

# Poisson Sigma Model on the Sphere

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**Abstract:** We evaluate the path integral of the Poisson sigma model on the sphere and study the correlators of quantum observables. We argue that for the path integral to be well-defined the corresponding Poisson structure should be unimodular. The construction of the finite dimensional BV theory is presented and we argue that it is responsible for the leading semiclassical contribution. For a (twisted) generalized Kähler manifold we discuss the gauge fixed action for the Poisson sigma model. Using the localization we prove that for the holomorphic Poisson structure the semiclassical result for the correlators is indeed the full quantum result.

## 1. Introduction

The Poisson sigma model (PSM), introduced in [24,43], is a topological two-dimensional field theory with target a Poisson manifold  $M$ , whose Poisson tensor we will denote by  $\alpha$  throughout. Recently PSM has attracted a lot of attention due to its role in the deformation quantization [6]. In particular the star product is given by a semiclassical expansion of the path integral of the PSM over the disk. In the present paper we study the PSM defined over the sphere.

Let us start with a brief reminder of PSM. Take  $\Sigma$  to be a two-dimensional oriented compact manifold without boundary. The starting point is the classical action functional  $S$  defined on the space of vector bundle morphisms  $\hat{X}: T\Sigma \rightarrow T^*M$  from the tangent bundle  $T\Sigma$  to the cotangent bundle  $T^*M$  of the Poisson manifold  $M$ . Such a map  $\hat{X}$  is given by its base map  $X: \Sigma \rightarrow M$  and the linear map  $\eta$  between fibers, which may also be regarded as a section in  $\Gamma(\Sigma, \text{Hom}(T\Sigma, X^*(T^*M)))$ . The pairing  $\langle \cdot, \cdot \rangle$  between the cotangent and tangent space at each point of  $M$  induces a pairing between the differential forms on  $\Sigma$  with values in the pull-backs  $X^*(T^*M)$  and  $X^*(TM)$  respectively. It is defined as pairing of the values and the exterior product of differential forms. Then the

action functional  $S$  of the theory is

$$S(X, \eta) = \int_{\Sigma} \langle \eta, dX \rangle + \frac{1}{2} \langle \eta, (\alpha \circ X) \eta \rangle. \tag{1.1}$$

Here  $\eta$  and  $dX$  are viewed as one-forms on  $\Sigma$  with the values in the pull-back of the cotangent and tangent bundles of  $M$  correspondingly. Thus, in local coordinates, we can rewrite the action (1.1) as follows:

$$S(X, \eta) = \int_D \eta_{\mu} \wedge dX^{\mu} + \frac{1}{2} \alpha^{\mu\nu}(X) \eta_{\mu} \wedge \eta_{\nu}. \tag{1.2}$$

The variation of the action gives rise to the following equations of motion:

$$d\eta_{\rho} + \frac{1}{2} (\partial_{\rho} \alpha^{\mu\nu}) \eta_{\mu} \wedge \eta_{\nu} = 0, \quad dX^{\mu} + \alpha^{\mu\nu} \eta_{\nu} = 0. \tag{1.3}$$

In covariant language these equations are equivalent to the statement that the bundle morphism  $\hat{X}$  is a Lie algebroid morphism from  $T\Sigma$  (with standard Lie algebroid structure) to  $T^*M$  (with Lie algebroid structure canonically induced by the Poisson structure). The action (1.2) is invariant under the infinitesimal gauge transformations

$$\delta_{\beta} X^{\mu} = \alpha^{\mu\nu} \beta_{\nu}, \quad \delta_{\beta} \eta_{\mu} = -d\beta_{\mu} - (\partial_{\mu} \alpha^{\nu\rho}) \eta_{\nu} \beta_{\rho}, \tag{1.4}$$

which form a closed algebra only on-shell (i.e., modulo the equations of motion (1.3)).

In order to quantize the PSM we have to resolve the Batalin–Vilkovisky (BV) formalism [3] which we will review later. In what follows we will concentrate mainly on the case when the world-sheet  $\Sigma$  is the two-sphere  $S^2$ . Our goal is to calculate a leading term for PSM correlators on  $S^2$ . We will argue that the notion of unimodularity appears naturally in the construction of the correlators. Indeed our construction is very similar to the one presented in [41] and is a generalization of the correlators for A- and B-models (see [23] for review). It is not surprising since the notion of generalized Calabi-Yau manifold given in [20] is a complex version of the notion of unimodularity of a Lie algebroid. In particular the unimodularity of the Poisson manifold is a real analog of the generalized Calabi-Yau condition. Previously in a different context the path integral for PSM and related models was also discussed in [4, 19, 32].

In the second part of the paper we consider a particular gauge fixing which involves a choice of an (almost) complex structure. The whole setup is realized on (twisted) generalized Kähler manifolds. For these gauge fixed models there exists a residual BRST symmetry which allows to use the localization. Thus we are able to produce examples where the leading term is a full answer for the quantum theory.

The paper is organized as follows. In Sect. 2 we review basic concepts of BV formalism. Section 3 is devoted to overview of BV treatment of PSM. In particular we discuss the classical observables. In Sect. 4 we consider the truncation of the full BV theory to a finite dimensional BV theory which is responsible for the leading semiclassical contribution in the correlators. We discuss this finite dimensional BV theory in detail. In this context the unimodularity of the Poisson manifold arises naturally from the quantum master equation. In Sect. 5 the specific gauge fixing is discussed. Indeed the geometrical set-up we are using is the same as for the  $N = 2$  supersymmetric PSM [5]. We work out the details of gauge fixing and discuss the residual BRST transformations of the gauge fixed action and present the calculations of the correlators for the gauge fixed model. Finally Sect. 6 summarizes the results and discusses open issues.

In addition we have Appendices A and B where the relevant mathematical material is collected. The material presented there is not entirely original and furthermore we could not find appropriate references with all material. Many of the results presented in the Appendices are scattered throughout the literature. Moreover we would like to link two different languages used by different communities. In particular the notion of generalized Calabi-Yau manifold introduced by Hitchin [20] is related to the notion of unimodularity for a complex Lie algebroid.

Throughout the paper we use the language of graded manifolds which are supermanifolds with a  $\mathbb{Z}$ -refinement of  $\mathbb{Z}_2$ -grading, e.g. see [42] for the review.

## 2. Review of BV Formalism

In this section we briefly review the relevant concepts within the general BV framework. For further details the reader may consult the following reviews [8, 13, 18].

**Definition 1.** A graded algebra  $\mathcal{A}$  with an odd bracket  $\{, \}$  is called an odd Poisson algebra (Gerstenhaber algebra) if the bracket satisfies

$$\begin{aligned} \{f, g\} &= -(-1)^{(|f|+1)(|g|+1)}\{g, f\}, \\ \{f, \{g, h\}\} &= \{\{f, g\}, h\} + (-1)^{(|f|+1)(|g|+1)}\{g, \{f, h\}\}, \\ \{f, gh\} &= \{f, g\}h + (-1)^{(|f|+1)|g|}g\{f, h\}. \end{aligned}$$

Quite often such an odd Poisson bracket is called either a Gerstenhaber bracket or an antibracket.

**Definition 2.** A Gerstenhaber algebra  $(\mathcal{A}, \{, \})$  together with an odd  $\mathbb{R}$ -linear map

$$\Delta : \mathcal{A} \longrightarrow \mathcal{A},$$

which squares to zero  $\Delta^2 = 0$  and generates the bracket  $\{, \}$  as

$$\{f, g\} = (-1)^{|f|}\Delta(fg) + (-1)^{|f|+1}(\Delta f)g - f(\Delta g),$$

is called a BV-algebra.  $\Delta$  is called an odd Laplace operator (odd Laplacian).

The canonical example of BV algebra is given by the space of functions on  $W \oplus \Pi W^*$ , where  $W$  is a superspace,  $W^*$  is its dual and  $\Pi$  stands for the reversed parity functor.  $W \oplus \Pi W^*$  is equipped with an odd non-degenerate pairing. Let  $y^a$  be the coordinates on  $W$  (the fields) and  $y_a^+$  be the corresponding coordinates on  $\Pi W^*$  (the antifields). We denote the parity of  $y^a$  as  $(-1)^{|y^a|}$  and that of  $y_a^+$  as  $(-1)^{|y_a^+|} = (-1)^{|y^a|+1}$ . Then the odd Laplacian is defined as follows:

$$\Delta = (-1)^{|y_a|} \frac{\partial}{\partial y_a^+} \frac{\partial}{\partial y^a}. \tag{2.5}$$

It generates the canonical antibracket on  $C^\infty(W \oplus \Pi W^*)$ ,

$$\{f, g\} = (-1)^{|y_a|} \overleftarrow{\frac{\partial}{\partial y_a^+}} f \overrightarrow{\frac{\partial}{\partial y^a}} g + (-1)^{|y_a|} \overleftarrow{\frac{\partial}{\partial y^a}} f \overrightarrow{\frac{\partial}{\partial y_a^+}} g, \tag{2.6}$$

where we use the notation  $\overrightarrow{\partial}_v f = \partial_v f$  and  $\overleftarrow{\partial}_v f = (-1)^{|v||f|} \partial_v f$ . Indeed the bracket (2.6) is non degenerate and defines the canonical odd symplectic structure on  $W \oplus \Pi W^*$ .

A Lagrangian submanifold  $\mathcal{L} \subset W \oplus \Pi W^*$  is an isotropic supermanifold of maximal dimension. The volume form  $dy^1 \dots dy^n dy_1^+ \dots dy_n^+$  induces a well defined volume form on  $\mathcal{L}$ . Thus the integral

$$\int_{\mathcal{L}} f, \quad f \in C^\infty(W \oplus \Pi W^*) \tag{2.7}$$

is defined for any  $\mathcal{L}$ . The following is the main theorem of BV-formalism.

**Theorem 3.** *If  $\Delta f = 0$ , then  $\int_{\mathcal{L}} f$  depends only on the homology class of  $\mathcal{L}$ . Moreover  $\int_{\mathcal{L}} \Delta f = 0$  for any Lagrangian  $\mathcal{L}$ .*

The canonical example  $W \oplus \Pi W^*$  can be generalized to the cotangent bundle  $T^*[-1]\mathcal{M}$  of any graded manifold  $\mathcal{M}$  [44]. As a cotangent bundle,  $T^*[-1]\mathcal{M}$  is naturally equipped with an odd Poisson bracket that makes  $C^\infty(T^*[-1]\mathcal{M})$  a Gerstenhaber algebra according to Definition 1. The idea is that locally one can map  $T^*[-1]\mathcal{M}$  to  $W \oplus \Pi W^*$ , define the bracket on coordinates with (2.6) and then glue the patches in a consistent manner.

Now in order to define the odd Laplacian  $\Delta$  we need an integration over  $T^*[-1]\mathcal{M}$ . Namely, the choice of a volume form  $v$  on  $\mathcal{M}$  produces the corresponding volume form  $\mu_v$  on  $T^*[-1]\mathcal{M}$ . The divergence operator is defined as a map from the vector fields on  $T^*[-1]\mathcal{M}$  to  $C^\infty(T^*[-1]\mathcal{M})$  through the following integral relation

$$\int_{T^*[-1]\mathcal{M}} X(f) \mu_v = - \int_{T^*[-1]\mathcal{M}} \text{div}_{\mu_v} X f \mu_v, \quad \forall f \in C^\infty(T^*[-1]\mathcal{M}), \tag{2.8}$$

with  $X$  being a vector field. As one can easily check, for any function  $f$  and vector field  $X$  the divergence satisfies

$$\text{div}_{\mu_v}(fX) = f \text{div}_{\mu_v}(X) + (-1)^{|f||X|} X(f). \tag{2.9}$$

Now the odd Laplacian of  $f \in C^\infty(T^*[-1]\mathcal{M})$  is defined through the divergence of the corresponding Hamiltonian vector field as

$$\Delta_v f = \frac{(-1)^{|f|}}{2} \text{div}_{\mu_v} X_f, \quad \{f, g\} = X_f(g). \tag{2.10}$$

Indeed one can check that thanks to (2.9)  $\Delta_v$  generates the bracket and  $\Delta_v^2 = 0$ . Thus  $C^\infty(T^*[-1]\mathcal{M})$  is a BV-algebra according to Definition 2, see [29] for the explicit calculations. If the volume form is written in terms of an even density  $\rho_v$  as

$$\mu_v = \rho_v dy^1 \dots dy^n dy_1^+ \dots dy_n^+,$$

then the Laplacian can be written as

$$\Delta_v = (-1)^{|y_a|} \frac{\partial}{\partial y_a^+} \frac{\partial}{\partial y^a} + \frac{1}{2} \{\log \rho_v, -\}. \tag{2.11}$$

There exists a canonical way (up to a sign) of restricting a volume form  $\mu_v$  on  $T^*[-1]\mathcal{M}$  to a volume form on a Lagrangian submanifold  $\mathcal{L}$ . We denote such restriction as  $\sqrt{\mu_v}$  and consider the integrals of the form

$$\int_{\mathcal{L}} \sqrt{\mu_v} f, f \in C^\infty(T^*[-1]\mathcal{M}). \tag{2.12}$$

Thus Theorem 3 will remain true for the general case. In particular we are interested in the situation when the integrands in (2.12) are of the form

$$\int_{\mathcal{L}} \sqrt{\mu_v} \Psi e^S \equiv \langle \Psi \rangle, \tag{2.13}$$

where we assume naturally that  $\Delta_v(\Psi e^S) = 0$ . If  $\Psi = 1$  then we get the following relation:

$$\Delta_v(e^S) = 0 \iff \Delta_v S + \frac{1}{2}\{S, S\} = 0, \tag{2.14}$$

which is known as the *quantum master equation*. In the general case we have

$$\Delta_v(\Psi e^S) = 0 \iff \Delta_{(v,S)}\Psi = \Delta_v\Psi + \{S, \Psi\} = 0, \tag{2.15}$$

where we refer to  $\Delta_{(v,S)}$  as the quantum Laplacian. In the derivation of (2.15) we have used the quantum master equation (2.14). A function  $S$  that satisfies the quantum master equation is called a quantum BV action and  $\Psi$  satisfying (2.15) is a quantum observable. Indeed the quantum observables are elements of the cohomology  $H(\Delta_{(v,S)})$ ; by the above construction it is clear that  $S$  defines the isomorphism

$$H^\bullet(\Delta_v) \approx H^\bullet(\Delta_{(v,S)}). \tag{2.16}$$

If we change  $S$  to  $S/\hbar$ , we see that in the classical limit ( $\hbar \rightarrow 0$ )  $S$  must satisfy the classical master equation  $\{S, S\} = 0$  and the classical observables  $\Psi$  are such that  $\delta_{BV}\Psi \equiv \{S, \Psi\} = 0$ . Due to the classical master equation the vector field  $\delta_{BV}$  squares to zero and defines the cohomology  $H(\delta_{BV})$  of classical observables.

