

Intersection Theory from Duality and Replica

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Abstract: Kontsevich’s work on Airy matrix integrals has led to explicit results for the intersection numbers of the moduli space of curves. In this article we show that a duality between k -point functions on $N \times N$ matrices and N -point functions of $k \times k$ matrices, plus the replica method, familiar in the theory of disordered systems, allows one to recover Kontsevich’s results on the intersection numbers, and to generalize them to other models. This provides an alternative and simple way to compute intersection numbers with one marked point, and leads also to some new results.

1. Introduction

After Witten’s celebrated conjectures [1, 2] on the relation between intersection numbers on moduli spaces of curves and the KdV hierarchy, and Kontsevich’s proof [3], the literature on the subject, both from the point of view of mathematics or from its string theory relationship, has become considerable. We want to add here a new method, duality plus replica, which allows one to recover easily some results which are difficult to obtain by fancier methods, and provides some new results as well.

In a previous article [4] we have used explicit integral representations for the correlation functions [5–7] for a Gaussian unitary ensemble (GUE) of random matrices M in the presence of an external matrix source. The probability distribution for $N \times N$ Hermitian matrices is

$$P_A(M) = \frac{1}{Z_A} e^{-\frac{N}{2} \text{tr} M^2 - N \text{tr} M A}. \quad (1)$$

From this representation we have obtained the correlation functions of the ‘vertices’

$$V(k_1, \dots, k_n) = \frac{1}{N^n} \langle \text{tr} M^{k_1} \text{tr} M^{k_2} \dots \text{tr} M^{k_n} \rangle. \quad (2)$$

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As is well-known from Wick’s theorem (when the source A vanishes) such correlation functions are just the numbers of pairwise gluing of the legs of the vertex operators. The dual cells of these vertices are polygons, whose edges are pairwise glued. This generates orientable surfaces, discretized Riemann surfaces whose genus is related to the expansion in power of $1/N^2$.

Okounkov and Pandharipande [8–10] have shown that the intersection numbers, computed by Kontsevich [3], may be obtained by taking a simultaneous large N and large k_i limit. In our previous work we have used the exact integral representation valid for finite N of those vertex correlation functions, and obtained explicitly the scaling region for large k_i and large N by a simple saddle-point. This led to a practical way to compute intersection numbers from a pure Gaussian model, much simpler than Kontsevich’s Airy matrix model.

In this article we want to show that duality and replica may be used also to recover easily earlier results, to establish some new ones and give support to Witten’s conjecture. In order to make this article self-contained we have added appendices in which we rederive some of the steps leading to the representation that we are using:

(i) explicit formulae, valid for arbitrary N and arbitrary source matrix A for the average

$$U(s_1, \dots, s_k) = \langle \text{Tr}e^{s_1 M} \dots \text{Tr}e^{s_k M} \rangle \tag{3}$$

which rely on the Itzykson-Zuber formula [11, 12] (Appendix A);

(ii) a duality representation for the average of characteristic determinants

$$\langle \det(\lambda_1 - M) \dots \det(\lambda_k - M) \rangle$$

in terms of another GUE integral, but in which the random matrices are $k \times k$, whereas the initial problem involved $N \times N$ matrices (Appendix B) [13]. (This duality seems to be a simple reflection of the open string/closed string duality [14].) With the help of these two kinds of results we may proceed to the replica approach based on the simple relation

$$\lim_{n \rightarrow 0} \frac{1}{n} \frac{\partial}{\partial \lambda} [\det(\lambda - B)]^n = \text{tr} \frac{1}{\lambda - B}. \tag{4}$$

Therefore after applying these two steps we end up looking for an n goes to zero limit on matrix integrals whose size vanishes with n , a very different problem from the familiar large N limit. The Feynman graph representation of this matrix integral connects as usual with Riemann surfaces with marked points; letting the rank n go to zero selects the graphs of maximal genus for a given number of vertices.

