Commun. Math. Phys. 286, 681–696 (2009)
Digital Object Identifier (DOI) 10.1007/s00220-008-0587-1

Communications in
**Mathematical
Physics**

A Maximum Principle for the Muskat Problem for Fluids with Different Densities

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Received: 7 December 2007 / Accepted: 8 April 2008
Published online: 8 August 2008 –

Abstract: We study the fluid interface problem given by two incompressible fluids with different densities evolving by Darcy's law. This scenario is known as the Muskat problem for fluids with the same viscosities, being in two dimensions mathematically analogous to the two-phase Hele-Shaw cell. We prove in the stable case (the denser fluid is below) a maximum principle for the L^∞ norm of the free boundary.

1. Introduction

The Muskat problem models the fluid interface problem given by two fluids in a porous medium with different characteristics. The problem was proposed by Muskat (see [13]) in a study about the encroachment of water into oil in a porous medium. In this phenomena, Darcy's law is used to govern the dynamics of the different fluids [2]. This law is represented by the following formula:

$$\frac{\mu}{\kappa}v = -\nabla p - (0, 0, \mathrm{g}\,\rho),$$

where v is the velocity of the fluid, p is the pressure, μ is the dynamic viscosity, κ is the permeability of the isotropic medium, ρ is the liquid density and g is the acceleration due to gravity.

Saffman and Taylor [14] considered this problem in a study of the dynamics of the interface between two fluids with different viscosities and densities in a Hele–Shaw cell. In this physical scenario (see [11]) the fluid is trapped between two fixed parallel plates, that are close enough together, so that the fluid essentially only moves in two directions. The mean velocity of the fluid is given by

$$\frac{12\mu}{b^2}v = -\nabla p - (0, \mathrm{g}\,\rho),$$

where b is the distance between the plates. Darcy's law, in two dimensions, and the above formula become analogous if we consider the permeability of the medium κ equal to the constant $b^2/12$.

The Muskat problem and the two–phase Hele–Shaw flow have been extensively studied (see [4] and [12] and the references therein). These free boundary problems can be modeled with surface tension [9] using the Laplace–Young condition. In this case there is a jump of discontinuity in the pressure of the fluids across the interface proportional to the local curvature of the free boundary.

Here we shall consider the case without surface tension where the pressures are equal on the interface. Ambrose studies this scenario in [1], where he treats the two dimensional case with initial data satisfying

$$(\rho_2 - \rho_1)\mathrm{g}\cos(\theta(\alpha, 0)) + 2\frac{\mu_1 - \mu_2}{\mu_1 + \mu_2}U(\alpha, 0) > 0.$$

Here θ is the angle that the tangent to the curve forms with the horizontal, U is the normal velocity (given by the Birkhoff-Rott integral), ρ_i are the densities and μ_i are the viscosities of the fluids, for $i = 1, 2$. In this work he uses the arclength and the tangent angle formulation given by Hou, Lowengrub and Shelley in [12] to get energy estimates for the free boundary assuming that the arc-chord condition is satisfied locally in time. One of the authors shows in [10] that this is not enough to obtain local-existence for this kind of contour dynamics equations, since a regular interface could touch itself with order infinity and without satisfying the arc-chord condition.

Siegel, Caflisch and Howison analyze this problem in [15], where they show ill-posedness in an unstable case and global-in-time existence of small initial data in a stable case. They describe the two-dimensional dynamics of the incompressible flow as follows

$$v = -a\nabla p - (0, V),$$

where a takes two positive constant values

$$a_i = \frac{b^2}{12\mu_i}, \quad \text{for } i = 1, 2,$$

on each fluid, and V is a constant. With our notation, this case is equivalent to consider in the two-dimensional problem

$$\frac{\mu_i}{\kappa} = \frac{1}{a_i}, \qquad \text{and} \qquad \mathrm{g}\,\rho_i = \frac{V}{a_i}.$$

The results rely on the assumption that there is a jump of viscosities on the interface, say $\mu_1 \neq \mu_2$.

We shall study the fluid interface due to a jump of densities, hence $\mu_1 = \mu_2$. In order to simplify notation we can take $\kappa = b^2/12 = 1$ and $\mu_1 = \mu_2 = \mathrm{g} = 1$ without loss of generality. This case describes, among others, the dynamics of moist and dry regions in porous media. This scenario is treated by Dombre, Pumir and Siggia [8], but in a different context. They study the interface dynamics for convection in porous media where the density plays the role of the temperature. They analyze the unstable case, namely when the denser fluid (or the fluid with larger temperature) is above. They consider meromorphic initial conditions with complex poles and study the dynamics of these critical points.

It is well-known [12] that for these contour dynamics systems, the velocity in the tangential direction does not alter the shape of the interface. If we change the tangential component of the velocity, we only change the parametrization. In [7] we used this property to parameterize the interface as a function $(x, f(x,t))$, obtaining the following equations:

$$\begin{aligned} f_t(x,t) &= \frac{\rho_2-\rho_1}{4\pi} PV \int_{\mathbb{R}^2} \frac{(\nabla f(x,t)-\nabla f(x-y,t))\cdot y}{[|y|^2+(f(x,t)-f(x-y,t))^2]^{3/2}} dy, \\ f(x,0) &= f_0(x), \quad x\in\mathbb{R}^2 \end{aligned} \tag{1}$$

for a two-dimensional interface and

$$\begin{aligned} f_t(\alpha,t) &= \frac{\rho_2-\rho_1}{2\pi} PV \int_{\mathbb{R}} \frac{(\partial_x f(\alpha,t)-\partial_x f(\alpha-\beta,t))\beta}{\beta^2+(f(\alpha,t)-f(\alpha-\beta,t))^2} d\beta, \\ f(\alpha,0) &= f_0(\alpha), \quad \alpha\in\mathbb{R}, \end{aligned} \tag{2}$$

