

Local Geometry of the G_2 Moduli Space

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Abstract: We consider deformations of torsion-free G_2 structures, defined by the G_2 -invariant 3-form φ and compute the expansion of $*\varphi$ to fourth order in the deformations of φ . By considering M -theory compactified on a G_2 manifold, the G_2 moduli space is naturally complexified, and we get a Kähler metric on it. Using the expansion of $*\varphi$, we work out the full curvature of this metric and relate it to the Yukawa coupling.

1. Introduction

One of the possible approaches to M -theory is to consider compactifications of the 11-dimensional spacetimes of the form $M_4 \times X$, where M_4 is the 4-dimensional Minkowski space and X is a 7-dimensional manifold. If X is a compact manifold with G_2 holonomy, then this gives a vacuum solution of the low-energy effective theory, and moreover, since X has one covariantly constant spinor, the resulting theory in 4 dimensions has $N = 1$ supersymmetry. The physical content of the 4-dimensional theory is given by the moduli of G_2 holonomy manifolds. Such a compactification of M -theory is in many ways analogous to Calabi-Yau compactifications in String Theory, where much progress has been made through the study of the Calabi-Yau moduli spaces. In particular, as it was shown in [1] and [2], the moduli space of complex structures and the complexified moduli space of Kähler structures are both in fact, Kähler manifolds. Moreover, both have a *special geometry* - that is, both have a line bundle whose first Chern class coincides with the Kähler class. However until recently, the structure of the moduli space of G_2 holonomy manifolds has not been studied in that much detail. Generally, it turned out that the study of G_2 manifolds is quite difficult. Firstly, unlike in the Calabi-Yau case [3], there is no general theorem for existence of G_2 manifolds. Although there are constructions of compact G_2 manifolds such as those that can be found in [4] and [5], they are not explicit (a non-compact construction was also given in [6]). Another difficulty is that the G_2 -invariant 3-form which defines the G_2 -structure

and the metric corresponding to it are related in a non-linear fashion. This makes the study of G_2 manifolds more difficult from a computational point of view.

We first start with an overview of G_2 structures in Sect. 2, where we state the basic facts about G_2 manifolds and set up the notation. A G_2 -structure is defined by a G_2 -invariant 3-form φ , and in Sect. 3 we review some of the computational properties of φ and its Hodge dual $*\varphi$, which we will need later on. Since one of our main motivations to study G_2 manifolds comes from physics, in Sect. 4, we review the role of G_2 manifolds in M -theory, and in particular we consider the Kaluza-Klein compactification of the effective M -theory low-energy action on a G_2 manifold. It turns out that in the reduced action, the moduli of the M -theory 3-form C_{mnp} and the G_2 moduli naturally combine, to effectively give a complexification of the G_2 moduli space. Moreover, the metric on this complexified space turns out to be Kähler, and the Kähler potential is essentially the logarithm of the volume of the G_2 manifold.

The aim of this paper is to gain more information about the geometry of the moduli space, and so the aim is to compute the curvature of this Kähler metric. This involves calculation of the fourth derivative of the Kähler potential. The method which we use for this requires us to know the expansion of $*\varphi$ to third order in the deformations of φ . So in section 5, we in fact explicitly give the expansion of $*\varphi$ to fourth order in the deformations of φ . Previously, only the full expansion to first order was known [4], and only partially to second order [7]. However, there are approaches to calculating higher derivatives of the Kähler potential without explicitly computing an expansion of $*\varphi$ - for example the third derivative has been computed by de Boer et al in [8] and by Karigiannis and Leung in [9].

Finally, in section 6, we use our expansion of $*\varphi$ from section 5 to calculate the full curvature of the G_2 moduli space, and then the Ricci curvature as well. As it has already been noted in [8] and [9], the third derivative of the Kähler can be interpreted as a Yukawa coupling, and it bears a great resemblance to the Yukawa coupling encountered in the study of Calabi-Yau moduli spaces. At the end of section 6 we consider look at some properties of covariant derivatives on the moduli space.

2. Overview of G_2 Structures

We will first review the basics of G_2 structures on smooth manifolds. The main references for this section are [4, 7] and [10].

The 14-dimensional Lie group G_2 can be defined as a subgroup of $GL(7, \mathbb{R})$ in the following way. Suppose x^1, \dots, x^7 are coordinates on \mathbb{R}^7 and let $e^{ijk} = dx^i \wedge dx^j \wedge dx^k$. Then define φ_0 to be the 3-form on \mathbb{R}^7 given by

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}. \tag{2.1}$$

Then G_2 is defined as the subgroup of $GL(7, \mathbb{R})$ which preserves φ_0 . Moreover, it also fixes the standard Euclidean metric

$$g_0 = (dx^1)^2 + \dots + (dx^7)^2 \tag{2.2}$$

on \mathbb{R}^7 and the 4-form $*\varphi_0$ which is the corresponding Hodge dual of φ_0 :

$$*\varphi_0 = e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247}. \tag{2.3}$$

Now suppose X is a smooth, oriented 7-dimensional manifold. A G_2 structure Q on X is a principal subbundle of the frame bundle F , with fibre G_2 . However we can also uniquely define Q via 3-forms on X . Define a 3-form φ to be *positive* if we locally can choose coordinates such that φ is written in the form (2.1) - that is for every $p \in X$ there is an isomorphism between $T_p X$ and \mathbb{R}^7 such that $\varphi|_p = \varphi_0$. Using this isomorphism, to each positive φ we can associate a metric g and a Hodge dual $*\varphi$ which are identified with g_0 and $*\varphi_0$ under this isomorphism, and the associated metric is written (2.2). It is shown in [4] that there is a 1 – 1 correspondence between positive 3-forms φ and G_2 structures Q on X .

So given a positive 3-form φ on X , it is possible to define a metric g associated to φ and this metric then defines the Hodge star, which in turn gives the 4-form $*\varphi$. Thus although $*\varphi$ looks linear in φ , it actually is not, so sometimes we will write $\psi = *\varphi$ to emphasize that the relation between φ and $*\varphi$ is very non-trivial.

In general, any G -structure on a manifold X induces a splitting of bundles of p -forms into subbundles corresponding to irreducible representations of G . The same is of course true for G_2 -structure. From [4] we have the following decomposition of the spaces of p -forms Λ^p :

$$\Lambda^1 = \Lambda_7^1, \tag{2.4a}$$

$$\Lambda^2 = \Lambda_7^2 \oplus \Lambda_{14}^2, \tag{2.4b}$$

$$\Lambda^3 = \Lambda_1^3 \oplus \Lambda_7^3 \oplus \Lambda_{27}^3, \tag{2.4c}$$

$$\Lambda^4 = \Lambda_1^4 \oplus \Lambda_7^4 \oplus \Lambda_{27}^4, \tag{2.4d}$$

$$\Lambda^5 = \Lambda_7^5 \oplus \Lambda_{14}^5, \tag{2.4e}$$

$$\Lambda^6 = \Lambda_7^6. \tag{2.4f}$$

Here each Λ_k^p corresponds to the k -dimensional irreducible representation of G_2 . Moreover, for each k and p , Λ_k^p and Λ_k^{7-p} are isomorphic to each other via Hodge duality, and also Λ_7^p are isomorphic to each other for $n = 1, 2, \dots, 6$. Note that φ and $*\varphi$ are G_2 -invariant, so they generate the 1-dimensional sectors Λ_1^3 and Λ_1^4 , respectively.

Define the standard inner product on Λ^p , so that for p -forms α and β ,

$$\langle \alpha, \beta \rangle = \frac{1}{p!} \alpha_{a_1 \dots a_p} \beta^{a_1 \dots a_p}. \tag{2.5}$$

This is related to the Hodge star, since

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \text{vol}, \tag{2.6}$$

where vol is the invariant volume form given locally by

$$\text{vol} = \sqrt{\det g} dx^1 \wedge \dots \wedge dx^7. \tag{2.7}$$

Then it turns out that the decompositions (2.4) are orthogonal with respect to (2.5). This will be seen easily when we consider these decompositions in more detail in the next section.

As we already know, the metric g on a manifold with G_2 structure is determined by the invariant 3-form φ . It is in fact possible to write down an explicit relationship between φ and g . Let u and v be vector fields on X . Then

$$\langle u, v \rangle \text{vol} = \frac{1}{6} (u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi. \tag{2.8}$$

Here \lrcorner denotes interior multiplication, so that

$$(u \lrcorner \varphi)_{bc} = u^a \varphi_{abc}. \tag{2.9}$$

The definition (2.8) is rather indirect because vol depends on g via (2.7). To make more sense of it, rewrite in components

$$g_{ab} \sqrt{\det g} = \frac{1}{144} \varphi_{amn} \varphi_{bpq} \varphi_{rst} \hat{\varepsilon}^{mnpqrst}, \tag{2.10}$$

where $\hat{\varepsilon}^{mnpqrst}$ is the alternating symbol with $\varepsilon^{12\dots 7} = +1$. Define

$$B_{ab} = \frac{1}{144} \varphi_{amn} \varphi_{bpq} \varphi_{rst} \hat{\varepsilon}^{mnpqrst} \tag{2.11}$$

so that then, after taking the determinant of (2.10) we get

$$g_{ab} = (\det B)^{-\frac{1}{9}} B_{ab}. \tag{2.12}$$

This gives a direct definition, but because $\det s$ may be awkward to compute, (2.12) is not always the most practical definition. For us, it will be more useful to take the trace of (2.10) with respect to g , which gives

$$\sqrt{\det g} = \frac{1}{7} \text{Tr} B, \tag{2.13}$$

and hence

$$g_{ab} = \frac{7B_{ab}}{\text{Tr} B}. \tag{2.14}$$

Although this is also an indirect definition, it is sometimes easier to handle this expression.

There are in fact a total of 16 torsion classes of G_2 structures, each of which places certain restrictions on $d\varphi$ or $d * \varphi$ [11]. One of the most important classes of manifolds with G_2 structure are manifolds with G_2 holonomy. The group G_2 appears as one of two exceptional holonomy groups - the other one is $Spin(7)$ for 8-dimensional manifolds. The list of possible holonomy groups is limited and they were fully classified by Berger [12]. Specifically, if (X, g) is a simply-connected Riemannian manifold which is neither locally a product nor is symmetric, the only possibilities are shown in the table below.

Dimension	Holonomy	Type of Manifold
$2k$	$U(k)$	Kähler
$2k$	$SU(k)$	Calabi-Yau
$4k$	$Sp(k)$	HyperKähler
$4k$	$Sp(k) Sp(1)$	Quaternionic
7	G_2	Exceptional
8	$Spin(7)$	Exceptional

It turns out that the holonomy group $Hol(X, g) \subseteq G_2$ if and only if X has a torsion-free G_2 structure [4]. In this case, the invariant 3-form φ satisfies

$$d\varphi = d * \varphi = 0 \tag{2.15}$$

and equivalently, $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection of g . So in fact, in this case φ is harmonic. Moreover, if $Hol(X, g) \subseteq G_2$, then X is Ricci-flat.

For a torsion-free G_2 structure, the decompositions (2.4) carry over to de Rham cohomology [4], so that we have

$$H^2(X, \mathbb{R}) = H_7^2 \oplus H_{14}^2, \tag{2.16a}$$

$$H^3(X, \mathbb{R}) = H_1^3 \oplus H_7^3 \oplus H_{27}^3, \tag{2.16b}$$

$$H^4(X, \mathbb{R}) = H_1^4 \oplus H_7^4 \oplus H_{27}^4, \tag{2.16c}$$

$$H^5(X, \mathbb{R}) = H_7^5 \oplus H_{14}^5. \tag{2.16d}$$

Define the refined Betti numbers $b_k^p = \dim(H_k^p)$. Clearly, $b_1^3 = b_1^4 = 1$ and we also have $b_1 = b_7^k$ for $k = 1, \dots, 6$. Moreover, it turns out that $b_1 = 0$ if and only if $Hol(X, g) = G_2$. Therefore, in this case the H_7^k component vanishes in (2.16).

An example of a construction of a manifold with a torsion-free G_2 structure is to consider $X = Y \times S^1$, where Y is a Calabi-Yau 3-fold. Define the metric and a 3-form on X as

$$g_X = d\theta^2 \times g_Y, \tag{2.17}$$

$$\varphi = d\theta \wedge \omega + \text{Re } \Omega, \tag{2.18}$$

where θ is the coordinate on S^1 . This then defines a torsion-free G_2 structure, with

$$*\varphi = \frac{1}{2}\omega \wedge \omega - d\theta \wedge \text{Im } \Omega. \tag{2.19}$$

However, the holonomy of X in this case is $SU(3) \subset G_2$. From the Künneth formula we get the following relations between the refined Betti numbers of X and the Hodge numbers of Y :

$$\begin{aligned} b_7^k &= 1 \quad \text{for } k = 1, \dots, 6, \\ b_{14}^k &= h^{1,1} - 1 \quad \text{for } k = 2, 5, \\ b_{27}^k &= h^{1,1} + 2h^{2,1} \quad \text{for } k = 3, 4. \end{aligned}$$

3. Properties of φ

The invariant 3-form φ which defines a G_2 structure on the manifold X has a number of useful and interesting properties. In particular, contractions of φ and $\psi = *\varphi$ are very useful in computations. From [7, 13] and [14], we have

$$\varphi_{abc}\varphi_{mn}{}^c = g_{am}g_{bn} - g_{an}g_{bm} + \psi_{abmn}, \tag{3.20}$$

$$\varphi_{abc}\psi_{mnp}{}^c = 3(g_{a[m}\varphi_{np]b} - g_{b[m}\varphi_{np]a}). \tag{3.21}$$

Essentially, these identities can be derived straight from the definitions of φ and $\psi = *\varphi$ in flat space - (2.1) and (2.3) respectively. For more details, please refer to [7] and [13]. Note that we are using a different convention to [13], and hence some of the signs are different.

