(Co)cyclic (Co)homology of Bialgebroids: An Approach via (Co)monads

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Abstract: For a (co)monad T_l on a category $\mathcal{M}$, an object X in $\mathcal{M}$, and a functor $\Pi : \mathcal{M} \to \mathcal{C}$, there is a (co)simplex $Z^* := \Pi T_l^{*+1} X$ in $\mathcal{C}$. The aim of this paper is to find criteria for para-(co)cyclicity of Z^*. Our construction is built on a distributive law of T_l with a second (co)monad T_r on $\mathcal{M}$, a natural transformation $i : \Pi T_l \to \Pi T_r$, and a morphism $w : T_r X \to T_l X$ in $\mathcal{M}$. The (symmetrical) relations i and w need to satisfy are categorical versions of Kaygun's axioms of a transposition map. Motivation comes from the observation that a (co)ring T over an algebra R determines a distributive law of two (co)monads $T_l = T \otimes_R (-)$ and $T_r = (-) \otimes_R T$ on the category of R-bimodules. The functor Π can be chosen such that $Z^n = T \widehat{\otimes}_R \cdots \widehat{\otimes}_R T \widehat{\otimes}_R X$ is the cyclic R-module tensor product. A natural transformation $i : T \widehat{\otimes}_R (-) \to (-) \widehat{\otimes}_R T$ is given by the flip map and a morphism $w : X \otimes_R T \to T \otimes_R X$ is constructed whenever T is a (co)module algebra or coring of an R-bialgebroid. The notion of a stable anti-Yetter-Drinfel'd module over certain bialgebroids, the so-called $\times_R$-Hopf algebras, is introduced. In the particular example when T is a module coring of a $\times_R$-Hopf algebra $\mathcal{B}$ and X is a stable anti-Yetter-Drinfel'd $\mathcal{B}$-module, the para-cyclic object Z_* is shown to project to a cyclic structure on $T^{\otimes_R *+1} \otimes_{\mathcal{B}} X$. For a $\mathcal{B}$-Galois extension $S \subseteq T$, a stable anti-Yetter-Drinfel'd $\mathcal{B}$-module T_S is constructed, such that the cyclic objects $\mathcal{B}^{\otimes_R *+1} \otimes_{\mathcal{B}} T_S$ and $T^{\widehat{\otimes}_S *+1}$ are isomorphic. This extends a theorem by Jara and Ştefan for Hopf Galois extensions. As an application, we compute Hochschild and cyclic homologies of a groupoid with coefficients in a stable anti-Yetter-Drinfel'd module, by tracing it back to the group case. In particular, we obtain explicit expressions for (coinciding relative and ordinary) Hochschild and cyclic homologies of a groupoid. The latter extends results of Burghelea on cyclic homology of groups.

Contents

Introduction

Cyclic cohomology of Hopf algebras originates from the work [CM98] of Connes and Moscovici on the index theory of transversally elliptic operators. Their local index formula in [CM95] gives a generalization of the Chern character to non-commutative geometry. In order to give a geometrical interpretation of the non-commutative Chern character in terms of non-commutative foliations, in [CM98] a cocyclic structure was constructed on a cosimplex $Z_{CM}^* = H^{\otimes *}$, associated to the coalgebra underlying a Hopf algebra H over a field $\mathbb{K}$. The cocyclic operator was given in terms of a so-called modular pair in involution.

In the subsequent years the Connes-Moscovici cocyclic module was placed in a broader and broader context. In [KR03] (see also [HKRS2]) to any (co)module algebra T of a Hopf algebra H, and any H-(co)module X, there was associated a para-cyclic module with components $T^{\otimes *+1} \otimes X$. Dually, for any (co)module coalgebra T of a Hopf algebra H, and any H-(co)module X, there is a para-cocyclic module with components $T^{\otimes *+1} \otimes X$. The Connes-Moscovici cosimplex Z_{CM}^* turns out to be isomorphic to a quotient of the para-cocyclic module associated to the regular module coalgebra $T := H$ and an H-comodule defined on $\mathbb{K}$. For bialgebras, the Connes-Moscovici construction was generalized in [Kay05].

In the papers [HKRS1] and [JŞ], a modular pair in involution was proven to be equivalent to a stable anti-Yetter-Drinfel'd module structure on the ground field $\mathbb{K}$. In [HKRS2], the para-cocyclic module $T^{\otimes *+1} \otimes X$, associated to an H-module coalgebra T and a stable anti-Yetter-Drinfel'd H-module X, was shown to project to a cocyclic object whose components are the H-module tensor products $T^{\otimes *+1} \otimes_H X$. The way in which the para-cocyclic object $H^{\otimes *+1}$ projects to the Connes-Moscovici cosimplex Z_{CM}^*, is an example of this scenario. Dually, the para-cyclic module, associated to an H-comodule algebra, was proven to have a cyclic submodule.

In the spirit of [JŞ], one can follow a dual approach. That is, para-cocyclic modules can be constructed for (co)module algebras of Hopf algebras, and para-cyclic modules for (co)module coalgebras, in both cases with coefficients in H-(co)modules. Taking coefficients in a stable anti-Yetter-Drinfel'd module, it was shown in [JŞ] that in this case the para-cyclic object associated to a module coalgebra possesses a cyclic quotient.

In [KR05] an isomorphism was proven between the cyclic quotient of the para-cyclic object in [JŞ] of H as an H-module coalgebra on one hand, and the cyclic subobject of the para-cyclic object in [HKRS2] of H as an H-comodule algebra on the other.

Constructions in Sect. 2 of the current paper follow the root in [JŞ]. Since this framework is dual to that suggested in [HKRS2] (cf. also [Kay06]), some might like to call it a *dual* Hopf (co)cyclic theory. However, we do not use this somewhat involved terminology in the paper, but remind the reader of the difference between the two possible dual approaches.

In [Kay06] Kaygun proposed a unifying approach to the para-(co)cyclic objects corresponding to a (co)module (co)algebra of a Hopf algebra. Starting with a (co)algebra T and an object X in a symmetric monoidal category $\mathcal{S}$, he introduced the notion of a transposition map. It is a morphism $w : X \otimes T \to T \otimes X$ in $\mathcal{S}$, satisfying conditions reminiscent to half of the axioms of a distributive law in [Be]. Any transposition map w was shown to determine a para-(co)cyclic structure on the (co)simplex $T^{\otimes *+1} \otimes X$ in $\mathcal{S}$. In particular, canonical transposition maps were constructed for (co)module (co)algebras T and (co)modules X of a bialgebra.

Connes and Moscovici's index theory of transversally elliptic operators lead beyond cyclic homology of Hopf algebras. In dealing with the general, non-flat case, in [CM01] certain bialgebroids (in fact $\times_R$-Hopf algebras) arose naturally. Bialgebroids can be thought of as a generalization of bialgebras to a non-commutative base algebra R, while $\times_R$-Hopf algebras generalize Hopf algebras. There are a few papers in the literature, e.g. [KR04] and [Ra], attempting to extend Hopf cyclic theory to non-commutative base algebras. However, an understanding of the subject, comparable to that in the classical case of a commutative base ring (or field), is missing yet. The aim of the current paper is to give a universal construction of para-(co)cyclic (co)simplices, including examples coming from (co)module algebras and (co)module corings for bialgebroids.

When replacing bialgebras over a commutative ring $\mathbb{K}$ by bialgebroids over a non-commutative $\mathbb{K}$-algebra R, the monoidal category of $\mathbb{K}$-modules becomes replaced by the monoidal category of R-bimodules. Indeed, (co)module algebras of an R-bialgebroid are in particular algebras, and (co)module corings are coalgebras, in the category of R-bimodules. The main difference is that the category of $\mathbb{K}$-modules is symmetric. In contrast, the category of R-bimodules is not even braided in general. Hence Kaygun's elegant theory [Kay06], formulated in a symmetric monoidal category $\mathcal{S}$, is not applicable.

Our key observation is that the role the symmetry plays in Kaygun's work is that it defines a compatible natural transformation i between the two (co)monads $T \otimes (-)$ and $(-) \otimes T$ on the symmetric monoidal category $\mathcal{S}$, induced by a (co)algebra T in $\mathcal{S}$. Note that these (co)monads on $\mathcal{S}$ are connected by a trivial distributive law. Guided by this observation, in Sect. 1 we start with a distributive law of two (co)monads T_l and T_r on any category $\mathcal{M}$. In addition, we allow for the presence of a functor $\mathbf{\Pi} : \mathcal{M} \to \mathcal{C}$ (it is the identity functor on $\mathcal{S}$ in [Kay06]). Then, for any object X in $\mathcal{M}$, there is a (co)simplex in $\mathcal{C}$, given at degree n by $\mathbf{\Pi} T_l^{n+1} X$. In Sects. 1.C and 1.E we show that it is para-(co)cyclic provided that there exist a natural transformation $i : \mathbf{\Pi} T_l \to \mathbf{\Pi} T_r$ and a morphism $T_r X \to T_l X$ in $\mathcal{M}$, satisfying symmetrical conditions generalizing Kaygun's axioms of a transposition map. Examples of this situation are collected in Sect. 1.D. Among other (classical) examples, we show that Škoda's functorial construction in [Šk] of a para-cyclic object in the category of endofunctors, Majid and Akrami's para-cyclic modules associated to a ribbon algebra in [AM], and Rangipour's cyclic module in [Ra] determined by a coring, fit our framework. It is discussed in Sects. 2.A and 2.B how the general results in Sects. 1.C and 1.E cover the particular cases when the two

(co)monads $T_l = T \otimes_R (-)$ and $T_r = (-) \otimes_R T$ are induced by a (co)module algebra or (co)module coring T of an R-bialgebroid $\mathcal{B}$, the functor Π is defined via the coequalizer of the R-actions in a bimodule, and X is a $\mathcal{B}$-(co)module. The components of the resulting para-(co)cyclic module are cyclic R-module tensor products $T \widehat{\otimes}_R \cdots \widehat{\otimes}_R T \widehat{\otimes}_R X$. In this way we obtain examples which extend both some para-(co)cyclic objects in [Kay06] and [HKRS2] and their cyclic duals.

By the above procedure, we associate a para-cyclic object in particular to a module coring C and a comodule X of an R-bialgebroid $\mathcal{B}$. Following [JŞ], in Sect. 2.C we look for situations in which it projects to the $\mathcal{B}$-module tensor product $(C \otimes_R \cdots \otimes_R C) \otimes_{\mathcal{B}} X$. Restricting at this point our study to $\times_R$-Hopf algebras $\mathcal{B}$, we define stable anti-Yetter-Drinfel'd modules for $\mathcal{B}$. In parallel to the case of Hopf algebras [JŞ, Theorem 4.13], [HKRS2, Theorem 2.1], we prove cyclicity of the simplex $(C \otimes_R \cdots \otimes_R C) \otimes_{\mathcal{B}} X$, whenever X is a stable anti-Yetter-Drinfel'd module.

The simplest example of a cyclic simplex is associated to an algebra extension $S \subseteq T$. Its components are given by the $n + 1$ fold cyclic tensor product $T^{\widehat{\otimes}_S n+1}$; face and degeneracy maps are determined by the algebra structure of T and the cyclic map is given by the cyclic permutation of the tensor factors. In Sect. 2.D, for a Galois extension $S \subseteq T$ by a $\times_R$-Hopf algebra $\mathcal{B}$, we construct a stable anti-Yetter-Drinfel'd module $T_S := T / \{ s \cdot t - t \cdot s \mid s \in S, \ t \in T \}$. We prove that the cyclic simplices $T^{\widehat{\otimes}_S n+1}$ and $\mathcal{B}^{\otimes_R n+1} \otimes_{\mathcal{B}} T_S$ are isomorphic. This extends [JŞ, Theorem 3.7].

A most fundamental class of examples of bialgebroids (in fact $\times_R$ Hopf algebras) is given by algebras (over fields), generated by a groupoid of finitely many objects. As an application of our abstract theory, we compute explicitly the relative Hochschild and cyclic homologies of such a groupoid, with coefficients in a stable anti-Yetter-Drinfel'd module. By our results, any Galois extension by the groupoid provides us with a stable anti-Yetter-Drinfel'd module. In particular, the groupoid algebra $\mathcal{B}$ is a Galois extension of its base algebra R. Applying the isomorphism of the simplices $\mathcal{B}^{\widehat{\otimes}_R n+1}$ and $\mathcal{B}^{\otimes_R n+1} \otimes_{\mathcal{B}} \mathcal{B}_R$, we obtain the R-relative cyclic homology of $\mathcal{B}$. Since R is a separable algebra, it is equal to ordinary cyclic homology of $\mathcal{B}$, hence our results extend those by Burghelea on the cyclic homology of groups [Burg]. Similar formulae were obtained by Crainic for cyclic homology of *étale* groupoids [Cra]. Observe that any groupoid (with an arbitrary set of objects can be obtained as a direct limit of groupoids with finite sets of objects. Certainly, the algebra generated by a groupoid with infinitely many objects is no longer unital. However, one can still consider its cyclic homology, as a homology of Connes' complex, associated to a presimplicial object. Since the homology functor commutes with direct limits, we can extend our formula of cyclic homology to arbitrary groupoids.

Throughout the paper $\mathbb{K}$ denotes a commutative and associative unital ring. The term $\mathbb{K}$-*algebra* means an associative and unital algebra over $\mathbb{K}$.

1. The (Co)cyclic Object Associated to a Transposition Map

In this first section we establish a general categorical framework – in terms of *admissible septuples* and their *transposition maps* – to produce para-cocyclic, and dually, para-cyclic objects.

1.A. Notation and conventions. In the 2-category CAT we denote horizontal composition (of functors) by juxtaposition, while $\circ$ is used for vertical composition (of natural
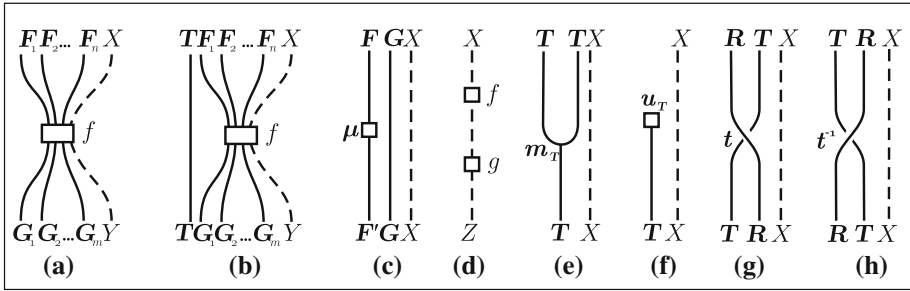
Fig. 1. Diagrammatic representation of morphisms in a category

transformations). That is, for two functors $F : \mathcal{C} \to \mathcal{C}'$, $G : \mathcal{C}' \to \mathcal{C}''$ and an object X in $\mathcal{C}$, instead of $G(F(X))$ we write GFX. For two natural transformations $\mu : F \to F'$ and $\nu : G \to G'$ we write $G'\mu X \circ \nu FX : GFX \to G'F'X$ instead of $G'(\mu_X) \circ \nu_{F(X)}$. In equalities of natural transformations we shall omit the object X in our formulae.

Inspired by the diagrammatic computation in a 2-category (in particular CAT), we shall use a graphical representation of morphisms in a category. For functors $F_1, \ldots, F_n, G_1, \ldots, G_m$, which can be composed to $F_1 F_2 \cdots F_n : \mathcal{D}_1 \to \mathcal{C}$ and $G_1 G_2 \cdots G_m : \mathcal{D}_2 \to \mathcal{C}$, and objects X in $\mathcal{D}_1$ and Y in $\mathcal{D}_2$, a morphism $f : F_1 F_2 \cdots F_n X \to G_1 G_2 \cdots G_m Y$ will be represented vertically, with the domain up, as in Fig. 1(a). Furthermore, for a functor $T : \mathcal{C} \to \mathcal{C}'$, the morphism $T f$ will be drawn as in (b). Keeping the notation from the first paragraph of this section, the picture representing μGX is shown in diagram (c). The composition $g \circ f$ of the morphisms $f : X \to Y$ and $g : Y \to Z$ will be represented as in diagram (d). For the multiplication m_T and the unit u_T of a monad T on $\mathcal{C}$ (see Definition 1.1), and an object X in $\mathcal{C}$, to draw $m_T X$ and $u_T X$ we shall use the diagrams (e) and (f), while for a distributive law $t : RT \to TR$ (see Definition 1.3) tX will be drawn as in the picture (g). If t is invertible, the representation of $t^{-1} X$ is shown in diagram (h). For simplifying diagrams containing only natural transformations, we shall always omit the last string that corresponds to an object in the category. That is, we work with diagrams in CAT whenever it is possible.

We shall use the following method to perform computations with such diagrams. In view of associativity of composition, any diagram, representing a well-defined composition of morphisms, can be thought of as a tower with several layers. Any part of the diagram, corresponding to a layer, can be substituted with any other equivalent representation of it. Usually, equivalent representations are obtained from formulas that define the notions that we deal with, or equations that have been previously proved.

1.B. Monads and distributive laws. Monads represent the main ingredient in our approach to cyclic (co)homology. The definition of monads traces back to Godement's book [Go], where they are called "standard constructions". In the literature they are also called "triples", see for example [EM].

Definition 1.1. *A* **monad** *on a category $\mathcal{C}$ is a triple (T, m_T, u_T), where $T : \mathcal{C} \to \mathcal{C}$ is a functor, $m_T : T^2 \to T$ and $u_T : Id_{\mathcal{C}} \to T$ are natural transformations such that the first two diagrams in Fig. 2, expressing associativity and unitality, are commutative. We call m_T and u_T the* multiplication *and the* unit *of the monad T, respectively.*

For two monads (T, m_T, u_T) and $(T', m_{T'}, u_{T'})$ on $\mathcal{C}$, we say that a natural transformation $\varphi : T \to T'$ is a morphism of monads *if the last two diagrams in Fig. 2 are commutative.*
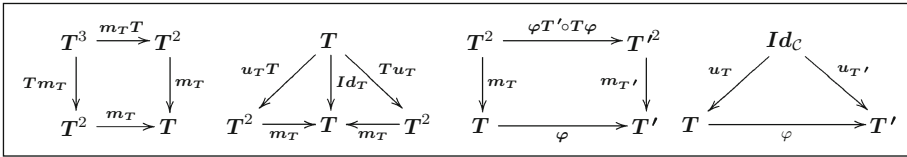
Fig. 2. Monads and morphisms of monads

Example 1.2. Let $(\mathcal{C}, \otimes, \boldsymbol{a}, \boldsymbol{l}, \boldsymbol{r}, \mathbf{1})$ be a monoidal category with unit object $\mathbf{1}$, associativity constraint $\boldsymbol{a}$ and unit constraints $\boldsymbol{l}, \boldsymbol{r}$. For details about monoidal categories the reader is referred to [Kass, Chap. XI]. An algebra in $\mathcal{C}$ is a triple (T, m_T, u_T) such that $m_T : T \otimes T \to T$ defines an associative multiplication on T with unit $u_T : \mathbf{1} \to T$. To such an algebra one can associate two monads $\boldsymbol{T_l} := T \otimes (-)$ and $\boldsymbol{T_r} := (-) \otimes T$ on $\mathcal{C}$. The multiplication $\boldsymbol{m_{T_l}}$ and the unit $\boldsymbol{u_{T_l}}$ of $\boldsymbol{T_l}$ are given by

$$\boldsymbol{m_{T_l}} X := (m_T \otimes X) \circ \boldsymbol{a}_{T,T,X}^{-1} \qquad \text{and} \qquad \boldsymbol{u_{T_l}} X := (u_T \otimes X) \circ \boldsymbol{l}_X^{-1},$$

for every X in $\mathcal{C}$. Analogously, for X in $\mathcal{C}$, $\boldsymbol{m_{T_r}} X$ and $\boldsymbol{u_{T_r}} X$ are defined by

$$\boldsymbol{m_{T_r}} X := (X \otimes m_T) \circ \boldsymbol{a}_{X,T,T} \qquad \text{and} \qquad \boldsymbol{u_{T_r}} X := (X \otimes u_T) \circ \boldsymbol{r}_X^{-1}.$$

A homomorphism $\varphi : T \to T'$ of algebras in $\mathcal{C}$ induces monad morphisms $\boldsymbol{\varphi_l} : \boldsymbol{T_l} \to \boldsymbol{T_l'}$ and $\boldsymbol{\varphi_r} : \boldsymbol{T_r} \to \boldsymbol{T_r'}$. For example, $\boldsymbol{\varphi_l} X := \varphi \otimes \mathrm{id}_X$, for any object X in $\mathcal{C}$.

A particular case of these constructions, which is very important for our work, is obtained when we take $\mathcal{C}$ to be the category R-Mod-R of bimodules over an ordinary $\mathbb{K}$-algebra R (i.e. R is an algebra in the category of $\mathbb{K}$-modules, where $\mathbb{K}$ is a commutative ring). The category R-Mod-R is monoidal with respect to the R-module tensor product $\otimes_R$. The unit object is R. An algebra in R-Mod-R is called an *R-ring*. R-rings (T, m_T, φ) are in bijective correspondence with $\mathbb{K}$-algebra maps $\varphi : R \to T$. Indeed, for an algebra (T, m_T, φ) in R-Mod-R, composition of the canonical epimorphism $T \otimes_{\mathbb{K}} T \to T \otimes_R T$ with $m_T : T \otimes_R T \to T$ defines a $\mathbb{K}$-algebra structure on T such that φ is a $\mathbb{K}$-algebra homomorphism. Conversely, via a $\mathbb{K}$-algebra homomorphism $\varphi : R \to T$, T becomes an R-bimodule. Multiplication of T induces a morphism m_T from $T \otimes_R T$ to T, which makes T an associative algebra in R-Mod-R. The unit of T is φ. (With a slight abuse of notation, we denote both multiplication maps $T \otimes_R T \to T$ and $T \otimes_{\mathbb{K}} T \to T$ by the same symbol m_T.) Consequently, a $\mathbb{K}$-algebra homomorphism $\varphi : R \to T$ defines two monads $T \otimes_R (-)$ and $(-) \otimes_R T$ on R-Mod-R.

Distributive laws were introduced by J. Beck [Be]. They give a way to compose two monads in order to obtain a monad.

Definition 1.3. *A* **distributive law** *between two monads* $(\boldsymbol{R}, \boldsymbol{m_R}, \boldsymbol{u_R})$ *and* $(\boldsymbol{T}, \boldsymbol{m_T}, \boldsymbol{u_T})$ *is a natural transformation* $\boldsymbol{t} : \boldsymbol{RT} \to \boldsymbol{TR}$ *satisfying the four conditions in Fig. 3.*

Remark 1.4. Since we are using for the first time the diagrammatic representation of morphisms, let us write out explicitly the first and the third relations in Fig. 3. They read as

$$\boldsymbol{t} \circ \boldsymbol{m_R} \boldsymbol{T} = \boldsymbol{T} \boldsymbol{m_R} \circ \boldsymbol{t} \boldsymbol{R} \circ \boldsymbol{R} \boldsymbol{t}, \qquad\qquad \boldsymbol{t} \circ \boldsymbol{u_R} \boldsymbol{T} = \boldsymbol{T} \boldsymbol{u_R}.$$
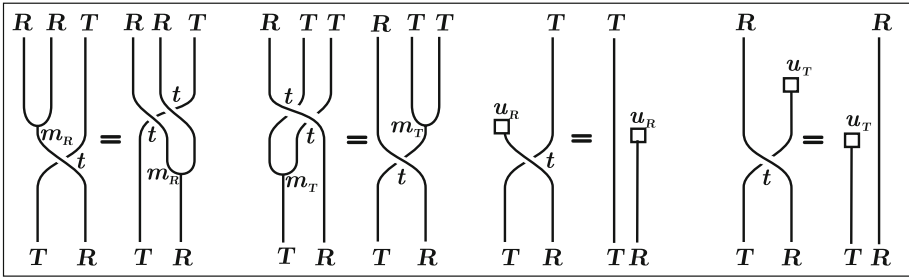
Fig. 3. The definition of distributive laws

Example 1.5. Let T be an algebra in a monoidal category as in Example 1.2. Keeping the notation from Example 1.2, the natural transformation $t : T_r T_l \to T_l T_r$, given by

$$tX := a_{T,X,T} \tag{1.1}$$

for any object X in $\mathcal{C}$, is a distributive law. Note that the first equality in Fig. 3 follows by the Pentagon Axiom [Kass, p. 282, Diagram (2.6)], applied to the quadruple (T, X, T, T). Similarly, by applying the Pentagon Axiom for (T, T, X, T) we deduce the second equality in the definition of distributive laws. The fourth equality in Fig. 3 is a consequence of $l_{X \otimes T} \circ a_{1,X,T} = l_X \otimes T$, see [Kass, Lemma XI.2.2]. The other relation in the above cited lemma can be used to prove that the third equality in Fig. 3 holds too.

Example 1.6. Let $\mathcal{C}$ be a braided monoidal category with braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$. For the definition and properties of braided monoidal categories see [Kass, Chap. XIII]. If R and T are algebras in $\mathcal{C}$ then

$$tX := a_{T,R,X} \circ (c_{R,T} \otimes X) \circ a^{-1}_{R,T,X}$$

defines a distributive law $t : R_l T_l \to T_l R_l$, where R_l and T_l are constructed as in Example 1.2. Obviously, $t^{-1} : T_l R_l \to R_l T_l$ is also a distributive law.

1.C. Admissible septuples and transposition maps. The main result. In this section we introduce admissible septuples and transposition morphisms of them. We show that to these data one associates functorially para-cocyclic objects. Our aim is twofold. On one hand, in this way we obtain a very general but at the same time technically very simple framework. In particular, it can be used to associate para-cocyclic objects to (co)module algebras of bialgebroids, cf. Sect. 2.A. On the other hand, the resulting setting will be easily dualized to describe in Sect. 2.B the situation dual to that in Sect. 2.A, i.e. the (para-)cyclic objects associated to (co)module corings of bialgebroids.

Definition 1.7. *An **admissible septuple** $\mathcal{S} := (\mathcal{M}, \mathcal{C}, T_l, T_r, \Pi, t, i)$ is defined by the following data:*

- *Two categories $\mathcal{M}$ and $\mathcal{C}$;*
- *Two monads T_l and T_r on $\mathcal{M}$;*
- *A functor $\Pi : \mathcal{M} \to \mathcal{C}$;*
- *A distributive law $t : T_r T_l \to T_l T_r$;*
- *A natural transformation $i : \Pi T_l \to \Pi T_r$.*

These data are assumed to satisfy the relations

$$i \circ \Pi u_{T_l} = \Pi u_{T_r} \quad and \quad i \circ \Pi m_{T_l} = \Pi m_{T_r} \circ i T_r \circ \Pi t \circ i T_l. \qquad (1.2)$$

Examples of admissible septuples will be given in Sect. 1.D, where also several applications of the main result of this section, Theorem 1.10, will be indicated.