for a one-dimensional interface. We point out that with these formulations the arc-chord condition is satisfied locally in time if local-existence for the systems is reached. This avoids a kind of singularity in the fluid when the interface collapses (see [6] for example). We also proved that when the denser fluid is below the other fluid, $\rho_2 > \rho_1$, the problem is well-posed given local-existence and uniqueness for the systems (1) and (2). When the less dense fluid is below, $\rho_2 < \rho_1$, we prove ill-posedness showing that Eqs. (1) and (2) are ill-posed. We get this result using global solutions of (2) in the stable case, $\rho_2 > \rho_1$, for small initial data in a similar way as in [15].

If we neglect the terms of order two in (1), the linearized equation is obtained. It reads

$$\begin{aligned} f_t &= \frac{\rho_1-\rho_2}{2}(R_1\partial_{x_1} f + R_2\partial_{x_2} f) = \frac{\rho_1-\rho_2}{2}\Lambda f, \\ f(x,0) &= f_0(x), \end{aligned} \tag{3}$$

where R_1 and R_2 are the Riesz transforms (see [16]) and the operator Λf is defined by the Fourier transform $\widehat{\Lambda f}(\xi) = |\xi|\hat{f}(\xi)$. In the stable case $\rho_1 < \rho_2$ (the greater density is below), the linear equation is dissipative and it is clear that the following maximum principle is reached:

$$\|f\|_{L^\infty}(t) \le \|f_0\|_{L^\infty}.$$

We devote Sects. 3 and 4 to derive similar estimates for the nonlinear systems (1) and (2). To this end, we follow the evolution of the maximum of the absolute value of $f(x,t)$. This technique was used by one of the authors in [5] in a family of dissipative transport equations for incompressible fluids. Also, we would like to cite the work of A. Constantin and J. Echer where they study the shallow water equation in the same way. By a similar approach, in Sect. 5, we obtain a global bound on the derivative for small initial data.

2. Parameterizing the Interface in Terms of a Function

In this section we briefly explain how to parameterize the free boundary in terms of a function (see [7] for more details). The way of writing the nonlocal equation is crucial in order to check the evolution of the maximum of the absolute value of the function which yields the maximum principle.

In our case, Darcy's law can be written as follows:

$$v(x_1, x_2, x_3, t) = -\nabla p(x_1, x_2, x_3, t) - (0, 0, \rho(x_1, x_2, x_3, t)), \tag{4}$$

where $(x_1, x_2, x_3) \in \mathbb{R}^3$ are the spatial variables and $t \geq 0$ denotes the time. Here ρ is defined by

$$\rho(x_1, x_2, x_3, t) = \begin{cases} \rho_1 & \text{in } \Omega_1(t) \\ \rho_2 & \text{in } \Omega_2(t), \end{cases}$$

with $\rho_1, \rho_2 \geq 0$ constants and $\rho_1 \neq \rho_2$. The sets $\Omega_j(t)$ are defined by

$$\Omega_1(t) = \{x_3 > f(x_1, x_2, t)\}$$

and

$$\Omega_2(t) = \{x_3 < f(x_1, x_2, t)\},$$

$f(x_1, x_2, t)$ being the fluid interface. If we apply the curl operator to Darcy's law twice then the pressure disappears. Considering the incompressibility of the fluid, we have curl curl $v = -\Delta v$, and we can express the velocity in terms of the density as follows:

$$v = (\partial_{x_1} \Delta^{-1} \partial_{x_3} \rho, \partial_{x_2} \Delta^{-1} \partial_{x_3} \rho, -\partial_{x_1} \Delta^{-1} \partial_{x_1} \rho - \partial_{x_2} \Delta^{-1} \partial_{x_2} \rho). \tag{5}$$

The density ρ has a jump of discontinuity on the free boundary, therefore the gradient of the function is given by a Dirac distribution δ,

$$\nabla \rho = (\rho_2 - \rho_1)(\partial_{x_1} f(x_1, x_2, t), \partial_{x_2} f(x_1, x_2, t), -1)\delta(x_3 - f(x_1, x_2, t)). \tag{6}$$

Using the kernels for $\partial_{x_1} \Delta^{-1}$ and $\partial_{x_2} \Delta^{-1}$ we obtain

$$v(x_1, x_2, x_3, t) = -\frac{\rho_2 - \rho_1}{4\pi} PV \int_{\mathbb{R}^2} \frac{(y_1, y_2, \nabla f(x - y, t) \cdot y)}{[|y|^2 + (x_3 - f(x - y, t))^2]^{3/2}} dy, \tag{7}$$

where $x_3 \neq f(x, t)$, $x = (x_1, x_2)$ and $y = (y_1, y_2)$. The principal value is taken at infinity (see [16]). The vorticity is at the same level as the gradient of the density, so it is determined by a delta function. This forces the velocity to have a discontinuity on the free boundary. Just checking the incompressibility of the fluid in the sense of the distributions, we obtain that this discontinuity is in the tangential direction, i.e. it does not affect the shape of the interface (see [7]). Ignoring the tangential terms we obtain that the velocity on the free boundary is given by

$$v(x, f(x, t), t) = -\frac{\rho_2 - \rho_1}{4\pi} PV \int_{\mathbb{R}^2} \frac{(y_1, y_2, \nabla f(x - y, t) \cdot y)}{[|y|^2 + (f(x, t) - f(x - y, t))^2]^{3/2}} dy. \tag{8}$$

