M_n\omega$. Moreover, (4.18) and (4.33) imply that $x_{M_{n+1}-1} \in C$, and since $\theta_{M_{n+1}-1} \in I_{n+1} - \omega \subset \mathbb{T}\backslash(I_0 \cup (I_0 + \omega))$, it follows from Lemma 3.2 that $x_{M_n+1} \in C$. Thus

$$\Phi^{M_{n+1}-M_n}(\Gamma) \subset (I_{n+1} - M_n\omega) \times C \quad \text{and} \tag{4.34}$$
$$\Phi^{M_{n+1}}(\Gamma) \subset I_n \times C. \tag{4.35}$$

Note that $(4.12)_{n+1}$ now follows from $(ii)_n$ and (4.34).

Next, if we think of $x_0 = x_0(\theta, \alpha) = x \in C$, where $\theta \in I_{n+1} - M_{n+1}\omega$ and $\alpha \in \mathcal{A}_{n+1}$, it follows from (4.32) and Lemma 3.5 that

$$|\partial_\theta x_{M_{n+1}}|, |\partial_\alpha x_{M_{n+1}}| < \lambda^{-1/4}. \tag{4.36}$$

Indeed, if $\theta_0 \in I_{n+1} - M_{n+1}\omega$, then $\theta_{M_{n+1}} \in I_n \subset I_0$ and hence $\theta_{M_{n+1}-k} \notin I_0 \cup (I_0 + \omega)$ for (at least) $k = 1, 2, \ldots, M_0$, and thus $|\partial_\theta c(\theta_{M_{n+1}-k}, \alpha)| < 1/\sqrt{\lambda}$ for $k \in [1, M_0]$ by Lemma 3.2. Since $M_0 \gg 10 \log \lambda$, the use of Lemma 3.5 is possible. By proceeding as in the proof of the basic step, Lemma 4.1, making use of the estimates (4.35) and (4.36), we get $(4.13)_{n+1}$.

It remains to check $(4.14)_{n+1}$. From $(ii)_n$ we know that if $\Lambda_y = (I_n - M_n\omega) \times \{y\}$, $y \in C$, then $\Phi^{M_n+1}(\Lambda_y) = \{(\theta, \phi(\theta, \alpha, y) : \theta \in I_n + \omega)\}$, where $\phi : (I_n + \omega) \times \mathcal{A}_n \times C$ satisfy

$$|\partial_\theta \phi| > \lambda^{1/7} \quad \text{and} \quad |\partial_\alpha \phi| < const(\omega). \tag{4.37}$$

Moreover, for every $y \in C$ there is an $\alpha(y) \in \mathcal{A}_n$ such that $\phi(\alpha(y), \alpha(y), y) = 1/2$. By (4.37) it is clear that it is unique. By the definition of $\mathcal{A}_{n+1}$ above, we have $\alpha(y) \in \mathcal{A}_{n+1}$. If we let $\psi(\alpha, y) = \phi(\alpha, \alpha, y)$ for $\alpha \in \mathcal{A}_{n+1}$ and $y \in C$, and use the estimates (4.37), it follows from the implicit function theorem that the function $C \ni y \mapsto \alpha(y) \in \mathcal{A}_{n+1}$ is (at least) continuous.

Consider now the curve $\Delta : (\alpha(y) - (M_n + 1)\omega, y), y \in C$. We know that $\alpha(y) \in \mathcal{A}_{n+1} \subset I_{n+1} + \omega$, so Δ divides the box $(I_{n+1} - M_n\omega) \times C$ into two pieces. Since (4.34) holds, the curve $\Phi^{M_{n+1}-M_n}(\Gamma)$ must intersect Δ. By construction we have $\Phi^{M_n+1}(\alpha(y) - (M_n + 1)\omega, y) = 1/2$ if $y \in C$ and $\alpha = \alpha(y)$. This gives $(4.14)_{n+1}$.