By [We, p. 281], to every monad $T_l : \mathcal{M} \to \mathcal{M}$ and object X in $\mathcal{M}$ one can associate a cosimplicial object of components $T_l^{n+1} X$ in $\mathcal{M}$. Thus in particular an admissible septuple $\mathcal{S}$ in Definition 1.7 determines a cosimplicial object in $\mathcal{M}$. It can be transported to $\mathcal{C}$ via the functor $\Pi : \mathcal{M} \to \mathcal{C}$ in Definition 1.7. The resulting cosimplex in $\mathcal{C}$ will be denoted by $Z^*(\mathcal{S}, X)$. By construction, $Z^n(\mathcal{S}, X) = \Pi T_l^{n+1} X$ and, for every $k \in \{0, \ldots, n\}$, the coface maps $d_k : \Pi T_l^n X \to \Pi T_l^{n+1} X$ and the codegeneracy maps $s_k : \Pi T_l^{n+2} X \to \Pi T_l^{n+1} X$ are given by

$$d_k := \Pi T_l^k u_{T_l} T_l^{n-k} X, \qquad s_k := \Pi T_l^k m_{T_l} T_l^{n-k} X. \qquad (1.3)$$

Our aim is to construct a category $\mathcal{W}_{\mathcal{S}}$ such that $Z^*(\mathcal{S}, -)$ can be regarded as a functor from $\mathcal{W}_{\mathcal{S}}$ to the category of para-cocyclic objects in $\mathcal{C}$. Observe that, for an admissible septuple $\mathcal{S}$ in Definition 1.7, the distributive law t is lifted to a natural transformation $t_n : \Pi T_r T_l^n \to \Pi T_l^n T_r$,

$$t_n := \Pi T_l^{n-1} t \circ \Pi T_l^{n-2} t T_l \circ \cdots \circ \Pi T_l t T_l^{n-2} \circ \Pi t T_l^{n-1}. \qquad (1.4)$$

Definition 1.8. *Let* $\mathcal{S} := (\mathcal{M}, \mathcal{C}, T_l, T_r, \Pi, t, i)$ *be an admissible septuple. We say that an arrow* $w : T_r X \to T_l X$ *in* $\mathcal{M}$ *is a* **transposition morphism** *with respect to* $\mathcal{S}$ *if*

$$w \circ u_{T_r} X = u_{T_l} X \quad and \quad w \circ m_{T_r} X = m_{T_l} X \circ T_l w \circ t X \circ T_r w. \qquad (1.5)$$

The category of pairs (X, w)*, with* $w : T_r X \to T_l X$ *a transposition morphism of* $\mathcal{S}$*, will be denoted by* $\mathcal{W}_{\mathcal{S}}$*. A morphism from* (X, w) *to* (X', w') *is an arrow* $f : X \to X'$ *in* $\mathcal{M}$ *such that* $T_l f \circ w = w' \circ T_r f$.

Morphisms $w : T_r X \to T_l X$ satisfying (1.5), for a distributive law $t : T_r T_l \to T_l T_r$, were termed t-*algebras* in [Burr]. Based on [Burr, Prop. I.1.1], transposition morphisms can be characterized as in Proposition 1.9 below. Recall that a module of a monad (T, m_T, u_T) on a category $\mathcal{M}$ is a pair (Y, ϱ), consisting of an object Y and a morphism $\varrho : T Y \to Y$ in $\mathcal{M}$, such that $\varrho \circ T \varrho = \varrho \circ m_T Y$ and $\varrho \circ u_T Y = \mathrm{Id}_Y$ (i.e. ϱ is associative and unital). A morphism of T-modules $(Y, \varrho) \to (Y', \varrho')$ is a morphism $f : Y \to Y'$ in $\mathcal{M}$, such that $f \circ \varrho = \varrho' \circ T f$.

Proposition 1.9. *Consider an admissible septuple* $\mathcal{S} := (\mathcal{M}, \mathcal{C}, T_l, T_r, \Pi, t, i)$*. There is a bijective correspondence between objects* (X, w) *in the category* $\mathcal{W}_{\mathcal{S}}$ *and* T_r*-modules of the form* $(T_l X, \varrho)$*, satisfying*

$$m_{T_l} X \circ T_l \varrho \circ t T_l X = \varrho \circ T_r m_{T_l} X. \qquad (1.6)$$

Moreover, a morphism $f : X \to X'$ *in* $\mathcal{M}$ *is a morphism in* $\mathcal{W}_{\mathcal{S}}$ *if and only if* $T_l f$ *is a* T_r*-module morphism.*

Proof. Similarly to the proof of [Burr, Prop. I.1.1] one checks that, for an object (X, w) in $\mathcal{W}_{\mathcal{S}}$, an associative and unital T_r-action on $T_l X$ satisfying (1.6) is given by $\varrho_w := m_{T_l} X \circ T_l w \circ t X : T_r T_l X \to T_l X$. Conversely, note that for a T_r-module $(T_l X, \varrho)$, (1.6) is equivalent to $\varrho = m_{T_l} X \circ T_l \varrho \circ T_l T_r u_{T_l} X \circ t X$. With this identity at hand,

the pair $(X, w_\varrho := \varrho \circ \boldsymbol{T_r u_{T_l}} X)$ is checked to be an object in $\mathcal{W}_\mathcal{S}$. A straightforward computation shows that the two constructions are mutual inverses. A morphism $\boldsymbol{T_l} f$ is a morphism of $\boldsymbol{T_r}$-modules $(\boldsymbol{T_l} X, \varrho_w) \rightarrow (\boldsymbol{T_l} X', \varrho_{w'})$ if

$$\boldsymbol{m_{T_l}} X' \circ \boldsymbol{T_l} w' \circ \boldsymbol{T_l T_r} f \circ t X = \boldsymbol{m_{T_l}} X' \circ \boldsymbol{T_l T_l} f \circ \boldsymbol{T_l} w \circ t X. \quad (1.7)$$

If f is a morphism in $\mathcal{W}_\mathcal{S}$ then (1.7) obviously holds. In order to prove the converse implication, compose both sides of (1.7) with $\boldsymbol{T_r u_{T_l}} X$ on the right. $\quad \square$

Theorem 1.10. *Consider an admissible septuple $\mathcal{S}$ and a transposition map $w : \boldsymbol{T_r} X \rightarrow \boldsymbol{T_l} X$ in $\mathcal{W}_\mathcal{S}$. The cosimplicial object $Z^*(\mathcal{S}, X)$ is para-cocyclic with respect to*

$$w_n := \boldsymbol{\Pi T_l}^n w \circ t_n X \circ \boldsymbol{i T_l}^n X. \quad (1.8)$$

We shall denote this para-cocyclic object by $Z^*(\mathcal{S}, w)$. For a morphism $f : (X, w) \rightarrow (X', w')$ in $\mathcal{W}_\mathcal{S}$, the morphisms $\boldsymbol{\Pi T_l}^{n+1} f : Z^n(\mathcal{S}, w) \rightarrow Z^n(\mathcal{S}, w')$ determine a morphism of para-cocyclic objects.

Proof. In Fig. 4 we show that the morphism (1.8) is compatible with the coface maps, that is

$$w_n \circ d_0 = d_n \quad \text{and} \quad w_n \circ d_k = d_{k-1} \circ w_{n-1} \quad (1.9)$$

for any $k \in \{1, \ldots, n\}$. The proof of the first equation is given in three steps in the left picture. To simplify the diagrams, we draw the n strings representing $\boldsymbol{T_l}^n$ as a black stripe. For the first equality we used the compatibility between $\boldsymbol{i}$ and the unit of $\boldsymbol{T_l}$, that is the first equation in (1.2). Next we applied n times the compatibility relation between the distributive law t and the unit of $\boldsymbol{T_r}$, i.e. the third equality in Fig. 3. The first relation in (1.5) implies the third equality. The second relation in (1.9) follows in a similar way, as it is shown in the right picture in Fig. 4. Note that the leftmost black stripe represents $\boldsymbol{T_l}^{k-1}$ and the other one represents $\boldsymbol{T_l}^{n-k}$. Since $\boldsymbol{u_{T_l}}$ is a natural transformation, the box representing it can be pushed down along the string until it meets the crossing t. By the fourth identity in Fig. 3, one can push $\boldsymbol{u_{T_l}}$ under the string in the crossing. To conclude the proof of this equality, we use once again that $\boldsymbol{u_{T_l}}$ is a natural transformation to move it to the bottom of the diagram.
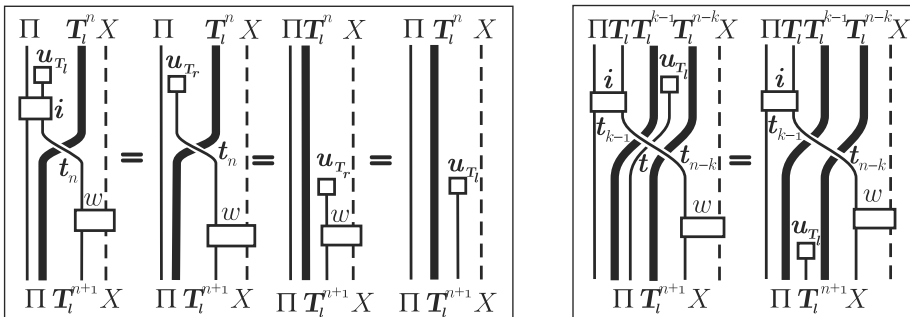

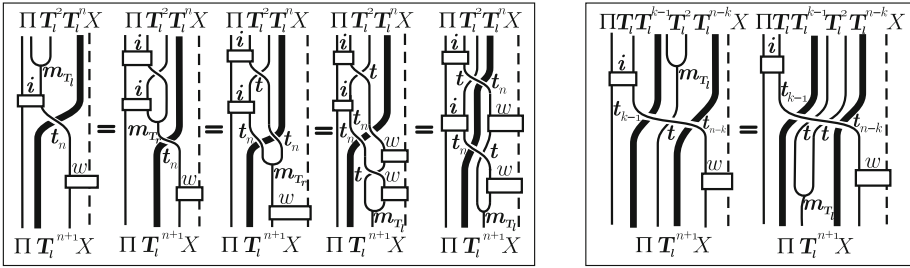**Fig. 4.** The proof of the relations (1.9)

Fig. 5. The proof of the relations (1.10)

Next we prove that the morphism (1.8) and the codegeneracy maps are compatible too, that is

$$w_n \circ s_0 = s_n \circ (w_{n+1})^2 \quad \text{and} \quad w_n \circ s_k = s_{k-1} \circ w_{n+1} \qquad (1.10)$$

for any $k \in \{1, \ldots, n\}$. The proof of the first relation can be found in the left picture in Fig. 5. As before, the black stripe represents T_l^n. The morphisms corresponding to the first two diagrams are equal in view of the second equation in (1.2). By applying n times the first identity in Fig. 3, it follows that the second and the third diagrams represent the same morphism. Taking into account the second relation in (1.5) we got the penultimate equality, while for the last one we used that i is a natural transformation.

The other relation in (1.10) immediately follows by the second identity in Fig. 3 and the fact m_{T_l} is natural (see the second picture in Fig. 5).

Since the coface and codegeneracy morphisms (1.3) are defined in terms of natural transformations, the morphisms $\Pi T_l^{n+1} f : Z^n(\mathcal{S}, X) \to Z^n(\mathcal{S}, X')$ determine a morphism $Z^*(\mathcal{S}, f) : Z^*(\mathcal{S}, X) \to Z^*(\mathcal{S}, X')$ of cosimplicial objects, for any morphism $f : X \to X'$ in $\mathcal{M}$. It follows immediately from the definition of a morphism in $\mathcal{W}_\mathcal{S}$ that if $f : (X, w) \to (X', w')$ is a morphism in $\mathcal{W}_\mathcal{S}$ then $Z^*(\mathcal{S}, f)$ is a morphism of para-cocyclic objects $Z^*(\mathcal{S}, w) \to Z^*(\mathcal{S}, w')$. □

Corollary 1.11. *Let $\mathcal{S}$ be an admissible septuple as in Definition 1.7 and let $w : T_r X \to T_l X$ be a transposition morphism in $\mathcal{W}_\mathcal{S}$. Consider the corresponding para-cocyclic morphism w_n in (1.8). If the coequalizer*

$$Z^n(\mathcal{S}, w) \underset{\mathrm{Id}_{Z^n(\mathcal{S},w)}}{\overset{(w_n)^{n+1}}{\rightrightarrows}} Z^n(\mathcal{S}, w) \longrightarrow \widehat{Z}^n(\mathcal{S}, w)$$

exists in $\mathcal{C}$, for every non-negative integer n, then it defines a cocyclic cosimplex $\widehat{Z}^*(\mathcal{S}, w)$.

Proof. Let $k \in \{0, \ldots, n\}$. It follows by (1.9) that d_k satisfies $d_k \circ (w_n)^{n+1} = (w_{n+1})^{n+2} \circ d_k$. Similarly, by (1.10), the codegeneracy morphism s_k satisfies $s_k \circ (w_n)^{n+1} = (w_{n-1})^n \circ s_k$. Hence d_k and s_k determine coface morphisms $\widehat{d}_{k_X}$ and codegeneracy morphisms $\widehat{s}_k$ on $\widehat{Z}^n(\mathcal{S}, X)$. Together with the projection $\widehat{w}_n$ of w_n onto $\widehat{Z}^n(\mathcal{S}, X)$ they define a cocyclic object $(\widehat{Z}^*(\mathcal{S}, w), \widehat{d}^*, \widehat{s}^*, \widehat{w}^*)$. □

1.D. Examples of admissible septuples and transposition maps. Applications. In this section we shall apply Theorem 1.10 to several examples of admissible septuples. In this way we shall show that the most known (co)cyclic objects in the literature can be obtained as direct applications of the result obtained in Sect. 1.C.

A functorial construction of a para-cyclic object in a category of endofunctors, of a somewhat similar flavour to that in Theorem 1.10, was proposed in [Šk]. The following example is its dual version.

Example 1.12. Let $T = (T, m, u)$ be a monad on a category $\mathcal{M}$ and $t : TT \to TT$ be a distributive law. Assume that t satisfies the Yang-Baxter relation

$$tT \circ Tt \circ tT = Tt \circ tT \circ Tt \tag{1.11}$$

of natural transformations $TTT \to TTT$, and $m \circ t \circ t = m$. As a consequence of (1.11), also $T^0 = (T, m \circ t, u)$ is a monad, and t can be regarded as a distributive law $T^0 T \to T T^0$. Furthermore, the datum $\mathcal{S} := (\mathcal{M}, \mathcal{M}, T, T^0, \mathrm{Id}_{\mathcal{M}}, t, \mathrm{Id}_T)$ (where $\mathrm{Id}_{\mathcal{M}}$ denotes the identity functor $\mathcal{M} \to \mathcal{M}$ and Id_T is the identity natural transformation $T \to T$) is an admissible septuple. For any object X in $\mathcal{M}$, the identity morphism Id_{TX} is a transposition morphism. The corresponding para-cocyclic morphism is t_n in (1.4).

The simplest example of an admissible septuple can be obtained by starting with a morphism $\varphi : R \to T$ of $\mathbb{K}$-algebras. As in Example 1.2, we define two monads on the category $\mathcal{M} := R\text{-Mod-}R$ by $T_l := T \otimes_R (-)$ and $T_r := (-) \otimes_R T$. The category $\mathcal{C}$ is, by definition, the category Mod-$\mathbb{K}$ of $\mathbb{K}$-modules. The functor Π is constructed below.

Definition 1.13. *On the objects $X \in R\text{-Mod-}R$, the functor $\Pi : R\text{-Mod-}R \to Mod\text{-}\mathbb{K}$ is defined as a coequalizer*

$$X \otimes_{\mathbb{K}} R \underset{x \otimes r \mapsto r \cdot x}{\overset{x \otimes r \mapsto x \cdot r}{\rightrightarrows}} X \xrightarrow{\ pX\ } \Pi X .$$

For a morphism $f : X \to Y$ of R-bimodules, Πf is the unique $\mathbb{K}$-linear map such that $pY \circ f = \Pi f \circ pX$. Hence p can be interpreted as a natural epimorphism from the forgetful functor $U : R\text{-Mod-}R \to Mod\text{-}\mathbb{K}$ to Π.

Remark 1.14. An R-bimodule X can be considered as a left or right module for the enveloping algebra $R^e := R \otimes_{\mathbb{K}} R^{op}$ of R. In terms of the functor Π in Definition 1.13, the *cyclic tensor product* $X \widehat{\otimes}_R Y$ of two R-bimodules X and Y is defined by $X \widehat{\otimes}_R Y := \Pi(X \otimes_R Y) \cong X \otimes_{R^e} Y$. With this interpretation in mind, the $\mathbb{K}$-module $\Pi X \cong R \otimes_{R^e} X$ can be seen as the cyclic tensor product of R and X. For R-bimodules $X_1, \ldots, X_n$, the n-fold cyclic module tensor product is defined as

$$X_1 \widehat{\otimes}_R \cdots \widehat{\otimes}_R X_n := \Pi(X_1 \otimes_R \cdots \otimes_R X_n) = (X_1 \otimes_R \cdots \otimes_R X_k)$$
$$\widehat{\otimes}_R (X_{k+1} \otimes_R \cdots \otimes_R X_n),$$

for $k \in \{1, \ldots, n-1\}$. It is generated by the cyclic tensor monomials $x_1 \widehat{\otimes}_R \cdots \widehat{\otimes}_R x_n$. It is well known that the symmetry $c_{X,Y} : X \otimes_{\mathbb{K}} Y \to Y \otimes_{\mathbb{K}} X$ induces a natural isomorphism $i_{X,Y} : X \widehat{\otimes}_R Y \cong Y \widehat{\otimes}_R X$. In particular, there is a natural isomorphism $i_{X_1, \ldots, X_n} : X_1 \widehat{\otimes}_R \cdots \widehat{\otimes}_R X_n \to X_2 \widehat{\otimes}_R \cdots \widehat{\otimes}_R X_n \widehat{\otimes}_R X_1$, that maps a generator $x_1 \widehat{\otimes}_R \cdots \widehat{\otimes}_R x_n$ to $x_2 \widehat{\otimes}_R \cdots \widehat{\otimes}_R x_n \widehat{\otimes}_R x_1$.

As we have noticed in Example 1.5, the associativity constraint of the monoidal category R-Mod-R defines a distributive law $t : T_r T_l \to T_l T_r$. Thus, in this particular case that we are investigating, tX is the canonical isomorphism $(T \otimes_R X) \otimes_R T \cong T \otimes_R (X \otimes_R T)$, for any X in R-Mod-R. Let us define $iX : \Pi T_l X \to \Pi T_r X$ by $iX := i_{T,X}$, as in Remark 1.14. In terms of these natural transformations we can give one of the main examples of admissible septuples.

Proposition 1.15. *Let $\varphi : R \to T$ be a morphism of $\mathbb{K}$-algebras. The following data:*

- *the categories $\mathcal{M} := R$-Mod-R and $\mathcal{C} := $ Mod-$\mathbb{K}$,*
- *the monads $T_l := T \otimes_R (-)$ and $T_r := (-) \otimes_R T$,*
- *the functor $\Pi : \mathcal{M} \to \mathcal{C}$, $\Pi X := R \otimes_{R^e} X$,*
- *the natural transformation $tX : (T \otimes_R X) \otimes_R T \to T \otimes_R (X \otimes_R T)$ defined by the canonical isomorphism;*
- *the natural transformation $iX : T \widehat{\otimes}_R X \to X \widehat{\otimes}_R T$, $t \widehat{\otimes}_R x \mapsto x \widehat{\otimes}_R t$,*

define an admissible septuple $\mathcal{S}_T$.

Proof. Let X be an R-bimodule. By definition,

$$\Pi T_l{}^n X = R \otimes_{R^e} (T^{\otimes_R n} \otimes_R X) \cong T^{\widehat{\otimes}_R n} \widehat{\otimes}_R X \quad \text{and} \quad \Pi T_r{}^n X \cong X \widehat{\otimes}_R T^{\widehat{\otimes}_R n}.$$

Via these identifications, $iT_l X = i_{T,T,X}$ and $iT_r X = i_{T,X,T}$. So the conditions (1.2) take the form

$$i_{T,X} \circ (\varphi \widehat{\otimes}_R X) = X \widehat{\otimes}_R \varphi \quad \text{and} \quad i_{T,X} \circ (m_T \widehat{\otimes}_R X) = (X \widehat{\otimes}_R m_T) \circ i_{T \otimes_R T, X},$$

identities which are obvious. □

Let $\mathcal{S}_T$ be the admissible septuple associated to an algebra morphism $\varphi : R \to T$. A morphism of R-bimodules $w : X \otimes_R T \to T \otimes_R X$ is a transposition map in $\mathcal{W}_{\mathcal{S}_T}$ if, and only if, it satisfies the conditions

$$w \circ (X \otimes_R \varphi) = \varphi \otimes_R X \quad \text{and} \quad w \circ (X \otimes_R m_T)$$
$$= (m_T \otimes_R X) \circ (T \otimes_R w) \circ (w \otimes_R T), \quad (1.12)$$

where $m_T : T \otimes_R T \to T$ denotes the multiplication map $t \otimes_R t' \mapsto tt'$. Note in passing the similarity of conditions (1.12) to some of those defining an entwining structure over R. (For the definition of entwining structures see [BMa, Def. 2.1], and for a reformulation over an arbitrary base algebra R see [BB, Sect. 2.3].) By Proposition 1.9, there is a bijective correspondence between transposition maps $w : X \otimes_R T \to T \otimes_R X$ on one hand, and right T-actions on $T \otimes_R X$, which are left T-module maps with respect to the left T-action $t' \cdot (t \otimes_R x) = t't \otimes_R x$, on the other hand.

Theorem 1.16. *Let $\mathcal{S}_T$ be the admissible septuple associated to an algebra morphism $\varphi : R \to T$. Let $w : X \otimes_R T \to T \otimes_R X$ be a transposition map in $\mathcal{W}_{\mathcal{S}_T}$, that is, a morphism of R-bimodules satisfying (1.12). Then there is a cocyclic quotient $\widehat{Z}^*(\mathcal{S}_T, w)$ of $T^{\widehat{\otimes}_R *+1} \widehat{\otimes}_R X$ such that its cocyclic structure is induced by the para-cocyclic morphisms $w_n : T^{\widehat{\otimes}_R n+1} \widehat{\otimes}_R X \to T^{\widehat{\otimes}_R n+1} \widehat{\otimes}_R X$,*

$$w_n := \left(T^{\widehat{\otimes}_R n} \widehat{\otimes}_R w \right) \circ i_{T,\dots,T,X}, \quad (1.13)$$

where $i_{T,\dots,T,X}$ is the $\mathbb{K}$-linear map defined in Remark 1.14.

Proof. Apply Theorem 1.10 for $\mathcal{S} = \mathcal{S}_T$. It yields a para-cocyclic object $Z^*(\mathcal{S}_T, w)$ whose para-cocyclic operator is given in formula (1.8). For $\mathcal{S}_T$, the map $\boldsymbol{t}_n$ is the identity morphism of $T^{\widehat{\otimes}_R n+1} \widehat{\otimes}_R X$, cf. (1.4). Hence w_n satisfies (1.13). We conclude the proof by applying Corollary 1.11. $\square$

Corollary 1.17. *Let $\mathcal{S}_T$ be the admissible septuple associated to a $\mathbb{K}$-algebra homomorphism $\varphi : R \to T$ as in Proposition 1.15. Then the canonical isomorphism $w_T : R \otimes_R T \to T \otimes_R R$ is a transposition map in $\mathcal{W}_{\mathcal{S}_T}$. The corresponding cocyclic cosimplex $Z^*(\mathcal{S}_T, w_T)$ has in degree n the $\mathbb{K}$-module $Z^n(\mathcal{S}_T, w_T) = T^{\widehat{\otimes}_R n+1}$. The coface and codegeneracy maps are*

$$d_k(t_0 \widehat{\otimes}_R t_1 \widehat{\otimes}_R \cdots \widehat{\otimes}_R t_{n-1}) = t_0 \widehat{\otimes}_R t_1 \widehat{\otimes}_R \cdots \widehat{\otimes}_R t_{k-1} \widehat{\otimes}_R 1_T \widehat{\otimes}_R t_k \widehat{\otimes}_R \cdots \widehat{\otimes}_R t_{n-1},$$
$$s_k(t_0 \widehat{\otimes}_R t_1 \widehat{\otimes}_R \cdots \widehat{\otimes}_R t_{n+1}) = t_0 \widehat{\otimes}_R t_1 \widehat{\otimes}_R \cdots \widehat{\otimes}_R t_{k-1} \widehat{\otimes}_R t_k t_{k+1} \widehat{\otimes}_R t_{k+2} \widehat{\otimes}_R \cdots \widehat{\otimes}_R t_{n+1},$$

where $k \in \{0, \ldots, n\}$. The cocyclic operator is given by

$$w_n(t_0 \widehat{\otimes}_R t_1 \widehat{\otimes}_R \cdots \widehat{\otimes}_R t_n) = t_1 \widehat{\otimes}_R t_2 \widehat{\otimes}_R \cdots \widehat{\otimes}_R t_n \widehat{\otimes}_R t_0.$$

Remark 1.18. As $u_T : k \to T$, the unit of a $\mathbb{K}$-algebra T is an algebra map, we can apply Proposition 1.15 to get an admissible septuple $\mathcal{S}_{u_T}$. Corresponding transposition maps were also considered by Kaygun in [Kay06] to construct cocyclic $\mathbb{K}$-modules. His approach should be considered, however, dual to ours (see related remarks in the Introduction).

It follows by an observation in [Burr, p. 11] that for the admissible septuple $\mathcal{S}_T$, associated to an algebra morphism $\varphi : R \to T$ in Proposition 1.15, any R-T bimodule Y admits a transposition morphism $w_Y : Y \otimes_R T \to T \otimes_R Y$, $y \otimes_R t \mapsto 1_T \otimes_R y \cdot t$. In particular, for any R-bimodule X, the pair $(X \otimes_R T, (u_T \otimes_R X \otimes_R T) \circ (X \otimes_R m_T))$ is an object in $\mathcal{W}_{\mathcal{S}_T}$. Corresponding para-cocyclic objects are given in the following

Example 1.19. Let $\mathcal{S}_T$ be the admissible septuple associated to a $\mathbb{K}$-algebra homomorphism $\varphi : R \to T$ as in Proposition 1.15. For any R-T bimodule Y, there is a para-cocyclic cosimplex $Z^*(\mathcal{S}_T, w_Y)$, given in degree n by the $\mathbb{K}$-module $Z^n(\mathcal{S}_T, w_Y) = T^{\widehat{\otimes}_R n+1} \widehat{\otimes}_R Y$. The coface and codegeneracy maps are

$$d_k(t_0 \widehat{\otimes}_R t_1 \widehat{\otimes}_R \cdots \widehat{\otimes}_R t_{n-1} \widehat{\otimes}_R y) = t_0 \widehat{\otimes}_R t_1 \widehat{\otimes}_R \cdots \widehat{\otimes}_R t_{k-1} \widehat{\otimes}_R 1_T$$
$$\widehat{\otimes}_R t_k \widehat{\otimes}_R \cdots \widehat{\otimes}_R t_{n-1} \widehat{\otimes}_R y,$$
$$s_k(t_0 \widehat{\otimes}_R t_1 \widehat{\otimes}_R \cdots \widehat{\otimes}_R t_{n+1} \widehat{\otimes}_R y) = t_0 \widehat{\otimes}_R t_1 \widehat{\otimes}_R \cdots \widehat{\otimes}_R t_{k-1} \widehat{\otimes}_R t_k t_{k+1}$$
$$\widehat{\otimes}_R t_{k+2} \widehat{\otimes}_R \cdots \widehat{\otimes}_R t_{n+1} \widehat{\otimes}_R y,$$

where $k \in \{0, \ldots, n\}$. The para-cocyclic operator is given by

$$w_n(t_0 \widehat{\otimes}_R t_1 \widehat{\otimes}_R \cdots \widehat{\otimes}_R t_n \widehat{\otimes}_R y) = t_1 \widehat{\otimes}_R \cdots \widehat{\otimes}_R t_n \widehat{\otimes}_R 1_T \widehat{\otimes}_R y \cdot t_0.$$

Note that w_n is degenerate in the sense that the cocyclic quotient of $Z^*(\mathcal{S}_T, w_Y)$ (cf. Corollary 1.11) is given by $\widehat{Z}^n(\mathcal{S}_T, w_Y) = R \widehat{\otimes}_R Y$, in every degree n.

