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Nonlinear Dynamical Stability of Newtonian Rotating and Non-rotating White Dwarfs and Rotating Supermassive Stars

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Abstract: We prove general nonlinear stability and existence theorems for rotating star solutions which are axi-symmetric steady- state solutions of the compressible isentropic Euler-Poisson equations in 3 spatial dimensions. We apply our results to rotating white dwarf and high density supermassive (extreme relativistic) stars, stars which are in convective equilibrium and have uniform chemical composition. Also, we prove nonlinear dynamical stability of non-rotating white dwarfs with general perturbation without any symmetry restrictions. This paper is a continuation of our earlier work ([26]).

Contents

1. Introduction

The motion of a compressible isentropic perfect fluid with self-gravitation is modeled by the Euler-Poisson equations in three space dimensions (cf. [5]):

$$\begin{cases} \rho_t + \nabla\cdot(\rho\mathbf{v}) = 0, \\ (\rho\mathbf{v})_t + \nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v}) + \nabla p(\rho) = -\rho\nabla\Phi, \\ \Delta\Phi = 4\pi\rho. \end{cases} \tag{1.1}$$

Here ρ, $\mathbf{v} = (v_1, v_2, v_3)$, $p(\rho)$ and Φ denote the density, velocity, pressure and gravitational potential, respectively. The gravitational potential is given by

$$\Phi(x) = -\int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|} dy = -\rho * \frac{1}{|x|}, \tag{1.2}$$

where $*$ denotes convolution. System (1.1) is used to model the evolution of a Newtonian gaseous star ([5]). In the study of time-independent solutions of system (1.1), there are two cases, non-rotating stars and rotating stars. An important question concerns the stability of such solutions. Physicists call such star solutions stable provided that they are minima of an associated energy functional ([37], p.305 & [33]). Mathematicians, on the other hand, consider dynamical nonlinear stability via solutions of the Cauchy problem. The main purpose of this paper is to prove a general theorem which relates these two notions and shows that for a wide class of Newtonian rotating stars, minima of the energy functional are in fact, *dynamically* stable. This is done for various equations of state $p = p(\rho)$ which includes polytropes, supermassive, and white dwarf stars.

For non-rotating stars, Rein ([32]) has proved nonlinear stability under various hypotheses on the equation of state, including in particular, polytropes where $p = k\rho^{\gamma}$, $\gamma > 4/3$; his theory applies to neither white dwarf nor supermassive stars. In a recent paper, [26], we studied nonlinear stability of *rotating* polytropic stars, where $p = k\rho^{\gamma}$, $\gamma > 4/3$. In this paper, we generalize these results to rotating white dwarf and supermassive stars, thereby completing the nonlinear stability theory for rotating (and non-rotating) compressible Newtonian stars.[1]

Our main theorem applies to minimizers of an energy functional with a total mass constraint. The crucial hypotheses are that the infimum of the energy functional in the requisite class, be finite and negative. This is verified for both white dwarf and supermassive stars by combining a scaling technique used by Rein ([31]), together with our method in [26] where we use some particular solutions of the Euler-Poisson equations in order to simplify the energy functional. It should be noticed that neither the scaling technique in [31] nor the method in [26] using particular solutions of Euler-Poisson equations apply to white dwarf stars directly. As a by-product of our method, we prove the existence of a minimizer for the energy functional, which is a rotating white dwarf star solution, in a class of functions having less symmetry than those solutions obtained in [1] and [10]. The method in [1] and [10] is to construct a specific minimizing sequence of the energy functional, each element in the sequence being a local minimizer of the energy functional. In contrast, our method is to show that *any* minimizing sequence of the energy functional must be compact (cf. Theorem 3.1 below). This fact is crucial for both existence and stability results.

For a white dwarf star (a star in which gravity is balanced by electron degeneracy pressure), the pressure function $p(\rho)$ obeys the following asymptotics ([5], Chap. 10):

$$\begin{cases} p(\rho) = c_1\rho^{4/3} - c_2\rho^{2/3} + \cdots, & \rho \to \infty, \\ p(\rho) = d_1\rho^{5/3} - d_2\rho^{7/3} + O(\rho^3), & \rho \to 0, \end{cases} \tag{1.3}$$

[1] In all cases under consideration, stability is only "conditional" because no global in time solutions have been constructed so far for compressible Euler-type equations in three spatial dimensions; this is a major open problem. In the stability result in [32], it was assumed that the solutions of the Cauchy problem for the evolutionary Euler-Poisson equations exist and preserve the total mass and energy. In general, shock waves appear in compressible fluid flows. In the presence of shock waves, the total energy should be non-increasing in time due to the entropy condition. We prove the conservation of total mass for general weak solutions and the non-increase of the total energy for entropy weak solutions if the weak solutions are in certain L^p spaces (see Theorem 3.2). Those two properties are important for our stability analysis.

where c_1, c_2, d_1 and d_2 are positive constants. The existence theory for non-rotating white dwarf stars is classical provided the mass M of the star is not greater than a critical mass M_c ($M \le M_c$) ([5]). For rotating white dwarf stars with prescribed total mass and angular momentum distribution, Auchumuty and Beals ([1]) proved that if the angular momentum distribution is nonnegative, then existence holds if $M \le M_c$. Friedman and Turkington ([10]) proved existence for any mass provided that the angular momentum distribution is everywhere positive; see Li ([21]), Chanillo & Li ([6]) and Luo & Smoller ([25]) for related results for rotating star solutions with prescribed constant angular velocity. To the best of our knowledge, our stability theorem in this paper for rotating and non-rotating white dwarf stars with $M \le M_c$ is the first nonlinear dynamical stability theorem for such stars.

For a supermassive star (a star which is supported by the pressure of radiation rather than that of matter; sometimes called an extreme relativistic degenerate star [33]), the pressure $p(\rho)$ is given by ([37]):

$$p(\rho) = k\rho^{\gamma},\ \gamma = 4/3, \tag{1.4}$$

where $k > 0$ is a constant. For non-rotating spherically symmetric solutions for supermassive stars, Weinberg ([37]) showed that the total energy vanishes; thus to quote Weinberg ([37], p. 327) "the polytrope with $\gamma = 4/3$ is trembling between stability and instability", and he remarks that one needs to use general relativity to settle this stability problem. For rotating supermassive star solutions, we show here that the energy is negative $E < 0$ due to the rotational kinetic energy (see (4.26) below). Thus the stability problem falls within the framework of Newtonian mechanics and so our general stability theorem applies to show that *rotating* supermassive stars are nonlinearly stable, provided that $M \le M_c$.

For the stability of both white dwarfs and supermassive stars, we require that the total mass of each one lies below a corresponding critical mass, a "Chandrasekhar" limit. We show that this holds because the pressure function for both is of the order $\rho^{4/3}$ as $\rho \to \infty$.

The above dynamical stability results for rotating stars apply for axi-symmetric perturbations with some restrictions on angular momentum. For non-rotating stars, G. Rein ([32]) proved nonlinear dynamical stability for general perturbations. However, his result does not apply to white dwarf stars. For non-rotating white dwarf stars, the problem was formulated by Chandrasekhar [4] in 1931 (and also in [8] and [16]) and leads to an equation for the density which was called the " Chandrasekhar equation " by Lieb and Yau in [22]. This equation predicts the gravitational collapse at some critical mass ([4] and [5]). This gravitational collapse was also verified by Lieb and Yau ([22]) as the limit of Quantum Mechanics. In Sect. 5, we prove the nonlinear dynamical stability for non-rotating white dwarf stars with general perturbations without any symmetry assumption provided that the total mass is below some critical mass.

Other related results besides those mentioned above for compressible fluid rotating stars can be found in [2, 3, 9, and 25].

The linearized stability and instability for non-rotating and rotating stars were discussed by Lin ([23]), Lebovitz ([18]) and Lebovitz & Lifschitz ([19]). Related nonlinear stability and instability results for galaxies, globular and gaseous stellar objects can be found in Guo & Rein ([12,13]) and Jang ([11]). Related results for the Euler- Poisson equations of self-gravitating fluids can be found in [7, 15, 28 and 36].

2. Rotating Star Solutions

We now introduce some notation which will be used throughout this paper. We use $\int$ to denote $\int_{\mathbb{R}^3}$, and use $||\cdot||_q$ to denote $||\cdot||_{L^q(\mathbb{R}^3)}$. For any point $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, let

$$r(x) = \sqrt{x_1^2 + x_2^2},\ z(x) = x_3,\ B_R(x) = \{y \in \mathbb{R}^3,\ |y - x| < R\}. \tag{2.1}$$

For any function $f \in L^1(\mathbb{R}^3)$, we define the operator B by

$$Bf(x) = \int \frac{f(y)}{|x-y|}dy = f * \frac{1}{|x|}. \tag{2.2}$$

Also, we use ∇ to denote the spatial gradient, i.e., $\nabla = \nabla_x = (\partial_{x_1},\ \partial_{x_2},\ \partial_{x_3})$. C will denote a generic positive constant.

A rotating star solution $(\tilde{\rho}, \tilde{\mathbf{v}}, \tilde{\Phi})(r, z)$, where $r = \sqrt{x_1^2 + x_2^2}$ and $z = x_3$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, is an *axi-symmetric* time-independent solution of system (1.1), which models a star rotating about the x_3- axis. Suppose the angular momentum (per unit mass), $J(m_{\tilde{\rho}}(r))$ is prescribed, where

$$m_{\tilde{\rho}}(r) = \int_{\sqrt{x_1^2+x_2^2}<r} \tilde{\rho}(x)dx = \int_0^r 2\pi s \int_{-\infty}^{+\infty} \tilde{\rho}(s, z)dsdz, \tag{2.3}$$

is the mass in the cylinder $\{x = (x_1, x_2, x_3) : \sqrt{x_1^2 + x_2^2} < r\}$, and J is a given function. In this case, the velocity field $\tilde{\mathbf{v}}(x) = (v_1, v_2, v_3)$ takes the form

$$\tilde{\mathbf{v}}(x) = (-\frac{x_2 J(m_{\tilde{\rho}}(r))}{r^2}, \frac{x_1 J(m_{\tilde{\rho}}(r))}{r^2}, 0).$$

Substituting this in (1.1), we find that $\tilde{\rho}(r, z)$ satisfies the following two equations:

$$\begin{cases} \partial_r p(\tilde{\rho}) = \tilde{\rho}\partial_r(B\tilde{\rho}) + \tilde{\rho}L(m_{\tilde{\rho}}(r))r^{-3}, \\ \partial_z p(\tilde{\rho}) = \tilde{\rho}\partial_z(B\tilde{\rho}), \end{cases} \tag{2.4}$$

where the operator B is defined in (2.2), and

$$L(m_{\tilde{\rho}}) = J^2(m_{\tilde{\rho}})$$

is the square of the angular momentum. We define

$$A(\rho) = \rho \int_0^\rho \frac{p(s)}{s^2}ds. \tag{2.5}$$

It is easy to verify that (cf. [1]) (2.4) is equivalent to

$$A'(\tilde{\rho}(x)) + \int_{r(x)}^\infty L(m_{\tilde{\rho}}(s)s^{-3}ds - B\tilde{\rho}(x) = \lambda, \qquad \text{where } \tilde{\rho}(x) > 0, \tag{2.6}$$

for some constant λ. Here $r(x)$ and $z(x)$ are as in (2.1). Let M be a positive constant and let W_M be the set of functions ρ defined by

$$\begin{aligned} W_M = \{&\rho : \mathbb{R}^3 \to \mathbb{R},\ \rho \text{ is axisymmetric},\ \rho \geq 0, a.e., \\ &\int \rho(x)dx = M,\ \int \left(A(\rho(x)) + \frac{\rho(x)L(m_\rho(r(x)))}{r(x)^2} + \rho(x)B\rho(x)\right)dx < +\infty\}. \end{aligned}$$

For $\rho \in W_M$, we define the **energy functional** F by

$$F(\rho) = \int [A(\rho(x)) + \frac{1}{2}\frac{\rho(x)L(m_\rho(r(x)))}{r(x)^2} - \frac{1}{2}\rho(x)B\rho(x)]dx. \tag{2.7}$$

In (2.7), the first term denotes the potential energy, the middle term denotes the rotational kinetic energy and the third term is the gravitational energy.

For a white dwarf star, the pressure function $p(\rho)$ satisfies the following conditions:

$$\lim_{\rho\to 0+}\frac{p(\rho)}{\rho^{4/3}} = 0, \ \lim_{\rho\to\infty}\frac{p(\rho)}{\rho^{4/3}} = \mathfrak{K}, \ p'(\rho) > 0 \text{ as } \rho > 0, \tag{2.8}$$

where $\mathfrak{K}$ is a finite positive constant. Assuming that the function $L \in C^1[0, M]$ and satisfies

$$L(0) = 0, \ L(m) \geq 0, \ for\ 0 \leq m \leq M, \tag{2.9}$$

Auchmuty and Beals (cf. [1]) proved the existence of a minimizer of the functional $F(\rho)$ in the class of functions $W_{M,S} = W_M \cap W_{sym}$, where

$$W_{sym} = \{\rho : \mathbb{R}^3 \to \mathbb{R}, \ \rho(x_1, x_2, -x_3) = \rho(x_1, x_2, x_3), \ x_i \in \mathbb{R}, i = 1, \ 2, \ 3\}. \tag{2.10}$$

Their result is given in the following theorem.