If we want to parameterize the free boundary in terms of a function, it is necessary that the velocity $v = (v_1, v_2, v_3)$ satisfies $v_1 = v_2 = 0$, since otherwise the points on the plane are not fixed and they would depend on time. If we add the following tangential terms to (8):

$$\frac{\rho_2-\rho_1}{4\pi} PV \int_{\mathbb{R}^2} \frac{y_1}{[|y|^2+(f(x,t)-f(x-y,t))^2]^{3/2}} dy(1,0,\partial_{x_1} f(x,t)),$$
$$\frac{\rho_2-\rho_1}{4\pi} PV \int_{\mathbb{R}^2} \frac{y_2}{[|y|^2+(f(x,t)-f(x-y,t))^2]^{3/2}} dy(0,1,\partial_{x_2} f(x,t)),$$

we do not alter the interface and we obtain

$$v(x,f(x,t),t) = \frac{\rho_2-\rho_1}{4\pi}(0,0,PV \int_{\mathbb{R}^2} \frac{(\nabla f(x,t)-\nabla f(x-y,t))\cdot y}{[|y|^2+(f(x,t)-f(x-y,t))^2]^{3/2}} dy). \quad (9)$$

Then the contour equation is given by

$$\begin{aligned} f_t(x,t) &= \frac{\rho_2-\rho_1}{4\pi} PV \int_{\mathbb{R}^2} \frac{(\nabla f(x,t)-\nabla f(x-y,t))\cdot y}{[|y|^2+(f(x,t)-f(x-y,t))^2]^{3/2}} dy, \\ f(x,0) &= f_0(x). \end{aligned} \quad (10)$$

This formula is well defined for a periodic interface and for a free boundary near planar at infinity. In both cases it presents a principal value only at infinity. If we suppose that the function $f(x,t)$ only depends on x_1, integrating in x_2, the contour equation in the 2-D case is

$$\begin{aligned} f_t(x,t) &= \frac{\rho_2-\rho_1}{2\pi} PV \int_{\mathbb{R}} \frac{(\partial_x f(x,t)-\partial_x f(x-\alpha,t))\alpha}{\alpha^2+(f(x,t)-f(x-\alpha,t))^2} d\alpha, \\ f(x,0) &= f_0(x); \quad x \in \mathbb{R}. \end{aligned} \quad (11)$$

We check in [7] that as long as this equation is satisfied we obtain weak solutions of the following system:

$$\begin{aligned} &\rho_t + v\cdot\nabla\rho = 0, \\ &v = -\nabla p - (0,0,\rho), \quad \operatorname{div} v = 0. \end{aligned} \quad (12)$$

3. Two Dimensional Case (1-D Interface)

Next we shall show that the L^∞ norm of the system (11) decreases in time in the stable case ($\rho_2 > \rho_1$). We shall consider the set Ω equal to $\mathbb{R}$ or $\mathbb{T}$. The following theorem is the main result of the section.

Theorem 3.1. *Let $f_0 \in H^k(\Omega)$ with $k \geq 3$ and $\rho_2 > \rho_1$. Then the unique solution to the system (11) satisfies the following inequality*:

$$\|f\|_{L^\infty}(t) \leq \|f_0\|_{L^\infty}.$$

Proof. For $f_0 \in H^k$ with $k \geq 3$, we prove in [7] that there exists a time $T > 0$ such that the unique solution $f(x,t)$ to (11) belongs to $C^1([0,T];H^k)$. In particular we have $f(x,t) \in C^1([0,T]\times\Omega)$, hence the Rademacher theorem shows that the functions

$$M(t) = \max_x f(x,t),$$

and

$$m(t) = \min_x f(x,t),$$

are differentiable at almost every t. In the non periodic case, we also notice that by the Riemann-Lebesgue lemma there always exists a point $x_t \in \mathbb{R}$ where

$$|f(x_t,t)| = \max_x |f(x,t)|,$$

since $f(\cdot,t) \in H^s$ with $s > 1/2$ implies that $f(x,t)$ tends to 0 when $|x| \to \infty$. First, we suppose that this point x_t satisfies that $0 < f(x_t,t) = M(t)$ (a similar argument can be used for $m(t) = f(x_t,t) < 0$). Let us consider a point in which $M(t)$ is differentiable, then we have

$$\begin{aligned} M'(t) &= \lim_{h\to 0^+} \frac{M(t+h) - M(t)}{h} \\ &= \lim_{h\to 0^+} \frac{f(x_{t+h}, t+h) - f(x_t,t)}{h} \\ &= \lim_{h\to 0^+} \frac{f(x_{t+h}, t+h) - f(x_t, t+h)}{h} + \frac{f(x_t, t+h) - f(x_t,t)}{h}. \end{aligned}$$