The article is organized as follows: In Sect. 2 we first recall the main formulae (i) for the explicit integral representation of k -point functions in a Gaussian plus source, $N \times N$, Hermitian matrix integral, (ii) for the N - k duality. The derivations are reproduced in Appendices A and B respectively. Specializing to a constant matrix source, we are led to Kontsevich’s cubic model in the appropriate scaling limit. In Sect. 3 we derive the theorem (20) which gives explicitly the sourceless k -point functions in the zero-replica limit. This allows one to recover the known results for the intersection numbers, such as $\langle \tau_{3g-2} \rangle_g = \frac{1}{(24)^g g!}$ ($g = 0, 1, 2, \dots$). In Sect. 4 we show that an appropriate tuning of the matrix source may be used to generate generalizations of the Kontsevich model to higher powers than cubic, thereby providing generalized intersection numbers $\langle \prod \tau_{m,j} \rangle$

for moduli of curves with ‘spin’ j . For instance from the quartic generalized model we find

$$\langle \tau_{\frac{8g-5-j}{3}, j} \rangle_g = \frac{1}{(12)^g g!} \frac{\Gamma(\frac{g+1}{3})}{\Gamma(\frac{2-j}{3})},$$

where $j = 0$ for $g = 3m + 1$ and $j = 1$ for $g = 3m$. (For $g = 3m + 2$, the intersection numbers are zero.)

2. The Duality Plus Replica Strategy

We first consider the average of products of characteristic polynomials, defined as

$$\begin{aligned} F_k(\lambda_1, \dots, \lambda_k) &= \frac{1}{Z_N} \langle \prod_{\alpha=1}^k \det(\lambda_\alpha - M) \rangle_{A, M} \\ &= \frac{1}{Z_N} \int dM \prod_{i=1}^k \det(\lambda_i \cdot \mathbf{I} - M) e^{-\frac{N}{2} \text{tr} M^2 + N \text{tr} M A}, \end{aligned} \tag{5}$$

where M is an $N \times N$ Hermitian random matrix, A a given Hermitian matrix, whose eigenvalues are (a_1, \dots, a_n) and Z_N the normalization constant of the probability measure (for $A = 0$). We have shown earlier [13] that this correlation function has also a dual expression. This duality interchanges N , the size of the random matrix, with k , the number of points in F_k , as well as the matrix source A with the diagonal matrix $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_k)$. This duality reads [13]

$$F_k(\lambda_1, \dots, \lambda_k) = \frac{1}{Z_k} \int dB \prod_{j=1}^N [\det(a_j - iB)] e^{-\frac{N}{2} \text{tr}(B - i\Lambda)^2}, \tag{6}$$

where $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_k)$ and B is a $k \times k$ Hermitian matrix. (The normalization is now on GUE ensembles of Hermitian $k \times k$ matrices $Z_k = \int dB \text{Exp}(-\frac{1}{2} \text{tr} B^2)$). The derivation is reproduced in Appendix B.

If we specialize this formula to a source A equal to the unit matrix, a trivial shift for the original $N \times N$ matrices M , which has the effect of making the support of Wigner’s semi-circle law, for the asymptotic density of eigenvalues of M , to lie between 0 and 2 rather than $(-1, +1)$, the formula (6) involves

$$\begin{aligned} \det(1 - iB)^N &= \exp[N \text{tr} \ln(1 - iB)] \\ &= \exp[-iN \text{tr} B + \frac{N}{2} \text{tr} B^2 + i \frac{N}{3} \text{tr} B^3 + \dots]. \end{aligned} \tag{7}$$

The linear term in B in (7), combined with the linear term of the exponent of (6), shifts Λ by one. The B^2 terms in (6) cancel. In a scale in which the initials λ_k are close to one, or more precisely $N^{2/3}(\lambda_k - 1)$ is finite, the large N asymptotics of (6) is given by matrices B of order $N^{-1/3}$. Then the higher terms in (7) are negligible and we are left with terms linear and cubic in the exponent, namely

$$F_k(\lambda_1, \dots, \lambda_k) = e^{\frac{N}{2} \text{tr} \Lambda^2} \int dB e^{i \frac{N}{3} \text{tr} B^3 + iN \text{tr} B(\Lambda - 1)}. \tag{8}$$