Theorem 2.1 ([1]). *If the pressure function p satisfies (2.8) (for either $0 < \mathfrak{K} < +\infty$ or $\mathfrak{K} = +\infty$) and (2.9) holds, then there exists a constant $M_c > 0$ depending on the constant $\mathfrak{K}$ in (2.8) (if $\mathfrak{K} = +\infty$ then $M_c = +\infty$, if $0 < \mathfrak{K} < +\infty$, then $0 < M_c < +\infty$) such that, if*

$$M < M_c, \tag{2.11}$$

then there exists a function $\hat{\rho}(x) \in W_{M,S}$ which minimizes $F(\rho)$ in $W_{M,S}$. Moreover, if

$$G = \{x \in \mathbb{R}^3 : \ \hat{\rho}(x) > 0\}, \tag{2.12}$$

then $\bar{G}$ is a compact set in $\mathbb{R}^3$, and $\hat{\rho} \in C^1(G) \cap C^\beta(\mathbb{R}^3)$ for some $0 < \beta < 1$. Furthermore, there exists a constant $\mu < 0$ such that

$$\begin{cases} A'(\hat{\rho}(x)) + \int_{r(x)}^\infty L(m_{\hat{\rho}}(s)s^{-3}ds - B\hat{\rho}(x) = \mu, & x \in G, \\ \int_{r(x)}^\infty L(m_{\hat{\rho}}(s)s^{-3}ds - B\hat{\rho}(x) \geq \mu, & x \in \mathbb{R}^3 - G. \end{cases} \tag{2.13}$$

Remark 1. When $0 < \mathfrak{K} < \infty$, the constant $0 < M_c < +\infty$ in (2.11) is called critical mass. The critical mass was first found by Chandrasekhar (cf. [5]) in the study of non- rotating white dwarf stars. When $0 < \mathfrak{K} < \infty$, it was proved by Friedman and Turkington ([10]) that, if the angular momentum satisfies the following condition

$$J \in C^1([0, M]), \ J'(m) \geq 0, \text{ for } 0 \leq m \leq M, J(0) = 0, \ J(m) > 0 \ \text{ for } 0 < m \leq M, \tag{2.14}$$

where J is the angular momentum, then the condition (2.11) can be removed, i.e., the above theorem holds for any positive total mass M.

In this paper, we are interested in minimizers of functional F in the *larger* class W_M. By the same argument as in [1], it is easy to prove the following theorem on the regularity of a minimizer.

Theorem 2.2. *Suppose that the pressure function p satisfies:*

$$\lim_{\rho\to 0+}\frac{p(\rho)}{\rho^{6/5}}=0,\ \lim_{\rho\to\infty}\frac{p(\rho)}{\rho^{6/5}}=\infty,\ p'(\rho)>0 \text{ as } \rho>0, \tag{2.15}$$

and the angular momentum satisfies (2.9). *Let $\tilde\rho$ be a minimizer of the energy functional F in W_M and let*

$$\Gamma=\{x\in\mathbb{R}^3:\ \tilde\rho(x)>0\}, \tag{2.16}$$

then $\tilde\rho\in C(\mathbb{R}^3)\cap C^1(\Gamma)$. Moreover, there exists a constant λ such that

$$\begin{cases} A'(\tilde\rho(x))+\int_{r(x)}^{\infty}L(m_{\tilde\rho}(s)s^{-3}ds-B\tilde\rho(x)=\lambda, & x\in\Gamma,\\ \int_{r(x)}^{\infty}L(m_{\tilde\rho}(s)s^{-3}ds-B\tilde\rho(x)\ge\lambda, & x\in\mathbb{R}^3-\Gamma.\end{cases} \tag{2.17}$$

We call such a minimizer $\tilde\rho$ a *rotating star* solution with total mass M and angular momentum $\sqrt{L(m)}$.

3. General Existence and Stability Theorems

For the angular momentum, besides the condition (2.9), we also assume that it satisfies the following conditions:

$$L(am)\ge a^{4/3}L(m),\ 0<a\le 1,\ 0\le m\le M, \tag{3.1}$$

$$L'(m)\ge 0,\qquad 0\le m\le M. \tag{3.2}$$

Condition (3.2) is called the Sölberg stability criterion ([35]).

3.1. Compactness of minimizing sequence. In this section, we first establish a compactness result for the minimizing sequences of the functional F. This compactness result is crucial for the existence and stability analyses.

Theorem 3.1. *Suppose that the square of the angular momentum L satisfies* (2.9), (3.1) *and* (3.2)*, and the pressure function p satisfies the following conditions*:

$$p\in C^1[0,+\infty),\int_0^1\frac{p(\rho)}{\rho^2}d\rho<+\infty,\ \lim_{\rho\to\infty}\frac{p(\rho)}{\rho^\gamma}=K,\ p(\rho)\ge 0,\ p'(\rho)>0\ \text{ for } \rho>0, \tag{3.3}$$

where $0<K<+\infty$ *and* $\gamma\ge 4/3$. *If*

(1)

$$\inf_{\rho\in W_M}F(\rho)<0, \tag{3.4}$$

and

(2) *for* $\rho \in W_M$,

$$\int [A(\rho)(x) + \frac{1}{2}\frac{\rho(x)L(m_\rho(r(x)))}{r(x)^2}]dx \le C_1 F(\rho) + C_2, \tag{3.5}$$

for some positive constants C_1 *and* C_2, *then the following hold:*

(a) *If* $\{\rho^i\} \subset W_M$ *is a minimizing sequence for the functional* F, *then there exist a sequence of vertical shifts* $a_i\mathbf{e_3}$ ($a_i \in \mathbb{R}$, $\mathbf{e_3} = (0, 0, 1)$), *a subsequence of* $\{\rho^i\}$, *(still labeled* $\{\rho^i\}$*), and a function* $\tilde{\rho} \in W_M$, *such that for any* $\epsilon > 0$ *there exists* $R > 0$ *with*

$$\int_{|x|\ge R} T\rho^i(x)dx \le \epsilon, \quad i \in \mathbb{N}, \tag{3.6}$$

and

$$T\rho^i(x) \rightharpoonup \tilde{\rho},\ weakly\ in\ L^\gamma(\mathbb{R}^3),\ as\ i \to \infty, \tag{3.7}$$

where $T\rho^i(x) := \rho^i(x + a_i\mathbf{e_3})$.
Moreover

(b)

$$\nabla B(T\rho^i) \to \nabla B(\tilde{\rho})\ strongly\ in\ L^2(\mathbb{R}^3),\ as\ i \to \infty. \tag{3.8}$$

(c) $\tilde{\rho}$ *is a minimizer of* F *in* W_M.

Thus $\tilde{\rho}$ is a rotating star solution with total mass M and angular momentum $\sqrt{L}$.

Remark 2. i) The assumption (3.4) is crucial for our compactness and stability analysis. The physical meaning of this is that the gravitational energy, the negative part of the energy F, should be greater than the positive part, which means the gravitation should be strong enough to hold the star together. In Sect. 4, we will verify this assumption. Roughly speaking, in addition to (3.3), if we require

$$\lim_{\rho\to 0+} \frac{p(\rho)}{\rho^{\gamma_1}} = \alpha, \tag{3.9}$$

for some constants $\gamma_1 > 4/3$ and $0 < \alpha < +\infty$, then (3.4) holds for the following cases:

(a) When $\gamma = 4/3$ (where γ is the constant in (3.3)), if the total mass M is less than a "critical mass" M_c, then (3.4) holds. This case includes white dwarf stars. For a white dwarf star, $\gamma_1 = 5/3$.

(b) When $\gamma > 4/3$, (3.4) holds for arbitrary positive total mass M. This generalizes our previous result in [26] for the polytropic stars with $p(\rho) = \rho^\beta$, $\beta > 4/3$.
It should be noted that (3.9) does not apply to supermassive star, i.e. $p(\rho) = k\rho^{4/3}$. For the supermassive star, in order that (3.4) holds, in addition to requiring that the total mass is less than a "critical mass", we also require that the angular momentum (per unit mass) J is not identically zero.

ii) Assumption (2) in the above theorem implies that the functional F is bounded below, i.e.,

$$\inf_{\rho \in W_M} F(\rho) > -\infty. \tag{3.10}$$

We will verify this assumption in Sect. 4 (see Theorem 4.1).

iii) The inequality (3.6) is crucial for the compactness result (3.8). One of the difficulties in the analysis is the loss of compactness because we consider the problem in an unbounded space, $\mathbb{R}^3$. The inequality (3.6) means the masses of the elements in the minimizing sequence $T\rho^i(x)$ "almost" concentrate in a ball $B_R(0)$.

iv) It is easy to verify that the functional F is invariant under any vertical shift, i.e., if $\rho(\cdot) \in W_M$, then $\bar{\rho}(x) =: \rho(x + a\mathbf{e_3}) \in W_M$ and $F(\bar{\rho}) = F(\rho)$ for any $a \in \mathbb{R}$. Therefore, if $\{\rho^i\}$ is a minimizing sequence of F in W_M, then $\{T\rho^i\} =:= \rho^i(x + a_i\mathbf{e_3})$ is also a minimizing sequence in W_M.

Theorem 3.1 is proved in a sequence of lemmas with some modifications of the arguments in [26]. We only sketch the proofs of those lemmas and Theorem 3.1. Complete details can be followed as in [26]. We first give some inequalities which will be used later. We begin with Young's inequality (see [14], p. 146.)

Lemma 3.1. *If $f \in L^p \cap L^r$, $1 \le p < q < r \le +\infty$, then*

$$||f||_q \le ||f||_p^a ||f||_r^{1-a}, \qquad a = \frac{q^{-1} - r^{-1}}{p^{-1} - r^{-1}}. \tag{3.11}$$

The following two lemmas are proved in [1].

Lemma 3.2. *Suppose the function $f \in L^1(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$. If $1 < q \le 3/2$, then $Bf =: f * \frac{1}{|x|}$ is in $L^r(\mathbb{R}^3)$ for $3 < r < 3q/(3-2q)$, and*

$$||Bf||_r \le C\left(||f||_1^b ||f||_q^{1-b} + ||f||_1^c ||f||_q^{1-c}\right), \tag{3.12}$$

for some constants $C > 0$, $0 < b < 1$, and $0 < c < 1$. If $q > 3/2$, then $Bf(x)$ is a bounded continuous function, and satisfies (3.12) with $r = \infty$.

Lemma 3.3. *For any function $f \in L^1(\mathbb{R}^3) \cap L^{4/3}(\mathbb{R}^3)$, $\nabla Bf \in L^2(\mathbb{R}^3)$. Moreover,*

$$|\int f(x)Bf(x)dx| = \frac{1}{4\pi}||\nabla Bf||_2^2 \le C\left(\int |f|^{4/3}(x)dx\right)\left(\int |f|(x)dx\right)^{2/3}, \tag{3.13}$$

for some constant C.

We also need the following lemma.

Lemma 3.4. *Suppose that the pressure function p satisfies (3.3) and that (3.5) holds. Let $\{\rho^i\} \subset W_M$ be a minimizing sequence for the functional F. Then there exists a constant $C > 0$ such that*

$$\int [(\rho^i)^\gamma(x) + \frac{1}{2}\frac{\rho^i(x)L(m_{\rho^i}(r(x)))}{r(x)^2}]dx \le C, \quad \text{for all } i \ge 1, \tag{3.14}$$

where $\gamma \ge 4/3$ is the constant in (3.3). So, the sequence $\{\rho^i\}$ is bounded in $L^\gamma(\mathbb{R}^3)$.