If  $\mathcal{M}$  is a finite dimensional manifold then everything is well-defined. However in field theory one deals with  $\mathcal{M}$  being infinite dimensional. In fact,  $\mathcal{M}$  is usually the space of the physical fields, ghosts and Lagrange multipliers, that is infinite dimensional. We extend this set of fields by adding antifields such that together they form  $T^*[-1]\mathcal{M}$ , where an odd Poisson bracket is well-defined on a large enough class of functions, as described above. However there are no well-defined measure on  $\mathcal{M}$  and thus there are no well-defined odd Laplace operators. In the physics literature, the naive Laplacian of the form (2.6) is used. Moreover the field theory suffers from the problems with renormalization which can be resolved within the perturbative setup.

### 3. BV Formalism for PSM

The quantization of PSM requires the machinery of the BV formalism. In this section we set the notation and give a background information on the BV treatment of PSM. We mainly review the relevant results from [6] and [7]. Furthermore we discuss the classical observables.

3.1. *BV action.* The PSM action (1.2) has gauge symmetries which do not close off-shell. Therefore one should resort to the BV formalism. We may organize the fields, ghosts and antifields into superfields  $(X, \eta)$  which correspond to the components of supermap  $T[1]\Sigma \rightarrow T^*[1]M$ . Introducing the local coordinates on  $\Sigma$  and  $M$  the superfields read as

$$X^\mu = X^\mu + \theta^\alpha \eta_\alpha^{+\mu} - \frac{1}{2} \theta^\alpha \theta^\beta \beta_{\alpha\beta}^{+\mu},$$

$$\eta_\mu = \beta_\mu + \theta^\alpha \eta_{\alpha\mu} + \frac{1}{2} \theta^\alpha \theta^\beta X_{\alpha\beta\mu}^+,$$

with  $\theta$  being the odd coordinate on  $\Pi T\Sigma$ ,  $\alpha, \beta$  are labels for local coordinates on  $\Sigma$  and  $\mu$  are labels for local coordinates on  $M$ . In the expansion  $\beta$  is a ghost with the ghost number 1, while  $\eta^+, \beta^+$  and  $X^+$  are antifields of ghost number  $-1, -2$  and  $-1$  respectively. The full BV action reads

$$S_{BV} = \int d^2\theta d^2u \left( \eta_\mu D X^\mu + \frac{1}{2} \alpha^{\mu\nu}(X) \eta_\mu \eta_\nu \right), \tag{3.17}$$

where  $D = \theta^\alpha \partial_\alpha$ . An elegant way to derive this action is to use the AKSZ formalism [1] as done in [7]. On  $T^*[-1]\mathcal{M}$  the odd symplectic structure is

$$\omega = \int_\Sigma (\delta X \wedge \delta X^+ + \delta \eta \wedge \delta \eta^+ + \delta \beta \wedge \delta \beta^+), \tag{3.18}$$

where  $\mathcal{M}$  is an infinite dimensional manifold corresponding to the fields  $(X, \eta, \beta)$ . The action (3.17) satisfies both classical and naive quantum master equations [6]. The corresponding BRST operator  $\delta_{BV}$  acts on the superfields as follows:

$$\delta_{BV} X^\mu = D X^\mu + \alpha^{\mu\nu}(X) \eta_\nu, \tag{3.19}$$

$$\delta_{BV} \eta_\mu = D \eta_\mu + \frac{1}{2} \partial_\mu \alpha^{\nu\rho}(X) \eta_\nu \eta_\rho. \tag{3.20}$$

In component the BV action (3.17) has the form

$$S_{BV} = \int_\Sigma \eta_\mu \wedge d X^\mu + \frac{1}{2} \alpha^{\mu\nu}(X) \eta_\mu \wedge \eta_\nu + X_\mu^+ \alpha^{\mu\nu}(X) \beta_\nu - \eta^{+\mu} \wedge (d\beta_\mu + \partial_\mu \alpha^{\rho\nu}(X) \eta_\rho \beta_\nu) - \frac{1}{2} \beta^{+\mu} \partial_\mu \alpha^{\rho\nu}(X) \beta_\rho \beta_\nu - \frac{1}{4} \eta^{+\mu} \wedge \eta^{+\nu} \partial_\mu \partial_\nu \alpha^{\rho\sigma}(X) \beta_\rho \beta_\sigma. \tag{3.21}$$

The component version of the BV transformations (3.19)–(3.20) is

$$\delta_{BV} X^\mu = \alpha^{\mu\nu}(X) \beta_\nu, \tag{3.22}$$

$$\delta_{BV} \eta^{+\mu} = -d X^\mu - \alpha^{\mu\nu}(X) \eta_\nu - \partial_\nu \alpha^{\mu\rho}(X) \eta^{+\nu} \beta_\rho, \tag{3.23}$$

$$\delta_{BV} \beta^{+\mu} = -d \eta^{+\mu} - \alpha^{\mu\nu}(X) X_\nu^+ + \frac{1}{2} \partial_\nu \partial_\rho \alpha^{\mu\sigma}(X) \eta^{+\nu} \wedge \eta^{+\rho} \beta_\sigma + \partial_\rho \alpha^{\mu\nu}(X) \eta^{+\rho} \wedge \eta_\nu + \partial_\rho \alpha^{\mu\nu}(X) \beta^{+\rho} \beta_\nu, \tag{3.24}$$

$$\delta_{BV} \beta_\mu = \frac{1}{2} \partial_\mu \alpha^{v\rho}(X) \beta_v \beta_\rho, \tag{3.25}$$

$$\delta_{BV} \eta_\mu = -d\beta_\mu - \partial_\mu \alpha^{v\rho}(X) \eta_v \beta_\rho - \frac{1}{2} \partial_\mu \partial_v \alpha^{\rho\sigma}(X) \eta^{+v} \beta_\rho \beta_\sigma, \tag{3.26}$$

$$\begin{aligned} \delta_{BV} X_\mu^+ &= d\eta_\mu + \partial_\mu \alpha^{v\rho}(X) X_v^+ \beta_\rho - \partial_\mu \partial_v \alpha^{\rho\sigma}(X) \eta^{+v} \wedge \eta_\rho \beta_\sigma + \frac{1}{2} \partial_\mu \alpha^{v\rho}(X) \eta_v \wedge \eta_\rho \\ &\quad - \frac{1}{4} \partial_\mu \partial_v \partial_\rho \alpha^{\sigma\tau}(X) \eta^{+v} \wedge \eta^{+\rho} \beta_\sigma \beta_\tau - \frac{1}{2} \partial_\mu \partial_v \alpha^{\rho\sigma}(X) \beta^{+v} \beta_\rho \beta_\sigma. \end{aligned} \tag{3.27}$$

3.2. *Classical observables.* Next we consider the classical observables for PSM. By an observable we mean a BRST invariant operator which is not BRST exact.

Let us take the antisymmetric multivector field  $w \in \Gamma(\wedge^p TM)$  and construct the superfield  $w^{\mu_1 \dots \mu_p}(X) \eta_{\mu_1} \dots \eta_{\mu_p}$ . Using (3.19)–(3.20) we calculate the BRST transformation of this superfield

$$\begin{aligned} \delta_{BV}(w^{\mu_1 \dots \mu_p} \eta_{\mu_1} \dots \eta_{\mu_p}) &= D(w^{\mu_1 \dots \mu_p} \eta_{\mu_1} \dots \eta_{\mu_p}) \\ &\quad - \frac{1}{2}([\alpha, w]_s)^{\mu_0 \mu_1 \dots \mu_p} \eta_{\mu_0} \eta_{\mu_1} \dots \eta_{\mu_p}. \end{aligned} \tag{3.28}$$

The last term on the right-hand side vanishes if  $d_{LP} w = [\alpha, w]_s = 0$ . Moreover we do not want the superfield  $w^{\mu_1 \dots \mu_p} \eta_{\mu_1} \dots \eta_{\mu_p}$  to be BRST exact. Thus we have to take  $w$  to be an element in the Lichnerowicz-Poisson cohomology  $H_{LP}^\bullet(M)$ . Now assuming  $[w] \in H_{LP}^\bullet(M)$ , we can interpret (3.28) in components. The superfield has the expansion

$$w^{\mu_1 \dots \mu_p} \eta_{\mu_1} \dots \eta_{\mu_p} = O_0^p + \theta^\alpha (O_1^{p-1})_\alpha + \frac{1}{2} \theta^\alpha \theta^\beta (O_2^{p-2})_{\alpha\beta}$$

on which the BRST differential  $\delta_{BV}$  acts as

$$\delta_{BV}(w^{\mu_1 \dots \mu_p} \eta_{\mu_1} \dots \eta_{\mu_p}) = \delta_{BV} O_0^p - \theta^\alpha \delta_{BV} (O_1^{p-1})_\alpha + \frac{1}{2} \theta^\alpha \theta^\beta \delta_{BV} (O_2^{p-2})_{\alpha\beta}.$$

The operator  $D = \theta^\alpha \partial_\alpha$  acts on the component fields as the de Rham differential. Thus for  $[w] \in H_{LP}^\bullet(M)$  the condition (3.28) implies the descent equations for the components

$$\delta_{BV} O_0^p = 0, \quad \delta_{BV} O_1^{p-1} = -dQ_0^p, \quad \delta_{BV} O_2^{p-2} = dQ_1^{p-1}. \tag{3.29}$$

More explicitly for a nontrivial element  $[w] \in H_{LP}^p(M)$  we can formally define the cocycles

$$O_0^p(w) = w^{\mu_1 \dots \mu_p} \beta_{\mu_1} \dots \beta_{\mu_p}, \tag{3.30}$$

$$O_1^{p-1}(w) = \partial_\rho w^{\mu_1 \dots \mu_p} \eta^{+\rho} \beta_{\mu_1} \dots \beta_{\mu_p} + p w^{\mu_1 \mu_2 \dots \mu_p} \eta_{\mu_1} \beta_{\mu_2} \dots \beta_{\mu_p}, \tag{3.31}$$

$$\begin{aligned} O_2^{p-2}(w) &= -\frac{1}{2} \partial_\rho \partial_\sigma w^{\mu_1 \dots \mu_p} \eta^{+\rho} \wedge \eta^{+\sigma} \beta_{\mu_1} \dots \beta_{\mu_p} - \partial_\rho w^{\mu_1 \dots \mu_p} \beta^{+\rho} \beta_{\mu_1} \dots \beta_{\mu_p} \\ &\quad - p \partial_\rho w^{\mu_1 \dots \mu_p} \eta^{+\rho} \wedge \eta_{\mu_1} \beta_{\mu_2} \dots \beta_{\mu_p} + p w^{\mu_1 \dots \mu_p} X_{\mu_1}^+ \beta_{\mu_2} \dots \beta_{\mu_p} \\ &\quad + p(p-1) w^{\mu_1 \dots \mu_p} \eta_{\mu_1} \wedge \eta_{\mu_2} \beta_{\mu_3} \dots \beta_{\mu_p}, \end{aligned} \tag{3.32}$$

where in  $O_i^{p-i}(w)$  the upper index stands for the ghost number and the lower index for the degree of the differential form on  $\Sigma$ .  $Q_i^{p-i}(w)$  satisfy (3.29) and thus  $O_0^p(w)$  are BRST-invariant local observables labeled by the elements of the Lichnerowicz-Poisson cohomology  $H_{LP}^\bullet(M)$ . From  $O_i^{p-i}(w)$  with  $i > 0$  we can construct BRST-invariant non-local observables as integrals

$$W(w, c_i) = \int_{c_i} O_i^{p-i}(w), \tag{3.33}$$

where  $c_i$  is an  $i$ -cycle on  $\Sigma$ . These observables depend only on the homology class of  $c_i$ . The antibracket  $\{, \}$  of two non-local observables

$$\{W(w, \Sigma), W(\lambda, \Sigma)\} = -W([w, \lambda]_s, \Sigma) \tag{3.34}$$

gets mapped into the Schouten bracket between the multivector fields [6].