Since $f(x, t+h)$ takes its maximum value at $x = x_{t+h}$, it follows

$$M'(t) \geq \lim_{h\to 0^+} \frac{f(x_t, t+h) - f(x_t,t)}{h} = f_t(x_t,t).$$

Computing for $h > 0$,

$$\begin{aligned} M'(t) &= \lim_{h\to 0^+} \frac{M(t) - M(t-h)}{h} \\ &= \lim_{h\to 0^+} \frac{f(x_t,t) - f(x_{t-h}, t-h)}{h} \\ &= \lim_{h\to 0^+} \frac{f(x_t, t-h) - f(x_{t-h}, t-h)}{h} + \frac{f(x_t,t) - f(x_t, t-h)}{h} \\ &\leq \lim_{h\to 0^+} \frac{f(x_t,t) - f(x_t, t-h)}{h} \\ &\leq f_t(x_t,t), \end{aligned}$$

and we obtain finally

$$M'(t) = f_t(x_t,t). \tag{13}$$

If we take the value $x = x_t$ in Eq. (11), the above identity yields

$$M'(t) = -\frac{\rho_2 - \rho_1}{2\pi} PV \int_{\mathbb{R}} \frac{\partial_x f(x_t - \alpha, t)\alpha}{\alpha^2 + ((f(x_t,t) - f(x_t - \alpha, t)))^2} d\alpha,$$

using the fact that $\partial_x f(x_t,t) = 0$. Integrating by parts

$$\begin{aligned} M'(t) &= -\frac{\rho_2 - \rho_1}{2\pi} PV \int_{\mathbb{R}} \frac{\partial_\alpha (f(x_t,t) - f(x_t - \alpha, t))}{\alpha} \frac{1}{1 + \left(\frac{f(x_t,t) - f(x_t - \alpha, t)}{\alpha}\right)^2} d\alpha \\ &= I_1 + I_2, \end{aligned}$$

where

$$I_1 = -\frac{\rho_2-\rho_1}{2\pi} PV \int_{\mathbb{R}} \frac{f(x_t,t)-f(x_t-\alpha,t)}{\alpha^2} \frac{1}{1+\left(\frac{f(x_t,t)-f(x_t-\alpha,t)}{\alpha}\right)^2} d\alpha,$$

and

$$I_2 = -\frac{\rho_2-\rho_1}{2\pi} \int_{\mathbb{R}} 2 \frac{\left(\frac{f(x_t,t)-f(x_t-\alpha,t)}{\alpha}\right)^2}{\left(1+\left(\frac{f(x_t,t)-f(x_t-\alpha,t)}{\alpha}\right)^2\right)^2} \partial_\alpha \left(\frac{f(x_t,t)-f(x_t-\alpha,t)}{\alpha}\right) d\alpha.$$

Using the function

$$G(x) = -\frac{x}{1+x^2} + \arctan x,$$

we can write I_2 as follows:

$$I_2 = -\frac{\rho_2-\rho_1}{2\pi} PV \int_{\mathbb{R}} \partial_\alpha\, G\left(\frac{f(x_t,t)-f(x_t-\alpha,t)}{\alpha}\right) d\alpha.$$

Integrating we obtain

$$\begin{aligned} I_2 = -\frac{\rho_2-\rho_1}{2\pi} [G\left(\lim_{\alpha\to+\infty} \frac{f(x_t,t)-f(x_t-\alpha,t)}{\alpha}\right) \\ -G\left(\lim_{\alpha\to-\infty} \frac{f(x_t,t)-f(x_t-\alpha,t)}{\alpha}\right)] = 0. \end{aligned}$$

The I_1 term is equal to

$$I_1 = -\frac{\rho_2-\rho_1}{2\pi} PV \int_{\mathbb{R}} \frac{M(t)-f(x_t-\alpha,t)}{\alpha^2+(M(t)-f(x_t-\alpha,t))^2} d\alpha \le 0,$$

so that $M'(t) \le 0$ for almost every t. In a similar way we obtain for $m(t)$ the following inequality

$$m'(t) = -\frac{\rho_2-\rho_1}{2\pi} PV \int_{\mathbb{R}} \frac{m(t)-f(x_t-\alpha,t)}{\alpha^2+(m(t)-f(x_t-\alpha,t))^2} d\alpha \ge 0,$$

for almost every t. Integrating in time we conclude the argument and obtain the maximum principle.

In the periodic case, $\Omega = \mathbb{T}$, the maximum principle leads to the following decay estimates of the L^∞ norm. □

Proposition 3.2. *Let $f_0 \in H^k(\mathbb{T})$ with $k \geq 3$ and $\rho_2 > \rho_1$. If*

$$\int_{\mathbb{T}} f_0(x)dx = 0,$$

then the unique solution to the system (11) satisfies the following inequality

$$\|f\|_{L^\infty}(t) \leq \|f_0\|_{L^\infty} e^{-(\rho_2-\rho_1)C(\|f_0\|_{L^\infty})t},$$

with $C(\|f_0\|_{L^\infty}) > 0$.

Proof. Suppose that

$$\int_{\mathbb{T}} f_0(x)dx = 0.$$

We can write (11) as follows:

$$f_t(x,t) = \frac{\rho_2-\rho_1}{2\pi} PV \int_{\mathbb{R}} \partial_x \arctan\left(\frac{f(x,t)-f(x-\alpha,t)}{\alpha}\right) d\alpha,$$

and therefore we have

$$\begin{aligned}\int_{\mathbb{T}} f_t(x,t)dx &= \frac{\rho_2-\rho_1}{2\pi} \int_{\mathbb{T}} PV \int_{\mathbb{R}} \partial_x \arctan\left(\frac{f(x,t)-f(x-\alpha,t)}{\alpha}\right) d\alpha dx\\ &= \frac{\rho_2-\rho_1}{2\pi} PV \int_{\mathbb{R}} \int_{\mathbb{T}} \partial_x \arctan\left(\frac{f(x,t)-f(x-\alpha,t)}{\alpha}\right) dx d\alpha\\ &= 0.\end{aligned}$$