Proof. By (3.5), we know that

$$\int [A(\rho^i)(x) + \frac{1}{2}\frac{\rho^i(x)L(m_{\rho^i}(r(x)))}{r(x)^2}]dx \leq C, \quad \text{for all } i \geq 1, \tag{3.15}$$

for any minimizing sequence $\{\rho^i\} \subset W_M$ for the functional F, where we have used that $\{F(\rho^i)\}$ is bounded from above since it converges to $\inf_{W_M} F$. It is easy to verify that, by virtue of (3.3) and (2.5),

$$\lim_{\rho\to\infty} \frac{A(\rho)}{\rho^\gamma} = \frac{K}{\gamma - 1}, \ A(\rho) > 0 \quad \text{for } \rho > 0. \tag{3.16}$$

Therefore, there exits a constant $\rho^* > 0$ such that

$$\alpha A(\rho) \geq \rho^\gamma, \qquad \text{for } \rho \geq \rho^*, \tag{3.17}$$

where $\alpha = \frac{2(\gamma-1)}{K}$. Hence, for $\rho \in W_M$,

$$\begin{aligned} \int \rho^\gamma dx &\leq \int_{\rho<\rho^*} (\rho^*)^{\gamma-1}\rho dx + \alpha \int_{\rho\geq\rho^*} A(\rho)dx \\ &\leq (\rho^*)^{\gamma-1}M + \alpha \int A(\rho)dx. \end{aligned} \tag{3.18}$$

Applying this inequality to ρ^i, we conclude that the sequence $\{\rho^i\}$ is bounded in $L^\gamma(\mathbb{R}^3)$ by using (3.15). □

For any $M > 0$, we let

$$f_M = \inf_{\rho\in W_M} F(\rho). \tag{3.19}$$

Lemma 3.5. *If (3.1) holds, then $f_{\bar{M}} \geq (\bar{M}/M)^{5/3} f_M$ for every $M > \bar{M} > 0$.*

Proof. The proof follows from a scaling argument as in [31] and [26]. Take $a = (M/\bar{M})^{1/3}$ and let $\bar{\rho}(x) = \rho(ax)$ for any $\rho \in W_M$. It is easy to verify that $\bar{\rho} \in W_{\bar{M}}$. Moreover, for $r \geq 0$, it is easy to verify (as in [26]) that

$$m_{\bar{\rho}}(r) = \frac{1}{a^3} m_\rho(ar). \tag{3.20}$$

Since L satisfies (3.1) and $a > 1$, we have

$$L(m_{\bar{\rho}}(r)) \geq \frac{1}{a^4} L(m_\rho(ar)). \tag{3.21}$$

Thus, as in [26], we can show that

$$\int \frac{\bar{\rho}(x)L(m_{\bar{\rho}}(r(x)))}{r(x)^2} dx \geq \frac{1}{a^5} \int \frac{\rho(x)L(m_\rho(r(x)))}{r(x)^2} dx. \tag{3.22}$$

Therefore, since $a \geq 1$, it follows from (3.21) and (3.22) that

$$\begin{aligned} F(\bar\rho) &\geq a^{-3}\int A(\rho)dx - \frac{a^{-5}}{2}\int \rho B\rho dx + \frac{a^{-5}}{2}\int \frac{\rho(x)L(m_\rho(r(x)))}{r(x)^2}dx \\ &\geq a^{-5}\left(\int A(\rho)dx - \frac{1}{2}\int \rho B\rho dx + \frac{1}{2}\int \frac{\rho(x)L(m_\rho(r(x)))}{r(x)^2}dx\right) \\ &= (\bar M/M)^{5/3}F(\rho). \end{aligned} \tag{3.23}$$

Since $\rho \to \bar\rho$ is one-to-one between W_M and $W_{\bar M}$, this proves the lemma. □

Lemma 3.6. *Let $\{\rho^i\} \subset W_M$ be a minimizing sequence for F. Then there exist constants $r_0 > 0$, $\delta_0 > 0$, $i_0 \in \mathbf{N}$ and $x^i \in \mathbb{R}^3$ with $r(x^i) \leq r_0$, such that*

$$\int_{B_1(x^i)} \rho^i(x)dx \geq \delta_0,\ i \geq i_0. \tag{3.24}$$

Proof. First, since $\lim_{i\to\infty} F(\rho^i) \to f_M$ and $f_M < 0$ (see (3.4)), for large i,

$$-\frac{f_M}{2} \leq -F(\rho^i) \leq \frac{1}{2}\int \rho^i B\rho^i dx. \tag{3.25}$$

For any i, let

$$\delta_i = \sup_{x\in\mathbb{R}^3}\int_{|y-x|<1} \rho^i(y)dy. \tag{3.26}$$

Now

$$\int \rho^i B\rho^i(x)dx \tag{3.27}$$

$$\begin{aligned} &= \int_{\mathbb{R}^3} \rho^i(x)\left\{\int_{|y-x|<1} + \int_{1<|y-x|<r} + \int_{|y-x|>r}\right\}\frac{\rho^i(y)}{|y-x|}dydx \\ &=: D_1 + D_2 + D_3, \end{aligned} \tag{3.28}$$

and $D_3 \leq M^2 r^{-1}$. The shell $1 < |y-x| < r$ can be covered by at most Cr^3 balls of radius 1, so $D_2 \leq CM\delta_i r^3$. By using Hölder's inequality and applying (3.12) to the restriction of ρ^i to $\{y : |y-x| < 1\}$, we get

$$\begin{aligned} D_1 &\leq \|\rho^i\|_{4/3}\left\|\int_{|y-x|<1}\frac{\rho^i(y)}{|y-x|}dy\right\|_4 \\ &\leq C\|\rho^i\|_{4/3}\left(\|\chi_{B_1(x)}\rho^i\|_1^b\|\rho^i\|_{4/3}^{1-b} + \|\chi_{B_1(x)}\rho^i\|_1^c\|\rho^i\|_{4/3}^{1-c}\right) \\ &\leq C\|\rho^i\|_{4/3}\left(\delta_i^b\|\rho^i\|_{4/3}^{1-b} + \delta_i^c\|\rho^i\|_{4/3}^{1-c}\right), \end{aligned} \tag{3.29}$$

where $0 < b < 1$ and $0 < c < 1$. Now since $\{\|\rho^i\|_\gamma\}$ is bounded, it follows that $\{\|\rho^i\|_{4/3}\}$ is bounded due to the fact $\gamma \geq 4/3$ in view of (3.11) and $\|\rho^i\|_1 = M$; this gives $D_1 \leq C(\delta_i^b + \delta_i^c)$. It follows that we could choose r so large that the above estimates give $\int \rho^i B\rho^i(x)dx < -f_M$ *if δ_i were small enough*. This would contradict (3.25). So there exists $\delta_0 > 0$ such that $\delta_i \geq \delta_0$ for large i. Thus, as i is large, there exist $x^i \in \mathbb{R}^3$ and $i_0 \in \mathbb{N}$ such that

$$\int_{B_1(x^i)} \rho^i(x)dx \geq \delta_0, \quad i \geq i_0. \tag{3.30}$$

We now prove that there exists $r_0 > 0$ independent of i such that x^i must satisfy $r(x^i) \leq r_0$ for i large. Namely, since ρ^i has mass at least δ_0 in the unit ball centered at x^i, and is axially symmetric, it has mass $\geq Cr(x^i)\delta_0$ in the torus obtained by revolving this ball around the x_3-axis (or z- axis).Therefore $r(x^i) \leq (C\delta_0)^{-1}M$. □

In order to prove Theorem 3.1, we will need the following lemma.

Lemma 3.7. *Let $\{f^i\}$ be a bounded sequence in $L^\gamma(\mathbb{R}^3)$ ($\gamma \geq 4/3$) and suppose*

$$f^i \rightharpoonup f^0 \;\; weakly \; in \; L^\gamma(\mathbb{R}^3).$$

Then

(a) *For any $R > 0$,*

$$\nabla B(\chi_{B_R(0)} f^i) \to \nabla B(\chi_{B_R(0)} f^0) \;\; strongly \; in \; L^2(\mathbb{R}^3),$$

where χ is the characteristic function.

(b) *If in addition $\{f^i\}$ is bounded in $L^1(\mathbb{R}^3)$, $f^0 \in L^1(\mathbb{R}^3)$, and for any $\epsilon > 0$ there exist $R > 0$ and $i_0 \in \mathbf{N}$ such that*

$$\int_{|x|>R} |f^i(x)|dx < \epsilon, \qquad i \geq i_0, \tag{3.31}$$

then

$$\nabla B f^i \to \nabla B f^0 \; strongly \; in \; L^2(\mathbb{R}^3).$$

Proof. This lemma follows easily from the proof of Lemma 3.7 in [31], due to the following observation:
The map: $\rho \in L^\gamma(\mathbb{R}^3) \mapsto \chi_{B_R(0)}\nabla B(\chi_{B_R(0)}\rho)$ is compact for any $R > 0$, if $\gamma \geq 4/3$, where χ denotes the characteristic function. □

With the above lemmas, the proof of Theorem 3.1 is similar to that in [26]. So we only outline the main steps.

Proof of Theorem 3.1.

Step 1. Splitting. We begin with a splitting as in [31]. For $\rho \in W_M$, for any $0 < R_1 < R_2$, we have

$$\rho = \rho\chi_{|x|\leq R_1} + \rho\chi_{R_1<|x|\leq R_2} + \rho\chi_{|x|>R_2} =: \rho_1 + \rho_2 + \rho_3, \tag{3.32}$$

where again χ is the characteristic function. It is easy to verify that

$$\begin{aligned}\int \frac{\rho(x)L(m_\rho(r(x)))}{r^2(x)}dx &= \sum_{j=1}^{3}\int \frac{\rho_j(x)L(m_{\rho_j}(r(x)))}{r^2(x)}dx \\ &\quad + \sum_{j=1}^{3}\int \frac{\rho_j(x)(L(m_\rho(r(x))) - L(m_{\rho_j}(r(x)))}{r^2(x)}dx, \\ &\geq \sum_{j=1}^{3}\int \frac{\rho_j(x)L(m_{\rho_j}(r(x)))}{r^2(x)}dx. \end{aligned} \tag{3.33}$$

In the last inequality above, we have used (3.2). So, we have

$$F(\rho) \geq \sum_{j=1}^{3} F(\rho_j) - \sum_{1\leq i<j\leq 3} I_{ij}, \tag{3.34}$$

where

$$I_{ij} = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |x-y|^{-1}\rho_i(x)\rho_j(y)dxdy, \qquad 1 \leq i < j \leq 3.$$

If we choose $R_2 > 2R_1$ in the splitting (3.32), then

$$I_{13} \leq \frac{C}{R_2}. \tag{3.35}$$

By (3.12) and (3.13), we have

$$\begin{aligned} &I_{12} + I_{23} \\ &= \frac{1}{4\pi}\int \nabla(B\rho_1 + B\rho_3)\cdot\nabla B\rho_2 dx \leq C\|\nabla(B\rho_1 + B\rho_3)\|_2\|\nabla B\rho_2\|_2 \\ &\leq CM^{1/3}\|\rho_1 + \rho_3\|_{4/3}^{2/3}\|\nabla B\rho_2\|2 \leq CM^{1/3}\|\rho\|_{4/3}^{2/3}\|\nabla B\rho_2\|_2. \end{aligned} \tag{3.36}$$

Using Lemma 3.5, (3.4), (3.34), (3.35) and (3.36), and following an argument as in the proof of Theorem 3.1 in [31], we can show that

$$\begin{aligned} &f_M - F(\rho) \\ &\leq (1 - (\frac{M_1}{M})^{5/3} - (\frac{M_2}{M})^{5/3} - (\frac{M_3}{M})^{5/3}) f_M + C(R_2^{-1} + M^{1/3}\|\rho\|_{4/3}^{2/3}||\nabla B\rho_2||_2) \\ &\leq Cf_M M_1 M_3 + C(R_2^{-1} + M^{1/3}\|\rho\|_{4/3}^{2/3}||\nabla B\rho_2||_2), \end{aligned} \tag{3.37}$$

by choosing $R_2 > 2R_1$ in the splitting (3.32), where $M_i = \int \rho_i(x)dx$ $(i = 1, 2, 3.)$

Step 2. Compactness. Let $\{\rho^i\}$ be a minimizing sequence of F in W_M. By Lemma 3.6, we know that there exists $i_0 \in \mathbf{N}$ and $\delta_0 > 0$ independent of i such that

$$\int_{a_i\mathbf{e_3}+B_{R_0(0)}} \rho^i(x)dx \geq \delta_0, \qquad if\ i \geq i_0, \tag{3.38}$$

where $a_i = z(x^i)$ and $R_0 = r_0 + 1$, x^i and r_0 are those quantities in Lemma 3.6, $\mathbf{e_3} = (0, 0, 1)$. Having proved (3.38), we can follow the argument in the proof of Theorem 3.1 in [31] to verify (3.31) for

$$f^i(x) = T\rho^i(x) =: \rho^i(\cdot + a_i\mathbf{e_3})$$

by using (3.34) and (3.38) and choosing suitable R_1 and R_2 in the splitting (3.32). We sketch this as follows. The sequence $T\rho^i =: \rho^i(\cdot + a_i\mathbf{e_3})$, $i \geq i_0$, is a minimizing sequence of F in W_M (see Remark 2 after Theorem 3.1). We rewrite (3.38) as

$$\int_{B_{R_0}(0)} T\rho^i(x)dx \geq \delta_0,\ i \geq i_0. \tag{3.39}$$

Applying (3.37) with $T\rho^i$ replacing ρ, and noticing that $\{T\rho^i\}$ is bounded in $L^\gamma(\mathbb{R}^3)$ (see Lemma 3.4) (so $\{\|T\rho^i\|_{4/3}\}$ is bounded if $\gamma \geq 4/3$ in view of (3.11) and the fact $\|\rho^i\|_1 = M$), we obtain, if $R_2 > 2R_1$,

$$-Cf_M M_1^i M_3^i \leq C(R_2^{-1} + ||\nabla BT\rho_2^i||_2) + F(T\rho^i) - f_M, \tag{3.40}$$

where $M_1^i = \int T\rho_1^i(x)dx = \int_{|x|<R_1} T\rho^i(x)dx,$, $M_3^i = \int T\rho_3^i(x)dx = \int_{|x|>R_2} T\rho^i(x)dx$ and $T\rho_2^i = \chi_{R_1<|x|\leq R_2} T\rho^i$. Since $\{T\rho^i\}$ is bounded in $L^\gamma(\mathbb{R}^3)$, there exists a subsequence, still labeled by $\{T\rho^i\}$, and a function $\tilde{\rho} \in W_M$ such that

$$T\rho^i \rightharpoonup \tilde{\rho} \text{ weakly in } L^\gamma(\mathbb{R}^3).$$