*3.3. General comments on the path integral.* The main task is to calculate the correlation functions of observables which can be represented as the path integral expression

$$\langle W(w_1, c_{i_1}) \dots W(w_n, c_{i_n}) \rangle = \int_{\mathcal{L}} \mathcal{D}X \mathcal{D}\eta W(w_1, c_{i_1}) \dots W(w_n, c_{i_n}) e^{\frac{i}{\hbar} S_{BV}}. \tag{3.35}$$

For this integral to make sense at least perturbatively we have to integrate not over the whole functional space but over the ‘‘Lagrangian’’ submanifold  $\mathcal{L}$ . The choice of  $\mathcal{L}$  is called the gauge fixing and it is typically generated by a gauge fixing fermion  $\Psi$ . The path integral (3.35) is invariant under the deformations of the Lagrangian submanifold  $\mathcal{L}$ .

However due to the absence of any well-defined measure on the space of fields we cannot treat this integral non-perturbatively. Despite this difficulty we can address and even sometimes solve it completely from the different direction, namely by reducing to an appropriate finite dimensional problem. We would expect that the correlator (3.35) has a well-defined expansion in non-negative powers of  $\hbar$ . In particular there will be a leading term in this expansion which we can evaluate by consistent reduction of the full theory to a finite dimensional BV theory for which all objects can be defined. This reduction will produce the leading terms in the correlators. Indeed for some models these terms correspond to a full quantum result. In Sect. 4 we will consider the finite dimensional BV theory responsible for leading terms in the correlators on  $S^2$ .

In Sect. 5 we present the details for a concrete choice of  $\mathcal{L}$ . The gauge fixed theory will have residual BRST symmetry which allows us to localize the infinite dimensional integrals to finite dimensional.

### 4. The Reduced BV Theory

In this section we consider a consistent truncation of the infinite dimensional BV theory to a finite dimensional one, that computes the contribution of constant configurations. We conjecture that this reduced BV theory controls the leading contribution into the path integral in the limit  $\hbar \rightarrow 0$ .



This procedure can be considered as a reduction of  $BV$ -manifolds and for a Riemann surface  $\Sigma_g$  of genus  $g$  the truncation can be organized in the following fashion. We define the submanifold  $\mathcal{C}$  of the whole space of fields by requiring that all fields are closed forms

$$dX = 0, \quad d\beta = 0, \quad d\eta = 0, \quad d\eta^+ = 0, \quad dX^+ = 0, \quad d\beta^+ = 0. \quad (4.36)$$

These equations define a set of first class constraints (the conditions  $dX^+ = d\beta^+ = 0$  are redundant since  $X^+$  and  $\beta^+$  are the top forms), i.e.  $\mathcal{C}$  is a coisotropic submanifold. The gauge transformations generated by the constraints (4.36) shift the field by an exact form. Therefore the reduced  $BV$  space is obtained by going to the cohomology of  $\Sigma_g$ . The reduced variables are then defined by the integration of the fields over all cycles of  $\Sigma_g$ . Thus zero-forms  $X$  and  $\beta$  are constants, and we use the same symbols to indicate the reduced coordinates. For one-forms we choose the basis  $\{c_a\}$  in  $H_1(\Sigma_g, \mathbb{R}) = H^1(\Sigma_g, \mathbb{R})$  and introduce the reduced coordinates

$$\eta_a = \int_{c_a} \eta, \quad \eta_a^+ = \int_{c_a} \eta^+.$$

While two-forms  $X^+$  and  $\beta^+$  are integrated over whole  $\Sigma$  and give

$$X^+ = \int_{\Sigma_g} X^+, \quad \beta^+ = \int_{\Sigma_g} \beta^+.$$

All the  $BV$  structure goes to the quotient and defines a finite dimensional  $BV$  manifold. The space  $H^1(\Sigma_g, \mathbb{R})$  is symplectic with the structure  $\omega^{ab}$ . Therefore on the reduced finite dimensional manifold, the odd symplectic structure (3.18) reads

$$\omega = dX^\mu dX_\mu^+ + \omega^{ab} d\eta_a d\eta_b^+ + d\beta_\mu d\beta^{+\mu}. \quad (4.37)$$

Moreover, the  $BV$  action  $S_{BV}$  defined in (3.21) when restricted to  $\mathcal{C}$  depends only on the reduced variables, i.e. it is a pull-back of a function on the reduced manifold. We use the same notation  $S_{BV}$  for it.

However we are interested in zero genus case, and we leave for future investigations the case of genus  $g > 0$ . In this situation the corresponding finite dimensional  $BV$  manifold is  $\mathcal{F} = T^*[-1]T^*[1]M$ , where the odd symplectic structure is written in the coordinates  $z = (X^\mu, \beta_\mu, X_\mu^+, \beta^{+\mu})$  as

$$\omega = dX^\mu dX_\mu^+ + d\beta_\mu d\beta^{+\mu}. \quad (4.38)$$

The degree of the coordinates is the one induced from the corresponding fields. Under a coordinate change  $\tilde{X}^i(X^\mu)$ , the new coordinates  $\tilde{z} = (\tilde{X}^i, \tilde{\beta}_i, \tilde{X}^{+i}, \tilde{\beta}^{+i})$  are

$$\tilde{\beta}_i = T_i^\mu \beta_\mu, \quad \tilde{\beta}^{+i} = T_\mu^i \beta^{+\mu}, \quad \tilde{X}_i^+ = X_\mu^+ T_i^\mu - \beta^{+\mu} \beta_\nu \frac{\partial T_j^v}{\partial Y^i} (T^{-1})_\mu^j, \quad (4.39)$$

where  $T_i^\mu = \partial X^\mu / \partial \tilde{X}^i$ . The  $BV$  action (3.21) becomes

$$S_{BV} = X_\mu^+ \alpha^{\mu\nu}(X) \beta_\nu - \frac{1}{2} \beta^{+\mu} \partial_\mu \alpha^{\rho\nu}(X) \beta_\rho \beta_\nu, \quad (4.40)$$

which obviously satisfies the classical master equation. In the following discussion we will analyze this finite dimensional BV theory and claim that it gives the leading contribution to PSM correlators. Later using a particular gauge fixing we will confirm this statement.

In addition to the BV reduction described above we can provide a different heuristic argument in the support of our construction. The action (4.40) can be understood as a leading term in the effective BV theory with the “constant” maps as IR degrees of freedom. The reader may consult [31, 40] for the explanation the effective actions within the BV framework.

*4.1. Integration on finite dimensional BV manifold.* We start by defining the integration over  $\mathcal{F} = T^*[-1]T^*[1]M$ . This will allow us to define an odd Laplacian which is necessary for a proper BV description, according to the lines outlined in Sect. 2.

Integration on  $\mathcal{F}$  can be defined by putting together berezinian integration in the odd directions of  $X_\mu^+$  and  $\beta_\mu$  and fiberwise integration in the even directions of  $\beta^{+\mu}$ . Let us choose a volume form  $\Omega = \Omega_{\mu_1 \dots \mu_n} dX^{\mu_1} \dots dX^{\mu_n} = \rho_\Omega dX^1 \dots dX^n$  on  $M$ .

We introduce the volume form  $\mu_\Omega = \rho_\Omega^4 Dz$ , where  $Dz = dX^1 \dots d\beta_1 \dots dX_1^+ \dots d\beta^{+1} \dots$  is the coordinate volume form. Since under the change of coordinates (4.39) the coordinate volume form transforms as

$$D\tilde{z} = \text{Ber} \frac{\partial \tilde{z}}{\partial z} Dz, \quad \text{Ber} \begin{pmatrix} I_{00} & I_{01} \\ I_{10} & I_{11} \end{pmatrix} = \frac{\det(I_{00} - I_{01} I_{11}^{-1} I_{10})}{\det I_{11}},$$

it is simple to check that  $\mu_\Omega$  is well defined. By applying (2.11), we get

$$\Delta_\Omega = \frac{\partial}{\partial X_\mu^+} \frac{\partial}{\partial X^\mu} - \frac{\partial}{\partial \beta^{+\mu}} \frac{\partial}{\partial \beta_\mu} + 2\{\log \rho_\Omega, -\}.$$

The restriction to  $\mathcal{F}$  of local and the non-local observables (3.32) associated to multi-vector fields defines the corresponding observables on the reduced manifold  $\mathcal{F}$ . Namely, to  $w \in \Gamma(\wedge^p TM)$  we associate the local observable

$$O_0^p(w) = w^{\mu_1 \dots \mu_p} \beta_{\mu_1} \dots \beta_{\mu_p}, \tag{4.41}$$

and the non-local one

$$O_2^{p-2}(w) = -\partial_\rho w^{\mu_1 \dots \mu_p} \beta^{+\rho} \beta_{\mu_1} \dots \beta_{\mu_p} + p w^{\mu_1 \dots \mu_p} X_{\mu_1}^+ \beta_{\mu_2} \dots \beta_{\mu_p}. \tag{4.42}$$

It is straightforward to check that they are covariant under the transformation of coordinates (4.39). The antibracket defined by the odd symplectic structure (4.37) between local and non-local observables can be expressed in terms of the Schouten bracket; let  $w \in \Gamma(\wedge^p TM)$ ,  $\lambda \in \Gamma(\wedge^\ell TM)$ , then we have that

$$\begin{aligned} \{O_2^{p-2}(w), O_0^\ell(\lambda)\} &= -O_0^{p+\ell-1}([w, \lambda]_S) \\ \{O_2^{p-2}(w), O_2^{\ell-2}(\lambda)\} &= -O_2^{p+\ell-3}([w, \lambda]_S), \end{aligned} \tag{4.43}$$

in analogy with (3.34). The odd Laplacian  $\Delta_\Omega$  acts on this observable as follows:

$$\Delta_\Omega O_2^{p-2}(w) = -2(D_\Omega(w))^{\mu_1 \dots \mu_{p-1}} \beta_{\mu_1} \dots \beta_{\mu_{p-1}} = -2O_0^{p-1}(D_\Omega(w)), \tag{4.44}$$

where  $D_\Omega$  is the divergence associated to the volume form  $\Omega$  defined in Appendix A. The  $BV$ -differential also descends to the reduced manifold as  $\delta_{BV}(F) = \{S_{BV}, F\}$ , for any  $F \in C^\infty(\mathcal{F})$ .

The action  $S_{BV} = 1/2 O_2^0(\alpha)$  defined in (4.40) satisfies the quantum master equation (2.14) if the following holds:

$$\Delta_\Omega S_{BV} + \frac{1}{2} \{S_{BV}, S_{BV}\} = 0 \iff D_\Omega \alpha = 0, \quad [\alpha, \alpha]_s = 0, \tag{4.45}$$

where  $[\cdot, \cdot]_s$  is the Schouten bracket on multivector fields, see Appendix A for the definitions. Thus the classical and quantum master equations have to be satisfied simultaneously. The geometrical meaning of the quantum master equation is clear: the volume form  $\Omega$  must be invariant under the flow of the hamiltonian vector fields of  $\alpha$ . The existence of such a volume form is equivalent to the unimodularity of the Poisson tensor, see the discussion in Appendix A. More generally, we may say that the action (4.40) is the zero order in  $\hbar$  of the solution of the quantum master equation if and only if  $\alpha$  is Poisson and unimodular<sup>1</sup>. If  $\Omega$  is not an invariant form then the unimodularity of  $\alpha$  implies

$$D_\Omega \alpha = -d_{LP} f, \tag{4.46}$$

for some function  $f(X)$ . This would correspond to the addition to

$$S_{BV} + 2\hbar f(X).$$

Equivalently this amounts to the redefinition  $\Omega$  by  $e^{\hbar f} \Omega$ . In what follows we set  $\hbar = 1$ .

By applying formulas (4.43), we see that for any  $w \in \Gamma(\Lambda^* TM)$  we have

$$\Delta_{(\Omega, \alpha)} O_0^p(w) = 0 \iff d_{LP}(w) = 0, \tag{4.47}$$

and thus the local observable associated to  $w$  is a quantum observable iff  $d_{LP} w = 0$ . The non-local observable  $O_2^{p-2}(w)$  will be quantum if the following holds:

$$\begin{aligned} \Delta_\Omega \left( O_2^{p-2}(w) e^{S_{BV}} \right) = 0 &\iff \Delta_{(\Omega, \alpha)}(O_2^{p-2}(w)) = 0 \\ &\iff D_\Omega w = 0, \quad d_{LP} w = 0. \end{aligned} \tag{4.48}$$

Moreover, by applying (4.43) we see that local and nonlocal observables form a subcomplex of the quantum laplacian  $\Delta_{(\Omega, \alpha)} = \Delta_\Omega + \delta_{BV}$ . See the next subsection for the discussion of these observables.

Finally we can evaluate the path integral. We have to choose a Lagrangian submanifold  $\mathcal{L}$  and the most obvious choice is  $\mathcal{L} = \{X^+ = 0, \beta^+ = 0\}$ . In order to compensate the odd integration we have to insert into the path integral the local observables

$$\int_{\mathcal{L}} O_0^{p_1}(w_1) \dots O_0^{p_k}(w_k) e^{S_{BV}} = tr_\Omega(w_1 \wedge \dots \wedge w_k), \tag{4.49}$$

where the trace map is defined in Appendix B. This expression is non-zero only if  $p_1 + \dots + p_k = d$ . With this choice of lagrangian submanifold, the nonlocal observables are identically zero.