Integrating in time we obtain

$$\int_{\mathbb{T}} f(x,t)dx = 0, \quad \forall t \geq 0. \tag{14}$$

As we showed in the proof of the previous theorem, we have

$$\frac{d}{dt}\|f\|_{L^\infty}(t) = -\frac{\rho_2-\rho_1}{2\pi} PV \int_{\mathbb{R}} \frac{\|f\|_{L^\infty}(t) - f(x_t-\alpha,t)}{\alpha^2 + (\|f\|_{L^\infty}(t) - f(x_t-\alpha,t))^2} d\alpha,$$

for almost every t. Applying the maximum principle, for $|\alpha| \leq r$ we get

$$\alpha^2 + (\|f\|_{L^\infty}(t) - f(x_t-\alpha,t))^2 \leq r^2 + 4\|f_0\|^2_{L^\infty},$$

and

$$\begin{aligned}\frac{d}{dt}\|f\|_{L^\infty}(t) &\leq -\frac{\rho_2-\rho_1}{2\pi} PV \int_{|\alpha|\leq r} \frac{\|f\|_{L^\infty}(t) - f(x_t-\alpha,t)}{\alpha^2 + (\|f\|_{L^\infty}(t) - f(x_t-\alpha,t))^2} d\alpha\\ &\leq -\frac{\rho_2-\rho_1}{2\pi} \frac{2r}{r^2+4\|f_0\|^2_{L^\infty}} \|f\|_{L^\infty}(t)\\ &\quad + \frac{\rho_2-\rho_1}{2\pi} \frac{1}{r^2+4\|f_0\|^2_{L^\infty}} \int_{|\alpha|\leq r} f(x_t-\alpha)d\alpha.\end{aligned}$$

If we take $r = n\pi$ for $n \in \mathbb{N}$, from (14) we obtain

$$\frac{d}{dt}\|f\|_{L^\infty}(t) \leq -\frac{\rho_2-\rho_1}{2\pi}\frac{2n\pi}{n^2\pi^2+4\|f_0\|_{L^\infty}^2}\|f\|_{L^\infty}(t),$$

and integrating in time we conclude the proof. □

For $\Omega = \mathbb{R}$ we obtain the following result.

Proposition 3.3. *Let $f_0 \in H^k(\mathbb{R})$ with $k \geq 3$ and $\rho_2 > \rho_1$. If $f_0(x) \leq 0$ or $f_0(x) \geq 0$, then the unique solution to the system (11) satisfies the following inequality*

$$\|f\|_{L^\infty}(t) \leq \frac{\|f_0\|_{L^\infty}}{1+(\rho_2-\rho_1)C(\|f_0\|_{L^\infty}, \|f_0\|_{L^1})t},$$

with $C(\|f_0\|_{L^\infty}, \|f_0\|_{L^1}) > 0$.

Proof. Let us consider $f_0(x) \geq 0$ (the argument is similar to $f_0(x) \leq 0$). Our maximum principle shows that

$$m'(t) = -\frac{\rho_2-\rho_1}{2\pi}PV\int_{\mathbb{R}}\frac{m(t)-f(x_t-\alpha,t)}{\alpha^2+(m(t)-f(x_t-\alpha,t))^2}d\alpha \geq 0,$$

for almost every t. Hence, if $f_0(x) \geq 0$, then $f(x,t) \geq 0$. In a similar way as in the previous result, we have

$$\int_{\mathbb{R}} f_t(x,t)dx = 0,$$

and therefore

$$\int_{\mathbb{R}} f(x,t)dx = \int_{\mathbb{R}} f_0(x)dx.$$

Since f is nonnegative, we control the L^1 norm of the solution, hence $\|f\|_{L^1}(t) = \|f_0\|_{L^1}$. We have $\|f\|_{L^\infty}(t) = f(x_t,t)$, and

$$\frac{d}{dt}\|f\|_{L^\infty}(t) = -I,$$

with

$$I = \frac{\rho_2-\rho_1}{2\pi}PV\int_{\mathbb{R}}\frac{f(x_t,t)-f(x_t-\alpha,t)}{\alpha^2+(f(x_t,t)-f(x_t-\alpha,t))^2}d\alpha,$$

for almost every t. If we consider the interval $[-r,r]$ for $r > 0$,

$$U_1 = \{\alpha \in [-r,r] : f(x_t,t)-f(x_t-\alpha,t) \geq f(x_t,t)/2\},$$

and

$$U_2 = \{\alpha \in [-r,r] : f(x_t,t)-f(x_t-\alpha,t) < f(x_t,t)/2\},$$

we get

$$I \geq \frac{\rho_2-\rho_1}{2\pi}PV\int_{U_1}\frac{f(x_t,t)-f(x_t-\alpha,t)}{\alpha^2+(f(x_t,t)-f(x_t-\alpha,t))^2}d\alpha \geq \frac{\rho_2-\rho_1}{2\pi}\frac{f(x_t,t)/2}{r^2+4\|f_0\|_{L^\infty}^2}|U_1|.$$

In order to estimate $|U_1|$, we use that $|U_1| = 2r - |U_2|$, and

$$\|f_0\|_{L^1} = \int_{\mathbb{R}} f(x_t - \alpha, t)d\alpha \geq \int_{U_2} f(x_t - \alpha, t)d\alpha \geq \frac{f(x_t, t)}{2}|U_2|,$$

which implies the lower bound $|U_1| \geq 2(r - \|f_0\|_{L^1}/f(x_t, t))$. This estimate yields

$$I \geq \frac{\rho_2 - \rho_1}{2\pi}\frac{f(x_t, t)/2}{r^2 + 4\|f_0\|^2_{L^\infty}}|U_1| \geq \frac{\rho_2 - \rho_1}{2\pi}\frac{rf(x_t, t) - \|f_0\|_{L^1}}{r^2 + 4\|f_0\|^2_{L^\infty}},$$

and this function reaches its maximum at

$$r = \left(\|f_0\|_{L^1} + \sqrt{\|f_0\|^2_{L^1} + 4\|f_0\|^2_{L^\infty} f^2(x_t, t)}\right)/f(x_t, t).$$