So finally for this edge behavior problem, in which the matrix B is of order $N^{-1/3}$, and $\lambda - 1$ of order $N^{-2/3}$ we can rescale B and $\lambda - 1$ to get rid of N . The result is nearly identical to the matrix Airy integral, namely Kontsevich’s model [3], which gives the intersection numbers of moduli of curves. The original Kontsevich partition function was defined as

$$Z = \frac{1}{Z'} \int dB e^{-\text{tr} \Lambda B^2 + \frac{1}{3} \text{tr} B^3}, \tag{9}$$

where $Z' = \int dB e^{-\text{tr} B^2}$. The shift $B \rightarrow B + i \Lambda$, eliminates the B^2 term and one recovers (8) up to a trivial rescaling.

Let us apply this to a one-point function. In view of the replica limit we specialize those formulae to

$$\langle [\det(\lambda - M)]^n \rangle_{A, M} = \langle [\det(1 - iB)]^N \rangle_{\Lambda, B}, \tag{10}$$

where B is an $n \times n$ random Hermitian matrix. For this edge problem we have chosen for the source matrix A , the N -dimensional unit matrix, whereas Λ is a multiple of the $n \times n$ identity matrix : $\Lambda = \text{diag}(\lambda, \dots, \lambda)$. Note that the average in the l.h.s. of (10) is meant for the $N \times N$ GUE ensemble, whereas in the r.h.s. the average is performed on $n \times n$ Hermitian matrices with the weight (8) or (9).

The strategy that we will use is thus to take the integral (9) for $\Lambda = \lambda \times \mathbf{1}$ and expand it in powers of the cubic term. The formulae that will be established, being exact for finite n , are easily continued in n and allow one to take the $n \rightarrow 0$ limit. The method relies on explicit exact representations of Gaussian averages [7, 15] in the presence of an external matrix source (in Appendix A the main steps of the derivation have been recalled). As a result we will obtain formulae for quantities such as

$$\lim_{n \rightarrow 0} \frac{1}{n} \langle (\text{tr} B^k)^1 \rangle$$

computed with the (normalized) Gaussian weight $Z^{-1} \exp(-\frac{1}{2} \text{tr} B^2)$ on $n \times n$ Hermitian matrices; thereby this procedure gives the values of the intersection numbers of the moduli of curves for a number of cases.

3. Evolution Operators and Replica Limit

We will rely on explicit expressions for

$$U_A(s_1, \dots, s_k) = \frac{1}{n} \langle \text{tr} e^{s_1 B} \dots \text{tr} e^{s_k B} \rangle \tag{11}$$

for a probability measure on $n \times n$ Hermitian matrices in the presence of an Hermitian matrix source A , whose eigenvalues are a_1, \dots, a_n . The average is thus defined with the normalized weight

$$P_A(B) = \frac{1}{Z} e^{-\frac{1}{2} \text{tr} B^2 + \text{tr} AB}. \tag{12}$$

Then one has (see Appendix A) for the one-point function

$$U_A(s) = \frac{1}{n} \langle \text{tr} e^{sB} \rangle = \frac{e^{\frac{s^2}{2\lambda}}}{\text{ns}} \oint \frac{du}{2i\pi} e^{su/\lambda} \prod_1^n \left(\frac{u - a_\alpha + s}{u - a_\alpha} \right) \tag{13}$$

the contour, in the complex u -plane, circling around the n poles a_α . We have chosen the normalization $U_A(0) = 1$. This formula simplifies for a vanishing external source (although a non-vanishing source simplifies the derivation) to

$$U(s) = \frac{e^{\frac{s^2}{2\lambda}}}{ns} \oint \frac{du}{2i\pi} e^{su/\lambda} \left(1 + \frac{s}{u}\right)^n. \tag{14}$$