This proves (3.7). By (3.39), we know that M_1^i in (3.40) satisfies $M_1^i \geq \delta_0$ for $i \geq i_0$ by choosing $R_1 \geq R_0$ where R_0 is the constant in (3.39). Therefore, by (3.40) and the fact that $f_M < 0$ (cf. (3.4)) , we have

$$-Cf_M\delta_0 M_3^i \leq CR_2^{-1} + C||\nabla B\tilde{\rho}_2||_2 + C||\nabla BT\rho_2^i - \nabla B\tilde{\rho}_2||_2) + F(T\rho^i) - f_M, \tag{3.41}$$

where $\tilde{\rho}_2 = \chi_{|x|>R_2}\tilde{\rho}$. Given any $\epsilon > 0$, by the same argument as [31], we can increase $R_1 > R_0$ such that the second term on the right hand side of (3.41) is small, say less than $\epsilon/4$. Next choose $R_2 > 2R_1$ such that the first term is small. Now that R_1 and R_2 are fixed, the third term on the right hand side of (3.41) converges to zero by Lemma 3.7(a). Since $\{T\rho^i\}$ is a minimizing sequence of F in W_M, we can make $F(T\rho^i) - f_M$ small by taking i large. Therefore, for i sufficiently large, we can make

$$M_3^i =: \int_{|x|>R_2} T\rho^i(x)dx < \epsilon. \tag{3.42}$$

This verifies (3.31) in Lemma 3.7 for $f^i = T\rho^i$. By weak convergence we have that for any $\epsilon > 0$ there exists $R > 0$ such that

$$M - \epsilon \leq \int_{B_R(0)} \tilde{\rho}(x)dx \leq M,$$

which implies $\tilde{\rho} \in L^1(\mathbb{R}^3)$ with $\int \tilde{\rho}dx = M$. Therefore, by Lemma 3.7(b), we have

$$||\nabla BT\rho^i - \nabla B\tilde{\rho}||_2 \to 0, \qquad i \to +\infty. \tag{3.43}$$

This proves (3.8). Equation (3.6) in Theorem 3.1 follows from (3.42) by taking $R = R_2$.

Step 3. Lower Semi-Continuity. Let $\{\rho^i\}$ be a minimizing sequence of the energy functional F, and let $\tilde{\rho}$ be a weak limit of $\{T\rho^i\}$ in $L^\gamma(\mathbb{R}^3)$. We will prove that $\tilde{\rho}$ is a minimizer of F in W_M; that is

$$F(\tilde{\rho}) \le \liminf_{i\to\infty} F(T\rho^i). \tag{3.44}$$

By (3.3), there exist positive constants C and ρ^* such that

$$A'(\rho) \le C\rho^{\gamma-1},\ for\ \rho \ge \rho^*, \tag{3.45}$$

where $\gamma \ge 4/3$ is the constant in (3.3). Since $\tilde{\rho} \in L^\gamma$ and $\int \tilde{\rho}dx = M$, we can conclude $A'(\tilde{\rho}) \in L^{\gamma'}$, where $L^{\gamma'}$ is the dual space of L^γ, i.e., $\gamma' = \frac{\gamma}{\gamma-1}$. In view of (2.5) and (3.3), we have

$$A''(\rho) = p'(\rho)/\rho > 0, \qquad \text{for } \rho > 0, \tag{3.46}$$

so that

$$\int A(T\rho^i)dx \ge \int A(\tilde{\rho})dx + \int A'(\tilde{\rho})(T\rho^i - \tilde{\rho}),\ \text{for } i \ge 1. \tag{3.47}$$

Since $A'(\tilde{\rho}) \in L^{\gamma'}$ and $T\rho^i$ weakly converges to $\tilde{\rho}$ in L^γ,

$$\int A'(\tilde{\rho})(T\rho^i - \tilde{\rho}) \to 0,\ \text{as } i \to +\infty. \tag{3.48}$$

Therefore,

$$\int A(\tilde{\rho})dx \le \liminf_{i\to\infty} \int A(T\rho^i)dx. \tag{3.49}$$

Next, following the proof in [26], we can show that

$$\liminf_{i\to\infty} \int \frac{T\rho^i(x)L(m_{T\rho^i}(r(x)) - \tilde{\rho}(x)L(m_{\tilde{\rho}}(r(x))}{r^2(x)}dx \ge 0, \tag{3.50}$$

by showing that the mass function

$$m_{\tilde{\rho}}(r) =: \int_{\sqrt{x_1^2+x_2^2} \le r} \tilde{\rho}(x)dx$$

is continuous for $r \ge 0$, and using (3.6). Then (3.44) follows from (3.43), (3.49) and (3.50).

3.2. Stability. In this section, we assume that the pressure function p satisfies

$$p \in C^1[0, +\infty), \quad \lim_{\rho\to 0+} \frac{p(\rho)}{\rho^{6/5}} = 0, \quad \lim_{\rho\to\infty} \frac{p(\rho)}{\rho^{\gamma}} = K, \quad p'(\rho) > 0 \quad \text{for } \rho > 0. \tag{3.51}$$

where $0 < K < +\infty$ and $\gamma \geq 4/3$ are constants. It should be noticed that (3.51) implies both (2.15) and (3.3). We consider the Cauchy problem for (1.1) with the initial data

$$\rho(x, 0) = \rho_0(x), \quad \mathbf{v}(x, 0) = \mathbf{v}_0(x). \tag{3.52}$$

We begin by giving the definition of a weak solution.

Definition 3.1. *Let* $\rho\mathbf{v} = \mathbf{m}$. *The triple* $(\rho, \mathbf{m}, \Phi)(x, t)$ $(x \in \mathrm{R}^3, t \in [0, T])$ $(T > 0)$ *and* Φ *given by* (1.2)*, with* $\rho \geq 0$, $p(\rho)$, $\mathbf{m}$, $\mathbf{m}\otimes\mathbf{m}/\rho$ *and* $\rho\nabla\Phi$ *being in* $L^1(\mathrm{R}^3 \times [0, T])$, *is called a weak solution of the Cauchy problem* (1.1) *and* (3.52) *on* $\mathrm{R}^3 \times [0, T]$ *if for any Lipschitz continuous test function* ψ *with compact support in* $\mathrm{R}^3 \times [0, T]$,

$$\int_0^T \int (\rho\psi_t + \mathbf{m}\cdot\nabla\psi + p(\rho)\nabla\psi)\, dxdt + \int \rho_0(x)\psi(x, 0)dx = 0, \tag{3.53}$$

and

$$\begin{aligned}\int_0^T \int \left(\mathbf{m}\psi_t + (p(\rho)\mathbb{I} + \frac{\mathbf{m}\otimes\mathbf{m}}{\rho})\nabla\psi\right) dxdt + \int \mathbf{m}_0(x)\psi(x, 0)dx \\ = \int_0^T \int \rho\nabla\Phi\psi\, dxdt,\end{aligned} \tag{3.54}$$

where $\mathbb{I}$ *is the* 3×3 *unit matrix.*

The total energy of system (1.1) *at time* t *is*

$$E(t) = E(\rho(t), \mathbf{v}(t)) = \int \left(A(\rho) + \frac{1}{2}\rho|\mathbf{v}|^2\right)(x, t)dx - \frac{1}{8\pi}\int |\nabla\Phi|^2(x, t)dx, \tag{3.55}$$

where as before,

$$A(\rho) = \rho \int_0^{\rho} \frac{p(s)}{s^2} ds. \tag{3.56}$$

For a solution of (1.1) without shock waves, the total energy is conserved, i.e., $E(t) = E(0)$ $(t \geq 0)$(cf. [35]). For solutions with shock waves, the energy should be non-increasing in time, so that for all $t \geq 0$,

$$E(t) \leq E(0), \tag{3.57}$$

due to the entropy conditions, which is described below.

Definition 3.2. *A weak solution (defined above) on* $\mathbb{R}^3 \times [0, T]$ *is called an entropy weak solution of* (1.1) *if it satisfies the following "entropy inequality":*

$$\partial_t\eta + \sum_{j=1}^{3} \partial_{x_j} q_j + \rho \sum_{j=1}^{3} \eta_{m_j}\Phi_{x_j} \leq 0, \tag{3.58}$$

in the sense of distributions; i.e.,

$$\int_0^T \int_{\mathbb{R}^3} \left(\eta\beta_t + \mathbf{q} \cdot \nabla\beta - \rho \sum_{j=1}^{3} \eta_{m_j} \Phi_{x_j} \beta \right) dxdt + \int_{\mathbb{R}^3} \beta(x,0)\eta(x,0)dx \geq 0, \tag{3.59}$$

for any nonnegative Lipschitz continuous test function β with compact support in $[0, T) \times \mathbb{R}^3$. *Here the "entropy" function η and "entropy flux" functions q_j and $\mathbf{q}$, are defined by*

$$\begin{cases} \eta = \frac{|\mathbf{m}|^2}{2\rho} + \rho \int_0^\rho \frac{p(s)}{s^2} ds, \\ q_j = \frac{|\mathbf{m}|^2 m_j}{2\rho^2} + m_j \int_0^\rho \frac{p'(s)}{s} ds, \\ \mathbf{q} = (q_1,\ q_2,\ q_3). \end{cases} \tag{3.60}$$

Remark 3. The inequality (3.58) is motivated by the second law of thermodynamics ([17]), and plays an important role in shock wave theory ([34]). For smooth solutions, the inequality in (3.58) can be replaced by equality.

Some properties of entropy weak solutions are given in the following theorem.

Theorem 3.2. *If* $(\rho, \mathbf{m}) \in L^\infty([0, T]; L^1(\mathbb{R}^3))$ *satisfies the first equation in* (1.1) *in the sense of distributions, then*

$$\int_{\mathbb{R}^3} \rho(x,t)dx = \int_{\mathbb{R}^3} \rho(x,0)dx =: M, \qquad 0 < t < T. \tag{3.61}$$

Let $(\rho, \mathbf{m}, \Phi)$ *be a weak solution defined in Definition* 3.1. *Suppose* $(\rho, \mathbf{m}, \Phi)$ *satisfies the entropy condition* (3.58), $\rho \in L^\infty([0, T]; L^1(\mathbb{R}^3)) \cap L^\infty([0, T]; L^r(\mathbb{R}^3))$ *for some r satisfying $r > 3/2$ and $r \geq \gamma$ ($\gamma \geq 4/3$ is the constant in* 3.51*),* $\mathbf{m} \in L^\infty([0, T]; L^s(\mathbb{R}^3))$ $(s > 3)$*,* $(\eta, \mathbf{q}) \in L^\infty([0, T]; L^1(\mathbb{R}^3))$*, where η and $\mathbf{q}$ are given in* (4.3). *Moreover, we assume that* $(\rho, \mathbf{m})$ *has the following additional regularity:*

$$\lim_{h \to 0} \int_0^t \int_{\mathbb{R}^3} |\rho(x, \tau + h) - \rho(x, \tau)| dxd\tau = 0, \quad t \in (0, T), a.e. \tag{3.62}$$

Then

$$E(t) \leq E(0), \qquad 0 < t < T, \tag{3.63}$$

where $E(t)$ is defined in (3.55).

The proof of this theorem is the same as that for Theorem 5.1 in [26], so we omit it.

Remark 4. The local existence of smooth solutions of the Cauchy problem (1.1) and (3.52) can be found in [29]. The local existence of solutions with shock fronts for the equations of compressible fluids can be found in [27]. The global existence of solutions for compressible fluids in three dimensions has been a major open problem. It would be possible to prove the global existence of entropy weak solutions with symmetry, by using some ideas for compressible Euler equations as in [20]. In this paper, we consider the weak solutions of the Cauchy problem satisfying some physically reasonable properties.

