Commun. Math. Phys. 285, 991–1004 (2009)
Digital Object Identifier (DOI) 10.1007/s00220-008-0562-x

Communications in
Mathematical Physics

W-Algebra $W(2, 2)$ and the Vertex Operator Algebra $L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0)$

Wei Zhang, Chongying Dong⋆

Department of Mathematics, University of California, Santa Cruz, CA 95064, USA.
E-mail: dong@math.ucsc.edu

Received: 27 November 2007 / Accepted: 28 February 2008
Published online: 27 June 2008 – © Springer-Verlag 2008

Abstract: In this paper the W-algebra $W(2, 2)$ and its representation theory are studied. It is proved that a simple vertex operator algebra generated by two weight 2 vectors is either a vertex operator algebra associated to an irreducible highest weight $W(2, 2)$-module or a tensor product of two simple Virasoro vertex operator algebras. Furthermore, we show that any rational, C_2-cofinite and simple vertex operator algebra whose weight 1 subspace is zero, weight 2 subspace is 2-dimensional and with central charge $c = 1$ is isomorphic to $L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0)$.

1. Introduction

Motivated partially by the problem of classification of rational vertex operator algebras with central charge $c = 1$ and by the Frenkel-Lepowsky-Meurman's uniqueness conjecture on the moonshine vertex operator algebra $V^\natural$ [FLM], we give a characterization of the vertex operator algebra $L(1/2, 0) \otimes L(1/2, 0)$ in terms of the central charge and the dimensions of weights 1 and 2 subspaces in this paper. Here $L(1/2, 0)$ is the vertex operator algebra associated to the irreducible highest weight module for the Virasoro algebra with central charge $1/2$ which is the smallest central charge among the discrete unitary series for the Virasoro algebra.

The classification of rational conformal field theories with $c = 1$ at character level has been achieved in the physics literature under the assumption that the sum of the square of the norm of the irreducible characters is a modular function over the full modular group [K]. But the classification of rational vertex operator algebras with $c = 1$ remains open. If a vertex operator algebra $V = \sum_{n \geq 0} V_n$ with $\dim V_0 = 1$ is rational and C_2-cofinite, then V_1 is a reductive Lie algebra and its rank is less than or equal to the effective central charge $\tilde{c}$ [DM1]. Also, the vertex operator subalgebra generated by V_1 is a tensor product of vertex operator algebras associated to integrable highest weight

⋆ Supported by NSF grants and a research grant from the Committee on Research, UC Santa Cruz.

modules for affine Kac-Moody algebras and the lattice vertex operator algebra [DM2]. In the case that $c = \tilde{c} = 1$, we can classify the vertex operator algebras with dim $V_1 \neq 0$. Since V_1 is a reductive Lie algebra whose rank is less than or equal to 1, we immediately see that V_1 is either 1-dimensional or 3-dimensional, as a result, V is isomorphic to a vertex operator algebra associated to a rank 1 lattice. So one can assume that $V_1 = 0$. There are two cases: dim $V_2 > 1$ and dim $V_2 = 1$. The $L(1/2, 0) \otimes L(1/2, 0)$ is the only known such vertex operator algebra whose weight two subspace is not one-dimensional. So a characterization of $L(1/2, 0) \otimes L(1/2, 0)$ can be regarded as a part of a program of classification of rational vertex operator algebras with $c = 1$.

The vertex operator algebra $L(1/2, 0) \otimes L(1/2, 0)$ plays an important role in the study of the moonshine vertex operator algebra $V^\natural$. The moonshine vertex operator algebra $V^\natural$ which is fundamental in shaping the field of vertex operator algebra was constructed as a bosonic orbifold theory based on the Leech lattice [FLM]. The discovery of existence of $L(1/2, 0)^{\otimes 48}$ inside the moonshine vertex operator algebra $V^\natural$ [DMZ] opens a different way to study $V^\natural$. This leads to the theory of code and framed vertex operator algebras [M2, DGH]. This discovery is also essential in a proof that $V^\natural$ is holomorphic [D], a new construction of $V^\natural$ [M3], proofs of weak versions of the Frenkel-Lepowsky-Meurman's uniqueness conjecture on $V^\natural$ [DGL, LY] and a study of $V^\natural$ in terms of conformal nets [KL]. There is no doubt that a characterization of $L(1/2, 0) \otimes L(1/2, 0)$ will be very helpful in the study of the structure of $V^\natural$ and the Frenkel-Lepowsky-Meurman's uniqueness conjecture.

The $W(2, 2)$ and its highest weight modules enter the picture naturally during our discussion on $L(1/2, 0) \otimes L(1/2, 0)$. The W-algebra $W(2, 2)$ is an extension of the Virasoro algebra and also has a very good highest weight module theory (see Sect. 2). Its highest weight modules produce a new class of vertex operator algebras. In contrast to the Virasoro algebra case, this class of vertex operator algebras are always irrational. From this point of view, this class of vertex operator algebras are not interesting.

The $W(2, 2)$ and associated vertex operator algebras are also closely related to the classification of the simple vertex operator algebra with two generators. It is well known that each homogeneous subspace V_n of a vertex operator algebra $V = \sum_{n\in\mathbb{Z}} V_n$ is some kind of algebra under the product $u \cdot v = u_{n-1}v$ for $u, v \in V_n$, where u_{n-1} is the component operator of $Y(u, z) = \sum_{m\in\mathbb{Z}} u_m z^{-m-1}$. If a vertex operator algebra $V = \sum_{n\geq 0} V_n$ with dim $V_0 = 1$ is rational and C_2-cofinite, then V_1 and the vertex operator subalgebra generated by V_1 are well understood [DM1]. So it is natural to turn our attention to V_2. This is still a very hard problem even with $V_1 = 0$. A simple vertex operator algebra V satisfying $V_1 = 0$ is called the *moonshine type*. The V_2 in this case is a commutative nonassociative algebra. The simple vertex operator algebras of the moonshine type with dim $V_2 = 2$ and generated by V_2 are also classified in this paper. There are two families of such algebras. One of this family consists of the tensor product of two vertex operator algebras associated to the irreducible highest weight modules for the Virasoro algebra and the other family consists of the vertex operator algebras associated to the highest weight modules for the W-algebra $W(2, 2)$.

The paper is organized as follows. We define and study the W-algebra $W(2, 2)$ in Sect. 2. In particular we use the bilinear form on Verma modules $V(c, h_1, h_2)$ to determine the irreducible quotient modules $L(c, h_1, h_2)$ for $W(2, 2)$ for most c and h_i. In Sect. 3 we classify the simple vertex operator algebras of the moonshine type generated by two weight 2 vectors. Section 4 is devoted to the characterization of rational vertex operator algebra $L(1/2, 0) \otimes L(1/2, 0)$. The main idea is to use the modular invariance of the graded characters of the irreducible modules [Z] to control the growth of the graded dimensions of the vertex operator algebra.