Consider the product $\psi_{abcd}\psi^{mnpq}$. Expanding ψ as the Hodge star of φ and then using the usual identity for a product of Levi-Civita tensors and then applying (3.20) gives

$$\psi_{abcd}\psi^{mnpq} = 24\delta_a^m\delta_b^n\delta_c^p\delta_d^q + 72\psi_{[ab}^{[mn}\delta_c^p\delta_d^q] - 16\varphi_{[abc}\varphi^{[mnp}\delta_d^q]}. \tag{3.22}$$

Contracting over d and q gives

$$\psi_{abcd}\psi^{mnpd} = 6\delta_a^m\delta_b^n\delta_c^p + 9\psi_{[ab}^{[mn}\delta_c^p] - \varphi_{abc}\varphi^{mnp}, \tag{3.23}$$

which agrees with the expression given in [14]. Of course the above relations can be further contracted to obtain

$$\varphi_{abc}\varphi_m^{bc} = 6g_{am}, \tag{3.24}$$

$$\varphi_{abc}\psi_{mn}^{bc} = 4\varphi_{amn}, \tag{3.25}$$

$$\psi_{abcd}\psi_{mn}^{cd} = 4g_{am}g_{bn} - 4g_{an}g_{bm} + 2\psi_{abmn}. \tag{3.26}$$

Contracting even further, we are left with

$$\varphi_{abc}\varphi^{abc} = 42, \tag{3.27}$$

$$\varphi_{abc}\psi_m^{abc} = 0, \tag{3.28}$$

$$\psi_{abcd}\psi_m^{bcd} = 24g_{am}, \tag{3.29}$$

$$\psi_{abcd}\psi^{abcd} = 168. \tag{3.30}$$

The relations (3.27) and (3.30) both yield $|\varphi|^2 = 7$ in the inner product (2.5). So in fact we have

$$V = \frac{1}{7} \int \varphi \wedge *\varphi, \tag{3.31}$$

where V is the volume of the manifold X .

Now look in more detail at the decompositions (2.4). We are in particular interested in decompositions of 2-forms and 3-forms since the decompositions for 4-forms and 5-forms are derived from these via Hodge duality. From [7] and [10], we have

$$\Lambda_7^2 = \{\omega \lrcorner \varphi : \omega \text{ a vector field}\}, \tag{3.32}$$

$$\Lambda_{14}^2 = \left\{ \alpha = \frac{1}{2} \alpha_{ab} dx^a \wedge dx^b : (\alpha_{ab}) \in \mathfrak{g}_2 \right\}, \tag{3.33}$$

$$\Lambda_1^3 = \{f\varphi : f \text{ a smooth function}\}, \tag{3.34}$$

$$\Lambda_7^3 = \{\omega \lrcorner *\varphi : \omega \text{ a vector field}\}, \tag{3.35}$$

$$\Lambda_{27}^3 = \left\{ \chi \in \Omega^3 : \chi \wedge \varphi = 0 \text{ and } \chi \wedge *\varphi = 0 \right\}. \tag{3.36}$$

Following [7], it is enough to consider what happens in \mathbb{R}^7 in order to understand these decompositions. Consider first the Lie algebra $\mathfrak{so}(7)$, which is the space of antisymmetric 7×7 matrices. For a vector $\omega \in \mathbb{R}^7$, define the map $\rho_\varphi : \mathbb{R}^7 \rightarrow \mathfrak{so}(7)$ by $\rho_\varphi(\omega) = \omega \lrcorner \varphi$, and this map is clearly injective. Conversely, define the map $\tau_\varphi : \mathfrak{so}(7) \rightarrow \mathbb{R}^7$ such that $\tau_\varphi(\alpha_{ab})^c = \frac{1}{6} \varphi^c{}_{ab} \alpha^{ab}$. From (3.24), we get that

$$\tau_\varphi(\rho_\varphi(\omega)) = \omega,$$

so that τ_φ is a partial inverse of ρ_φ . Now the Lie algebra \mathfrak{g}_2 can be defined as the kernel of τ_φ [13], that is

$$\mathfrak{g}_2 = \ker \tau_\varphi = \left\{ \alpha \in \mathfrak{so}(7) : \varphi_{abc} \alpha^{bc} = 0 \right\}. \tag{3.37}$$

This further implies that we get the following decomposition of $\mathfrak{so}(7)$:

$$\mathfrak{so}(7) = \mathfrak{g}_2 \oplus \rho_\varphi(\mathbb{R}^7). \tag{3.38}$$

The group G_2 acts via the adjoint representation on the 14-dimensional vector space \mathfrak{g}_2 and via the natural, vector representation on the 7-dimensional space $\rho_\varphi(\mathbb{R}^7)$. This is a G_2 -invariant irreducible decomposition of $\mathfrak{so}(7)$ into the representations **7** and **14**. Hence follows the decomposition of Λ^2 (2.4a and also the characterizations (3.32) and (3.33)).

Following [7] again, let us look at Λ_{27}^3 in more detail. Consider $Sym^2((\mathbb{R}^7)^*)$ - the space of symmetric 2-tensors and define a map $i_\varphi : Sym^2((\mathbb{R}^7)^*) \rightarrow \Lambda^3((\mathbb{R}^7)^*)$ by

$$i_\varphi(h)_{abc} = h_{[a}^d \varphi_{bc]d}. \tag{3.39}$$

Clearly,

$$i_\varphi(g)_{abc} = \varphi_{abc}.$$

Now, we can decompose $Sym^2((\mathbb{R}^7)^*) = \mathbb{R}g \oplus Sym_0^2((\mathbb{R}^7)^*)$, where $\mathbb{R}g$ is the set of symmetric tensors proportional to the metric g and $Sym_0^2((\mathbb{R}^7)^*)$ is the set of traceless symmetric tensors. This is a G_2 -invariant irreducible decomposition of $Sym^2((\mathbb{R}^7)^*)$ into 1-dimensional and 27-dimensional components. The map i_φ is also G_2 -invariant and is injective on each summand of this decomposition. Looking at the first summand, we get that $i_\varphi(\mathbb{R}g) = \Lambda_1^3((\mathbb{R}^7)^*)$. Now look at the second summand and consider $i_\varphi(Sym_0^2((\mathbb{R}^7)^*))$. This is 27-dimensional and irreducible, so by dimension count it follows easily that $i_\varphi(Sym_0^2((\mathbb{R}^7)^*)) = \Lambda_{27}^3((\mathbb{R}^7)^*)$. All of this carries over to 3-forms on our G_2 manifold X , and so we get

$$\Lambda_{27}^3 = \left\{ \chi \in \Lambda^3 : \chi_{abc} = h_{[a}^d \varphi_{bc]d} \text{ for } h_{ab} \text{ traceless and symmetric} \right\}. \tag{3.40}$$

From the identities for contraction of φ and $*\varphi$, it is possible to see that this is equivalent to the description (3.36) of Λ_{27}^3 . Thus we see that 1-dimensional components correspond to scalars, 7-dimensional components correspond to vectors and 27-dimensional components correspond to traceless symmetric matrices.

Now suppose we have $\chi \in \Lambda^3$, then it is always useful to be able to compute the different projections of χ into Λ_1^3 , Λ_7^3 and Λ_{27}^3 . Denote these projections by π_1 , π_7 and π_{27} , respectively. As shown in Appendix 1, we have the following relations:

$$\pi_1(\chi) = a\varphi \text{ where } a = \frac{1}{42} (\chi_{abc} \varphi^{abc}) = \frac{1}{7} \langle \chi, \varphi \rangle \text{ and } |\pi_1(\chi)|^2 = 7a^2, \tag{3.41}$$

$$\pi_7(\chi) = \omega \lrcorner * \varphi \text{ where } \omega^a = -\frac{1}{24} \chi_{mnp} \psi^{mnpa} \text{ and } |\pi_7(\chi)|^2 = 4|\omega|^2, \tag{3.42}$$

$$\pi_{27}(\chi) = i_\varphi(h) \text{ where } h_{ab} = \frac{3}{4} \chi_{mn\{a} \varphi_{b\}^{mn} \text{ and } |\pi_{27}(\chi)|^2 = \frac{2}{9} |h|^2. \tag{3.43}$$

Here $\{a\ b\}$ denotes the traceless symmetric part.

4. G_2 manifolds in M -Theory

Special holonomy manifolds play a very important role in string and M -theory because of their relation to supersymmetry. In general, if we compactify string or M -theory on a manifold of special holonomy X the preservation of supersymmetry is related to existence of covariantly constant spinors (also known as parallel spinors). In fact, if all bosonic fields except the metric are set to zero, and a supersymmetric vacuum solution is sought, then in both string and M -theory, this gives precisely the equation

$$\nabla \xi = 0 \tag{4.44}$$

for a spinor ξ . As lucidly explained in [15], condition (4.44) on a spinor immediately implies special holonomy. Here ξ is invariant under parallel transport, and is hence invariant under the action of the holonomy group $Hol(X, g)$. This shows that the spinor representation of $Hol(X, g)$ must contain the trivial representation. For $Hol(X, g) = SO(n)$, this is not possible since the spinor representation is reducible, so $Hol(X, g) \subset SO(n)$. In particular, Calabi-Yau 3-folds with $SU(3)$ holonomy admit two covariantly constant spinors and G_2 holonomy manifolds admit only one covariantly constant spinor.

Consider the bosonic action of eleven-dimensional supergravity [16], which is supposed to describe low-energy M -theory:

$$S = \frac{1}{2} \int d^{11}x (-\hat{g})^{\frac{1}{2}} R^{(11)} - \frac{1}{4} \int G \wedge *G - \frac{1}{12} \int C \wedge G \wedge G, \tag{4.45}$$

where \hat{g} is the metric on the 11-dimensional space M and C is a 3-form potential with field strength $G = dC$. From (4.45), the equation of motion for C is found to be

$$d * G = \frac{1}{2} G \wedge G. \tag{4.46}$$

Suppose we fix $M = M_4 \times X$, where M_4 is the 4-dimensional Minkowski space and X is a space with holonomy equal to G_2 . Then M is Ricci flat, so from Einstein's equation, G has to vanish. However, it turns out that the assumption that $G_X = G|_X = 0$ is not an obvious one to make. In fact, as explained in [17], Dirac quantization on X gives a shifted quantization condition and gives the statement

$$\left[\frac{G_X}{2\pi} \right] - \frac{\lambda}{2} \in H^4(X, \mathbb{Z}), \tag{4.47}$$

where $\left[\frac{G_X}{2\pi} \right]$ is the cohomology class of $\frac{G_X}{2\pi}$ and $\lambda = \frac{1}{2} p_1(X)$, where $p_1(X)$ is the first Pontryagin class on X . So if λ were not even in $H^4(X, \mathbb{Z})$, then the ansatz $G_X = 0$ would not be consistent. Nonetheless, it was shown in [18] that if X is a seven dimensional spin manifold (or in particular G_2 holonomy manifold), then in fact λ is even, and setting $G_X = 0$ is consistent.