We conclude that in the present finite dimensional  $BV$ -theory the action (4.40) satisfies the quantum master equation if the Poisson tensor  $\alpha$  is unimodular. This is equivalent

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<sup>1</sup> Within the general BV framework it can be shown that the modular class corresponds to the first obstruction to the existence of a quantum master action [36].

to the requirement that there exists a trace map  $tr_\Omega$  satisfying two properties in Theorem 9 of Appendix A. In Appendix A we present the mathematical discussion of these properties. Below we present “physical” derivation of those identities. The first property of  $tr_\Omega$  from Theorem 9 is a consequence of the quantum master equation for  $S_{BV}$  (i.e., the unimodularity of Poisson structure  $\alpha$ ). Namely we have the following chain of relations

$$\begin{aligned} tr_\Omega \left( d_{LP}(w) \wedge \lambda - (-1)^{|w|+1} w \wedge d_{LP}(\lambda) \right) &= tr_\Omega (d_{LP}(w \wedge \lambda)) \\ &= -2 \int_{\mathcal{L}} \{e^{S_{BV}}, O_0^{|w|+|\lambda|}(w \wedge \lambda)\} = -2 \int_{\mathcal{L}} \Delta_\Omega \left( e^{S_{BV}} O_0^{|w|+|\lambda|}(w \wedge \lambda) \right) = 0. \end{aligned}$$

This property implies that the trace map  $tr_\Omega$  descends to the Lichnerowicz–Poisson cohomology  $H_{LP}^\bullet(M)$ . The second property in Theorem 9 is a simple consequence of the fundamental BV Theorem 3. To be specific for the multivector fields  $w, \lambda$  we have the following relations:

$$\begin{aligned} tr_\Omega \left( D_\Omega(w) \wedge \lambda - (-1)^{|w|} w \wedge D_\Omega(\lambda) \right) &= \int_{\mathcal{L}} \left( O_0^{|w|-1}(D_\Omega w) O_0^{|\lambda|}(\lambda) - (-1)^{|w|} O_0^{|w|}(w) O_0^{|\lambda|-1}(D_\Omega \lambda) \right) \\ &= -2 \int_{\mathcal{L}} \Delta_\Omega \left( O_2^{|w|-2}(w) O_0^{|\lambda|}(\lambda) - O_0^{|w|}(w) O_2^{|\lambda|-2}(\lambda) \right) = 0, \end{aligned}$$

where (4.45) and (4.48) have been used. This property implies that the trace descends to the cohomology of  $D_\Omega$ . The cohomology of  $D_\Omega$  on the multivectors  $H^\bullet(D_\Omega)$  is isomorphic to the de Rham cohomology  $H_{dR}^\bullet(M)$ .

In the present context it is worthwhile to mention another interesting property of the trace map  $tr_\Omega$  on multivector fields. For the unimodular Poisson structure  $\alpha$  there is the following relation:

$$e^{-\alpha} D_\Omega e^\alpha = d_{LP} + D_\Omega, \tag{4.50}$$

where  $e^\alpha$  acts on the multivector field  $w$  as

$$e^\alpha w = w + \alpha \wedge w + \frac{1}{2} \alpha \wedge \alpha \wedge w + \dots,$$

and  $D_\Omega e^\alpha = 0$  is used. The relation (4.50) implies the isomorphism of cohomologies,  $H^\bullet(d_{LP} + D_\Omega) \approx H_{dR}^\bullet(M)$ . Moreover the trace map  $tr_\Omega$  descends to the cohomology  $H^\bullet(d_{LP} + D_\Omega)$ .

**4.2. Maurer-Cartan equation and formal Frobenius manifolds.** In this subsection we comment on the relation between the BV setting described above and the construction of Frobenius manifolds from BV-manifolds which appeared previously in mathematical works, in particular in the papers by Barannikov and Kontsevich [2] and by Manin [38,39]. Our observations have preliminary and speculative character. We plan to come back to this subject elsewhere.

The BV theory discussed in the previous section can be deformed by adding to the solution (4.40) of the quantum master equation any observable of ghost number 0.

Take  $w(t) \in \Gamma(\wedge^2 TM[[t]])$  with  $t$  being a formal parameter of degree zero such that  $w = w(0)$ . Consider the deformed action

$$S_{BV}(t) = S_{BV} + \frac{t}{2} O_2^0(w(t)). \tag{4.51}$$

Obviously,  $S_{BV}(t)$  satisfies the quantum master equation if and only if  $\alpha + tw(t)$  is an unimodular Poisson structure with the invariant volume form  $\Omega$ . This is equivalent to the Maurer–Cartan equation for  $w(t)$ ,

$$d_{LP}w(t) + \frac{t}{2}[w(t), w(t)]_S = 0, \quad D_\Omega w(t) = 0. \tag{4.52}$$

At the infinitesimal level this means  $d_{LP}w = D_\Omega w = 0$  and thus  $O_2^0(w)$  is a quantum non-local observable. However it is natural to allow the volume form  $\Omega$  to vary and use the argument presented around Eq. (4.46). Therefore we can describe the infinitesimal deformations as follows:

$$d_{LP}w = 0, \quad D_\Omega w + d_{LP}f = 0, \tag{4.53}$$

with  $w + f \in \Gamma(\wedge^2 TM \oplus \wedge^0 TM)$ , where  $w$  corresponds to the deformations of unimodular Poisson structure and  $f$  to the deformations of the volume form. Eqs. (4.53) can be equivalently rewritten as follows:

$$(d_{LP} + D_\Omega)(w + f) = e^{-\alpha} D_\Omega e^\alpha (w + f) = 0, \tag{4.54}$$

where we assume that  $\Omega$  is an invariant volume form for  $\alpha$ . In BV theory the deformation will be trivial if it is in the image of the quantum Laplacian  $\Delta_{(\Omega, \alpha)}$ . However the question is to understand the geometrical description of these trivial BV deformations. For example, the diffeomorphisms give a trivial deformation of the BV theory. Namely for  $w = \mathcal{L}_\xi \alpha = d_{LP}(\xi)$  and  $f = D_\Omega \xi$  for  $\xi \in \Gamma(TM)$  the deformation is trivial,

$$\frac{1}{2} O_2^0(w) + 2O_0^0(f) = -\Delta_{(\Omega, \alpha)} O_2^{-1}(\xi).$$

However the formula (4.54) suggests that the deformations is trivial if

$$w + f = (d_{LP} + D_\Omega)\xi = e^{-\alpha} D_\Omega(e^\alpha \xi), \tag{4.55}$$

with  $\xi \in \Gamma(\wedge^\bullet TM)$ , not just simply a vector. One has to show that the corresponding deformations of the BV theory are trivial. Unfortunately we are unable to do it in all generality. Nevertheless we give some plausible arguments in its favor and analyze the problem in special cases.

The linear space of deformations defined as the condition (4.54) modulo the identification (4.55) would be interpreted as the tangent space to some kind of modular space of unimodular Poisson structures (if such space exists). The crucial point motivated by the BV consideration is that the Poisson tensors may be equivalent even if they are not diffeomorphic. Indeed the equivalence relation (4.55) looks very natural in terms of the pure spinor description (see Appendix B for the details). The unimodular Poisson structure can be described in terms of closed pure spinor  $\rho = e^\alpha \Omega$ . The deformation of the pure spinor would be given by

$$\delta\rho = (w + f) \cdot \rho,$$

where the finite deformation is  $e^{\alpha+w} e^f \Omega$ . The property (4.54) implies that  $d(\delta\rho) = 0$ . If the deformation satisfies (4.55) then

$$\delta\rho = (w + f) \cdot \rho = -d(\xi \cdot \rho),$$

where we used Theorem 13 in Appendix B. Thus we look at the deformations of closed pure spinor modulo exact ones which correspond to the subspace of the de Rham cohomology group, namely

$$\{[(w + f) \cdot \rho] \in H_{dR}^\bullet(M), (w + f) \in \Gamma(\wedge^2 TM \oplus \wedge^0 TM)\},$$

where we deal with the alternative grading of the differential forms, see Appendix B. Following standard terminology, we refer to the corresponding space of deformations of the BV theory modulo the trivial ones as the *geometric moduli space*.

Let us get back to the BV theory. More generally we want to understand the subspace of the cohomology of the quantum Laplacian spanned by non-local observables

$$H_{nonloc}(\Delta_{(\Omega,\alpha)}) = \{[O_2(w)] \in H(\Delta_{(\Omega,\alpha)}), w \in \Gamma(\wedge^\bullet TM)\}.$$

In particular we want to understand if it is finite dimensional and moreover related to the de Rham cohomology  $H_{dR}(M) \approx H(D_\Omega) \approx H(d_{LP} + D_\Omega)$ . We are unable to answer this question in all generality. However we can analyze two special cases which give a positive answer.

Let us discuss first the case of the trivial Poisson structure,  $\alpha = 0$ . In this case a quantum non-local observable  $O_2^{p-2}(w)$  corresponds to the multivector field  $w \in \Gamma(\wedge^p TM)$  such that  $D_\Omega w = 0$ . Then we can show that  $O_2^{p-2}(D_\Omega v)$ ,  $D_\Omega v \in \wedge^p TM$ , is trivial. In fact it is always possible to write  $v = \sum_i f_i D_\Omega \lambda_i$ , for some  $f_i \in C^\infty(M)$  and  $\lambda_i \in \Gamma(\wedge^{p+2} TM)$ . This is obviously equivalent to say that the de Rham differential finitely generates the module of forms. Then using the basic properties of the antibracket we arrive to

$$\begin{aligned} O_2^{p-2}(D_\Omega v) &= \sum_i O_2^{p-2}([f_i, D_\Omega \lambda_i]_s) = - \sum_i \{O_2^{-2}(f_i), O_2^{p-1}(D_\Omega \lambda_i)\} \\ &= -\Delta_\Omega \left( \sum_i O_2^{-2}(f_i) O_2^{p-1}(D_\Omega \lambda_i) \right). \end{aligned} \tag{4.56}$$

Therefore the correspondence  $w \rightarrow O_2^{p-2}(w)$  defines a surjection from  $H(D_\Omega)$  to  $H_{nonloc}(\Delta_\Omega)$ . Thus the corresponding geometrical moduli space is finite dimensional.

Next consider the case of the non-trivial Poisson structure  $\alpha$  such that two differentials  $(d_{LP}, D_\Omega)$  satisfy the  $\partial\bar{\partial}$ -lemma, *i.e.*

$$\text{Im}d_{LP} D_\Omega = \text{Im}d_{LP} \cap \text{Ker}D_\Omega = \text{Ker}d_{LP} \cap \text{Im}D_\Omega. \tag{4.57}$$

The condition (4.57) is satisfied for a large class of symplectic manifolds obeying the strong Lefschetz property (see [39]). However the  $\partial\bar{\partial}$ -lemma does not hold for a generic Poisson manifold since  $H_{LP}(M)$  is infinite dimensional. One of the consequences of the  $\partial\bar{\partial}$ -lemma is the isomorphism of the cohomologies,  $H_{LP}(M) \approx H_{dR}(M)$ . The extreme example of the failure for this lemma is the trivial Poisson structure. Consider  $w \in \Gamma(\wedge^p TM)$  which defines a trivial class in  $(d_{LP} + D_\Omega)$ -cohomology, *i.e.*

$w = d_{LP}\xi_{p-1} + D_{\Omega}\xi_{p+1}$ ,  $0 = d_{LP}\xi_{k-1} + D_{\Omega}\xi_{k+1}$  for  $k \neq p$ . After straightforward calculation we arrive at the following relation:

$$O_2^{p-2}(w) = -2\Delta_{(\Omega,\alpha)}(O_2(\xi_{p-1}) + 4O_0(\xi_{p-3})) + O_2(D_{\Omega}\xi_{p+1}).$$

Since  $D_{\Omega}\xi_{p+1} \in \text{Im}D_{\Omega} \cap \text{Ker}d_{LP} = \text{Im}D_{\Omega}d_{LP}$ , there exists  $v_p$  such that  $D_{\Omega}\xi_{p+1} = D_{\Omega}d_{LP}v_p$  and  $O_2(D_{\Omega}\xi_{p+1}) = 2\Delta_{(\Omega,\alpha)}O_2(D_{\Omega}v_p)$ . Thus we conclude that also in this case the correspondence  $w \rightarrow O_2^{p-2}(w)$  defines a surjective map from the finite dimensional space  $H_{dR}^p(M, \alpha)$  to  $H_{nonloc}(\Delta_{(\Omega,\alpha)})$  where  $H_{dR}^p(M)$  is defined as follows:

$$H_{dR}^p(M, \alpha) = \{[w \cdot \rho] \in H_{dR}^{\bullet}(M), w \in \Gamma(\wedge^p TM)\}.$$

Motivated by these two examples we conjecture that the space  $H_{nonloc}(\Delta_{(\Omega,\alpha)})$  is finite dimensional. Thus in general the action  $S_{BV}$  can be deformed for arbitrary ghost number, mimicking the construction of Frobenius manifolds of [2] and [38]. Let  $\{w_k \in \Gamma(\wedge^{p_k} TM)\}$  define a basis  $\{O_2^{p_k-2}(w_k)\}$  of  $H_{nonloc}(\Delta_{(\Omega,\alpha)})$ . We introduce the formal variables  $\{t_k\}$  of degree  $2 - p_k$  and extend the full BV machinery to  $\mathcal{F} \otimes \mathbb{R}[[t_k]]$ . Clearly  $S(t) = S_{BV} + \sum_k t_k O_2^{p_k-2}(w_k)$  the quantum master equation solves at the infinitesimal level. Interpreting  $H_{nonloc}(\Delta_{(\Omega,\alpha)})$  as the tangent space of the *extended moduli space* the main problem is to find a finite deformation, *i.e.* a solution of the Maurer-Cartan equation

$$\delta_{BV}S(t) + \frac{1}{2}\{S(t), S(t)\} = 0. \tag{4.58}$$

In [2, 38, 39] the solution of such an equation is discussed within the BV setup. The main difference with the setup in [2, 38, 39] is the requirement of  $\partial\bar{\partial}$ -lemma that we want to avoid because it excludes the non-symplectic cases. Is it possible to solve the Maurer-Cartan equation (4.58) in this context? The  $\partial\bar{\partial}$ -lemma provides the isomorphism between the spaces of the classical and quantum observables. While for the generic unimodular Poisson manifold, the space of classical observables is infinite dimensional and the space of quantum observables is expected to be finite dimensional.