2. W-Algebra $W(2,2)$

The W-algebra $W(2,2)$ considered in this paper is an infinite dimensional Lie algebra with basis L_m, W_m, C for $m \in \mathbb{Z}$ and Lie brackets

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}C, \tag{2.1}$$

$$[L_m, W_n] = (m-n)W_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}C, \tag{2.2}$$

$$[W_m, W_n] = 0 \tag{2.3}$$

for $m, n \in \mathbb{Z}$, where C is a central element. Since the adjoint action of L_0 is semi-simple with integral eigenvalues, $W(2,2)$ is a $\mathbb{Z}$-graded algebra and has the triangular decomposition

$$\begin{aligned} W_{(n)} &= \mathbb{C}L_{-n} \oplus \mathbb{C}W_{-n} \quad \text{for } n \neq 0, \\ W_{(0)} &= \mathbb{C}L_0 \oplus \mathbb{C}W_0 \oplus \mathbb{C}C, \\ W(2,2) &= W_+ \oplus W_{(0)} \oplus W_-, \end{aligned}$$

where $W_+ = \oplus_{n\geq 1} W_{(n)}$, $W_- = \oplus_{n\geq 1} W_{(-n)}$. In this section we study the highest weight modules for this algebra and the corresponding vertex operator algebras.

Let $c, h_1, h_2 \in \mathbb{C}$, and we denote by $V(c, h_1, h_2)$ the Verma module for $W(2,2)$ with central charge c and highest weight (h_1, h_2). Then $V(c, h_1, h_2) = U(W(2,2))/I_{c,h_1,h_2}$, where I_{c,h_1,h_2} is the left ideal of the universal enveloping algebra $U(W(2,2))$ generated by L_m, W_m, $C - c$, $L_0 - h_1$ and $W_0 - h_2$ for positive m. The $V(c, h_1, h_2)$ can also be realized as an induced module as in the case of Virasoro algebra. By PBW theorem $V(c, h_1, h_2)$ has basis

$$\{W_{-m_1}\cdots W_{-m_s}L_{-n_1}\cdots L_{-n_t}\mathbf{1} | m_1 \geq \cdots \geq m_s \geq 1, n_1 \geq \cdots \geq n_t \geq 1\},$$

where $\mathbf{1} = 1 + I_{c,h_1,h_2}$. Then $V(c, h_1, h_2)$ is graded by the L_0-eigenvalues:

$$V(c, h_1, h_2) = \bigoplus_{n\geq 0} V(c, h_1, h_2)_{n+h_1},$$

where

$$V(c, h_1, h_2)_{n+h_1} = \{v \in V(c, h_1, h_2) | L_0 v = (n + h_1)v\}$$

is spanned by $W_{-m_1}\cdots W_{-m_s}L_{-n_1}\cdots L_{-n_t}\mathbf{1}$ with $m_1 + \cdots + m_s + n_1 + \cdots + n_t = n$. Note that $V(c, h_1, h_2)_{h_1} = \mathbb{C}\mathbf{1}$. A highest weight $W(2,2)$-module is a quotient module of the Verma module with the same central charge and highest weight. It is standard that $V(c, h_1, h_2)$ has a unique maximal submodule $J(c, h_1, h_2)$ so that $L(c, h_1, h_2) = V(c, h_1, h_2)/J(c, h_1, h_2)$ is an irreducible highest weight module.

As in the case of Virasoro algebra, there is an anti-involution α for $W(2,2)$ defined by $\alpha(L_n) = L_{-n}$, $\alpha(W_n) = W_{-n}$, $\alpha(C) = C$. The α can be extended to an anti-involution of $U(W(2,2))$. So we get a symmetric bilinear form $(,)$ on $V(c, h_1, h_2)$ by

$$(A\mathbf{1}, B\mathbf{1})\mathbf{1} = P_{h_1}(\alpha(A)B\mathbf{1}) \tag{2.4}$$

for $A, B \in U(W(2,2))$, where P_{h_1} is the projection from $V(c, h_1, h_2)$ to $V(c, h_1, h_2)_{h_1}$. Then the bilinear form is invariant in the sense:

$$(L_m u, v) = (u, L_{-m} v), \quad (W_m u, v) = (u, W_{-m} v), \quad (\mathbf{1}, \mathbf{1}) = 1 \tag{2.5}$$

for $u, v \in V(c, h_1, h_2)$ and $m \in \mathbb{Z}$. Moreover, the radical of this bilinear form is exactly the maximal submodule $J(c, h_1, h_2)$.

Let X be a proper submodule of $V(c, h_1, h_2)$. Then X is a submodule of $J(c, h_1, h_2)$ and the bilinear form $(,)$ on $V(c, h_1, h_2)$ induces an invariant symmetric bilinear form $(,)$ on the quotient module $V(c, h_1, h_2)/X$.

As in the classical case we need to answer the basic question: What is $J(c, h_1, h_2)$? We first consider the case $(c, h_1, h_2) = (c, 0, 0)$. Clearly, $L(0, 0, 0) = \mathbb{C}$. So we now assume that $c \neq 0$. Note that $U(W(2,2))L_{-1}\mathbf{1} + U(W(2,2))W_{-1}\mathbf{1}$ is a proper submodule of $V(c, 0, 0)$.

Theorem 2.1. *If $c \neq 0$, then $J(c, 0, 0) = U(W(2,2))L_{-1}\mathbf{1} + U(W(2,2))W_{-1}\mathbf{1}$ and $L(c, 0, 0)$ has a basis*

$$\{W_{-m_1} \cdots W_{-m_s} L_{-n_1} \cdots L_{-n_t}\mathbf{1} | m_1 \geq \cdots \geq m_s > 1, n_1 \geq \cdots \geq n_t > 1\}, \tag{2.6}$$

where $\mathbf{1}$ is the canonical highest weight vector of $L(c, 0, 0)$.

Proof. Set $\bar{V}(c, 0, 0) = V(c, 0, 0)/(U(W(2,2))L_{-1}\mathbf{1} + U(W(2,2))W_{-1}\mathbf{1})$ and let S be the set consisting of vectors given by (2.6) with $\mathbf{1}$ being the canonical highest weight vector of $\bar{V}(c, 0, 0)$. Then S forms a basis of $\bar{V}(c, 0, 0)$ by PBW theorem. For $n \geq 0$ we set $S_n = S \cap \bar{V}(c, 0, 0)_n$. We prove the irreducibility of $\bar{V}(c, 0, 0)$ by showing that the Gram matrix of $(,)$ with respect to the basis S_n of $\bar{V}(c, 0, 0)_n$ is nondegenerated for all $n \geq 0$.

For short we set

$$u(m_1, \ldots, m_s; n_1, \ldots, n_t) = W_{-m_1} \cdots W_{-m_s} L_{-n_1} \cdots L_{-n_t}\mathbf{1}$$

with $m_1 \geq \cdots \geq m_s > 1, n_1 \geq \cdots \geq n_t > 1$.

