Commun. Math. Phys. 284, 391–406 (2008)
Digital Object Identifier (DOI) 10.1007/s00220-008-0568-4

Communications in **Mathematical Physics**

Invariant Measures Satisfying an Equality Relating Entropy, Folding Entropy and Negative Lyapunov Exponents*

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Received: 10 October 2007 / Accepted: 30 March 2008
Published online: 18 July 2008 –

Abstract: In this paper we prove that, for a C^2 (non-invertible but non-degenerate) map on a compact manifold, an invariant measure satisfies an equality relating entropy, folding entropy and negative Lyapunov exponents if and, under a condition on the Jacobian of the map, only if the measure has absolutely continuous conditional measures on the stable manifolds.

1. Introduction

Let M be a connected compact Riemannian manifold without boundary, $f : M \to M$ a C^2 non-invertible map and μ an f-invariant measure. The entropy production $e_\mu(f)$ of the dynamical system (f, μ) is defined by Ruelle [10] as

$$e_\mu(f) := F_\mu(f) - \int \log|\det T_x f|\, d\mu,$$

where $F_\mu(f) := H_\mu(\epsilon \mid f^{-1}\epsilon)$ with ϵ being the partition of M into single points and it is called the *folding entropy* of (f, μ). Let $h_\mu(f)$ be the (measure-theoretic) entropy of (f, μ) and, for μ-a.e. x, let $-\infty \le \lambda_1(x) < \lambda_2(x) < \cdots < \lambda_{r(x)}(x) < +\infty$ be the Lyapunov exponents of f at x with $m_i(x)$ denoting the multiplicity of $\lambda_i(x)$. Under a set of conditions on degenerate points of the map, Liu [3] proved the following inequality conjectured by Ruelle [10]:

$$h_\mu(f) \le F_\mu(f) - \int \sum_i \lambda_i(x)^- m_i(x)\, d\mu \tag{1.1}$$

* This work is supported by National Basic Research Program of China (973 Program) (2007CB814800).

(where $a^- := \min\{a, 0\}$). When μ satisfies the Pesin entropy formula

$$h_\mu(f) = \int \sum_i \lambda_i(x)^+ m_i(x)\, d\mu \tag{1.2}$$

with $a^+ := \max\{a, 0\}$ (such a measure μ is sometimes called an SRB measure), one obtains the non-negativity of the entropy production since

$$\begin{aligned} e_\mu(f) &= F_\mu(f) - \int \log|\det T_x f|\, d\mu \\ &= F_\mu(f) - \int \sum_i \lambda_i(x)^- m_i(x)\, d\mu - h_\mu(f) \\ &\quad + h_\mu(f) - \int \sum_i \lambda_i(x)^+ m_i(x)\, d\mu \\ &\geq 0. \end{aligned}$$

In this article, we further investigate the question when $e_\mu(f) = 0$ or $e_\mu(f) > 0$. We show that, when f has no degenerate points, the formula

$$h_\mu(f) = F_\mu(f) - \int \sum_i \lambda_i(x)^- m_i(x)\, d\mu \tag{1.3}$$

holds if and, under a somewhat restrictive condition on the Jacobian of (f, μ), only if μ has absolutely continuous conditional measures on the stable manifolds of (f, μ).

This paper is organized in the following way. Section 2 is devoted to the definitions and statement of the results. The rest of the sections are devoted to the proofs.

2. Definitions and Statement of the Results

Let $f : M \to M$ be a C^2 non-invertible map such that $T_x f$ is non-degenerate at every $x \in M$ (i.e. $\det T_x f \neq 0$ at every $x \in M$), and let μ be an invariant measure of f. Choose a Borel set Λ such that $\mu(\Lambda) = 1$, $f\Lambda \subset \Lambda$ and every point $x \in \Lambda$ is regular in the sense of Oseledec, that is, there exist a sequence of subspaces of $T_x M$,

$$\{0\} = V_0(x) \subset V_1(x) \subset \cdots \subset V_{r(x)}(x) = T_x M,$$

such that

$$\lim_{n\to+\infty} \frac{1}{n} \log|T_x f^n v| = \lambda_i(x)$$

for all $v \in V_i(x)\setminus V_{i-1}(x)$, $1 \leq i \leq r(x)$.

Set $I = \{x \in \Lambda : \lambda_i(x) \geq 0 \text{ for all } 1 \leq i \leq r(x)\}$ and $\Delta = \Lambda\setminus I$. For $x \in I$, define $W^s(x) = \{x\}$. For $x \in \Delta$, define

$$W^s(x) = \{y \in M : \limsup_{n\to+\infty} \frac{1}{n} \log d(f^n y, f^n x) < 0\}$$

($\log 0 := -\infty$) and call it the *stable manifold* of f at x. The arguments in Liu and Qian [5, Sects. III.1-III.3] restricted to a deterministic map show that, for μ-a.e. $x \in \Delta$, there exist a sequence of $C^{1,1}$ embedded k-dimensional discs $\{W(f^n x)\}_{n=0}^{+\infty}$ ($k = \dim E^s(x)$ and $E^s(x) = \bigcup_{\lambda_i(x)<0} V_i(x)$) such that $fW(f^n x) \subset W(f^{n+1}x)$ for all $n \geq 0$ and

$$W^s(x) = \bigcup_{n=0}^{+\infty} f^{-n} W(f^n x).$$

Let $V^s(x)$ denote the arc connected component of $W^s(x)$ which contains x. It is a $C^{1,1}$ immersed submanifold of M.

Let $\mathcal{B}_\mu(M)$ denote the completion of the Borel σ-algebra of M with respect to μ so that $(M, \mathcal{B}_\mu(M), \mu)$ constitutes a Lebesgue space.

Definition 2.1. *A measurable partition ξ of $(M, \mathcal{B}_\mu(M), \mu)$ is said to be subordinate to the W^s-manifolds of (f, μ) if for μ-a.e. x one has $\xi(x) \subset W^s(x)$ ($\xi(x)$ denotes the element of ξ which contains x) and $\xi(x)$ contains an open neighborhood of x in $V^s(x)$ (with respect to the submanifold topology of $V^s(x)$).*

Definition 2.2. *μ is said to have absolutely continuous conditional measures (abbreviated as accm) on the stable manifolds if for every measurable partition ξ subordinate to the W^s-manifolds of (f, μ) one has for μ-a.e. x,*

$$\mu_x^\xi \ll \lambda_x^s$$

where μ_x^ξ is the conditional measure of μ on $\xi(x)$ and λ_x^s denotes the Lebesgue measure on $W^s(x)$ induced by its inherited Riemannian structure as a submanifold of M ($\lambda_x^s := \delta_x$ if $W^s(x) = \{x\}$).

Theorem 2.3 (Sufficiency). *Let (f, μ) be as given at the beginning of Sect. 2. If μ has accm on the stable manifolds, then the equality (1.3) holds true.*

Remark 2.4. If f has no negative Lyapunov exponents at μ-a.e. x, one has

$$h_\mu(f) = F_\mu(f).$$

This follows directly from the inequality (1.1) and the fact that $h_\mu(f) \geq H_\mu(\epsilon | f^{-1}\epsilon) = F_\mu(f)$.