Next we are going to associate an admissible septuple to every ribbon algebra. Recall that a *ribbon algebra* is an algebra (T, m_T, u_T) in a braided monoidal category $\mathcal{M}$ together with an automorphism $\sigma : T \to T$ in $\mathcal{M}$ such that

$$\sigma \circ u_T = u_T \quad \text{and} \quad \sigma \circ m_T = m_T \circ (\sigma \otimes \sigma) \circ \boldsymbol{c}_{T,T}^2. \tag{1.14}$$

Ribbon algebras appeared in [AM], where they are used to define cyclic homology of quasialgebras (non-associative algebras that are obtained by a cochain twist). We shall show that the ribbon automorphism σ can be used to define a certain admissible septuple. For, we start with an algebra (T, m_T, u_T) and an automorphism $\sigma : T \to T$ in a braided monoidal category. It is easy to see that T is also an associative and unital algebra with respect to $m'_T := m_T \circ c_{T,T}$ and $u'_T := u_T$. To make a distinction between T and the new algebra, the latter one will be denoted by T'.

Consider the monads T'_l and T_l on $\mathcal{M}$, defined as in Example 1.2. In the following, T'_l will play the role of T_r in the definition of an admissible septuple. We have seen in Example 1.6 that $c_{T,T} : T \otimes T \to T \otimes T$ induces a distributive law $t : T'_l T_l \to T_l T'_l$,

$$t X := a_{T,T,X} \circ (c_{T,T} \otimes X) \circ a^{-1}_{T,T,X},$$

where X is an arbitrary object in $\mathcal{M}$. Furthermore, we take $\mathcal{C} = \mathcal{M}$ and $\Pi = \mathrm{Id}_{\mathcal{M}}$. By definition, the natural transformation $i : T_l \to T'_l$ is

$$i X := (\sigma \otimes X).$$

It is not difficult to prove that the relations in (1.2) hold if, and only if, the identities in (1.14) are satisfied. Thus, we have the following

Proposition 1.20. *The algebra (T, m_T, u_T) in a braided monoidal category $\mathcal{M}$ is a ribbon algebra with ribbon automorphism σ if and only if $\mathcal{S}_{T,\sigma} := (\mathcal{M}, \mathcal{M}, T_l, T'_l, \mathrm{Id}_{\mathcal{M}}, t, i)$, the septuple constructed above, is admissible.*

Let (T, m_T, u_T) be a ribbon algebra with ribbon automorphism σ. In view of Proposition 1.20, we can speak about transposition morphisms with respect to $\mathcal{S}_{T,\sigma}$. A morphism $w : T \otimes X \to T \otimes X$ in $\mathcal{M}$ is a transposition map in $\mathcal{W}_{\mathcal{S}_{T,\sigma}}$ if, and only if

$$w \circ (u_T \otimes X) = u_T \otimes X, \tag{1.15}$$

$$w \circ (m_T \otimes X) \circ (c_{T,T} \otimes X) = (m_T \otimes X) \circ (T \otimes w) \circ (c_{T,T} \otimes X) \circ (T \otimes w). \tag{1.16}$$

Note that, in the second equation, we omitted the associativity constraints, to make the formula as short as possible. In fact, in view of the Coherence Theorem, we can omit bracketing in any equality involving morphisms in an arbitrary monoidal category $\mathcal{M}$. (For MacLane's Coherence Theorem consult e.g. [MacL, Theorem 1, p. 162] or [Mj, pp. 420–421]).

Relations (1.15) and (1.16) already appeared in the definition of braided twistors, structures that are used to construct new associative and unitary multiplications on $T \otimes T$. For details the reader is referred to [LPvO].

As an application of Theorem 1.10 we get Proposition 1.21 below. Note that, for simplifying the formulae of coface, codegeneracy and para-cocyclic morphisms, we omitted the associativity and unit constraints.

Proposition 1.21. *Let (T, m_T, u_T, σ) be a ribbon algebra in a braided monoidal category $\mathcal{M}$. For every object X in $\mathcal{M}$, the sequence $Z^n(\mathcal{S}_{T,\sigma}, X) := T^{\otimes n+1} \otimes X$ defines a cosimplicial object, with respect to the coface and codegeneracy morphisms*

$$d_k := T^{\otimes k} \otimes u_T \otimes T^{\otimes n-k} \otimes X \quad and \quad s_k := T^{\otimes k} \otimes m_T \otimes T^{\otimes n-k} \otimes X,$$

where $k \in \{0, \ldots, n\}$. *Moreover, if $w : T \otimes X \to T \otimes X$ is a transposition map in* $\mathcal{W}_{\mathcal{S}_{T,\sigma}}$, *then $Z^*(\mathcal{S}_{T,\sigma}, X)$ admits a para-cocyclic structure with respect to the operator*

$$w_n := (T^{\otimes n} \otimes w) \circ (T^{\otimes n-1} \otimes c_{T,T} \otimes X) \circ \cdots \circ$$
$$\times (c_{T,T} \otimes T^{\otimes n-1} \otimes X) \circ (\sigma \otimes T^{\otimes n} \otimes X).$$

We shall denote this para-cocyclic object by $Z^*(\mathcal{S}_{T,\sigma}, w)$.

Remarks 1.22. (i) Every ribbon algebra (T, σ) in a braided category $(\mathcal{M}, \otimes, a, l, r, c, 1)$ can be seen as an algebra with ribbon element σ^{-1} in the opposite braided category of $\mathcal{M}$. Recall that the opposite of the braided category $\mathcal{M}$ is $(\mathcal{M}, \otimes, a, l, r, \widetilde{c}, 1)$, where

$$\widetilde{c}_{X,Y} = c_{Y,X}^{-1}.$$

(ii) To every (para)-cocyclic object with invertible para-cocyclic morphism, there corresponds a (para)-cyclic object, namely its cyclic dual. Roughly speaking, the cyclic dual is obtained by interchanging the coface and codegeneracy morphisms and inverting the para-cocyclic operator. The interested reader can find the definition of the cyclic dual in [KR05]. The cyclic dual of $Z^*(\mathcal{S}_{T,\sigma^{-1}}, \mathrm{Id}_T)$ is, modulo a sign in the formula of w_n, the cyclic object in [AM, Theorem 4]. Note that, via the identification $T \otimes 1 \cong T$, the identity morphism Id_T can be regarded as a transposition map in $\mathcal{W}_{\mathcal{S}_{T,\sigma^{-1}}}$. Thus, for an arbitrary w in $\mathcal{W}_{\mathcal{S}_{T,\sigma^{-1}}}$, the cyclic dual of $Z^*(\mathcal{S}_{T,\sigma^{-1}}, w)$ may be interpreted as a generalization of cyclic homology introduced in [AM].

Other examples of para-cocyclic objects, obtained as applications of Theorem 1.10, will be discussed in Theorems 2.4 and Theorem 2.7.

1.E. The dual construction. In this section we turn to studying the situation dual to that in Sect. 1.C, i.e. application of Theorem 1.10 to the opposite categories $\mathcal{C}^{op}$ and $\mathcal{M}^{op}$. By $\mathcal{M}^{op}$ we mean the category with the same classes of objects and morphisms in $\mathcal{M}$, with composition opposite to that in $\mathcal{M}$. Note that any diagram expressing an identity of morphisms in $\mathcal{M}$, yields a diagram in $\mathcal{M}^{op}$, by interchanging the top and the bottom. In particular, a *comonad* on $\mathcal{M}$ is a monad on $\mathcal{M}^{op}$. That is, a triple $(T_l, \Delta_{T_l}, \varepsilon_{T_l})$, consisting of a functor $T_l : \mathcal{M} \to \mathcal{M}$ and natural transformations $\Delta_{T_l} : T_l \to T_l^2$ and $\varepsilon_{T_l} : T_l \to Id_{\mathcal{M}}$. Their compatibility axioms are obtained by reversing the arrows in the first two diagrams in Fig. 2. For two comonads $(T_l, \Delta_{T_l}, \varepsilon_{T_l})$ and $(T_r, \Delta_{T_r}, \varepsilon_{T_r})$ on a category $\mathcal{M}$, a *dual distributive law* is a distributive law for the monads T_l and T_r on $\mathcal{M}^{op}$. That is, a natural transformation $t : T_l T_r \to T_r T_l$ such that the relations encoded in the up-down mirror images of the diagrams in Fig. 3 hold.

To dualize admissible septuples we need two comonads T_l and T_r on a category $\mathcal{M}$, a dual distributive law $t : T_l T_r \to T_r T_l$, a covariant functor $\Pi : \mathcal{M} \to \mathcal{C}$ and a natural transformation $i : \Pi T_r \to \Pi T_l$ that satisfy the identities

$$\Pi \varepsilon_{T_l} \circ i = \Pi \varepsilon_{T_r} \quad \text{and} \quad \Pi \Delta_{T_l} \circ i = i T_l \circ \Pi t \circ i T_r \circ \Pi \Delta_{T_r}. \qquad (1.17)$$

Such a *dual admissible septuple* $\mathcal{S}^0 = (\mathcal{M}, \mathcal{C}, T_l, T_r, \Pi, t, i)$ determines a simplicial object $Z_*(\mathcal{S}^0, X)$ in $\mathcal{C}$, which in degree n is given by $Z_n(\mathcal{S}^0, X) = \Pi T_l^{n+1} X$. Its face maps $d_k : \Pi T_l^{n+1} X \to \Pi T_l^n X$ and degeneracy maps $s_k : \Pi T_l^{n+1} X \to \Pi T_l^{n+2} X$ are

$$d_k := \Pi T_l^k \varepsilon_{T_l} T_l^{n-k} X, \qquad s_k := \Pi T_l^k \Delta_{T_l} T_l^{n-k} X,$$

for any $k \in \{0, \dots, n\}$. An arrow $w : T_l X \to T_r X$ in $\mathcal{M}$ is a *transposition morphism* with respect to the dual admissible septuple $\mathcal{S}^0$ if, and only if

$$\boldsymbol{\varepsilon}_{T_r} X \circ w = \boldsymbol{\varepsilon}_{T_l} X \quad \text{and} \quad \boldsymbol{\Delta}_{T_r} X \circ w = T_r w \circ t X \circ T_l w \circ \boldsymbol{\Delta}_{T_l} X. \quad (1.18)$$

Morphisms between transpositions maps can be easily defined by duality. The category of transposition maps with respect to $\mathcal{S}^0$ will be denoted by $\mathcal{W}_{\mathcal{S}^0}$.

Note that t can be lifted to a natural transformation $t_n : \boldsymbol{\Pi} T_l^n T_r \to \boldsymbol{\Pi} T_r T_l^n$,

$$t_n^0 := \boldsymbol{\Pi} t T_l^{n-1} \circ \boldsymbol{\Pi} T_l t T_l^{n-2} \circ \cdots \circ \boldsymbol{\Pi} T_l^{n-2} t T_l \circ \boldsymbol{\Pi} T_l^{n-1} t. \quad (1.19)$$

Now we can state, for future reference, the dual of Theorem 1.10.

Theorem 1.23. *Consider a dual admissible septuple $\mathcal{S}^0$ as above and a transposition morphism $w : T_l X \to T_r X$ in $\mathcal{W}_{\mathcal{S}^0}$. The simplex $Z_*(\mathcal{S}^0, X)$ is para-cyclic with para-cyclic morphism*

$$w_n := i T_l^n X \circ t_n^0 X \circ \boldsymbol{\Pi} T_l^n w. \quad (1.20)$$

We shall denote this cyclic object by $Z_*(\mathcal{S}^0, w)$. For a morphism $f : (X, w) \to (X', w')$ in $\mathcal{W}_{\mathcal{S}^0}$, the morphisms $\boldsymbol{\Pi} T_l^{n+1} f : Z_n(\mathcal{S}^0, w) \to Z_n(\mathcal{S}^0, w')$ determine a morphism of para-cyclic objects.

Dually to Example 1.12, we have the following

Example 1.24. Let $T = (T, \boldsymbol{\Delta}, \boldsymbol{\varepsilon})$ be a comonad on a category $\mathcal{M}$ and $t : TT \to TT$ a dual distributive law. Assume that t satisfies the Yang-Baxter relation 1.11 and $t \circ t \circ \boldsymbol{\Delta} = \boldsymbol{\Delta}$. Then $T^0 = (T, t \circ \boldsymbol{\Delta}, \boldsymbol{\varepsilon})$ is a comonad, and t can be regarded as a distributive law $T^0 T \to T T^0$. Furthermore, the datum $\mathcal{S} := (\mathcal{M}, \mathcal{M}, T^0, T, \mathrm{Id}_{\mathcal{M}}, t, \mathrm{Id}_T)$ is a dual admissible septuple. For any object X in $\mathcal{M}$, the identity morphism Id_{TX} is a transposition morphism. The corresponding para-cyclic morphism is t_n^0 in (1.19). Note that if in addition t is an invertible morphism in $\mathcal{M}$ then its properties assumed above are equivalent to the premises in [Šk, Theorem 1].

Let R be an algebra over a commutative ring $\mathbb{K}$. It was recalled in Example 1.2 that R-Mod-R, the category of R-bimodules, is monoidal with respect to the tensor product $\otimes_R$ and unit object R. By definition, an *R-coring* (C, Δ, ϵ) is a coalgebra in $(R\text{-Mod-}R, \otimes_R, R)$.

Proposition 1.25. *Let $(C, \Delta_C, \epsilon_C)$ be an R-coring. The following data:*

- *the category $\mathcal{C} := \mathrm{Mod}\text{-}\mathbb{K}$ of $\mathbb{K}$-modules and the category $\mathcal{M} := R\text{-Mod-}R$ of R-bimodules,*
- *the comonads $T_l := C \otimes_R (-)$ and $T_r := (-) \otimes_R C$ on R-Mod-R,*
- *the functor $\boldsymbol{\Pi} : R\text{-Mod-}R \to \mathrm{Mod}\text{-}\mathbb{K}$, $M \mapsto R \otimes_{R^e} M$ in Definition 1.13,*
- *the trivial dual distributive law $t X : C \otimes_R (X \otimes_R C) \to (C \otimes_R X) \otimes_R C$,*
- *the natural morphism $i X : X \widehat{\otimes}_R C \to C \widehat{\otimes}_R X$, given by the flip map $x \widehat{\otimes}_R c \mapsto c \widehat{\otimes}_R x$,*

define a dual admissible septuple $\mathcal{S}_C$.

Proof. We have to check the identities in (1.17). Recall that, for any R-bimodule X, the cyclic tensor product $\Pi(X) := R \otimes_{R^e} X$ is isomorphic to the quotient of X modulo the $\mathbb{K}$-submodule $[X, R]$ generated by all commutators $[x, r]$, where $x \in X$ and $r \in R$. Hence $\widehat{rx} = \widehat{xr}$, where $\widehat{z}$ denotes the class of z in the quotient module, for any $z \in X$. To prove the first relation in (1.17), note that

$$\left(\Pi \boldsymbol{\varepsilon}_{T_l} \circ i\right)(x \widehat{\otimes}_R c) = \widehat{\varepsilon_C(c) x} = \widehat{x \varepsilon_C(c)} = \Pi \boldsymbol{\varepsilon}_{T_r}(x \widehat{\otimes}_R c).$$

For the coproduct in the coring C we use a Sweedler type notation, namely we write $\Delta_C(c) = c_{(1)} \otimes_R c_{(2)}$, with implicit summation understood. A straightforward computation yields

$$\left(\boldsymbol{\Pi \Delta}_{T_l} \circ i\right)(x \widehat{\otimes}_R c) = c_{(1)} \widehat{\otimes}_R c_{(2)} \widehat{\otimes}_R x = \left(i T_l \circ \Pi t \circ i T_r \circ \Pi \Delta_{T_r}\right)(x \widehat{\otimes}_R c),$$

for any $x \in X$ and $c \in C$. Thus the second relation in (1.17) is also proven. $\square$

Let $\mathcal{S}_C$ be the dual admissible septuple associated to an R-coring $(C, \Delta_C, \epsilon_C)$. In this particular case, a map of R-bimodules $w : C \otimes_R X \to X \otimes_R C$ is a transposition map in $\mathcal{W}_{\mathcal{S}_C}$ if, and only if,

$$(X \otimes_R \varepsilon_C) \circ w = \varepsilon_C \otimes_R X \quad \text{and} \quad (X \otimes_R \Delta_C) \circ w$$
$$= (w \otimes_R C) \circ (C \otimes_R w) \circ (\Delta_C \otimes_R X). \tag{1.21}$$

Theorem 1.26. *Let $\mathcal{S}_C$ be the dual admissible septuple associated to an R-coring, as in Proposition 1.25. Let $w : C \otimes_R X \to X \otimes_R C$ be a transposition map in $\mathcal{W}_{\mathcal{S}_C}$, that is, a morphism of R-bimodules satisfying (1.21). Then there is a cyclic subobject $\widehat{Z}_*(\mathcal{S}_C, w)$ of $C^{\widehat{\otimes}_R *+1} \widehat{\otimes}_R X$ whose cyclic structure is a restriction of the para-cyclic morphism $w_n : C^{\widehat{\otimes}_R n+1} \widehat{\otimes}_R X \to C^{\widehat{\otimes}_R n+1} \widehat{\otimes}_R X$,*

$$w_n := i_{C,\ldots,C,X}^{-1} \circ \left(C^{\widehat{\otimes}_R n} \widehat{\otimes}_R w\right),$$

where $i_{C,\ldots,C,X}$ is the isomorphism constructed in Remark 1.14.

Proof. Proceed as in the proof of Theorem 1.16. $\square$

Dually to Corollary 1.17, the following holds.

Corollary 1.27. *Let $\mathcal{S}_C$ be the dual admissible septuple associated to an R-coring, as in Proposition 1.25. Then the canonical isomorphism $w_C : C \otimes_R R \to R \otimes_R C$ is a transposition map in $\mathcal{W}_{\mathcal{S}_C}$. The corresponding cyclic object $Z_*(\mathcal{S}_C, w_C)$ has in degree n the $\mathbb{K}$-module $Z_n(\mathcal{S}_C, w_C) = C^{\widehat{\otimes}_R n+1}$. The face and degeneracy maps are*

$$d_k(c_0 \widehat{\otimes}_R c_1 \widehat{\otimes}_R \cdots \widehat{\otimes}_R c_n) = \begin{cases} c_0 \widehat{\otimes}_R \cdots \widehat{\otimes}_R \epsilon_C(c_k) c_{k+1} \widehat{\otimes}_R \cdots \widehat{\otimes}_R c_n, & \text{for } 0 \leq k < n, \\ c_0 \widehat{\otimes}_R c_1 \widehat{\otimes}_R \cdots \widehat{\otimes}_R c_{n-2} \widehat{\otimes}_R c_{n-1} \epsilon_C(c_n), & \text{for } k = n, \end{cases}$$

$$s_k(c_0 \widehat{\otimes}_R c_1 \widehat{\otimes}_R \cdots \widehat{\otimes}_R c_n) = c_0 \widehat{\otimes}_R \cdots \widehat{\otimes}_R c_{k-1} \widehat{\otimes}_R \Delta_C(c_k) \widehat{\otimes}_R c_{k+1} \widehat{\otimes}_R$$
$$\cdots \widehat{\otimes}_R c_n, \quad \text{for } 0 \leq k \leq n.$$

The cyclic operator is defined by

$$w_n(c_0 \widehat{\otimes}_R \cdots \widehat{\otimes}_R c_n) = c_n \widehat{\otimes}_R c_0 \widehat{\otimes}_R \cdots \widehat{\otimes}_R c_{n-1}.$$

A symmetrical version of the construction in Corollary 1.27 is described in [Ra, Prop. 3.1].

Example 1.28. Let $\varphi : S \to T$ be a homomorphism of algebras over a commutative ring $\mathbb{K}$. It determines Sweedler's T-coring $T \otimes_S T$, where on $T \otimes_S T$ we take the obvious T-bimodule structure. The coproduct $\Delta_{T \otimes_S T}$ and the counit $\epsilon_{T \otimes_S T}$ are respectively defined by

$$\Delta_{T \otimes_S T} : T \otimes_S T \to (T \otimes_S T) \otimes_T (T \otimes_S T), \quad \Delta_{T \otimes_S T}(t \otimes_S t') = t \otimes_S 1_T \otimes_S t',$$
$$\epsilon_{T \otimes_S T} : T \otimes_S T \to T, \quad\quad\quad\quad\quad\quad\quad \epsilon_{T \otimes_S T}(t \otimes_S t') = tt',$$

where in the definition of $\Delta_{T \otimes_R T}$ we identified $(T \otimes_S T) \otimes_T (T \otimes_S T)$ and $T \otimes_S T \otimes_S T$.

For S-bimodules T and X, let $v : T \otimes_S X \to X \otimes_S T$ be an S-bimodule map. For $t \in T$ and $x \in X$ we shall use the notation $v(t \otimes_S x) = x_v \otimes_S t_v$, where in the right hand side implicit summation is understood.

Corollary 1.29. *Let $\varphi : S \to T$ be a homomorphism of algebras over a commutative ring $\mathbb{K}$, X be an S-bimodule and $v : T \otimes_S X \to X \otimes_S T$ be an S-bimodule map satisfying*

$$v \circ (\varphi \otimes_S X) = X \otimes_S \varphi \quad and \quad v \circ (m_T \otimes_S X) = (X \otimes_S m_T) \circ (v \otimes_S T) \circ (T \otimes_S v).$$

There is a cyclic object $Z_*(T/S, v)$, with $Z_n(T/S, v) = T^{\hat{\otimes}_S n+1} \hat{\otimes}_S X$ whose face and degeneracy maps are

$$d_k(t_0 \hat{\otimes}_S t_1 \hat{\otimes}_S \cdots \hat{\otimes}_S t_n \hat{\otimes}_S x)$$
$$= \begin{cases} t_0 \hat{\otimes}_S t_1 \hat{\otimes}_S \cdots \hat{\otimes}_S t_k t_{k+1} \hat{\otimes}_S \cdots \hat{\otimes}_S t_n \hat{\otimes}_S x, & for\ 0 \le k < n, \\ (t_n)_v t_0 \hat{\otimes}_S t_1 \hat{\otimes}_S \cdots \hat{\otimes}_S t_{n-1} \hat{\otimes}_S (x)_v, & for\ k = n, \end{cases}$$
$$s_k(t_0 \hat{\otimes}_S t_1 \hat{\otimes}_S \cdots \hat{\otimes}_S t_n \hat{\otimes}_S x)$$
$$= t_0 \hat{\otimes}_S t_1 \hat{\otimes}_S \cdots \hat{\otimes}_S t_k \hat{\otimes}_S 1_T \hat{\otimes}_S \cdots \hat{\otimes}_S t_n \hat{\otimes}_S x, \quad for\ 0 \le k \le n.$$

The para-cyclic map is given by

$$v_n(t_0 \hat{\otimes}_S \cdots \hat{\otimes}_S t_n \hat{\otimes}_S x) = (t_n)_v \hat{\otimes}_S t_0 \hat{\otimes}_S \cdots \hat{\otimes}_S t_{n-1} \hat{\otimes}_S (x)_v.$$

Proof. In terms of the map v, we can equip $X \otimes_S T$ with a T-bimodule structure by

$$t_1(x \otimes_S t)t_2 = v(t_1 \otimes_S x)tt_2.$$

Moreover,

$$v \otimes_S T : T \otimes_S X \otimes_S T \cong (T \otimes_S T) \otimes_T (X \otimes_S T) \to (X \otimes_S T) \otimes_T (T \otimes_S T)$$
$$\cong X \otimes_S T \otimes_S T$$

is a transposition map for Sweedler's T-coring $T \otimes_S T$ and the T-bimodule $X \otimes_S T$, in the sense of (1.18). Consequently, we can apply Corollary 1.27 to Sweedler's coring in Example 1.28. One proves that the corresponding para-cyclic object has $Z_*(T/S, v)$ as an underlying simplicial structure and v_* as a para-cyclic map.   □

Remarks 1.30. (i) Let us take $X = S$. The canonical isomorphism $v : T \otimes_S S \to S \otimes_S T$ satisfies the hypothesis of Corollary 1.29. The corresponding para-cyclic (in fact cyclic) object was used in [JŞ] to define relative cyclic homology. Moreover, this cyclic object and the cocyclic object $Z^*(\mathcal{S}_T, w_T)$ in Corollary 1.17 are (cyclic) dual to each other. In the particular case when $R = k$ and $\varphi = u_T$, we rediscover the cyclic object introduced

by A. Connes in order to define the cyclic homology of an algebra, cf. [We, p. 330]. Thus $Z^*(\mathcal{S}_T, w_T)$ is the cyclic dual of Connes' cyclic object.

(ii) Note that the construction of $Z_*(T/S, v)$ can be performed for any algebra T in a symmetric monoidal category $\mathcal{M}$, by replacing everywhere $\widehat{\otimes}_S$ with $\otimes$, the tensor product in $\mathcal{M}$. Therefore, para-cyclic objects in [Kay06] and [HKRS2] are examples of this type.

(iii) If $w : X \otimes_S T \to T \otimes_S X$ is an invertible S-bimodule map satisfying (1.12), then $v = w^{-1}$ satisfies the relations in Corollary 1.29. Conversely, in the case when the morphism v in the above construction is invertible, then its inverse is a transposition map in the sense of (1.12). As a matter of fact, the corresponding para-cocyclic object in Theorem 1.16 is a cyclic dual of the para-cyclic object in Corollary 1.29. This suggests a categorical approach to cyclic duality, details of which will be studied elsewhere.

Other examples of para-cyclic objects, obtained as applications of Theorem 1.23, will be discussed in Theorems 2.9 and Theorem 2.11.

2. Cyclic (Co)homology of Bialgebroids

In this section we apply the categorical framework, obtained in Sect. 1, to examples provided by (co)module algebras and (co)module corings of bialgebroids, and analyze the structure of the resulting para-(co)cyclic objects.

2.A. (Co)module algebras of bialgebroids. In this section we consider admissible sep-tuples $\mathcal{S}_T$, coming from a $\mathbb{K}$-algebra homomorphism $\varphi : R \to T$ as in Proposition 1.15. As we have seen in Proposition 1.15, $\mathcal{S}_T$ determines a cosimplex

$$Z^n(\mathcal{S}_T, X) = T^{\widehat{\otimes}_R n+1} \widehat{\otimes}_R X, \tag{2.1}$$

for any R-bimodule X. Coface and codegeneracy maps are given by

$$\widehat{d}_k = T^{\widehat{\otimes}_R k} \widehat{\otimes}_R \varphi \widehat{\otimes}_R T^{\widehat{\otimes}_R n-k} \widehat{\otimes}_R X \quad \text{and} \quad \widehat{s}_k = T^{\widehat{\otimes}_R k} \widehat{\otimes}_R$$
$$\times m_T \widehat{\otimes}_R T^{\widehat{\otimes}_R n-k} \widehat{\otimes}_R X, \tag{2.2}$$

where $k = 0, \ldots, n$ and m_T denotes the multiplication map $T \otimes_R T \to T$. Furthermore, by Theorem 1.16, the cosimplex $Z^*(\mathcal{S}_T, X)$ is para-cocyclic provided that there is a transposition map $w : X \otimes_R T \to T \otimes_R X$. Conditions (1.12) characterizing a transposition map are reminiscent of some of the axioms of an entwining structure (over an algebra R), cf. [BB, Sect. 2.3]. Main examples of entwining structures over non-commutative algebras arise from Doi-Koppinen data of bialgebroids (in the sense of [BCM]). In a similar manner, the aim of this section is to construct canonical transposition maps in the case when T is a (co)module algebra of an R-bialgebroid $\mathcal{B}$ and X is a $\mathcal{B}$-(co)module.