Let

$$P = \{(m_1, \ldots, m_s) | s \geq 1, m_1 \geq \cdots \geq m_s > 1\}$$

which is a set of partitions of n without 1. We define a total order on P so that

$$(m_1, \ldots, m_s) > (n_1, \ldots, n_t)$$

if there exists $1 \leq k \leq s$ such that $m_i = n_i$ for $i < k$ and $m_k > n_k$. For $n \geq 0$ we define a total order for S_n as follows:

$$u(m_1, \ldots, m_s; n_1, \ldots, n_t) > u(k_1, \ldots, k_p; l_1, \ldots, l_q)$$

if (a) $\sum m_i < \sum k_j$ (if there is no m term, the $\sum m_i$ is understood to be 0 and similarly for $\sum k_j$), or (b) $\sum m_i = \sum k_j$ and $(m_1, \ldots, m_s) > (k_1, \ldots, k_p)$ or (c) $\sum m_i = \sum k_j$, $(m_1, \ldots, m_s) = (k_1, \ldots, k_p)$ and $(n_1, \ldots, n_t) < (l_1, \ldots, l_q)$. For example, S_6 is ordered in the following way from the largest to the smallest:

$$L_{-2}^3\mathbf{1},\ L_{-3}^2\mathbf{1},\ L_{-4}L_{-2}\mathbf{1},\ L_{-6}\mathbf{1},\ W_{-2}L_{-2}^2\mathbf{1},\ W_{-2}L_{-4}\mathbf{1},\ W_{-3}L_{-3}\mathbf{1},$$
$$W_{-4}L_{-2}\mathbf{1},\ W_{-2}^2L_{-2}\mathbf{1},\ W_{-6}\mathbf{1},\ W_{-4}W_{-2}\mathbf{1},\ W_{-3}^2\mathbf{1},\ W_{-2}^3\mathbf{1}.$$

Observe that if $(m_1,\ldots,m_s)\in P$ and $m\geq m_1$ then

$$L_mW_{-m_1}\cdots W_{-m_s}\mathbf{1}=\frac{m^3-m}{12}c\frac{\partial}{\partial W_{-m}}W_{-m_1}\cdots W_{-m_s}\mathbf{1}.$$

Using (2.4) and (2.5) we immediately see that

$$(L_{-n_1}\cdots L_{-n_t}\mathbf{1},W_{-m_1}\cdots W_{-m_s}\mathbf{1})\mathbf{1}=L_{n_t}\cdots L_{n_1}W_{-m_1}\cdots W_{-m_s}\mathbf{1}=0$$

if $(m_1,\ldots,m_s)<(n_1,\ldots,n_t)$. Since $c\neq 0$,

$$(L_{-m_1}\cdots L_{-m_s}\mathbf{1},W_{-m_1}\cdots W_{-m_s}\mathbf{1})\mathbf{1}=L_{m_s}\cdots L_{m_1}W_{-m_1}\cdots W_{-m_s}\mathbf{1}\neq 0.$$

Let $u=W_{-m_1}\cdots W_{-m_s}L_{-n_1}\cdots L_{-n_t}\mathbf{1}$, $v=W_{-k_1}\cdots W_{-k_p}L_{-l_1}\cdots L_{-l_q}\mathbf{1}\in S_n$. Again using (2.4) and (2.5) we have

$$\begin{aligned}(u,v)\mathbf{1}&=L_{n_t}\cdots L_{n_1}W_{-k_1}\cdots W_{-k_p}W_{m_s}\cdots W_{m_1}L_{-l_1}\cdots L_{-l_q}\mathbf{1}\\&=L_{l_q}\cdots L_{l_1}W_{-m_1}\cdots W_{-m_s}W_{k_p}\cdots W_{k_1}L_{-n_1}\cdots L_{-n_t}\mathbf{1},\end{aligned}$$

where we have used the fact that W_m commute with each other (2.3). It is clear that $(u,v)=0$ if $\sum m_i>\sum l_j$ or $\sum k_i>\sum n_j$. If $\sum m_i=\sum l_j$ and $\sum n_i=\sum k_j$ we have

$$(u,v)=(W_{-m_1}\cdots W_{-m_s}\mathbf{1},L_{-l_1}\cdots L_{-l_q}\mathbf{1})(W_{-k_1}\cdots W_{-k_p}\mathbf{1},L_{-n_1}\cdots L_{-n_t}\mathbf{1}).$$

This implies that $(u,v)=0$ if either $(l_1,\ldots,l_q)>(m_1,\ldots,m_s)$ or $(n_1,\ldots,n_t)>(k_1,\ldots,k_p)$.

Now let s be the cardinality of S_n we can label vectors in S_n by $u_1,\ldots,u_s$ such that $u_i>u_j$ if $i>j$. Set $A=(a_{ij})$, where $a_{ij}=(u_{s+1-i},u_j)$ for $i,j=1,\ldots,s$. Note that if $u_i=W_{-m_1}\cdots W_{-m_s}L_{-n_1}\cdots L_{-n_t}\mathbf{1}$, then $u_{s+1-i}=W_{-n_1}\cdots W_{-n_t}L_{-m_1}\cdots_{-m_s}\mathbf{1}$.

It immediately follows that $a_{ii}=(u_{s+1-i},u_i)\neq 0$ for all i. We prove next that $a_{ij}=0$ if $i>j$. Let $u_i=W_{-m_1}\cdots W_{-m_s}L_{-n_1}\cdots L_{-n_t}\mathbf{1}$ and $u_j=W_{-k_1}\cdots W_{-k_p}L_{-l_1}\cdots L_{-l_q}\mathbf{1}$. So

$$u_{s+1-i}=u(n_1,\ldots,n_t;m_1,\ldots,m_s),u_j=u(k_1,\ldots,k_p;l_1,\ldots,l_q).$$

Since $u_i>u_j$ then either (a) $\sum m_i<\sum k_j$, or (b) $\sum m_i=\sum k_j$ and $(m_1,\ldots,m_s)>(k_1,\ldots,k_p)$ or (c) $\sum m_i=\sum k_j$, $(m_1,\ldots,m_s)=(k_1,\ldots,k_p)$ and $(n_1,\ldots,n_t)<(l_1,\ldots,l_q)$. From the discussion above, it is obvious that $(u_{s+1-i},u_j)=0$ in all cases. That is, $a_{ij}=0$ if $i>j$. As a result, the Gram matrix A is an upper triangular matrix with every entry in the diagonal being nonzero. This shows that $\bar{V}(c,0,0)$ is irreducible and $L(c,0,0)=\bar{V}(c,0,0)$. □

Remark 2.2. Although $W(2,2)$ is an extension of the Virasoro algebra, the representation theory for $W(2,2)$ is different from that for the Virasoro algebra in a fundamental way. For $W(2,2)$, the structure of $L(c,0,0)$ for $c\neq 0$ is uniform and simple. But for the Virasoro algebra, the situation is totally different. Let $L(c,h)$ be the irreducible highest weight module for the Virasoro algebra with central charge c and highest weight h. In the case $c\neq c_{s,t}=1-6(s-t)^2/st$, where s,t are two coprime positive integers $1<s<t$, then $L(c,0)=\bar{V}(c,0)$, where $\bar{V}(c,0)=V(c,0)/U(Vir)L_{-1}v$ and v is a nonzero highest weight vector of the Verma module $V(c,0)$ (see [FF]). The structure of $L(c_{s,t},0)$ is much more complicated. On the other hand, from the point of view of vertex operator algebra, $L(c_{s,t},0)$ is a rational vertex operator algebra for all $c_{s,t}$ but $L(c,0)$ is not if $c\neq c_{s,t}$ (see [FZ] and [W]).