In order to prove that μ having accm on W^s-manifolds is necessary for the equality (1.3), we make further assumptions. Recall now the notion of Jacobian of measure-preserving transformations (Parry [7]). Let $T : (X, \mathcal{A}, \nu) \to (Y, \mathcal{B}, \rho)$ be a measure-preserving transformation between two probability spaces. Assume that there is a countable partition of X (ν-mod 0) into measurable sets $\alpha = \{A_i\}$ such that for each A_i the map $T_i := T|_{A_i} : A_i \to Y$ is *absolutely continuous* (with respect to ν and ρ), i.e.,

(i) T_i is injective;
(ii) $T_i A$ is measurable if A is a measurable subset of A_i;
(iii) $\rho(T_i A) = 0$ if $A \subset A_i$ is measurable and $\nu(A) = 0$.

(i) and (ii) allow us to define a measure ν_{T_i} on each A_i by $\nu_{T_i}(A) := \rho(T_i A)$ for measurable $A \subset A_i$. By (iii), $\nu_{T_i} \ll \nu$. Define

$$J_T(x) = \frac{d\nu_{T_i}}{d\nu}(x) \quad \text{if } x \in A_i.$$

Clearly the definition of J_T is ν-mod 0 independent of the choice of the partition α. J_T is called the *Jacobian* of T. It is clear that

$$J_T(x) \geq 1, \quad \nu - \text{a.e. } x. \tag{2.1}$$

When $(X, \mathcal{A}, \nu)$ and $(Y, \mathcal{B}, \rho)$ are both Lebesgue spaces and ϵ is the partition of Y into single points, Parry [7, Lemma 10.5] gives a very useful property of the Jacobian

$$-\log \nu_x^{T^{-1}\epsilon}(\{x\}) = \log J_T(x), \quad \nu - \text{a.e. } x. \tag{2.2}$$

For a C^1 measure-preserving map $g : (M, \mathcal{B}(M), \nu) \hookleftarrow$ with $\nu(\Sigma) = 0$, where $\Sigma = \{x \in M : \det T_x g = 0\}$, it is always possible to define the Jacobian of g on a measurable set of full ν-measure. In fact, since $T_x g$ is non-degenerate for any $x \in M \backslash \Sigma$, a countable Borel partition $\alpha = \{A_i\}$ of $M \backslash \Sigma$ satisfying (i)–(ii) above clearly exists. Let

$$\Gamma = \{x \in M \backslash \Sigma : \nu_x^{g^{-1}\epsilon}(\{x\}) > 0\}.$$

Clearly $\nu(\Gamma) = 1$ and it is easy to check that $g|_{A_i \cap \Gamma} : A_i \cap \Gamma \to M$ and ν satisfy (iii) for each A_i and hence J_g is well defined on Γ. Moreover, if $\nu(\Sigma \cup g\Sigma) = 0$, then, since g preserves ν, one has for ν-a.e. $y \in M$,

$$\sum_{z : gz = y} \frac{1}{J_g(z)} = 1. \tag{2.3}$$

We now make an assumption on the Jacobian of $f : (M, \mu) \hookleftarrow$ which seems rather restrictive.

(H) There is a Hölder continuous function $J_f : M \to [1, +\infty)$ such that

$$\mu(fB) = \int_B J_f(y)\, d\mu(y) \tag{2.4}$$

for any Borel $B \subset M$ which is so that $f : B \to fB$ is injective.

Assumption (H) clearly implies that (2.3) is true for every $y \in M$. Actually we need the following weaker conditions.

(H)' For μ-a.e. x, $J_f(y)$ is well defined on $V^s(x)$, $\prod_{k=0}^{+\infty} \frac{J_f(f^k x)}{J_f(f^k y)}$ converges and is bounded away from 0 and $+\infty$ on any given neighborhood of x in $V^s(x)$ whose d^s-diameter is finite, where d^s is the distance along $V^s(x)$; moreover, (2.3) is true λ_x^s almost everywhere on $V^s(x)$.

Assumption (H) clearly implies (H)'. The author does not know how often (H)' is satisfied, but it is an almost necessary condition for μ having accm on the W^s-manifolds (see Subsect. 4.1). In some particular cases, $J_f = l$ is constant everywhere, where $l = \# f^{-1}\{x\}$ for any $x \in M$.

Example 2.5. Let $f : M \to M$ be a C^1 map so that $T_x f$ is non-degenerate for every $x \in M$. Let $l = \# f^{-1}\{x\}$ for all $x \in M$. Take $x_0 \in M$. Let μ_k be the probability so that $\mu_k(\{z\}) = \frac{1}{l^k}$ for any $z \in f^{-k}\{x_0\}$. Let μ be any weak limit point of $\{\frac{1}{n} \sum_{k=0}^{n-1} \mu_k\}_{n \geq 0}$. Then μ is an f-invariant measure and $f : (M, \mu) \hookleftarrow$ has constant Jacobian $J_f = l$ which satisfies (2.4).

Theorem 2.6 (Necessity). *Let (f, μ) be as given at the beginning of Sect. 2 and assume (H) or (H)'. If (1.3) holds true, then μ has accm on the stable manifolds.*

Guess 2.7. If μ is hyperbolic (that is, $\lambda_i(x) \neq 0, 1 \le i \le r(x)$ for μ-a.e. x), has property (H) or (H)' and satisfies both formulae (1.2) and (1.3), then μ is absolutely continuous with respect to the Lebesgue measure on M.

Note that, when μ satisfies the Pesin formula (1.2), the entropy production $e_\mu(f) = 0$ if and only if μ satisfies the formula (1.3). Hence, Guess 2.7 implies that, if μ is moreover hyperbolic and satisfies (H) or (H)', $e_\mu(f) = 0$ if (see Remark 2.9 below) and only if μ is absolutely continuous with respect to the Lebesgue measure on M.

Remark 2.8. The referee indicated to the author the following outline of an argument which *hopefully* can confirm that Guess 2.7 is true (a rigorous proof is, however, still lacking). In the natural extension or inverse limit system $(\bar{f}, \bar{\mu})$ of (f, μ), $\bar{\mu}$ has absolutely continuous conditional measures along the unstable manifolds ([8]). On M, the stable foliation is absolutely continuous. To describe the transverse measure, one should take a transversal T and project μ on T along local stable leaves. By projection from $\bar{\mu}$, the measure μ is an average of the projections from the natural extension of the conditional measures on unstable manifolds. Each of these projections is an absolutely continuous measure on the projection of the corresponding unstable manifold, which is transversal to the stable foliation. By absolute continuity of the stable foliation, these are carried to T into an absolutely continuous measure on T. The transversal measure on T, which is an average of these last ones, is also absolutely continuous. Now, by Theorem 2.6, the measure μ has moreover absolutely continuous conditional measures on the stable manifolds. Pairing with an absolutely continuous measure on transversals yields an absolutely continuous measure on M.