Using the maximum principle

$$\begin{aligned} I &\geq \frac{\rho_2 - \rho_1}{8\pi}\frac{\|f_0\|_{L^1} f^2(x_t, t)}{\|f_0\|^2_{L^1} + 2\|f_0\|_{L^1}\|f_0\|^2_{L^\infty} + 2\|f_0\|^4_{L^\infty}} \\ &\geq (\rho_2 - \rho_1)C(\|f_0\|_{L^1}, \|f_0\|_{L^\infty})f^2(x_t, t). \end{aligned}$$

Finally, we obtain

$$\frac{d}{dt}\|f\|_{L^\infty}(t) \leq -(\rho_2 - \rho_1)C(\|f_0\|_{L^1}, \|f_0\|_{L^\infty})\|f\|^2_{L^\infty}(t),$$

which ends the proof. □

4. Three Dimensional Case (2-D Interface)

In this section, by applying the same technique, we extend the maximum principle for the three dimensional stable case. We consider the set Ω to be the plane or the periodic setting.

Theorem 4.1. *Let $f_0 \in H^k(\Omega)$ for $k \geq 4$, and $\rho_2 > \rho_1$. Then the unique solution to* (10) *satisfies that*

$$\|f\|_{L^\infty}(t) \leq \|f_0\|_{L^\infty}.$$

Proof. From [7] we know that there exists a time $T > 0$ and a unique solution $f(x, t) \in C^1([0, T]; H^k(\Omega))$ of (10). In the case $\Omega = \mathbb{R}^2$, there always exists a point $x_t \in \mathbb{R}^2$ where $|f(x, t)|$ reaches its maximum due to the fact that $f(\cdot, t) \in H^s$ with $s > 1$. Suppose that this point is for $M(t) = f(x_t, t) > 0$. A similar argument can be used for $m(t) = f(x_t, t) < 0$. By the Rademacher theorem, the function $M(t)$ is differentiable almost everywhere and by a similar argument as before we obtain

$$M'(t) = f_t(x_t, t), \tag{15}$$

for almost every t. Using Eq. (10) and the fact that $\nabla f(x_t, t) = 0$, we have

$$M'(t) = \frac{\rho_2 - \rho_1}{4\pi} PV \int_{\mathbb{R}^2} \frac{-\nabla f(y, t) \cdot (x_t - y)}{[|x_t - y|^2 + (f(x_t, t) - f(y, t))^2]^{3/2}} dy.$$

Integrating by parts

$$
\begin{aligned}
M'(t)&\\
&= \frac{\rho_2-\rho_1}{4\pi} PV\int_{\mathbb{R}^2} \nabla_y (f(x_t,t)-f(y,t)) \cdot \frac{x_t-y}{|x_t-y|^3}\left(1+\left(\frac{f(x_t,t)-f(y,t)}{|x_t-y|}\right)^2\right)^{-3/2} dy\\
&= -\frac{\rho_2-\rho_1}{4\pi} PV\int_{\mathbb{R}^2} (f(x_t,t)-f(y,t)) \left(\mathrm{div}_y \frac{x_t-y}{|x_t-y|^3}\right)\left(1+\left(\frac{f(x_t,t)-f(y,t)}{|x_t-y|}\right)^2\right)^{-3/2} dy\\
&\quad -\frac{\rho_2-\rho_1}{4\pi} PV\int_{\mathbb{R}^2} \frac{f(x_t,t)-f(y,t)}{|x_t-y|}\frac{x_t-y}{|x_t-y|^2} \cdot \nabla_y \left(1+\left(\frac{f(x_t,t)-f(y,t)}{|x_t-y|}\right)^2\right)^{-3/2} dy\\
&= J_1+J_2.
\end{aligned}
$$

We have

$$
J_2 = -\frac{\rho_2-\rho_1}{4\pi} PV\int_{\mathbb{R}^2} \nabla_y (\ln|x_t-y|)\cdot \nabla_y H\left(\frac{f(x_t,t)-f(y,t)}{|x_t-y|}\right) dy,
$$

where

$$
H(x) = \frac{x^3}{(1+x^2)^{3/2}}.
$$

The identity $\Delta_y(\ln|x_t-y|)/4\pi = \delta(x_t)$ and the following limit

$$
\lim_{y\to x_t} \frac{f(x_t,t)-f(y,t)}{|x_t-y|} = \lim_{y\to x_t} \frac{f(x_t,t)-f(y,t)-\nabla f(x_t,t)\cdot(x_t-y)}{|x_t-y|} = 0,
$$

show that

$$
J_2 = \frac{\rho_2-\rho_1}{4\pi} PV\int_{\mathbb{R}^2} \Delta_y (\ln|x_t-y|) H\left(\frac{f(x_t,t)-f(y,t)}{|x_t-y|}\right) dy = (\rho_2-\rho_1)H(0),
$$

and consequently $J_2 = 0$. The J_1 term is equal to

$$
J_1 = -\frac{\rho_2-\rho_1}{4\pi} PV\int_{\mathbb{R}^2} \frac{M(t)-f(y,t)}{[|x_t-y|^2+(M(t)-f(y,t))^2]^{3/2}} dy \le 0,
$$

which implies that $M'(t)\le 0$ for almost every t. For $m(t)$ we have $m'(t)\ge 0$. □

As in the previous section, using this maximum principle we get the following decay of the L^∞ norm.