Next we discuss the vertex operator algebras associated to the highest weight modules for $W(2,2)$. Let $\mathbf{1}$ be the canonical highest weight vector of $V(c,0,0)$. From the axiom of vertex operator algebra we must modulo out the submodule generated by $L_{-1}\mathbf{1}$. From the commutator relation (2.2) we know that W_n should be the component operators of a weight two vector. That is, there is a weight two vector x such that

$$Y(x,z)=\sum_{n\in\mathbb{Z}} x_n z^{-n-1}=\sum_{n\in\mathbb{Z}} W_n z^{-n-2}.$$

Moreover, we must modulo out the submodule generated by $W_{-1}\mathbf{1}$, since $L_{-1}W_0\mathbf{1}=[L_{-1},W_0]\mathbf{1}+W_0L_{-1}\mathbf{1}=-W_{-1}\mathbf{1}+W_0L_{-1}\mathbf{1}$ by (2.2).

A $W(2,2)$-module M is restricted if for any $w\in M$, $L_m w=W_m w=0$ if m is sufficiently large. Recall the weak module, admissible module and ordinary module from [DLM1].

Theorem 2.3. *Assume that $c\neq 0$. Then*

(1) *There is a unique vertex operator algebra structure on $L(c,0,0)$ with the vacuum vector $\mathbf{1}$ and the Virasoro element $\omega=L_{-2}\mathbf{1}$. Moreover, $L(c,0,0)$ is generated by ω and $x=W_{-2}\mathbf{1}$ with $Y(\omega,z)=\sum_{n\in\mathbb{Z}}L_nz^{-n-2}$ and $Y(x,z)=\sum_{n\in\mathbb{Z}}W_nz^{-n-2}$.*
(2) *If M is a restricted $W(2,2)$-module with central charge c, then M is a weak $L(c,0,0)$-module with $Y_M(\omega,z)=\sum_{n\in\mathbb{Z}}L_nz^{-n-2}$ and $Y_M(x,z)=\sum_{n\in\mathbb{Z}}W_nz^{-n-2}$. In particular, any quotient module of $V(c,h_1,h_2)$ is an ordinary module for $L(c,0,0)$.*
(3) *Any irreducible admissible $L(c,0,0)$-module is ordinary.*
(4) *$\{L(c,h_1,h_2)|h_i\in\mathbb{C}\}$ gives a complete list of irreducible $L(c,0,0)$-modules up to isomorphism.*

Proof. (1) and (2) are fairly standard following from the local system theory (see [L2,LL]). (3) and (4) follow from the fact that any irreducible admissible module for $L(c,0,0)$ is an irreducible highest weight module for $W(2,2)$. □

We now turn our attention to the Verma module $V(c,h_1,h_2)$ in general. As in general highest weight module theory, we want to know when $V(c,h_1,h_2)=L(c,h_1,h_2)$ is irreducible.

Theorem 2.4. *The Verma module $V(c,h_1,h_2)$ is irreducible if and only if $\frac{m^2-1}{12}c+2h_2\neq 0$ for any nonzero integer m.*

Proof. The proof is similar to that of Theorem 2.1. Note that

$$V(c,h_1,h_2)=\oplus_{n\geq 0}V(c,h_1,h_2)_{h_1+n}.$$

By PBW theorem, $V(c,h_1,h_2)_{h_1+n}$ has a basis S consisiting of vectors

$$W_{-m_1}\cdots W_{-m_s}L_{-n_1}\cdots L_{-n_t}\mathbf{1},$$

where $m_1\geq\cdots\geq m_s>0, n_1\geq\cdots\geq n_t>0, \sum m_i+\sum n_j=n$. We also define a total order on S_n as before. Let $S_n=\{u_1,\ldots,u_s\}$ and $u_i<u_j$ if $i<j$. Set $A_n=(a_{ij})$, where $a_{ij}=(u_{s+1-i},u_j)$. Then $V(c,h_1,h_2)=L(c,h_1,h_2)$ if and only if $\det A_n\neq 0$ for all $n>0$.

Note that if $m\geq m_1\geq\cdots\geq m_s>0$,

$$L_mW_{-m_1}\cdots W_{-m_s}\mathbf{1}=\left(\frac{m^3-m}{12}c+2mh_2\right)\frac{\partial}{\partial W_{-m}}W_{-m_1}\cdots W_{-m_s}\mathbf{1}.$$

If $\frac{m^2-1}{12}c+2h_2 \neq 0$ for all $0 \neq m \in \mathbb{Z}$, we see immediately that the same argument used in the proof of Theorem 2.1 works for $V(c, h_1, h_2)$. That is, A_n is an upper triangular matrix with every entry in the diagonal being nonzero for all $n \geq 0$, and $V(c, h_1, h_2)$ is irreducible in this case.

If $\frac{m^2-1}{12}c + 2h_2 = 0$ for some $0 < m$, then A_m is still an upper triangular matrix and one of the entries in the diagonal is $(L_{-m}\mathbf{1}, W_{-m}\mathbf{1})$ which is the coefficient of $\mathbf{1}$ in $L_m W_{-m}\mathbf{1} = 0$. As a result, $\det A_m = 0$. The proof is complete. □

It is definitely interesting to determine the $J(c, h_1, h_2)$ if $\frac{m^2-1}{12}c + 2h_2 = 0$ for some nonzero integer m. But this will be a problem which has nothing to do with the characterization of $L(1/2, 0) \otimes L(1/2, 0)$ in this paper. We will not further go in this direction.

3. Vertex Operator Algebras of the Moonshine Type

Motivated by the moonshine vertex operator algebra $V^\natural$ [FLM], we call a vertex operator algebra $V = \oplus_{n\in\mathbb{Z}} V_n$ the *moonshine type* if $V_1 = 0$. In this section we classify the simple vertex operator algebras V of the moonshine type such that V is generated by V_2 and V_2 is 2-dimensional.

Note that $V_0 = \mathbb{C}\mathbf{1}$ is 1-dimensional for the moonshine type vertex operator algebra V and $V_n = 0$ if $n < 0$ by Lemma 7.1 of [DGL]. Since $V_1 = 0$ and V_0 is 1-dimensional, there is a unique symmetric, nondegenerate invariant bilinear form (,) on V such that $(\mathbf{1}, \mathbf{1}) = 1$ (see [L1]). Then for any $u, v, w \in V$,

$$(Y(u, z)v, w) = (v, Y(e^{L(1)z}(-z^{-2})^{L(0)}u, z^{-1})w)$$

and

$$(u, v)\mathbf{1} = \mathrm{Res}_z z^{-1} Y(e^{L(1)z}(-z^{-2})^{L(0)}u, z^{-1})v.$$

In particular, the restriction of the form to each homogeneous subspace V_n is nondegenerate and

$$(u_{n+1}v, w) = (v, u_{-n+1}w)$$

for all $u, v \in V_2$ and $w \in V$.