### 5. Gauge Fixing

In this section we perform the gauge fixing by choosing an appropriate Lagrangian submanifold. In particular we use a complex structure for the gauge fixing.

*5.1. Geometrical setup.* Let us start from the description of the relevant geometric setup. It turns out to be very convenient to consider the  $N = 2$  supersymmetric PSM [5]. The existence of the extended supersymmetry for PSM requires a generalized complex structure

$$\mathcal{J} = \begin{pmatrix} J & P \\ L & -J^t \end{pmatrix}, \tag{5.59}$$

such that  $[\mathcal{R}, \mathcal{J}] = 0$ , where

$$\mathcal{R} = \begin{pmatrix} 1_d & \alpha \\ 0 & -1_d \end{pmatrix}. \tag{5.60}$$

These conditions can be worked out completely. To be specific  $L = 0$ ,  $J$  is a complex structure and moreover the  $(2, 0) + (0, 2)$  part of  $\alpha$

$$P = \frac{1}{2}(J\alpha + \alpha J^t), \tag{5.61}$$

is a holomorphic Poisson structure. If we switch to the complex coordinates with the labels  $(i, \bar{i})$  then the  $(2, 0)$ -part  $\alpha^{ij}$  is a holomorphic Poisson structure if the following holds:

$$\partial_{\bar{k}}\alpha^{ij} = 0, \quad \alpha^{il}\partial_l\alpha^{jk} + \alpha^{jl}\partial_l\alpha^{ki} + \alpha^{kl}\partial_l\alpha^{ij} = 0. \tag{5.62}$$

Indeed the geometrical setup we will use can be summarized as follows: a Poisson manifold  $(M, \alpha, J)$  with a complex structure  $J$  such that  $(2, 0)$ -part of  $\alpha$  is holomorphic. The fact that  $(2, 0)$ -part is Poisson itself follows from this.

It may look at first that the geometry we just described is somewhat exotic. However that is not the case and this Poisson geometry is always realized on (twisted) generalized Kähler manifolds [37, 15, 21]. The (twisted) generalized Kähler manifold can be characterized as a bihermitian geometry  $(g, J_+, J_-)$ , where  $J_{\pm}$  are two complex structures and  $g$  is a metric which is hermitian with respect to both complex structures. In addition there are certain integrability conditions on two-forms  $gJ_{\pm}$ . The (twisted) generalized Kähler manifold has two real Poisson structures  $\pi_{\pm} = (J_{\pm} \pm J_-)g^{-1}$  [37]. Moreover their  $(2, 0)$ -part with respect to  $J_+$  (or  $J_-$ ) is a holomorphic Poisson structure with respect to  $J_+$  (or  $J_-$ ), [21].

*5.2. Gauge fixed action.* Let us assume that the Poisson manifold  $(M, \alpha)$  admits a complex structure  $J$  such that the  $(2, 0)$ -part of  $\alpha$  is a holomorphic Poisson structure and the world-sheet  $\Sigma$  is equipped with a complex structure. We will concentrate our attention on the case of the two-sphere where the complex structure is unique. Introducing the complex coordinates on  $M$  and  $\Sigma$  we define the following Lagrangian submanifold in the space of (anti)fields:

$$\eta_{zi} = 0, \quad \eta_{\bar{z}\bar{i}} = 0, \quad \eta_z^{+i} = 0, \quad \eta_{\bar{z}}^{+\bar{i}} = 0, \quad X^+ = 0, \quad \beta^+ = 0, \tag{5.63}$$

where  $(i, \bar{i})$  stand for the complex coordinates on  $M$  and  $(z, \bar{z})$  are the complex coordinates on  $\Sigma$ . The odd symplectic structure (3.18) is zero on (5.63). Equivalently we could write the conditions (5.63) using the projectors constructed out of  $J$  and complex structure on  $\Sigma$ , in the same fashion as in [47]. Indeed we do not need to assume that  $J$  is integrable, it is enough for  $J$  to be an almost complex structure. However in what follows we are in the geometrical setup described in the previous subsection. In this case many calculations simplify drastically.

Assuming the gauge (5.63) the gauge fixed action is

$$S_{GF} = i \int_{\Sigma} d^2\sigma \left[ \eta_{z\bar{i}}\partial_{\bar{z}}X^{\bar{i}} - \eta_{\bar{z}i}\partial_zX^i + \alpha^{\bar{i}j}\eta_{z\bar{i}}\eta_{\bar{z}j} + \eta_{\bar{z}}^{+i}(\partial_z\beta_i + \partial_i\alpha^{\bar{l}s}\eta_{z\bar{l}}\beta_s) - \eta_z^{+\bar{i}}(\partial_{\bar{z}}\beta_{\bar{i}} + \partial_{\bar{i}}\alpha^{\bar{l}s}\eta_{z\bar{l}}\beta_s) - \partial_{\bar{i}}\partial_j\alpha^{k\bar{l}}\eta_z^{+\bar{i}}\eta_{\bar{z}}^{+j}\beta_k\beta_{\bar{l}} \right], \tag{5.64}$$



which is just the action (3.21) restricted to (5.63). The action (5.64) is invariant under the following BRST transformations:

$$\delta X^i = \alpha^{ij} \beta_j + \alpha^{i\bar{j}} \beta_{\bar{j}}, \tag{5.65}$$

$$\delta X^{\bar{i}} = \alpha^{\bar{i}j} \beta_j + \alpha^{\bar{i}\bar{j}} \beta_{\bar{j}}, \tag{5.66}$$

$$\delta \eta_{\bar{z}}^{+i} = -\partial_{\bar{z}} X^i - \alpha^{ij} \eta_{\bar{z}j} - \partial_k \alpha^{i\bar{j}} \eta_{\bar{z}}^{+k} \beta_{\bar{j}} - \partial_k \alpha^{ij} \eta_{\bar{z}}^{+k} \beta_j, \tag{5.67}$$

$$\delta \eta_{z\bar{i}}^{+} = -\partial_z X^{\bar{i}} - \alpha^{\bar{i}j} \eta_{z\bar{j}} - \partial_{\bar{k}} \alpha^{\bar{i}j} \eta_z^{+k} \beta_j - \partial_{\bar{k}} \alpha^{\bar{i}\bar{j}} \eta_z^{+k} \beta_{\bar{j}}, \tag{5.68}$$

$$\delta \beta_i = \partial_i \alpha^{k\bar{j}} \beta_k \beta_{\bar{j}} + \frac{1}{2} \partial_i \alpha^{kj} \beta_k \beta_j, \tag{5.69}$$

$$\delta \beta_{\bar{i}} = \partial_{\bar{i}} \alpha^{\bar{k}j} \beta_{\bar{k}} \beta_j + \frac{1}{2} \partial_{\bar{i}} \alpha^{\bar{k}\bar{j}} \beta_{\bar{k}} \beta_{\bar{j}}, \tag{5.70}$$

$$\begin{aligned} \delta \eta_{z\bar{i}} = & -\partial_z \beta_{\bar{i}} - \partial_{\bar{i}} \alpha^{\bar{k}l} \eta_{z\bar{k}} \beta_l - \partial_{\bar{i}} \alpha^{\bar{k}\bar{l}} \eta_{z\bar{k}} \beta_{\bar{l}} - \partial_{\bar{i}} \partial_s \alpha^{\bar{k}\bar{l}} \eta_z^{+s} \beta_k \beta_{\bar{l}} \\ & - \frac{1}{2} \partial_{\bar{i}} \partial_s \alpha^{\bar{k}\bar{l}} \eta_z^{+s} \beta_{\bar{k}} \beta_{\bar{l}}, \end{aligned} \tag{5.71}$$

$$\begin{aligned} \delta \eta_{\bar{z}i} = & -\partial_{\bar{z}} \beta_i - \partial_i \alpha^{k\bar{l}} \eta_{\bar{z}k} \beta_{\bar{l}} - \partial_i \alpha^{kl} \eta_{\bar{z}k} \beta_l - \partial_i \partial_s \alpha^{k\bar{l}} \eta_{\bar{z}}^{+s} \beta_{\bar{k}} \beta_l \\ & - \frac{1}{2} \partial_i \partial_s \alpha^{kl} \eta_{\bar{z}}^{+s} \beta_k \beta_l, \end{aligned} \tag{5.72}$$

which are nilpotent only on-shell. The existence of such residual BRST symmetry within BV formalism is discussed in [18, 1].

Next using the gauge fixed action (5.64) we can calculate the path integral explicitly on the sphere. In particular let us perform the one-loop calculation around the constant map. We take a classical solution  $\eta = 0$  and  $X = x_0$  with  $x_0$  being a constant and the rest of the fields are zero. Consider the fluctuations around this configuration

$$X = x_0 + X_f, \quad \eta = 0 + \eta_f, \quad \beta = 0 + \beta_f, \quad \eta^+ = 0 + \eta_f^+, \tag{5.73}$$

where naturally by  $\eta$  and  $\eta^+$  we understand only non-vanishing components  $(\eta_{\bar{z}i}, \eta_{z\bar{i}})$  and  $(\eta_{\bar{z}}^{+i}, \eta_z^{+\bar{i}})$  correspondingly. We take the expansion (5.73) and plug it into the gauge fixed action (5.64) while keeping only up to the quadratic terms in the fluctuations. The bosonic part of the resulting action can be written schematically as

$$\frac{1}{2} \begin{pmatrix} X & \eta \end{pmatrix} \begin{pmatrix} 0 & D \\ -D & A \end{pmatrix} \begin{pmatrix} X \\ \eta \end{pmatrix}, \tag{5.74}$$

where  $A$  is a part composed from the Poisson tensor  $\alpha$  and  $D$  is a first order differential operator

$$D = \begin{pmatrix} \partial_z & 0 \\ 0 & -\partial_z \end{pmatrix}.$$

While the fermionic part of the corresponding action is written as

$$\eta^t D \beta, \tag{5.75}$$

with the same  $D$ . We can perform easily the gaussian integral over the bosonic (5.74) and the fermionic parts (5.75). The integration produces the ratio of determinants of  $D$  which is exactly 1. Thus the result of this gaussian integration is just one. However the integration over zero modes of  $D$  will remain. The fields  $\eta$  and  $\eta^+$  do not have any zero modes since there are no (anti)holomorphic 1-forms on the sphere. While  $\beta$  have constant zero modes and  $X$  does as well. These zero modes give an integration over the finite dimensional graded manifold  $T^*[1]M$  which is defined by choosing a volume form  $\Omega$  on  $M$ . In order to compensate the odd integration we have to insert the local observables into the path integral. Thus the final result for the correlators of local observables is

$$\langle O_0^{p_1}(w_1) \dots O_0^{p_k}(w_k) \rangle = tr_\Omega(w_1 \wedge \dots \wedge w_k), \tag{5.76}$$

where the trace map  $tr_\Omega$  is defined in the Appendix and the correlator agrees with (4.49). Since the number of zero modes for  $\beta$  corresponds to the dimensionality of  $M$  we have that the correlator (5.76) is non-zero only if  $p_1 + \dots + p_k = d$ . Moreover if we require that the correlator is invariant under the BRST symmetry (5.65)–(5.72) then the Poisson tensor  $\alpha$  should be unimodular and  $\Omega$  is the corresponding invariant volume form. To prove this we need to remember how BRST symmetry (5.65)–(5.72) acts on the local observables and the theorem 8 from the Appendix A. Notice that as far as the fields  $X$  and  $\beta$  concern the action of BV symmetry (3.22)–(3.27) and the BRST symmetry (5.65)–(5.72) is the same. Since the local observables are constructed from  $X$  and  $\beta$  only we can apply the discussion of Subsect. 3.2 to the analysis of BRST invariant observables in the present setup.

We conclude that the present calculation is in complete agreement with our previous analysis within the finite dimensional BV framework. Although the unimodularity of  $\alpha$  is argued completely differently, now through the BRST invariance of the zero-mode measure. The answer (5.76) is just the leading contribution into the full quantum correlator.

Finally we comment when the geometry required for the present gauge fixing is compatible with the unimodularity. Indeed for a generalized Calabi-Yau manifold the corresponding Poisson structure is always unimodular [16]. Thus as a possible example, we may consider the generalized Kähler geometry where one of the generalized complex structures satisfies a generalized Calabi-Yau condition. Actually the gauge fixing can be performed for a generalized Calabi-Yau manifold by itself with the use of an almost generalized complex structure. However we have to stress that unimodularity of Poisson structure is a real condition and indeed much weaker than the generalized Calabi-Yau condition.