Remark 2.9. For any C^2 measure-preserving map $f : (M, \mu) \hookleftarrow$ (possibly with degenerate points), if μ is absolutely continuous with respect to the Lebesgue, then μ satisfies both formulae (1.2) and (1.3) and hence $e_\mu(f) = 0$. In fact, μ satisfying (1.2) is proved in Liu [4]. On the other hand,

$$J_f(x) = \frac{\phi(fx)}{\phi(x)} |\det T_x f|, \quad \mu\text{-a.e. } x,$$

where $\phi = d\mu/d\mathrm{Leb}$. By (2.2),

$$-\log \mu_x^{f^{-1}\epsilon}(\{x\}) = \log J_f(x), \quad \mu\text{-a.e. } x.$$

By Liu [4], $\log |\det T_x f| \in L^1(M, \mu)$ (this follows from the fact that $\mu \ll$Leb.) and $\log \frac{\phi\circ f}{\phi} d\mu = 0$. Hence

$$\begin{aligned} H_\mu(\epsilon | f^{-1}\epsilon) &= \int -\log \mu_x^{f^{-1}\epsilon}(\{x\}) d\mu(x) \\ &= \int \log \frac{\phi(fx)}{\phi(x)} d\mu + \int \log |\det T_x f| d\mu \\ &= \int \sum_i \lambda_i(x)^- m_i(x)\, d\mu + \int \sum_i \lambda_i(x)^+ m_i(x)\, d\mu, \end{aligned}$$

and thus

$$F_\mu(f) - \int \sum_i \lambda_i(x)^- m_i(x)\, d\mu = \int \sum_i \lambda_i(x)^+ m_i(x)\, d\mu = h_\mu(f).$$

3. Proof of Theorem 2.3

By Liu [3], (1.1) holds true for (f, μ) since we assume here that f has no degenerate points. It remains to prove

$$h_\mu(f) \geq F_\mu(f) - \int \sum_i \lambda_i(x)^- m_i(x)\, d\mu. \tag{3.1}$$

With a little modification of the sets chosen in Sect. 2, take a Borel set Λ with $\mu(\Lambda) = 1$, $f\Lambda \subset \Lambda$ and $\Lambda = I \cup \Delta$ so that, for every $x \in I$, $\lambda_i(x) \geq 0$ for all i and $W^s(x) = \{x\}$, and, for every $x \in \Delta$, $\lambda_1(x) < 0$ and $W^s(x) = \bigcup_{n=0}^{+\infty} f^{-n} W(f^n x)$. We may take Δ so that $W^s(x) \subset \Delta$ for every $x \in \Delta$.

Lemma 3.1. *There exists a measurable partition η of $(\Delta, \mu|_\Delta)$ which has the following properties:*

(1) $f^{-1}\eta \leq \eta$ (meaning that $(f^{-1}\eta)(x) \supset \eta(x)$ for μ-a.e. $x \in \Delta$);
(2) η is subordinate to the W^s-manifolds of (f, μ);
(3) for every Borel set B the function

$$P_B(x) = \lambda_x^s(\eta(x) \cap B)$$

is measurable and is μ almost everywhere finite on Δ.

The proof of this lemma is omitted here since it is almost the same as that of Liu and Qian [5, Prop. IV.2.1] restricted to the case of a deterministic map.

Property (3) just above allows one to define a σ-finite Borel measure λ^* on Δ by

$$\lambda^*(B) = \int \lambda_x^s(\eta(x) \cap B)\, d\mu$$

for each Borel $B \subset \Delta$. From the assumption of μ having accm on W^s-manifolds it follows that $\mu \ll \lambda^*$. Put

$$h = \frac{d\mu}{d\lambda^*}.$$

By arguments similar to Ledrappier and Strelcyn [1, Prop. 4.1] or [5, Prop. IV.2.2] we have for μ-a.e. $x \in \Delta$,

$$h = \frac{d\mu_x^\eta}{d\lambda_x^s} \quad \lambda_x^s \text{ almost everywhere on } \eta(x). \tag{3.2}$$

Let $x \in \Delta$ and consider the measure-preserving map between Lebesgue spaces

$$f_x := f|_{(f^{-1}\eta)(x)} : \left((f^{-1}\eta)(x),\ \mu_x^{f^{-1}\eta}\right) \longrightarrow \left(\eta(fx),\ \mu_{fx}^\eta\right).$$

Since $T_x f$ is non-degenerate at every $x \in M$ and $\mu_x^{f^{-1}\eta} \ll \lambda_x^s$ and $\mu_{fx}^\eta \ll \lambda_{fx}^s$, we know that f_x admits a Jacobian which, using (3.2), is given by

$$J_{f_x}(z) = \frac{1}{\mu_x^{f^{-1}\eta}(\eta(z))} \cdot \frac{h(fz)}{h(z)} \cdot |\det(T_z f|_{E^s(z)})|$$

for $\mu_x^{f^{-1}\eta}$-a.e. $z \in (f^{-1}\eta)(x)$. With a bit of abuse of notations, let ϵ be the partition of $\eta(x)$ into single points. By (2.2), for $\mu_x^{f^{-1}\eta}$-a.e. $z \in (f^{-1}\eta)(x)$,

$$-\log(\mu_x^{f^{-1}\eta})_z^{(f_x)^{-1}\epsilon}(\{z\}) = \log J_{f_x}(z)$$

which, by the transitivity of conditional measures, implies that for μ-a.e. $z \in \Delta$,

$$-\log \mu_z^{f^{-1}\epsilon}(\{z\}) = -\log \mu_z^{f^{-1}\eta}(\eta(z)) + \log \frac{h(fz)}{h(z)} + \log|\det(T_z f|_{E^s(z)})|. \quad (3.3)$$

Since $\int -\log \mu_z^{f^{-1}\epsilon}(\{z\})d\mu(z) \leq l$ (to recall, $l = \#f^{-1}\{x\}$ for all $x \in M$), we know that the left hand of (3.3) is μ-integrable. Since, by Ruelle inequality [11], $h_\mu(f) < +\infty$, we know that $-\log \mu_z^{f^{-1}\eta}(\eta(z)) \in L^1(\mu)$ since $H_\mu(\eta|f^{-1}\eta) \leq h_\mu(f)$. The last term in the right side of (3.3) is clearly integrable. Thus

$$\int \log \frac{h(fz)}{h(z)} d\mu(z) = 0.$$

Taking integration of the two sides of (3.3), we have

$$\int_\Delta -\log \mu_z^{f^{-1}\epsilon}(\{z\})d\mu = \int_\Delta -\log \mu_z^{f^{-1}\eta}(\eta(z))d\mu + \int_\Delta \sum_i \lambda_i(z)^- m_i(z)\, d\mu.$$

Letting $\eta = \epsilon$ on I, the above equality clearly holds true with Δ being replaced by I. Thus

$$H_\mu(\epsilon|f^{-1}\epsilon) = H_\mu(\eta|f^{-1}\eta) + \int_M \sum_i \lambda_i(z)^- m_i(z)\, d\mu,$$

which implies

$$h_\mu(f) \geq H_\mu(\eta|f^{-1}\eta) = F_\mu(f) - \int_M \sum_i \lambda_i(z)^- m_i(z)\, d\mu.$$

4. Proof of Theorem 2.6

In this section we largely use the strategy of Ledrappier and Young [2] which deals with unstable manifolds of diffeomorphisms. Our maps under consideration are non-invertible and unstable manifolds can not be defined for the system $f : (M, \mu) \hookleftarrow$ (but can be defined for its inverse limit system). We deal with stable manifolds and use the Jacobian and the inverse limit space.