This representation leads to a simple continuation in n , and to an expansion in powers of n . For instance it gives

$$\lim_{n \rightarrow 0} U(s) = \frac{e^{\frac{s^2}{2\lambda}}}{s} \oint \frac{du}{2i\pi} e^{su/\lambda} \log \left(1 + \frac{s}{u}\right). \tag{15}$$

This last contour integral reduces to the integral of the discontinuity of the logarithm, giving readily

$$\lim_{n \rightarrow 0} U(s) = \frac{\sinh\left(\frac{s^2}{2\lambda}\right)}{\left(\frac{s^2}{2\lambda}\right)}, \tag{16}$$

and thus

$$\lim_{n \rightarrow 0} \frac{1}{n} \langle \text{tr} B^k \rangle = \frac{4k!}{\lambda^{2k} 4^k (2k+1)!}. \tag{17}$$

Note that in terms of Feynman diagrams with double lines, the limit $n \rightarrow 0$ selects the diagrams with one single internal index all along the lines of the diagrams. Those diagrams correspond to a surface of maximum genus for a given number of vertices.

The same strategy works for higher point-functions. The k -point function is given (for a vanishing source) by

$$\begin{aligned} U(s_1, \dots, s_k) &= \frac{1}{n} \langle \text{tr} e^{s_1 B} \dots \text{tr} e^{s_k B} \rangle \\ &= (-1)^{k(k-1)/2} e^{\sum_1^k \frac{s_i^2}{2\lambda}} \oint \prod_1^k \frac{du_i}{2i\pi} e^{\sum_1^k (u_i s_i / \lambda)} \prod_1^k \left(1 + \frac{s_i}{u_i}\right)^n \det \frac{1}{u_i + s_i - u_j}. \end{aligned} \tag{18}$$

Again the continuation to non-integer n is straightforward and leads to

$$\begin{aligned} \lim_{n \rightarrow 0} U(s_1, \dots, s_k) &= (-1)^{k(k-1)/2} e^{\sum_1^k \frac{s_i^2}{2\lambda}} \\ &\times \oint \prod_1^k \frac{du_i}{2i\pi} e^{\sum_1^k (u_i s_i / \lambda)} \sum_1^k \log \left(1 + \frac{s_i}{u_i}\right) \det \frac{1}{u_i + s_i - u_j}. \end{aligned} \tag{19}$$

The calculation of the contour integrals is more cumbersome, but all the integration can be done explicitly to the end and give the following theorem, which is a remarkably compact result (proof is given in Appendix C).

Theorem.

$$\lim_{n \rightarrow 0} U(s_1, \dots, s_k) = \frac{\lambda}{\sigma^2} \prod_1^k 2 \sinh \frac{\sigma s_i}{2\lambda}, \tag{20}$$

with $\sigma = s_1 + \dots + s_k$.

This function is of course the generating function for the $n = 0$ limit of $\frac{1}{n} \langle \text{tr} B^{p_1} \dots \text{tr} B^{p_k} \rangle$ by expanding in the s_i 's. Selecting the coefficients of equal powers for every s_i , for instance of $(s_1 \dots s_k)^3$, we find

$$\lim_{n \rightarrow 0} \frac{1}{n} \langle (\text{tr} B^3)^{4g-2} \rangle = \frac{3^{3g-2} 2^{-2g} (6g-4)!}{\lambda^{6g-3}} \binom{4g-2}{g} \tag{21}$$

all other powers of $\lim_{n \rightarrow 0} \frac{1}{n} \langle (\text{tr} B^3)^k \rangle$ vanishing unless $k = 2 \pmod{4}$. This leads to the intersection number of the moduli of curves with one marked point. Indeed, following Kontsevich, these numbers are given by

$$\frac{1}{n} \log Z = \sum_l t_l \langle \tau_l \rangle \tag{22}$$

with

$$t_l = (-2)^{-(4l+2)/6} \prod_0^{l-1} (2m+1) \left(\frac{2}{\lambda}\right)^{2m+1} \tag{23}$$

from which the above result (21) provides

$$\langle \tau_{3g-2} \rangle_g = \frac{1}{(24)^g g!} \quad (g = 0, 1, 2, \dots). \tag{24}$$

These numbers agree with the values of the intersection numbers computed earlier by Kontsevich, Witten and others [1, 3, 16, 17].