*5.3. Relation to A-model.* If we assume that  $\alpha^{ij} = 0$  and  $\alpha$  is invertible, then we are on the Kähler manifold where  $\omega = \alpha^{-1}$  is the Kähler form and  $g = -\omega J$  is the hermitian metric. Due to the fact that  $\alpha$  is invertible we can perform the integration over  $\eta_{z\bar{i}}$  and  $\eta_{\bar{z}i}$  in the path integral with the gauge fixed action (5.64). Introducing the following notation:

$$\psi^i = -i g^{i\bar{j}} \beta_{\bar{j}}, \quad \psi^{\bar{i}} = i g^{\bar{j}j} \beta_j, \quad \psi_z^i = -i \eta_z^{+i}, \quad \psi_{\bar{z}}^{\bar{i}} = -i \eta_{\bar{z}}^{+\bar{i}}, \tag{5.77}$$

the result of the integration of  $\eta$  is

$$S_A = \int d^2\sigma \left[ \partial_z X^{\bar{i}} g_{\bar{i}j} \partial_z X^j + i \psi_z^{\bar{i}} g_{\bar{i}j} \nabla_{\bar{z}} \psi^j + i \psi_{\bar{z}}^i g_{i\bar{j}} \nabla_z \psi^{\bar{k}} - \mathcal{R}_{\rho\bar{i}j\bar{n}} \psi_z^j \psi_{\bar{z}}^{\bar{i}} \psi^{\rho} \psi^{\bar{n}} \right], \tag{5.78}$$

where we adopted the following notation:

$$\nabla_{\bar{z}}\psi^k = \partial_{\bar{z}}\psi^k + \Gamma^k_{nl}\partial_{\bar{z}}X^n\psi^l, \quad \nabla_z\psi^{\bar{k}} = \partial_z\psi^{\bar{k}} + \Gamma^{\bar{k}}_{\bar{n}l}\partial_zX^{\bar{n}}\psi^{\bar{l}} \tag{5.79}$$

with  $\Gamma$  being the Levi–Civita connection and  $\mathcal{R}$  the corresponding Riemann tensor. The first term in the action (5.78) can be rewritten as

$$\partial_{\bar{z}}X^{\bar{i}}g_{\bar{i}j}\partial_zX^j = \frac{1}{2}\sqrt{hh^{\alpha\beta}}\partial_\alpha X^{\bar{i}}g_{\bar{i}j}\partial_\beta X^j + \frac{1}{2}\epsilon^{\alpha\beta}\partial_\alpha X^{\bar{i}}(ig_{\bar{i}j})\partial_\beta X^j, \tag{5.80}$$

where the last term is a topological, the pull-back of the Kähler form  $\omega$ . The BRST transformations (5.65)–(5.72) become

$$\delta X^i = \psi^i, \quad \delta X^{\bar{i}} = \psi^{\bar{i}}, \quad \delta\psi^i = 0, \quad \delta\psi^{\bar{i}} = 0, \tag{5.81}$$

$$\delta\psi_z^{+i} = i\partial_{\bar{z}}X^i + \Gamma^i_{lk}\psi_z^k\psi^l, \quad \delta\psi_z^{+\bar{i}} = i\partial_zX^{\bar{i}} + \Gamma^{\bar{i}}_{\bar{l}k}\psi_z^{\bar{k}}\psi^{\bar{l}}. \tag{5.82}$$

The action (5.78) with the BRST transformations (5.82) corresponds to the topological sigma model [47] on the Kähler manifold which corresponds to the A-twist of  $N = (2, 2)$  supersymmetric sigma model [48]. Previously the BV treatment of the A-model has been discussed in [1]. Here we presented the improved analysis of the relation between the BV-formulation of PSM and the A-model.

Any symplectic manifold with symplectic structure  $\omega$  is unimodular with the volume form given by  $\Omega = \omega^{d/2}$ . Moreover there exists a natural isomorphism between the Lichnerowicz–Poisson cohomology and the de Rham cohomology,  $H_{LP}^*(M) \approx H_{dR}(M)$  which is provided by the symplectic structure  $\omega$ . Therefore the observable corresponding to a multivector field can be mapped into the observable corresponding to the differential form through the identification (5.77). Thus the correlator (5.76) can be rewritten as

$$tr_\Omega(w_1 \wedge \dots \wedge w_k) = \int_M (\sharp w_1) \wedge \dots \wedge (\sharp w_k), \tag{5.83}$$

where  $\sharp w_l$  is a differential form corresponding to a multivector field  $w_l$  constructed through the map  $\sharp : \wedge^* TM \rightarrow \wedge^* T^*M$  defined by the symplectic structure  $\omega$ . Indeed the correlator (5.83) is the standard one for the A-model and can be interpreted as the intersection number of the Poincaré dual cycles to  $\sharp w_l$ . In the full quantum theory the correlator (5.83) gets corrections from the holomorphic maps on which the theory is localized. These instanton corrections are related to the Gromov–Witten invariants. This is well-developed subject, see [23] for a review.

*5.4. Zero Poisson structure.* As a next example we consider the case of zero Poisson structure,  $\alpha = 0$ . In this case the gauge fixed action (5.64) is of the form

$$S_{GF} = i \int_\Sigma d^2\sigma \left[ \eta_{z\bar{i}}\partial_{\bar{z}}X^{\bar{i}} - \eta_{\bar{z}i}\partial_zX^i + \eta_z^{+i}\partial_z\beta_i - \eta_z^{+\bar{i}}\partial_z\beta_{\bar{i}} \right], \tag{5.84}$$

while the BRST transformations (5.65)–(5.72) become

$$\delta X^i = 0, \quad \delta X^{\bar{i}} = 0, \quad \delta\eta_z^{+i} = -\partial_{\bar{z}}X^i, \quad \delta\eta_z^{+\bar{i}} = -\partial_zX^{\bar{i}}, \tag{5.85}$$

$$\delta\beta_i = 0, \quad \delta\beta_{\bar{i}} = 0, \quad \delta\eta_{z\bar{i}} = -\partial_z\beta_{\bar{i}}, \quad \delta\eta_{\bar{z}i} = -\partial_{\bar{z}}\beta_i. \tag{5.86}$$

Now these transformations are nilpotent off-shell. The action (5.84) is reminiscent of the action obtained through the infinite volume limit of the A-model [14]. However our BRST symmetry differs from the one discussed in [14] and thus these are different theories. As well the action (5.84) with the symmetries (5.85)–(5.86) has appeared in different context in [52] as a specific gauge fixed version of the ‘‘Hitchin sigma model’’ [51].

Next we argue that the correlator (5.76) is a full quantum answer for the PSM with  $\alpha = 0$ . We can use the BRST symmetry (5.85)–(5.86) to localize the theory on the holomorphic maps,  $\partial_{\bar{z}}X^i = 0$ . Namely we can add to the action (5.84) the BRST exact term

$$-t\delta \int_{\Sigma} d^2\sigma \left( \eta_z^{+i} g_{i\bar{j}} \partial_{\bar{z}}X^j + \eta_{\bar{z}}^{+i} g_{i\bar{j}} \partial_z X^{\bar{j}} \right) = t \int_{\Sigma} d^2\sigma \left( \partial_z X^{\bar{i}} g_{i\bar{j}} \partial_{\bar{z}}X^j + \partial_{\bar{z}}X^i g_{i\bar{j}} \partial_z X^{\bar{j}} \right), \tag{5.87}$$

where  $t$  is any real number and this exact term is positive definite. The addition of this exact term to the action cannot change the theory and the result is independent from the parameter  $t$ . By sending  $t$  to infinity the dominant contribution to the path integral will come from the holomorphic maps,  $\partial_{\bar{z}}X^i = 0$  and  $\partial_z X^{\bar{i}} = 0$ . Moreover we can perform the integration over  $\eta$  which imposes the conditions  $\partial_{\bar{z}}X^i = 0$  and  $\partial_z X^{\bar{i}} = 0$  which together with the BRST argument imply that only the constant maps contribute to the path integrals. Thus in the evaluation of the path integral on the sphere with the insertion of local observables the only remaining integration is the integration over  $M$  and the corresponding zero modes of  $\beta$ . On the sphere there will be no zero modes for  $\eta$  and  $\eta^+$ .

Thus we have proven that for the PSM with zero Poisson structure the leading result (5.76) for the correlators of local observables is indeed exact. Actually this should not be a surprise since the Poisson tensor controls  $\hbar$ -corrections. In the general action (3.17) the fields can be rescaled in such way that  $\hbar$  appears in front of  $\alpha$  only.

*5.5. Holomorphic Poisson structure.* Another interesting case is when there exists such a complex structure  $J$  that  $\alpha$  is a holomorphic Poisson structure. In other words the  $(1, 1)$ -part of  $\alpha$  vanishes and thus the gauge fixed action (5.64) is independent of  $\alpha$ . The gauge fixed action for the holomorphic Poisson structure is the same as (5.84) for the zero Poisson structure. However the Poisson structure enters into the BRST transformations. For the case of holomorphic Poisson structure the transformations (5.65)–(5.72) become

$$\delta X^i = \alpha^{ij} \beta_j, \tag{5.88}$$

$$\delta X^{\bar{i}} = \alpha^{\bar{i}\bar{j}} \beta_{\bar{j}}, \tag{5.89}$$

$$\delta \eta_z^{+i} = -\partial_z X^i - \alpha^{ij} \eta_{\bar{z}j} - \partial_k \alpha^{ij} \eta_z^{+k} \beta_j, \tag{5.90}$$

$$\delta \eta_{\bar{z}}^{+i} = -\partial_{\bar{z}} X^{\bar{i}} - \alpha^{\bar{i}\bar{j}} \eta_{z\bar{j}} - \partial_{\bar{k}} \alpha^{\bar{i}\bar{j}} \eta_{\bar{z}}^{+k} \beta_{\bar{j}}, \tag{5.91}$$

$$\delta \beta_i = \frac{1}{2} \partial_i \alpha^{kj} \beta_k \beta_j, \tag{5.92}$$

$$\delta \beta_{\bar{i}} = \frac{1}{2} \partial_{\bar{i}} \alpha^{\bar{k}\bar{j}} \beta_{\bar{k}} \beta_{\bar{j}}, \tag{5.93}$$

$$\delta\eta_{z\bar{i}} = -\partial_z\beta_{\bar{i}} - \partial_{\bar{i}}\alpha^{\bar{k}\bar{l}}\eta_{z\bar{k}}\beta_{\bar{l}} - \frac{1}{2}\partial_{\bar{i}}\partial_{\bar{s}}\alpha^{\bar{k}\bar{l}}\eta_z^{+\bar{s}}\beta_{\bar{k}}\beta_{\bar{l}}, \tag{5.94}$$

$$\delta\eta_{\bar{z}i} = -\partial_{\bar{z}}\beta_i - \partial_i\alpha^{kl}\eta_{\bar{z}k}\beta_l - \frac{1}{2}\partial_i\partial_s\alpha^{kl}\eta_{\bar{z}}^{+s}\beta_k\beta_l. \tag{5.95}$$

These transformations are nilpotent  $\delta^2 = 0$  *off-shell* and the action (5.84) is invariant under them. Indeed there is not a single BRST transformation but a whole family. In the transformations (5.88)–(5.95) we can put a complex parameter  $t \in \mathbb{C}$  in front of all terms containing  $\alpha^{ij}$  and correspondingly  $\bar{i}$  in front of terms with  $\alpha^{\bar{i}\bar{j}}$ . This would define a complex family of the BRST transformations  $\delta_t$  which are nilpotent  $\delta_t^2 = 0$  *off-shell* and the action (5.84) is invariant under  $\delta_t$ .

We can repeat the argument from the previous subsection. Using the localization with respect to  $\delta_t$  for any  $t$  (including zero) and the integration over  $\eta$  we arrive at the conclusion that the path integral is localized on the constant maps. Thus again the correlator (5.76) of local observables is a full quantum result.

The example of holomorphic Poisson structure is provided by the hyperKähler manifold which admits a holomorphic symplectic structure with respect to the appropriate complex structure. Therefore the A-model on the hyperKähler manifold can be localized to constant maps and the semi-classical result is exact. However our results are applicable for the wide class of Poisson holomorphic manifold, e.g. the Del Pezzo surfaces, the Poisson Fano varieties,  $\mathbb{C}P^2$ , etc. These examples have attracted a lot attention recently, especially in the context of generalized complex geometry (see [33, 16] for the general discussion and examples [22, 17]).

One may observe that the PSM for a holomorphic Poisson manifold has striking similarities with the B-model [41] defined for the following generalized complex structure:

$$\begin{pmatrix} J & \alpha \\ 0 & -J^t \end{pmatrix}, \tag{5.96}$$

where  $\alpha = \alpha^{(2,0)} + \alpha^{(0,2)}$  is the real part of a holomorphic Poisson structure. However to define the B-model we need a closed pure spinor

$$\rho = e^{\alpha^{(2,0)}}\Omega,$$

where  $\Omega$  is a closed holomorphic volume form. Indeed this condition gives the holomorphic analog of unimodularity. However for the PSM discussed above we need a real version of unimodularity of  $\alpha$  which is a weaker condition on a real volume form. Thus the unimodular deformations of holomorphic Poisson structure cannot be mapped to the corresponding deformations of generalized Calabi-Yau structure corresponding to (5.96). Therefore for a given geometrical setup the B-model and PSM are two different models, with different moduli dependence.