4.1. Increasing partitions subordinate to W^s-manifolds and the necessity. We first assume that (f, μ) is ergodic. Let $-\infty < \lambda_1 < \lambda_2 < \cdots < \lambda_r < +\infty$ be the Lyapunov exponents of (f, μ) with m_i being the multiplicity of λ_i. If $\lambda_1 \geq 0$, $W^s(x) = \{x\}$ for μ-a.e. $x \in M$ and the conditional measure of μ on $W^s(x)$ is δ_x, Theorem 2.6 is trivial in this case (cf. Remark 2.4). We will assume that $\lambda_1 < 0$ and $\lambda_1 < \lambda_2 < \cdots < \lambda_s < 0$ are all the negative exponents.

Let $\bar{M} = \{\bar{x} = (\cdots, x_{-1}, x_0, x_1, \cdots) : x_i \in M, fx_i = x_{i+1}, i \in \mathbf{Z}\}$ be the inverse limit space of (M, f), $\pi : \bar{M} \to M, \bar{x} \mapsto x_0$ the natural projection, $\tau : \bar{M} \to \bar{M}$ the left shift transformation, and $\bar{\mu}$ the unique τ-invariant measure such that $\pi\bar{\mu} = \mu$.

Proposition 4.1.1. *There exists a measurable partition ξ of $(M, \mathcal{B}_\mu(M), \mu)$ with the following properties:*

(1) $f^{-1}\xi \leq \xi$, and ξ is subordinate to the W^s-manifolds of (f, μ);
(2) $\bigvee_{n=0}^{+\infty} \tau^n(\pi^{-1}\xi)$ is the partition of $\bar{M}$ into single points.

This result and its proof are similar to Lemma 3.1 (see [5, Prop. IV.2.1] for details). We will give an outline of the proof here since it produces certain additional properties of the partition that will be useful in the next subsection. This is similar to [2, Lemma 3.1.1].
Outline of construction. There is a measurable set S with the following properties:

(a) $\mu(S) > 0$;
(b) S is the disjoint union of a continuous family of embedded discs $\{D_\alpha\}$ where each D_α is an open neighborhood of x_α in $V^s(x_\alpha)$;
(c) For μ-a.e. x, there is an open neighborhood U_x of x in $V^s(x)$ such that, for each $n \geq 0$, either $f^n U_x \cap S = \emptyset$ or $f^n U_x \subset D_\alpha$ for some α;
(d) There is $\gamma > 0$ such that: i) the d^s-diameter of every D_α in S is less than γ; ii) if $x, y \in S$ are such that $y \in V^s(x)$ and $d^s(x, y) > \gamma$, then x, y lie on distinct D_α-discs.

Let $\hat{\xi}$ be the partition of M defined by

$$\hat{\xi}(x) = \begin{cases} D_\alpha & \text{if } x \in D_\alpha, \\ M \backslash S & \text{if } x \notin S. \end{cases}$$

Then $\xi := \hat{\xi}^- = \bigvee_{n=0}^{+\infty} f^{-n}\hat{\xi}$ is the partition we desire. □

The partitions whose construction is just outlined have the following alternate characterization: There is a set S satisfying (a)–(d) such that, if $\sigma = \bigvee_{n=0}^{+\infty} f^{-n}\{S, M\backslash S\}$, then, for every $x \in M$, $y \in \xi(x)$ if and only if $y \in \sigma(x)$ and $d^s(f^n y, f^n x) \leq \gamma$ whenever $f^n x \in S$.

Proposition 4.1.2. *Let ξ be a partition given in Proposition 4.1.1. Then*

$$h_\mu(f) = H_\mu(\xi | f^{-1}\xi).$$

A discussion of the proof of this proposition will be given in Subsect. 4.2. We first show

$$H_\mu(\xi|f^{-1}\xi) = F_\mu(f) - \sum_i \lambda_i^- m_i \implies \mu_x^\xi \ll \lambda_x^s \text{ for } \mu\text{−a.e. } x. \tag{4.1.1}$$

Let $D^s(x) = |\det(T_x f|_{E^s(x)})|$. *Suppose* we know that $\mu_x^\xi \ll \lambda_x^s$ for μ-a.e. x. Then $d\mu_x^\xi = h d\lambda_x^s$ μ almost everywhere for some function h (see (3.2)). By Liu [4, Proof of Claim 2.1], this function must satisfy

$$\mu_x^{f^{-1}\xi}(\xi(x)) = \frac{1}{J_f(y)} \cdot \frac{h(fy)}{h(y)} \cdot |\det(T_y f|_{E^s(y)})|$$

for λ_x^s-a.e. $y \in \xi(x)$ and hence

$$\begin{aligned}\frac{h(y)}{h(x)} &= \frac{h(fy)}{h(fx)} \cdot \frac{J_f(x)}{J_f(y)} \cdot \frac{|\det(T_y f|_{E^s(y)})|}{|\det(T_x f|_{E^s(x)})|} \\ &= \cdots\cdots \\ &= \prod_{k=0}^{+\infty} \frac{J_f(f^k x)}{J_f(f^k y)} \cdot \frac{D^s(f^k y)}{D^s(f^k x)} \\ &=: \Delta(x, y)\end{aligned}$$

as long as $\frac{h(f^k y)}{h(f^k x)} \to 1$ as $k \to +\infty$ and $\Delta(x, y)$ is well defined. A candidate for h is then

$$h(y) = \frac{\Delta(x, y)}{\int_{\xi(x)} \Delta(x, y) d\lambda_x^s(y)}, \quad \forall y \in \xi(x). \tag{4.1.2}$$

Also note that the same observation holds if we replace ξ by $f^{-m}\xi$ for $m \geq 0$.

Lemma 4.1.3. *Let* $m \geq 0$. *There exists a measurable function* $h_m : M \to (0, +\infty)$ *such that for* μ*-a.e.* x*,*

$$h_m(y) = \frac{\Delta(x, y)}{\int_{(f^{-m}\xi)(x)} \Delta(x, y) d\lambda_x^s(y)}, \quad \forall y \in (f^{-m}\xi)(x). \tag{4.1.3}$$

This lemma follows from our assumption (H) or (H)' in Sect. 2 and the fact that $E^s(y)$ is Lipschitz continuous along each $V^s(x)$. The detailed proof is the same as that of [5, Lemma VI.8.2] and is omitted here.