Verifying $(iii)_{n+1}$. Assume that $x_0 \in [1/100, 99/100]$ and $\theta_0 \notin I_0 \cup (I_0 + \omega)$, and let $N = N(\theta_0; I_{n+1})$. We need to prove that

$$x_k \notin [1/100, 99/100] \quad \text{and} \quad k \in [0, N] \Rightarrow \theta_k \in G_n. \tag{4.38}$$

Let (4.38)$[T]$ denote the above condition with N replaced by $T \geq 0$. Let $0 < s_1 < s_2 < \cdots < s_r = N$ be the times $k \in [0, N]$ when $\theta_k \in I_n$.

From $(iii)_n$ we get (4.38)$[s_1]$. Assume now that (4.38)$[s_l]$ holds for some $0 \leq l < r$. Since (4.18) holds, we get from (4.38)$[s_l]$ that $x_{s_l-1} \in [1/100, 99/100]$. Since (4.20) holds, it therefore follows from Lemma 3.4 that there is a k, $3 \leq k \leq 2K_n - 7$, such that $x_{s_l+k} \in [1/100, 99/100]$. Proceeding exactly as in the verification of $(i)_{n+1}$ above, we get that $x_{s_l+2K_n} \in [1/100, 99/100]$. Note that we could have $x_{s_l+k} \notin [1/100, 99/100]$ for $k \in [0, 2K_n - 1]$. For such k we have $\theta_{s_l+k} \in \bigcup_{m=0}^{2K_n}(I_n + m\omega) \subset G_n$. Since $\theta_{s_l+2K_n} \in I_n + 2K_n$ and since $(I_n + 2K_n) \cap (I_0 \cup (I_0 + \omega)) = \emptyset$ by $(4.7)_n$, we can apply $(iii)_n$ to the point $(\theta_{s_l+2K_n}, x_{s_l+2K_n})$ and get

$$x_k \notin [1/100, 99/100] \quad \text{and} \quad k \in [s_l + 2K_n, s_{l+1}] \Rightarrow \theta_k \in G_{n-1}.$$

Summing up, this shows that $(iii)_{n+1}$ holds.

This finishes the proof of Proposition 4.2. □

5. Proof of Main Theorem

We now have all the pieces needed for the proof of the Main Theorem. The proof will consist of several lemmas. We begin by defining the main objects.

From now on we assume that $\lambda > \max\{\lambda_1, \lambda_2\}$ is sufficiently large so that the (finitely many) conditions below hold true. By using Lemma 4.1 and Proposition 4.2 we inductively get a nested sequence of closed intervals $I_0' \supset I_1 \supset I_2 \supset \ldots$ and integers $M_0 < M_1 < M_2 < \ldots$, $K_0 < K_1 < K_2 < \ldots$ satisfying the estimates $(4.5\text{–}4.6)_n$ for $n = 1, 2, 3, \ldots$. Moreover, we get closed non-degenerate intervals $\mathcal{A}_0 \supset \mathcal{A}_1 \supset \mathcal{A}_2 \supset \ldots$ such that

$$\mathcal{A}_n \cap (I_n + \omega) \supset \mathcal{A}_{n+1} \quad n = 0, 1, 2, \ldots, \tag{5.1}$$

and $(i - iii)_n$ in Proposition 4.2 hold for all n.

Let $\theta_c \in \mathbb{T}$ be the unique point such that

$$\bigcap_{n \geq 0} I_n = \{\theta_c\}.$$

We now fix the parameter $\alpha \in \mathbb{T}$ as

$$\bigcap_{n\geq 0} \mathcal{A}_n = \{\alpha\}.$$

This is the α appearing in the statement of the Main Theorem; we have "fine tuned it on infinitely many scales". Note that by (5.1) we have

$$\alpha = \theta_c + \omega.$$

From now on α is fixed like this. In the rest of the paper we are going to verify that the mapping Φ_α (with the above α) has the required properties.