Bialgebroids can be thought of as a generalization of bialgebras to arbitrary, non-commutative base algebras. The first form of the structure that is known today as a left bialgebroid was introduced by Takeuchi in [Ta] under the name $\times_R$-bialgebra. Another definition and the name 'bialgebroid' was proposed by Lu in [Lu]. The two definitions were proven to be equivalent in [BMi]. 'Left' and 'right' versions of bialgebroids were defined in [KSz].

Definition 2.1. *Consider an algebra R over a commutative ring $\mathbb{K}$. A* **left bialgebroid**
*$\mathcal{B}$ over R consists of the data $(B, \xi, \zeta, \Delta, \epsilon)$. Here B is a $\mathbb{K}$-algebra and ξ and ζ are
$\mathbb{K}$-algebra homomorphisms $R \to B$ and $R^{op} \to B$, respectively, such that their ranges
are commuting subalgebras in B. In terms of the maps ξ and ζ, B can be equipped with
an R-bimodule structure as*

$$r_1 \cdot b \cdot r_2 := \xi(r_1)\zeta(r_2)b, \quad for \ r_1, r_2 \in R \ and \ b \in B.$$

By definition, the coproduct $\Delta : B \to B \otimes_R B$ and the counit $\epsilon : B \to R$ equip this
bimodule with an R-coring structure. Between the algebra and coring structures of B the
following compatibility axioms are required. For the coproduct we introduce the index
notation $\Delta(b) = b_{(1)} \otimes_R b_{(2)}$, where implicit summation is understood:

  (i) $b_{(1)}\zeta(r) \otimes_R b_{(2)} = b_{(1)} \otimes_R b_{(2)}\xi(r)$, for $r \in R$ and $b \in B$.
  (ii) $\Delta(1_B) = 1_B \otimes_R 1_B$ and $\Delta(bb') = b_{(1)}b'_{(1)} \otimes_R b_{(2)}b'_{(2)}$, for $b, b' \in B$.
  (iii) $\epsilon(1_B) = 1_R$ and $\epsilon(bb') = \epsilon\big(b\xi(\epsilon(b'))\big)$, for $b, b' \in B$.

    Axiom (i) in Definition 2.1 needs to be imposed in order for the second condition
in axiom (ii) to make sense. Axiom (iii) implies that also $\epsilon(bb') = \epsilon\big(b\zeta(\epsilon(b'))\big)$, for
$b, b' \in B$. It follows by the R-module map properties, unitality and multiplicativity of
the coproduct Δ that

$$\Delta\big(\xi(r_1)\zeta(r_2)b\xi(r_3)\zeta(r_4)\big) = \xi(r_1)b_{(1)}\xi(r_3) \otimes_R \zeta(r_2)b_{(2)}\zeta(r_4), \tag{2.3}$$

for $r_1, r_2, r_3, r_4 \in R$ and $b \in \mathcal{B}$. Since the coproduct is coassociative, the Sweedler-
Heynemann index notation can be used. That is, for the iterated power of the coproduct
we write $(\Delta \otimes_R \mathcal{B} \otimes_R \cdots \otimes_R \mathcal{B}) \circ \cdots \circ (\Delta \otimes_R \mathcal{B}) \circ \Delta(b) = b_{(1)} \otimes_R \cdots \otimes_R b_{(n-1)} \otimes_R b_{(n)}$,
for any positive integer n and $b \in \mathcal{B}$.
    Note that the axioms in Definition 2.1 are not invariant under changing the multipli-
cation in B to the opposite multiplication. Definition 2.1 has a symmetrical counterpart,
known as a *right bialgebroid*. For the details we refer to [KSz].
    Definition 2.1 is motivated by the following result of Schauenburg. Consider two
algebras R and B over a commutative ring $\mathbb{K}$ and two $\mathbb{K}$-algebra homomorphisms $\xi :
R \to B$ and $\zeta : R^{op} \to B$, whose ranges are commuting subalgebras of B. Clearly, in
this setting any (left or right) B module can be equipped with an R bimodule structure
using the maps ξ and ζ. For example, for a left B-module V with action $\triangleright : B \otimes_{\mathbb{K}} V \to V$,
one can define an R-bimodule structure as

$$r_1 \otimes_{\mathbb{K}} v \otimes_{\mathbb{K}} r_2 \mapsto \xi(r_1)\zeta(r_2) \triangleright v.$$

With respect to the resulting R-actions, B-module maps are R-bimodule maps. That
is, there exists a forgetful functor from the category of (left or right) B-modules to the
category of R-bimodules.

Theorem 2.2. [Sch98, Theorem 5.1] *Consider two algebras R and B over a commutative
ring $\mathbb{K}$ and two $\mathbb{K}$-algebra homomorphisms $\xi : R \to B$ and $\zeta : R^{op} \to B$, whose
ranges are commuting subalgebras of B. There exists a right (resp. left) bialgebroid
$(B, \xi, \zeta, \Delta, \epsilon)$ if and only if the forgetful functor from the category of right (resp. left)
B-modules to the category of R-bimodules is strict monoidal. That is, R is a right (resp.
left) B-module and the R-module tensor product of two right (resp. left) B-modules is a
right (resp. left) B-module.*

Similarly to the case of a bialgebra, in Theorem 2.2 for a left R-bialgebroid $(B, \xi, \zeta, \Delta, \epsilon)$ the following B-actions are used on R, and on the R-module tensor product of two left B-modules V and W, respectively.

$$b \triangleright r := \epsilon\big(b\xi(r)\big) \quad \text{and} \quad b \triangleright (v \otimes_R w) := b_{(1)} \triangleright v \otimes_R b_{(2)} \triangleright w, \qquad (2.4)$$

for $r \in R$, $v \otimes_R w \in V \otimes_R W$ and $b \in B$. It was proven in [Sch98, Theorem 5.1] that the diagonal action in the second equation in (2.4) is meaningful by axiom (i) in Definition 2.1.

In light of Theorem 2.2, one can speak about right (resp. left) module algebras of a right (resp. left) bialgebroid B, i.e. about algebras in the monoidal category of right (resp. left) B-modules.

Definition 2.3. *Consider an algebra R over a commutative ring $\mathbb{K}$ and a left R-bialgebroid B. A **left B-module algebra** is a $\mathbb{K}$-algebra and left B-module T, with B-action $\triangleright : B \otimes_{\mathbb{K}} T \to T$, such that the multiplication in T is R-balanced and*

$$b \triangleright 1_T = \xi\big(\epsilon(b)\big) \triangleright 1_T \quad \text{and} \quad b \triangleright (tt') = (b_{(1)} \triangleright t)(b_{(2)} \triangleright t'), \qquad (2.5)$$

for $b \in B$ and $t, t' \in T$.

Note that for a left module algebra T of a left R-bialgebroid $B = (B, \xi, \zeta, \Delta, \epsilon)$, there is a canonical $\mathbb{K}$-algebra homomorphism $R \to T$, $r \mapsto \xi(r) \triangleright 1_T$. Hence there is a corresponding admissible septuple as in Proposition 1.15.

A left comodule of a left R-bialgebroid $B = (B, \xi, \zeta, \Delta, \epsilon)$ is defined as a left comodule of the underlying R-coring (B, Δ, ϵ). That is, a left R-module X, together with a left R-module map $X \to B \otimes_R X$, $x \mapsto x_{[-1]} \otimes_R x_{[0]}$ (where implicit summation is understood), satisfying coassociativity and counitality axioms. Note that a left B-comodule X can be equipped with an R-bimodule structure by introducing a right R-action

$$x \cdot r := \epsilon\big(x_{[-1]}\xi(r)\big) \cdot x_{[0]}, \qquad \text{for } r \in R \text{ and } x \in X. \qquad (2.6)$$

With respect to the resulting bimodule structure, B-comodule maps are R-bimodule maps. In particular, the left B-coaction on X is an R-bimodule map in the sense that, for $r, r' \in R$ and $x \in X$,

$$(r \cdot x \cdot r')_{[-1]} \otimes_R (r \cdot x \cdot r')_{[0]} = \xi(r)x_{[-1]}\xi(r') \otimes_R x_{[0]}. \qquad (2.7)$$

Furthermore, for any $x \in X$ and $r \in R$,

$$x_{[-1]} \otimes_R x_{[0]} \cdot r = x_{[-1]}\zeta(r) \otimes_R x_{[0]}. \qquad (2.8)$$

Theorem 2.4. *Let R be an algebra over a commutative ring $\mathbb{K}$ and let B be a left bialgebroid over R. Consider a left B-module algebra T with B-action $\triangleright$ and a left B-comodule X with coaction $x \mapsto x_{[-1]} \otimes_R x_{[0]}$ (where implicit summation is understood). Then a transposition map for the admissible septuple S_T, associated via Proposition 1.15 to the $\mathbb{K}$-algebra map $R \to T$, $r \mapsto \xi(r) \triangleright 1_T$, is given by*

$$w : X \otimes_R T \to T \otimes_R X, \qquad x \otimes_R t \mapsto x_{[-1]} \rhd t \otimes_R x_{[0]}. \tag{2.9}$$

Hence the cosimplex (2.1) admits a para-cocyclic structure

$$w_n(t_0 \,\widehat{\otimes}_R\, \cdots \,\widehat{\otimes}_R\, t_n \,\widehat{\otimes}_R\, x) = t_1 \,\widehat{\otimes}_R\, \cdots \,\widehat{\otimes}_R\, t_n \,\widehat{\otimes}_R\, x_{[-1]} \rhd t_0 \,\widehat{\otimes}_R\, x_{[0]}. \tag{2.10}$$

Proof. The map (2.9) is a well defined left R-module homomorphism by (2.7). Its right R-module map property follows by (2.8). Conditions (1.12) follow by definition (2.5) of a module algebra as it is shown below. Denote by φ the algebra homomorphism $R \to T$, $r \mapsto \xi(r) \rhd 1_T = \zeta(r) \rhd 1_T$. Omitting canonical isomorphisms $R \otimes_R X \cong X \cong X \otimes_R R$, we have

$$\big(w \circ (X \otimes_R \varphi)\big)(x) = x_{[-1]} \rhd 1_T \otimes_R x_{[0]} = \zeta\big(\epsilon(x_{[-1]})\big) \rhd 1_T \otimes_R x_{[0]} = 1_T$$
$$\otimes_R \epsilon(x_{[-1]}) \cdot x_{[0]} = 1_T \otimes_R x = \big((\varphi \otimes_R X)\big)(x),$$
$$\big((m_T \otimes_R X) \circ (T \otimes_R w) \circ (w \otimes_R T)\big)(x \otimes_R t \otimes_R t')$$
$$= (x_{[-1]} \rhd t)(x_{[0][-1]} \rhd t') \otimes_R x_{[0][0]} =$$
$$(x_{[-1](1)} \rhd t)(x_{[-1](2)} \rhd t') \otimes_R x_{[0]}$$
$$= x_{[-1]} \rhd tt' \otimes_R x_{[0]} = \big(w \circ (X \otimes_R m_T)\big)(x \otimes_R t \otimes_R t'). \qquad \square$$

Analogously to (2.6), also a right comodule V of a right R-bialgebroid $\mathcal{B} = (B, \xi, \zeta, \Delta, \epsilon)$ can be equipped with an R-bimodule structure by introducing a left R-action,

$$r \cdot v := v^{[0]} \cdot \epsilon\big(\xi(r)v^{[1]}\big), \qquad \text{for } r \in R \text{ and } v \in V, \tag{2.11}$$

where $v \mapsto v^{[0]} \otimes_R v^{[1]}$ denotes the right $\mathcal{B}$-coaction on V, with implicit summation understood. Hence there exists a forgetful functor from the category of right $\mathcal{B}$-comodules to the category of R-bimodules. With this observation in mind, the next theorem follows by a symmetrical form of [Sch98, Prop. 5.6].

Theorem 2.5. *Consider an algebra R over a commutative ring $\mathbb{K}$ and a right R-bialgebroid $\mathcal{B}$. Then the forgetful functor from the category of right $\mathcal{B}$-comodules to the category of R-bimodules is strict monoidal. That is, R is a right $\mathcal{B}$-comodule and the R-module tensor product of two right $\mathcal{B}$-comodules is a right $\mathcal{B}$-comodule.*

Similarly to a bialgebra, in Theorem 2.5 for a right R-bialgebroid $\mathcal{B} = (B, \xi, \zeta, \Delta, \epsilon)$ the following $\mathcal{B}$-coactions on R, and on the R-module tensor product $V \otimes_R W$ of two right $\mathcal{B}$-comodules V and W, are used.

$$R \to R \otimes_R \mathcal{B} \cong \mathcal{B}, \quad r \mapsto \xi(r) \text{ and } V \otimes_R W \to V \otimes_R W \otimes_R \mathcal{B},$$
$$v \otimes_R w \mapsto v^{[0]} \otimes_R w^{[0]} \otimes_R v^{[1]}w^{[1]}.$$

The coaction on $V \otimes_R W$ is well defined by the right bialgebroid versions of properties (2.7) and (2.8), i.e. the identities

$$(r \cdot v \cdot r')^{[0]} \otimes_R (r \cdot v \cdot r')^{[1]} = v^{[0]} \otimes_R \xi(r)v^{[1]}\xi(r') \qquad \text{and}$$
$$r \cdot v^{[0]} \otimes_R v^{[1]} = v^{[0]} \otimes_R \zeta(r)v^{[1]}, \tag{2.12}$$

for $r, r' \in R$ and $v \in V$. Symmetrically, the forgetful functors from the category of left comodules of a right R-bialgebroid, and from the categories of right or left comodules of a left R-bialgebroid, to the category of R-bimodules are strict (anti-)monoidal.

In light of Theorem 2.5, one can speak about right comodule algebras of a right bialgebroid $\mathcal{B}$, i.e. about algebras in the monoidal category of right $\mathcal{B}$-comodules.

Definition 2.6. *Consider an algebra R over a commutative ring $\mathbb{K}$ and a right R-bialgebroid $\mathcal{B}$. A* right $\mathcal{B}$-comodule algebra *is a $\mathbb{K}$-algebra and right $\mathcal{B}$-comodule T, with coaction $t \mapsto t^{[0]} \otimes_R t^{[1]}$, such that the multiplication in T is R-balanced and, for $t, t' \in T$,*

$$1_T^{[0]} \otimes_R 1_T^{[1]} = 1_T \otimes_R 1_T \quad and \quad (tt')^{[0]} \otimes_R (tt')^{[1]} = t^{[0]} t'^{[0]} \otimes_R t^{[1]} t'^{[1]}. \quad (2.13)$$

For example, the constituent algebra in a right bialgebroid $\mathcal{B}$ is itself a (so-called *right regular*) right $\mathcal{B}$-comodule algebra via the coaction given by the coproduct in $\mathcal{B}$.

Note that the second condition in (2.13) is meaningful since the multiplication in T is R-balanced and the second condition in (2.12) holds. For a right comodule algebra T of a right R-bialgebroid $\mathcal{B}$, there is a canonical $\mathbb{K}$-algebra homomorphism $R \to T$, $r \mapsto r \cdot 1_T = 1_T \cdot r$, in terms of which $r \cdot t = (r \cdot 1_T)t$ and $t \cdot r = t(r \cdot 1_T)$. Hence there is a corresponding admissible septuple as in Proposition 1.15.

Theorem 2.7. *Let R be an algebra over a commutative ring $\mathbb{K}$ and let $\mathcal{B}$ be a right bialgebroid over R. Consider a right $\mathcal{B}$-comodule algebra T, with coaction $t \mapsto t^{[0]} \otimes_R t^{[1]}$ (where implicit summation is understood) and a right $\mathcal{B}$-module X with action $\lhd$. Then a transposition map for the admissible septuple $\mathcal{S}_T$, associated via Proposition 1.15 to the $\mathbb{K}$-algebra map $R \to T$, $r \mapsto r \cdot 1_T = 1_T \cdot r$, is given by*

$$w : X \otimes_R T \to T \otimes_R X, \quad x \otimes_R t \mapsto t^{[0]} \otimes_R x \lhd t^{[1]}.$$

Hence the cosimplex (2.1) admits a para-cocyclic structure

$$w_n(t_0 \,\widehat{\otimes}_R\, \cdots \,\widehat{\otimes}_R\, t_n \,\widehat{\otimes}_R\, x) = t_1 \,\widehat{\otimes}_R\, \cdots \,\widehat{\otimes}_R\, t_n \,\widehat{\otimes}_R\, t_0^{[0]} \,\widehat{\otimes}_R\, x \lhd t_0^{[1]}.$$

Proof. The map w is a well defined R-bimodule homomorphism by (2.12). Conditions (1.12) follow by definition (2.13) of a comodule algebra. $\square$

2.B. (Co)module corings of bialgebroids. In this section we consider comonads $T_l := C \otimes_R (-)$ on the category of R-bimodules as in Proposition 1.25, determined by a coring (C, Δ, ϵ) over an algebra R. Let Π be the functor in Definition 1.13. By Proposition 1.25, for any R-bimodule X, there is an associated simplicial module $Z_*(\mathcal{S}_C, X)$, which in degree n is given by

$$Z_n(\mathcal{S}_C, X) = C^{\widehat{\otimes}_R\, n+1} \,\widehat{\otimes}_R\, X. \quad (2.14)$$

Face and degeneracy maps in degree n are

$$\widehat{d_k} = C^{\widehat{\otimes}_R\, k} \,\widehat{\otimes}_R\, \epsilon_C \,\widehat{\otimes}_R\, C^{\widehat{\otimes}_R\, n-k} \,\widehat{\otimes}_R\, X \quad \text{and}$$
$$\widehat{s_k} = C^{\widehat{\otimes}_R\, k} \,\widehat{\otimes}_R\, \Delta_C \,\widehat{\otimes}_R\, C^{\widehat{\otimes}_R\, n-k} \,\widehat{\otimes}_R\, X, \quad (2.15)$$

for $k \in \{0, \ldots, n\}$. As we have seen in Proposition 1.25, we can choose a second comonad $T_r := (-) \otimes_R C$ on the category of R-bimodules, and a natural transformation $i : (-) \,\widehat{\otimes}_R\, C \to C \,\widehat{\otimes}_R (-)$, given by the flip map. Taking the trivial natural transformation $t : C \otimes_R (-) \otimes_R C \to C \otimes_R (-) \otimes_R C$, the conditions in (1.17) hold. By Theorem 1.26, the simplex $Z_*(\mathcal{S}_C, X)$ is para-cyclic provided that there exists a morphism $w : C \otimes_R X \to X \otimes_R C$, satisfying (1.21). Note the similarity of conditions (1.21) to some of the axioms of an entwining structure over R. Similarly to the way Doi-Koppinen data

determine entwining structures, in this section we construct dual transposition maps in the case when C is a (co)module coring of a bialgebroid $\mathcal{B}$ and X is a $\mathcal{B}$-(co)module.

In light of (a symmetrical version of) Theorem 2.5, one can speak about left comodule corings of a left bialgebroid $\mathcal{B}$, i.e. about coalgebras in the monoidal category of left $\mathcal{B}$-comodules.

Definition 2.8. *Consider an algebra R over a commutative ring $\mathbb{K}$ and a left R-bialgebroid $\mathcal{B} = (B, \xi, \zeta, \Delta, \epsilon)$. A left $\mathcal{B}$-comodule coring is an R-coring and left $\mathcal{B}$-comodule C, with one and the same underlying R-bimodule structure, such that for $c \in C$,*

$$\xi\big(\epsilon_C(c)\big) = \zeta\big(\epsilon_C(c_{[0]})\big)c_{[-1]} \quad and \quad c_{[-1]} \underset{R}{\otimes} c_{[0](1)} \underset{R}{\otimes} c_{[0](2)}$$
$$= c_{(1)[-1]}c_{(2)[-1]} \underset{R}{\otimes} c_{(1)[0]} \underset{R}{\otimes} c_{(2)[0]}, \tag{2.16}$$

where ϵ_C is the counit and $\Delta_C : c \mapsto c_{(1)} \otimes_R c_{(2)}$ is the coproduct of C and $c \mapsto c_{[-1]} \otimes_R c_{[0]}$ is the $\mathcal{B}$-coaction on C.

The second condition in (2.16) is meaningful by (2.7) and (2.8).

Theorem 2.9. *Let R be an algebra over a commutative ring $\mathbb{K}$ and let $\mathcal{B}$ be a left bialgebroid over R. Consider a left $\mathcal{B}$-comodule coring C with coaction $c \mapsto c_{[-1]} \otimes_R c_{[0]}$, and a left $\mathcal{B}$-module X with action $\triangleright$. Then a transposition map $w : C \otimes_R X \to X \otimes_R C$ for the dual admissible septuple $\mathcal{S}_C$, associated via Proposition 1.25 to the R-coring C, is given by $w(c \otimes_R x) := c_{[-1]} \triangleright x \otimes_R c_{[0]}$. Hence the simplex (2.14) admits a para-cyclic structure*

$$w_n(c_0 \widehat{\otimes}_R \cdots \widehat{\otimes}_R c_n \widehat{\otimes}_R x) = c_{n[0]} \widehat{\otimes}_R c_0 \widehat{\otimes}_R \cdots \widehat{\otimes}_R c_{n-1} \widehat{\otimes}_R c_{n[-1]} \triangleright x.$$

Proof. The map w is a well defined R-bimodule homomorphism by (2.7) and (2.8). Conditions (1.21) follow by definition (2.16) of a comodule coring. $\square$

In light of Theorem 2.2, one can speak about right (resp. left) module corings of a right (resp. left) bialgebroid $\mathcal{B}$, i.e. about coalgebras in the monoidal category of right (resp. left) $\mathcal{B}$-modules.

Definition 2.10. *Consider an algebra R over a commutative ring $\mathbb{K}$ and a right R-bialgebroid $\mathcal{B} = (B, \xi, \zeta, \Delta, \epsilon)$. A right $\mathcal{B}$-module coring is an R-coring and right $\mathcal{B}$-module C, with one and the same underlying R-bimodule structure, such that for $c \in C$ and $b \in B$,*

$$\epsilon_C(c \triangleleft b) = \epsilon\big(\xi(\epsilon_C(c))b\big) \quad and$$
$$(c \triangleleft b)^{(1)} \otimes_R (c \triangleleft b)^{(2)} = c^{(1)} \triangleleft b^{(1)} \otimes_R c^{(2)} \triangleleft b^{(2)}, \tag{2.17}$$

where ϵ_C is the counit and $\Delta_C : c \mapsto c^{(1)} \otimes_R c^{(2)}$ is the coproduct in C, the symbol $\triangleleft$ denotes the $\mathcal{B}$-action on C and for the coproduct in $\mathcal{B}$ the index notation $\Delta : b \mapsto b^{(1)} \otimes_R b^{(2)}$ is used, with implicit summation understood.

For example, the constituent R-coring in a right R-bialgebroid $\mathcal{B}$ is itself a (so called *right regular*) right $\mathcal{B}$-module coring via the action given by the product in $\mathcal{B}$.

Theorem 2.11. *Let R be an algebra over a commutative ring $\mathbb{K}$ and let $\mathcal{B}$ be a right bialgebroid over R. Consider a right $\mathcal{B}$-module coring C with action $\triangleleft$, and a right $\mathcal{B}$-comodule X with coaction $x \mapsto x^{[0]} \otimes_R x^{[1]}$. Then a transposition map $w : C \otimes_R X \to X \otimes_R C$, for the dual admissible septuple $\mathcal{S}_C$, associated via Proposition 1.25 to the R-coring C, is given by $w(c \otimes_R x) := x^{[0]} \otimes_R c \triangleleft x^{[1]}$. Hence the simplex (2.14) admits a para-cyclic structure,*

$$w_n(c_0 \,\widehat{\otimes}_R\, \cdots \,\widehat{\otimes}_R\, c_n \,\widehat{\otimes}_R\, x) = c_n \triangleleft x^{[1]} \,\widehat{\otimes}_R\, c_0 \,\widehat{\otimes}_R\, \cdots \,\widehat{\otimes}_R\, c_{n-1} \,\widehat{\otimes}_R\, x^{[0]}. \quad (2.18)$$

Proof. The map w is a well defined R-bimodule homomorphism by (2.12). Conditions (1.21) follow by definition (2.17) of a module coring. □

2.C. Stable anti-Yetter-Drinfel'd modules of $\times_R$-Hopf algebras. For a right module coring C and right comodule X of a right bialgebroid $\mathcal{B} = (B, \xi, \zeta, \Delta, \epsilon)$ over a $\mathbb{K}$-algebra R, there is a $\mathbb{K}$-module isomorphism

$$C^{\,\widehat{\otimes}_R n+1} \widehat{\otimes}_R X \cong C^{\,\otimes_R n+1} \otimes_{R^e} X.$$

Assume that X has an additional left $\mathcal{B}$-module structure. In this case, corresponding to the $\mathbb{K}$-algebra homomorphism $R^e \to B$, $r \otimes_{\mathbb{K}} r' \mapsto \xi(r)\zeta(r') = \zeta(r')\xi(r)$, there is a canonical epimorphism $C^{\,\widehat{\otimes}_R n+1} \widehat{\otimes}_R X \to C^{\,\otimes_R n+1} \otimes_{\mathcal{B}} X$, where $C^{\,\otimes_R n+1}$ is understood to be a right $\mathcal{B}$-module via the diagonal action $(c_1 \otimes_R \cdots \otimes_R c_{n+1}) \triangleleft b := c_1 \triangleleft b^{(1)} \otimes_R \cdots \otimes_R c_{n+1} \triangleleft b^{(n+1)}$, given by the iterated coproduct in $\mathcal{B}$. It is a well defined action by (a symmetrical version of) [Sch98, Theorem 5.1].

Lemma 2.12. *Let R be an algebra over a commutative ring $\mathbb{K}$ and let $\mathcal{B}$ be a right bialgebroid over R. Consider a right $\mathcal{B}$-module coring C and a left $\mathcal{B}$-module right $\mathcal{B}$-comodule X. Then the simplex in Theorem 2.11 projects to a simplex $C^{\,\otimes_R n+1} \otimes_{\mathcal{B}} X$.*

Proof. Since the coproduct Δ_C and the counit ϵ_C of C are right $\mathcal{B}$-module maps by definition, face and degeneracy maps of the simplex in Example 1.27 are right $\mathcal{B}$-module maps with respect to the diagonal action. Hence we can take their tensor product with the identity map on X over the algebra $\mathcal{B}$, yielding a simplex as stated. □

The task of this section is to find criteria for the cyclicity of the quotient simplex in Lemma 2.12. In order to do so, some restriction on the involved bialgebroid is needed.

Definition 2.13. [Sch00, Theorem and Definition 3.5]. *Let R be an algebra over a commutative ring $\mathbb{K}$. A right R-bialgebroid $\mathcal{B} = (B, \xi, \zeta, \Delta, \epsilon)$ is said to be a right $\times_R$-**Hopf algebra** provided that the map*

$$\vartheta : \mathcal{B} \otimes_{R^{op}} \mathcal{B} \to \mathcal{B} \otimes_R \mathcal{B}, \qquad b \otimes_{R^{op}} b' \mapsto bb'^{(1)} \otimes_R b'^{(2)} \qquad (2.19)$$

is bijective. In the domain of the map in (2.19), R^{op}-module structures are given by right and left multiplication by $\zeta(r)$, for $r \in R$. In the codomain of the map in (2.19), R-module structures are given by right multiplication by $\xi(r)$ and $\zeta(r)$, for $r \in R$.