Lemma 4.1.4. *For* μ*-a.e.* x *one has*

$$\int_{(f^{-1}\xi)(x)} \Delta(x, y) d\lambda_x^s(y) = \frac{J_f(x)}{D^s(x)} \int_{\xi(fx)} \Delta(fx, y) d\lambda_{fx}^s(y). \tag{4.1.4}$$

Proof. Let $y_0 \in \xi(fx)$. Since $T_z f$ is assumed to be non-degenerate at any $z \in M$, there is an open neighborhood U_{y_0} of y_0 in M such that $f^{-1}U_{y_0} = \bigcup_{i=1}^l V_{z_i}$, where $z_i \in f^{-1}\{y_0\}$ and V_{z_i} is an open neighborhood of z_i so that $f_i := f|_{V_{z_i}} : V_{z_i} \to U_{y_0}$ is a diffeomorphism. For any Borel set $B \subset U_{y_0} \cap \xi(fx)$, put $C_i = V_{z_i} \cap f^{-1}B$. Then

$$\begin{aligned}&\int_{f^{-1}B} \Delta(x, z)\, d\lambda_x^s(z) \\ &= \sum_{i=1}^l \int_{C_i} \Delta(x, z)\, d\lambda_x^s(z) \\ &= \sum_{i=1}^l \int_B \Delta(x, f_i^{-1}y) |\det(T_y f_i^{-1}|_{E^s(y)})|\, d\lambda_{fx}^s(y) \\ &= \sum_{i=1}^l \int_B \frac{\Delta(x, f_i^{-1}y)}{|\det(T_{f_i^{-1}y} f|_{E^s(f_i^{-1}y)})|}\, d\lambda_{fx}^s(y)\end{aligned}$$

$$= \sum_{i=1}^{l} \int_B \prod_{k=0}^{+\infty} \frac{J_f(f^k x)}{J_f(f^k(f_i^{-1} y))} \cdot \frac{D^s(f^k(f_i^{-1} y))}{D^s(f^k x)} \cdot \frac{1}{D^s(f_i^{-1} y)} \, d\lambda^s_{fx}(y)$$

$$= \frac{J_f(x)}{D^s(x)} \int_B \sum_{i=1}^{l} \frac{1}{J_f(f_i^{-1} y)} \Delta(fx, y) \, d\lambda^s_{fx}(y)$$

$$= \frac{J_f(x)}{D^s(x)} \int_B \Delta(fx, y) \, d\lambda^s_{fx}(y),$$

the last equality uses (2.3). Taking a finite cover of $\xi(fx)$ by open sets of the type of U_{y_0}, we get (4.1.4). □

Let $(\bar{M}, \tau, \bar{\mu})$ be the inverse limit system of (M, f, μ) and set $\bar{\xi} = \pi^{-1}\xi$. We now define a Borel probability $\bar{\nu}$ on $\bar{M}$ by letting $\bar{\nu} = \bar{\mu}$ on $\mathcal{B}(\bar{\xi})$, the σ-algebra generated by $\bar{\xi}$, and by introducing a conditional measure $\bar{\nu}^{\bar{\xi}}_{\bar{x}}$ on $\bar{\xi}(\bar{x})$ (where $\bar{x} = (\cdots, x_{-1}, x_0, x_1, \cdots) \in \bar{M}$) in the following way. For every cylinder set

$$\bar{C} = \{\bar{y} = (\cdots, y_{-1}, y_0, y_1, \cdots) \in \bar{M} : \; y_i \in A_i, \; i = -p, \cdots, -1, 0, 1, \cdots, q\},$$

let

$$C = \{y \in (f^{-p}\xi)(x_{-p}) : \; y \in A_{-p}, fy \in A_{-p+1}, \cdots, f^{p+q} y \in A_q\}$$

and define

$$\bar{\nu}^{\bar{\xi}}_{\bar{x}}(\bar{C}) = \frac{\int_C \Delta(x_{-p}, y) \, d\lambda^s_{x_{-p}}(y)}{\int_{(f^{-p}\xi)(x_{-p})} \Delta(x_{-p}, y) \, d\lambda^s_{x_{-p}}(y)}.$$

From Lemma 4.1.4 and its proof we know that for any Borel $B \subset \xi(fx)$,

$$\frac{\int_{f^{-1}B} \Delta(x, y) \, d\lambda^s_x(y)}{\int_{(f^{-1}\xi)(x)} \Delta(x, y) \, d\lambda^s_x(y)} = \frac{\int_B \Delta(fx, y) \, d\lambda^s_{fx}(y)}{\int_{\xi(fx)} \Delta(fx, y) \, d\lambda^s_{fx}(y)}.$$

This implies that $\bar{\nu}^{\bar{\xi}}_{\bar{x}}$ is well defined.

Replacing ξ with $f^{-(m-1)}\xi$ in Lemma 4.1.4, we get for $m \geq 1$,

$$\int_{(f^{-m}\xi)(x)} \Delta(x, y) \, d\lambda^s_x(y) = \frac{J_f(x) \cdots J_f(f^{m-1}x)}{D^s(x) \cdots D^s(f^{m-1}x)} \int_{\xi(f^m x)} \Delta(f^m x, y) d\lambda^s_{f^m x}(y). \tag{4.1.5}$$

Lemma 4.1.5. *For $m \geq 1$,*

$$\frac{1}{m} \int_{\bar{M}} -\log \bar{\nu}^{\bar{\xi}}_{\bar{x}}((\tau^m \bar{\xi})(\bar{x})) \, d\bar{\mu}(\bar{x}) = F_\mu(f) - \sum_i \lambda_i^- m_i.$$

Proof. Put $L(\bar{x}) = \int_{\xi(x_0)} \Delta(x_0, y)\, d\lambda^s_{x_0}(y)$. Then

$$q_m(\bar{x}) := \bar{\nu}^{\bar{\xi}}_{\bar{x}}((\tau^m\bar{\xi})(\bar{x})) = \frac{\int_{\xi(x_{-m})} \Delta(x_{-m}, y)\, d\lambda^s_{x_{-m}}(y)}{\int_{(f^{-m}\xi)(x_{-m})} \Delta(x_{-m}, y)\, d\lambda^s_{x_{-m}}(y)}$$
$$= \frac{L(\tau^{-m}\bar{x})}{L(\bar{x})} \cdot \frac{D^s(x_{-m}) \cdots D^s(x_{-1})}{J_f(x_{-m}) \cdots J_f(x_{-1})}.$$

Since $q_m(\bar{x}) \le 1$ and $\log D^s$ and $\log J_f$ are μ-integrable, we know $\int \log^+ \frac{L\circ\tau^{-m}}{L} d\bar{\mu} < +\infty$ and hence $\int \log \frac{L\circ\tau^{-m}}{L}\, d\bar{\mu} = 0$. This yields

$$\int -\log q_m(\bar{x})\, d\bar{\mu} = \int \sum_{j=1}^{m} \log J_f(x_{-j})\, d\bar{\mu}(\bar{x}) - \int \sum_{j=1}^{m} \log D^s(x_{-j})\, d\bar{\mu}(\bar{x})$$
$$= m\left[\int \log J_f(x)\, d\mu(x) - \int \log D^s(x)\, d\mu(x)\right]$$
$$= m\left[F_\mu(f) - \sum_i \lambda_i^- m_i\right].$$

This proves the lemma. □

Lemma 4.1.6. $\frac{1}{m} H_{\bar{\mu}}(\tau^m\bar{\xi} \mid \bar{\xi}) = F_\mu(f) - \sum_i \lambda_i^- m_i$ *implies* $\bar{\nu} = \bar{\mu}$ *on* $\mathcal{B}(\tau^m\bar{\xi})$.