We define

$$\Theta_\infty = \bigcap_{n\geq 0} \Theta_n = \mathbb{T}\setminus \bigcup_{i=0}^{\infty} \bigcup_{-M_i}^{M_i} (I_i + m\omega).$$

Since

$$(2M_0+1)|I_0| \leq 3\lambda^{1/(14\tau)}\lambda^{-1/7}$$

and

$$(2M_k+1)|I_k| \leq 5(5/4)^{K_{k-1}/(2\tau)}(4/5)^{K_{k-1}}, \quad k \geq 1,$$

and since $\tau \geq 1$, it follows that $|\Theta_\infty| > 0$. In fact, $|\Theta_\infty| \to |\mathbb{T}| = 1$ as $\lambda \to \infty$. Recall the extreme growth of the numbers K_k. We also let

$$G_\infty = \bigcup_{n\geq 0} G_n = \bigcup_{i=0}^{\infty} \bigcup_{m=0}^{3K_i} (I_i + m\omega).$$

By definition we have (recall that $K_j \ll M_j$ for all j)

$$G_\infty \cap \Theta_\infty = \emptyset.$$

Next we let

$$\begin{aligned}\Theta^* = (&\{\theta \in \mathbb{T} : \theta - k\omega \in \Theta_\infty \text{ for infinitely many } k \geq 0\}\\ &\cap\{\theta \in \mathbb{T} : \theta + k\omega \in \Theta_\infty \text{ for infinitely many } k \geq 0\})\\ &\setminus(\{\theta_c + k\omega : k \in \mathbb{Z}\} \cup \{k\omega : k \in \mathbb{Z}\}).\end{aligned}$$

Since $|\Theta_\infty| > 0$, it follows by ergodicity ($\theta \mapsto \theta + \omega$ is ergodic) that $|\Theta^*| = 1$, i.e., it has full Lebesgue measure. By definition Θ^* is invariant under rotation by ω:

$$\Theta^* \pm \omega = \Theta^*.$$

The next two lemmas are direct consequences of the definition of Θ^*.

Lemma 5.1. *If $\theta_0 \in \Theta^*$ and $x_0 \in (0,1)$, then $x_k \in (0,1)$ for all $k \geq 0$.*

Proof. Recall that $c(\theta) = 4$ only for $\theta = 0, \alpha$. Since $\alpha = \theta_c + \omega$, it thus follows from the definition of Θ^* that $c(\theta) = 3/2 + (5/2)f(\theta) \in (3/2, 4)$ for all $\theta \in \Theta^*$. This clearly implies that if $x \in (0,1)$, then $c(\theta)p(x) \in (0,1)$. $\square$

Lemma 5.2. *If* $\theta_0 \in \Theta^*$, *then* $\sup_{n\geq 0} N(\theta_0; I_n) = \infty$.

Proof. If $\sup_{n\geq 0} N(\theta_0; I_n) = N < \infty$, then $\theta_N \in \bigcap_{n\geq 0} I_n = \{\theta_c\}$. □

Using this lemma, together with $(iii)_n$ in Proposition 4.2, which holds for each n, and recalling that $\alpha \in \mathcal{A}_n$ for all n, we get the following

Lemma 5.3. *If* $x_0 \in [1/100, 99/100]$ *and* $\theta_0 \in \Theta^* \backslash (I_0 \cup (I_0 + \omega))$, *then*

$$x_k \notin [1/100, 99/100] \quad \text{and} \quad k \geq 0 \Rightarrow \theta_k \in G_\infty.$$

Furthermore, by $(i)_n$, which holds for each n, we get

Lemma 5.4. *If* $\theta_0 \in \Theta_\infty$ *and* $x_0, y_0 \in C$, *then*

$$|x_k - y_k| \leq (3/5)^{k/2} |x_0 - y_0| \quad \text{for all } k > 0; \quad \text{and}$$
$$x_k \notin C \text{ and } k \geq 0 \Rightarrow \theta_k \in G_\infty.$$

The next lemma shows that any point $(\theta, x) \in \Theta^* \times (0, 1)$ "ends up well" after a finite time.

Lemma 5.5. *If* $\theta_0 \in \Theta^*$ *and* $x_0 \in (0, 1)$, *then there is a* $t \geq 0$ *such that* $\theta_t \in \Theta_\infty$ *and* $x_t \in C$.