The V_2 is a commutative and associative algebra with the product $ab = a_1 b$ for $a, b \in V_2$ and the identity $\frac{\omega}{2}$ (cf. [FLM]). The V_2 is called the Griess algebra of V. Note that for $a, b \in V_2$ we have $(a, b)\mathbf{1} = a_3 b$. Moreover, the form on V_2 is associative. That is, $(ab, c) = (a, bc)$ for $a, b, c \in V_2$.

Theorem 3.1. *Let V be a simple vertex operator algebra of the moonshine type with central charge $c \neq 0$ such that V is generated by V_2 and V_2 is 2-dimensional. Then V is isomorphic to $L(c_1, 0) \otimes L(c_2, 0)$ for some nonzero complex numbers c_1, c_2 such that $c_1 + c_2 = c$ if V_2 is semisimple, and isomorphic to $L(c, 0, 0)$ if V_2 is not semisimple.*

Proof. Assume that V_2 is a 2-dimensional semisimple commutative associative algebra with the identity $\omega/2$. Then $\omega/2$ is a sum of two primitive idempotents $\omega^1/2$ and $\omega^2/2$. It follows from [M1] that ω^1 and ω^2 are Virasoro vectors. Let

$$Y(\omega^i, z) = \sum_{n\in\mathbb{Z}} L^i(n) z^{-n-2}$$

for $i = 1, 2$. Then

$$[L^i(m), L^i(n)] = (m-n)L^i(m+n) + \frac{m^3 - m}{12}\delta_{m+n,0}c_i$$

for all $m, n \in \mathbb{Z}$, where $c_i \in \mathbb{C}$ is the central charge of ω^i. Since $\frac{\omega^i}{2}\frac{\omega^j}{2} = \delta_{i,j}$ we see that $(\omega^1)_3\omega^2 = (\omega^1, \omega^2)\mathbf{1} = 0$ by using the invariant property of the bilinear form. This implies that

$$[L^1(m), L^2(n)] = 0$$

for all $m, n \in \mathbb{Z}$ and $c_1 + c_2 = c$. Then $V = \langle\omega^1\rangle \otimes \langle\omega^2\rangle$, where $\langle\omega^i\rangle$ is the vertex operator subalgebra of V generated by ω^i (with a different Virasoro vector). Note that $\langle\omega^i\rangle$ is a quotient of $\bar{V}(c_i, 0)$. Since V is simple we immediately have that $\langle\omega^i\rangle$ is isomorphic to $L(c_i, 0)$. As a result, V is isomorphic to $L(c_1, 0) \otimes L(c_2, 0)$ in this case.

It remains to deal with the case that V_2 is not semisimple. In this case the Jacobson radical J of V_2 is 1-dimensional. Assume that $J = \mathbb{C}x$. Then $x^2 = 0$ and $(x, x) = (\omega/2, x^2) = 0$. Using the skew symmetry $Y(x, z)x = e^{L(-1)z}Y(x, -z)x$ we see that

$$x_0x = -x_0x + L(-1)x_1x = -x_0x + L(-1)x^2 = -x_0x.$$

This implies $x_0x = 0$. As a consequence, we see the component operators x_n of $Y(x, z)$ commute with each other. That is, $[x_m, x_n] = 0$ for all $m, n \in \mathbb{Z}$.

Note that $(\omega, \omega)\mathbf{1} = L(2)\omega = \frac{c}{2}\mathbf{1}$. Since the form $(,)$ on V_2 is nondegenerate, we may choose x so that $(\omega, x) = c/2$. Set $W(m) = x_{m+1}$ for $m \in \mathbb{Z}$. Then we have the following commutator formula

$$[L(m), W(n)] = (m-n)W(m+n) + \frac{m^3 - m}{12}\delta_{m,-n}c.$$

This exactly says that the operators $L(m), W(m), c$ generate a copy of $W(2, 2)$ and V is an irreducible highest weight module for $W(2, 2)$. Hence V is isomorphic to $L(c, 0, 0)$, as desired. □

Remark 3.2. Theorem 3.1 is the main reason we introduce and study the Lie algebra $W(2, 2)$ and its highest weight modules. The vertex operator algebra $L(c, 0, 0)$ will be used in the next section when we characterize the rational vertex operator algebra $L(1/2, 0) \otimes L(1/2, 0)$.

4. Characterization of $L(1/2, 0) \otimes L(1/2, 0)$

In this section we give a characterization for the vertex operator algebra $L(1/2, 0) \otimes L(1/2, 0)$.

We first recall some basic facts about a rational vertex operator algebra following [DLM1]. A vertex operator algebra V is called rational if any admissible module is completely reducible. It is proved in [DLM1] (also see [Z]) that if V is rational then there are only finitely many irreducible admissible modules $M^1, \ldots, M^k$ up to isomorphism such that

$$M^i = \oplus_{n\geq 0} M^i_{\lambda_i + n},$$

where $\lambda_i \in \mathbb{Q}$, $M^i_{\lambda_i} \neq 0$ and each $M^i_{\lambda_i+n}$ is finite dimensional (see [AM] and [DLM2]). Let λ_{min} be the minimum of λ_i's. The effective central charge $\tilde{c}$ is defined as $c - 24\lambda_{min}$. A vertex operator algebra is called C_2-cofinite if $C_2(V)$ has finite codimension, where $C_2(V) = \langle u_{-2}v | u, v \in V\rangle$.

Let $f(z) = q^{\lambda}\sum_{n\geq 0} a_n q^n$ be either a formal power series in z or a complex function. We say that the coefficients of $f(z)$ satisfy the *polynomial growth condition* if there exist positive numbers A and α such that $|a_n| \leq An^{\alpha}$ for all n.

For each M^i we define the q-character of M^i by

$$\mathrm{ch}_q M^i = q^{-c/24}\sum_{n\geq 0}(\dim M^i_{\lambda_i+n})q^{n+\lambda_i}.$$

Then $\mathrm{ch}_q M^i$ converges to a holomorphic function on the upper half plane if V is C_2-cofinite [Z]. Using the modular invariance result from [Z] and results on vector valued modular forms from [KM] we have (see [DM1])

Lemma 4.1. *Let V be rational and C_2-cofinite. For each i, the coefficients of $\eta(q)^{\tilde{c}}\mathrm{ch}_q M^i$ satisfy the polynomial growth condition where*

$$\eta(q) = q^{1/24}\prod_{n\geq 1}(1-q^n).$$

We also need some basic facts about the highest weight modules for the Virasoro algebra (see [FF,FQS,GKO,FZ,W]).