### 6. Conclusions

In this work we have attempted to study the Poisson sigma model beyond the perturbative expansion. The main lesson is that the quantum theory requires the corresponding Poisson tensor  $\alpha$  to be unimodular. We argued this additional property of  $\alpha$  in different ways. In the BV framework the unimodularity is related to the quantum master equation, which requires additional care in its definition. Moreover for the specific gauge fixing

we obtained the unimodularity as from the requirement of the BRST invariance of the zero mode measure.

Alternatively one can provide a different heuristic argument<sup>2</sup> for the unimodularity of the Poisson tensor coming from the perturbative analysis as in [6]. In the perturbative expansion all integrals are absolutely convergent except those containing tadpole diagrams. One may try to regularize the tadpoles by point-splitting using the vector field with no zeros on  $\Sigma$ . However such a vector does not exist on  $S^2$  and thus the tadpoles should be dealt with differently. Since the tadpoles correspond to the bidifferential operators involving the divergence of the Poisson tensor then the unimodularity is the way to eliminate them.

The unimodularity of the Poisson tensor reformulated in terms of pure spinors allows us to treat the PSM exactly in the same fashion as A- and B-models [23] together with their generalized complex relatives [25, 26, 34, 41]. Indeed the Poisson structure defines a real analog of the generalized complex structure and the unimodularity of  $\alpha$  is a real analog of the generalized Calabi-Yau condition. We believe that it is important that all these models can be treated uniformly and there is an intricate interrelation between all these models.

There are several open questions we would like to address in the future, in particular the generalization of the construction of Frobenius manifolds from [2] and [38] for the case when the  $\partial\bar{\partial}$ -lemma fails, as in a generic Poisson case. Also we plan to use further the localization for PSM along the lines presented in Sect. 5. There is an indication that the Gromov-Witten story can be generalized for PSM defined over the generalized Kähler manifold. Furthermore it would be interesting to develop the present analysis for PSM for the higher genus surfaces.

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### A. The Multivector Calculus

Throughout the Appendices A and B we consider mainly the case of the compact manifold  $M$ . This condition can be relaxed if we require the appropriate integrals to be defined and integration by parts should work without any boundary contributions.

In this Appendix we review the relevant structures on the multivector fields  $\Gamma(\wedge^\bullet TM)$  over a smooth manifold  $M$ . For further details the reader may consult the textbook by Vaisman [45].

The Lie bracket on the vector fields can be extended to a bracket on the multivectors. This bracket is called the Schouten bracket. In local coordinates the multivector fields  $P$  and  $Q$  are written as

$$\begin{aligned}
 P &= P^{\mu_1 \dots \mu_p} \partial_{\mu_1} \wedge \dots \wedge \partial_{\mu_p}, \\
 Q &= Q^{\mu_1 \dots \mu_q} \partial_{\mu_1} \wedge \dots \wedge \partial_{\mu_q},
 \end{aligned}$$

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<sup>2</sup> We thank Alberto Cattaneo for sharing this argument with us. Also see [12] for the related discussion and another interesting work [9] on the relation between the deformation quantization and unimodularity.

and their Schouten bracket is defined by the following expression:<sup>3</sup>

$$[P, Q]_s = (p P^{\mu_1 \dots \mu_{p-1} \rho} \partial_\rho Q^{\mu_p \dots \mu_{q+p-1}} - q \partial_\rho P^{\mu_1 \dots \mu_p} Q^{\rho \mu_{p+1} \dots \mu_{q+p-1}}) \partial_{\mu_1} \wedge \dots \wedge \partial_{\mu_{q+p-1}}. \tag{A.1}$$

The algebra  $(\Gamma(\wedge^\bullet TM), \wedge, [, ]_s)$  is a Gerstenhaber algebra (see Definition 1).

If further we specify a volume form  $\Omega$  on  $M$  and a closed one-form  $\lambda$  then we can introduce an operator  $D_{\Omega, \lambda}$ ,

$$D_{\Omega, \lambda} P = \mathbf{div}_\Omega P + i_\lambda P,$$

where  $\mathbf{div}$  is a divergence operator defined by  $\Omega$  and  $i_\lambda$  is a contraction with one-form  $\lambda$ . In local coordinates with the volume form written as  $\Omega = \rho dx^1 \wedge \dots \wedge dx^d$  the divergence operator is

$$(\mathbf{div}_\Omega P)^{\mu_2 \dots \mu_p} = -p \frac{1}{\rho} \partial_{\mu_1} (\rho P^{\mu_1 \mu_2 \dots \mu_d}).$$

Equivalently, in coordinate free notation, the divergence can be written as

$$\mathbf{div}_\Omega P = - *^{-1} d * P,$$

where  $*P = i_P \Omega$  provides a map from  $\Gamma(\wedge^p TM)$  to differential forms and  $d$  is de Rham differential.

Assuming that  $d\lambda = 0$  we have  $(D_{\Omega, \lambda})^2 P = 0$  and moreover

$$[P, Q]_s = (-1)^p D_{\Omega, \lambda}(P \wedge Q) + (-1)^{p+1} (D_{\Omega, \lambda} P) \wedge Q - P \wedge D_{\Omega, \lambda} Q. \tag{A.2}$$

Indeed  $D_{\Omega, \lambda}$  is the most general operator which generates the Schouten bracket [49]. Therefore the algebra  $(\Gamma(\wedge^\bullet TM), \wedge, [, ]_s, D_{\Omega, \lambda})$  is a BV algebra (see Definition 2).

**Definition 4.** The bivector  $\alpha \in \Gamma(\wedge^2 TM)$  is called a Poisson structure if it satisfies

$$[\alpha, \alpha]_s = 0.$$

The manifold with such  $\alpha$  is called a Poisson manifold.

The Poisson structure defines a Lichnerowicz–Poisson differential  $d_{LP}$  on multivector fields

$$d_{LP} P \equiv [\alpha, P]_s, \quad P \in \Gamma(\wedge^\bullet TM).$$

The corresponding cohomology  $H_{LP}^\bullet(M)$  is called the Lichnerowicz–Poisson cohomology group.

We assume that  $M$  is orientable and thus we can choose a volume form  $\Omega$ . Then we can study how the Hamiltonian vector fields  $X_f = \alpha(df)$ ,  $f \in C^\infty(M)$  act on  $\Omega$ . In particular there exists a vector field  $\phi_\Omega$  such that

$$\mathcal{L}_{X_f} \Omega = \phi_\Omega(f) \Omega.$$

$\phi_\Omega$  is named the modular vector field. Indeed the vector field  $\phi_\Omega$  defines a class  $[\phi_\Omega] \in H_{LP}^1(M)$ . This class is independent of  $\Omega$ ,

$$\mathcal{L}_{X_f}(e^g \Omega) = \left( \phi_\Omega + \frac{1}{2} d_{LP} g \right) (f) e^g \Omega,$$

and  $[\phi_\Omega]$  is called the Poisson modular class.

<sup>3</sup> Our definition differs by the overall factor  $(-1)^{p-1}$  compared to the one in [45].

**Definition 5.** A Poisson manifold  $(M, \alpha)$  is called unimodular [46] if  $[\phi_\Omega] = 0$ . In other words there exists such  $\Omega$  that  $\mathcal{L}_{X_f}\Omega = 0$  for any Hamiltonian vector field  $X_f$ . We refer to such  $\Omega$  as an invariant volume form.

For a Poisson manifold  $(M, \alpha)$  we can introduce a (Koszul-)Brylinski differential  $\delta_B$  on the differential forms  $\Omega^\bullet(M)$

$$\delta_B = i_\alpha d - di_\alpha,$$

where  $i_\alpha$  is a contraction with a Poisson tensor  $\alpha$  and  $d$  is a de Rham differential [30].

**Theorem 6.** A Poisson manifold  $(M, \alpha)$  is unimodular if and only if there exists a volume form  $\Omega$  such that  $\delta_B\Omega = 0$  or alternatively  $D_\Omega\alpha = 0$ .

*Proof.* We use notation  $D_\Omega \equiv D_{\Omega,0}$ . The proof of the theorem follows straightforwardly from the relation  $\delta_B\Omega = -i_{\phi_\Omega}\Omega$ . This relation arises from the definition of the modular vector field  $\phi_\Omega$  given above and the following identities:

$$d(i_{X_f}\Omega) = -df \wedge \delta_B\Omega, \quad \phi_\Omega(f)\Omega = df \wedge i_{\phi_\Omega}\Omega.$$

Moreover using the definition of  $D_\Omega$  the modular vector field can also be defined using the divergence operator with respect to  $\Omega$  as  $D_\Omega\alpha = -\phi_\Omega$ . For more details and the related discussion the reader may consult [28,46].  $\square$

Thus we refer to an unimodular Poisson manifold as a triple  $(M, \alpha, \Omega)$ , where  $\Omega$  is a volume form which is closed under the Brylinski differential.

**Definition 7.** For a manifold  $M$  with a volume form  $\Omega$  we define a trace map over the multivector fields

$$tr_\Omega : \Gamma(\wedge^{top} TM) \rightarrow \mathbb{R}$$

as follows:

$$tr_\Omega(P) = \int_M \Omega \wedge i_P\Omega.$$

**Theorem 8.** For a Poisson manifold  $(M, \alpha)$  with a trace map  $tr_\Omega$  the relation

$$tr_\Omega(d_{LP}P \wedge Q) = (-1)^{p+1}tr_\Omega(P \wedge d_{LP}Q)$$

is satisfied if and only if  $(M, \alpha)$  is an unimodular and  $\Omega$  is an invariant volume form.

*Proof.* To prove this statement we use the formulas from Vaisman’s textbook [45]. The relation in the theorem is equivalent to the following statement:

$$\int_M \Omega \wedge i_{(d_{LP}W)}\Omega = 0, \quad W \in \Gamma(\wedge^{d-1} TM).$$

For this to hold it would be enough to show that  $\Omega \wedge i_{(d_{LP}W)}\Omega$  is an exact  $d$ -form. Using the Lichnerowicz definition of the Schouten bracket (see the formula (1.16) in [45]) we rewrite

$$\Omega \wedge i_{(d_{LP}W)}\Omega = -\Omega \wedge i_W\delta_B\Omega + (-1)^{d-1}\Omega \wedge \delta_B(i_W\Omega).$$



Assuming that the one-form  $i_W\Omega = fdg$  and using the properties of the Brylinski differential, we recast the two terms in the above expression as follows:

$$\begin{aligned}
 -\Omega \wedge i_W\delta_B\Omega &= (-1)^{d-1} f\mathcal{L}_{X_g}\Omega, \\
 (-1)^{d-1}\Omega \wedge \delta_B(fdg) &= (-1)^d\{g, f\}\Omega = (-1)^d\mathcal{L}_{X_g}(f\Omega) + (-1)^{d-1}f\mathcal{L}_{X_g}\Omega.
 \end{aligned}$$

To derive the first relation we have used  $\delta_B\Omega = -i_{\phi_\Omega}\Omega$ . If we require that the above forms are exact for any  $g$  and  $f$  then the manifold should be unimodular and  $\Omega$  is an invariant volume form. Since any one form can be written as the sum of the terms like  $fdg$  we can extend our proof for a generic situation.  $\square$

We can summarize the relevant properties of an unimodular Poisson manifold in the following theorem.

**Theorem 9.** *If  $(M, \alpha, \Omega)$  is a unimodular Poisson manifold then  $(\Gamma(\wedge^\bullet TM), \wedge, [, ]_S, D_\Omega, d_{LP})$  is a graded differential BV algebra such that*

$$D_\Omega d_{LP} + d_{LP} D_\Omega = 0.$$

Moreover there exists a trace map  $tr_\Omega$  such that

$$\begin{aligned}
 tr_\Omega(d_{LP}P \wedge Q) &= (-1)^{p+1}tr_\Omega(P \wedge d_{LP}Q), \\
 tr_\Omega(D_\Omega P \wedge Q) &= (-1)^p tr_\Omega(P \wedge D_\Omega Q).
 \end{aligned}$$

*Proof.* The first part of the theorem has been discussed in [28,49]. We have explained most of the statements already. The relation between  $d_{LP}$  and  $D_\Omega$  is derived as follows:

$$D_\Omega d_{LP}P = D_\Omega(D_\Omega(\alpha \wedge P) - \alpha \wedge D_\Omega P) = -D_\Omega(\alpha \wedge D_\Omega P) = -d_{LP}D_\Omega P,$$

where we use the unimodularity,  $D_\Omega\alpha = 0$ . The property of the trace with respect to the divergence operator  $D_\Omega$  is valid for any manifold with a volume form and is just a simple consequence of the Stokes theorem for the differential forms.  $\square$

## B. Poisson Geometry and Pure Spinors

In this Appendix we reformulate the previous Appendix in a different language. This allows us to put the whole formalism into the wider context which is related to generalized geometry on the sum  $TM \oplus T^*M \equiv T \oplus T^*$  of the tangent and cotangent bundles. Below we review very briefly the notions of generalized complex structure, generalized Calabi-Yau condition and their real analogs. For more details we refer the reader to the reviews [15,16,50].