We consider axi-symmetric initial data, which takes the form

$$\begin{aligned}\rho_0(x) &= \rho(r,z),\\ \mathbf{v}_0(x) &= v_0^r(r,z)\mathbf{e}_r + v_0^\theta(r,z)\mathbf{e}_\theta + v_0^3(\rho,z)\mathbf{e}_3.\end{aligned} \tag{3.64}$$

Here $r=\sqrt{x_1^2+x_2^2}$, $z=x_3$, $x=(x_1,x_2,x_3)\in\mathbb{R}^3$ (as before), and

$$\mathbf{e}_r=(x_1/r,x_2/r,0)^{\mathrm{T}},\ \mathbf{e}_\theta=(-x_2/r,x_1/r,\ 0)^{\mathrm{T}},\ \mathbf{e}_3=(0,0,1)^{\mathrm{T}}. \tag{3.65}$$

We seek axi-symmetric solutions of the form

$$\rho(x,t)=\rho(r,z,t),$$
$$\mathbf{v}(x,t)=v^r(r,z,t)\mathbf{e}_r+v^\theta(r,z,t)\mathbf{e}_\theta+v^3(r,z,t)\mathbf{e}_3, \tag{3.66}$$
$$\Phi(x,t)=\Phi(r,z,t)=-B\rho(r,z,t). \tag{3.67}$$

We call a vector field $\mathbf{u}(x,t)=(u_1,u_2,u_3)(x)$ $(x\in\mathbb{R}^3)$ axi-symmetric if it can be written in the form

$$\mathbf{u}(x)=u^r(r,z)\mathbf{e}_r+u^\theta(r,z)\mathbf{e}_\theta+u^3(\rho,z)\mathbf{e}_3.$$

For the velocity field $\mathbf{v}=(v_1,v_2,v_3)(x,t)$, we define the angular momentum (per unit mass) $j(x,t)$ about the x_3-axis at (x,t), $t\geq 0$, by

$$j(x,t)=x_1v_2-x_2v_1. \tag{3.68}$$

For an axi-symmetric velocity field

$$\mathbf{v}(x,t)=v^r(r,z,t)\mathbf{e}_r+v^\theta(r,z,t)\mathbf{e}_\theta+v^3(\rho,z,t)\mathbf{e}_3, \tag{3.69}$$

$$v_1=\frac{x_1}{r}v^r-\frac{x_2}{r}v^\theta,\ \ v_2=\frac{x_2}{r}v^r+\frac{x_1}{r}v^\theta,\ v_3=v^3, \tag{3.70}$$

so that

$$j(x,t)=rv^\theta(r,z,t). \tag{3.71}$$

In view of (3.69) and (3.71), we have

$$|\mathbf{v}|^2=|v^r|^2+\frac{j^2}{r^2}+|v^3|^2. \tag{3.72}$$

Therefore, the total energy at time t can be written as

$$\begin{aligned}E(\rho(t),\mathbf{v}(t)) &= \int A(\rho)(x,t)dx+\frac{1}{2}\int\frac{\rho j^2(x,t)}{r^2(x)}dx\\ &\quad-\frac{1}{8\pi}\int|\nabla B\rho|^2(x,t)dx+\frac{1}{2}\int\rho(|v^r|^2+|v^3|^2)(x,t)dx.\end{aligned} \tag{3.73}$$

There is an important conserved quantity for the Euler-Poisson equations (1.1); namely the angular momentum. In order to describe these, we define D_t, the non-vacuum region at time $t\geq 0$ of the solution by

$$D_t=\{x\in\mathbb{R}^3:\rho(x,t)>0\}. \tag{3.74}$$

We will make the following assumption of the conservation of angular momentum for the axi-symmetric solutions of the Cauchy problem (1.1), which is motivated by physical considerations, cf. [35]).

A1) For any $t \geq 0$, there exists a measurable subset $G_t \subset D_t$ with $meas(D_t - G_t) = 0$ ($meas$ denotes Lebsegue measure) such that, for any $x \in G_t$, the angular momentum $j(x, t)$ defined in (3.68) only depends on the mass in the cylinder with radius $r(x)$, i.e.,

$$j(x,t) = j_t(m_{\rho_t}(r(x)), \tag{3.75}$$

where

$$m_{\rho_t}(r(x) = \int_{\sqrt{y_1^2+y_2^2} \leq r(x)} \rho(y,t)dy, \quad y = (y_1, y_2, y_3) \in \mathbb{R}^3.$$

Moreover, for $t \geq 0$ and $x \in G_t$, there exists a point $x_0(t) \in G_0$ satisfying

$$m_{\rho_t}(r(x)) = m_{\rho_0}(r(x_0(t))), \tag{3.76}$$

and

$$j(x,\ t) = j_t(m_{\rho_t}(r(x)) = j_0(m_{\rho_0}(r(x_0(t))). \tag{3.77}$$

Remark 5. For axi-symmetric motion, we have formally

$$\frac{Dj}{Dt} = 0, \tag{3.78}$$

where $\frac{Dj}{Dt}$ is the material derivative, i. e., $\frac{Dj}{Dt} := \frac{\partial j}{\partial t} + \mathbf{v} \cdot \nabla j$. This means that the angular momentum (per unit mass) is transported by the fluids. On the other hand, by the conservation of mass, the mass enclosed within any material volume cannot change as we follow the volume in its motion ([35], p. 47)). Mathematically, this means that, for any point $x_0 \in G_0$, along the particle path $x = \psi(t)$ satisfying $\frac{d\psi}{dt} = \mathbf{v}(\psi(t), t)$ and $\psi(0) = x_0$,

$$m_{\rho(t)}(r(\psi(t))) = m_{\rho_0}(r(x_0))$$

and

$$j(\psi(t), t) = j(x_0, 0).$$

Also, we need a technical assumption; namely,
A2)

$$\lim_{r \to 0+} \frac{L(m_{\rho(t)}(r) + m_{\tilde{\rho}}(r))m_{\sigma(t)}(r)}{r^2} = 0, \tag{3.79}$$

for $t \geq 0$, where $\sigma(t) = \rho(t) - \tilde{\rho}$ and L is the distribution of the square of angular momentum for the rotating star solution.

Remark 6. Equation (3.79) can be understood as follows. For any $\rho \in W_M$, we have $\lim_{r\to 0+} m_\rho(r) = 0$. Therefore $\lim_{r\to 0+} L(m_{\rho(t)}(r) + m_{\tilde{\rho}}(r)) = L(0) = 0$, so if we define

$$\hat{\rho}(s,t) - \hat{\tilde{\rho}}(s) = \int_{-\infty}^{+\infty} (\rho(s,z,t) - \tilde{\rho}(s,z))dz,$$

then if

$$\frac{m_{\sigma(t)}(r)}{r^2} = \frac{\int_0^r (2\pi s(\hat{\rho}(s,t) - \hat{\tilde{\rho}}(s))ds}{r^2} \in L^\infty(0,\delta) \ for\ some\ \delta > 0, \tag{3.80}$$

(3.79) will hold. If $\hat{\rho}(\cdot,t) - \hat{\tilde{\rho}}(\cdot) \in L^\infty(0,\delta)$, then (3.80) holds. This can be assured by assuming that $\rho(r,z,t) - \tilde{\rho}(r,z) \in L^\infty((0,\delta) \times \mathbb{R} \times \mathbb{R}^+)$ and decays fast enough in the z direction. For example, when $\rho(x,t) - \tilde{\rho}(x)$ has compact support in $\mathbb{R}^3$ and $\rho(\cdot,t) - \tilde{\rho}(\cdot) \in L^\infty(\mathbb{R}^3)$, then (3.79) holds.

We next make some assumptions on the initial data; namely, we assume that the initial data is such that the initial total mass and angular momentum are the same as those of the rotating star solution (those two quantities are conserved quantities). Therefore, we require

I_1)

$$\int \rho_0(x)dx = \int \tilde{\rho}(x)dx = M. \tag{3.81}$$

Moreover we assume

I_2) For the initial angular momentum $j(x,0) = rv_0^\theta(r,z) =: j_0(r,z)$ ($r = \sqrt{x_1^2 + x_2^2}$, $z = x_3$ for $x = (x_1, x_2, x_3)$, we assume $j(x,0)$ only depends on the total mass in the cylinder $\{y \in \mathbb{R}^3, r(y) \le r(x)\}$, i.e. ,

$$j(x,0) = j_0\left(m_{\rho_0}(r(x))\right). \tag{3.82}$$

(This implies that we require that $v_0^\theta(r,z)$ only depends on r.) Finally, we assume that the initial profile of the angular momentum per unit mass is the same as that of the rotating star solution, i. e.,

I_3)

$$j_0^2(m) = L(m), \qquad 0 \le m \le M, \tag{3.83}$$

where $L(m)$ is the profile of the square of the angular momentum of the rotating star defined in Sect. 2.

In order to state our stability result, we need some notation. Let λ be the constant in Theorem 2.2, i.e.,

$$\begin{cases} A'(\tilde{\rho}(x)) + \int_{r(x)}^\infty L(m_{\tilde{\rho}}(s))s^{-3}ds - B\tilde{\rho}(x) = \lambda, \ x \in \Gamma, \\ \int_{r(x)}^\infty L(m_{\tilde{\rho}})(s))s^{-3}ds - B\tilde{\rho}(x) \ge \lambda, \qquad x \in \mathbb{R}^3 - \Gamma, \end{cases} \tag{3.84}$$

with A defined in (3.56) and Γ defined in (2.16).

For $\rho \in W_M$, we define,

$$d(\rho,\tilde\rho)=\int [A(\rho)-A(\tilde\rho)]+(\rho-\tilde\rho)\{\int_{r(x)}^{\infty}\frac{L(m_{\tilde\rho}(s))}{s^3}ds-\lambda-B\tilde\rho\}dx. \quad (3.85)$$

For $x\in\Gamma$, in view of the convexity of the function A (cf. (3.46)) and (3.84), we have,

$$\begin{aligned}&(A(\rho)-A(\tilde\rho))(x)+(\int_{r(x)}^{\infty}\frac{L(m_{\tilde\rho}(s))}{s^3}ds-\lambda-B\tilde\rho(x))(\rho-\tilde\rho)\\&=(A(\rho)-A(\tilde\rho)-A'(\tilde\rho)(\rho-\tilde\rho))(x)\ge 0.\end{aligned} \quad (3.86)$$

For $x\in\mathbb{R}^3-\Gamma$, $\tilde\rho(x)=0$, so we have $A(\tilde\rho)(x))=0$. This is because since $A(0)=0$ due to $p(0)=0$ (cf. (3.3)) and (2.5). Therefore, by (3.84), we have, for $\rho\in W_M$ and $x\in\mathbb{R}^3-\Gamma$,

$$\begin{aligned}&(A(\rho)-A(\tilde\rho))(x)+(\int_{r(x)}^{\infty}\frac{L(m_{\tilde\rho}(s))}{s^3}ds-\lambda-B\tilde\rho(x))(\rho-\tilde\rho)\\&=A(\rho)\ge 0.\end{aligned} \quad (3.87)$$

Thus, for $\rho\in W_M$,

$$d(\rho,\tilde\rho)\ge 0. \quad (3.88)$$

We also define

$$\begin{aligned}d_1(\rho,\tilde\rho)=&\frac12\int\frac{\rho(x)L(m_\rho(r(x)))-\tilde\rho(x)L(m_{\tilde\rho}(r(x))}{r^2(x)}dx\\&-\int\int_{r(x)}^{\infty}s^{-3}L(m_{\tilde\rho}(s))ds(\rho(x)-\tilde\rho(x))dx,\end{aligned} \quad (3.89)$$

for $\rho\in W_M$. We shall show later that $d_1\ge 0$. Our main stability result in this paper is the following global-in- time stability theorem.

Theorem 3.3. *Suppose that the pressure function satisfies* (3.51)*, and both* (3.4)*,* (3.5) *hold. Let* $\tilde\rho$ *be a minimizer of the functional* F *in* W_M*, and assume that it is unique up to a vertical shift. Assume that* I_1*)-* I_3*), [*(3.81)*–*(3.83)*] hold. Moreover, assume that the angular momentum of the rotating star solution* $\tilde\rho$ *satisfies* (2.9)*,* (3.1) *and* (3.2)*. Let* $(\rho,\mathbf{v},\Phi)(x,t)$ *be an entropy weak solution of the Cauchy problem* (1.1) *and* (3.52) *satisfying* (3.61) *and* (3.63) *with axi-symmetry. If the angular momentum* j *satisfies Assumption A1) and Assumption A2) holds, then for every* $\epsilon>0$*, there exists a number* $\delta>0$ *such that if*

$$\begin{aligned}&d(\rho_0,\tilde\rho)+\frac{1}{8\pi}||\nabla B\rho_0-\nabla B\tilde\rho||_2^2+|d_1(\rho_0,\tilde\rho)|\\&+\frac12\int\rho_0(x)(|v_0^r|^2+|v_0^3|^2)(x)dx<\delta,\end{aligned} \quad (3.90)$$

then for every $t>0$*, there is a vertical shift* $a(t)\mathbf{e_3}$ *(*$a(t)\in\mathbb{R}$*,* $\mathbf{e_3}=(0,0,1)$*) such that,*

$$\begin{aligned}&d(\rho(t),T^{a(t)}\tilde\rho)+\frac{1}{8\pi}||\nabla B\rho(t)-\nabla BT^{a(t)}\tilde\rho||_2^2+|d_1(\rho(t),T^{a(t)}\tilde\rho)|\\&+\frac12\int\rho(x,t)(|v^r(x,t)|^2+|v^3(x,t)|^2)dx<\epsilon,\end{aligned} \quad (3.91)$$

where $T^{a(t)}\tilde\rho(x)=:\tilde\rho(x+a(t)\mathbf{e_3})$*.*

Remark 7. The above stability results of rotating star solutions apply for axi-symmetric perturbations. For the stability of non-rotating star solutions, we can consider general perturbations without axi- symmetry. Also, Assumptions A1)- A2) and I2)-I3) in the above theorem are used to control the angular momentum, for the stability of non-rotating stars, those assumptions are not needed. Moreover, the uniqueness assumption for minimizers of the energy functional is not needed for non-rotating star solutions since this uniqueness was proved in [22]. We give a general result of the stability for non-rotating white dwarf stars in Sect. 5, for which the stability results of non-rotating stars in [32] do not apply.