The notion of a $\times_R$-Hopf algebra extends that of a Hopf algebra. Indeed, if $\mathcal{B}$ is a bialgebra over a *commutative* ring R, with coproduct $b \mapsto b^{(1)} \otimes_R b^{(2)}$, then the map (2.19) is bijective if and only if $\mathcal{B}$ is a Hopf algebra. In this case the inverse is given in terms of the antipode S as $\vartheta^{-1}(b \otimes_R b') := bS(b'^{(1)}) \otimes_R b'^{(2)}$.

For an algebra R, consider a right $\times_R$-Hopf algebra $\mathcal{B}$. Since the map ϑ in (2.19) is a left $\mathcal{B}$-module map, its inverse is determined by the restriction $\vartheta^{-1}(1_{\mathcal{B}} \otimes_R b) =: b_- \otimes_{R^{op}} b_+$, where implicit summation is understood. Lemma 2.14, which is essentially a symmetrical version of [Sch00, Prop. 3.7], collects properties of the map $b \mapsto b_- \otimes_{R^{op}} b_+$.

Lemma 2.14. *For an algebra R, consider a right $\times_R$-Hopf algebra $\mathcal{B} = (B, \xi, \zeta, \Delta, \epsilon)$. Write $\Delta(b) =: b^{(1)} \otimes_R b^{(2)}$ for the coproduct and in terms of the map (2.19) put $b_- \otimes_{R^{op}} b_+ := \vartheta^{-1}(1_{\mathcal{B}} \otimes_R b)$. The following identities hold, for $b, b' \in \mathcal{B}$ and $r \in R$.*

(i) $b_- b_+{}^{(1)} \otimes_R b_+{}^{(2)} = 1_{\mathcal{B}} \otimes_R b$,
(ii) $b^{(1)} b^{(2)}{}_- \otimes_{R^{op}} b^{(2)}{}_+ = 1_{\mathcal{B}} \otimes_{R^{op}} b$,
(iii) $(bb')_- \otimes_{R^{op}} (bb')_+ = b'_- b_- \otimes_{R^{op}} b_+ b'_+$,
(iv) $1_{\mathcal{B}-} \otimes_{R^{op}} 1_{\mathcal{B}+} = 1_{\mathcal{B}} \otimes_{R^{op}} 1_{\mathcal{B}}$,
(v) $b_- \otimes_{R^{op}} b_+{}^{(1)} \otimes_R b_+{}^{(2)} = b^{(1)}{}_- \otimes_{R^{op}} b^{(1)}{}_+ \otimes_R b^{(2)}$,
(vi) $b_-{}^{(1)} \otimes_R b_-{}^{(2)} \otimes_{R^{op}} b_+ = b_{+-} \otimes_R b_- \otimes_{R^{op}} b_{++}$,
(vii) $b = \zeta\big(\epsilon(b_-)\big) b_+$,
(viii) $b_- b_+ = \xi\big(\epsilon(b)\big)$,
(x) $\zeta(r) b_- \otimes_{R^{op}} b_+ = b_- \otimes_{R^{op}} b_+ \zeta(r)$.

Next Definition 2.15 extends [JŞ, Definition 4.1] or [HKRS1, Definition 2.1].

Definition 2.15. *For an algebra R, consider a right $\times_R$-Hopf algebra $\mathcal{B} = (B, \xi, \zeta, \Delta, \epsilon)$. Let X be a right $\mathcal{B}$-comodule and left $\mathcal{B}$-module. Denote the right $\mathcal{B}$-coaction on X by $x \mapsto x^{[0]} \otimes_R x^{[1]}$, for $x \in X$ (where implicit summation is understood) and denote the left $\mathcal{B}$-action by $b \triangleright x$, for $b \in \mathcal{B}$ and $x \in X$. We say that X is an* **anti-Yetter-Drinfel'd module** *provided that the following conditions hold.*

(i) *The R-bimodule structures of X, underlying its module and comodule structures, coincide. That is, for $x \in X$ and $r \in R$,*

$$x \cdot r = \zeta(r) \triangleright x \quad and \quad r \cdot x = \xi(r) \triangleright x, \tag{2.20}$$

where $x \cdot r$ denotes the right R-action on the right $\mathcal{B}$-comodule X and $r \cdot x$ is the canonical left R-action (2.11) coming from the right $\mathcal{B}$-coaction.

(ii) *For $b \in \mathcal{B}$ and $x \in X$,*

$$(b \triangleright x)^{[0]} \otimes_R (b \triangleright x)^{[1]} = b^{(1)}{}_+ \triangleright x^{[0]} \otimes_R b^{(2)} x^{[1]} b^{(1)}{}_-, \tag{2.21}$$

where for the coproduct Δ and the inverse of the map (2.19) the respective index notations (with implicit summation), $\Delta(b) = b^{(1)} \otimes_R b^{(2)}$, and $\vartheta^{-1}(1_{\mathcal{B}} \otimes_R b) = b_- \otimes_{R^{op}} b_+$ are used, for $b \in \mathcal{B}$.

The anti-Yetter-Drinfel'd module X is said to be **stable** *if in addition, for any $x \in X$,*

$$x^{[1]} \triangleright x^{[0]} = x. \tag{2.22}$$

We need to show that condition (ii) in Definition 2.15 is meaningful, i.e. the expression on the right hand side of (2.21) is well defined. This follows by the following

Lemma 2.16. *For an algebra R, consider a right $\times_R$-Hopf algebra $\mathcal{B} = (B, \xi, \zeta, \Delta, \epsilon)$. Let X be a right $\mathcal{B}$-comodule and left $\mathcal{B}$-module. Keeping the notations in Definition 2.15, assume that axiom (i) in Definition 2.15 holds. Then the following hold:*

(1) *Considering $\mathcal{B}$ as a left R-module via ζ, the R-module tensor product $X \otimes_R \mathcal{B}$ is a left $\mathcal{B}$-module via the action*

$$b' \blacktriangleright (x \otimes_R b) := b'_+ \rhd x \otimes_R bb'_-.$$

(2) *$X \otimes_R \mathcal{B}$ is a $\mathcal{B}$-$\mathcal{B}^{op}$ bimodule, via the left $\mathcal{B}$-action in part* (i) *and the right $\mathcal{B}^{op}$-action*

$$(x \otimes_R b) \blacktriangleleft b' := x \otimes_R b'b.$$

(3) *For any elements $x \in X$ and $r \in R$,*

$$\xi(r) \blacktriangleright \left(x^{[0]} \otimes_R x^{[1]}\right) = \left(x^{[0]} \otimes_R x^{[1]}\right) \blacktriangleleft \zeta(r).$$

Proof. By the first condition in (2.20) and Lemma 2.14 (ix), there is a well defined map

$$\mathcal{B} \otimes_{\mathbb{K}} (X \otimes_R \mathcal{B}) \to X \otimes_R \mathcal{B}, \qquad b' \otimes_{\mathbb{K}} (x \otimes_R b) \mapsto b'_+ \rhd x \otimes_R b\, b'_-.$$

It is an associative left $\mathcal{B}$-action by part (iii) of Lemma 2.14 and it is unital by part (iv). This proves claim (1). Claim (2) is obvious. Claim (3) follows by the following computation, for $r \in R$ and $x \in X$:

$$\begin{aligned}
\xi(r) \blacktriangleright \left(x^{[0]} \otimes_R x^{[1]}\right) &= \xi(r) \rhd x^{[0]} \otimes_R x^{[1]} = x^{[0][0]} \cdot \epsilon(\zeta(r)x^{[0][1]}) \otimes_R x^{[1]} \\
&= x^{[0]} \otimes_R x^{[1](2)}\zeta\left(\epsilon(\zeta(r)x^{[1](1)})\right) = x^{[0]} \otimes_R \zeta(r)x^{[1]} \\
&= \left(x^{[0]} \otimes_R x^{[1]}\right) \blacktriangleleft \zeta(r).
\end{aligned}$$

In the first equality we used that, by unitality and the right R-module map property of the coproduct, the map (2.19) satisfies $\vartheta(1_{\mathcal{B}} \otimes_{R^{op}} \xi(r)) = 1_{\mathcal{B}} \otimes_R \xi(r)$. Hence $\xi(r)_- \otimes_{R^{op}} \xi(r)_+ = 1_{\mathcal{B}} \otimes_{R^{op}} \xi(r)$. The second equality follows by the second condition in (2.20). The third equality follows by coassociativity of the $\mathcal{B}$-coaction on X. The fourth equality follows by counitality of Δ and the right bialgebroid version of (2.3), i.e. the identity

$$\Delta\left(\xi(r_1)\zeta(r_2)b\xi(r_3)\zeta(r_4)\right) = \zeta(r_2)b^{(1)}\zeta(r_4) \otimes_R \xi(r_1)b^{(2)}\xi(r_3), \qquad (2.23)$$

for $r_1, r_2, r_3, r_4 \in R$ and $b \in \mathcal{B}$. $\square$

Using the notations in Lemma 2.16, the right-hand side of (2.21) is equal to the well defined expression $b^{(1)} \blacktriangleright \left(x^{[0]} \otimes_R x^{[1]}\right) \blacktriangleleft b^{(2)}$.

Remark 2.17. Note that a Hopf algebra H over a commutative ring $\mathbb{K}$ is a $\times_{\mathbb{K}}$-Hopf algebra. The bialgebroid structure is given by the equal source and target maps $\mathbb{K} \to H$, $\kappa \mapsto \kappa 1_H$, and the coproduct and counit in H. The canonical map (2.19) has an inverse $\vartheta^{-1}(h' \otimes_{\mathbb{K}} h) = h'S(h^{(1)}) \otimes_{\mathbb{K}} h^{(2)}$, where $h \mapsto h^{(1)} \otimes_{\mathbb{K}} h^{(2)}$ is the usual Sweedler index notation for the coproduct, with implicit summation understood. That is, $h_- \otimes_{\mathbb{K}} h_+ = S(h^{(1)}) \otimes_{\mathbb{K}} h^{(2)}$. Clearly, in this case (2.20) becomes the trivial condition

$$x\kappa = (\kappa 1_H) \rhd x = \kappa x,$$

for $\kappa \in \mathbb{K}$ and any element x of a left H-module right H-comodule X. This condition simply expresses the requirement that the left and right $\mathbb{K}$-actions on X are equal, and are induced by the H-module structure. The second condition (2.21) in Definition 2.15 reduces to

$$(b \rhd x)^{[0]} \otimes (b \rhd x)^{[1]} = b^{(2)} \rhd x^{[0]} \otimes b^{(3)}x^{[1]}S(b^{(1)}),$$

for $b \in H$ and $x \in X$, which is the defining property of a (left-right) anti-Yetter-Drinfel'd module X of a Hopf algebra H in [JŞ, Def. 4.1] or [HKRS1, Def. 2.1]. So we conclude that Def. 2.15 generalizes these definitions.

Example 2.18. For an algebra R, consider a right $\times_R$-Hopf algebra $\mathcal{B} = (B, \xi, \zeta, \Delta, \epsilon)$. Note that left B-actions $\triangleright$ on R, satisfying $\zeta(r')\triangleright r = rr'$, are in bijective correspondence with maps $\chi : B \to R$, obeying the following properties, for $r \in R$, $b, b' \in B$:

$$\chi(\zeta(r)b) = \chi(b)r \qquad \chi(bb') = \chi\big(b\zeta(\chi(b'))\big), \qquad \chi(1_B) = 1_R.$$

Indeed, in terms of such a map χ, one can put $b \triangleright r := \chi(b\zeta(r))$. Furthermore, right B-coactions on R, with underlying right regular R-module structure, are in bijective correspondence with grouplike elements in B, i.e. $g \in B$ such that $\Delta(g) = g \otimes_R g$ and $\epsilon(g) = 1_R$. Indeed, in terms of a grouplike element g, a right B-coaction on R is given by $r \mapsto 1_R \otimes_R g\xi(r)$. One checks that the left B-module determined by χ and the right B-comodule determined by g combine into an anti-Yetter-Drinfel'd module on R if and only if, for $r \in R$ and $b \in B$,

$$\epsilon\big(\xi(r)g\big) = \chi\big(\xi(r)\big), \qquad \text{and} \qquad g\xi\big(\chi(b)\big) = b^{(2)}gb^{(1)}{}_-\zeta\big(\chi(b^{(1)}{}_+)\big).$$

The anti-Yetter-Drinfel'd module R is stable if in addition $\chi\big(g\xi(r)\big) = r$, for all $r \in R$. The pair (χ, g) with these properties generalizes the notion of a *modular pair in involution* for a Hopf algebra in [CM01] or a weak Hopf algebra in [V].

Proposition 2.19. *Let $\mathcal{B}$ be a right $\times_R$-Hopf algebra over an algebra R. Consider a right $\mathcal{B}$-module coring C with $\mathcal{B}$-action $\triangleleft$:*

(1) *For an anti-Yetter-Drinfel'd module X of $\mathcal{B}$, the para-cyclic object in Theorem 2.11 projects to a para-cyclic structure on $C^{\otimes_R n+1} \otimes_\mathcal{B} X$.*

(2) *For a stable anti-Yetter-Drinfel'd module X of $\mathcal{B}$, the para-cyclic object $C^{\otimes_R n+1} \otimes_\mathcal{B} X$ in part (1) is cyclic, in which case it will be denoted by $Z_*(C, M)$.*

Proof. We need to show that the composite map

$$C^{\widehat{\otimes}_R n+1} \widehat{\otimes}_R X \xrightarrow{\ w_n\ } C^{\widehat{\otimes}_R n+1} \widehat{\otimes}_R X \longrightarrow C^{\otimes_R n+1} \otimes_\mathcal{B} X$$

is $\mathcal{B}$-balanced, i.e. that

$$\big(c_n \triangleleft b^{(n+1)} x^{[1]} \underset{R}{\otimes} c_0 \triangleleft b^{(1)} \underset{R}{\otimes} \cdots \underset{R}{\otimes} c_{n-1} \triangleleft b^{(n)}\big) \underset{\mathcal{B}}{\otimes} x^{[0]}$$
$$= \big(c_n \triangleleft (b \triangleright x)^{[1]} \underset{R}{\otimes} c_0 \underset{R}{\otimes} \cdots \underset{R}{\otimes} c_{n-1}\big) \underset{\mathcal{B}}{\otimes} (b \triangleright x)^{[0]},$$

for $b \in \mathcal{B}$, $x \in X$ and $c_0 \otimes_R \cdots \otimes_R c_n \in C^{\otimes_R n+1}$. By counitality of the coproduct in $\mathcal{B}$, the left hand side is equal to

$$c_n \triangleleft b^{(n+2)} x^{[1]} \otimes_R c_0 \triangleleft b^{(2)} \zeta\big(\epsilon(b^{(1)})\big) \otimes_R c_1 \triangleleft b^{(3)} \otimes_R \cdots \otimes_R c_{n-1} \triangleleft b^{(n+1)} \otimes_\mathcal{B} x^{[0]}$$
$$= c_n \triangleleft b^{(n+2)} x^{[1]} \xi\big(\epsilon(b^{(1)})\big) \otimes_R c_0 \triangleleft b^{(2)} \otimes_R c_1 \triangleleft b^{(3)}$$
$$\quad \otimes_R \cdots \otimes_R c_{n-1} \triangleleft b^{(n+1)} \otimes_\mathcal{B} x^{[0]}$$
$$= c_n \triangleleft b^{(n+2)} x^{[1]} b^{(1)}{}_- b^{(1)}{}_+ \otimes_R c_0 \triangleleft b^{(2)} \otimes_R c_1 \triangleleft b^{(3)} \otimes_R \cdots$$
$$\quad \otimes_R c_{n-1} \triangleleft b^{(n+1)} \otimes_\mathcal{B} x^{[0]}$$
$$= c_n \triangleleft b_+{}^{(n+2)} x^{[1]} b_- b_+{}^{(1)} \otimes_R c_0 \triangleleft b_+{}^{(2)} \otimes_R \cdots \otimes_R c_{n-1} \triangleleft b_+{}^{(n+1)} \otimes_\mathcal{B} x^{[0]}$$
$$= \big(c_n \triangleleft b_+{}^{(2)} x^{[1]} b_- \otimes_R c_0 \otimes_R \cdots \otimes_R c_{n-1}\big) \triangleleft b_+{}^{(1)} \otimes_\mathcal{B} x^{[0]}$$
$$= \big(c_n \triangleleft b_+{}^{(2)} x^{[1]} b_- \otimes_R c_0 \otimes_R \cdots \otimes_R c_{n-1}\big) \otimes_\mathcal{B} b_+{}^{(1)} \triangleright x^{[0]}$$
$$= \big(c_n \triangleleft (b \triangleright x)^{[1]} \otimes_R c_0 \otimes_R \cdots \otimes_R c_{n-1}\big) \otimes_\mathcal{B} (b \triangleright x)^{[0]}.$$

The second equality follows by part (viii) in Lemma 2.14 and the third one follows by part (v). The last equality is a consequence of Lemma 2.14 (v) and (2.21). This proves that the para-cyclic map w_n in Theorem 2.11 factors to a map $\widehat{w}_n : C^{\otimes_R n+1} \otimes_{\mathcal{B}} X \to C^{\otimes_R n+1} \otimes_{\mathcal{B}} X$, hence it defines a para-cyclic structure on the simplex in Lemma 2.12. This completes the proof of part (1). Furthermore, $(\widehat{w}_n)^{n+1}$ takes an element $\big(c_0 \otimes_R \cdots \otimes_R c_n\big) \otimes_{\mathcal{B}} x \in C^{\otimes_R n+1} \otimes_{\mathcal{B}} X$ to

$$\big(c_0 \triangleleft x^{[1](1)} \underset{R}{\otimes} \cdots \underset{R}{\otimes} c_n \triangleleft x^{[1](n+1)}\big) \underset{\mathcal{B}}{\otimes} x^{[0]} = \big(c_0 \underset{R}{\otimes} \cdots \underset{R}{\otimes} c_n\big) \triangleleft x^{[1]} \underset{\mathcal{B}}{\otimes} x^{[0]}$$

$$= \big(c_0 \underset{R}{\otimes} \cdots \underset{R}{\otimes} c_n\big) \underset{\mathcal{B}}{\otimes} x^{[1]} \triangleright x^{[0]}.$$

Hence if X is a stable anti-Yetter-Drinfel'd module, i.e. condition (2.22) holds, then $(\widehat{w}_n)^{n+1}$ is the identity map. Thus we have claim (2) proven. □

Remark 2.20. Let $\mathcal{B} = (B, \xi, \zeta, \Delta, \epsilon)$ be a right bialgebroid over an algebra R. Consider $\mathcal{B}$ as an R^{op}-bimodule via right multiplications by ξ and ζ. This bimodule has an R^{op}-coring structure with counit ϵ and coproduct $\Delta_{cop} : b \mapsto b^{(2)} \otimes_{R^{op}} b^{(1)}$, co-opposite to Δ. Together with the opposite algebra B^{op}, they constitute a left R^{op}-bialgebroid $\mathcal{B}^{op}_{cop} = (B^{op}, \xi, \zeta, \Delta_{cop}, \epsilon)$. A right $\mathcal{B}$-module algebra T determines a left $\mathcal{B}^{op}_{cop}$-module algebra T^{op}, canonically. Furthermore, a right $\mathcal{B}$-comodule X can be looked at as a left $\mathcal{B}^{op}_{cop}$-comodule. Application of Theorem 2.4 to the left R^{op}-bialgebroid $\mathcal{B}^{op}_{cop}$, the left $\mathcal{B}^{op}_{cop}$-module algebra T^{op} and the left $\mathcal{B}^{op}_{cop}$-comodule X, yields a para-cocyclic cosimplex that has in degree n,

$$\big(T^{op}\big)^{\widehat{\otimes}_{R^{op}} n+1} \widehat{\otimes}_{R^{op}} X \cong T^{\widehat{\otimes}_R n+1} \widehat{\otimes}_R X,$$

where the isomorphism is given by reversing the order of the factors, i.e.

$$t_0 \widehat{\otimes}_{R^{op}} t_1 \widehat{\otimes}_{R^{op}} \cdots \widehat{\otimes}_{R^{op}} t_n \widehat{\otimes}_{R^{op}} x \quad \mapsto \quad t_n \widehat{\otimes}_R t_{n-1} \widehat{\otimes}_R \cdots \widehat{\otimes}_R t_0 \widehat{\otimes}_R x.$$

The resulting coface and codegeneracy maps on $T \widehat{\otimes}_R \cdots \widehat{\otimes}_R T \widehat{\otimes}_R X$ are the maps in (2.2) and the para-cocyclic map comes out as

$$t_0 \widehat{\otimes}_R \cdots \widehat{\otimes}_R t_n \widehat{\otimes}_R x \quad \mapsto \quad t_n \triangleleft x^{[1]} \widehat{\otimes}_R t_0 \widehat{\otimes}_R \cdots \widehat{\otimes}_R t_{n-1} \widehat{\otimes}_R x^{[0]}. \quad (2.24)$$

By the right $\mathcal{B}$-module map property of the maps $\varphi : R \to T$, $r \mapsto 1_T \triangleleft \xi(r)$, and the multiplication map $m_T : T \otimes_R T \to T$, the coface and codegeneracy maps (2.2) project to $T^{\otimes_R n+1} \otimes_{\mathcal{B}} X$. Note moreover that the para-cocyclic map (2.24) and the para-cyclic map (2.18) are of the same form. Hence it follows by the computation in the proof of Proposition 2.19 that also the para-cocyclic map (2.24) projects to $T^{\otimes_R n+1} \otimes_{\mathcal{B}} X$, whenever X is an anti-Yetter-Drinfel'd module of $\mathcal{B}$. That is, in this case $T^{\otimes_R *+1} \otimes_{\mathcal{B}} X$ is a para-cocyclic object, which is cocyclic if the anti-Yetter-Drinfel'd module X is stable.

2.D. Galois extensions of $\times_R$-Hopf algebras.

For a right $\times_R$-Hopf algebra $\mathcal{B}$ over an algebra R, consider a right comodule algebra T with coaction $t \mapsto t^{[0]} \otimes_R t^{[1]}$ (where implicit summation is understood). The subalgebra S of *coinvariants* in T consists of those elements $s \in T$ for which $s^{[0]} \otimes_R s^{[1]} = s \otimes_R 1_{\mathcal{B}}$. To the inclusion map $S \hookrightarrow T$ one can associate a cyclic simplex

$$Z_*(T/S) = T^{\widehat{\otimes}_S *+1} \quad (2.25)$$

as in Corollary 1.29. On the other hand, it follows by Proposition 2.19 that, regarding $\mathcal{B}$ as a right $\mathcal{B}$-module coring, for any stable anti-Yetter-Drinfel'd module X of $\mathcal{B}$ there is a cyclic simplex

$$Z_*(B, X) = \mathcal{B}^{\otimes_R *+1} \otimes_\mathcal{B} X, \tag{2.26}$$

where $\mathcal{B}^{\otimes_R n+1}$ is understood to be a right $\mathcal{B}$-module via the diagonal action. In this section, under the additional assumption that T is a $\mathcal{B}$-*Galois extension* of S, we construct a stable anti-Yetter-Drinfel'd module $X := T/\{ st - ts \mid s \in S, \ t \in T \} \cong S \otimes_{S^e} T$ of $\mathcal{B}$, such that the cyclic simplices (2.25) and (2.26) are isomorphic. This extends [JŞ, Theorem 3.7].

In a right comodule algebra T of a right bialgebroid $\mathcal{B}$ over an algebra R, we denote the coaction by $\varrho : t \mapsto t^{[0]} \otimes_R t^{[1]}$, where implicit summation is understood. For the iterated power of the coaction we write $(\varrho \otimes_R \mathcal{B}^{\otimes_R n-1}) \circ \cdots \circ (\varrho \otimes_R \mathcal{B}) \circ \varrho(t) =: t^{[0]} \otimes_R \cdots \otimes_R t^{[n-1]} \otimes_R t^{[n]}$.

Definition 2.21. *Let $\mathcal{B}$ be a right bialgebroid over an algebra R. A right $\mathcal{B}$-comodule algebra T is said to be a $\mathcal{B}$-Galois extension of its coinvariant subalgebra S if the canonical map*

$$\mathrm{can} : T \otimes_S T \to T \otimes_R \mathcal{B}, \qquad t' \otimes_S t \mapsto t' t^{[0]} \otimes_R t^{[1]} \tag{2.27}$$

is bijective.

For example, if $\mathcal{B} = (B, \xi, \zeta, \Delta, \epsilon)$ is a right $\times_R$-Hopf algebra, then the right regular $\mathcal{B}$-comodule algebra is a $\mathcal{B}$-Galois extension of the coinvariant subalgebra $\zeta(R) \cong R^{op}$.

Let $\mathcal{B}$ be a right R-bialgebroid and T a right $\mathcal{B}$-comodule algebra. It follows by the right R-module map property of a right $\mathcal{B}$-coaction and (2.13) that, for a coinvariant $s \in S$ and $r \in R$,

$$((1_T \cdot r)s)^{[0]} \otimes_R ((1_T \cdot r)s)^{[1]} = s \otimes_R \xi(r) = (s(1_T \cdot r))^{[0]} \otimes_R (s(1_T \cdot r))^{[1]}.$$

Hence, applying the counit of $\mathcal{B}$ to the second factor on both sides, we conclude that the elements $s \in S$ commute with $1_T \cdot r$, for all $r \in R$. Hence $T \otimes_R \mathcal{B}$ is a right S-module, with action

$$(t \otimes_R b) \cdot s := ts \otimes_R b.$$

Consider a right $\times_R$-Hopf algebra $\mathcal{B}$ over an algebra R, and a $\mathcal{B}$-Galois extension $S \subseteq T$. As in Lemma 2.14, in terms of the maps (2.19) and (2.27), introduce the index notations

$$\mathrm{can}^{-1}(1_T \otimes_R b) =: b^{\{-\}} \otimes_S b^{\{+\}} \qquad \text{and} \qquad \vartheta^{-1}(1_\mathcal{B} \otimes_R b) =: b_- \otimes_{R^{op}} b_+, \tag{2.28}$$

for $b \in \mathcal{B}$, where in both cases implicit summation is understood. The following lemma is a right bialgebroid version of [H, Lemma 4.1.21]. It extends Lemma 2.14 (vi).