Proposition 4.2. *Let c be a complex number.*

(1) $\bar{V}(c,0)$ *is a vertex operator algebra and* $L(c,0)$ *is a simple vertex operator algebra.*
(2) *If $c \neq c_{s,t} = 1 - 6(s-t)^2/st$ for all coprime positive integers s, t with $1 < s < t$, then $\bar{V}(c,0) = L(c,0)$, and $L(c,0)$ is not rational. In this case, the q-character of $L(c,0)$ is equal to $\frac{q^{-c/24}}{\prod_{n>1}(1-q^n)}$ and the coefficients grow faster than any polynomials.*
(3) *If $c = c_{s,t}$ for some s, t, then $\bar{V}(c,0) \neq L(c,0)$, and $L(c,0)$ is rational.*

From now on we assume that V is a rational and C_2-cofinite vertex operator algebra of the moonshine type such that $c = \tilde{c} = 1$ and $\dim V_2 = 2$. We have already mentioned in Sect. 3 that V_2 is a commutative associative algebra with identity $\frac{\omega}{2}$.

Lemma 4.3. *The V_2 is a semisimple associative algebra. That is, V_2 is a direct sum of two ideals isomorphic to $\mathbb{C}$.*

Proof. Suppose that V_2 is not semisimple. Recall from the proof of Theorem 3.1 that the Jacobson radical $J = \mathbb{C}x$ is one-dimensional. We assume that $(\omega, x) = 1$. Then the component operator $W(n)$ of $Y(x,z) = \sum_{n\in\mathbb{Z}} W(n)z^{-n-2}$ and the component operator of the $Y(\omega, z)$ generate a copy of the W-algebra $W(2,2)$ with central charge 1.

Let U be the vertex operator subalgebra of V generated by V_2. Then U is a highest weight $W(2,2)$-module with highest weight vector $\mathbf{1}$ such that W_n acts as $W(n)$ and L_n acts as $L(n)$ for all $n \in \mathbb{Z}$. Since $L(-1)\mathbf{1} = W(-1)\mathbf{1} = 0$, we see that U is a quotient of $\bar{V}(c,0,0)$. From Theorem 2.1, $\bar{V}(c,0,0) = L(c,0,0)$ is irreducible and U is isomorphic to $L(1,0,0)$. Furthermore,

$$\mathrm{ch}_q U = \frac{q^{-1/24}}{\prod_{n>1}(1-q^n)^2}.$$

Note that $\mathrm{ch}_q U \le \mathrm{ch}_q V$, that is, the coefficients of $\mathrm{ch}_q U$ are less than or equal to the corresponding coefficients of $\mathrm{ch}_q V$. Note that if $|q| < 1$, $\mathrm{ch}_q U$ and $\mathrm{ch}_q V$ are convergent. So as functions we also have $\mathrm{ch}_q U \le \mathrm{ch}_q V$ for $q \in (0, 1)$. Then $\eta(q)\mathrm{ch}_q U \le \eta(q)\mathrm{ch}_q V$ as functions for $q \in (0, 1)$ since $\eta(q)$ is positive. By Lemma 4.1, the coefficients of $\eta(q)\mathrm{ch}_q V$ satisfy the polynomial growth condition. On the other hand, the coefficients of $\eta(q)\mathrm{ch}_q U = \frac{1-q}{\prod_{n>1}(1-q^n)}$ grow faster than any polynomial in n. Thus $\eta(q)\mathrm{ch}_q U$ should be much bigger than $\eta(q)\mathrm{ch}_q V$ as q goes close to 1. This is a contradiction. □

Again from the proof of Theorem 3.1, we can write $\omega = \omega^1 + \omega^2$ so that $\omega^1/2$ and $\omega^2/2$ are the primitive idempotents. The ω^1 and ω^2 are Virasoro vectors with central charges c_1 and c_2 such that $c_1 + c_2 = 1$. Let $L^i(n)$ be as in Sect. 3. Then we have two commutative Virasoro algebras:

$$[L^i(m), L^j(n)] = \delta_{i,j}\left((m-n)L^i(m+n) + \frac{m^3-m}{12}\delta_{m+n,0}c_i\right)$$

for $m, n \in \mathbb{Z}$ and $i, j = 1, 2$. As before we denote by U the vertex operator subalgebra of V generated by V_2. Then $U = \langle\omega^1\rangle \otimes \langle\omega^2\rangle$, where $\langle\omega^i\rangle$ is the vertex operator subalgebra of V generated by ω^i (with a different Virasoro vector). Then $\langle\omega^i\rangle$ is a quotient of $\bar{V}(c_i, 0)$.

Lemma 4.4. *If* $c \neq 0$, *then the coefficients of* $\mathrm{ch}_q L(c, 0)$ *does not satisfy the polynomial growth condition.*

Proof. If $c \neq c_{s,t}$ for any coprime integers $1 < s < t$, then $\mathrm{ch}_q L(c, 0) = \frac{q^{-c/24}}{\prod_{n>1}(1-q^n)}$ by Proposition 4.2 and the result is clear. We now assume that $c = c_{s,t}$ for some s, t. Suppose that the coefficients of

$$\mathrm{ch}_q L(c_{s,t}, 0) = q^{-c/24}\sum_{n\ge 0} a_n q^n$$

satisfy the polynomial growth condition. Then there exists a positive integer A and α such that $a_n \le An^\alpha$ for all $n \ge 0$.

Let m be a positive integer such that $m \ge \alpha$. Then

$$\frac{1}{(1-q)^{m+1}} = \sum_{n\ge 0}\binom{-m-1}{n}(-1)^n q^n,$$

where

$$\binom{-m-1}{n} = \frac{(-m-1)(-m-2)\cdots(-m-n)}{n!} = \binom{m+n}{m}(-1)^n.$$

Thus

$$\frac{1}{(1-q)^{m+1}} = \sum_{n\ge 0}\binom{m+n}{m}q^n.$$

Since $\binom{m+n}{m}$ is greater than $\frac{n^m}{m!}$ we see that

$$q^{c/24}\mathrm{ch}_q L(c_{s,t}, 0) \le m!A\frac{1}{(1-q)^{m+1}}$$

as formal power series.

We next prove that there exists a positive integer k such that $kc_{s,t} \neq c_{s_1,t_1}$ for any coprime integers $1 < s_1 < t_1$. To see this we need to examine the equation

$$1-\frac{6(s_1-t_1)^2}{s_1t_1} = k\left(1-\frac{6(s-t)^2}{st}\right)$$

which is equivalent to

$$st(13s_1t_1-6s_1^2-6t_1^2) = s_1t_1k(13st-6s^2-6t^2).$$

Then both s_1 and t_1 are factors of $6st$. So there are only finitely many s_1, t_1 satisfy this equation. This implies that such k exists.

Consider a vertex operator algebra $L(c,0)^{\otimes k}$ which contains the vertex operator subalgebra $\bar{V}(kc,0) = L(kc,0)$ as $kc \neq c_{s_1,t_1}$ for any s_1, t_1. So

$$q^{kc/24}\mathrm{ch}_q L(kc,0) \le q^{kc/24}(\mathrm{ch}_q L(c,0)^{\otimes k}) = q^{kc/24}(\mathrm{ch}_q L(c,0))^k \le (m!A)^k\frac{1}{(1-q)^{(m+1)k}}$$

and the coefficients of $q^{kc/24}\mathrm{ch}_q L(kc,0)$ satisfy the polynomial growth condition.