Remark 8. The integral terms in (3.90) and (3.91) can be understood as follows; namely for rotating stars, the velocity has no r or z components, so it is natural that these terms be small.

Remark 9. Without the uniqueness assumption for the minimizer of F in W_M, we can have the following type of stability result, as observed in [32] for the non-rotating star solutions. Suppose the assumptions in Theorem 3.3 hold. Let $\mathcal{S}_M$ be the set of all minimizers of F in W_M and $(\rho, \mathbf{v}, \Phi)(x,t)$ be an axi-symmetric weak entropy solution of the Cauchy problem (1.1) and (3.52) satisfying (3.61) and (3.63). Then for every $\epsilon > 0$, there exists a number $\delta > 0$ such that if

$$\inf_{\tilde{\rho}\in\mathcal{S}_M}\left[d(\rho_0,\tilde{\rho}) + \frac{1}{8\pi}||\nabla B\rho_0 - \nabla B\tilde{\rho}||_2^2 + |d_1(\rho_0,\tilde{\rho})|\right] + \frac{1}{2}\int \rho_0(x)(|v_0^r|^2 + |v_0^3|^2)(x)dx < \delta, \tag{3.92}$$

then for every $t > 0$, there is a vertical shift $a(t)\mathbf{e_3}$ ($a \in \mathbb{R}$, $\mathbf{e_3} = (0,0,1)$) such that

$$\inf_{\tilde{\rho}\in\mathcal{S}_M}\left[d(\rho(t), T^{a(t)}\tilde{\rho}) + \frac{1}{8\pi}||\nabla B\rho(t) - \nabla BT^{a(t)}\tilde{\rho}||_2^2 + |d_1(\rho(t), T^{a(t)}\tilde{\rho})|\right] + \frac{1}{2}\int \rho(x,t)(|v^r(x,t)|^2 + |v^3(x,t)|^2)(x)dx < \epsilon, \tag{3.93}$$

where $T^{a(t)}\tilde{\rho}(x) =: \tilde{\rho}(x + a(t)\mathbf{e_3})$. In the case of non-rotating stars, i.e. $L = 0$, the uniqueness of minimizers of the energy functional was proved by Lieb and Yau in [22]. There has been no uniqueness results for the case of rotating stars. It might be expected that this problem can be solved by using some ideas in [22].

The proof of Theorem 3.3 follows from several lemmas. The proofs of these lemmas are similar to those in [26], and therefore we only sketch them. First we have

Lemma 3.8. *Suppose the angular momentum of the rotating star solutions satisfies* (2.9), (3.1) *and* (3.2). *For any $\rho(x) \in W_M$, if*

$$\lim_{r\to 0+} L(m_\rho(r) + m_{\tilde{\rho}}(r))m_\sigma(r)r^{-2} = 0, \tag{3.94}$$

where $\sigma = \rho - \tilde{\rho}$, then

$$d_1(\rho,\tilde{\rho}) \geq 0, \tag{3.95}$$

where d_1 is defined by (3.89).

Proof. For an axi-symmetric function $f(x) = f(r,z)$ ($r = \sqrt{x_1^2 + x_2^2}$, $z = x_3$ for $x = (x_1, x_2, x_3)$), we let

$$\hat{f}(r) = 2\pi r \int_{-\infty}^{+\infty} f(r,z)dz, \tag{3.96}$$

$$m_f(r) = \int_{\{x:\sqrt{x_1^2+x_2^2}\le r\}} f(x)dx = \int_0^r \hat{f}(s)ds, \tag{3.97}$$

so that

$$m'_f(r) = \hat{f}(r). \tag{3.98}$$

In order to show (3.95), we let

$$\sigma(x) = (\rho - \tilde{\rho})(x), \tag{3.99}$$

and for $0 \le \alpha \le 1$, we define

$$\begin{aligned} Q(\alpha) = \frac{1}{2}\int \frac{(\tilde{\rho}+\alpha\sigma)(x)L(m_{\tilde{\rho}+\alpha\sigma}(r(x))) - \tilde{\rho}(x)L(m_{\tilde{\rho}}(r(x)))}{r^2(x)}dx \\ -\alpha \int\int_{r(x)}^{\infty} s^{-3} L(m_{\tilde{\rho}}(s))ds\sigma(x)dx. \end{aligned} \tag{3.100}$$

Then

$$Q(0) = 0,\ \ Q(1) = d_1(\rho,\ \tilde{\rho}). \tag{3.101}$$

By the assumption that $L'(m) \ge 0$ for $0 \le m \le M$ (cf. (3.2)) and (3.94), we can show that

$$Q'(\alpha) = \int_0^{+\infty} \hat{\sigma}(r) \int_r^{\infty} s^{-3}(L(m_{\tilde{\rho}+\alpha\sigma}(s)) - L(m_{\tilde{\rho}}(s)))dsdr, \tag{3.102}$$

and therefore

$$Q(0) = Q'(0) = 0. \tag{3.103}$$

This is done by interchanging the order of integration and integrating by parts (details can be found in [26]). Differentiating (3.103) again and interchanging the order of integration, we get

$$\frac{d^2Q(\alpha)}{d\alpha^2} = \alpha \int_0^{+\infty} s^{-3} L'(m_{\tilde{\rho}+\alpha\sigma}(s))(m_\sigma(s))^2 ds. \tag{3.104}$$

Therefore, if $L'(m) \ge 0$ for $0 \le m \le M$, then

$$\frac{d^2Q(\alpha)}{d\alpha^2} \ge 0,\ \ for\ 0 \le \alpha \le 1. \tag{3.105}$$

This, together with (3.103) and (3.101), yields $d_1(\rho, \tilde{\rho}) = Q(1) \ge 0$. □

Lemma 3.9. *Let* $(\rho, \mathbf{v})$ *be a solution of the Cauchy problem* (1.1) *and* (3.52) *as stated in Theorem* 3.3*, then*

$$\begin{aligned} &E(\rho, \mathbf{v})(t) - F(\tilde{\rho}) \\ &= d(\rho(t), \tilde{\rho}) + d_1(\rho(t), \tilde{\rho}) - \frac{1}{8\pi}||\nabla B\rho(\cdot, t) - \nabla B\tilde{\rho}||_2^2 \\ &+\frac{1}{2}\int \rho(|v^r|^2 + |v^3|^2)(x, t)dx. \end{aligned} \tag{3.106}$$

Proof. From (3.75) and (3.77) in A1), we have, for $x \in G_t = \{x|\rho(x, t) > 0\}$,

$$\begin{aligned} j^2(x, t) &= (j_t(m_{\rho_t}(r(x)))^2 = (j_0(m_{\rho_0}(r(x_0(t))))^2 = L(m_{\rho_0}(r(x_0(t)) \\ &= L(m_{\rho_t}(r(x))). \end{aligned} \tag{3.107}$$

Therefore, by (3.73), we have

$$\begin{aligned} E(\rho(t), \mathbf{v}(t)) &= \int A(\rho)(x, t)dx + \frac{1}{2}\int \frac{\rho(x, t)L(m_{\rho(t)}(r(x))}{r^2(x)}dx \\ &-\frac{1}{8\pi}\int |\nabla B\rho|^2(x, t)dx + \frac{1}{2}\int \rho(|v^r|^2 + |v^3|^2)(x, t)dx. \end{aligned} \tag{3.108}$$

Equation (3.106) follows from (3.108) and the following identities:

$$\begin{aligned} &(||\nabla B\rho(\cdot, t)||_2^2 - ||\nabla B\tilde{\rho}||_2^2) \\ &= ||\nabla(B\rho(\cdot, t)) - \nabla B\tilde{\rho})||_2^2 + 2\int \nabla B\tilde{\rho}(x) \cdot (\nabla B\rho(x, t) - \nabla B\tilde{\rho}(x))dx \\ &= ||\nabla(B\rho(\cdot, t)) - \nabla B\tilde{\rho})||_2^2 - 8\pi\int B\tilde{\rho}(x)(\rho(x, t) - \tilde{\rho}(x))dx, \end{aligned}$$

and

$$\int \rho(x, t)dx = \int \tilde{\rho}(x)dx = M.$$

□

Having established these lemmas, the proof of Theorem 3.3 is similar to the proof of Theorem 3.1 in [26]. We sketch it as follows.

Proof of Theorem 3.3 . Assume the theorem is false. Then there exist $\epsilon_0 > 0$, $t_n > 0$ and initial data $\rho_n(x, 0) \in W_M$ and $\mathbf{v}_n(x, 0)$ such that for all $n \in \mathbb{N}$,

$$\begin{aligned} &d(\rho_n(0), \tilde{\rho}) + d_1(\rho_0, \tilde{\rho}) + \frac{1}{8\pi}||\nabla B\rho_n(0) - \nabla B\tilde{\rho}||_2^2 \\ &+\frac{1}{2}\int \rho_n(x, 0)(|v_n^r(x, 0)|^2 + |v_n^3(x, 0)|^2)(x)dx < \frac{1}{n}, \end{aligned} \tag{3.109}$$

but for any $a(t_n) \in \mathbb{R}$,

$$\begin{aligned} &d(\rho_n(t_n), T^{a(t_n)}\tilde{\rho}) + d_1(\rho_n(t_n), T^{a(t_n)}\tilde{\rho}) + \frac{1}{8\pi}||\nabla B\rho_n(t_n) - \nabla BT^{a(t_n)}\tilde{\rho}||_2^2 \\ &+\frac{1}{2}\int \rho_n(x, t_n)(|v_n^r(x, t_n)|^2 + |v_n^3(x, t_n)|^2)(x)dx \geq \epsilon_0. \end{aligned} \tag{3.110}$$

By (3.106) and (3.109), we have

$$\lim_{n\to\infty} E(\rho_n(0), \mathbf{v}_n(0)) = F(\tilde{\rho}). \tag{3.111}$$

Since $E(\rho_n(t), \mathbf{v}_n(t))$ is non-increasing in time,

$$\lim_{n\to\infty} \sup F(\rho_n(t_n)) \le \lim_{n\to\infty} E(\rho_n(t_n), \mathbf{v}_n(t_n)) \le \lim_{n\to\infty} E(\rho_n(0), \mathbf{v}_n(0)) = F(\tilde{\rho}). \tag{3.112}$$

Therefore $\{\rho_n(\cdot, t_n)\} \subset W_M$ is a minimizing sequence for the functional F. We then can apply Theorem 3.1 to conclude that there exists a sequence $\{a_n\} \subset \mathbb{R}$ such that up to a subsequence,

$$||\nabla(B\rho_n(t_n) - BT^{a_n}\tilde{\rho})||_2 \to 0, \tag{3.113}$$

as $n \to \infty$; this is where we use the assumption that the minimizer is unique up to a vertical shift. By (3.106), the fact that the energy is non-increasing in time, and $F(T^a\rho) = F(\rho)$, we have for any $\rho \in W_M$ and $a \in \mathbb{R}$,

$$\begin{aligned}
&E(\rho_n(t_n), \mathbf{v}_n(t_n)) - F(T^{a_n}\tilde{\rho})\\
&= d(\rho_n(t_n), T^{a_n}\tilde{\rho}) + d_1(\rho(t_n), T^{a_n}\tilde{\rho})\\
&\quad -\frac{1}{8\pi}||\nabla(B\rho_n(t_n) - BT^{a_n}\tilde{\rho})||_2^2\\
&\quad +\frac{1}{2}\int \rho_n(|v_n^r|^2 + |v_n^3|^2)(x, t_n)dx\\
&\le E(\rho_n(0), \mathbf{v}_n(0)) - F(T^{a_n}\tilde{\rho})\\
&= E(\rho_n(0), \mathbf{v}_n(0)) - F(\tilde{\rho}) \to 0,
\end{aligned} \tag{3.114}$$

as $n \to \infty$. Since

$$||\nabla B\rho_n(t_n) - \nabla BT^{a_n}\tilde{\rho}||_2 \to 0,$$

as $n \to \infty$, $d(\rho_n(t_n), \tilde{\rho}) \ge 0$,

$$\begin{aligned}
&d(\rho_n(t_n), T^{a_n}\tilde{\rho}) + d_1(\rho(t_n), T^{a_n}\tilde{\rho})\\
&\quad +\frac{1}{8\pi}||\nabla(B\rho_n(t_n) - T^{a_n}B\tilde{\rho})||_2^2\\
&\quad +\frac{1}{2}\int \rho_n(|v_n^r|^2 + |v_n^3|^2)(x, t_n)dx \to 0,
\end{aligned} \tag{3.115}$$

as $n \to \infty$. This contradicts (3.110), and completes the proof.