Proposition 4.2. *Let $f_0 \in H^k(\mathbb{T}^2)$ with $k\ge 4$ and $\rho_2 > \rho_1$. If*

$$
\int_{\mathbb{T}^2} f_0(x)dx = 0,
$$

then the unique solution to the system (11) satisfies the following inequality

$$
\|f\|_{L^\infty}(t) \le \|f_0\|_{L^\infty} e^{-(\rho_2-\rho_1)C(\|f_0\|_{L^\infty})t},
$$

with $C(\|f_0\|_{L^\infty})>0$

Proof. We can write (10) as follows:

$$f_t(x,t) = \frac{\rho_2-\rho_1}{4\pi} PV\int_{\mathbb{R}^2} \frac{y}{|y|^2}\cdot\nabla_x P\left(\frac{f(x)-f(x-y)}{|y|}\right)dy,$$
$$f(x,0) = f_0(x),$$

with

$$P(x) = \frac{x}{\sqrt{1+x^2}}.$$

Checking the evolution of the integral of f on $\mathbb{T}^2$, we obtain

$$\int_{\mathbb{T}^2} f(x,t)dx = 0. \tag{16}$$

The proof of the previous theorem shows that

$$\frac{d}{dt}\|f\|_{L^\infty}(t) = -\frac{\rho_2-\rho_1}{4\pi} PV\int_{\mathbb{R}^2} \frac{\|f\|_{L^\infty}(t)-f(y,t)}{[|x_t-y|^2+(\|f\|_{L^\infty}(t)-f(y,t))^2]^{3/2}}dy,$$

for almost every t. If we consider $x_t - y\in[-n\pi,n\pi]\times[-n\pi,n\pi] = A_n$, with $n\in\mathbb{N}$, we have

$$|x_t-y|^2+(\|f\|_{L^\infty}(t)-f(x_t-\alpha,t))^2 \leq 2(n\pi)^2+4\|f_0\|^2_{L^\infty}.$$

Using (16), the above inequality gives

$$\begin{aligned}\frac{d}{dt}\|f\|_{L^\infty}(t) &\leq -\frac{\rho_2-\rho_1}{4\pi} PV\int_{(x_t-y)\in A_n} \frac{\|f\|_{L^\infty}(t)-f(y,t)}{[|x_t-y|^2+(\|f\|_{L^\infty}(t)-f(y,t))^2]^{3/2}}dy\\ &\leq -\frac{\rho_2-\rho_1}{4\pi}\frac{(2n\pi)^2}{[2(n\pi)^2+4\|f_0\|^2_{L^\infty}]^{3/2}}\|f\|_{L^\infty}(t),\end{aligned}$$

and the desired estimate follows. □

Proposition 4.3. *Let $f_0\in H^k(\mathbb{R}^2)$ with $k\geq 4$ and $\rho_2>\rho_1$. If $f_0(x)\leq 0$ or $f_0(x)\geq 0$, then the unique solution to the system* (11) *satisfies the following inequality:*

$$\|f\|_{L^\infty}(t) \leq \frac{\|f_0\|_{L^\infty}}{(1+(\rho_2-\rho_1)C(\|f_0\|_{L^\infty},\|f_0\|_{L^1})t)^2},$$

with $C(\|f_0\|_{L^\infty},\|f_0\|_{L^1})>0$.

Proof. Let us consider $f_0(x)\geq 0$, the same estimate is obtained for $f_0(x)\leq 0$. We know that $f(x,t)\geq 0$ and $\|f\|_{L^1}(t)=\|f_0\|_{L^1}$. We have $\|f\|_{L^\infty}(t) = f(x_t,t)$ and

$$\frac{d}{dt}\|f\|_{L^\infty}(t) = -J$$

for almost every t, with

$$J = \frac{\rho_2-\rho_1}{4\pi} PV\int_{\mathbb{R}^2} \frac{f(x_t,t)-f(y,t)}{[|x_t-y|^2+(f(x_t,t)-f(y,t))^2]^{3/2}}dy.$$

If we define the set $B_r(x_t) = \{y : |x_t - y| \le r\}$ for $r > 0$,

$$V_1 = \{y \in B_r(x_t) : f(x_t, t) - f(y, t) \ge f(x_t, t)/2\},$$

and

$$V_2 = \{y \in B_r(x_t) : f(x_t, t) - f(y, t) < f(x_t, t)/2\},$$

we get

$$J \ge \frac{\rho_2 - \rho_1}{4\pi} \frac{f(x_t, t)/2}{[r^2 + 4\|f_0\|^2_{L^\infty}]^{3/2}} |V_1|.$$

Using that $|V_1| = \pi r^2 - |V_2|$ and

$$\|f_0\|_{L^1} \ge \int_{V_2} f(y, t)dy \ge \frac{f(x_t, t)}{2} |V_2|,$$

we can estimate from below $|V_1| \ge \pi r^2 - 2\|f_0\|_{L^1}/f(x_t, t)$. Then

$$J \ge \frac{\rho_2 - \rho_1}{8\pi} \frac{\pi r^2 f(x_t, t) - 2\|f_0\|_{L^1}}{[r^2 + 4\|f_0\|^2_{L^\infty}]^{3/2}}.$$

Taking

$$r = \left(\frac{2\|f_0\|_{L^1}/\pi + 1}{f(x_t, t)}\right)^{1/2},$$

we find

$$\begin{aligned} J &\ge \frac{\rho_2 - \rho_1}{8\pi} \frac{\pi (f(x_t, t))^{3/2}}{[1 + 2\|f_0\|_{L^1}/\pi + 4\|f_0\|^2_{L^\infty} f(x_t, t)]^{3/2}} \\ &\ge \frac{\rho_2 - \rho_1}{8} \frac{(f(x_t, t))^{3/2}}{[1 + 2\|f_0\|_{L^1}/\pi + 4\|f_0\|^3_{L^\infty}]^{3/2}}. \end{aligned}$$