Clearly the method allows one to compute more than that. For instance one can derive as well

$$\lim_{n \rightarrow 0} \frac{1}{n} \langle (\text{tr} B^4)^{2p-1} \rangle = \frac{2^{2p-2}}{\lambda^{4p-2}} (4p-3)! \binom{2p-1}{p}. \tag{25}$$

These vertices appear in the higher Airy functions and they are related to the intersection numbers of Witten's top Chern class [1, 3].

4. Application to Intersection Numbers of Top Chern Class (p-Spin Curves)

Up to now we have considered an external source matrix A which was a multiple of identity. Let us now choose a source matrix in which half of the eigenvalues of A are $-a$ and the other half $+a$. Then the asymptotic density of states has for support two disconnected segments of the real axis (for $a > 1$) with a gap in-between. In the limit in which a approaches one the gap closes, and for $a(1)$ the support is made of one single segment. For the critical closing gap case $a = 1$ the density of state $\rho(\lambda)$ vanishes as $\lambda^{1/3}$ at the center of the band [18, 19]. We will show that the correlation functions for this critical case correspond to a higher Airy matrix model, related to the intersection numbers of Witten's top Chern class ($p = 3$ spin curves). The formula (20) derived in the previous section allows us to compute also those intersection numbers for one marked point. With the help of the previous duality and replica method, we shall show that these numbers are given by the Fourier transform of the one point function, which is given by an Airy function [18].

This may be extended to higher Airy cases ($p > 3$) for the n -point functions. The details are left to a future publication.

The partition function for the p^{th} generalization of Kontsevich’s Airy matrix-model is defined as

$$Z = \frac{1}{Z_0} \int dB \exp\left[\frac{1}{p+1} \text{tr}(B^{p+1} - \Lambda^{p+1}) - \text{tr}(B - \Lambda)\Lambda^p\right] \tag{26}$$

normalized by the Gaussian part of the integrand (after the shift $B \rightarrow B + \Lambda$ which cancels the linear terms in B of the exponent):

$$Z_0 = \int dB \exp\left[\sum_{j=0}^{p-1} \text{tr} \frac{1}{2} \Lambda^j B \Lambda^{p-j-1} B\right]. \tag{27}$$

The ‘free energy’, i.e. the logarithm of the partition function, is the generating function of the generalized intersection numbers $\langle \prod \tau_{m,j} \rangle$ for moduli of curves with ‘spin’ j [3, 20]

By taking the external source $a_i = \pm 1$ in the formula of the duality (6), and by expanding the exponent up to order B^4 in the large N limit, we obtain

$$\begin{aligned} \langle \prod_{i=1}^N \det(a_i - iB) \rangle &= \langle [\det(1 + B^2)]^{\frac{N}{2}} \rangle \\ &= \int dB e^{-\frac{N}{4} \text{tr} B^4 - iN \text{tr} B \Lambda}, \end{aligned} \tag{28}$$

which coincides with the above partition function Z in the case $p = 3$ after trivial scalings. For $p > 3$, we also obtain the partition function Z in (26) with appropriate choice of the external source [18].