Lemma 2.22. *Consider a right $\times_R$-Hopf algebra $\mathcal{B} = (B, \xi, \zeta, \Delta, \epsilon)$ over an algebra R, and a $\mathcal{B}$-Galois extension $S \subseteq T$. Using the notations in (2.28), for any $b \in \mathcal{B}$ the following pentagonal equation holds in $(T \otimes_R \mathcal{B}) \otimes_S T$:*

$$b^{\{-\}[0]} \otimes_R b^{\{-\}[1]} \otimes_S b^{\{+\}} = b_+{}^{\{-\}} \otimes_R b_- \otimes_S b_+{}^{\{+\}}.$$

Proof. The second condition in (2.12) implies that there is a well defined bijection

$$\mathrm{can}_{13} : (T \otimes_R \mathcal{B}) \otimes_S T \to T \otimes_R (\mathcal{B} \otimes_{R^{op}} \mathcal{B}),$$
$$(t' \otimes_R b) \otimes_S t \mapsto t't^{[0]} \otimes_R (b \otimes_{R^{op}} t^{[1]}),$$

where in the R^{op}-module tensor product $\mathcal{B} \otimes_{R^{op}} \mathcal{B}$ the R^{op}-actions $R^{op} \otimes_{\mathbb{K}} \mathcal{B} \otimes_{\mathbb{K}} R^{op} \to \mathcal{B}, r_1 \otimes_{\mathbb{K}} b \otimes_{\mathbb{K}} r_2 \mapsto \zeta(r_2) b \zeta(r_1)$ are used, and $\mathcal{B} \otimes_{R^{op}} \mathcal{B}$ is meant to be a left R-module via the action $r \cdot (b \otimes_{R^{op}} b') = b \otimes_{R^{op}} b' \zeta(r)$. A straightforward computation using (2.13) shows that the S-bimodule map and right R-module map (2.27), and the left R-module map (2.19) satisfy

$$(\mathrm{can} \otimes_R \mathcal{B}) \circ (T \otimes_S \mathrm{can}) = (T \otimes_R \vartheta) \circ \mathrm{can}_{13} \circ (\mathrm{can} \otimes_S T).$$

Hence, by bijectivity of all involved maps,

$$(\mathrm{can} \otimes_S T) \circ (T \otimes_S \mathrm{can}^{-1}) = \mathrm{can}_{13}^{-1} \circ (T \otimes_R \vartheta^{-1}) \circ (\mathrm{can} \otimes_R \mathcal{B}).$$

Application of this identity to $1_T \otimes_S 1_T \otimes_R b$ yields the claim. □

For an algebra extension $S \subseteq T$, T has a canonical S-bimodule structure. Hence application of the functor $\mathbf{\Pi} : S\text{-Mod-}S \to \text{Mod-}\mathbb{K}$ in Definition 1.13 to T yields a $\mathbb{K}$-module $\mathbf{\Pi} T \cong S \widehat{\otimes}_S T$.

Proposition 2.23. *Consider a right $\times_R$-Hopf algebra $\mathcal{B}$ over an algebra R, and a $\mathcal{B}$-Galois extension $S \subseteq T$. Then the quotient*

$$T_S := S \widehat{\otimes}_S T \tag{2.29}$$

is a stable anti-Yetter-Drinfel'd module.

Proof. Since the S-, and R-actions on T commute (cf. second paragraph following Definition 2.21), there is a unique R-bimodule structure on T_S such that the epimorphism $p_T : T \to T_S$ is an R-bimodule map. Furthermore, by the S-bimodule map property of the coaction $\varrho : t \mapsto t^{[0]} \otimes_R t^{[1]}$ in T, the map $(p_T \otimes_R \mathcal{B}) \circ \varrho : T \to T_S \otimes_R \mathcal{B}$ coequalizes the left and right S-actions on T. Hence there exists a unique right $\mathcal{B}$-comodule structure on T_S such that $p_T : T \to T_S$ is a right $\mathcal{B}$-comodule map.

The algebra map $S \hookrightarrow T$ equips T with an S-bimodule structure. The center $(T \otimes_S T)^S$ of the S-bimodule $T \otimes_S T$ is an algebra, with multiplication $(\sum_i u_i \otimes_S u'_i)(\sum_j v_j \otimes_S v'_j) = \sum_{i,j} v_j u_i \otimes_S u'_i v'_j$. Recall from [JŞ, Sect. 2.2] that for any any T-bimodule M, the quotient $S \widehat{\otimes}_S M \cong M / \{ s \cdot m - m \cdot s \mid s \in S, m \in M \}$ is a left $(T \otimes_S T)^S$-module via the action

$$(T \otimes_S T)^S \otimes_{\mathbb{K}} M_S \to M_S, \qquad (\sum_i u_i \otimes_S u'_i) \otimes_{\mathbb{K}} p_M(m) \mapsto \sum_i p_M(u'_i m u_i),$$

where $p_M : M \to S \widehat{\otimes}_S M$ denotes the canonical epimorphism. In particular, T_S is a left $(T \otimes_S T)^S$-module.

On the other hand, for a Galois extension $S \subseteq T$ by a right R-bialgebroid $\mathcal{B}$, using the notation in (2.28), the map

$$\mathcal{B} \to (T \otimes_S T)^S, \qquad b \mapsto b^{\{-\}} \otimes_S b^{\{+\}} \tag{2.30}$$

is an algebra homomorphism. Indeed, by the S-bimodule map property of the coaction on T it follows that

$$\mathrm{can}(sb^{\{-\}} \otimes_S b^{\{+\}}) = s \otimes_R b = \mathrm{can}(b^{\{-\}} \otimes_S b^{\{+\}}s).$$

Hence, by bijectivity of can, $b^{\{-\}} \otimes_S b^{\{+\}} \in (T \otimes_S T)^S$, for all $b \in \mathcal{B}$. The map (2.30) is unital and multiplicative since by (2.13) for all $b, b' \in \mathcal{B}$,

$$\mathrm{can}(1_T \otimes_S 1_T) = 1_T \otimes_S 1_{\mathcal{B}} \quad \text{and} \quad \mathrm{can}(b'^{\{-\}}b^{\{-\}} \otimes_S b^{\{+\}}b'^{\{+\}}) = 1_T \otimes_R bb',$$

and can is bijective. This proves that T_S is a left $\mathcal{B}$-module, with a so-called Miyashita-Ulbrich type action

$$b \triangleright p_T(t) = p_T(b^{\{+\}}tb^{\{-\}}). \tag{2.31}$$

It remains to check the compatibility conditions in Definition 2.15 between the $\mathcal{B}$-module and $\mathcal{B}$-comodule structures on T_S. It follows by the R-bimodule map property of (2.27) that

$$\begin{aligned}
\zeta(r)^{\{-\}} \otimes_S \zeta(r)^{\{+\}} &= r \cdot 1_T \otimes_S 1_T \\
\text{and} \quad \xi(r)^{\{-\}} \otimes_S \xi(r)^{\{+\}} &= 1_T \otimes_S 1_T \cdot r.
\end{aligned} \tag{2.32}$$

Hence

$$\begin{aligned}
\zeta(r) \triangleright p_T(t) &= p_T\big(t(r \cdot 1_T)\big) = p_T(t \cdot r) = p_T(t) \cdot r \text{ and} \\
\xi(r) \triangleright p_T(t) &= p_T\big((1_T \cdot r)t\big) = p_T(r \cdot t) = r \cdot p_T(t).
\end{aligned}$$

Furthermore, for $b \in \mathcal{B}$ and $t \in T$,

$$\begin{aligned}
(b \triangleright p_T(t))^{[0]} \otimes_R (b \triangleright p_T(t))^{[1]} &= p_T\left(b^{\{+\}[0]}t^{[0]}b^{\{-\}[0]}\right) \otimes_R b^{\{+\}[1]}t^{[1]}b^{\{-\}[1]} \\
&= p_T\left(b^{(1)\{+\}}t^{[0]}b^{(1)\{-\}[0]}\right) \otimes_R b^{(2)}t^{[1]}b^{(1)\{-\}[1]} \\
&= p_T\left(b^{(1)}{}_+^{\{+\}}t^{[0]}b^{(1)}{}_+^{\{-\}}\right) \otimes_R b^{(2)}t^{[1]}b^{(1)}{}_- \\
&= b^{(1)}{}_+ \triangleright p_T(t)^{[0]} \otimes_R b^{(2)}p_T(t)^{[1]}b^{(1)}{}_- .
\end{aligned}$$

The first equality follows by (2.31), the second condition in (2.13) and the comodule map property of p_T. The second equality is a consequence of the right $\mathcal{B}$-comodule map property of (2.27), hence of the map (2.30). The third equality results from the application of Lemma 2.22. The last equality follows by (2.31) and the comodule map property of p_T again. Thus we proved that T_S is an anti-Yetter-Drinfel'd module. Finally, by the comodule map property of p_T and the identity

$$t^{[0]}t^{[1]\{-\}} \otimes_S t^{[1]\{+\}} = \mathrm{can}^{-1}\big(\mathrm{can}(1_T \otimes_S t)\big) = 1_T \otimes_S t, \qquad \text{for } t \in T, \quad (2.33)$$

it follows that

$$p_T(t)^{[1]} \triangleright p_T(t)^{[0]} = t^{[1]} \triangleright p_T(t)^{[0]} = p_T\big(t^{[1]\{+\}}t^{[0]}t^{[1]\{-\}}\big) = p_T(t).$$

That is, the anti-Yetter-Drinfel'd module T_S is stable. $\square$

Lemma 2.24. *For an algebra R over a commutative ring $\mathbb{K}$, consider a right R-bialgebroid $\mathcal{B}$ and a $\mathcal{B}$-Galois extension $S \subseteq T$. Using the notation in (2.28), for any non-negative integer n there exist S-bimodule isomorphisms $T^{\otimes_S n+1} \cong T \otimes_R \mathcal{B}^{\otimes_R n}$,*

$$\alpha_n(t_0 \underset{S}{\otimes} \cdots \underset{S}{\otimes} t_n) = t_0 t_1^{[0]} t_2^{[0]} \cdots t_n^{[0]} \underset{R}{\otimes} t_1^{[1]} t_2^{[1]} \cdots t_n^{[1]} \underset{R}{\otimes} t_2^{[2]} t_3^{[2]} \cdots t_n^{[2]} \underset{R}{\otimes}$$
$$\cdots \underset{R}{\otimes} t_{n-1}^{[n-1]} t_n^{[n-1]} \underset{R}{\otimes} t_n^{[n]},$$
$$\alpha_n^{-1}(t \underset{R}{\otimes} b_1 \underset{R}{\otimes} \cdots \underset{R}{\otimes} b_n) = t \, b_1^{\{-\}} \underset{S}{\otimes} b_1^{\{+\}} b_2^{\{-\}} \underset{S}{\otimes} b_2^{\{+\}} b_3^{\{-\}} \underset{S}{\otimes}$$
$$\cdots \underset{S}{\otimes} b_{n-1}^{\{+\}} b_n^{\{-\}} \underset{S}{\otimes} b_n^{\{+\}}.$$

Projections of the above isomorphisms yield $\mathbb{K}$-module isomorphisms $\widehat{\alpha}_n : T^{\widehat{\otimes}_S n+1} \to T_S \otimes_R \mathcal{B}^{\otimes_R n}$, where T_S is the R-bimodule (2.29).

Proof. It follows by the S-bimodule map property of the right $\mathcal{B}$-coaction on T that α_n is a well defined S-bimodule map. The to-be-inverse α_n^{-1} is well defined by (2.32) and Lemma 2.14 (iii). We prove by induction that the maps α_n and α_n^{-1} are mutual inverses. For $n = 0$, both α_0 and α_0^{-1} are equal to the identity map on T, hence they are mutual inverses. It follows by the second condition in (2.13) that, for all values of n,

$$\alpha_{n+1} = (\alpha_n \otimes_R \mathcal{B}) \circ (T^{\otimes_S n} \otimes_S \mathrm{can}) \quad \text{and} \quad \alpha_{n+1}^{-1} = (T^{\otimes_S n} \otimes_S \mathrm{can}^{-1}) \circ (\alpha_n^{-1} \otimes_R \mathcal{B}).$$

Hence if α_n^{-1} is the inverse of α_n then α_{n+1} is also an S-bimodule isomorphism with inverse α_{n+1}^{-1}. Applying the functor in Definition 1.13, from the category of S-bimodules to the category of $\mathbb{K}$-modules, it takes α_n to the required $\mathbb{K}$-module isomorphism $\widehat{\alpha}_n : T^{\widehat{\otimes}_S n+1} \to T_S \otimes_R \mathcal{B}^{\otimes_R n}$. $\square$

Lemma 2.25. *Let $\mathcal{B}$ be a right $\times_R$-Hopf algebra over an algebra R. For the inverse of the canonical map (2.19) use the index notation in (2.28). Then, for any non-negative integer n, there exist right $\mathcal{B}$-module isomorphisms $\mathcal{B}^{\otimes_R n} \otimes_{R^{op}} \mathcal{B} \cong \mathcal{B}^{\otimes_R n+1}$,*

$$\beta_n\big(b_1 \otimes_R \cdots \otimes_R b_n \otimes_{R^{op}} b'\big) = b_1 b'^{(1)} \otimes_R \cdots \otimes_R b_n b'^{(n)} \otimes_R b'^{(n+1)},$$
$$\beta_n^{-1}\big(b_1 \otimes_R \cdots \otimes_R b_n \otimes_R b'\big) = b_1 b'_-{}^{(1)} \otimes_R \cdots \otimes_R b_n b'_-{}^{(n)} \otimes_{R^{op}} b'_+.$$

Proof. For any right $\mathcal{B}$-module N, the $\mathcal{B}$-bimodule isomorphism (2.19) induces a right $\mathcal{B}$-module isomorphism $N \otimes_{\mathcal{B}} \vartheta : N \otimes_{\mathcal{B}} \mathcal{B} \otimes_{R^{op}} \mathcal{B} \cong N \otimes_{R^{op}} \mathcal{B} \to N \otimes_{\mathcal{B}} \mathcal{B} \otimes_R \mathcal{B} \cong N \otimes_R \mathcal{B}$. Consider $\mathcal{B}^{\otimes_R n}$ as a right $\mathcal{B}$-module via the diagonal action. Then $\beta_n = \mathcal{B}^{\otimes_R n} \otimes_{\mathcal{B}} \vartheta$ is a right $\mathcal{B}$-module isomorphism as stated. $\square$

Theorem 2.26. *Let $\mathcal{B}$ be a right $\times_R$-Hopf algebra over an algebra R and let $S \subseteq T$ be a $\mathcal{B}$-Galois extension. Consider the right regular $\mathcal{B}$-module coring and the stable anti-Yetter-Drinfel'd module T_S in Prop. 2.23. Then the associated cyclic simplex $Z_*(B, T_S)$ in Proposition 2.19 is isomorphic to the cyclic simplex $Z_*(T/S)$ in Corollary 1.29.*

Proof. In terms of the maps in Lemmas 2.24 and 2.25, for any non-negative integer n, one constructs a $\mathbb{K}$-module isomorphism ω_n as the composition of the following morphisms

$$T^{\widehat{\otimes}_S n+1} \xrightarrow{\widehat{\alpha}_n} T_S \underset{R}{\otimes} \mathcal{B}^{\otimes_R n} \longrightarrow \mathcal{B}^{\otimes_R n} \underset{R^{op}}{\otimes} T_S \xrightarrow{\cong} \mathcal{B}^{\otimes_R n} \underset{R^{op}}{\otimes} \mathcal{B} \underset{\mathcal{B}}{\otimes} T_S \xrightarrow{\beta_n \otimes_{\mathcal{B}} T_S} \mathcal{B}^{\otimes_R n+1} \underset{\mathcal{B}}{\otimes} T_S \, .$$

Explicitly,

$$\omega_n\big(t_0 \mathbin{\widehat{\underset{S}{\otimes}}} t_1 \mathbin{\widehat{\underset{S}{\otimes}}} \cdots \mathbin{\widehat{\underset{S}{\otimes}}} t_n\big) = \big(t_1^{[1]} t_2^{[1]} \cdots t_n^{[1]} \underset{R}{\otimes} t_2^{[2]} \cdots t_n^{[2]} \underset{R}{\otimes} \cdots \underset{R}{\otimes} t_n^{[n]} \underset{R}{\otimes} 1_{\mathcal{B}}\big)$$
$$\underset{\mathcal{B}}{\otimes} p_T(t_0 t_1^{[0]} t_2^{[0]} \cdots t_n^{[0]}), \tag{2.34}$$

where $p_T : T \to T_S \cong S \widehat{\otimes}_S T$ denotes the canonical epimorphism. We show that (2.34) is a homomorphism of cyclic simplices.

Denote the counit in $\mathcal{B}$ by ϵ. By multiplicativity of the coproduct in $\mathcal{B}$, for any integer $0 \le k < n$,

$$\big((\mathcal{B}^{\otimes_R k} \underset{R}{\otimes} \epsilon \underset{R}{\otimes} \mathcal{B}^{\otimes_R n-k}) \underset{\mathcal{B}}{\otimes} T_S\big) \circ \omega_n(t_o \mathbin{\widehat{\underset{S}{\otimes}}} \cdots \mathbin{\widehat{\underset{S}{\otimes}}} t_n) =$$
$$= (t_1^{[1]} \cdots t_n^{[1]} \underset{R}{\otimes} \cdots \underset{R}{\otimes} t_k^{[k]} \cdots t_n^{[k]} \underset{R}{\otimes} t_{k+2}^{[k+1]} \cdots t_n^{[k+1]} \underset{R}{\otimes}$$
$$\cdots \underset{R}{\otimes} t_n^{[n-1]} \underset{R}{\otimes} 1_{\mathcal{B}}) \underset{\mathcal{B}}{\otimes} p_T(t_0 t_1^{[0]} \cdots t_n^{[0]})$$
$$= \omega_{n-1}(t_o \mathbin{\widehat{\underset{S}{\otimes}}} \cdots \mathbin{\widehat{\underset{S}{\otimes}}} t_{k-1} \mathbin{\widehat{\underset{S}{\otimes}}} t_k t_{k+1} \mathbin{\widehat{\underset{S}{\otimes}}} t_{k+2} \mathbin{\widehat{\underset{S}{\otimes}}} \cdots \mathbin{\widehat{\underset{S}{\otimes}}} t_n).$$

Furthermore, using the form of the (diagonal) right $\mathcal{B}$-action on $\mathcal{B}^{\otimes_R n}$ (in the second equality) and the form of the left $\mathcal{B}$-action (2.31) on T_S (in the third equality), one computes

$$\big((\mathcal{B}^{\otimes_R n} \underset{R}{\otimes} \epsilon) \underset{\mathcal{B}}{\otimes} T_S\big) \circ \omega_n(t_o \mathbin{\widehat{\underset{S}{\otimes}}} \cdots \mathbin{\widehat{\underset{S}{\otimes}}} t_n)$$
$$= \big(t_1^{[1]} \cdots t_n^{[1]} \underset{R}{\otimes} t_2^{[2]} \cdots t_n^{[2]} \underset{R}{\otimes} \cdots \underset{R}{\otimes} t_n^{[n]}\big) \underset{\mathcal{B}}{\otimes} p_T(t_0 t_1^{[0]} \cdots t_n^{[0]})$$
$$= \big(t_1^{[1]} \cdots t_{n-1}^{[1]} \underset{R}{\otimes} t_2^{[2]} \cdots t_{n-1}^{[2]} \underset{R}{\otimes} \cdots \underset{R}{\otimes} t_{n-1}^{[n-1]} \underset{R}{\otimes} 1_{\mathcal{B}}\big) \triangleleft t_n^{[1]} \underset{\mathcal{B}}{\otimes} p_T(t_0 t_1^{[0]} \cdots t_n^{[0]})$$
$$= \big(t_1^{[1]} \cdots t_{n-1}^{[1]} \underset{R}{\otimes} t_2^{[2]} \cdots t_{n-1}^{[2]} \underset{R}{\otimes} \cdots \underset{R}{\otimes} t_{n-1}^{[n-1]} \underset{R}{\otimes} 1_{\mathcal{B}}\big) \underset{\mathcal{B}}{\otimes}$$
$$p_T(t_n^{[1]\{+\}} t_0 t_1^{[0]} \cdots t_n^{[0]} t_n^{[1]\{-\}})$$
$$= \big(t_1^{[1]} \cdots t_{n-1}^{[1]} \underset{R}{\otimes} t_2^{[2]} \cdots t_{n-1}^{[2]} \underset{R}{\otimes} \cdots \underset{R}{\otimes} t_{n-1}^{[n-1]} \underset{R}{\otimes} 1_{\mathcal{B}}\big) \underset{\mathcal{B}}{\otimes} p_T(t_n t_0 t_1^{[0]} \cdots t_{n-1}^{[0]})$$
$$= \omega_{n-1}(t_n t_o \mathbin{\widehat{\underset{S}{\otimes}}} t_1 \mathbin{\widehat{\underset{S}{\otimes}}} \cdots \mathbin{\widehat{\underset{S}{\otimes}}} t_{n-1}), \tag{2.35}$$

where the penultimate equality follows by (2.33). This proves that the map ω_n is compatible with the face maps.

Denote the coproduct in $\mathcal{B}$ by Δ. It follows by its multiplicativity that, for any integer $0 \le k < n$,

$$\big((\mathcal{B}^{\otimes_R k} \underset{R}{\otimes} \Delta \underset{R}{\otimes} \mathcal{B}^{\otimes_R n-k}) \underset{\mathcal{B}}{\otimes} T_S\big) \circ \omega_n(t_o \mathbin{\widehat{\underset{S}{\otimes}}} \cdots \mathbin{\widehat{\underset{S}{\otimes}}} t_n)$$
$$= (t_1^{[1]} \cdots t_n^{[1]} \underset{R}{\otimes} \cdots \underset{R}{\otimes} t_{k+1}^{[k+1]} \cdots t_n^{[k+1]} \underset{R}{\otimes} t_{k+1}^{[k+2]} \cdots t_n^{[k+2]} \underset{R}{\otimes}$$
$$\cdots \underset{R}{\otimes} t_n^{[n+1]} \underset{R}{\otimes} 1_{\mathcal{B}}) \underset{\mathcal{B}}{\otimes} p_T(t_0 t_1^{[0]} \cdots t_n^{[0]})$$
$$= \omega_{n+1}(t_o \mathbin{\widehat{\underset{S}{\otimes}}} \cdots \mathbin{\widehat{\underset{S}{\otimes}}} t_k \mathbin{\widehat{\underset{S}{\otimes}}} 1_T \mathbin{\widehat{\underset{S}{\otimes}}} t_{k+1} \mathbin{\widehat{\underset{S}{\otimes}}} \cdots \mathbin{\widehat{\underset{S}{\otimes}}} t_n).$$

Furthermore, by unitality of Δ and of the $\mathcal{B}$-coaction on T,

$$\big((\mathcal{B}^{\otimes_R n} \underset{R}{\otimes} \Delta) \underset{\mathcal{B}}{\otimes} T_S\big) \circ \omega_n(t_o \mathbin{\widehat{\underset{S}{\otimes}}} \cdots \mathbin{\widehat{\underset{S}{\otimes}}} t_n)$$
$$= \big(t_1^{[1]} t_2^{[1]} \cdots t_n^{[1]} \underset{R}{\otimes} t_2^{[2]} \cdots t_n^{[2]} \underset{R}{\otimes} \cdots \underset{R}{\otimes} t_n^{[n]} \underset{R}{\otimes} 1_{\mathcal{B}} \underset{R}{\otimes} 1_{\mathcal{B}}\big) \underset{\mathcal{B}}{\otimes} p_T(t_0 t_1^{[0]} t_2^{[0]} \cdots t_n^{[0]})$$
$$= \omega_{n+1}(t_o \mathbin{\widehat{\underset{S}{\otimes}}} \cdots \mathbin{\widehat{\underset{S}{\otimes}}} t_n \mathbin{\widehat{\underset{S}{\otimes}}} 1_T).$$

This proves that the map ω_n is compatible with the degeneracy maps.

Finally, by similar steps as in (2.35), the cyclic map $\widehat{w}_n$ in Proposition 2.19 is checked to satisfy

$$
\begin{aligned}
\widehat{w}_n &\circ \omega_n(t_o \mathbin{\widehat{\underset{S}{\otimes}}} \cdots \mathbin{\widehat{\underset{S}{\otimes}}} t_n) \\
&= \left(t_0^{[1]} t_1^{[1]} t_2^{[1]} \cdots t_n^{[1]} \underset{R}{\otimes} t_1^{[2]} \cdots t_n^{[2]} \underset{R}{\otimes} \cdots \underset{R}{\otimes} t_{n-1}^{[n]} t_n^{[n]} \underset{R}{\otimes} t_n^{[n+1]}\right) \\
&\quad \underset{B}{\otimes} p_T(t_0^{[0]} t_1^{[0]} t_2^{[0]} \cdots t_n^{[0]}) \\
&= \left(t_0^{[1]} t_1^{[1]} t_2^{[1]} \cdots t_{n-1}^{[1]} \underset{R}{\otimes} t_1^{[2]} \cdots t_{n-1}^{[2]} \underset{R}{\otimes} \cdots \underset{R}{\otimes} t_{n-1}^{[n]} \underset{R}{\otimes} 1_{\mathcal{B}}\right) \\
&\quad \underset{B}{\otimes} p_T(t_n t_0^{[0]} t_1^{[0]} t_2^{[0]} \cdots t_{n-1}^{[0]}) \\
&= \omega_n(t_n \mathbin{\widehat{\underset{S}{\otimes}}} t_0 \mathbin{\widehat{\underset{S}{\otimes}}} \cdots \mathbin{\widehat{\underset{S}{\otimes}}} t_{n-1}).
\end{aligned}
$$

Hence ω_n is compatible with the cyclic maps as well, which proves the claim. $\quad\square$

3. Cyclic Homology of Groupoids

Consider a groupoid $\mathcal{G}$ (i.e. a small category in which all morphisms are invertible) with a *finite* set $\mathcal{G}^0$ of objects and an arbitrary set $\mathcal{G}^1$ of morphisms. Via the map associating to $x \in \mathcal{G}^0$ the identity morphism $x \to x$, we consider $\mathcal{G}^0$ as a subset of $\mathcal{G}^1$. Composition in $\mathcal{G}$ is denoted by $\circ$ while the source and target maps $\mathcal{G}^1 \to \mathcal{G}^0$ are denoted by s and t, respectively. For any field $\mathbb{K}$, the $\mathbb{K}$-vector space $B := \mathbb{K}\mathcal{G}^1$, spanned by the elements of $\mathcal{G}^1$, has a right $\times_R$-Hopf algebra structure over the commutative base algebra $R := \mathbb{K}\mathcal{G}^0$. Structure maps are the following. Multiplication in the $\mathbb{K}$-algebra B is given on basis elements $g, g' \in \mathcal{G}^1$ by $g \circ g'$, if g and g' are composable, i.e. $s(g) = t(g')$, and zero otherwise. We denote by juxtaposition this product, linearly extended to all elements of B. The unit element is $1_B = \sum_{x \in \mathcal{G}^0} x$. Similarly, R is a commutative $\mathbb{K}$-algebra with minimal orthogonal idempotents $\{\, x \in \mathcal{G}^0 \,\}$. Both algebra maps ξ and $\zeta : R \to B$ are induced by the inclusion map $\mathcal{G}^0 \hookrightarrow \mathcal{G}^1$. That is, B is an R-module (or R-bimodule, with coinciding left and right actions) via multiplication on the right. The coproduct is diagonal on the basis elements $g \in \mathcal{G}^1$, i.e. $\Delta(g) := g \otimes_R g$. The counit ϵ maps $g \in \mathcal{G}^1$ to $s(g) \in \mathcal{G}^0$. The canonical map (2.19) has the explicit form on the generating set $\{ g \otimes_R g' \mid g, g' \in \mathcal{G}^1 \}$,

$$
\vartheta : B \otimes_R B \to B \otimes_R B, \qquad g \otimes_R g' \mapsto gg' \otimes_R g'.
$$

(Note that R-module structures in the domain and codomain are different, cf. Definition 2.13.) It obviously has an inverse $\vartheta^{-1}(g \otimes_R g') = gg'^{-1} \otimes_R g'$.