Finally, the following estimate is obtained

$$\frac{d}{dt}\|f\|_{L^\infty}(t) \le -(\rho_2 - \rho_1) C(\|f_0\|_{L^1}, \|f_0\|_{L^\infty}) \|f\|^{3/2}_{L^\infty}(t).$$

□

5. Small Initial Data

In the two-dimensional case, we prove in [7] that if the following quantity of the initial data is small:

$$\sum |\xi||\hat{f}(\xi)|,$$

then there is global-in-time solution of the system (11). The aim of this section is to show that if initially the L^∞ norm of the first derivative is less than one then it continues less than one for all time.

Lemma 5.1. *Let $f_0 \in H^s$ with $s \geq 3$, and $\|\partial_x f_0\|_{L^\infty} < 1$. Then the unique solution of the system* (11) *satisfies*

$$\|\partial_x f\|_{L^\infty}(t) < 1.$$

Proof. We consider the following term in (11):

$$K = -\frac{\rho_2 - \rho_1}{2\pi} PV \int_{\mathbb{R}} \frac{\partial_x f(x-\alpha, t)\alpha}{\alpha^2 + (f(x,t) - f(x-\alpha,t))^2} d\alpha,$$

we can integrate by parts and get

$$\begin{aligned}
K &= -\frac{\rho_2 - \rho_1}{2\pi} PV \int_{\mathbb{R}} \frac{\partial_\alpha (f(x,t) - f(x-\alpha,t))}{\alpha} \frac{1}{1 + \left(\frac{f(x,t) - f(x-\alpha,t)}{\alpha}\right)^2} d\alpha \\
&= -\frac{\rho_2 - \rho_1}{2\pi} PV \int_{\mathbb{R}} \frac{f(x,t) - f(x-\alpha,t)}{\alpha^2} \frac{1}{1 + \left(\frac{f(x,t) - f(x-\alpha,t)}{\alpha}\right)^2} d\alpha \\
&\quad -\frac{\rho_2 - \rho_1}{2\pi} \int_{\mathbb{R}} 2 \frac{\left(\frac{f(x,t) - f(x-\alpha,t)}{\alpha}\right)^2}{\left(1 + \left(\frac{f(x,t) - f(x-\alpha,t)}{\alpha}\right)^2\right)^2} \partial_\alpha \left(\frac{f(x,t) - f(x-\alpha,t)}{\alpha}\right) d\alpha \\
&= L_1 + L_2.
\end{aligned}$$

As we showed before

$$L_2 = -\frac{\rho_2 - \rho_1}{2\pi} PV \int_{\mathbb{R}} \partial_\alpha G\left(\frac{f(x,t) - f(x-\alpha,t)}{\alpha}\right) d\alpha = 0,$$

so $K = L_1$. Making a change of variables we find the following equivalent system:

$$f_t(x,t) = \frac{\rho_2 - \rho_1}{2\pi} PV \int_{\mathbb{R}} \frac{\partial_x f(x,t)(x-\alpha) - (f(x,t) - f(\alpha,t))}{(x-\alpha)^2 + (f(x,t) - f(\alpha,t))^2} d\alpha.$$

Taking one derivative in this formula, we have

$$\partial_x f_t(x) = N_1(x) + N_2(x), \tag{17}$$

with

$$N_1(x) = \frac{\rho_2 - \rho_1}{2\pi} PV\int_{\mathbb{R}} \frac{\partial_x^2 f(x)(x-\alpha)}{(x-\alpha)^2+(f(x)-f(\alpha))^2} d\alpha,$$

$$N_2(x) = -\frac{\rho_2 - \rho_1}{2\pi} PV\int_{\mathbb{R}} \frac{\partial_x f(x) - \triangle_\alpha f(x)}{(x-\alpha)^2} Q(x,\alpha) d\alpha,$$

where

$$Q(x,\alpha) = 2\,\frac{1+\partial_x f(x)\triangle_\alpha f(x)}{(1+(\triangle_\alpha f(x))^2)^2},$$

and

$$\triangle_\alpha f(x) = \frac{f(x)-f(\alpha)}{x-\alpha}.$$

Next, we set

$$M(t) = \|\partial_x f\|_{L^\infty}(t),$$

then $M(t) = \max_x \partial_x f(x,t) = \partial_x f(x_t,t)$, where x_t is the trajectory of the maximum. Similar conclusions are obtained for $m(t) = \min_x \partial_x f(x,t)$. Using the Rademacher theorem as in the previous section, we have that $M'(t) = \partial_x f_t(x_t,t)$ and $\partial_x^2 f(x_t,t) = 0$. Therefore by taking $x = x_t$ in (17) we get

$$M'(t) = N_2(x_t),$$

since $N_1(x_t) = 0$. The inequality

$$|\triangle_\alpha f(x_t)| \leq M(t),$$

shows that for $M(t) < 1$ the integral $N_2(x_t) \leq 0$, and therefore $M'(t) \leq 0$. If $M(0) < 1$, using the theorem of local existence, we have that for short time $M(t) < 1$ which implies $M'(t) \leq 0$ for almost every t. Consequently we obtain $M(t) < 1$. In the case of $m(t)$ we find $m(t) > -1$. □

Acknowledgement. The authors were partially supported by the grant MTM2005- 05980 of the MEC (Spain) and the grant PAC-05-005-2 of the JCLM (Spain).

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Communicated by P. Constantin