On the other hand we know from Proposition 4.2 that

$$q^{kc/24}\mathrm{ch}_q L(kc,0) = \frac{1}{\prod_{n>1}(1-q^n)}$$

whose coefficients satisfy the exponential growth condition. This is a contradiction. The proof is complete. □

Lemma 4.5. *Let ω^i and c_i be as before. Then $c_i = c_{s_i,t_i}$ for some coprime integers $1 < s_i < t_i$ and $\langle\omega^i\rangle$ is isomorphic to $L(c_{s_i,t_i},0)$ for $i = 1, 2$.*

Proof. Recall that U is the vertex operator subalgebra of V generated by V_2. First we note that as formal power series, $\mathrm{ch}_q U \le \mathrm{ch}_q V$. Let $U^i = \langle\omega^i\rangle$. Then $U = U^1 \otimes U^2$ and $\mathrm{ch}_q U^1 ch_q U^2 \le \mathrm{ch}_q V$. Since $\mathrm{ch}_q U^i \ge \mathrm{ch}_q L(c_i,0)$ for $i = 1, 2$ we have

$$\eta(q)\mathrm{ch}_q L(c_1,0)\mathrm{ch}_q L(c_2,0) \le \eta(q)\mathrm{ch}_q U^1\mathrm{ch}_q U^2 \le \eta(q)\mathrm{ch}_q V$$

as functions for $q \in (0,1)$.

Assume that $\mathrm{ch}_q U^1 = \frac{q^{-c_1/24}}{\prod_{n>1}(1-q^n)}$. Then

$$\eta(q)\mathrm{ch}_q U \ge \eta(q)\frac{q^{-c_1/24}}{\prod_{n>1}(1-q^n)}\mathrm{ch}_q L(c_2,0)$$

as functions for $q \in (0,1)$. That is,

$$\eta(q)\mathrm{ch}_q U \ge q^{c_2/24}(1-q)\mathrm{ch}_q L(c_2,0).$$

From the proof of Lemma 4.4 we see that if the coefficients of $(1-q)\mathrm{ch}_q L(c_2,0)$ satisfy the polynomial growth condition, so does the coefficients of $\mathrm{ch}_q L(c_2,0)$. But this is impossible by Lemma 4.4. Thus the coefficients of $(1-q)\mathrm{ch}_q L(c_2,0)$ does not satisfy the polynomial growth condition. On the other hand, $q^{c_2/24}(1-q)\mathrm{ch}_q L(c_2,0) \le \eta(q)\mathrm{ch}_q V$ as functions for $q \in (0,1)$ and the coefficients of $\eta(q)\mathrm{ch}_q V$ satisfy the polynomial growth condition. This is a contradiction.

By Proposition 4.2 we see immediately that $c_i = c_{s_i,t_i}$ for some s_i, t_i and $\langle\omega_i\rangle$ is isomorphic to $L(c_{s_i,t_i},)$ for $i = 1, 2$. □

Lemma 4.6. *Let $c_i = c_{s_i,t_i}$ as in Lemma 4.5. Then both c_1 and c_2 are $1/2$.*

Proof. We need to solve the equation

$$1 - \frac{6(s_1 - t_1)^2}{s_1 t_1} + 1 - \frac{6(s_2 - t_2)^2}{s_2 t_2} = 1$$

for two pairs of coprime integers $1 < s_i < t_i$. That is,

$$\frac{s_1}{t_1} + \frac{t_1}{s_1} + \frac{s_2}{t_2} + \frac{t_2}{s_2} = \frac{25}{6}.$$

Let $x = \frac{s_1}{t_1}$ and $y = \frac{s_2}{t_2}$. Then the equation becomes

$$x + \frac{1}{x} + y + \frac{1}{y} = \frac{25}{6}.$$

The following argument using the elliptic curve is due to N. Elkies and we thank him and A. Ryba for communicating the solution to us. The equation $x + \frac{1}{x} + y + \frac{1}{y} = \frac{25}{6}$ gives an elliptic curve. Multiply the equation by $6xy$ to get

$$E : 6xy^2 + 6x^2 y + 6x + 6y = 25xy.$$

Putting one of the Weierstrass points at infinity yields the curve

$$Y^2 + XY = X^3 - 1070X + 7812$$

which has rank 0 over $\mathbb{Q}$. So every rational points in E is a torsion point. So $E/\mathbb{Q}$ has at most 16 torsion points. Note that the curve has 8 obvious symmetries, generated by the involutions taking (x, y) to $(1/x, y)$, $(x, 1/y)$, and (y, x). Here are the rational points in E : four from $(\frac{3}{4}, \frac{3}{4})$, four from $(1, \frac{2}{3})$, four from $(-1, 6)$ and four from infinity.

Since we assume that $1 < s_i < t_i$ and s_i, t_i are coprime, we immediately see that the only solution interesting to us is $(\frac{3}{4}, \frac{3}{4})$. This is, $c_i = \frac{1}{2}$ for $i = 1, 2$. □

Here is a characterization of $L(1/2, 0) \otimes L(1/2, 0)$.

Theorem 4.7. *If V is a simple, rational and C_2-cofinite vertex operator algebra of the moonshine type such that $c = \tilde{c} = 1$ and $\dim V_2 = 2$, then V is isomorphic to $L(1/2, 0) \otimes L(1/2, 0)$.*

Proof. By Lemmas 4.5 and 4.6, the vertex operator subalgebra U generated by V_2 of V is isomorphic to $L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0)$ which is rational and has 9 inequivalent irreducible modules $L(\frac{1}{2}, h_1) \otimes L(\frac{1}{2}, h_2)$ for $h_i \in \{0, \frac{1}{2}, \frac{1}{16}\}$ (see [DMZ,W]).Thus V is a direct sum of irreducible $L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0)$-modules. Note that $h_1 + h_2 \in \mathbb{Z}$ if and only if $h_1 = h_2 = 0$ or $h_1 = h_2 = \frac{1}{2}$. So only $L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0)$ and $L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{1}{2}, \frac{1}{2})$ can possibly occur in V as $L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0)$-modules. Since $\dim V_0 = 1$ and $V_1 = 0$, we immediately see that V is isomorphic to $L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0)$. □

We certainly believe that Theorem 4.7 is false if we do not assume $c=\tilde{c}$. One can construct a counter example involving the permutation orbifolds [BDM] modulo the following rational orbifold theory conjecture: If V is a rational vertex operator algebra and A is a finite automorphism group of V then the fixed point vertex operator subalgebra V^A is rational. Let $U=L(c_{3,5},0)^{\otimes 5}$ and $W=L(1/2,0)^{\otimes 8}$. Then both U and V are rational vertex operator algebras with central charges -3 and 4 respectively. Let G be the cyclic group generated by the permutation $(1,2,3,4,5)$ and H the cyclic group generated by $(1,2,3,4,5,6,7,8)$. Then G and H act obviously on U and W as automorphisms. The tensor product $U^G\otimes W^H$ is a counter example.