In this final section we apply the theory developed in the earlier sections to the groupoid bialgebroid $\mathcal{B}$ and its stable anti-Yetter-Drinfel'd modules. In this way, we obtain expressions for Hochschild and cyclic homologies of a groupoid with finitely many objects. Describing then any groupoid as a direct limit of groupoids with finitely many objects, we extend the computation of cyclic homology to arbitrary groupoids. Similar arguments don't seem to apply in the case of Hochschild homology.

3.A. Anti-Yetter-Drinfel'd modules for groupoids. The subject of this section is a complete characterization of (stable) anti-Yetter-Drinfel'd modules of a groupoid bialgebroid. As a first step, we study comodules.

Proposition 3.1. *Let $\mathcal{G}$ be a small groupoid with a finite set of objects. Let $\mathcal{B}$ be the groupoid R-bialgebroid associated to $\mathcal{G}$. Then any right $\mathcal{B}$-comodule M has a direct sum decomposition $M \cong \oplus_{g \in \mathcal{G}^1} M_g$ (as an R-module) such that the R-action $\cdot$ and the $\mathcal{B}$-coaction ϱ satisfy the conditions*

   (i) $m \cdot x = \delta_{x,s(g)} m$ *and*
  (ii) $\varrho(m) = m \otimes_R g,$

for $x \in \mathcal{G}^0$, $g \in \mathcal{G}^1$ and $m \in M_g$. Conversely, on an R-module $M \cong \oplus_{g \in \mathcal{G}^1} M_g$, which is subject to condition (i)*, there is a unique right $\mathcal{B}$-coaction satisfying* (ii)*.*

Proof. Recall that by definition R acts on B by right multiplication. For any $g \in \mathcal{G}^1$ there is an R-module map $\chi_g : B \to R$, $h \mapsto \delta_{g,h} s(g)$. Introduce the map

$$\pi_g := (M \otimes_R \chi_g) \circ \varrho : M \to M.$$

We claim that M is isomorphic to a direct sum of the R-modules $M_g := \mathrm{Im}(\pi_g)$. Since B is a free $\mathbb{K}$-module, there exist (non-unique) elements $\{m_g \mid g \in \mathcal{G}^1\}$ in M, in terms of which $\varrho(m) = \sum_{g \in \mathcal{G}^1} m_g \otimes_R g$, hence $\pi_g(m) = m_g \cdot s(g)$. By construction, for a given element $m \in M$ there are only finitely many elements $g \in \mathcal{G}^1$ such that $m_g \neq 0$, hence $\pi_g(m) \neq 0$. By coassociativity of ϱ,

$$\sum_{g,h \in \mathcal{G}^1} (m_h)_g \otimes_R g \otimes_R h = \sum_{g \in \mathcal{G}^1} m_g \otimes_R g \otimes_R g.$$

Hence applying $M \otimes_R \chi_{g'} \otimes_R \chi_{h'}$, for some $g', h' \in \mathcal{G}^1$, we conclude that $(m_{h'})_{g'} \cdot s(g')s(h') = \delta_{g',h'} m_{g'} \cdot s(g')$, i.e. $\pi_{g'} \circ \pi_{h'} = \delta_{g',h'} \pi_{g'}$. By counitality of ϱ, for all $m \in M$,

$$\sum_{g \in \mathcal{G}^1} \pi_g(m) = \sum_{g \in \mathcal{G}^1} m_g \cdot s(g) = m.$$

This proves the direct sum decomposition of M as an R-module. Condition (i) follows by the computation, for $m \in M$, $g \in \mathcal{G}^1$ and $x \in \mathcal{G}^0$,

$$\pi_g(m) \cdot x = m_g \cdot s(g)x = \delta_{s(g),x} m_g \cdot s(g) = \delta_{s(g),x} \pi_g(m).$$

Moreover, for $m \in M$,

$$\varrho(m) = \sum_{g \in \mathcal{G}^1} m_g \otimes_R g = \sum_{g \in \mathcal{G}^1} m_g \otimes_R gs(g) = \sum_{g \in \mathcal{G}^1} m_g \cdot s(g) \otimes_R g$$
$$= \sum_{g \in \mathcal{G}^1} \pi_g(m) \otimes_R g.$$

Hence, by orthogonality of the projections π_g,

$$\varrho(\pi_h(m)) = \sum_{g \in \mathcal{G}^1} \pi_g(\pi_h(m)) \otimes_R g = \pi_h(m) \otimes_R h,$$

which proves condition (ii). Conversely, assume that for an R-module $M \cong \oplus_{g \in \mathcal{G}^1} M_g$ condition (i) holds. Put $\varrho_g : M_g \to M_g \otimes_R B$, $m \mapsto m \otimes_R g$. One can check that it makes M_g to a right $\mathcal{B}$-comodule. By universality of a direct sum, this defines a unique right $\mathcal{B}$-coaction ϱ on M, such that condition (ii) holds.   □

Note that property (ii) in Proposition 3.1 characterizes uniquely the elements m of a component M_g. Indeed, if for $m \in M$, $\varrho(m) = m \otimes_R g$, then by counitality of the coaction ϱ we obtain $\pi_g(m) = m \cdot s(g) = m$.

We are ready to characterize (stable) anti-Yetter-Drinfel'd modules of groupoids.

Theorem 3.2. *Let $\mathcal{G}$ be a small groupoid with a finite set of objects. Let $\mathcal{B}$ be the groupoid bialgebroid associated to $\mathcal{G}$. A left $\mathcal{B}$-module $M \cong \oplus_{g \in \mathcal{G}^1} M_g$, with action $\triangleright$, is an anti-Yetter-Drinfel'd $\mathcal{B}$-module if and only if the following conditions hold.*

(i) *For $x \in \mathcal{G}^0$, $g \in \mathcal{G}^1$ and $m \in M_g$, $\delta_{x,s(g)} m = x \triangleright m = \delta_{x,t(g)} m$.*
(ii) *For $g, h \in \mathcal{G}^1$ and $m \in M_g$, the element $h \triangleright m$ is zero if $hgh^{-1} = 0$ in B and it belongs to $M_{hgh^{-1}}$ if $hgh^{-1} \neq 0$ in B.*

The anti-Yetter-Drinfel'd $\mathcal{B}$-module M is stable if and only if in addition $g \triangleright m = m$, for all $g \in \mathcal{G}^1$ and $m \in M_g$.

Proof. By Proposition 3.1 there is a unique $\mathcal{B}$-coaction on M corresponding to the given direct sum decomposition. For $g \in \mathcal{G}^1$, it takes $m \in M_g$ to $m \otimes_R g$. The R-bimodule structure corresponding to this coaction comes out, for $x, y \in \mathcal{G}^0$, $g \in \mathcal{G}^1$ and $m \in M_g$, as

$$x \cdot m \cdot y = m \cdot \epsilon(xgy) = \delta_{s(g),y} \, \delta_{t(g),x} \, m \cdot s(g) = \delta_{s(g),y} \, \delta_{t(g),x} \, m.$$

Hence axiom (2.20) of an anti-Yetter-Drinfel'd module translates to condition (i) in the theorem. A straightforward computation shows that for a groupoid bialgebroid, axiom (2.21) of an anti-Yetter-Drinfel'd module takes the form

$$\varrho(h \triangleright m) = h \triangleright m \otimes_R hgh^{-1}, \tag{3.1}$$

for $h, g \in \mathcal{G}^1$ and $m \in M_g$. If $hgh^{-1} = 0$ in B, then the right-hand side of (3.1) vanishes. Since ϱ is a monomorphism of R-modules (split by $M \otimes_R \epsilon$), this is equivalent to $h \triangleright m = 0$. If $hgh^{-1} \neq 0$ in B, then (3.1) is equivalent to $h \triangleright m \in M_{hgh^{-1}}$ (cf. Proposition 3.1, and discussions following it). This completes the proof. □

As a consequence of Theorem 3.2 (condition (i)), in the direct sum decomposition of an anti-Yetter-Drinfel'd module M of a groupoid bialgebroid, M_g is non-zero for only those elements $g \in \mathcal{G}^1$ for which $s(g) = t(g)$, i.e. which are *loops*. That is, introducing the notation $\mathcal{L}(\mathcal{G}) := \{ g \in \mathcal{G}^1 \mid s(g) = t(g) \}$, one can write $M \cong \oplus_{l \in \mathcal{L}(\mathcal{G})} M_l$.

Our next aim is to decompose an anti-Yetter-Drinfel'd module M of a groupoid bialgebroid $\mathcal{B}$ as a direct sum *of anti-Yetter-Drinfel'd modules*. For a loop $l \in \mathcal{L}(\mathcal{G})$, denote by $[l]$ the orbit of l in $\mathcal{L}(\mathcal{G})$ for the adjoint action, that is, the set of different non-zero elements of the form glg^{-1}, as g runs through $\mathcal{G}^1$. This gives a ($\mathcal{G}$-invariant) partition $\mathcal{T}(\mathcal{G})$ of $\mathcal{L}(\mathcal{G})$. Using Theorem 3.2 one concludes that $M_{[l]} := \oplus_{l' \in [l]} M_{l'}$ is an anti-Yetter-Drinfel'd $\mathcal{B}$-module, and

$$M \cong \bigoplus_{[l] \in \mathcal{T}(\mathcal{G})} M_{[l]}, \tag{3.2}$$

as anti-Yetter-Drinfel'd modules. Let us give an alternative description of the anti-Yetter-Drinfel'd module $M_{[l]}$. Introduce the following subalgebras of the groupoid algebra B. For $l \in \mathcal{L}(\mathcal{G})$, let B_l be the group algebra of the centralizer $\mathcal{G}_l^1$ of l in the group $\{ l' \in \mathcal{L}(\mathcal{G}) \mid s(l') = s(l) \}$. That is,

$$B_l := \mathbb{K}\mathcal{G}_l^1 \equiv \mathbb{K}\{ l' \in \mathcal{L}(\mathcal{G}) \mid l'll'^{-1} = l \}. \tag{3.3}$$

The algebra B_l possesses a unit (the element $s(l) = t(l)$), different from the unit element of B. For $x \in \mathcal{G}^0$, let $B(x)$ denote the vector space spanned by the elements $g \in \mathcal{G}^1$, such that $s(g) = x$. Consider the group bialgebra structure of B_l. Clearly, for any $l \in \mathcal{L}(\mathcal{G})$, the component M_l in the direct sum decomposition of an anti-Yetter-Drinfel'd $\mathcal{B}$-module M is an anti-Yetter-Drinfel'd module of the group bialgebra B_l (in the sense of [JŞ], cf. Theorem 3.2). Moreover, $B(s(l))$ is a B-B_l bimodule and a right B_l-module coalgebra (with coproduct induced by the map $g \mapsto g \otimes_{\mathbb{K}} g$, for $g \in \mathcal{G}^1$ such that $s(g) = s(l)$.)

Lemma 3.3. *Let $\mathcal{G}$ be a small groupoid with a finite set of objects. Let $\mathcal{B}$ be the groupoid bialgebroid over a field $\mathbb{K}$, associated to $\mathcal{G}$. Keeping the notation introduced above, $B(s(l))$ is free as a right B_l-module, for any $l \in \mathcal{L}(\mathcal{G})$.*

Proof. Since any morphism in a groupoid is invertible, the right action of the group $\mathcal{G}_l^1$ on the set $\{ g \in \mathcal{G}^1 \mid s(g) = s(l) \}$ is faithful in the sense that $gl' = g$ implies $l' = s(l)$. Hence the claim follows by the fact that $\mathbb{K}X$ is a free module for a group algebra $\mathbb{K}G$ whenever G acts faithfully on the set X. Indeed, fix a $\mathbb{K}$-basis $\{ e_x \mid x \in X \}$ of $\mathbb{K}X$, such that $e_x \cdot g = e_{x \cdot g}$. Fix a section f of the canonical epimorphism from X to the set of G-orbits X/G. By construction, $\{ e_{f(\mathcal{O})} \mid \mathcal{O} \in X/G \}$ is a generating set of the $\mathbb{K}G$-module $\mathbb{K}X$. It is also linearly independent over $\mathbb{K}G$, by the following reasoning. Assume that, for some coefficients $a_{\mathcal{O}} = \sum_{g \in G} \alpha_{\mathcal{O},g} \, g \in \mathbb{K}G$,

$$0 = \sum_{\mathcal{O} \in X/G} e_{f(\mathcal{O})} \cdot a_{\mathcal{O}} = \sum_{\mathcal{O} \in X/G} \sum_{g \in G} \alpha_{\mathcal{O},g} \, e_{f(\mathcal{O}) \cdot g}.$$

Since G acts on X faithfully, in the above sum each element $e_{f(\mathcal{O})}$ appears exactly once. Hence $\alpha_{\mathcal{O},g} = 0$, for all $\mathcal{O} \in X/G$ and $g \in G$. Thus we have the claim proven. $\square$

Proposition 3.4. *Let $\mathcal{G}$ be a small groupoid with a finite set of objects. Let $\mathcal{B}$ be the groupoid bialgebroid associated to $\mathcal{G}$. Let M be an anti-Yetter-Drinfel'd $\mathcal{B}$-module. Keeping the notation introduced above, there is an isomorphism of anti-Yetter-Drinfel'd $\mathcal{B}$-modules*

$$M_{[l]} \cong B(s(l)) \otimes_{B_l} M_l, \quad \text{for all } l \in \mathcal{L}(\mathcal{G}).$$

Proof. First we construct a left B-module isomorphism

$$\varphi_l : B(s(l)) \otimes_{B_l} M_l \to M_{[l]}, \qquad g \otimes_{B_l} m \mapsto g \triangleright m. \tag{3.4}$$

For any $g \in \mathcal{G}^1$ such that $s(g) = s(l)$, consider the map

$$\psi_g : M_{glg^{-1}} \to B(s(l)) \otimes_{B_l} M_l, \qquad g \mapsto g \otimes_{B_l} g^{-1} \triangleright m.$$

In order to see that the map ψ_g does not depend on g, only on glg^{-1}, choose another element $h \in \mathcal{G}^1$, such that $glg^{-1} = hlh^{-1}$. Note that this implies in particular $t(g) = t(h)$. Then, for $m \in M_{glg^{-1}}$,

$$\begin{aligned}
\psi_h(m) &= h \otimes_{B_l} h^{-1} \triangleright m = h \otimes_{B_l} h^{-1} t(h) \triangleright m = h \otimes_{B_l} h^{-1} t(g) \triangleright m \\
&= h \otimes_{B_l} h^{-1} g g^{-1} \triangleright m \\
&= h h^{-1} g \otimes_{B_l} g^{-1} \triangleright m = t(h) g \otimes_{B_l} g^{-1} \triangleright m = t(g) g \otimes_{B_l} g^{-1} \triangleright m \\
&= g \otimes_{B_l} g^{-1} \triangleright m = \psi_g(m),
\end{aligned}$$

where in the fifth equality we used that $h^{-1}g$ is an element of B_l. Thus we conclude by universality of a direct sum on the existence of a map $\psi : M_{[l]} \cong \oplus_{l' \in [l]} M_{l'} \to B(s(l)) \otimes_{B_l} M_l$, mapping $m \in M_{glg^{-1}}$ to $\psi_g(m) = g \otimes_{B_l} g^{-1} \rhd m$. A straightforward computation shows that ψ is the inverse of φ_l in (3.4).

Next we show that $B(s(l)) \otimes_{B_l} M_l$ is an anti-Yetter-Drinfel'd module with respect to the direct sum decomposition $B(s(l)) \otimes_{B_l} M_l \cong \oplus_{l' \in [l]} \psi(M_{l'})$, hence (3.4) is an isomorphism of anti-Yetter-Drinfel'd modules, as stated. For $m \in M_l$, $l' \in \mathcal{L}(\mathcal{G})$, $h, h' \in \mathcal{G}^1$ and $y \in \mathcal{G}^0$, such that $hlh^{-1} = l'$ and $s(h') = t(h)$,

$$y \rhd (h \otimes_{B_l} m) = yh \otimes_{B_l} m = \delta_{t(h), y} h \otimes_{B_l} m = \delta_{s(l'), y} h \otimes_{B_l} m,$$

$$h' \rhd (h \otimes_{B_l} m) = h'h \otimes_{B_l} m \in \psi(M_{h'l'h'^{-1}}).$$

In view of Theorem 3.2 this implies that $B(s(l)) \otimes_{B_l} M_l$ is an anti-Yetter-Drinfel'd module with respect to the given decomposition, hence it completes the proof. $\square$

3.B. Hochschild and cyclic homology with coefficients. In view of Proposition 2.19, there is a cyclic simplex associated to a groupoid bialgebroid $\mathcal{B}$, the right regular $\mathcal{B}$-module coring and any stable anti-Yetter-Drinfel'd $\mathcal{B}$-module M. At degree n, it is given by $Z_n(B, M) = B^{\otimes_R n+1} \otimes_B M$ (where B acts on $B^{\otimes_R n+1}$ via the diagonal right action). In this section we compute its Hochschild and cyclic homologies.

With an eye on the decomposition of M in Sect. 3.A, computations start with the following

Lemma 3.5. *Let $\mathcal{G}$ be a small groupoid with a finite set of objects. Let $\mathcal{B}$ be the groupoid bialgebroid over a field $\mathbb{K}$, associated to $\mathcal{G}$. Let M be an anti-Yetter-Drinfel'd $\mathcal{B}$-module. Using notations introduced in Sect. 3.A, there is an isomorphism of right B_l-modules,*

$$B^{\otimes_R n+1} \otimes_B B(x) \cong B(x)^{\otimes_{\mathbb{K}} n+1},$$

for all $l \in \mathcal{L}(\mathcal{G})$ and $x := s(l)$. Here $B^{\otimes_R n+1} \otimes_B B(x)$ is understood to be a right B_l-module via the last factor and the group algebra B_l acts on $B(x)^{\otimes_{\mathbb{K}} n+1}$ via the diagonal action.

Proof. Since $R = \mathbb{K}\mathcal{G}^0$ is a separable $\mathbb{K}$-algebra, $B^{\otimes_R n+1}$ is isomorphic to the subspace $B^{\times n+1}$ of $B^{\otimes_{\mathbb{K}} n+1}$, spanned by those elements $g_0 \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} g_n$ for which $s(g_i) = s(g_{i+1})$, for all $i = 0 \cdots n$. Thus it suffices to prove $B^{\times n+1} \otimes_B B(x) \cong B(x)^{\otimes_{\mathbb{K}} n+1}$. We claim that the right B_l-module map

$$B(x)^{\otimes_{\mathbb{K}} n+1} \to B^{\times n+1} \otimes_B B(x), \quad g_0 \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} g_n \mapsto (g_0 \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} g_n) \otimes_B x \quad (3.5)$$

is an isomorphism. Since the map

$$B^{\times n+1} \otimes_{\mathbb{K}} B(x) \to B(x)^{\otimes_{\mathbb{K}} n+1}, \quad (g_0 \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} g_n) \otimes_{\mathbb{K}} h \mapsto g_0 h \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} g_n h$$

factorizes through $B^{\times n+1} \otimes_B B(x)$, it defines a unique map

$$B^{\times n+1} \otimes_B B(x) \to B(x)^{\otimes_{\mathbb{K}} n+1}, \quad (g_0 \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} g_n) \otimes_B h \mapsto g_0 h \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} g_n h.$$

$$(3.6)$$

Obviously, (3.5) and (3.6) are mutual inverses. $\square$

Proposition 3.6. *Let $\mathcal{G}$ be a small groupoid with a finite set of objects. Let $\mathcal{B}$ be the groupoid bialgebroid over a field $\mathbb{K}$, associated to $\mathcal{G}$. Let M be a stable anti-Yetter-Drinfel'd $\mathcal{B}$-module, with decomposition (3.2). For all $l \in \mathcal{L}(\mathcal{G})$, the cyclic simplex $Z_*(B, M_{[l]}) = B^{\otimes_R *+1} \otimes_B M_{[l]}$ is isomorphic to $Z_*(B(s(l)), M_l) = B(s(l))^{\otimes_{\mathbb{K}} *+1} \otimes_{B_l} M_l$, corresponding (as in [JŞ, Remark 4.16]) to the module coalgebra $B(s(l))$, and stable anti-Yetter-Drinfel'd module M_l, of the group bialgebra B_l.*

Proof. Combining the isomorphisms in Proposition 3.4 and Lemma 3.5, we obtain an isomorphism of vector spaces

$$B(s(l))^{\otimes_{\mathbb{K}} n+1} \otimes_{B_l} M_l \cong B^{\otimes_R n+1} \otimes_B B(s(l)) \otimes_{B_l} M_l \cong B^{\otimes_R n+1} \otimes_B M_{[l]}$$
$$(g_0 \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} g_n) \otimes_{B_l} m \qquad \longmapsto \qquad (g_0 \otimes_R \cdots \otimes_R g_n) \otimes_B m,$$

with inverse $B^{\otimes_R n+1} \otimes_B M_{hlh^{-1}} \ni (g_0 \otimes_R \cdots \otimes_R g_n) \otimes_B m \mapsto (g_0 h \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} g_n h) \otimes_{B_l} h^{-1} \triangleright m$. It is left to the reader to check that it is an isomorphism of cyclic objects. $\square$

Hochschild and cyclic homologies of a group, with coefficients in a stable anti-Yetter-Drinfel'd module, were computed in [JŞ, Corollary 5.13]. Hence in what follows we relate Hochschild and cyclic homologies of the cyclic object $Z_*(B(s(l)), M_l) \cong Z_*(B, M_{[l]})$ in Proposition 3.6 to the known respective homology of $Z_*(B_l, M_l)$.

Lemma 3.7. *Let (C, Δ, ϵ) be a coalgebra over a field $\mathbb{K}$. Consider the corresponding simplex $C^{\otimes_{\mathbb{K}} *+1}$ (with face maps ∂_i induced by ϵ and degeneracy maps μ_i induced by Δ). Then the associated complex $\widetilde{C}_*(C)$ is acyclic, i.e. $H_n(\widetilde{C}_*(C)) = \delta_{n,0} \mathbb{K}$, for any non-negative integer n. Moreover, if (C, Δ, ϵ) is a right module coalgebra of a $\mathbb{K}$-Hopf algebra H and free as a right H-module, then $\widetilde{C}_*(C)$ is a free resolution of $\mathbb{K}$ in the category of right H-modules.*

Proof. We need to show that the chain complex

$$\cdots \xrightarrow{\delta_{n+1}} C^{\otimes_{\mathbb{K}} n+1} \xrightarrow{\delta_n} C^{\otimes_{\mathbb{K}} n} \longrightarrow \cdots \xrightarrow{\delta_1} C \xrightarrow{\delta_0 = \epsilon} \mathbb{K} \longrightarrow 0$$

is exact, where $\delta_n = \sum_{i=0}^n (-1)^i \partial_i$. Indeed, the map $C \otimes_{\mathbb{K}} \epsilon$ is surjective, having a section Δ. Since C is a faithfully flat module of the field $\mathbb{K}$, this implies surjectivity of ϵ. Take an element $g \in C$ such that $\epsilon(g) = 1_{\mathbb{K}}$. One easily checks that, for $z \in C^{\otimes_{\mathbb{K}} n+1}$ such that $\delta_n(z) = 0$, $\delta_{n+1}(g \otimes_{\mathbb{K}} z) = z - (g \otimes_{\mathbb{K}} \delta_n(z)) = z$.

Assume now that (C, Δ, ϵ) is a right module coalgebra of a $\mathbb{K}$-Hopf algebra H. Then $\widetilde{C}_*(C)$ is an acyclic complex in the category of right H-modules. It remains to show that $C^{\otimes_{\mathbb{K}} n+1}$ is a free H-module whenever C is. Indeed, in this case $C^{\otimes_{\mathbb{K}} n+1}$ is free as a right module of $H^{\otimes_{\mathbb{K}} n+1}$ via factorwise action. By [JŞ, Lemma 2.10] (cf. Lemma 2.25), $H^{\otimes_{\mathbb{K}} n+1}$ is free as a right H-module via the diagonal action. Hence $C^{\otimes_{\mathbb{K}} n+1} \cong C^{\otimes_{\mathbb{K}} n+1} \otimes_{H^{\otimes_{\mathbb{K}} n+1}} H^{\otimes_{\mathbb{K}} n+1}$ is a free right H-module. $\square$

Lemma 3.8. *Let H be a Hopf algebra over a field $\mathbb{K}$ and let $i : C \to C'$ be a morphism of right H-module coalgebras. Let M be a stable anti-Yetter-Drinfel'd H-module. If both C and C' are free as right H-modules, then the induced morphism $i_* : Z_*(C, M) \to Z_*(C', M)$ of cyclic objects gives rise to isomorphisms both of Hochschild and cyclic homologies. That is,*

$$\mathrm{HH}_*(C, M) \cong \mathrm{HH}_*(C', M) \quad and \quad \mathrm{HC}_*(C, M) \cong \mathrm{HC}_*(C', M).$$

Proof. By Lemma 3.7, both $\widetilde{C}_*(C)$ and $\widetilde{C}_*(C')$ are free resolutions of $\mathbb{K}$ over H. Thus $i^{\otimes_{\mathbb{K}} *+1} : \widetilde{C}_*(C) \to \widetilde{C}_*(C')$ is a quasi-isomorphism. Then so is $i^{\otimes_{\mathbb{K}} *+1} \otimes_H M : \widetilde{C}_*(C) \otimes_H M \to \widetilde{C}_*(C') \otimes_H M$, yielding an isomorphism

$$\mathrm{HH}_n(C, M) \equiv \mathrm{H}_n(\widetilde{C}_*(C) \otimes_H M) \cong \mathrm{H}_n(\widetilde{C}_*(C') \otimes_H M) \equiv \mathrm{HH}_n(C', M),$$

for all non-negative integers n. The isomorphism of Hochschild homologies implies the isomorphism of cyclic homologies, see e.g. (a dual form of) [Lo, 2.2.3]. ☐

Theorem 3.9. *Let $\mathcal{G}$ be a small groupoid with a finite set of objects. Let $\mathcal{B}$ be the groupoid bialgebroid over a field $\mathbb{K}$, associated to $\mathcal{G}$. Let M be a stable anti-Yetter-Drinfel'd $\mathcal{B}$-module, with decomposition* (3.2). *For all $l \in \mathcal{L}(\mathcal{G})$,*

$$\mathrm{HH}_*(B, M_{[l]}) \cong \mathrm{HH}_*(B_l, M_l) \quad and \quad \mathrm{HC}_*(B, M_{[l]}) \cong \mathrm{HC}_*(B_l, M_l).$$

Proof. By Proposition 3.6, the cyclic objects $Z_*(B, M_{[l]})$ and $Z_*(B(s(l)), M_l)$ are isomorphic. Hence $\mathrm{HH}_*(B, M_{[l]}) \cong \mathrm{HH}_*(B(s(l)), M_l)$. Furthermore, the right B_l-module coalgebra $B(s(l))$ is a free right B_l-module by Lemma 3.3. Hence we conclude by Lemma 3.8 that the inclusion map $B_l \hookrightarrow B(s(l))$ induces an isomorphism $\mathrm{HH}_*(B(s(l)), M_l) \cong \mathrm{HH}_*(B_l, M_l)$. Combination of these isomorphisms proves the theorem. ☐

Let us assume now that $\mathbb{K}$ is a field of characteristic zero. In view of Theorem 3.9, one can compute also

$$\mathrm{HH}_*(B, M) \cong \bigoplus_{[l] \in \mathcal{T}(\mathcal{G})} \mathrm{HH}_*(B_l, M_l) \quad and \quad \mathrm{HC}_*(B, M) \cong \bigoplus_{[l] \in \mathcal{T}(\mathcal{G})} \mathrm{HC}_*(B_l, M_l).$$

Since B_l is a group algebra of $\mathcal{G}_l^1$ (cf. (3.3)), $\mathrm{HH}_*(B_l, M_l) = H_*(\mathcal{G}_l^1, M_l)$ is the group homology of $\mathcal{G}_l^1$ with coefficients in M_l. Applying [JŞ, Corollary 5.13], for the cyclic homology we get

$$\mathrm{HC}_*(B_l, M_l) = \begin{cases} \bigoplus_{i \geq 0} \mathrm{H}_{*-2i}\big(\mathcal{G}_l^1 / \langle l \rangle, M_l\big), & \text{if } l \text{ has finite order} \\ \mathrm{H}_*\big(\mathcal{G}_l^1 / \langle l \rangle, M_l\big), & \text{if } l \text{ has infinite order.} \end{cases}$$

3.C. Cyclic homology of groupoids. The results in Sect. 3.B can be specialized further to stable anti-Yetter-Drinfel'd modules provided by groupoid Galois extensions, cf. Proposition 2.23. This enables us, in particular, to compute ordinary (i.e. non-relative) Hochschild and cyclic homologies of a groupoid.