So overall the simplest, Ricci-flat vacuum solutions are given by

$$\langle \hat{g} \rangle = \eta \times g_7, \tag{4.48}$$

$$\langle C \rangle = 0, \tag{4.49}$$

$$\langle G \rangle = 0, \tag{4.50}$$

where $\langle \cdot \rangle$ denotes the vacuum expectation value and g_7 is some metric with G_2 holonomy while η is the standard metric on the four dimensional Minkowski space. However, we know that a G_2 structure and hence the metric g_7 is defined by a G_2 -invariant 3-form φ_0 , so we have

$$\langle \varphi \rangle = \varphi_0. \tag{4.51}$$

Now consider small fluctuations about the vacuum,

$$\hat{g} = \langle \hat{g} \rangle + \delta \hat{g}, \tag{4.52}$$

$$C = \langle C \rangle + \delta C = \delta C, \tag{4.53}$$

$$\varphi = \langle \varphi \rangle + \delta \varphi = \varphi_0 + \delta \varphi. \tag{4.54}$$

So a Kaluza-Klein ansatz for C can be written as

$$C = \sum_{N=1}^{b_3} c^N(x) \phi_N + \sum_{I=1}^{b_2} A^I(x) \wedge \alpha_I, \tag{4.55}$$

where $\{\phi_N\}$ are a basis for harmonic 3-forms on X , $\{\alpha_I\}$ are a basis for harmonic 2-forms on X , $c^N(x)$ are scalars on M_4 and $A^I(x)$ are 1-forms on M_4 which describe the fluctuations of C . Also b_2 and b_3 are the Betti numbers of X . Since we assume that X has holonomy equal to G_2 , $b_1 = 0$, so in (4.55) we do not have a contribution from harmonic 1-forms on X . Now, deformations of the metric on X are encoded in the deformations of φ and since φ is harmonic on X , we parameterize φ as

$$\varphi = \sum_{N=1}^{b_3} s^N(x) \phi_N. \tag{4.56}$$

Overall, in 4 dimensions we get b_3 real scalars c^N and b_3 real scalars s^N . Together these combine into b_3 massless complex scalars z^N :

$$z^N = \frac{1}{2} (s^N + i c^N). \tag{4.57}$$

In the 4-dimensional supergravity theory this gives b_3 massless chiral superfields. The 1-forms A^I in (4.55) give rise to b_2 massless Abelian gauge fields, and together with superpartners arising from the gravitino fields, these form b_2 massless vector superfields [15]. Thus overall, in four dimensions the effective low energy theory is $\mathcal{N} = 1$ supergravity coupled to b_2 abelian vector supermultiplets and b_3 massless chiral supermultiplets. The physical theory is not very interesting from a phenomenological point of view, since the gauge group is abelian and there are no charged particles. However the combination (4.57) proves to be very useful for studying the moduli space of G_2 manifolds, since it provides a natural, physically motivated complexification of the pure G_2 moduli space—something very similar to the complexified Kähler cone used in the study of Calabi-Yau moduli spaces.

Let us now use our Kaluza-Klein ansatz to reduce the 11-dimensional action (4.45) to 4 dimensions. Here we follow [19,20] and [14]. The term which interests us is the kinetic term for the z^N . The kinetic term for the c^N , $L_{kin}(c)$ comes from the reduction

of the $G \wedge *G$ term in (4.45). After switching to the Einstein frame by $g_{\mu\nu} \rightarrow V^{-1}g_{\mu\nu}$ we immediately see this gives us

$$L_{kin}(c) = -\frac{1}{4V} \partial_\mu c^M \partial^\mu c^N \int_X \phi_M \wedge * \phi_N. \tag{4.58}$$

The kinetic term for the s^M appears from the reduction of the $R^{(11)}$ term in (4.45). This is less straightforward than the derivation of $L_{kin}(c)$, but the calculation was shown explicitly in [14]. From the general properties of the Ricci scalar we can decompose the eleven-dimensional Einstein-Hilbert action as

$$\int d^{11}x (-\hat{g})^{\frac{1}{2}} R^{(11)} = \int d^{11}x (-\hat{g})^{\frac{1}{2}} V \left(R^{(4)} + R^{(7)} + \frac{1}{4V} (\partial_\mu g_{mn} \partial^\mu g^{mn} - \text{Tr}(\partial_\mu g) \text{Tr}(\partial^\mu g)) \right). \tag{4.59}$$

Then, using deformation properties of the G_2 metric g_{mn} from Sect. 5, and switching to the Einstein frame $g_{\mu\nu} \rightarrow V^{-1}g_{\mu\nu}$, we eventually get

$$L_{kin}(s) = -\frac{1}{4V} \partial_\mu s^M \partial^\mu s^N \int_X \phi_M \wedge * \phi_N. \tag{4.60}$$

The kinetic term of the dimensionally reduced action is in general given in the Einstein frame by

$$L_{kin} = -G_{M\bar{N}} \partial_\mu z^M \partial^\mu \bar{z}^N. \tag{4.61}$$

Comparing (4.61) with (4.58) and (4.60), we can read off the moduli space metric $G_{M\bar{N}}$ as

$$G_{M\bar{N}} = \frac{1}{V} \int_X \phi_M \wedge * \phi_{\bar{N}}. \tag{4.62}$$

Note that the Hodge star implicitly depends on the coordinates z^M , so this metric is quite non-trivial.

The bosonic part of fully reduced 4-dimensional Lagrangian is given in this case by [21, 20]

$$L = -G_{M\bar{N}} \partial_\mu z^M \partial^\mu \bar{z}^N - \frac{1}{4} \text{Re} h_{IJ} F_{mn}^I F^{Jmn} + \frac{1}{4} \text{Im} h_{IJ} F_{mn}^I * F^{Jmn}, \tag{4.63}$$

where $G_{M\bar{N}}$ is as in (4.62), and

$$F_{mn}^I = \partial_m A_n^I - \partial_n A_m^I.$$

The couplings $\text{Re} h_{IJ}$ and $\text{Im} h_{IJ}$ are given by

$$\text{Re} h_{IJ}(s) = \frac{1}{2} \int \alpha_I \wedge * \alpha_J = -\frac{1}{2} s^M \int \alpha_I \wedge \alpha_J \wedge \phi_M, \tag{4.64}$$

$$\text{Im} h_{IJ}(c) = -\frac{1}{2} c^M \int \alpha_I \wedge \alpha_J \wedge \phi_M. \tag{4.65}$$

To get the second equality in (4.64) we have used that $H^2 = H_{14}^2$ for manifolds with G_2 holonomy and that for a 2-form α , $2 * \pi_7(\alpha) - * \pi_{14}(\alpha) = \alpha \wedge \varphi$. Proof of this fact can be found in [10].

5. Deformations of G_2 Structures

As we already know, the G_2 structure on X and the corresponding metric g are all determined by the invariant 3-form φ . Hence, deformations of φ will induce deformations of the metric. These deformations of metric will then also affect the deformation of $\ast\varphi$. Since the relationship (2.8) between g and φ is non-linear, the resulting deformations of the metric are highly non-trivial, and in general it is not possible to write them down in closed form. However, as shown by Karigiannis in [10], metric deformations can be made explicit when the 3-form deformations are either in Λ_1^3 or Λ_7^3 . We now briefly review some of these results.

First suppose

$$\tilde{\varphi} = f\varphi. \tag{5.1}$$

Then from (2.10) we get

$$\begin{aligned} \tilde{g}_{ab}\sqrt{\det \tilde{g}} &= \frac{1}{144}\tilde{\varphi}_{amn}\tilde{\varphi}_{bpq}\tilde{\varphi}_{rst}\hat{\varepsilon}^{mnpqrst} \\ &= f^3g_{ab}\sqrt{\det g}. \end{aligned} \tag{5.2}$$

After taking the determinant on both sides, we obtain

$$\det \tilde{g} = f^{\frac{14}{3}} \det g. \tag{5.3}$$

Substituting (5.3) into (5.2), we finally get

$$\tilde{g}_{ab} = f^{\frac{2}{3}}g_{ab}, \tag{5.4}$$

and hence

$$\tilde{\ast}\tilde{\varphi} = f^{\frac{4}{3}}\ast\varphi. \tag{5.5}$$

Therefore, a scaling of φ gives a conformal transformation of the metric. Hence deformations of φ in the direction Λ_1^3 also give infinitesimal conformal transformation. Suppose $f = 1 + \varepsilon a$, then to fourth order in ε , we can write

$$\tilde{\ast}\tilde{\varphi} = \left(1 + \frac{4}{3}a\varepsilon + \frac{2}{9}a^2\varepsilon^2 - \frac{4}{81}a^3\varepsilon^3 + \frac{5}{243}a^4\varepsilon^4 + O(\varepsilon^5)\right)\ast\varphi. \tag{5.6}$$

Now, suppose in general that $\tilde{\varphi} = \varphi + \varepsilon\chi$ for some $\chi \in \Lambda^3$. Then using (2.8) for the definition of the metric associated with $\tilde{\varphi}$,

$$\begin{aligned} \widetilde{\langle u, v \rangle} \widetilde{\text{vol}} &= \frac{1}{6}(u \lrcorner \tilde{\varphi}) \wedge (v \lrcorner \tilde{\varphi}) \wedge \tilde{\varphi} \\ &= \frac{1}{6}(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi \\ &\quad + \frac{1}{6}\varepsilon [(u \lrcorner \chi) \wedge (v \lrcorner \varphi) \wedge \varphi + (u \lrcorner \varphi) \wedge (v \lrcorner \chi) \wedge \varphi + (u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \chi] \\ &\quad + \frac{1}{6}\varepsilon^2 [(u \lrcorner \chi) \wedge (v \lrcorner \chi) \wedge \varphi + (u \lrcorner \varphi) \wedge (v \lrcorner \chi) \wedge \chi + (u \lrcorner \chi) \wedge (v \lrcorner \varphi) \wedge \chi] \\ &\quad + \frac{1}{6}\varepsilon^3 (u \lrcorner \chi) \wedge (v \lrcorner \chi) \wedge \chi. \end{aligned} \tag{5.7}$$

After some manipulations, we can rewrite this as:

$$\begin{aligned} \widetilde{\langle u, v \rangle} \widetilde{\text{vol}} &= \frac{1}{6} (u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi \\ &\quad + \frac{1}{2} \varepsilon [(u \lrcorner \chi) \wedge * (v \lrcorner \varphi) + (v \lrcorner \chi) \wedge * (u \lrcorner \varphi)] \\ &\quad + \frac{1}{2} \varepsilon^2 (u \lrcorner \chi) \wedge (v \lrcorner \chi) \wedge \varphi \\ &\quad + \frac{1}{6} \varepsilon^3 (u \lrcorner \chi) \wedge (v \lrcorner \chi) \wedge \chi. \end{aligned} \quad (5.8)$$

Rewriting (5.8) in local coordinates, we get

$$\begin{aligned} \tilde{g}_{ab} \frac{\sqrt{\det \tilde{g}}}{\sqrt{\det g}} &= g_{ab} + \frac{1}{2} \varepsilon \chi_{mn(a} \varphi_{b)}^{mn} + \frac{1}{8} \varepsilon^2 \chi_{amn} \chi_{bpq} \psi^{mnpq} \\ &\quad + \frac{1}{24} \varepsilon^3 \chi_{amn} \chi_{bpq} (*\chi)^{mnpq}. \end{aligned} \quad (5.9)$$

Now suppose the deformation is in the Λ_7^3 direction. This implies that

$$\chi = \omega \lrcorner * \varphi \quad (5.10)$$

for some vector field ω . Look at the first order term. From (3.41) and (3.43) we see that this is essentially a projection onto $\Lambda_1^3 \oplus \Lambda_{27}^3$ —the traceless part gives the Λ_{27}^3 component and the trace gives the Λ_1^3 component. Hence this term vanishes for $\chi \in \Lambda_7^3$. For the third order term, it is more convenient to study it in (5.8). By looking at

$$\omega \lrcorner ((u \lrcorner \omega \lrcorner * \varphi) \wedge (v \lrcorner \omega \lrcorner * \varphi) \wedge * \varphi) = 0,$$

we immediately see that the third order term vanishes. So now we are left with

$$\begin{aligned} \tilde{g}_{ab} \sqrt{\det \tilde{g}} &= \left(g_{ab} + \frac{1}{8} \varepsilon^2 \omega^c \omega^d \psi_{camn} \psi_{dbpq} \psi^{mnpq} \right) \sqrt{\det g} \\ &= \left(g_{ab} \left(1 + \varepsilon^2 |\omega|^2 \right) - \varepsilon^2 \omega_a \omega_b \right) \sqrt{\det g}, \end{aligned} \quad (5.11)$$

where we have used the contraction identity for ψ (3.26) twice. Taking the determinant of (5.11) gives

$$\sqrt{\det \tilde{g}} = \left(1 + \varepsilon^2 |\omega|^2 \right)^{\frac{2}{3}} \sqrt{\det g}, \quad (5.12)$$

$$\tilde{g}_{ab} = \left(1 + \varepsilon^2 |\omega|^2 \right)^{-\frac{2}{3}} \left(\left(g_{ab} \left(1 + \varepsilon^2 |\omega|^2 \right) - \varepsilon^2 \omega_a \omega_b \right) \right), \quad (5.13)$$

and eventually,

$$\tilde{*}\tilde{\varphi} = \left(1 + \varepsilon^2 |\omega|^2 \right)^{-\frac{1}{3}} \left(*\varphi + *\varepsilon (\omega \lrcorner * \varphi) + \varepsilon^2 \omega \lrcorner * (\omega \lrcorner \varphi) \right). \quad (5.14)$$

The details of these last steps can be found in [10]. Notice that to first order in ε , both $\sqrt{\det g}$ and g_{ab} remain unchanged under this deformation. Now let us examine the last term in (5.14) in more detail. Firstly, we have

$$\omega \lrcorner * (\omega \lrcorner \varphi) = * (\omega^\flat \wedge (\omega \lrcorner \varphi))$$

and

$$\begin{aligned} (\omega^\flat \wedge (\omega \lrcorner \varphi))_{mnp} &= 3\omega_{[m}\omega^a\varphi_{|a|n]p]} \\ &= 3i_\varphi(\omega \circ \omega), \end{aligned} \tag{5.15}$$

where $(\omega \circ \omega)_{ab} = \omega_a\omega_b$. Therefore, in (5.14), this term gives Λ_1^4 and Λ_{27}^4 components. So, we can write (5.14) as

$$\begin{aligned} \tilde{*}\tilde{\varphi} &= \left(1 + \varepsilon^2|\omega|^2\right)^{-\frac{1}{3}} \left(\left(1 + \frac{3}{7}\varepsilon^2|\omega|^2\right) * \varphi \right. \\ &\quad \left. + * \varepsilon(\omega \lrcorner * \varphi) + \varepsilon^2 * i_\varphi((\omega \circ \omega)_0) \right). \end{aligned} \tag{5.16}$$

Here $(\omega \circ \omega)_0$ denotes the traceless part of $\omega \circ \omega$, so that $i_\varphi((\omega \circ \omega)_0) \in \Lambda_{27}^3$ and thus, in (5.16), the components in different representations are now explicitly shown.