Proof. From Lemma 5.1 we know that $x_k \in (0, 1)$ for all $k \geq 0$. We first show that there is an $s \geq 0$ such that $x_s \in [1/100, 99/100]$ and $\theta_s \notin I_0 \cup (I_0 + \omega)$. There are two cases.

If $x_0 \notin [1/100, 99/100]$, let $q > 0$ be the smallest integer such that $x_q \in [1/100, 99/100]$. Such a q clearly exists since $x_k \in (0, 1)$ for all $k \geq 0$ and since Lemma 2.4 holds. If $\theta_q \notin I_0 \cup (I_0 + \omega)$ we are done. If $\theta_q \in I_0 \cup (I_0 + \omega)$, then $\theta_{q+2} \notin I_0 \cup (I_0 + \omega)$. Moreover, since $x_{q-1} \in (0, 1/100) \cup (99/100, 1)$, it follows from Lemma 3.3 that $x_{q+2} \in [1/100, 99/100]$.

If $x_0 \in [1/100, 99/100]$ and $\theta_0 \in I_0 \cup (I_0 + \omega)$, then $\theta_2 \notin I_0 \cup (I_0 + \omega)$. If $x_2 \in [1/100, 99/100]$ we are done. Otherwise we proceed as in the previous case.

We have thus shown that there is a $s \geq 0$ such that $x_s \in [1/100, 99/100]$ and $x_s \notin I_0 \cup (I_0 + \omega)$. From Lemma 5.3, applied to the point (θ_s, x_s), we hence get

$$x_k \notin [1/100, 99/100] \quad \text{and} \quad k \geq s \Rightarrow \theta_k \in G_\infty. \tag{5.2}$$

Let $r \geq s$ be such that $\theta_r \in \Theta_\infty - 20\omega$ (this is possible by the definition of Θ^*). Since $(\Theta_\infty - 20\omega) \cap G_\infty = \emptyset$ (see the definitions of Θ_∞ and G_∞ above), it follows from (5.2) that $x_r \in [1/100, 99/100]$. Moreover, since $(\Theta_\infty - j\omega) \cap (I_0 \cup (I_0 + \omega)) = \emptyset$ for $j \in [0, 20]$, it follows from Lemma 3.2 that $x_{r+20} \in C$. Thus, letting $t = r + 20$ finishes the proof. □

We now show that we have control on the contraction.

Lemma 5.6. *If* $\theta_0 \in \Theta^*$ *and* $x_0, y_0 \in (0, 1)$, *then*

$$|x_k - y_k| \leq const.(\theta_0, x_0, y_0)(3/5)^{k/2} |x_0 - y_0| \quad \text{for all } k > 0.$$

Furthermore,

$$\left| \frac{\partial x_n}{\partial x_0} \right| \leq const.(\theta_0, x_0)(3/5)^{k/2} \quad \text{for all } k > 0.$$

Proof. From Lemma 5.5 we get integers $s, t \geq 0$ such that $\theta_s \in \Theta_\infty$, $x_s \in C$, and $\theta_t \in \Theta_\infty$, $y_t \in C$. Moreover, from Lemma 5.4 we get the following:

$$x_k \notin C \text{ and } k \geq s \;\Rightarrow\; \theta_k \in G_\infty,$$
$$y_k \notin C \text{ and } k \geq t \;\Rightarrow\; \theta_k \in G_\infty.$$

Combining this gives us an $r \geq \max\{s, t\}$ such that $x_r, y_r \in C$ and $\theta_r \in \Theta_\infty$. Thus we can apply Lemma 5.3 and get

$$|x_k - y_k| \leq (3/5)^{(k-r)/2}|x_r - y_r| \quad \text{for all } k > r.$$

To get the second statement, we see that (we think of x_k as a function of x_r: $x_k = x_k(x_r)$ $k > r$)

$$\left|\frac{\partial x_k}{\partial x_r}\right| = \left|\lim_{h\to 0} \frac{x_k(x_r + h) - x_k(x_r)}{h}\right| \leq (3/5)^{(k-r)/2}, \quad k > r,$$

by Lemma 5.4, since $x_r \in C$. This finishes the proof. □

In the following lemma we construct the measurable function ψ mentioned in the Main Theorem.