Let $\mathcal{G}$ be a small groupoid with a finite set of objects. Let $\mathcal{B}$ be the groupoid bialgebroid over a field $\mathbb{K}$, associated to $\mathcal{G}$. Recall from [CaDGr, Sect. 3] that a Galois extension $S \subseteq T$ by $\mathcal{B}$ has an equivalent description as follows. T is a strongly $\mathcal{G}$-graded $\mathbb{K}$-algebra, that is, $T \cong \oplus_{g \in \mathcal{G}^1} T_g$ (as a vector space),

$$T_g T_{g'} = \begin{cases} T_{g \circ g'} & \text{if } s(g) = t(g') \\ 0 & \text{if } s(g) \neq t(g'), \end{cases}$$

$1_T = \sum_{x \in \mathcal{G}^0} 1_{T_x}$, and S is equal to the subalgebra $\oplus_{x \in \mathcal{G}^0} T_x$. Note that each direct summand T_g is an $R = \mathbb{K}\mathcal{G}^0$-module via $v \cdot x := v 1_{T_x} = \delta_{s(g),x} v$, for $v \in T_g$ and $x \in \mathcal{G}^0$.

By Proposition 2.23, $T_S := S \widehat{\otimes}_S T$ is a stable anti-Yetter-Drinfel'd module. In the following lemma its structure is investigated.

Lemma 3.10. *Let $\mathcal{G}$ be a small groupoid with a finite set of objects. Let $\mathcal{B}$ be the groupoid bialgebroid over a field $\mathbb{K}$, associated to $\mathcal{G}$. Let $S \subseteq T$ be a Galois extension by $\mathcal{B}$. Using the notations introduced in Sect. 3.A, there is an isomorphism of anti-Yetter-Drinfel'd modules*

$$T_S \cong \bigoplus_{l \in \mathcal{L}(\mathcal{G})} S \widehat{\otimes}_S T_l.$$

The (Miyashita-Ulbrich) action is given, for $g \in \mathcal{G}^1$, $l \in \mathcal{L}(\mathcal{G})$ and $v \in T_l$, by

$$g \triangleright p_{T_l}(v) = \sum_{i=1}^{n} p_{T_l}(b_i v a_i),$$

where $p_{T_l} : T_l \to S \widehat{\otimes}_S T_l$ denotes the canonical epimorphism and the elements $\{a_1, \ldots, a_n\} \subseteq T_{g^{-1}}$ and $\{b_1, \cdots, b_n\} \subseteq T_g$ satisfy $\sum_{i=1}^{n} a_i b_i = 1_{T_{s(g)}}$.

Proof. Since $T \cong \oplus_{g \in \mathcal{G}^1} T_g$ is $\mathcal{G}$-graded, each direct summand T_g is an S-bimodule. Hence

$$S \widehat{\otimes}_S T \cong \bigoplus_{g \in \mathcal{G}^1} S \widehat{\otimes}_S T_g.$$

We claim that only those elements $g \in \mathcal{G}^1$ give a non-zero contribution to this direct sum, for which $s(g) = t(g)$. Recall that $S \widehat{\otimes}_S T_g$ is isomorphic to the quotient of T_g with respect to the commutator subspace $[S, T_g] = \{qv - vq \mid q \in S, v \in T_g\}$. Take an element $g \in \mathcal{G}^1$ such that $s(g) \neq t(g)$ and an element $v \in T_g$. By strong grading of T, there exist elements $\{a_1, \cdots, a_n\} \subseteq T_{g^{-1}}$ and $\{b_1, \cdots, b_n\} \subseteq T_g$ such that $\sum_{i=1}^{n} a_i b_i = 1_{T_{s(g)}}$. Then

$$\sum_{i=1}^{n} [va_i, b_i] = \sum_{i=1}^{n} va_i b_i - b_i va_i = v 1_{T_{s(g)}} = v,$$

where the penultimate equality follows by the fact that, for all values of i, $b_i va_i \in T_g T_g T_{g^{-1}}$ is zero by the assumption that $s(g) \neq t(g)$. Since for all values of i, $va_i \in T_{g^{-1}} T_g = T_{s(g)} \subseteq S$, we conclude that $T_g \subseteq [S, T_g]$. Since the converse inclusion is obvious, we have $S \widehat{\otimes}_S T_g \cong T_g / [S, T_g] = 0$ proven.

In order to write down the Miyashita-Ulbrich action, consider again $g \in \mathcal{G}^1$ and elements $\{a_1, \cdots, a_n\} \subseteq T_{g^{-1}}$ and $\{b_1, \ldots, b_n\} \subseteq T_g$ such that $\sum_{i=1}^{n} a_i b_i = 1_{T_{s(g)}}$. The canonical map can : $T \otimes_S T \to T \otimes_R B$ satisfies

$$\text{can}(\sum_{i=1}^{n} a_i \otimes_S b_i) = \sum_{i=1}^{n} a_i b_i \otimes_R g = 1_{T_{s(g)}} \otimes_R g = 1_T \cdot s(g) \otimes_R g = 1_T \otimes_R g.$$

Hence the stated form of the Miyashita-Ulbrich action follows by (2.31).   □

Corollary 3.11. *Let $\mathcal{G}$ be a small groupoid with a finite set of objects. Let $\mathcal{B}$ be the groupoid bialgebroid over a field $\mathbb{K}$, associated to $\mathcal{G}$. Let $S \subseteq T$ be a Galois extension by $\mathcal{B}$. Using the notations introduced in Sect. 3.A, the S-relative Hochschild and cyclic homologies of T are given, respectively, by*

$$\text{HH}_*(T/S) = \bigoplus_{[l] \in \mathcal{T}(\mathcal{G})} \text{HH}_*(B_l, S \widehat{\otimes}_S T_l) \quad and \quad \text{HC}_*(T/S) = \bigoplus_{[l] \in \mathcal{T}(\mathcal{G})} \text{HC}_*(B_l, S \widehat{\otimes}_S T_l).$$

Proof. By Theorem 2.26, $\mathrm{HH}_*(T/S) \cong \mathrm{HH}_*(B, T_S)$ and $\mathrm{HC}_*(T/S) \cong \mathrm{HC}_*(B, T_S)$. Hence the claim follows by Theorem 3.9 and considerations following it, together with Lemma 3.10. $\square$

A particular example $R \subseteq B$ of a Galois extension by a groupoid $\mathcal{G}$ is provided by the inclusion of the base algebra $R = \mathbb{K}\mathcal{G}^0$ in the groupoid algebra $B = \mathbb{K}\mathcal{G}^1$. Applying Corollary 3.11 to it, we obtain formulae for the R-relative Hochschild and cyclic homologies of the groupoid $\mathcal{G}$. Since the base algebra $R = \mathbb{K}\mathcal{G}^0$ is separable over $\mathbb{K}$, relative homologies coincide with ordinary ones, cf. [Kad]. Therefore we obtain the following corollary, extending results in [Burg] on cyclic homology of groups. Similar expressions were derived also by Crainic in [Cra] for *étale* groupoids.

Corollary 3.12. *Let B be the groupoid algebra of a small groupoid $\mathcal{G}$ with a finite set of objects over a field $\mathbb{K}$ of characteristic zero. Using the notations introduced in Sect. 3.A we have*

$$\mathrm{HH}_*(B) = \bigoplus_{[l] \in \mathcal{T}(\mathcal{G})} \mathrm{H}_*(\mathcal{G}_l^1, \mathbb{K}),$$

$$\mathrm{HC}_*(B) = \left(\bigoplus_{\substack{[l] \in \mathcal{T}(\mathcal{G}) \\ \mathrm{ord}(l) < \infty}} \bigoplus_{i \geq 0} \mathrm{H}_{*-2i}(\mathcal{G}_l^1/\langle l \rangle, \mathbb{K}) \right) \bigoplus \left(\bigoplus_{\substack{[l] \in \mathcal{T}(\mathcal{G}) \\ \mathrm{ord}(l) = \infty}} \mathrm{H}_*(\mathcal{G}_l^1/\langle l \rangle, \mathbb{K}) \right).$$

Recall that a groupoid $\mathcal{G}$ is connected if, for any $x, y \in \mathcal{G}^0$, there exists at least one $g \in \mathcal{G}^1$ such that $s(g) = x$ and $t(g) = y$. For a connected groupoid $\mathcal{G}$ we fix an object $x \in \mathcal{G}^0$ and we denote by G the group of loops $l \in \mathcal{L}(\mathcal{G})$ such that $s(l) = t(l) = x$. (Since $\mathcal{G}$ is connected by assumption, different choices of x lead to isomorphic subgroups G of $\mathcal{G}$.) Let $\mathcal{T}(G)$ denote the set of conjugacy classes in G and let $\{g_\sigma \mid \sigma \in \mathcal{T}(G)\}$ be a transversal of $\mathcal{T}(G)$.

Lemma 3.13. *Let $\mathcal{G}$ be a connected groupoid. Keeping the above notation, there is an one-to-one correspondence between $\mathcal{T}(\mathcal{G})$ and $\mathcal{T}(G)$.*

Proof. We have to show that any orbit in $\mathcal{L}(\mathcal{G})$ for the adjoint action contains precisely one element of the set $\{g_\sigma \mid \sigma \in \mathcal{T}(G)\}$. That is, any loop l in $\mathcal{G}$ is equivalent to a certain g_σ and that two elements g_σ and g_τ are equivalent if, and only if $\sigma = \tau$. First, let us take $l \in \mathcal{L}(\mathcal{G})$. Since $\mathcal{G}$ is connected, there is a morphism g such that $s(g) = x$ and $t(g) = s(l)$. Let $l' := g^{-1} \circ l \circ g$. By construction, $l' \in G$, so there are $h \in G$ and $\sigma \in \mathcal{T}(G)$ such that $h \circ l' \circ h^{-1} = g_\sigma$. Since $l = (g \circ h^{-1}) \circ g_\sigma \circ (g \circ h^{-1})^{-1}$ it follows that l and g_σ are conjugated in $\mathcal{L}(\mathcal{G})$.

Let us take σ and τ in $\mathcal{T}(G)$ and assume that g_σ and g_τ define the same element in $\mathcal{T}(\mathcal{G})$. Then there is $g \in \mathcal{G}^1$ such that $g_\sigma = g \circ g_\tau \circ g^{-1}$. Since both loops g_σ and g_τ have the same source, we deduce that the source and the target of g must be x. Hence $g \in G$ and the conjugacy classes of g_σ and g_τ in $\mathcal{T}(G)$ are equal. In conclusion, $\sigma = \tau$. $\square$

Remark 3.14. Lemma 3.13 tells us, in particular, that $\{g_\sigma \mid \sigma \in \mathcal{T}(G)\}$ is also a transversal of $\mathcal{T}(\mathcal{G})$. Explicitly, a given element g_σ represents the following orbit $\widetilde{\sigma} \in \mathcal{T}(\mathcal{G})$. Recall that we defined the group G in terms of a fixed object x in $\mathcal{G}^0$. For every $y \in \mathcal{G}^0$ we pick up a fixed $g_y \in \mathrm{Hom}_\mathcal{G}(x, y)$. It defines a group isomorphism $G \to \mathrm{Hom}_\mathcal{G}(y, y)$, $h \mapsto g_y \circ h \circ g_y^{-1}$. It maps $\sigma \in \mathcal{T}(G)$ to the conjugacy class $\sigma_y = \{g_y \circ h \circ g_y^{-1} \mid h \in \sigma\}$ in the group $\mathrm{Hom}_\mathcal{G}(y, y)$. The orbit of g_σ for the adjoint $\mathcal{G}$-action is $\widetilde{\sigma} = \bigcup_{y \in \mathcal{G}^0} \sigma_y$.

Corollary 3.15. *Let $\mathcal{G}$ be a connected groupoid with finitely many objects. Let M be a stable anti-Yetter-Drinfel'd module over B, where B is the groupoid algebra of $\mathcal{G}$ over a field of characteristic zero.*

(i) $M_G := \bigoplus_{g \in G} M_g$ *is a stable anti-Yetter-Drinfel'd module over $\mathbb{K}G$ and the inclusions $\mathbb{K}G \subseteq B$ and $M_G \subseteq M$ induce isomorphisms*

$$\mathrm{HH}_*(B, M) \cong \mathrm{HH}_*(\mathbb{K}G, M_G) \quad and \quad \mathrm{HC}_*(B, M) \cong \mathrm{HC}_*(\mathbb{K}G, M_G).$$

(ii) *The inclusion $\mathbb{K}G \subseteq B$ induces isomorphisms*

$$\mathrm{HH}_*(B) \cong \mathrm{HH}_*(\mathbb{K}G) \quad and \quad \mathrm{HC}_*(B) \cong \mathrm{HC}_*(\mathbb{K}G).$$

Proof. (i) Obviously M_G is a stable anti-Yetter-Drinfel'd module over $\mathbb{K}G$. Consider the following diagram.

$$
\begin{array}{ccccc}
Z_*(\mathbb{K}G, M_G) & \longrightarrow & \bigoplus_{\sigma \in \mathcal{T}(G)} Z_*(\mathbb{K}G, M_\sigma) & \longrightarrow & \bigoplus_{\sigma \in \mathcal{T}(G)} Z_*(\mathbb{K}B_{g_\sigma}, M_{g_\sigma}) \\
\downarrow & & \downarrow & & \downarrow \\
Z_*(B, M) & \longrightarrow & \bigoplus_{\sigma \in \mathcal{T}(G)} Z_*(B, M_{\widetilde{\sigma}}) & \longrightarrow & \bigoplus_{\sigma \in \mathcal{T}(G)} Z_*(B(x), M_{g_\sigma})
\end{array}
$$

The leftmost morphism in the bottom row comes from the decomposition (3.2), while the other one is the direct sum of the arrows that were constructed in Proposition 3.6. Note that both maps are induced by appropriate inclusions and they are isomorphisms. The morphisms in the top row have the same properties, as any group can be regarded as a groupoid with one object. By definition, the vertical arrows are the canonical morphisms induced by inclusions, so they make the squares commutative. In view of the proof of Theorem 3.9, the rightmost vertical arrow gives isomorphisms both of Hochschild and cyclic homologies. Then also the leftmost vertical morphism does so.

(ii) The subalgebra $R := \mathbb{K}X$ of B is a separable $\mathbb{K}$-algebra. Hence $\mathrm{HH}_*(B) \cong \mathrm{HH}_*(B, R)$. By Lemma 3.10 we know that $M := R \widehat{\otimes}_R B$ is a stable anti-Yetter-Drinfel'd module over B. Since for any $g \in \mathcal{G}$, either $[R, \mathbb{K}g] = \mathbb{K}g$ or $[R, \mathbb{K}g] = 0$, depending on the fact that g is a loop or not, we get $M := \bigoplus_{g \in \mathcal{L}(\mathcal{G})} \mathbb{K}g$. Hence $M_G := \mathbb{K}G$. Obviously the action of B on $\mathbb{K}G$ induced from the Ulbrich-Miyashita action is the adjoint action of $\mathbb{K}G$ on itself. We conclude by applying the first part of the corollary and the isomorphisms $\mathrm{HH}_*(\mathbb{K}G) \cong \mathrm{HH}_*(\mathbb{K}G, (\mathbb{K}G)_{ad})$ and $\mathrm{HC}_*(\mathbb{K}G) \cong \mathrm{HC}_*(\mathbb{K}G, (\mathbb{K}G)_{ad})$, cf. [JŞ]. □

Our final aim is to compute $\mathrm{HC}_*(B)$, the ordinary cyclic homology of the groupoid algebra B of a groupoid $\mathcal{G}$ that may have an infinite number of objects. For such a groupoid, its groupoid algebra B is not unital anymore. Nevertheless, to define cyclic homology of B one can proceed as for unital algebras, cf. [Lo, Chap. 2, §2.1]. The point is that Connes' complex $C^\lambda(B)$ still exists, although it is now associated to a precyclic object, that is to a presimplicial structure endowed with cyclic operators. Here, by presimplicial object we mean a sequence of objects together only with face maps. The defining properties of face maps and cyclic operators are the same as in the definition of cyclic objects, neglecting of course the relations that involve the degeneracy maps. The key ingredient of our computation is a description of any groupoid as a direct limit of groupoids with finitely many objects. Since cyclic homology is defined as homology of Connes' complex and the homology functor commutes with direct limits, we obtain cyclic homology of an arbitrary groupoid as a direct limit. Note however that, for non-unital algebras, Hochschild homology is constructed in a different way. For the definition, see for example [Lo, Chap. 1, §1.2]. Therefore, we can not apply the same arguments to compute Hochschild homology of an arbitrary groupoid.

Theorem 3.16. *Let $\mathcal{G}$ be a connected groupoid. If B is the groupoid algebra over a field $\mathbb{K}$ of characteristic zero, then*

$$\mathrm{HC}_*(B) \cong \mathrm{HC}_*(\mathbb{K}G).$$

Proof. Let x be a given object in $\mathcal{G}^0$. Let G denote the group of loops $l \in \mathcal{L}(\mathcal{G})$ such that $s(l) = x$. We order the set

$$\mathfrak{X} := \{X \subset \mathcal{G}^0 \mid x \in X \text{ and } X \text{ is finite}\}$$

with respect to inclusion. Trivially $\mathfrak{X}$ is a direct system. For $X \in \mathfrak{X}$ we define $\mathcal{G}_X$ to be the full subgroupoid of $\mathcal{G}$ such that $\mathcal{G}_X^0 = X$ and we denote its groupoid algebra by B_X. Note that $B = \bigcup_{X \in \mathfrak{X}} B_X$, so $C_*^\lambda(B) = \bigcup_{X \in \mathfrak{X}} C_*^\lambda(B_X)$. As the direct limit in the category of vector spaces is exact, it follows that the homology functor and direct limit commute. Thus

$$\mathrm{HC}_*(B) \cong \mathrm{H}_*(C_*^\lambda(B)) \cong \mathrm{H}_*\Big(\varinjlim_{X \in \mathfrak{X}} C_*^\lambda(B_X)\Big) \cong \varinjlim_{X \in \mathfrak{X}} \mathrm{H}_*\big(C_*^\lambda(B_X)\big) \cong \varinjlim_{X \in \mathfrak{X}} \mathrm{HC}_*(B_X),$$

where the latter direct system is defined by the canonical maps $\mathrm{HC}_*(B_X) \to \mathrm{HC}_*(B_Y)$, with X, Y in $\mathfrak{X}$ such that $X \subset Y$. We claim that these maps are isomorphisms. Indeed, let us consider the following commutative diagram

$$
\begin{array}{ccc}
& Z_*(\mathbb{K}G) & \\
& \swarrow \qquad \searrow & \\
Z_*(B_X) & \longrightarrow & Z_*(B_Y)
\end{array}
$$

By the second part of Corollary 3.15, the oblique arrows induce isomorphisms in cyclic homology. Then, passing to cyclic homology, also the horizontal map yields an isomorphism. We can now conclude the proof of the theorem by remarking that, for any $X \in \mathfrak{X}$,

$$\mathrm{HC}_*(B_X) \cong \varinjlim_{X \in \mathfrak{X}} \mathrm{HC}_*(B_X).$$

Thus, taking $X = \{x\}$ we get the required isomorphism. □

The computation performed in Theorem 3.16 can be extended to an arbitrary (discrete) groupoid $\mathcal{G}$. Let $(\mathcal{G}_i)_{i \in I}$ be the connected components of $\mathcal{G}$. For each i we pick up $x_i \in \mathcal{G}_i^0$ and we denote the set of loops l with $s(l) = x_i$ by G_i. We have the following result.

Corollary 3.17. *Let $\mathcal{G}$ be a discrete groupoid. If B denotes the groupoid algebra of $\mathcal{G}$ over a field of characteristic zero, then*

$$\mathrm{HC}_*(B) \cong \bigoplus_{i \in I} \mathrm{HC}_*(\mathbb{K}G_i). \tag{3.7}$$

In addition, if $\mathcal{G}_i^0$ is a finite set for every $i \in I$, then a similar isomorphism holds in Hochschild homology.

Proof. For $i \in I$, let B_i be the groupoid algebra of $\mathcal{G}_i$. As a vector space, B is isomorphic to $\bigoplus_{i \in I} B_i$. Via this identification, the multiplication of B can be extended to the direct sum. It is easy to see that, for two families $(b'_i)_{i \in I}$ and $(b''_i)_{i \in I}$ in $\bigoplus_{i \in I} B_i$, we have

$$(b'_i)_{i \in I} \cdot (b''_i)_{i \in I} = (b'_i b''_i)_{i \in I}.$$

The corollary now follows by the isomorphism $\mathrm{HC}_*(\bigoplus_{i \in I} B_i) \cong \bigoplus_{i \in I} \mathrm{HC}_*(B_i)$. If I is a finite set, then this isomorphism can be found in [Lo, Exercise 2.2.1]. Since cyclic homology and direct limits commute, the isomorphism can be extended for an arbitrary set I.

Since for non-unital algebras Hochschild homology is constructed in a different way (cf. [Lo, Chap. 1, §1.2]), the above arguments can not be applied to deduce an isomorphism for Hochschild homology, analogous to (3.7). Nevertheless, in the case when $\mathcal{G}^0$ is finite, B is an unital algebra. Thus $\mathrm{HH}_*(B)$ can be computed as the Hochschild homology of Connes' cyclic object. The required isomorphism now follows by [We, Theorem 9.1.8], proceeding as for cyclic homology.   □

As an application of Corollary 3.17 we shall compute $\mathrm{HH}_*(B)$ and $\mathrm{HC}_*(B)$, where B is the groupoid algebra of the groupoid associated to a G-set X.

Throughout the remaining part of the paper we fix an arbitrary (discrete) group G that acts to the left on an arbitrary set X. The action of G on X maps a pair $(x, g) \in X \times G$ to ${}^g x \in X$. For a G-set X as above, one constructs a groupoid $\mathcal{G}$ as follows. By definition, the set of objects in $\mathcal{G}$ is $\mathcal{G}^0 := X$ while, for $x, y \in X$, we set

$$\mathrm{Hom}_{\mathcal{G}}(x, y) = \left\{ (x, g) \in X \times G \mid {}^g x = y \right\}.$$

Note that the source of (x, g) is x and its target is ${}^g x$. Thus the composition $(x, g) \circ (x', g')$ is defined if, and only if $x = {}^{g'} x'$ and, in this case

$$(x, g) \circ (x', g') := (x', gg').$$

The set of morphisms in $\mathcal{G}$ is $\mathcal{G}^1 = X \times G$. Therefore, the groupoid algebra B of $\mathcal{G}$ has $X \times G$ as a basis. To describe the multiplication on this basis let us recall some well-known facts about twisted semigroup algebras by a 2-cocycle. Let S be a semigroup. A function $\omega : S \times S \to \mathbb{K}$ is called a 2-cocycle if, for any p_1, p_2, p_3 in S,

$$\omega(p_1, p_2)\omega(p_1 p_2, p_3) = \omega(p_2, p_3)\omega(p_1, p_2 p_3).$$

The semigroup algebra of S is defined as in the group case: as a vector space it has S as a basis and the multiplication on this basis is given by the multiplication in S. We shall denote this algebra by $\mathbb{K}S$. The cocycle ω can be used to deform the multiplication of $\mathbb{K}S$ such that we get another associative algebra structure on the vector space $\mathbb{K}S$. Its multiplication is defined by

$$p_1 \cdot p_2 = \omega(p_1, p_2) p_1 p_2.$$

The resulting algebra will be denoted by $\mathbb{K}_\omega S$. Certainly, it is not unital in general. Still, even if S has no neutral element, the algebra $\mathbb{K}_\omega S$ may have a unit $e := \sum_{x \in X} x$, where X is an appropriate finite subset of S. In fact, it is easy to see that e is a unit element in the algebra $\mathbb{K}_\omega$ if, and only if ω is X-normalized, i.e. for any $p, q \in S$,

$$\sum_{\{x \in X \mid xp = q\}} \omega(x, p) = \delta_{p,q} = \sum_{\{x \in X \mid px = q\}} \omega(p, x).$$

Let us turn back to the groupoid algebra of $\mathcal{G}$, where X is a G-set. We define the semigroup $S := X \times G$ with the multiplication

$$(x, g)(y, h) = (y, gh).$$

It is not difficult to see that $\omega_X : S \times S \to \mathbb{K}$, given by

$$\omega_X\big((x, g), (y, h)\big) := \begin{cases} 0, & \text{if } {}^h y \neq x \\ 1, & \text{if } {}^h y = x \end{cases} \tag{3.8}$$

is a 2-cocycle, which is X-normalized if, and only if X is finite. Obviously, $B = \mathbb{K}_\omega S$ as non-unital algebras, in general. For a finite G-set X, this is an equality of unital algebras.

We denote the set of G-orbits in X by $\mathfrak{X}$. Let us remark that there is an one-to-one correspondence between $\mathfrak{X}$ and the set of connected components of $\mathcal{G}$. This bijection maps an orbit $\mathfrak{o} \in \mathfrak{X}$ to the full subgroupoid $\mathcal{G}_\mathfrak{o}$ defined uniquely such that $\mathcal{G}_\mathfrak{o}^0 = \mathfrak{o}$. We choose a transversal $\{x_\mathfrak{o} \in X \mid \mathfrak{o} \in \mathfrak{X}\}$ for $\mathfrak{X}$. Thus, any $x \in X$ is in the orbit of a certain $x_\mathfrak{o}$ and $x_{\mathfrak{o}'}$ and $x_{\mathfrak{o}''}$ are in the same orbit if, and only if $\mathfrak{o}' = \mathfrak{o}''$. Moreover, the set $G_\mathfrak{o}$ of loops l such that $s(l) = x_\mathfrak{o}$ is the stabilizer of $x_\mathfrak{o}$,

$$G_\mathfrak{o} := \{g \in G \mid {}^g x_\mathfrak{o} = x_\mathfrak{o}\}.$$

Hence, a direct application of Corollary 3.17 yields the following.

Theorem 3.18. *Let X be a G-set. If ω_X denotes the 2-cocycle in (3.8) and $\mathbb{K}$ is a field of characteristic zero, then*

$$\mathrm{HC}_*(\mathbb{K}_{\omega_X} G) = \bigoplus_{\mathfrak{o} \in \mathfrak{X}} \mathrm{HC}_*(\mathbb{K} G_\mathfrak{o}).$$

If in addition X is finite, then

$$\mathrm{HH}_*(\mathbb{K}_{\omega_X} G) = \bigoplus_{\mathfrak{o} \in \mathfrak{X}} \mathrm{HH}_*(\mathbb{K} G_\mathfrak{o}).$$

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