As we have seen above, in the cases when the deformations were in Λ_1^3 or Λ_{27}^3 directions, there were some simplifications, which make it possible to write down all results in a closed form. Now however we will look at deformations in the Λ_{27}^3 directions, and we will work to fourth order in ε . So suppose we have a deformation

$$\tilde{\varphi} = \varphi + \varepsilon\chi,$$

where $\chi \in \Lambda_{27}^3$. Now let us set up some notation. Define

$$\tilde{s}_{ab} = \frac{1}{144} \frac{1}{\sqrt{\det g}} \tilde{\varphi}_{amn} \tilde{\varphi}_{bpq} \tilde{\varphi}_{rst} \hat{g}^{mnpqrst} \tag{5.17}$$

$$= \tilde{g}_{ab} \frac{\sqrt{\det \tilde{g}}}{\sqrt{\det g}}. \tag{5.18}$$

From (2.10), the untilded s_{ab} is then just equal to g_{ab} . We can rewrite (5.18) as

$$(g_{ab} + \delta g_{ab}) \frac{\sqrt{\det \tilde{g}}}{\sqrt{\det g}} = g_{ab} + \delta s_{ab}, \tag{5.19}$$

where δg_{ab} is the deformation of the metric and δs_{ab} is the deformation of s_{ab} , which from (5.9) is given by

$$\delta s_{ab} = \frac{1}{2}\varepsilon \chi_{mn(a}\varphi_{b)}^{mn} + \frac{1}{8}\varepsilon^2 \chi_{amn} \chi_{bpq} \psi^{mnpq} + \frac{1}{24}\varepsilon^3 \chi_{amn} \chi_{bpq} (*\chi)^{mnpq}. \tag{5.20}$$

Also we introduce the following short-hand notation:

$$s_k = \text{Tr} \left((\delta s)^k \right), \tag{5.21}$$

$$t_k = \text{Tr} \left((\delta g)^k \right), \tag{5.22}$$

where the trace is taken using the original metric g . From (5.20), note that since $\chi \in \Lambda_{27}^3$, when taking the trace the first order term vanishes, and hence s_1 is second-order in ε .

Further, after taking the trace of (5.19) using g^{ab} and rearranging, we have

$$\sqrt{\frac{\det \tilde{g}}{\det g}} = \left(1 + \frac{1}{7}s_1\right) \left(1 + \frac{1}{7}t_1\right)^{-1}, \tag{5.23}$$

and hence

$$\tilde{g}_{ab} = \tilde{s}_{ab} \left(1 + \frac{1}{7}t_1 \right) \left(1 + \frac{1}{7}s_1 \right)^{-1}. \tag{5.24}$$

As shown in Appendix B, we can also expand $\det \tilde{g}$ as

$$\begin{aligned} \frac{\det \tilde{g}}{\det g} &= 1 + t_1 + \frac{1}{2} (t_1^2 - t_2) + \frac{1}{6} (t_1^3 - 3t_1t_2 + 2t_3) \\ &+ \frac{1}{24} (t_1^4 - 6t_1^2t_2 + 3t_2^2 + 8t_1t_3 - 6t_4) + O(|\delta g|^5), \end{aligned} \tag{5.25}$$

and hence

$$\begin{aligned} \sqrt{\frac{\det \tilde{g}}{\det g}} &= 1 + \frac{1}{2}t_1 + \left(\frac{1}{8}t_1^2 - \frac{1}{4}t_2 \right) + \left(\frac{1}{48}t_1^3 - \frac{1}{8}t_1t_2 + \frac{1}{6}t_3 \right) \\ &+ \left(\frac{1}{384}t_1^4 - \frac{1}{32}t_1^2t_2 + \frac{1}{32}t_2^2 + \frac{1}{12}t_1t_3 - \frac{1}{8}t_4 \right) + O(|\delta g|^5). \end{aligned} \tag{5.26}$$

Thus we can equate (5.23) and (5.26). Suppose t_1 is first order in ε . Then the only first order term in (5.26) is $\frac{1}{2}t_1$, but since s_1 is second-order, the only first order term in (5.23) is $-\frac{1}{7}t_1$. It therefore follows that first order terms vanish, and so in fact t_1 is also second-order in ε . This has profound consequences in that we can ignore some of the terms in (5.26), as they give terms higher than fourth order:

$$\sqrt{\frac{\det \tilde{g}}{\det g}} = 1 + \left(\frac{1}{2}t_1 - \frac{1}{4}t_2 \right) + \frac{1}{6}t_3 + \left(\frac{1}{8}t_1^2 - \frac{1}{8}t_1t_2 + \frac{1}{32}t_2^2 - \frac{1}{8}t_4 \right) + O(\varepsilon^5). \tag{5.27}$$

From (5.24) we can write down δg_{ab} to fourth order in ε in terms of t_1 and quantities related to δs_{ab} and from this get t_2, t_3 and t_4 in terms of t_1 and δs_{ab} . So we have

$$\begin{aligned} \delta g_{ab} &= g_{ab} \left(\left(\frac{1}{7}t_1 - \frac{1}{7}s_1 \right) + \left(\frac{1}{49}s_1^2 - \frac{1}{49}s_1t_1 \right) \right) \\ &+ \delta s_{ab} \left(1 + \left(\frac{1}{7}t_1 - \frac{1}{7}s_1 \right) \right) + O(\varepsilon^5) \end{aligned} \tag{5.28}$$

and then from this,

$$t_2 = s_2 + \frac{1}{7} \left(-s_1^2 + t_1^2 + 2t_1s_2 - 2s_1s_2 \right) + O(\varepsilon^5), \tag{5.29}$$

$$t_3 = s_3 + \frac{3}{7} (t_1s_2 - s_1s_2) + O(\varepsilon^5), \tag{5.30}$$

$$t_4 = s_4 + O(\varepsilon^5). \tag{5.31}$$

Substituting, (5.29)-(5.31) into (5.27), we obtain

$$\begin{aligned} \sqrt{\frac{\det \tilde{g}}{\det g}} &= 1 + \left(-\frac{1}{4}s_2 + \frac{1}{2}t_1 \right) + \frac{1}{6}s_3 + \left(-\frac{1}{8}s_4 - \frac{1}{8}s_2t_1 + \frac{1}{28}s_1^2 \right) \\ &+ \frac{1}{32}s_2^2 + \frac{5}{56}t_1^2 + O(\varepsilon^5). \end{aligned} \tag{5.32}$$

After expanding (5.23) to fourth order in ε and equating with (5.32), we are left with a quadratic equation for t_1 :

$$\begin{aligned} \frac{27}{392}t_1^2 + t_1 \left(\frac{9}{14} + \frac{1}{49}s_1 - \frac{1}{8}s_2 \right) + \left(-\frac{1}{7}s_1 - \frac{1}{4}s_2 + \frac{1}{6}s_3 \right. \\ \left. - \frac{1}{8}s_4 + \frac{1}{28}s_1^2 + \frac{1}{32}s_2^2 \right) + O(\varepsilon^5). \end{aligned} \quad (5.33)$$

Obviously there are two solutions, but it turns out that one of them has a term which is zero order in ε , so this does not fit our assumptions, and hence we are only left with one solution, which to fourth order in ε is given by

$$t_1 = \frac{2}{9}s_1 + \frac{7}{18}s_2 - \frac{7}{27}s_3 + \left(\frac{7}{36}s_4 + \frac{1}{81}s_1s_2 - \frac{11}{162}s_1^2 + \frac{7}{648}s_2^2 \right) + O(\varepsilon^5). \quad (5.34)$$

Now that we have $t_1 = \text{Tr}(\delta g)$, from (5.23) we have

$$\begin{aligned} \sqrt{\frac{\det \tilde{g}}{\det g}} = 1 + \left(\frac{1}{9}s_1 - \frac{1}{18}s_2 \right) + \frac{1}{27}s_3 + \left(\frac{1}{162}s_1^2 - \frac{1}{162}s_1s_2 \right. \\ \left. - \frac{1}{36}s_4 + \frac{1}{648}s_2^2 \right) + O(\varepsilon^5). \end{aligned} \quad (5.35)$$

Using this and (5.19) we can immediately get the deformed metric. The precise expression however is not very useful for us at this stage. What we want is to be able to calculate the Hodge star with respect to the deformed metric. So let α be a 3-form, and consider the Hodge dual of α with respect to the deformed metric:

$$\begin{aligned} (\tilde{*}\alpha)_{mnpq} &= \frac{1}{3!} \frac{1}{\sqrt{\det \tilde{g}}} \hat{\varepsilon}^{abcdrst} \tilde{g}_{ma} \tilde{g}_{nb} \tilde{g}_{pc} \tilde{g}_{qd} \alpha_{rst} \\ &= \frac{\sqrt{\det g}}{\sqrt{\det \tilde{g}}} (*\alpha)^{abcd} \tilde{g}_{ma} \tilde{g}_{nb} \tilde{g}_{pc} \tilde{g}_{qd} \\ &= \left(\frac{\det g}{\det \tilde{g}} \right)^{\frac{5}{2}} (*\alpha)^{abcd} \tilde{s}_{ma} \tilde{s}_{nb} \tilde{s}_{pc} \tilde{s}_{qd} \\ &= \left(\frac{\det g}{\det \tilde{g}} \right)^{\frac{5}{2}} \left((*\alpha)_{mnpq} + 4(*\alpha)_{[mnp}{}^d \delta s_{q]d} + 6(*\alpha)_{[mn}{}^{cd} \delta s_{p|c|} \delta s_{q]d} \right. \\ &\quad \left. + 4(*\alpha)_{[m}{}^{bcd} \delta s_{n|b|} \delta s_{p|c|} \delta s_{q]d} + (*\alpha)^{abcd} \delta s_{am} \delta s_{bn} \delta s_{cp} \delta s_{dq} \right). \end{aligned}$$

From (5.35), the prefactor $\left(\frac{\det g}{\det \tilde{g}} \right)^{\frac{5}{2}}$ is given to fourth order by

$$\begin{aligned} \left(\frac{\det g}{\det \tilde{g}} \right)^{\frac{5}{2}} = 1 + \left(-\frac{5}{9}s_1 + \frac{5}{18}s_2 \right) - \frac{5}{27}s_3 + \left(\frac{5}{36}s_4 \right. \\ \left. - \frac{25}{162}s_1s_2 + \frac{25}{162}s_1^2 + \frac{25}{648}s_2^2 \right) + O(\varepsilon^5). \end{aligned} \quad (5.36)$$

Finally, consider how $*\varphi$ deforms:

$$\begin{aligned}
 (\tilde{*}\tilde{\varphi})_{mnpq} &= \tilde{*}\varphi_{mnpq} + \varepsilon \tilde{*}\chi_{mnpq} \\
 &= \left(\frac{\det g}{\det \tilde{g}} \right)^{\frac{5}{2}} \left((*\varphi)_{mnpq} + 4(*\varphi)_{[mnp}{}^d \delta s_{q]d} + 6(*\varphi)_{[mn}{}^{cd} \delta s_{p|c|} \delta s_{q]d} \right. \\
 &\quad + 4(*\varphi)_{[m}{}^{bcd} \delta s_{n|b|} \delta s_{p|c|} \delta s_{q]d} + (*\varphi)^{abcd} \delta s_{am} \delta s_{bn} \delta s_{cp} \delta s_{dq} \\
 &\quad + \varepsilon (*\chi)_{mnpq} + 4\varepsilon (*\chi)_{[mnp}{}^d \delta s_{q]d} + 6\varepsilon (*\chi)_{[mn}{}^{cd} \delta s_{p|c|} \delta s_{q]d} \\
 &\quad \left. + 4\varepsilon (*\chi)_{[m}{}^{bcd} \delta s_{n|b|} \delta s_{p|c|} \delta s_{q]d} + O\left(\varepsilon^5\right) \right)
 \end{aligned} \tag{5.37}$$

We ignored the last term, because overall it is at least fifth order.