Lemma 5.7. *There is a measurable function* $\psi : \Theta^* \to (0, 1)$ *such that* $\psi(\theta) = c(\theta - \omega)p(\psi(\theta - \omega))$ *for all* $\theta \in \Theta^*$.

Proof. Let $\psi_n(\theta) = \pi_2(\Phi^n(\theta - n\omega, 1/100))$. From this we have $\psi_n(\theta) = c(\theta - \omega)p(\psi_{n-1}(\theta - \omega))$. We are going to show that $\psi_n(\theta)$ converges to a number $\psi(\theta)$ as $n \to \infty$ for all $\theta \in \Theta^*$. Then the function ψ is measurable, since the functions ψ_n are all continuous. Moreover, it is invariant:

$$\psi(\theta) = \lim_{n\to\infty} \psi_n(\theta) = \lim_{n\to\infty} c(\theta - \omega)p(\psi_{n-1}(\theta - \omega)) = c(\theta - \omega)p(\psi(\theta - \omega)).$$

We are thus left with the proof of the convergence.

Fix $\theta_0 \in \Theta^*$. Let $t > 0$ be a big integer such that

$$\theta_{-t+m} \in \Theta_\infty \quad \text{for } m \in [0, 20]. \tag{5.3}$$

Applying Lemma 5.4 to all the points in $\{\theta_{-t+20}\} \times C$ implies that

$$\Phi^{t-20}(\theta_{-t+20}, C) \subset \{\theta_0\} \times J_t, \tag{5.4}$$

where J_t is an interval of length $< (3/5)^{(t-20)/2}$.

Next, take any $n > t + 1$ and let $x_{-n} = 1/100$. This choice of x_{-n} implies that $\psi_n(\theta_0) = x_0$.

If $\theta_{-n} \notin I_0 \cup (I_0 + \omega)$, it follows from Lemma 5.3 (applied to the point (θ_{-n}, x_{-n})) that

$$x_k \notin [1/100, 99/100] \text{ and } k \geq -n \;\Rightarrow\; \theta_k \in G_\infty.$$

Since $G_\infty \cap \Theta_\infty = \emptyset$, we must have that $x_{-t} \in [1/100, 99/100]$.

If $\theta_{-n} \in I_0 \cup (I_0 + \omega)$, then since $x_{-n} = 1/100$ we get $x_{-n+2} \in [1/100, 99/100]$ (the same computation as in the proof of Lemma 3.3). In the same way as above we hence get that $x_{-t} \in [1/100, 99/100]$.

We thus know that $x_{-t} \in [1/100, 99/100]$, and by Lemma 3.2 and (5.3) we get that $\theta_{-t+20} \in \Theta_\infty$ and $x_{-t+20} \in C$. Therefore (5.4) implies that $x_0 \in J_t$. This shows that

$$|\psi_n(\theta_0) - \psi_m(\theta_0)| \leq |J_t| < (3/5)^{(t-20)/2} \text{ for all } m, n > t + 1.$$

Since t can be chosen arbitrarily large, we have hence shown that $\psi_n(\theta_0)$ is a Cauchy sequence, and thus there is a $\psi(\theta_0)$ such that $\psi_n(\theta_0) \to \psi(\theta_0)$ as $n \to \infty$. □

It remains to prove that the function Ψ is not continuous. That will be guaranteed by the following lemma.