The proof of this lemma is similar to that of [2, Lemma 6.1.3] and is omitted here. Noting that $\frac{1}{m} H_{\bar{\mu}}(\tau^m\bar{\xi} \mid \bar{\xi}) = \frac{1}{m} H_\mu(\xi \mid f^{-m}\xi) = h_\mu(f)$ and $\bigvee_{m=0}^{+\infty} \tau^m\bar{\xi}$ is the partition of $\bar{M}$ into single points, we know that $\bar{\nu} = \bar{\mu}$, and hence $\pi\bar{\nu} = \pi\bar{\mu} = \mu$. This completes the proof of the ergodic case of Theorem 2.6.

In what follows we complete the proof of Theorem 2.6 in the non-ergodic case. The arguments are similar to [2, Subsect. (6.2)] or [5, Subsect. VI.8.B] and are only outlined. Given (f, μ), by Rokhlin [9], there is a unique (μ-mod 0) measurable partition $\zeta = \{C\}$ of $(M, \mathcal{B}_\mu(M), \mu)$ such that $f^{-1}C = C$ for each $C \in \zeta$ and $f|_C : (C, \mu_C) \hookleftarrow$ is ergodic for μ_ζ-a.e. $C \in M/\zeta$, where μ_C is the conditional measure of μ on C and $(M/\zeta, \mu_\zeta)$ is the factor space of (M, μ) with respect to ζ. Suppose

$$h_\mu(f) = F_\mu(f) - \int \sum_i \lambda_i(x)^- m_i(x)\, d\mu.$$

Since

$$h_\mu(f) = \int_{M/\zeta} h_{\mu_C}(f)\, d\mu_\zeta(C)$$

and

$$F_\mu(f) - \int \sum_i \lambda_i(x)^- m_i(x)\, d\mu = \int_{M/\zeta} \left[F_{\mu_C}(f) - \int \sum_i \lambda_i(x)^- m_i(x) d\mu_C\right] d\mu_\zeta(C),$$

by the inequality (1.1), we have

$$h_{\mu_C}(f) = F_{\mu_C}(f) - \int \sum_i \lambda_i(x)^- m_i(x) d\mu_C$$

for μ_ζ-a.e. C. Let ξ be a measurable partition of M subordinate to the W^s-manifolds of (M, μ). Since, by [5, Lemma IV.2.2], ξ refines ζ, we have

$$\mu_x^\xi = (\mu_C)_x^\xi \quad \text{if } x \in C,$$

and hence $\mu_x^\xi \ll \lambda_x^s$ for μ-a.e. x.

4.2. Proof of Proposition 4.1.2 in the hyperbolic case. The proof follows largely the arguments of Ledrappier and Young [2], but we will use the inverse limit space and some modifications are necessary. A complete proof is quite long and it is in fact more similar to the arguments in [5, Chap. V] where a version of [2] for random diffeomorphisms is presented. In order to avoid similar arguments, we will only present a proof for the case when (f, μ) is hyperbolic, that is, (f, μ) does not have zero Lyapunov exponent. Though it is much simpler than that for the general case, such a presentation is sufficient for the reader to get the full flavor of the necessary modifications of [2] for the complete proof.

Lyapunov charts. Since $T_x f$ is assumed to be non-degenerate for any $x \in M$, there are $\rho_0, \ \rho_1 > 0$ such that, for any $x \in M$, $f_x := f|_{B(x,\rho_0)} : B(x, \rho_0) \to M$ is a diffeomorphism to the image which contains $B(fx, \rho_1)$. Let $f_x^{-1} : fB(x, \rho_0) \to B(x, \rho_0)$ denote the local inverse.

Assume (f, μ) is ergodic and let $\lambda_1 < \lambda_2 < \cdots < \lambda_r$ be all the Lyapunov exponents with $\lambda_i \neq 0$ for all i. Then there is a Borel set $\Gamma_0 \subset \underline{M}$ with $\bar{\mu}(\Gamma_0) = 1$ and for each $\bar{x} \in \Gamma_0$ there exists a measurable (in $\bar{x}$) splitting

$$T_{x_0} M = E_1(\bar{x}) \oplus E_2(\bar{x}) \oplus \cdots \oplus E_r(\bar{x})$$

such that for each $1 \le i \le r$,

$$\lim_{n \to \pm\infty} \frac{1}{n} \log |D(\bar{x}, n)v| = \lambda_i \quad \text{for } 0 \neq v \in E_i(\bar{x}),$$

where $D(\bar{x}, n) = T_{x_0} f^n$ for $n \ge 0$ and $D(\bar{x}, n) = (T_{x_n} f)^{-1} \circ \cdots \circ (T_{x_{-1}} f)^{-1}$ for $n < 0$.

Put $E^s(\bar{x}) = \oplus_{\lambda_i < 0} E_i(\bar{x})$, $E^u(\bar{x}) = \oplus_{\lambda_i > 0} E_i(\bar{x})$, $s = \dim E^s(\bar{x})$, $u = \dim E^u(\bar{x})$, $d = \dim M$.

For $(v^s, v^u) \in \mathbf{R}^s \times \mathbf{R}^u$, define $\|(v^s, v^u)\| = \max\{\|v^s\|_s, \|v^u\|_u\}$ where $\|\cdot\|_s$ and $\|\cdot\|_u$ are the usual Euclidean norms on $\mathbf{R}^s$ and $\mathbf{R}^u$ respectively. The closed disk in $\mathbf{R}^s$ of radius ρ centered at 0 is denoted by $\mathbf{R}^s(\rho)$ and $\mathbf{R}(\rho) := \mathbf{R}^s(\rho) \times \mathbf{R}^u(\rho)$.

Put $\lambda^- = \max\{ \lambda_i : \lambda_i < 0\}$ and $\lambda^+ = \min\{ \lambda_i : \lambda_i > 0\}$. Let $0 < \varepsilon < \min\{-\lambda^-/100, \ \lambda^+/100\}$ be given. Then there is a Borel set $\Gamma \subset \Gamma_0$ with $\bar{\mu}(\Gamma) = 1$ and $\tau\Gamma = \Gamma$ and there is a measurable function $l : \Gamma \to [1, +\infty)$ with $l(\tau^{\pm 1}\bar{x}) \le e^\varepsilon l(\bar{x})$ such that for each $\bar{x} \in \Gamma$ one can define an embedding $\Phi_{\bar{x}} : \mathbf{R}(l(\bar{x})^{-1}) \to M$ with the following properties:

i) $\Phi_{\bar{x}}(0) = x_0$, $T_0\Phi_{\bar{x}}$ takes $\mathbf{R}^s$, $\mathbf{R}^u$ to $E^s(\bar{x})$, $E^u(\bar{x})$ respectively.