So far, the only property of Λ_{27}^3 that we have used is that it is orthogonal to φ , thus in fact, up to this point everything applies to Λ_7^3 as well. Now however, let χ be of the form

$$\chi_{abc} = h_{[a}^d \varphi_{bc]d}, \tag{5.38}$$

where h_{ab} is traceless and symmetric, so that $\chi \in \Lambda_{27}^3$. Let us first introduce some further notation. Let h_1, h_2, h_3, h_4 be traceless, symmetric matrices, and introduce the following shorthand notation:

$$(\varphi h_1 h_2 \varphi)_{mn} = \varphi_{abm} h_1^{ad} h_2^{be} \varphi_{den}, \tag{5.39}$$

$$\varphi h_1 h_2 h_3 \varphi = \varphi_{abc} h_1^{ad} h_2^{be} h_3^{cf} \varphi_{def}, \tag{5.40}$$

$$(\psi h_1 h_2 h_3 \psi)_{mn} = \psi_{abcm} \psi_{defn} h_1^{ad} h_2^{be} h_3^{cf}, \tag{5.41}$$

$$\psi h_1 h_2 h_3 h_4 \psi = \psi_{abcm} \psi_{defn} h_1^{ad} h_2^{be} h_3^{cf} h_4^{mn}. \tag{5.42}$$

It is clear that all of these quantities are symmetric in the h_i and moreover $(\varphi h_1 h_2 \varphi)_{mn}$ and $(\psi h_1 h_2 h_3 \psi)_{mn}$ are both symmetric in indices m and n . Then, it can be shown that

$$\begin{aligned}
 \chi_{(a|mn|\varphi_b)}{}^{mn} &= \frac{4}{3} h_{ab}, \\
 \chi_{amn} \chi_{bpq} * \varphi^{mnpq} &= -\frac{4}{7} |\chi|^2 g_{ab} + \frac{16}{9} (h^2)_{\{ab\}} - \frac{4}{9} (\varphi h h \varphi)_{\{ab\}}, \\
 \chi_{amn} \chi_{bpq} * \chi^{mnpq} &= \frac{32}{189} \text{Tr}(h^3) g_{ab} - \frac{8}{9} (\varphi h h^2 \varphi)_{\{ab\}},
 \end{aligned}$$

where as before $\{a b\}$ denotes the traceless symmetric part. Using this and (5.20), we can now express δs_{ab} in terms of h :

$$\begin{aligned}
 \delta s_{ab} &= \frac{2}{3} \varepsilon h_{ab} + g_{ab} \left(-\frac{1}{14} \varepsilon^2 |\chi|^2 + \frac{4}{567} \varepsilon^3 \text{Tr}(h^3) \right) \\
 &\quad + \varepsilon^2 \left(\frac{2}{9} (h^2)_{\{ab\}} - \frac{1}{18} (\varphi h h \varphi)_{\{ab\}} \right) - \frac{\varepsilon^3}{27} (\varphi h h^2 \varphi)_{\{ab\}},
 \end{aligned} \tag{5.43}$$

and hence

$$s_1 = \text{Tr}(\delta s) = -\frac{1}{2}\varepsilon^2 |\chi|^2 + \frac{4}{81}\varepsilon^3 \text{Tr}(h^3), \tag{5.44}$$

$$s_2 = \text{Tr}(\delta s^2) = 2\varepsilon^2 |\chi|^2 + \varepsilon^3 \left(\frac{8}{27} \text{Tr}(h^3) - \frac{2}{27} (\phi h h h \phi) \right) + \varepsilon^4 \left(-\frac{1}{16} |\chi|^4 + \frac{7}{162} \text{Tr}(h^4) - \frac{2}{81} (\phi h h h^2 \phi) + \frac{1}{324} (\psi h h h h \psi) \right), \tag{5.45}$$

$$s_3 = \text{Tr}(\delta s^3) = \frac{8}{27}\varepsilon^3 \text{Tr}(h^3) + \varepsilon^4 \left(-\frac{3}{2} |\chi|^4 + \frac{8}{27} \text{Tr}(h^4) - \frac{2}{27} (\phi h h h^2 \phi) \right), \tag{5.46}$$

$$s_4 = \text{Tr}(\delta s^4) = \frac{16}{81}\varepsilon^4 \text{Tr}(h^4). \tag{5.47}$$

To get the full expression for $\tilde{\ast}\tilde{\varphi}$, (5.44)-(5.47) have to be substituted into the expression for the prefactor $\left(\frac{\det g}{\det \tilde{g}}\right)^{\frac{5}{2}}$ (5.36), and then both (5.36) and (5.43) have to be substituted into the expression for $\tilde{\ast}\tilde{\varphi}$ (5.37). Obviously, the expressions involved quickly become absolutely gargantuan. Thankfully, we were able to use Maple and the freely available package "Riegeom" [22] to help with these calculations. After all the substitutions, the resulting expression still has dozens of terms which are not of much use. In order for the expression for $\tilde{\ast}\tilde{\varphi}$ to be useful, the terms in it have to be separated according to which representation of G_2 they belong to. Thus the final step is to apply projections onto Λ_1^4 , Λ_7^4 and Λ_{27}^4 (3.41)-(3.43). When applying these projections, many of the terms have ϕ and ψ contracted in some way, so the contraction identities (3.20)-(3.23) have to be used to simplify the expressions. The package "Riegeom" lacks the ability to make such substitutions, so a few simple custom Maple programs based on "Riegeom" had to be written in order to facilitate these calculations. Overall, the expansion of $\tilde{\ast}\tilde{\varphi}$ to third order is

$$\begin{aligned} \tilde{\ast}\tilde{\varphi} = & \ast\varphi - \varepsilon \ast\chi + \varepsilon^2 \left(\frac{1}{6} \ast i_\varphi ((\phi h h \phi)_0) - \frac{1}{42} |\chi|^2 \ast\varphi \right) \\ & - \varepsilon^3 \left(\frac{2}{1701} (\phi h h h \phi) \ast\varphi + \frac{5}{24} |\chi|^2 \ast\chi - \frac{1}{18} \ast i_\varphi (h_0^3) \right) \\ & + \frac{1}{36} \ast i_\varphi ((\psi h h h \psi)_0) + \frac{1}{324} u_\lrcorner \ast\varphi + O(\varepsilon^4), \end{aligned} \tag{5.48}$$

where $(\phi h h \phi)_0$, h_0^3 and $(\psi h h h \psi)_0$ denote the traceless parts of $(\phi h h \phi)_{ab}$, $(h^3)_{ab}$ and $(\psi h h h \psi)_{ab}$, respectively, and

$$u^a = \psi^a_{mnp} \varphi_{rst} h^{mr} h^{ns} h^{pt}. \tag{5.49}$$

Although above we did all calculations to fourth order, we will really only need the expansion of $\tilde{\ast}\tilde{\varphi}$ to third order. However for possible future reference here is the G_2 singlet piece of the fourth order

$$\pi_1(\tilde{\ast}\tilde{\varphi})|_{\varepsilon^4} = \frac{5}{13608} (\psi h h h h \psi) + \frac{25}{2016} |\chi|^4 - \frac{5}{6804} \text{Tr}(h^4). \tag{5.50}$$

In fact, using the homogeneity property of $\varphi \wedge * \varphi$, it is possible to relate Λ_{27}^4 terms with a higher order Λ_1^4 term, so calculating higher order terms is also a way to make sure that all the coefficients are consistent.

Now that we have expansions of $\tilde{\varphi}$ for 1- and 27-dimensional deformations, it is not difficult to combine them together. Suppose we want to combine conformal transformation and 27-dimensional deformations. As in the case with 7-dimensional deformations consider

$$\tilde{\varphi} = \hat{\varphi} + \varepsilon \chi,$$

where $\hat{\varphi} = f \varphi$ and $\chi \in \Lambda_{27}^3$. Consider only up to second order in (5.48),

$$\tilde{*} \tilde{\varphi} = \tilde{*} \hat{\varphi} - \varepsilon \tilde{*} \chi + \varepsilon^2 \left(-\frac{1}{42} |\widehat{\chi}|^2 \tilde{*} \hat{\varphi} + \frac{1}{6} \tilde{*} i_{\hat{\varphi}} \left((\hat{\varphi} \hat{h} \hat{\varphi})_0 \right) \right) + O(\varepsilon^3).$$

Note that since $h_{ab} = \frac{3}{4} \chi_{mn\{a\varphi_b\}}^{mn}$,

$$\begin{aligned} \hat{h}_{ab} &= \frac{3}{4} \chi_{mn\{a\hat{\varphi}_b\}}^{mn} = \frac{3}{4} \hat{g}^{mr} \hat{g}^{ms} \chi_{mn\{a\hat{\varphi}_b\}rs} \\ &= f^{-\frac{1}{3}} h_{ab}, \end{aligned}$$

and hence

$$\begin{aligned} (\hat{\varphi} \hat{h} \hat{\varphi})_{ab} &= \hat{\varphi}_{abm} \hat{\varphi}_{den} \hat{h}^{ad} \hat{h}^{be} \\ &= f^{-\frac{4}{3}} (\varphi h h \varphi)_{ab}. \end{aligned}$$

Moreover,

$$i_{\hat{\varphi}} \left((\hat{\varphi} \hat{h} \hat{\varphi})_0 \right) = f^{-1} i_{\varphi} ((\varphi h h \varphi)_0).$$

Therefore, overall,

$$\begin{aligned} \tilde{*} \tilde{\varphi} &= f^{\frac{4}{3}} * \varphi - \varepsilon f^{\frac{1}{3}} * \chi + \varepsilon^2 \left(-\frac{1}{42} f^{-\frac{2}{3}} |\chi|^2 * \varphi \right. \\ &\quad \left. + \frac{1}{6} f^{-\frac{2}{3}} * i_{\varphi} ((\varphi h h \varphi)_0) \right) + O(\varepsilon^3). \end{aligned} \tag{5.51}$$

Let $f = 1 + \varepsilon a$, and expand in powers of ε to third order to get

$$\begin{aligned} \tilde{*} \tilde{\varphi} &= * \varphi + \varepsilon \left(\frac{4}{3} a * \varphi - * \chi \right) + \varepsilon^2 \left(\left(\frac{2}{9} a^2 - \frac{1}{42} |\chi|^2 \right) * \varphi \right. \\ &\quad \left. - \frac{1}{3} a * \chi + \frac{1}{6} * i_{\varphi} ((\varphi h h \varphi)_0) \right) \\ &+ \varepsilon^3 \left(\left(\frac{1}{63} a |\chi|^2 - \frac{4}{81} a^3 \right) * \varphi - \frac{1}{9} a * i_{\varphi} ((\varphi h h \varphi)_0) \right. \\ &\quad \left. + \left(\frac{1}{9} a^2 - \frac{5}{24} |\chi|^2 \right) * \chi \right) \\ &+ \varepsilon^3 \left(\frac{1}{18} * i_{\varphi} (h_0^3) - \frac{1}{36} * i_{\varphi} ((\psi h h h \psi)_0) \right. \\ &\quad \left. - \frac{2}{1701} (\varphi h h h \varphi) * \varphi - \frac{1}{324} u_{\perp} * \varphi \right) + O(\varepsilon^4). \end{aligned} \tag{5.52}$$

6. Moduli Space

In Sect. 4 we described how M -theory can be used to give a natural complexification of the G_2 moduli space—denote this space by $\mathcal{M}_{\mathbb{C}}$. The metric (4.62) on $\mathcal{M}_{\mathbb{C}}$ arises naturally from the Kaluza-Klein reduction of the M -theory action. As shown in [19], it turns out that this metric is in fact Kähler, with the Kähler potential K given by

$$K = -3 \log V, \tag{6.1}$$

where as before, V is the volume of X ,

$$V = \frac{1}{7} \int \varphi \wedge * \varphi.$$

Note that sometimes K is given with a different normalization factor. Here we follow [19], but in [20] and [9], in particular, a different convention is used.