Lemma 5.8. *There exists a set $\Theta_1^* \subset \Theta^*$ of full Lebesgue measure such that the following holds. If $\theta_0 \in \Theta_1^*$ and $x_0 \in (0, 1)$, then $\inf_{k\geq 0} x_k = 0$.*

Proof. We begin by proving the following statement. For any scale $n > 0$ there holds

$$\theta_0 \in I_n - M_n\omega \text{ and } x_0 \in C \;\Rightarrow\; x_{M_n+3} < |I_n| = (4/5)^{K_{n-1}}. \tag{5.5}$$

To prove this we use $(ii)_n$ in Proposition 4.2, which holds for each n. Before we start, recall that α was fixed as $\{\alpha\} = \bigcap_{n\geq 0} \mathcal{A}_n$. Fix $n > 0$ and take $\theta_0 \in I_n - M_n\omega$, $x_0 \in C$. Moreover, let $\Gamma = (I_n - M_n\omega) \times \{x_0\}$ be a horizontal line segment. By applying $(ii)_n$ we get

$$\Phi_a^{M_n+1}(\Gamma) = \{(\theta, \varphi(\theta)) : \theta \in I_n + \omega\} \quad (a \in \mathcal{A}_n),$$

where $\varphi : (I_n + \omega) \times \mathcal{A}_n \to (3/10, 99/100)$ satisfies $\lambda^{1/7} < |\partial_\theta \varphi| < \lambda^2$ and $|\partial_a \varphi| < const$. Moreover, there is an $\alpha_n \in \mathcal{A}_n \subset I_n + \omega$ such that $\varphi(\alpha_n, \alpha_n) = 1/2$. This implies that

$$\begin{aligned}|\varphi(\theta, \alpha) - 1/2| &= |\varphi(\theta, \alpha) - \varphi(\alpha_n, \alpha_n)| \\ &\leq \lambda^2|\theta - \alpha_n| + const|\alpha - \alpha_n| \quad \text{for all } \theta \in I_n + \omega.\end{aligned}$$

Since $\alpha, \alpha_n \in \mathcal{A}_n \subset I_n + \omega$ and since $x_{M_n+1} = \varphi(\theta_{M_n+1}, \alpha)$, it thus follows that

$$|x_{M_n+1} - 1/2| < \lambda^3|I_n|.$$

From the definition of c, we get a constant $c_1 > 0$ such that $c(\theta) > 4 - c_1\lambda(\theta - \alpha)^2$ for all θ sufficiently close to α (see the proof of Lemma 3.1). Since $\theta_{M_n+1}, \alpha \in I_n + \omega$, we have $|\theta_{M_n+1} - \alpha| \leq |I_n|$. Moreover, p can be written $p(x) = 1/4 - (x - 1/2)^2$. Thus

$$x_{M_n+2} = c(\theta_{M_n+1})p(x_{M_n+1}) > (4 - c_1\lambda|I_n|^2)(1/4 - \lambda^6|I_n|^2) > 1 - |I_n|/4.$$

This in turn shows that

$$x_{M_n+3} = c(\theta_{M_n+2})p(x_{M_n+2}) < 4p(1 - |I_n|/4) = 4p(|I_n|/4) < |I_n|.$$

Next we prove that the set $(I_n - M_n\omega) \cap G_\infty^c$ has a positive measure for each n. Here $G_\infty^c = \mathbb{T}\backslash G_\infty$. From (4.17) we have that $(I_n - M_n\omega) \cap G_n = \emptyset$, and by definition

$$G_\infty = G_n \cup \bigcup_{i=n+1}^{\infty} \bigcup_{m=0}^{3K_i} (I_i + m\omega).$$

From the estimates in (4.5), it follows that

$$|I_n| \gg \sum_{i=n+1}^{\infty} (3K_i + 1)|I_i|.$$

This shows that $|(I_n - M_n\omega) \cap G_\infty^c| > 0$.

We now define Θ_1^* to be the set of $\theta \in \Theta^*$ such that for each $n > 0$, there are infinitely many $k > 0$ such that $\theta + k\omega \in (I_n - M_n\omega) \cap G_\infty^c$. By ergodicity it is clear that this set has full Lebesgue measure.