4. Applications to White Dwarf and Supermassive Stars

In this section, we want to verify the assumptions (3.4) and (3.5) in Theorem 3.2 for both white dwarfs and supermassive stars. Once we verify (3.4) and (3.5), we can apply Theorems 3.1 and 3.3. We begin with the following theorem which verifies (3.5) for white dwarfs, supermassive stars, and polytropes with $\gamma > 4/3$, in both the rotating and non-rotating cases.

Theorem 4.1. *Assume that the pressure function p satisfies* (3.3). *Then there exists a constant $\mathfrak{M}_c$ satisfying $0 < \mathfrak{M}_c < \infty$ if $\gamma = 4/3$ and $\mathfrak{M}_c = \infty$ if $\gamma > 4/3$, such that if $M < \mathfrak{M}_c$, then* (3.5) *holds for $\rho \in W_M$.*

Proof. Using (3.13), we have, for $\rho \in W_M$,

$$\begin{aligned} F(\rho) &= \int [A(\rho) + \frac{1}{2}\frac{\rho(x)L(m_\rho(r(x)))}{r(x)^2} - \frac{1}{2}\rho B \rho] dx \\ &\geq \int [A(\rho) + \frac{1}{2}\frac{\rho(x)L(m_\rho(r(x)))}{r(x)^2}] dx - C \int \rho^{4/3} dx \left(\int \rho\, dx \right)^{2/3} \\ &= \int [A(\rho) + \frac{1}{2}\frac{\rho(x)L(m_\rho(r(x)))}{r(x)^2}] dx - C M^{2/3} \int \rho^{4/3} dx. \end{aligned} \tag{4.1}$$

Taking $p = 1, q = 4/3, r = \gamma$, and $a = \frac{\frac{3}{4}\gamma - 1}{\gamma - 1}$ (where $\gamma \geq 4/3$ is the constant in (3.3)) in Young's inequality (3.11), we obtain,

$$||\rho||_{4/3} \leq ||\rho||_1^a ||\rho||_\gamma^{1-a} = M^a ||\rho||_\gamma^{1-a}. \tag{4.2}$$

This, together with (3.16)–(3.18) yields

$$\begin{aligned} \int \rho^{4/3} dx &\leq M^{\frac{4}{3}a} (\int \rho^\gamma dx)^b \leq M^{\frac{4}{3}a} \left((\rho^*)^{\gamma-1} M + \alpha \int A(\rho) dx \right)^b \\ &\leq C \left(M^{\frac{4}{3}a + b} (\rho^*)^{1/3} + \alpha M^{\frac{4}{3}a} (\int A(\rho) dx)^b \right), \end{aligned} \tag{4.3}$$

where $b = \frac{1}{3(\gamma-1)}$, α and ρ^* are the constants in (3.17) and we have used the elementary inequality $(x + y)^b \leq C(x^b + y^b)$, for $x,\ y > 0,\ 0 < b < 1$, for some constant C. Therefore, (4.1) and (4.3) imply

$$\begin{aligned} \int [A(\rho) + \frac{1}{2}\frac{\rho(x)L(m_\rho(r(x)))}{r(x)^2}] dx \leq F(\rho) + C\alpha M^{\frac{4}{3}a + \frac{2}{3}} (\int A(\rho) dx)^b \\ + C M^{\frac{4}{3}a + b + \frac{2}{3}} (\rho^*)^{1/3}. \end{aligned} \tag{4.4}$$

If $\gamma > 4/3$, then $0 < b < 1$, if $\gamma = 4/3$, then $b = 1$. Therefore (4.4) implies (3.5). □

The next result shows that (3.4) holds for a wide class of (rotating or non-rotating) stars, including White Dwarfs.

Theorem 4.2. *Suppose that the pressure function p satisfies* (3.3) *and*

$$\lim_{\rho \to 0+} \frac{p(\rho)}{\rho^{\gamma_1}} = \beta, \tag{4.5}$$

for some constants $\gamma_1 > 4/3$ and $0 < \beta < +\infty$, and assume that the angular momentum (per unit mass) satisfies (2.9). *Then there exists $\mathbb{M}_c$ satisfying $0 < \mathbb{M}_c < +\infty$ if $\gamma = 4/3$ and $\mathbb{M}_c = +\infty$ if $\gamma > 4/3$ such that if $M < \mathbb{M}_c$, then* (3.4) *holds, where γ is the constant in* (3.3).

Remark 10. White dwarfs satisfy (3.3) and (4.5) with $\gamma = 4/3$ and $\gamma_1 = 5/3$.

Proof of Theorem 4.2. Due to (3.3) and (4.5), we can apply Theorem 2.1. Let $\hat{\rho}(x) \in W_{M,S}$ be a minimizer of $F(\rho)$ in $W_{M,S}$ as described in Theorem 2.1, and let

$$G = \{x \in \mathbb{R}^3 : \hat{\rho}(x) > 0\}.$$

Then $\bar{G}$ is a compact set in $\mathbb{R}^3$, and $\hat{\rho} \in C^1(G)$. Furthermore, there exists a constant $\mu < 0$ such that

$$\begin{cases} A'(\hat{\rho}(x)) + \int_{r(x)}^{\infty} L(m_{\hat{\rho}}(s)s^{-3}ds - B\hat{\rho}(x) = \mu, & x \in G, \\ \int_{r(x)}^{\infty} L(m_{\hat{\rho}}(s)s^{-3}ds - B\hat{\rho}(x) \geq \mu, & x \in \mathbb{R}^3 - G. \end{cases} \tag{4.6}$$

It follows from [1] that there exists $\hat{\rho} \in W_{M,S} \subset W_M$ such that $F(\hat{\rho}) = \inf_{\rho \in W_{M,S}} F(\rho)$. It is easy to verify that the triple $(\hat{\rho}, \hat{\mathbf{v}}, \hat{\Phi})$ is a time-independent solution of the Euler-Poisson equations (1.1) in the region $G = \{x \in \mathbb{R}^3$: $\hat{\rho}(x) > 0\}$, where $\hat{\mathbf{v}} = (-\frac{x_2 J(m_{\hat{\rho}}(r))}{r}, \frac{x_1 J(m_{\hat{\rho}}(r))}{r}, 0)$ and $\hat{\Phi} = -B\hat{\rho}$. Therefore

$$\nabla_x p(\hat{\rho}) = \hat{\rho}\nabla_x(B\hat{\rho}) + \hat{\rho}L(m_{\hat{\rho}})r(x)^{-3}\mathbf{e}_r, \ x \in G, \tag{4.7}$$

where $\mathbf{e}_r = (\frac{x_1}{r(x)}, \frac{x_2}{r(x)}, 0)$. Moreover, it is proved in [3] that the boundary ∂G of G is smooth enough to apply the Gauss-Green formula on G. Applying the Gauss-Green formula on G and noting that $\hat{\rho}|_{\partial G} = 0$, we obtain,

$$\int_G x \cdot \nabla_x p(\hat{\rho})dx = -3\int_G p(\hat{\rho})dx = -3\int p(\hat{\rho})dx. \tag{4.8}$$

As in [26], we have

$$\int_G x \cdot \hat{\rho}\nabla_x B\hat{\rho}dx = -\frac{1}{2}\int_G \hat{\rho}B\hat{\rho}dx = -\frac{1}{2}\int \hat{\rho}B\hat{\rho}dx. \tag{4.9}$$

Next, since $x \cdot \mathbf{e}_r = r(x)$, we have

$$\begin{aligned} &\int_G x \cdot \hat{\rho}(x)L(m_{\hat{\rho}}(r(x))r^{-3}(x)\mathbf{e}_r dx \\ &= \int_G \hat{\rho}(x)L(m_{\hat{\rho}}(r(x))r^{-2}(x)dx \\ &= \int \hat{\rho}(x)L(m_{\hat{\rho}}(r(x))r^{-2}(x)dx. \end{aligned} \tag{4.10}$$

Therefore, from (4.8)–(4.10) we have

$$-3\int p(\hat{\rho})dx = -\frac{1}{2}\int \hat{\rho}B\hat{\rho}dx + \int \hat{\rho}(x)L(m_{\hat{\rho}}(r(x))r^{-2}(x)dx. \tag{4.11}$$

Let $\bar{\rho}(x) = b^3\hat{\rho}(bx)$, for $b > 0$; then $\bar{\rho} \in W_M$. Also, it is easy to verify that the following identities hold,

$$\begin{aligned} \int \bar{\rho}B\bar{\rho}dx &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\bar{\rho}(x)\bar{\rho}(y)}{|x-y|}dxdy, \\ &= b\int\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\hat{\rho}(x)\hat{\rho}(y)}{|x-y|}dxdy = b\int \hat{\rho}B\hat{\rho}dx \end{aligned} \tag{4.12}$$

$$\int A(\bar{\rho})dx = b^{-3}\int A(b^3\hat{\rho}(x))dx. \tag{4.13}$$

Moreover, for $r \geq 0$,

$$\begin{aligned} m_{\bar{\rho}}(r) &= 2\pi\int_0^r s\int_{-\infty}^{\infty}\bar{\rho}(s,z)dsdz \\ &= 2\pi\int_0^r s\int_{-\infty}^{\infty}\hat{\rho}(bs,bz)dsdz \\ &= 2\pi\int_0^{br} s'\int_{-\infty}^{\infty}\rho(s',z')ds'dz' \\ &= m_\rho(br). \end{aligned} \tag{4.14}$$

Therefore,

$$\begin{aligned} \int\frac{\bar{\rho}(x)L(m_{\bar{\rho}}(r(x)))}{r(x)^2}dx &= \int\frac{b^3\hat{\rho}(x)L(m_{\hat{\rho}}(br(x)))}{r(x)^2}dx \\ &= b^2\int\frac{\hat{\rho}(x)L(m_{\hat{\rho}}(r(x)))}{r(x)^2}dx. \end{aligned} \tag{4.15}$$

It follows from (4.12)–(4.15) that

$$\begin{aligned} F(\bar{\rho}) = b^{-3}&\int A(b^3\hat{\rho})dx - \frac{1}{2}b\int\hat{\rho}B\hat{\rho}dx \\ &+\frac{b^2}{2}\int\frac{\hat{\rho}(x)L(m_{\hat{\rho}}(r(x)))}{r(x)^2}dx. \end{aligned} \tag{4.16}$$

Hence, (4.11) and (4.16) give

$$\begin{aligned} F(\bar{\rho}) = &\int\left(b^{-3}A(b^3\hat{\rho}) - 3bp(\hat{\rho}(x))\right)dx \\ &+\left(\frac{b^2}{2}-b\right)\int\frac{\hat{\rho}(x)L(m_{\hat{\rho}}(r(x)))}{r(x)^2}dx. \end{aligned} \tag{4.17}$$

In view of (2.9), we have

$$\left(\frac{b^2}{2}-b\right)\int\frac{\hat{\rho}(x)L(m_{\hat{\rho}}(r(x)))}{r(x)^2}dx \leq 0, \tag{4.18}$$

if $b > 0$ is small. It follows from (3.9) that

$$\frac{1}{2}\beta\rho^{\gamma_1} \leq p(\rho) \leq 2\beta\rho^{\gamma_1}, \text{ for small } \rho. \tag{4.19}$$

Thus, when b is small, since $\hat{\rho}$ is bounded, we have

$$\frac{\beta}{2(\gamma_1-1)}b^{3\gamma_1}(\hat{\rho})^{\gamma_1}(x) \leq A(b^3\hat{\rho}(x)) \leq \frac{2\beta}{\gamma_1-1}b^{3\gamma_1}(\hat{\rho})^{\gamma_1}(x), \tag{4.20}$$

for $x \in \mathbb{R}^3$. Hence, (4.18) and (4.19) imply

$$\int \left(b^{-3}A(b^3\hat{\rho}) - 3bp(\hat{\rho}(x))\right) dx \\ \leq \beta \int \left(\frac{2}{\gamma_1 - 1}b^{3\gamma_1-3} - \frac{3}{2}\right)(\hat{\rho})^{\gamma_1} dx. \tag{4.21}$$

Since $\gamma_1 > 4/3$, we have $3\gamma_1 - 3 > 1$. Therefore, we conclude that

$$\int \left(b^{-3}A(b^3\hat{\rho}) - 3bp(\hat{\rho}(x))\right) dx < 0, \tag{4.22}$$

for small b. Equation (3.4) follows from (4.17), (4.18) and (4.22). This completes the proof of Theorem 4.2. $\square$

We show next that if the angular momentum distribution is everywhere positive, we may apply the existence theorem of Friedman and Tarkington, [10], to conclude that (3.4) holds with no total mass restriction. This result applies also to White Dwarfs.

Theorem 4.3. *Suppose that the pressure function p satisfies* (3.3) *with $\gamma = 4/3$ and* (3.9) *holds. Assume that the angular momentum (per unit mass) $J(m) = \sqrt{L(m)}$ satisfies* (2.14)*, then* (3.4) *holds for any* $0 < M < +\infty$.