Let us show that K is indeed the Kähler potential for $G_{M\bar{N}}$. Clearly, V , K and $G_{M\bar{N}}$ only depend on the parameters s^M for the G_2 3-form—that is, only the real part s^M of the complex coordinates z^M on $\mathcal{M}_{\mathbb{C}}$. So let us for now just look at the s^M derivatives. Note that under a scaling $s^M \rightarrow \lambda s^M$, φ scales as $\varphi \rightarrow \lambda \varphi$ and from (5.5), $*\varphi$ scales as $*\varphi \rightarrow \lambda^{\frac{4}{3}} * \varphi$, and so V scales as

$$V \rightarrow \lambda^{\frac{7}{3}} V.$$

So V is homogeneous of order $\frac{7}{3}$ in the s^M , and hence

$$\begin{aligned} s^M \frac{\partial V}{\partial s^M} &= \frac{7}{3} V \\ &= \frac{1}{3} \int s^M \phi_M \wedge * \varphi, \end{aligned}$$

and thus,

$$\frac{\partial V}{\partial s^M} = \frac{1}{3} \int \phi_M \wedge * \varphi. \tag{6.2}$$

Hence,

$$\frac{\partial K}{\partial s^M} = -\frac{1}{V} \int \phi_M \wedge * \varphi. \tag{6.3}$$

Here the dependence on the s^M is encoded in V and in $*\varphi$, which depends non-linearly on the s^M . Thus we have,

$$\begin{aligned} \frac{\partial^2 K}{\partial z^M \partial \bar{z}^N} &= \frac{3}{V^2} \frac{\partial V}{\partial s^M} \frac{\partial V}{\partial s^N} - \frac{3}{V} \frac{\partial^2 V}{\partial s^M \partial s^N} \\ &= \frac{1}{3} \frac{1}{V^2} \left(\int \phi_M \wedge * \varphi \right) \left(\int \phi_N \wedge * \varphi \right) - \frac{1}{V} \int \phi_{(M} \wedge \partial_{N)} (*\varphi). \end{aligned}$$

As we know from Sect. 5, the first derivative of $*\varphi$ is given by

$$\partial_N (*\varphi) = \frac{4}{3} * \pi_1(\phi_N) + * \pi_7(\phi_N) - * \pi_{27}(\phi_N), \tag{6.4}$$

so therefore,

$$\int \phi_M \wedge \partial_N (*\varphi) = \frac{4}{3} \int (\pi_1(\varphi_M) \wedge *\pi_1(\varphi_N)) + \int (\pi_7(\varphi_M) \wedge *\pi_7(\varphi_N)) - \int (\pi_{27}(\varphi_M) \wedge *\pi_{27}(\varphi_N)).$$

Also using (3.41), we get

$$\frac{1}{3} \frac{1}{V^2} \left(\int \phi_M \wedge *\varphi \right) \left(\int \phi_N \wedge *\varphi \right) = \frac{7}{3} \frac{1}{V} \int \pi_1(\varphi_M) \wedge *\pi_1(\varphi_N). \tag{6.5}$$

Thus overall,

$$\frac{\partial^2 K}{\partial z^M \partial \bar{z}^N} = \frac{1}{V} \left(\int (\pi_1(\varphi_M) \wedge *\pi_1(\varphi_N)) - \int (\pi_7(\varphi_M) \wedge *\pi_7(\varphi_N)) + \int (\pi_{27}(\varphi_M) \wedge *\pi_{27}(\varphi_N)) \right). \tag{6.6}$$

Note that if $Hol(X) = G_2$ then all the seven-dimensional components vanish, and hence we get

$$\frac{\partial^2 K}{\partial z^M \partial \bar{z}^N} = \frac{1}{V} \int_X \phi_M \wedge *\phi_{\bar{N}} = G_{M\bar{N}}, \tag{6.7}$$

as claimed. Since the negative definite part of (6.6) vanishes, the resulting metric is positive definite.

In general, there is at least one other good candidate for the metric on the G_2 moduli space. The Hessian of V , rather than of $\log V$, can be used as a Kähler potential and gives a metric with signature $(1, b_{27}^3)$. This metric is in particular used in [23] and [9]. There are some advantages to using V as the Kähler potential, because some computations give more elegant results. However if we use the supergravity action as a starting point for the study of the moduli space, our choice of the Kähler potential is very natural.

Now we have a complex manifold $\mathcal{M}_{\mathbb{C}}$, equipped with the Kähler metric $G_{M\bar{N}}$, so it is now interesting to study the properties of this metric, and the geometry which it gives. We will use the metric $G_{M\bar{N}}$ to calculate the associated curvature tensor $\mathcal{R}_{M\bar{N}P\bar{Q}}$ of the manifold $\mathcal{M}_{\mathbb{C}}$. Note that calculation of the curvature of the moduli space but for a different choice of metric is done in [24].

Let us introduce local special coordinates on $\mathcal{M}_{\mathbb{C}}$. Let $\phi_0 = a\varphi$ and $\phi_\mu \in \Lambda_{27}^3$ for $\mu = 1, \dots, b_{27}^3$, so s^0 defines directions parallel to φ and s^μ define directions in Λ_{27}^3 . Since our metric is Kähler, the expression for $\mathcal{R}_{M\bar{N}P\bar{Q}}$ is given by

$$\mathcal{R}_{\bar{K}L\bar{M}N} = \partial_{\bar{M}} \partial_N \partial_L \partial_{\bar{K}} K - G^{R\bar{S}} (\partial_{\bar{M}} \partial_R \partial_{\bar{K}} K) (\partial_N \partial_L \partial_{\bar{S}} K). \tag{6.8}$$

Also define

$$A_{MNR} = \frac{\partial^3 K}{\partial z^M \partial z^N \partial z^R} \tag{6.9}$$

so that we can rewrite (6.8) as

$$\mathcal{R}_{\bar{K}L\bar{M}N} = \partial_{\bar{M}} \partial_N \partial_L \partial_{\bar{K}} K - G^{R\bar{S}} A_{\bar{M}R\bar{K}} A_{NL\bar{S}}. \tag{6.10}$$

Now it only remains to work out the third and fourth derivatives of K . Starting from (6.3) we find that

$$A_{MNR} = -\frac{1}{V} \int \phi_M \wedge \frac{\partial^2}{\partial s^N \partial s^R} (*\varphi) + \frac{1}{V^2} \left(\int \phi_{(M} \wedge *\varphi \right) \left(\int \phi_N \wedge \frac{\partial}{\partial s^R} (*\varphi) \right) - \frac{2}{9V^3} \left(\int \phi_M \wedge *\varphi \right) \left(\int \phi_N \wedge *\varphi \right) \left(\int \phi_R \wedge *\varphi \right), \quad (6.11)$$

and from the power series expansion of $*\varphi$ (5.52), we can extract the higher derivatives of $*\varphi$:

$$\partial_0 \partial_0 (*\varphi) = \frac{4}{9} a^2 * \varphi, \quad \partial_0 \partial_0 \partial_0 (*\varphi) = -\frac{8}{27} a^3 * \varphi, \quad (6.12a)$$

$$\partial_0 \partial_\mu (*\varphi) = -\frac{1}{3} a * \phi_\mu, \quad \partial_0 \partial_0 \partial_\mu (*\varphi) = \frac{2}{9} a^2 * \phi_\mu, \quad (6.12b)$$

$$\partial_\mu \partial_\nu (*\varphi) = -\frac{1}{21} \langle \phi_\mu, \phi_\nu \rangle * \varphi + \frac{1}{3} * i_\varphi ((\varphi h_\mu h_\nu \varphi)_0), \quad (6.12c)$$

$$\partial_0 \partial_\mu \partial_\nu (*\varphi) = \frac{2}{63} a \langle \phi_\mu, \phi_\nu \rangle * \varphi - \frac{2}{9} a * i_\varphi ((\varphi h_\mu h_\nu \varphi)_0), \quad (6.12d)$$

$$\begin{aligned} \partial_\mu \partial_\nu \partial_\kappa (*\varphi) &= -\frac{5}{4} \langle \phi_\mu, \phi_\nu \rangle * \phi_\kappa + \frac{1}{3} * i_\varphi ((h_\mu h_\nu h_\kappa)_0) \\ &\quad - \frac{1}{6} * i_\varphi ((\psi h_\mu h_\nu h_\kappa \psi)_0) - \frac{4}{567} (\varphi h_\mu h_\nu h_\kappa \varphi) * \varphi, \end{aligned} \quad (6.12e)$$

where h_μ, h_ν and h_κ are traceless symmetric matrices corresponding to the 3-forms ϕ_μ, ϕ_ν and ϕ_κ , respectively. Using these expressions, we can now write down all the components of A_{MNR} :

$$A_{\bar{0}\bar{0}\bar{0}} = -14a^3, \quad (6.13a)$$

$$A_{\bar{0}\bar{0}\bar{\mu}} = 0, \quad (6.13b)$$

$$A_{\bar{0}\bar{\mu}\bar{\nu}} = -\frac{2a}{V} \int \phi_\mu \wedge *\phi_\nu = -2a G_{\mu\bar{\nu}}, \quad (6.13c)$$

$$A_{\bar{\mu}\bar{\nu}\bar{\rho}} = -\frac{2}{27V} \int (\varphi h_{\bar{\mu}} h_{\bar{\nu}} h_{\bar{\rho}} \varphi) dV. \quad (6.13d)$$

Now also look at the fourth derivative of K . From (6.12), we get

$$\frac{\partial^4 K}{\partial z^0 \partial \bar{z}^0 \partial z^0 \partial \bar{z}^0} = 42a^4, \quad (6.14a)$$

$$\frac{\partial^4 K}{\partial z^0 \partial \bar{z}^0 \partial z^0 \partial \bar{z}^\mu} = 0, \quad (6.14b)$$

$$\frac{\partial^4 K}{\partial z^0 \partial \bar{z}^0 \partial z^\mu \partial \bar{z}^\nu} = \frac{4}{3} \frac{a^2}{V} \int \phi_\mu \wedge *\phi_\nu = \frac{4}{3} a^2 G_{\mu\bar{\nu}}, \quad (6.14c)$$

$$\frac{\partial^4 K}{\partial z^0 \partial \bar{z}^\mu \partial z^\nu \partial \bar{z}^\rho} = \frac{2}{9} \frac{a}{V} \int (\varphi h_\mu h_\nu h_\rho \varphi) \text{vol} = -3a A_{\bar{\mu}\bar{\nu}\bar{\rho}}, \quad (6.14d)$$

$$\begin{aligned}
\frac{\partial^4 K}{\partial z^\kappa \partial \bar{z}^\mu \partial z^\nu \partial \bar{z}^\rho} &= \frac{1}{3} (G_{\bar{\mu}\nu} G_{\kappa\bar{\rho}} + G_{\bar{\mu}\kappa} G_{\nu\bar{\rho}}) + \frac{1}{3} \frac{1}{V^2} \int \phi_\kappa \wedge * \phi_\nu \int \phi_{\bar{\mu}} \wedge * \phi_{\bar{\rho}} \\
&+ \frac{1}{27V} \int \left((\psi h_\kappa h_{\bar{\mu}} h_\nu h_{\bar{\rho}} \psi) - 2 \operatorname{Tr} (h_\kappa h_{\bar{\mu}} h_\nu h_{\bar{\rho}}) \right. \\
&\left. + \frac{5}{3} \operatorname{Tr} (h_{(\kappa} h_{\bar{\mu})}) \operatorname{Tr} (h_\nu h_{\bar{\rho}}) \right) \operatorname{vol}, \tag{6.14e}
\end{aligned}$$

Note that it can be shown using the identity (3.22) that

$$\psi h h h h \psi = 12 \left(\varphi h^2 h h \varphi \right) + 3 \operatorname{Tr} \left(h^2 \right)^2 - 6 \operatorname{Tr} \left(h^4 \right).$$

Now define

$$C_{MN} = \frac{\partial^2 K}{\partial z^M \partial \bar{z}^N}. \tag{6.15}$$

This is the second derivative of K but with pure indices, rather than the derivative with mixed indices which gives the metric $G_{M\bar{N}}$. Note that since $K = K(\operatorname{Im} z)$, we have

$$\frac{\partial^2 K}{\partial z^M \partial z^N} = \frac{\partial^2 K}{\partial z^M \partial \bar{z}^N} \tag{6.16}$$

so numerically, C_{MN} and $G_{M\bar{N}}$ are in fact equal, and in particular,

$$C_{\mu\nu} = \frac{1}{V} \int \phi_\mu \wedge * \phi_\nu. \tag{6.17}$$

So while C_{MN} is not technically part of the metric, it inherits some similar properties. This happens due to the fact that while the complexification of the moduli space comes naturally, the holomorphic structure is artificial to some extent, because the G_2 and C -field moduli do not really mix. Using this definition, we can rewrite (6.14e) as

$$\begin{aligned}
\frac{\partial^4 K}{\partial z^\kappa \partial \bar{z}^\mu \partial z^\nu \partial \bar{z}^\rho} &= \frac{1}{3} (G_{\bar{\mu}\nu} G_{\kappa\bar{\rho}} + G_{\bar{\mu}\kappa} G_{\nu\bar{\rho}}) + \frac{1}{3} C_{\bar{\mu}\bar{\rho}} C_{\kappa\nu} \\
&- \frac{1}{27V} \int \left(2 \operatorname{Tr} (h_\kappa h_{\bar{\mu}} h_\nu h_{\bar{\rho}}) - (\psi h_\kappa h_{\bar{\mu}} h_\nu h_{\bar{\rho}} \psi) \right. \\
&\left. - \frac{5}{3} \operatorname{Tr} (h_{(\kappa} h_{\bar{\mu})}) \operatorname{Tr} (h_\nu h_{\bar{\rho}}) \right) \operatorname{vol}.
\end{aligned}$$

Taking into account that $G^{0\bar{0}} = \frac{1}{7q^2}$ and $G^{0\bar{\mu}} = 0$, we have enough information to be able to write down the full expressions for the components of the curvature tensor:

$$\mathcal{R}_{0\bar{0}\bar{0}\bar{0}} = 14a^4, \tag{6.18}$$

$$\mathcal{R}_{0\bar{0}\bar{0}\bar{\mu}} = 0, \tag{6.19}$$

$$\mathcal{R}_{0\bar{0}\bar{\mu}\bar{\nu}} = 2a^2 G_{\mu\bar{\nu}}, \tag{6.20}$$

$$\mathcal{R}_{0\bar{\mu}\bar{\nu}\bar{\rho}} = -A_{\bar{\mu}\bar{\nu}\bar{\rho}} a, \tag{6.21}$$

$$\begin{aligned}
\mathcal{R}_{\kappa\bar{\mu}\nu\bar{\rho}} &= \frac{1}{3} (G_{\bar{\mu}\nu}G_{\kappa\bar{\rho}} + G_{\bar{\mu}\kappa}G_{\nu\bar{\rho}}) - G^{\tau\bar{\sigma}}A_{\bar{\mu}\tau\bar{\rho}}A_{\kappa\nu\bar{\sigma}} - \frac{5}{21}C_{\bar{\mu}\bar{\rho}}C_{\kappa\nu} \\
&+ \frac{1}{27V} \int \left((\psi h_{\kappa}h_{\bar{\mu}}h_{\nu}h_{\bar{\rho}}\psi) - 2 \operatorname{Tr} (h_{\kappa}h_{\bar{\mu}}h_{\nu}h_{\bar{\rho}}) \right. \\
&\left. + \frac{5}{3} \operatorname{Tr} (h_{(\kappa}h_{\bar{\mu})}) \operatorname{Tr} (h_{\nu}h_{\bar{\rho}}) \right) \operatorname{vol}.
\end{aligned} \tag{6.22}$$