The sum of tangent and cotangent bundles  $T \oplus T^*$  has a natural  $O(d, d)$  structure given by the natural pairing

$$\langle v + \xi, s + \lambda \rangle = \frac{1}{2}(i_v\lambda + i_s\xi),$$

where we adopt the notation  $(v + \xi), (s + \lambda) \in \Gamma(T \oplus T^*)$ . We are interested in a real (complex) Dirac structure which is defined as a maximally isotropic subbundle of  $T \oplus T^*$  (or  $(T \oplus T^*) \otimes \mathbb{C}$ ) and this subbundle is involutive with respect to the Courant bracket. The Dirac structure is an example of the Lie algebroid with the bracket originated from the restriction of the Courant bracket. In particular we are interested in the case when

tangent plus cotangent bundles (or its complexification) can be decomposed as a sum of two real (complex) Dirac structures

$$T \oplus T^* = L \oplus L^*, \quad (T \oplus T^*) \otimes \mathbb{C} = L \oplus L^*.$$

This decomposition gives us a real (complex) bialgebroid. Furthermore there is the structure a differential Gerstenhaber algebra [27,35]

$$(\Gamma(\wedge^\bullet L^*), \wedge, \{, \}, d_L),$$

where  $\{, \}$  is the extension of the Lie bracket from  $L^*$  to  $\wedge^\bullet L^*$  and  $d_L$  is the Lie algebroid differential. In the complex case it is natural to impose an extra condition, namely the dual space  $L^*$  is a complex conjugate of  $L$ . Thus the corresponding bialgebroid is

$$(T \oplus T^*) \otimes \mathbb{C} = L \oplus \bar{L}.$$

This special case corresponds to the notion of generalized complex structure [15,20].

Alternatively the Dirac structures can be described by means of the pure spinor lines. We define the action of a section  $(v + \xi) \in \Gamma(TM \oplus T^*M)$  on a differential form  $\rho \in \Gamma(\wedge^\bullet T^*M)$ ,

$$(v + \xi) \cdot \rho \equiv i_v \rho + \xi \wedge \rho,$$

which corresponds to the action of  $Cl(T \oplus T^*)$  on  $\wedge^\bullet T^*$ . Thus the differential forms form a natural representation of  $Cl(T \oplus T^*)$ . Consider the Dirac structure  $L$  and define a subbundle  $U_0$  of  $\wedge^\bullet T^*$  as follows:

$$L = \{(v + \xi) \in \Gamma(T \oplus T^*), (v + \xi) \cdot U_0 = 0\}.$$

We refer to  $U_0$  as a pure spinor line. The Dirac structure  $L$  induces the alternative grading on the differential forms

$$\wedge^\bullet T^* = \bigoplus_{k=0}^{\dim M} U_k, \quad U_k = (\wedge^k L^*) \cdot U_0,$$

where  $\cdot$  stands for the extension of  $Cl(T \oplus T^*)$  action to  $\wedge^\bullet T^*$ . The property that  $L$  is involutive under the Courant bracket is equivalent to the following:

$$d(\Gamma(U_0)) \subset \Gamma(U_1),$$

where  $d$  is de Rham differential. Indeed we can define a Dirac structure through the subbundle  $U_0$  of  $\wedge^\bullet T^*$  with the above properties. With respect to the alternative grading we can decompose the de Rham differential as follows:

$$d = \bar{\partial} + \partial, \quad \Gamma(U_{k-1}) \xleftarrow{\partial} \Gamma(U_k) \xrightarrow{\bar{\partial}} \Gamma(U_{k+1}),$$

such that  $\partial^2 = 0$  and  $\bar{\partial}^2 = 0$ . We borrow the notation from the generalized complex geometry and in the present context bar over  $\partial$  does not mean complex conjugation.

From now on we assume that the bundle  $U_0$  is trivial and there exists a global section, a pure spinor form  $\rho$  which defines  $L$  completely. The integrability of  $L$  is equivalent to the statement

$$d\rho = (v + \xi) \cdot \rho,$$

for some section  $(v + \xi) \in \Gamma(L^*)$ . Since for given  $L$  the pure spinor  $\rho$  is defined non uniquely, namely for any  $f \in C^\infty(M)$  the form  $e^f \rho$  is also a pure spinor. Thus there is a cohomology class  $[(v + \xi)] \in H^1(d_L)$ , which is just proportional to the modular class of the Lie algebroid [11]. Thus we arrive at the following theorem.

**Theorem 10.** *The Dirac structure  $L$  admits the description in terms of closed pure spinor if and only if the corresponding  $U_0$  bundle is trivial and the Lie algebroid  $L$  is unimodular.*

Since  $U_0$  is a line bundle then its triviality is analyzed differently in the real and complex cases. For instance, in the complex case we have to require the trivial first Chern class,  $c_1(U_0) = 0$ . In the generalized complex case  $(T \oplus T^*) \otimes \mathbb{C} = L \oplus \bar{L}$  the ability to describe  $L$  in terms of a closed pure spinor corresponds to the generalized Calabi-Yau condition, the notion introduced by Hitchin [20]. Thus the generalized Calabi-Yau condition is equivalent to two requirements,  $c_1(U_0) = 0$  and the unimodularity of Lie algebroid  $L$ .

From now on we assume that  $L$  admits the description in terms of a closed pure spinor  $\rho$ . For  $A \in \Gamma(\wedge^\bullet L^*)$  and a closed pure spinor  $\rho$  there are the following relations:

$$(d_L A) \cdot \rho = \bar{\partial}(A \cdot \rho), \quad (DA) \cdot \rho = \partial(A \cdot \rho),$$

where the last relation can be regarded as the definition of the operator  $D$  such that  $D^2 = 0$ . Indeed  $D$  generates the bracket  $\{, \}$  on  $\wedge^\bullet L^*$ . Therefore one can show that  $(\Gamma(\wedge^\bullet L^*), \wedge, \{, \}, D, d_L)$  is a differential BV-algebra [49, 26, 34]. In addition the closed pure spinor provides the isomorphisms of the cohomologies,  $H^\bullet(d_L) \approx H^\bullet(\bar{\partial})$  and  $H^\bullet(D) \approx H^\bullet(\partial)$ .

There exists an invariant form on spinors which, in the present context, corresponds to the Mukai pairing of the differential forms

$$(\rho, \phi) = \sum_j (-1)^j (\rho_{2j} \wedge \phi_{n-2j} + \rho_{2j+1} \wedge \phi_{n-2j-1}),$$

where  $n = \dim M$  and the forms decomposed by the standard degree  $\rho = \sum \rho_i$ ,  $\phi = \sum \phi_i$ . We can introduce the trace map as

$$tr_\rho(A) = \int_M (\rho, A \cdot \rho), \quad A \in \Gamma(\wedge^n L^*).$$

We can summarize these observations in the following theorem:

**Theorem 11.** *For a Lie bialgebroid  $T \oplus T^* = L \oplus L^*$  with  $L$  being a Dirac structure described by the closed pure spinor  $\rho$ ,*

$$(\Gamma(\wedge^\bullet L^*), \wedge, \{, \}, D, d_L)$$

*is a differential BV-algebra and there exists a trace map with the following properties:*

$$\begin{aligned} tr_\rho(d_L A \wedge B) &= (-1)^{|A|+1} tr_\rho(A \wedge d_L B), \\ tr_\rho(DA \wedge B) &= (-1)^{|A|} tr_\rho(A \wedge DB), \end{aligned}$$

where  $A, B$  are sections of  $\wedge^\bullet L^*$ .

*Proof.* The proof of this theorem is straightforward and the different elements of the proof are scattered in the literature, see [49, 26, 34]. Let us sketch the main idea behind the proof. For any differential form  $\rho \in \Gamma(\wedge^\bullet T^*)$  and any sections  $A, B \in \Gamma(T \oplus T^*)$ , there is the following identity:

$$A \cdot B \cdot d\rho = d(A \cdot B \cdot \rho) + B \cdot d(A \cdot \rho) - A \cdot d(B \cdot \rho) + [A, B]_c \cdot \rho - d\langle A, B \rangle \wedge \rho,$$

where  $[\cdot, \cdot]_c$  is the Courant bracket and  $\langle \cdot, \cdot \rangle$  is the natural pairing on  $T \oplus T^*$ . If we have a Lie bialgebroid  $T \oplus T^* = L \oplus L^*$  with  $L$  being a Dirac structure described by the closed pure spinor  $\rho$  then the above formula implies

$$d(A \cdot B \cdot \rho) + B \cdot d(A \cdot \rho) - A \cdot d(B \cdot \rho) + \{A, B\} \cdot \rho = 0,$$

where now  $A, B \in \Gamma(L^*)$  and  $\{, \}$  is a Lie bracket on  $L^*$ , which is a restriction of the Courant bracket to  $L^*$ . This formula can be extended to the general case when  $A, B$  are sections of definite degree in  $\Gamma(\wedge^\bullet L^*)$ . This extension together with the definition

$$(d_L + D)A \cdot \rho = d(A \cdot \rho), \quad \wedge^k L^* \xrightarrow{d_L} \wedge^{k+1} L^*, \quad \wedge^k L^* \xrightarrow{D} \wedge^{k-1} L^*$$

allow us to recover that  $D$  generates the bracket on  $\Gamma(\wedge^\bullet L^*)$  and moreover  $\Gamma(\wedge^\bullet L^*)$  is a differential BV algebra. The properties of the trace map can be proven easily also using the above properties.  $\square$

Using this language we now recast the previous definitions in Poisson geometry in a new language. Let us start from the following theorem.

**Theorem 12.** *The manifold  $M$  is a unimodular Poisson manifold if and only there exists a closed pure spinor of the form*

$$\rho = e^\alpha \Omega = \Omega + i_\alpha \Omega + \frac{1}{2} i_\alpha^2 \Omega + \dots,$$

where  $\alpha$  is a bivector and  $\Omega$  is a volume form.

*Proof.* If we have a unimodular Poisson manifold  $(M, \alpha, \Omega)$  then we can construct a pure spinor  $\rho = e^\alpha \Omega$  which satisfies

$$d\rho = \delta_B \Omega + \frac{1}{2} \delta_B (i_\alpha \Omega) + \dots = 0,$$

since  $\delta_B \Omega = 0$  and  $\delta_B i_\alpha = i_\alpha \delta_B$ . In the opposite direction we can start from a closed pure spinor  $\rho = e^\alpha \Omega$  which defines the following maximally isotropic subbundle of  $T \oplus T^*$ :

$$L = e^\alpha(T^*) = \{i_\xi \alpha + \xi : \xi \in \Gamma(T^*)\}.$$

Since  $\rho$  is closed,  $L$  is a Dirac structure and thus  $\alpha$  is an Poisson structure. Moreover the volume  $\Omega$  would be an invariant volume form with respect to the unimodular Poisson structure  $\alpha$ .  $\square$

Thus the Poisson structure on  $M$  gives the real Lie bialgebroid  $T \oplus T^* = e^\alpha(T^*) \oplus T$ . If the Poisson structure is unimodular then there exists a closed pure spinor  $\rho = e^\alpha \Omega$  and  $\Gamma(\wedge^\bullet T)$  is a differential BV algebra. Indeed the trace map  $tr_\Omega$  defined in the previous appendix coincides with the one defined here,  $tr_\rho$ , since only the top form part contributes in  $\rho$ .

On an unimodular Poisson manifold  $(M, \alpha, \Omega)$  with the pure spinor  $\rho = e^\alpha \Omega$  we can calculate the differentials  $\partial$  and  $\bar{\partial}$  associated with the alternative grading on the differential forms

$$\wedge^\bullet T^* = \bigoplus_{k=0}^{\dim M} (\wedge^k T) \cdot e^\alpha \Omega.$$

Indeed in this case we have  $\bar{\partial} = -\delta_B$  and  $\partial = d + \delta_B$ , see the following theorem:

**Theorem 13.** *For a unimodular Poisson manifold  $(M, \alpha, \Omega)$  with the closed pure spinor  $\rho = e^\alpha \Omega$ , the following relations hold:*

$$\begin{aligned} (D_\Omega P) \cdot \rho &= -(d + \delta_B)(P \cdot \rho), \\ (d_{LP} P) \cdot \rho &= \delta_B(P \cdot \rho). \end{aligned}$$

*Proof.* Let us start from the proof of the first relation. If  $\alpha = 0$  then this is just a definition of  $D_\Omega$  given in the previous appendix. In the general case  $\alpha \neq 0$  a simple calculation produces the following formula [10]:

$$d + \delta_B = e^\alpha de^{-\alpha},$$

which together with the definition of  $D_\Omega$  gives the desired relation.

Next we prove the second relation in the theorem. Using the fact that  $D_\Omega$  generates the Schouten bracket and the manifold is unimodular,  $D_\Omega \alpha = 0$ , we get

$$\begin{aligned} (d_{LP} P) \cdot \rho &= (D_\Omega(\alpha \wedge P) - \alpha \wedge D_\Omega P) \cdot \rho \\ &= -(d + \delta_B)(i_\alpha i_P \rho) + i_\alpha(d + \delta_B)(i_P \rho) = \delta_B(i_P \rho), \end{aligned}$$

where we used the previously proved relation and the property  $i_\alpha \delta_B = \delta_B i_\alpha$ .  $\square$

This theorem implies the isomorphism of certain cohomologies. For any Poisson manifold  $(M, \alpha)$  there are the following isomorphisms:

$$H_{dR}^\bullet(M) \approx H^\bullet(D_\Omega) \approx H^\bullet(d + \delta_B),$$

while for the unimodular Poisson manifold in addition we have

$$H_{LP}^\bullet(M) \approx H^\bullet(\delta_B).$$

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