The ‘free energy’, i.e. the logarithm of the partition function, is the generating function of the generalized intersection numbers $\langle \prod \tau_{m,j} \rangle$ for moduli of curves with ‘spin’ j [3, 20]

$$F = \sum_{d_{m,j}} \langle \prod_{m,j} \tau_{m,j}^{d_{m,j}} \rangle \prod_{m,j} \frac{t_{m,j}^{d_{m,j}}}{d_{m,j}!}, \tag{29}$$

where [3]

$$t_{m,j} = (-p)^{\frac{j-p-m(p+2)}{2(p+1)}} \prod_{l=0}^{m-1} (lp + j + 1) \text{tr} \frac{1}{\Lambda^{mp+j+1}}. \tag{30}$$

According to Witten [2] the intersection numbers is given by

$$\langle \tau_{m_1, j_1} \cdots \tau_{m_n, j_n} \rangle_g = \int_{\bar{M}_{g,n}} C_w(j_1, \dots, j_n) \psi_1^{m_1} \cdots \psi_n^{m_n} \tag{31}$$

with the condition which relates, for given p , the indices to the genus g of the surface

$$(p+1)(2g - 2 + n) = \sum_{i=1}^n (pm_i + j_i + 1). \tag{32}$$

The cohomology class $C_w(j_1, \dots, j_n)$ is Witten’s class (top Chern class), and ψ_1 is the first Chern class. Witten conjectured that F is the string solution of the p^{th} Gelfand-Dikii hierarchy [2,21]. The partition function Z of (26) is a model for the $(p,1)$ -quantum gravity, which is equivalent to the $(p - 1)$ -matrix model.

By retaining only the leading term of the limit $n \rightarrow 0$ limit, we restrict ourselves to surfaces with one marked point. Then the expansion of (29) is simply given by

$$\lim_{n \rightarrow 0} \frac{1}{n} F = \sum_{m,j} \langle \tau_{m,j} \rangle_g t_{m,j}. \tag{33}$$

In the leading order of the limit $n \rightarrow 0$, the matrix Λ is replaced by a scalar $\Lambda = \lambda \cdot I$.

After a simple rescaling of the Gaussian weight, we obtain, for instance in the case of $p = 3$,

$$Z = \frac{1}{Z_0} \int dB \exp\left[\frac{1}{36} \text{tr} B^4 + \frac{i}{3\sqrt{3}} \text{tr} \Lambda B^3 - \frac{1}{2} \text{tr} \Lambda^2 B^2\right], \tag{34}$$

$$Z_0 = \int dB \exp\left[-\frac{1}{2} \text{tr} \Lambda^2 B^2\right]. \tag{35}$$

In this $p = 3$ case, we obtain from (20),

$$\langle \tau_{\frac{8g-5-j}{3}, j} \rangle_g = \frac{1}{(12)^g g!} \frac{\Gamma(\frac{g+1}{3})}{\Gamma(\frac{2-j}{3})}, \tag{36}$$

where $j = 0$ for $g = 1, 4, 7, 10, \dots$ and $j = 1$ for $g = 3, 6, 9, \dots$. For $g = 2, 5, 8, \dots$, the intersection numbers are zero.

When the genus g is equal to three, the above formula gives $\langle \tau_{6,1} \rangle_{g=3} = \frac{1}{(12)^3 3! 3}$ which agrees with the value obtained earlier by Shadrin [22]. In the calculation of the intersection numbers for $p = 3$, in addition to $\langle (\text{tr} B^3)^p \rangle$ and $\langle (\text{tr} B^4)^q \rangle$ given here above in (21) and (25), one needs to compute mixed averages of the type $\langle (\text{tr} B^4)^k (\text{tr} B^3)^l \rangle$. Such averages are indeed required if one deals with the expansion of (34), but they are also contained in the explicit formula (20) for the generating function.

For general p , from (20), we have for $g = 1$, $\langle \tau_{1,0} \rangle_{g=1} = \frac{p-1}{24}$. To derive this result, we simply need to compute $\langle \text{tr} B^4 \rangle_{g=1} = 1$ and $\langle (\text{tr} B^3)^2 \rangle_{g=1} = 3$ with $t_{1,0} = -n/(p\lambda^{p+1})$.