Let us look at more detail at the expression for $A_{\mu\nu\bar{\rho}}$:

$$\begin{aligned}
A_{\bar{\mu}\nu\bar{\rho}} &= -\frac{2}{27V} \int \varphi h_{\bar{\mu}}h_{\nu}h_{\bar{\rho}}\varphi \operatorname{vol} \\
&= -\frac{2}{27V} \int \varphi_{abc}\varphi_{mnp}h_{\bar{\mu}}^{am}h_{\nu}^{bn}h_{\bar{\rho}}^{cp} \operatorname{vol}.
\end{aligned}$$

Define $h_{\mu}^a = h_{\mu}^a dx^m$. Then

$$\varphi_{abc}\varphi_{mnp}h_{\bar{\mu}}^{am}h_{\nu}^{bn}h_{\bar{\rho}}^{cp} \operatorname{vol} = 6\varphi_{abc}h_{\bar{\mu}}^a \wedge h_{\nu}^b \wedge h_{\bar{\rho}}^c \wedge *\varphi$$

and so,

$$A_{\bar{\mu}\nu\bar{\rho}} = -\frac{4}{9V} \int \varphi_{abc}h_{\bar{\mu}}^a \wedge h_{\nu}^b \wedge h_{\bar{\rho}}^c \wedge *\varphi. \tag{6.23}$$

This is the precise analogue of the Yukawa coupling which is defined on the Calabi-Yau moduli space. Similar expressions have appeared previously in [25, 8] and [9]. Similarly, we can write

$$\begin{aligned}
(\psi h_{\kappa}h_{\bar{\mu}}h_{\nu}h_{\bar{\rho}}\psi) \operatorname{vol} &= \psi_{abcd}\psi_{mnpq}h_{\kappa}^{am}h_{\bar{\mu}}^{bn}h_{\nu}^{cp}h_{\bar{\rho}}^{dq} \operatorname{vol} \\
&= 24 \left\langle \psi_{abcd}h_{\kappa}^a \wedge h_{\bar{\mu}}^b \wedge h_{\nu}^c \wedge h_{\bar{\rho}}^d, *\psi \right\rangle \operatorname{vol} \\
&= 24\psi_{abcd}h_{\kappa}^a \wedge h_{\bar{\mu}}^b \wedge h_{\nu}^c \wedge h_{\bar{\rho}}^d \wedge \varphi.
\end{aligned} \tag{6.24}$$

Hence, we can rewrite (6.22) as

$$\begin{aligned}
\mathcal{R}_{\kappa\bar{\mu}\nu\bar{\rho}} &= \frac{1}{3} (G_{\bar{\mu}\nu}G_{\kappa\bar{\rho}} + G_{\bar{\mu}\kappa}G_{\nu\bar{\rho}}) - G^{\tau\bar{\sigma}}A_{\bar{\mu}\tau\bar{\rho}}A_{\kappa\nu\bar{\sigma}} - \frac{5}{21}C_{\bar{\mu}\bar{\rho}}C_{\kappa\nu} \\
&+ \frac{8}{9} \frac{1}{V} \int \psi_{abcd}h_{\kappa}^a \wedge h_{\bar{\mu}}^b \wedge h_{\nu}^c \wedge h_{\bar{\rho}}^d \wedge \varphi \\
&+ \frac{1}{81} \frac{1}{V} \int (5 \operatorname{Tr} (h_{(\kappa}h_{\bar{\mu})}) \operatorname{Tr} (h_{\nu}h_{\bar{\rho}}) - 6 \operatorname{Tr} (h_{\kappa}h_{\bar{\mu}}h_{\nu}h_{\bar{\rho}})) \operatorname{vol}.
\end{aligned} \tag{6.25}$$

Note that because in the Λ_{27}^3 directions the first derivative of V vanishes, some of these terms which appear in the curvature expression can also be expressed as derivatives of V :

$$\begin{aligned}
\frac{\partial^3 V}{\partial \bar{z}^{\mu} \partial z^{\nu} \partial \bar{z}^{\rho}} &= -\frac{1}{3} A_{\bar{\mu}\nu\bar{\rho}} \\
\frac{\partial^4 V}{\partial z^{\kappa} \partial \bar{z}^{\mu} \partial z^{\nu} \partial \bar{z}^{\rho}} &= -\frac{8}{27} \int \psi_{abcd}h_{\kappa}^a \wedge h_{\bar{\mu}}^b \wedge h_{\nu}^c \wedge h_{\bar{\rho}}^d \wedge \varphi \\
&+ \frac{1}{243} \int (6 \operatorname{Tr} (h_{\kappa}h_{\bar{\mu}}h_{\nu}h_{\bar{\rho}}) - 5 \operatorname{Tr} (h_{(\kappa}h_{\bar{\mu})}) \operatorname{Tr} (h_{\nu}h_{\bar{\rho}})) \operatorname{vol}.
\end{aligned}$$

So alternatively, we can write

$$\mathcal{R}_{\kappa\bar{\mu}\nu\bar{\rho}} = \frac{1}{3} (G_{\bar{\mu}\nu}G_{\kappa\bar{\rho}} + G_{\bar{\mu}\kappa}G_{\nu\bar{\rho}}) - G^{\tau\bar{\sigma}} A_{\bar{\mu}\tau\bar{\rho}} A_{\kappa\nu\bar{\sigma}} - \frac{5}{21} C_{\bar{\mu}\bar{\rho}} C_{\kappa\nu} - \frac{3}{V} \frac{\partial^4 V}{\partial z^\kappa \partial \bar{z}^\mu \partial z^\nu \partial \bar{z}^\rho}.$$

Define

$$U_{\bar{M}} = \frac{3}{V} \frac{\partial^3 V}{\partial \bar{z}^M \partial z^N \partial \bar{z}^R} G^{N\bar{R}}. \tag{6.26}$$

Then

$$\partial_K U_{\bar{M}} = \frac{3}{V} \left(\frac{\partial^4 V}{\partial z^K \partial \bar{z}^M \partial z^N \partial \bar{z}^R} G^{N\bar{R}} - \frac{\partial^3 V}{\partial \bar{z}^M \partial z^N \partial \bar{z}^R} A_K^{N\bar{R}} \right). \tag{6.27}$$

We can use this to express the Ricci curvature

$$\mathcal{R}_{\kappa\bar{\mu}} = \left(\frac{1}{3} b^3(X) - \frac{1}{63} \right) G_{\kappa\bar{\mu}} - \partial_\kappa U_{\bar{\mu}}, \tag{6.28}$$

where $b^3(X) = b_{27}^3 + 1$ is the third Betti number of X . Also,

$$\mathcal{R}_{0\bar{\mu}} = -a A_{\bar{\mu}\nu\bar{\rho}} G^{\nu\bar{\rho}} = -\partial_0 U_{\bar{\mu}}, \tag{6.29}$$

$$\mathcal{R}_{0\bar{0}} = 2a^2 b^3(X). \tag{6.30}$$

Although here we have certain similarities with the structure of the Calabi-Yau moduli space, we are lacking a key feature of Calabi-Yau moduli space—a particular line bundle over the moduli space. For example, the holomorphic 3-form on a Calabi-Yau 3-fold defines a complex line bundle over the complex structure moduli space. In the G_2 case, we could try and see what happens if we look at the real line bundle L defined by φ over the complexified G_2 moduli space $\mathcal{M}_{\mathbb{C}}$. So consider the gauge transformations

$$\varphi \longrightarrow f(\operatorname{Re} z) \varphi, \tag{6.31}$$

where each $f(z)$ is a real number. Then, as in [8], define a covariant derivative \mathcal{D} on L by

$$\mathcal{D}_M \varphi = \partial_M \varphi + \frac{1}{7} (\partial_M K) \varphi. \tag{6.32}$$

Under the transformation (6.31),

$$\begin{aligned} V &\longrightarrow f^{\frac{7}{3}} V, \\ K &\longrightarrow K - 7 \log f, \end{aligned}$$

and so

$$\partial_M K \longrightarrow \partial_M K - \frac{7}{f} \partial_M f.$$

Hence

$$\mathcal{D}_M \varphi \longrightarrow f \mathcal{D}_M \varphi. \tag{6.33}$$

Moreover, from the expression for $\partial_M K$ (6.3), we find that

$$\mathcal{D}_0\varphi = 0 \quad \mathcal{D}_\mu\varphi = \partial_\mu\varphi.$$

So as noted in [8], this covariant derivative projects out the G_2 singlet contribution. It also gives a covariant way in which to extract the **27** contributions so we can use $\mathcal{D}_M\varphi$ when we just need to extract $\partial_\mu\varphi$. Also consider

$$\begin{aligned} \frac{1}{V} \langle\langle \mathcal{D}_M\varphi, *\mathcal{D}_{\bar{N}}\varphi \rangle\rangle &= \frac{1}{V} \int \mathcal{D}_M\varphi \wedge *\mathcal{D}_{\bar{N}}\varphi \\ &= G_{M\bar{N}} - \frac{1}{7} \partial_M K \partial_{\bar{N}} K. \end{aligned} \tag{6.34}$$

When one of the indices is equal to zero, the whole expression vanishes. However if both refer to the 27-dimensional components, then we just get $G_{\mu\bar{\nu}}$. A similar expression holds for C_{MN} .

More generally, we can extend the covariant to any quantity which transforms under (6.31). Suppose $Q(z)$ is a function on \mathcal{M}_C , which under (6.31) transforms as

$$Q(z) \longrightarrow f(z)^a Q(z).$$

Then define the covariant derivative on it by

$$\mathcal{D}_M Q = \partial_M Q + \frac{a}{7} (\partial_M K) Q. \tag{6.35}$$

From this we get

$$\begin{aligned} \mathcal{D}_M V &= 0, \\ \mathcal{D}_M (*\varphi) &= \partial_M (*\varphi) + \frac{1}{7} \frac{4}{3} (\partial_M K) (*\varphi), \end{aligned}$$

and in particular,

$$\mathcal{D}_0 (*\varphi) = 0 \quad \mathcal{D}_\mu (*\varphi) = - * (\partial_\mu\varphi)$$

so, in fact

$$\mathcal{D}_M (*\varphi) = - * \mathcal{D}_M\varphi.$$

Further we can extend \mathcal{D}_M to objects with moduli space indices by replacing ∂ by ∇ —the metric-compatible covariant derivative with respect to the moduli space metric $G_{M\bar{N}}$, for which the Christoffel symbols are given by

$$\Gamma_M^N Q = G^{N\bar{P}} \partial_M G_{\bar{P}Q} = A^N_{MQ}. \tag{6.36}$$

With these Christoffel symbols the covariant derivative of C_{MN} is hence

$$\nabla_Q C_{MN} = -A_{QMN}. \tag{6.37}$$

Then we also find that

$$\begin{aligned} \mathcal{D}_M \mathcal{D}_N \varphi &= \partial_M \left(\partial_N \varphi + \frac{1}{7} (\partial_N K) \varphi \right) - A^P_{NM} \mathcal{D}_P \varphi + \frac{1}{7} \partial_M K \mathcal{D}_N \varphi \\ &= \frac{1}{7} \left(C_{MN} - \frac{1}{7} \partial_M K \partial_N K \right) \varphi - A^P_{NM} \mathcal{D}_P \varphi + \frac{2}{7} \partial_{(M} K \mathcal{D}_{N)} \varphi \end{aligned} \quad (6.38)$$

$$= \frac{1}{7} \frac{1}{V} \langle \langle \mathcal{D}_M \varphi, * \mathcal{D}_N \varphi \rangle \rangle \varphi - A^P_{NM} \mathcal{D}_P \varphi + \frac{2}{7} \partial_{(M} K \mathcal{D}_{N)} \varphi, \quad (6.39)$$

and for mixed type derivatives, we have

$$\begin{aligned} \mathcal{D}_{\bar{M}} \mathcal{D}_N \varphi &= \partial_{\bar{M}} \left(\partial_N \varphi + \frac{1}{7} (\partial_N K) \varphi \right) + \frac{2}{7} \partial_{(\bar{M}} K \mathcal{D}_{N)} \varphi \\ &= \frac{1}{7} \frac{1}{V} \langle \langle \mathcal{D}_{\bar{M}} \varphi, * \mathcal{D}_N \varphi \rangle \rangle \varphi + \frac{2}{7} \partial_{(\bar{M}} K \mathcal{D}_{N)} \varphi \\ &= \frac{1}{7} \left(G_{\bar{M}N} \varphi + \frac{2}{7} (\partial_{\bar{M}} K \partial_N K) \varphi + \partial_{(\bar{M}} K \mathcal{D}_{N)} \varphi \right). \end{aligned}$$

Note that here the covariant derivatives commute, so this connection is in fact flat.