Proof. By the existence theorem in [10], if (2.14) is satisfied, then for any $0 < M < +\infty$, there exists $\tilde{\rho} \in W_{M,S}$ such that $F(\tilde{\rho}) = \inf_{\rho \in W_{M,S}} F(\rho)$. Also, all the properties of $\tilde{\rho}$ in Theorem 2.1 are satisfied. Moreover, the regularity of the boundary ∂G is smooth enough to apply the Gauss-Green formula (cf. [3]). The proof now follows exactly as in Theorem 4.2. $\square$

We finally turn to the case of rotating supermassive stars.

Theorem 4.4. *Consider a supermassive star; i.e.,*

$$p(\rho) = k\rho^{4/3}, \qquad k > 0 \text{ is a constant.} \tag{4.23}$$

If there exists $\hat{\rho} \in W_M$ such that $\hat{\rho} \in C^1(G) \cap C(\mathbb{R}^3)$ and $(\hat{\rho}, \hat{\mathbf{v}}$ is a steady state solution of the Euler-Poisson equation , where $\hat{\mathbf{v}} = (-\frac{x_2\sqrt{L}(m_{\hat{\rho}}(r))}{r}, \frac{x_1\sqrt{L}(m_{\hat{\rho}}(r))}{r}, 0)$, in an open bounded set $G \subset \mathbb{R}^3$ with the Lipschitz boundary ∂G, i.e.,

$$\begin{cases} \nabla_x p(\hat{\rho}) = \hat{\rho}\nabla_x(B\hat{\rho}) + \hat{\rho}L(m_{\hat{\rho}})r(x)^{-3}\mathbf{e}_r, \ x \in G, \\ \hat{\rho} = 0, \qquad x \in \mathbb{R}^3 - G. \end{cases} \tag{4.24}$$

then (3.4) *holds provided L satisfies* (2.9) *and*

$$L(m_0) > 0, \text{ for some } m_0 \in (0, M). \tag{4.25}$$

Remark 11. The existence of $\hat{\rho}$ described above is unknown. The significance of this theorem is that if there exists such a $\hat{\rho}$, which solves the Euler-Poisson equation, together with the induced velocity field $\hat{\mathbf{v}} = (-\frac{x_2\sqrt{L}(m_{\hat{\rho}}(r))}{r}, \frac{x_1\sqrt{L}(m_{\hat{\rho}}(r))}{r}, 0)$, then we can apply the stability theorem, Theorem 3.3.

Proof. Following along the same lines as (4.7)–(4.10), we obtain the same equality as (4.11). Therefore,

$$F(\hat{\rho}) = -\frac{1}{2}\int \hat{\rho}(x)L(m_{\hat{\rho}}(r(x))r^{-2}(x)dx, \tag{4.26}$$

in view of (4.23) and (4.11). Since $\hat{\rho} \in C^1(G) \cap C(\mathbb{R}^3)$ and $\hat{\rho} = 0$ for $x \in \mathbb{R}^3 - G$, it is easy to show that $m_{\hat{\rho}}(r)$ is continuous in r. Moreover, $m_{\hat{\rho}}(0) = 0$ and $m_{\hat{\rho}}(R) = M$, where $R = \max_{x\in\bar{G}}(r(x)$. Therefore, there exists $r_0 \in (0, M)$ such that

$$m_{\hat{\rho}}(r_0) = m_0, \tag{4.27}$$

where m_0 is the constant in (4.25). Thus,

$$L(m_{\hat{\rho}}(r_0)) > 0, \tag{4.28}$$

in view of (4.25). Since $m_{\hat{\rho}}(r)$ is continuous in r and $L(m)$ is continuous in m, we conclude that

$$\int \hat{\rho}(x)L(m_{\hat{\rho}}(r(x))r^{-2}(x)dx > 0. \tag{4.29}$$

The inequality (3.4) now follows from (4.26)). □

The preceding theorems, together with Theorem 3.3 show that polytropes ($p(\rho) = k\rho^{\gamma}$) with $\gamma > 4/3$ and White Dwarf stars, in both the rotating and non-rotating cases, as well as rotating supermassive stars are dynamically stable. Moreover, if the angular momentum distribution is not everywhere positive and the pressure p behaves asymptotically near infinity like $\rho^{4/3}$, then dynamic stability holds only under a (Chandrasekhar) mass restriction, $M \le M_c$.

5. Nonlinear Dynamical Stability of Non-Rotating White Dwarf Stars With General Perturbations

The dynamical stability results in Sect. 3 apply for axi-symmetric perturbations. Also, for the stability of rotating stars, Assumptions A1), A2) and I2), I3) are made in Theorem 3.3 to control the angular momentum. Moreover, the uniqueness of minimizers of the energy functional for rotating stars is not known. However, uniqueness for non-rotating stars was proved by Lieb and Yau in [22]. In this section, we prove a very general nonlinear dynamical stability for non-rotating white dwarf stars without Assumptions A1), A2) and I2), I3), and for general perturbations. For white dwarf stars, as mentioned before, the pressure function satisfies

$$p \in C^1[0, +\infty),\ \lim_{\rho\to 0+}\frac{p(\rho)}{\rho^{\gamma_1}} = \beta,\ \lim_{\rho\to\infty}\frac{p(\rho)}{\rho^{\gamma}} = K,\ p'(\rho) > 0 \quad \text{for } \rho > 0, \tag{5.1}$$

where $\gamma_1 > 4/3$, $0 < \beta < +\infty$ and $0 < K < +\infty$ are constants. In this section, we always assume that the pressure function satisfies (5.1). First, we define for $0 < M < +\infty$,

$$X_M = \{\rho : \mathbb{R}^3 \to \mathbb{R}, \rho \ge 0, a.e., \int \rho(x)dx = M,$$
$$\int [A(\rho(x)) + \frac{1}{2}\rho(x)B\rho(x)]dx < +\infty\}, \tag{5.2}$$

where $A(\rho)$ is the function given in (2.5). For $\rho \in X_M$, we define the **energy functional** G for non-rotating stars by

$$G(\rho) = \int [A(\rho(x)) - \frac{1}{2}\rho(x)B\rho(x)]dx. \tag{5.3}$$

We begin with the following theorem.

Theorem 5.1. *Suppose that the pressure function p satisfies (5.1). Let $\tilde{\rho}_N$ be a minimizer of the energy functional G in X_M and let*

$$\Gamma_N = \{x \in \mathbb{R}^3 : \ \tilde{\rho}_N(x) > 0\}, \tag{5.4}$$

then there exists a constant λ_N such that

$$\begin{cases} A'(\tilde{\rho}_N(x)) - B\tilde{\rho}_N(x) = \lambda_N, & x \in \Gamma_N, \\ -B\tilde{\rho}_N(x) \geq \lambda_N, & x \in \mathbb{R}^3 - \Gamma_N. \end{cases} \tag{5.5}$$

The proof of this theorem is well-known, cf. [32] or [1].

Remark 12. 1) We call the minimizer $\tilde{\rho}_N$ of the functional G in X_M a non-rotating star solution.
2) It follows from [22] that the minimizer $\tilde{\rho}_N$ of the functional G in X_M is actually radial, and has a compact support.

Similar to Theorem 3.1, we have the following compactness theorem.

Theorem 5.2. *Suppose that the pressure function p satisfies (5.1). There exists a constant M^c ($0 < M^c < \infty$) such that if $M < M^c$, then the following hold:*

(1)

$$\inf_{\rho \in X_M} G(\rho) < 0, \tag{5.6}$$

(2) *for $\rho \in X_M$,*

$$\int A(\rho)(x)dx \leq C_1 G(\rho) + C_2, \tag{5.7}$$

for some positive constants C_1 and C_2,

(3) *if $\{\rho^i\} \subset X_M$ is a minimizing sequence for the functional G, then there exist a sequence of translations $\{x^i\} \subset \mathbb{R}^3$, a subsequence of $\{\rho^i\}$, (still labeled $\{\rho^i\}$), and a function $\tilde{\rho}_N \in X_M$, such that for any $\epsilon > 0$ there exists $R > 0$ with*

$$\int_{|x| \geq R} T\rho^i(x)dx \leq \epsilon, \quad i \in \mathbb{N}, \tag{5.8}$$

and

$$T\rho^i(x) \rightharpoonup \tilde{\rho}_N, \ weakly\ in\ L^{4/3}(\mathbb{R}^3), \ as\ i \to \infty, \tag{5.9}$$

where $T\rho^i(x) := \rho^i(x + x^i)$.
Moreover

(4)

$$\nabla B(T\rho^i) \to \nabla B(\tilde{\rho}_N)\ strongly\ in\ L^2(\mathbb{R}^3),\ as\ i \to \infty, \tag{5.10}$$

and

(5) $\tilde{\rho}_N$ *is a minimizer of* G *in* X_M.

(6) *The minimizers of* G *in* X_M *are unique up to a translation* $\rho_N(x) \to \rho_N(x+y)$.

Proof. First, the proofs of (1) and (2) are the same as Theorems 4.1 and 4.2 by taking $L = 0$ (it is easy to check the axial symmetry is not used in the proof of Theorems 4.1 and 4.2 if $L = 0$). Lemmas 3.4, 3.5 and 3.7 still hold by taking $\gamma = 4/3$ and $L = 0$, and replacing W_M by X_M, F by G and f_M by $\inf_{\rho\in X_M} G(\rho)$. Also, it is easy to check that (3.25)–(3.29) in the proof of Lemma 3.6 still hold by replacing f_M by $\inf_{\rho\in X_M} G(\rho)$. Therefore, following the proof of Lemma 3.6, we conclude:
If $\{\rho^i\} \subset X_M$ is a minimizing sequence for G, then there exists constant $\delta_0 > 0$, $i_0 \in \mathbf{N}$ and $x^i \in \mathbb{R}^3$, such that

$$\int_{B_1(x^i)} \rho^i(x)dx \geq \delta_0,\ i \geq i_0.$$

Therefore, if we let

$$T\rho^i(x) := \rho^i(x + x^i), \tag{5.11}$$

then

$$\int_{B_1(0)} T\rho^i(x)dx \geq \delta_0,\ i \geq i_0.$$

This is similar to (3.39). Having established this inequality and the other analogues of Lemmas 3.4, 3.5 and 3.7, we can prove this theorem in a similar manner as the proof of Theorem 3.1. The uniqueness of minimizers is proved in [22]. □

For the stability, we consider the Cauchy problem (1.1) with the initial data (3.53). We *do not* assume that the initial data have any symmetry.

Let $\tilde{\rho}_N$ be the minimizer of G on X_M and λ_N be the constant in (5.5). For $\rho \in X_M$, we define

$$\begin{aligned} d(\rho, \tilde{\rho}_N) &= \int \{[A(\rho) - A(\tilde{\rho}_N)] - (\rho - \tilde{\rho}_N)(\lambda_N + B\tilde{\rho}_N\}dx, \\ &= \int \{[A(\rho) - A(\tilde{\rho}_N)] - B\tilde{\rho}_N(\rho - \tilde{\rho}_N)\}dx, \end{aligned} \tag{5.12}$$

where we have used the identity

$$\int \rho dx = \int \tilde{\rho}_N dx = M,$$

for $\rho \in X_M$. By a similar argument as (3.86)–(3.88), we have

$$d(\rho, \tilde{\rho}_N) \geq 0, \tag{5.13}$$

for any $\rho \in X_M$, in view of (4.6). Our nonlinear stability theorem of non-rotating white dwarf star solutions is the following theorem, which extends the results in [32].

Theorem 5.3. *Suppose that the pressure function satisfies* (5.1). *Let $\tilde{\rho}_N$ be the minimizer of the functional G in X_M. Let $(\rho, \mathbf{v}, \Phi)(x, t)$ be an entropy weak solution of the Cauchy problem* (1.1) *and* (3.52) *stated in Theorem* 3.2 *satisfying* (3.61) *and* (3.63). *If the initial data satisfies*

$$\int \rho_0(x) = \int \rho_N(x)dx = M,$$

then there exists a constant M^c $(0 < M^c < \infty)$ such that if $M < M^c$, then for every $\epsilon > 0$, there exists a number $\delta > 0$ such that if

$$d(\rho_0, \tilde{\rho}_N) + \frac{1}{8\pi}||\nabla B\rho_0 - \nabla B\tilde{\rho}_N||_2^2 + \frac{1}{2}\int \rho_0(x)(|v_0|^2)(x)dx < \delta, \tag{5.14}$$

then for every $t > 0$, there is a translation $y(t) \in \mathbb{R}^3$ such that,

$$d(\rho(t), T^{y(t)}\tilde{\rho}_N)+\frac{1}{8\pi}||\nabla B\rho(t)-\nabla BT^{y(t)}\tilde{\rho}_N||_2^2+\frac{1}{2}\int \rho(x,t)|v(x,t)|^2)dx<\epsilon, \tag{5.15}$$

where $T^{y(t)}\tilde{\rho}_N(x) =: \tilde{\rho}_N(x + y(t))$.

The proof of this theorem follows from the compactness result (Theorem 5.2), and the arguments as in the proof of Theorem 3.3 and in [32], and is thus omitted.

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