We end this paper with the following conjecture which strengthens Theorem 4.7.

Conjecture 4.8. If V is a simple, rational and C_2 cofinite vertex operator algebra of the moonshine type with $c=\tilde{c}=1$ and $\dim V_2>1$, then V is isomorphic to $L(1/2,0)\otimes L(1/2,0)$.

We have already mentioned that Theorem 4.7 is false without assuming $c=\tilde{c}$. This implies that Conjecture 4.8 is false without assuming $c=\tilde{c}$. Here we give a counter example to the conjecture without using the rational orbifold theory conjecture. Let $U=L(c_{3,5},0)^{\otimes 5}$ and $W=L(1/2,0)^{\otimes 8}$ as in the counter example before the conjecture. Then $V=U\otimes W$ is a rational, C_2-cofinite vertex operator algebra of the moonshine type and with $c=1$, $\tilde{c}=7$ (cf. [DM1]). It is clear that $\dim V_2=13$, and V is not isomorphic to $L(1/2,0)\otimes L(1/2,0)$.

It is essentially proved in [K] that if V is a rational vertex operator algebra such that $\sum_i|\chi_i(q)|^2$ is modular invariant where $\chi_i(q)$ are the q-character of the irreducible V-modules, then the q-character of V is equal to the character of one of the following vertex operator algebras V_L, V_L^+ and $V_{\mathbb{Z}\alpha}^G$, where L is any positive definite even lattice of rank 1, V_L^+ is the fixed points of the automorphism of V lifted from the -1 isometry of L, and $\mathbb{Z}\alpha$ is the root lattice of type A_1 such that $(\alpha,\alpha)=2$ and G is a finite subgroup of $SO(3)$ isomorphic to A_4, S_4 or A_5. It is widely believed that V_L, V_L^+ and $V_{\mathbb{Z}\alpha}^G$ should give a complete list of simple and rational vertex operator algebras with $c=\tilde{c}=1$. It is clear from the construction that if V is one of these vertex operator algebras of the moonshine type then $\dim V_2=2$. This should be very strong evidence for Conjecture 4.8. We remark that the assumption that $\sum_i|\chi_i(q)|^2$ is modular invariant in [K] is still an open problem in mathematics.

References

[AM] Anderson, G., Moore, G.: Rationality in conformal field theory. Commun. Math. Phys. **117**, 441–450 (1988)

[BDM] Dong, C., Barron, K., Mason, G.: Twisted sectors for tensor product voas associated to permutation groups. Commun. Math. Phys. **227**, 349–384 (2002)

[D] Dong, C.: Representations of the moonshine module vertex operator algebra. Contemp. Math. **175**, 27–36 (1994)

[DGH] Dong, C., Griess, R. Jr., Hoehn, G.: Framed vertex operator algebras, codes and the moonshine module. Commun. Math. Phys. **193**, 407–448 (1998)

[DGL] Dong, C., Griess, R. Jr., Lam, C.: Uniqueness results of the moonshine vertex operator algebra. Amer. J. Math. **129**, 583–609 (2007)

[DLM1] Dong, C., Li, H., Mason, G.: Twisted representations of vertex operator algebras. Math. Ann. **310**, 571–600 (1998)

[DLM2] Dong, C., Li, H., Mason, G.: Modular invariance of trace functions in orbifold theory and generalized moonshine. Commun. Math. Phys. **214**, 1–56 (2000)

[DM1] Dong, C., Mason, G.: Rational vertex operator algebras and the effective central charge. International Math. Research Notices **56**, 2989–3008 (2004)

[DM2] Dong, C., Mason, G.: *Integrability of C_2-cofinite vertex operator algebras*. International Math. Research Notices **2006** Article ID 80468, 15 pages, 2006
[DMZ] Dong, C., Mason, G., Zhu, Y.: Discrete series of the virasoro algebra and the moonshine module. Proc. Symp. Pure. Math., American Math. Soc. **56**(II), 295–316 (1994)
[FF] Feigin, B., Fuchs, D.: *Verma Modules over the Virasoro Algebra*. Lect. Notes in Math. **1060**, Berlin-Heidelberg-New York: Springer, pp. 230–245 (1984)
[FLM] Frenkel, I.B., Lepowsky, J., Meurman, A.: *Vertex Operator Algebras and the Monster*. Pure and Applied Math., Vol. **134**, New York: Academic Press, 1988
[FZ] Frenkel, I.B., Zhu, Y.: Vertex operator algebras associated to representations of affine and virasoro algebras. Duke Math. J. **66**, 123–168 (1992)
[FQS] Friedan, D., Qiu, Z., Shenker, S.: Details of the non-unitarity proof for highest weight representations of virasoro algebra. Commun. Math. Phys. **107**, 535–542 (1986)
[GKO] Goddard, P., Kent, A., Olive, D.: Unitary representations of the virasoro algebra and super-virasoro algebras. Commun. Math. Phys. **103**, 105–119 (1986)
[KL] Kawahigashi, Y., Longo, R.: Local conformal nets arising from framed vertex operator algebras. Adv. Math. **206**, 729–751 (2006)
[K] Kiritsis, E.: Proof of the completeness of the classification of rational conformal field theories with $c = 1$. Phys. Lett. B **217**, 427–430 (1989)
[KM] Knopp, M., Mason, G.: On vector-valued modular forms and their fourier coefficients. Acta Arith. **110**, 117–124 (2003)
[LY] Lam, C., Yamauchi, H.: *A characterization of the moonshine vertex operator algebra by means of Virasoro frame*. http://arXiv.org/list/math/0609718, 2006
[L1] Li, H.: Symmetric invariant bilinear forms on vertex operator algebras. J. Pure and Appl. Math. **96**, 279–297 (1994)
[L2] Li, H.: Local system of vertex operators, vertex superalgebras and modules. Pure and Appl. Math. **109**, 143–195 (1996)
[LL] Lepowsky, J., Li, H.: *Introduction to Vertex Operator Algebras, and Their Representations*. Progress in Math. Vol. **227**, Birkhäuser, Boston, 2004
[M1] Miyamoto, M.: Griess algebras and conformal vectors in vertex operator algebras. J. Algebra **179**, 523–548 (1996a)
[M2] Miyamoto, M.: Binary codes and vertex operator (super)algebras. J. Algebra **181**, 207–222 (1996b)
[M3] Miyamoto, M.: A new construction of the moonshine vertex operator algebra over the real number field. Ann. of Math. **159**, 535–596 (2004)
[W] Wang, W.: Rationality of virasoro vertex operator algebras. International Math. Research Notices **71**, 197–211 (1993)
[Z] Zhu, Y.: Modular invariance of characters of vertex operator algebras. J. Amer, Math. Soc. **9**, 237–302 (1996)

Communicated by Y. Kawahigashi