Now look at the third covariant derivative of φ :

$$\begin{aligned} \langle \langle \mathcal{D}_R \mathcal{D}_M \mathcal{D}_N \varphi, * \varphi \rangle \rangle &= \mathcal{D}_R \langle \langle \mathcal{D}_M \mathcal{D}_N \varphi, * \varphi \rangle \rangle - \langle \langle \mathcal{D}_M \mathcal{D}_N \varphi, \mathcal{D}_R * \varphi \rangle \rangle \\ &= \mathcal{D}_R \langle \langle \mathcal{D}_M \varphi, * \mathcal{D}_N \varphi \rangle \rangle + \langle \langle \mathcal{D}_M \mathcal{D}_N \varphi, * \mathcal{D}_R \varphi \rangle \rangle. \end{aligned} \quad (6.40)$$

First look at the second term in (6.40). Since $\mathcal{D}_R \varphi \in \Lambda^3_{27}$, we basically get the projection $\pi_{27}(\mathcal{D}_M \mathcal{D}_N \varphi)$:

$$\begin{aligned} \langle \langle \mathcal{D}_M \mathcal{D}_N \varphi, * \mathcal{D}_R \varphi \rangle \rangle &= -A^P_{NM} \langle \langle \mathcal{D}_P \varphi, * \mathcal{D}_R \varphi \rangle \rangle + \frac{2}{7} \partial_{(M} K \langle \langle \mathcal{D}_{N)} \varphi, * \mathcal{D}_R \varphi \rangle \rangle \\ &= -A_{MNR} + \frac{1}{7} A^P_{MN} \partial_R K \partial_P K + \frac{2}{7} C_{R(N} \partial_M) K - \frac{2}{49} \partial_R K \partial_M K \partial_N K. \end{aligned}$$

In the first term of (6.40), we have

$$\begin{aligned} \mathcal{D}_R \langle \langle \mathcal{D}_M \varphi, * \mathcal{D}_N \varphi \rangle \rangle &= V \mathcal{D}_R \left(\frac{1}{V} \langle \langle \mathcal{D}_M \varphi, * \mathcal{D}_N \varphi \rangle \rangle \right) \\ &= V \nabla_R \langle \langle \mathcal{D}_M \varphi, * \mathcal{D}_N \varphi \rangle \rangle \\ &= V \left(\nabla_R C_{MN} - \frac{1}{7} \nabla_R (\partial_M K \partial_N K) \right) \\ &= V \left(-A_{RMN} - \frac{2}{7} C_{R(M} \partial_{N)} K + \frac{2}{7} A^P_{R(M} \partial_{N)} K \partial_P K \right). \end{aligned}$$

Combining, we overall obtain

$$\frac{1}{V} \langle \langle \mathcal{D}_R \mathcal{D}_M \mathcal{D}_N \varphi, * \varphi \rangle \rangle = -2A_{RMN} - \frac{2}{49} \partial_R K \partial_M K \partial_N K + \frac{3}{7} A_{(MN}{}^P \partial_R) K \partial_P K. \quad (6.41)$$

Decomposing this into components, we have

$$\begin{aligned} \frac{1}{V} \langle \langle \mathcal{D}_\rho \mathcal{D}_\mu \mathcal{D}_\nu \varphi, *\varphi \rangle \rangle &= -2A_{\rho\mu\nu}, \\ \frac{1}{V} \langle \langle \mathcal{D}_0 \mathcal{D}_\mu \mathcal{D}_\nu \varphi, *\varphi \rangle \rangle &= 2C_{\mu\nu}, \\ \frac{1}{V} \langle \langle \mathcal{D}_0 \mathcal{D}_0 \mathcal{D}_\nu \varphi, *\varphi \rangle \rangle &= 0, \\ \frac{1}{V} \langle \langle \mathcal{D}_0 \mathcal{D}_0 \mathcal{D}_0 \varphi, *\varphi \rangle \rangle &= 0. \end{aligned}$$

Therefore, the quantity $\frac{1}{V} \langle \langle \mathcal{D}_\rho \mathcal{D}_\mu \mathcal{D}_\nu \varphi, *\varphi \rangle \rangle$ essentially gives the Yukawa coupling, again giving a result analogous to the case of Calabi-Yau moduli spaces.

7. Concluding Remarks

In this paper, we have computed the curvature of the complexified G_2 moduli space and found that while it has terms which are similar to the curvature of the Calabi-Yau moduli, there are a number of new terms. In future work it would be interesting to interpret these new terms geometrically. If we consider a 7-manifold of the form $CY_3 \times S^1$, where CY_3 is a Calabi-Yau 3-fold, then we can define a torsion-free G_2 structure on it. The relationship between the Calabi-Yau moduli space and the G_2 moduli space is however very non-trivial, because the complex structure moduli and the Kähler structure moduli become intertwined with each other. So it could turn out to be illuminating to try and relate the curvature of the G_2 moduli space to the curvatures of complex and Kähler moduli spaces. In that case, however, $b_2^3 = 1$, so in fact the second derivative of our Kähler potential would give a pseudo-Kähler metric with signature $(- + \dots +)$ (6.6). Moreover, the ansatz for the C -field (4.55) would also have to be different. Understanding how the Calabi-Yau moduli space is related to the G_2 moduli space could also enable us to find a manifestation of mirror symmetry from the G_2 perspective. Moreover, it would be interesting to see how existing approaches to mirror symmetry on G_2 manifolds (such as [26]) affect the geometric structures on the moduli space.

Another possible direction for further research is to look at G_2 manifolds in a slightly different way. Suppose we have type IIA superstrings on a non-compact Calabi-Yau 3-fold with a special Lagrangian submanifold which is wrapped by a $D6$ brane which also fills M_4 . Then, as explained in [27], from the M -theory perspective this looks like a S^1 bundle over the Calabi-Yau which is degenerate over the special Lagrangian submanifold, but this 7-manifold is still a G_2 manifold. The moduli space of this manifold will then be determined by the Calabi-Yau moduli and the special Lagrangian moduli. This possibly could provide more information about mirror symmetry on Calabi-Yau manifolds [28].

A. Appendix A: Projections of 3-Forms

Here will prove the formulae (3.41) to (3.43) which give the projections of 3-forms into 1-dimensional, 7-dimensional and 27-dimensional components. Let $\chi \in \Lambda^3$. Since Λ_1^3 , Λ_7^3 and Λ_{27}^3 are all orthogonal to each other, we immediately get

$$\pi_1(\chi) = a\varphi \text{ where } a = \frac{1}{42} \left(\chi_{abc} \varphi^{abc} \right) = \frac{1}{7} \langle \chi, \varphi \rangle \text{ and } |\pi_1(\chi)|^2 = 7a^2.$$

To work out $\pi_7(\chi)$, suppose

$$\pi_7(\chi) = u \lrcorner * \varphi,$$

then consider

$$\begin{aligned} (u \lrcorner * \varphi) \wedge * (v \lrcorner * \varphi) &= (u \lrcorner * \varphi) \wedge \varphi \wedge v^b \\ &= 4 * u^b \wedge v^b = 4 \langle u, v \rangle \text{vol}. \end{aligned} \quad (\text{A.1})$$

So this gives

$$|\pi_7(\chi)|^2 = 4 |\omega|^2. \quad (\text{A.2})$$

However (A.1) can also be expressed as

$$\begin{aligned} (u \lrcorner * \varphi) \wedge * (v \lrcorner * \varphi) &= \frac{1}{6} \pi_7(\chi)_{mnp} v_a \psi^{amnp} \text{vol} \\ &= -\frac{1}{6} \pi_7(\chi)_{mnp} \psi^{mnpa} v_a \text{vol}. \end{aligned} \quad (\text{A.3})$$

Equating (A.1) and (A.3), we get

$$u^a = -\frac{1}{24} \pi_7(\chi)_{mnp} \psi^{mnpa} = \omega^a.$$

Finally we look at $\pi_{27}(\chi)$. Consider

$$\chi_{abc} = \pi_1(\chi)_{abc} + \pi_7(\chi)_{abc} + h_{[a}^d \varphi_{bc]d}.$$

Then,

$$\pi_1(\chi)_{mn\{a} \varphi_b\}^{mn} = a \varphi_{mn\{a} \varphi_b\}^{mn} = 6g_{\{ab\}} = 0, \quad (\text{A.4})$$

$$\pi_7(\chi)_{mn\{a} \varphi_b\}^{mn} = \omega^p \psi_{p mn\{a} \varphi_b\}^{mn} = 4v^p \varphi_{p\{ab\}} = 0. \quad (\text{A.5})$$

Therefore,

$$\begin{aligned} \frac{3}{4} \chi_{mn\{a} \varphi_b\}^{mn} &= \frac{3}{4} h_{[m}^d \varphi_{n\{a} \varphi_b\}^{mn} \\ &= \frac{1}{2} h_m^d \varphi_{n\{a} \varphi_b\}^{mn} + \frac{1}{4} \varphi_{mnd} h_{\{a}^d \varphi_b\}^{mn} \\ &= \frac{1}{2} h_m^d \left(g_{\{ab\}} \delta_d^m - \delta_{\{a}^m g_{b\}d} - \psi_{\{ab\}d}{}^m \right) + \frac{3}{2} h_{ab} \\ &= h_{ab} \end{aligned} \quad (\text{A.6})$$

as required. Moreover,

$$\begin{aligned} |\pi_{27}(\chi)|^2 &= \frac{1}{6} h_{[a}^d \varphi_{bc]d} h^{ea} \varphi^bc_e \\ &= \frac{1}{18} h_a^d \varphi_{bcd} h^{ea} \varphi^bc_e + \frac{1}{9} h_c^d \varphi_{abd} h^{ea} \varphi^bc_e \\ &= \frac{1}{3} |h|^2 - \frac{1}{9} h_c^d h^{ea} \left(\delta_a^c g_{de} - g_{ae} \delta_q^c + * \varphi^c_{ade} \right) \\ &= \frac{2}{9} |h|^2. \end{aligned} \quad (\text{A.7})$$

B. Appendix B: Determinants

In this section, we will review deformations of determinants. Let I be the $n \times n$ identity matrix, and let h be a symmetric $n \times n$ matrix. Suppose $\lambda_1, \dots, \lambda_n$ are eigenvalues of h . Then

$$\begin{aligned} \det(I + \varepsilon h) &= \prod_{i=1}^n (1 + \varepsilon \lambda_i) \\ &= 1 + \varepsilon \sum_i \lambda_i + \varepsilon^2 \sum_{i < j} \lambda_i \lambda_j + \varepsilon^3 \sum_{i < j < k} \lambda_i \lambda_j \lambda_k \\ &\quad + \varepsilon^4 \sum_{i < j < k < l} \lambda_i \lambda_j \lambda_k \lambda_l + O(\varepsilon^5). \end{aligned} \tag{B.1}$$

Define

$$t_k = \sum_i \lambda_i^k = \text{Tr}(h^k).$$

Then from Newton’s identities we know that

$$\begin{aligned} \sum_i \lambda_i &= t_1, \\ \sum_{i < j} \lambda_i \lambda_j &= \frac{1}{2} (t_1^2 - t_2), \\ \sum_{i < j < k} \lambda_i \lambda_j \lambda_k &= \frac{1}{6} (t_1^3 - 3t_1 t_2 + 2t_3), \\ \sum_{i < j < k < l} \lambda_i \lambda_j \lambda_k \lambda_l &= \frac{1}{24} (t_1^4 - 6t_1^2 t_2 + 3t_2^2 + 8t_1 t_3 - 6t_4), \end{aligned}$$

and so we obtain

$$\begin{aligned} \det(I + \varepsilon h) &= 1 + \varepsilon t_1 + \frac{1}{2} \varepsilon^2 (t_1^2 - t_2) + \frac{1}{6} \varepsilon^3 (t_1^3 - 3t_1 t_2 + 2t_3) \\ &\quad + \frac{1}{24} \varepsilon^4 (t_1^4 - 6t_1^2 t_2 + 3t_2^2 + 8t_1 t_3 - 6t_4) + O(\varepsilon^5). \end{aligned} \tag{B.2}$$

Now, for a metric g , we get

$$\begin{aligned} \frac{\det(g + \varepsilon h)}{\det g} &= 1 + \varepsilon t_1 + \frac{1}{2} \varepsilon^2 (t_1^2 - t_2) + \frac{1}{6} \varepsilon^3 (t_1^3 - 3t_1 t_2 + 2t_3) \\ &\quad + \frac{1}{24} \varepsilon^4 (t_1^4 - 6t_1^2 t_2 + 3t_2^2 + 8t_1 t_3 - 6t_4) + O(\varepsilon^5), \end{aligned}$$

where the traces are now with respect to the metric g .

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