ii) Put $f_{\bar{x}} := \Phi_{\tau\bar{x}}^{-1} \circ f \circ \Phi_{\bar{x}}$ and $f_{\bar{x}}^{-1} := \Phi_{\tau^{-1}\bar{x}}^{-1} \circ f_{x_{-1}}^{-1} \circ \Phi_{\bar{x}}$, defined wherever they make sense. Then

$$\|T_0 f_{\bar{x}} v\| \le e^{\lambda^- + \varepsilon}\|v\| \quad \text{for } v \in \mathbf{R}^s$$

and

$$\|T_0 f_{\bar{x}} v\| \ge e^{\lambda^+ - \varepsilon}\|v\| \quad \text{for } v \in \mathbf{R}^u.$$

iii) Let $L(g)$ denote the Lipschitz constant of a map g. Then

$$L(f_{\bar{x}} - T_0 f_{\bar{x}}) \le \varepsilon, \quad L(f_{\bar{x}}^{-1} - T_0 f_{\bar{x}}^{-1}) \le \varepsilon$$

and

$$L(T.f_{\bar{x}}) \le l(\bar{x}), \quad L(T.f_{\bar{x}}^{-1}) \le l(\bar{x}).$$

iv) $\|f_{\bar{x}} v\| \le e^{\lambda}\|v\|$ and $\|f_{\bar{x}}^{-1} v\| \le e^{\lambda}\|v\|$ for all $v \in \mathbf{R}(e^{-\lambda-\varepsilon} l(\bar{x})^{-1})$, where $\lambda > 0$ is a number depending only on ε and the exponents. In particular, $f_{\bar{x}}^{-1}\mathbf{R}(e^{-\lambda-\varepsilon} l(\bar{x})^{-1}) \subset \mathbf{R}(l(\tau^{-1}\bar{x})^{-1})$.

v) For any $v,\ v' \in \mathbf{R}(l(\bar{x})^{-1})$ we have

$$K^{-1} d(\Phi_{\bar{x}} v,\ \Phi_{\bar{x}} v') \le \|v - v'\| \le l(\bar{x}) d(\Phi_{\bar{x}} v,\ \Phi_{\bar{x}} v')$$

for some universal constant $K > 0$.

The proof of the above facts is similar to [2, Appendix] or [5, Proof of Proposition VI.3.1] (by replacing ω with $\bar{x}$) and is omitted here. Any system of local charts $\{\Phi_{\bar{x}} : \bar{x} \in \Gamma\}$ satisfying i)–v) above will be referred to as (ε, l)-charts.

Let $\{\Phi_{\bar{x}} : \bar{x} \in \Gamma\}$ be a system of (ε, l)-charts and let $0 < \delta \le 1$ be a reduction factor. For $\bar{x} \in \Gamma$ define

$$S_\delta^s(\bar{x}) = \{z \in \mathbf{R}(l(\bar{x})^{-1}) : \|\Phi_{\tau^n\bar{x}}^{-1} \circ f^n \circ \Phi_{\bar{x}} z\| \le \delta l(\tau^n \bar{x})^{-1},\ \forall n \ge 0\}.$$

Then $\Phi_{\bar{x}} S_\delta^s(\bar{x}) \subset V^s(x_0)$ for $\bar{\mu}$-a.e. $\bar{x} \in \Gamma$. And, when $\delta > 0$ is small, $S_\delta^s(\bar{x})$ is the graph of a function $h_{\bar{x}} : \mathbf{R}^s(\delta l(\bar{x})^{-1}) \to \mathbf{R}^u(\delta l(\bar{x})^{-1})$ with $h_{\bar{x}}(0) = 0$ and $\|T.h_{\bar{x}}\| \le \frac{1}{3}$.

Partitions adapted to Lyapunov charts. A measurable partition $\mathcal{P}$ of $(\bar{M}, \bar{\mu})$ is said to be *adapted* to $(\{\Phi_{\bar{x}}\}, \delta)$ if for $\bar{\mu}$-a.e. $\bar{x} \in \Gamma$ one has $\pi\mathcal{P}^-(\bar{x}) \subset \Phi_{\bar{x}} S_\delta^s(\bar{x})$, where $\mathcal{P}^- = \bigvee_{n=0}^{+\infty} \tau^{-n}\mathcal{P}$.

Lemma 4.2.1. *Given* $\{\Phi_{\bar{x}}\}$ *and* $0 < \delta \le 1$, *there is a finite entropy partition* $\mathcal{P}$ *of* $(\bar{M}, \bar{\mu})$ *such that* $\mathcal{P}$ *is adapted to* $(\{\Phi_{\bar{x}}\}, \delta)$.

Proof. Fix some $l_0 > 0$ so that $\Lambda := \{\bar{x} \in \Gamma : l(\bar{x}) \le l_0\}$ has positive $\bar{\mu}$ measure. For $\bar{x} \in \Lambda$, let $r(\bar{x})$ be the smallest positive integer k such that $\tau^{-k}\bar{x} \in \Lambda$. Define $\psi : \bar{M} \to (0, +\infty)$ by

$$\psi(\bar{x}) = \begin{cases} \min\{\delta,\ \rho_0\} & \text{if } \bar{x} \notin \Lambda, \\ \min\{\delta l_0^{-2} e^{-(\lambda+\varepsilon) r(\bar{x})},\ \rho_0\} & \text{if } \bar{x} \in \Lambda. \end{cases}$$

Then ψ is defined $\bar{\mu}$ almost everywhere and $\log\psi$ is $\bar{\mu}$-integrable since $\int_\Lambda r(\bar{x}) d\bar{\mu} = 1$.

Take numbers $C > 0$ and $r_0 > 0$ such that, for any $0 < r \le r_0$, there exists a measurable partition α_r of M which satisfies

$$\operatorname{diam} \alpha_r(x) \le r \quad \text{for all } x \in M$$

and

$$|\alpha_r| \le C \left(\frac{1}{r}\right)^d ,$$

where $|\alpha_r|$ denotes the number of elements of α_r.

Put $U_n = \{\bar{x} \in \bar{M} : e^{-(n+1)} < \psi(\bar{x}) \le e^{-n}\}$. Define a partition $\mathcal{P}$ of $\bar{M}$ by requiring that $\mathcal{P} \ge \{U_n : n \ge 0\}$ and $\mathcal{P}|_{U_n} = \{\pi^{-1}A : A \in \alpha_{r_n}\}|_{U_n}$, where $r_n = e^{-(n+1)}$. Clearly

$$\operatorname{diam} \pi\mathcal{P}(\bar{x}) \le \psi(\bar{x}) \quad \text{for any } \bar{x} \in \bar{M},$$

and, by the $\bar{\mu}$-integrability of $\log \psi$ one has $H_{\bar{\mu}}(\mathcal{P}) < +\infty$ (see Mané [6]).