To continue, take $\theta_0 \in \Theta_1^*$ and $x_0 \in (0, 1)$. Lemma 5.5 gives us a $t \geq 0$ such that $\theta_t \in \Theta_\infty$ and $x_t \in C$. Therefore we can apply Lemma 5.4 and get

$$x_k \notin C \text{ and } k \geq t \Rightarrow \theta_k \in G_\infty. \tag{5.6}$$

By the definition of Θ_1^*, we have that for each scale $n > 0$, there is a $k_n > t$ such $\theta_{k_n} \in (I_n - M_n\omega) \cap G_\infty^c$. By (5.6) we must thus have $x_{k_n} \in C$, that is, we have $\theta_{k_n} \in I_n - M_n\omega$ and $x_{k_n} \in C$. Applying (5.5) to each point (θ_{k_n}, x_{k_n}) finishes the proof. □

Acknowledgements. This work was carried out while I was at the Department of Mathematics and Statistics at Queen's University, Canada.

References

1. Adomaitis, R., Kevrekidis, I.G., de la Llave, R.: A computer-assisted study of global dynamic transitions for a noninvertible system. Internat. J. Bifur. Chaos Appl. Sci. Engrg. **17**(4), 1305–1321 (2007)
2. Avila, A., Moreira, C.G.: Statistical properties of unimodal maps: the quadratic family. Ann. of Math. (2) **161**(2), 831–881 (2005)
3. Benedicks, M., Carleson, L.: The dynamics of the Hénon map. Ann. of Math. (2) **133**(1), 73–169 (1991)
4. Bezhaeva, Z.I., Oseledets, V.I.: On an example of a "strange nonchaotic attractor". (Russian) Funkt. Anal. i Prilozhen. **30**(4), 1–9 (1996)
5. Bjerklöv, K.: Dynamics of the quasi-periodic Schrödinger cocycle at the lowest energy in the spectrum. Commun. Math. Phys. **272**(2), 397–442 (2007)
6. Bjerklöv, K.: Positive Lyapunov exponent and minimality for the continuous 1-d quasi-periodic Schrödinger equations with two basic frequencies. Ann. Henri Poincaré **8**(4), 687–730 (2007)
7. Bjerklöv, K.: Positive Lyapunov exponent and minimality for a class of one-dimensional quasi-periodic Schrödinger equations. Ergo. Th. Dynam. Syst. **25**(4), 1015–1045 (2005)
8. Grebogi, C., Ott, E., Pelikan, S., Yorke, J.A.: Strange attractors that are not chaotic. Phys D. **13**(1-2), 261–268 (1984)
9. Herman, M.: Une méthode pour minorer les exposants de Lyapounov et quelques exemples montrant le caractère local d'un théorème d'Arnold et de Moser sur le tore de dimension 2. Comment. Math. Helv. **58**(3), 453–502 (1983)
10. Jäger, T.H.: The creation of strange non-chaotic attractors in non-smooth saddle-node bifurcations. To appear in Memoirs of the AMS
11. Jäger, T.H.: Strange non-chaotic attractors in quasiperiodically forced circle maps. Preprint
12. Johnson, R.A.: Ergodic theory and linear differential equations. J. Diff. Eqs. **28**(1), 23–34 (1978)
13. Jorba, À., Tatjer, J.C., Núñez, C., Obaya, R.: Old and new results on strange nonchaotic attractors. Int. J. Bifur. Chaos Appl. Sci. Eng. **17**(11), 3895–3928 (2007)
14. Keller, G.: A note on strange nonchaotic attractors. Fund. Math. **151**(2), 139–148 (1996)
15. Millionščikov, V.M.: Proof of the existence of irregular systems of linear differential equations with almost periodic coefficients. (Russian) Differencialnye Uravnenija **4**, 391–396 (1968)
16. Prasad, A., Mehra, V., Ramaswamy, R.: Strange nonchaotic attractors in the quasiperiodically forced logistic map. Phys. Rev. E **57**, 1576–1584 (1998)
17. Stark, J.: Regularity of invariant graphs for forced systems. Erg. Th. Dynam. Syst. **19**(1), 155–199 (1999)

18. Viana, M.: Multidimensional nonhyperbolic attractors. Inst. Hautes Etudes Sci. Publ. Math. No. 85, 63–96 (1997)
19. Vinograd R.E.: On a problem of N. P. Erugin. (Russian) Differencialnye Uravnenija **11**(4), 632–638, 763 (1975)

Communicated by A. Kupiainen