We now check that $\pi\mathcal{P}^-(\bar{x}) \subset \Phi_{\bar{x}}\mathbf{R}(\delta l(\bar{x})^{-1})$ for $\bar{\mu}$-a.e. $\bar{x} \in \bigcup_{n\ge 0} \tau^{-n}\Lambda$. This clearly implies that $\pi\mathcal{P}^-(\bar{x}) \subset \Phi_{\bar{x}} S^s_\delta(\bar{x})$ for $\bar{\mu}$-a.e. $\bar{x} \in \bar{M}$. First consider $\bar{x} \in \Lambda$. By the choice of $\mathcal{P}$, we have $\pi\mathcal{P}^-(\bar{x}) \subset \pi\mathcal{P}(\bar{x}) \subset B(x_0, \psi(\bar{x}))$ which is contained in $\Phi_{\bar{x}}\mathbf{R}(\delta l(\bar{x})^{-1})$ because $l(\bar{x})\psi(\bar{x}) = l(\bar{x}) \cdot \delta l_0^{-2} e^{-(\lambda+\varepsilon)r(\bar{x})} \le \delta l(\bar{x})^{-1}$. Suppose now $\bar{x} \notin \Lambda$ and $n > 0$ is the smallest positive integer n such that $\tau^n\bar{x} \in \Lambda$. Then

$$\begin{aligned}\pi\tau^n\mathcal{P}^-(\bar{x}) &\subset \pi\mathcal{P}^-(\tau^n\bar{x}) \subset B(x_n, \psi(\tau^n\bar{x}))\\ &\subset \Phi_{\tau^n\bar{x}}\mathbf{R}(\delta l(\tau^n\bar{x})^{-1} e^{-(\lambda+\varepsilon)r(\tau^n\bar{x})}).\end{aligned}$$

Now

$$\begin{aligned}\pi\mathcal{P}^-(\bar{x}) &\subset \Phi_{\bar{x}} f^{-1}_{\tau\bar{x}} \circ \cdots \circ f^{-1}_{\tau^{n-1}\bar{x}} \circ f^{-1}_{\tau^n\bar{x}}\mathbf{R}(\delta l(\tau^n\bar{x})^{-1} e^{-(\lambda+\varepsilon)r(\tau^n\bar{x})})\\ &\subset \Phi_{\bar{x}}\mathbf{R}(\delta l(\tau^n\bar{x})^{-1} e^{-(\lambda+\varepsilon)r(\tau^n\bar{x})} e^{\lambda n})\\ &\subset \Phi_{\bar{x}}\mathbf{R}(\delta l(\bar{x})^{-1}),\end{aligned}$$

since $n \le r(\tau^n\bar{x})$. This completes the proof. □

Proof of $H_\mu(\xi \mid f^{-1}\xi) = h_\mu(f)$. Fix arbitrarily $\kappa > 0$. Given $\{\Phi_{\bar{x}}\}$ and $0 < \delta \le 1$, take a finite entropy partition $\mathcal{P}$ of $(\bar{M}, \bar{\mu})$ such that $\mathcal{P}$ refines $\pi^{-1}\{S, M\backslash S\}$ (where S is the set given in the outline of the proof of Proposition 4.1.1), $\mathcal{P}$ is adapted to $(\{\Phi_{\bar{x}}\}, \delta)$ and $h_{\bar{\mu}}(\tau, \mathcal{P}) \ge h_{\bar{\mu}}(\tau) - \kappa = h_\mu(f) - \kappa$.

Put

$$\eta_1 = \bar{\xi} \vee \mathcal{P}^- \quad \text{and} \quad \eta_2 = \mathcal{P}^-$$

(recall that $\bar{\xi} = \pi^{-1}\xi$). Then

$$h_{\bar{\mu}}(\tau, \eta_2) = h_{\bar{\mu}}(\tau, \mathcal{P}) \tag{4.2.1}$$

and

$$h_{\bar{\mu}}(\tau, \eta_1) = H_{\bar{\mu}}(\bar{\xi} \mid \tau^{-1}\bar{\xi}). \tag{4.2.2}$$

The equality (4.2.1) is straightforward. The proof of (4.2.2) is similar to [2, Lemma 3.2.1] and we present it here for completeness.

$$\begin{aligned} h_{\bar{\mu}}(\tau,\ \eta_1) &= h_{\bar{\mu}}(\tau,\ \bar{\xi} \vee \tau^{-n}\mathcal{P}^-) \\ &= H_{\bar{\mu}}(\bar{\xi} \vee \tau^{-n}\mathcal{P}^- \mid \tau^{-1}\bar{\xi} \vee \tau^{-(n+1)}\mathcal{P}^-) \\ &= H_{\bar{\mu}}(\bar{\xi} \mid \tau^{-1}\bar{\xi} \vee \tau^{-(n+1)}\mathcal{P}^-) + H_{\bar{\mu}}(\mathcal{P}^- \mid \tau^n\bar{\xi} \vee \tau^{-1}\mathcal{P}^-), \end{aligned}$$

where the first term is $\le H_{\bar{\mu}}(\bar{\xi} \mid \tau^{-1}\bar{\xi})$ and the second term goes to 0 as $n \to +\infty$ since $\tau^n\bar{\xi}$ goes to the partition of $\bar{M}$ into single points. On the other hand,

$$h_{\bar{\mu}}(\tau,\ \eta_1) = h_{\bar{\mu}}(\tau,\ \bar{\xi} \vee \mathcal{P}) \ge h_{\bar{\mu}}(\tau,\ \bar{\xi}),$$

since $H_{\bar{\mu}}(\mathcal{P}) < +\infty$. This proves (4.2.2).

We now show that for sufficiently small $\delta > 0$ we have

$$\mathcal{P}^-(\bar{x}) = (\bar{\xi} \vee \mathcal{P}^-)(\bar{x}),\quad \bar{\mu}\text{–a.e. } \bar{x}, \tag{4.2.3}$$

which implies

$$H_{\bar{\mu}}(\bar{\xi} \mid \mathcal{P}^-) = 0. \tag{4.2.4}$$

In order to prove (4.2.3), it is sufficient to show that, if $\bar{y} \in \mathcal{P}^-(\bar{x})$, then $y_0 \in \xi(x_0)$. Since $\mathcal{P}$ refines $\pi^{-1}\{S, M\backslash S\}$ and $\bar{y} \in \mathcal{P}^-(\bar{x})$, it suffices to prove that $d^s(f^n y_0, f^n x_0) \le \gamma$ whenever $f^n x_0 \in S$. This is in fact true for all $n \ge 0$ since

$$\begin{aligned} d^s(f^n y_0, f^n x_0) &\le K \cdot \| f^n_{\bar{x}} \Phi^{-1}_{\bar{x}} y_0 - f^n_{\bar{x}} \Phi^{-1}_{\bar{x}} x_0 \| \\ &\le K \cdot e^{(\lambda^- + 2\varepsilon)n} \| \Phi^{-1}_{\bar{x}} y_0 - \Phi^{-1}_{\bar{x}} x_0 \| \le K \cdot 2\delta l(\bar{x})^{-1} \le \gamma. \end{aligned}$$

Then, by (4.2.4), we know that

$$H_\mu(\xi \mid f^{-1}\xi) = h_{\bar{\mu}}(\tau,\ \eta_1) \ge h_{\bar{\mu}}(\tau,\ \eta_2) \ge h_\mu(f) - \kappa.$$

Since $\kappa > 0$ is arbitrary, we get $H_\mu(\xi \mid f^{-1}\xi) = h_\mu(f)$.

Acknowledgement. The author expresses his sincere thanks to the anonymous referee for careful reading of the manuscript and, in particular, for indicating the arguments in Remark 2.8.

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Communicated by G. Gallavotti