In the case $p = 4$, we obtain from (20) by simple calculations up to order $g = 4$ as $\langle \tau_{1,0} \rangle_{g=1} = \frac{1}{8}$, $\langle \tau_{3,2} \rangle_{g=2} = \frac{9}{8^2 \cdot 5!}$, $\langle \tau_{6,0} \rangle_{g=3} = \frac{9}{8^3 \cdot 5!}$ and $\langle \tau_{8,2} \rangle_{g=4} = \frac{7 \cdot 11}{8^5 \cdot 5! \cdot 10}$.

Thus we find that the expression of (20) for the $n \rightarrow 0$ limit of $U(s_1, \dots, s_k)$ is a generating function for the intersection numbers for the moduli space of one-marked point, p spin, curves. This provides thus a simple algorithm for obtaining these numbers.

We have restricted this article to surfaces with one marked point. To go beyond one marked point, the analysis of higher orders in n is required. For two marked points, we need the order n^2 . However for genus one, we have two different kinds of intersection numbers, $\langle \tau_0 \tau_2 \rangle$ and $\langle \tau_1^2 \rangle$ (here $p = 2$). These two terms are distinguished by coupling to different combinations of the t_l parameters, namely $t_0 t_2$ and t_1^2 ; those combinations have different Λ dependence (30). Thus we need a matrix Λ which is no longer a multiple of the identity. If we go to higher orders in n with simply for Λ a multiple of the identity matrix, we obtain the values of the sum $\langle \tau_0 \tau_2 \rangle + \langle \tau_1^2 \rangle = \frac{1}{12}$, instead of that of the individual terms. We hope to be able to discuss higher marked points by extending the present approach to such cases.

Alternative approach. We shall now show that the duality (6) allows us to recover the same numbers from the one-point function of the closing gap problem. This is similar to the $p = 2$ case developed in previous articles [4,8].

The Fourier transform of the density of state, the one-point correlation function $U_A(s)$, is defined as

$$U_A(s) = \frac{1}{N} \int d\lambda e^{i\lambda s} \langle \text{tr} \delta(\lambda - M) \rangle_A, \tag{37}$$

which is known for arbitrary A and reduces for the choice $a_i = \pm 1$ (closing gap case) to

$$U_A(s) = \frac{1}{Ns} e^{Ns^2/2} \oint \frac{du}{2\pi i} \left(1 - \frac{s}{1-u}\right)^{\frac{N}{2}} \left(1 + \frac{s}{1+u}\right)^{\frac{N}{2}} e^{Nsu}. \tag{38}$$

In the range in which N is large and s of order $N^{-1/4}$, one can expand the integrand and the leading range is given by u 's of order $N^{-1/4}$ as well. Then

$$\begin{aligned} U_A(s) &= \frac{1}{Ns} e^{-\frac{N}{4}s^4} \int \frac{du}{2\pi i} e^{-Nsu^3 - \frac{3}{2}Ns^2u^2 - Ns^3u} \\ &= \frac{1}{Ns} \int \frac{du}{2\pi i} e^{-Nsu^3 - \frac{N}{4}s^3u}, \end{aligned} \tag{39}$$

where we have expanded the exponent up to order u^3 . The last equation has been obtained by the shift $u \rightarrow u - \frac{s}{2}$. The contour integral over u becomes an integral over the range $[-i\infty, +i\infty]$ in the large N limit. This integral is clearly related to an Airy function $Ai(x)$, defined by

$$Ai[\pm(3a)^{-\frac{1}{3}}x] = \frac{(3a)^{1/3}}{\pi} \int_0^\infty dt \cos(at^3 \pm xt). \tag{40}$$

We have thus

$$U_A(s) = \frac{1}{Ns(3Ns)^{\frac{1}{3}}} Ai(z) \tag{41}$$

with $z = -\frac{N}{4 \cdot 3^{\frac{1}{3}} N^{\frac{1}{3}}} s^{\frac{8}{3}}$. The expansion of the Airy function for small z gives the numbers

$$\begin{aligned} U_A(s) &= \frac{1}{Ns(3Ns)^{\frac{1}{3}} 3^{\frac{2}{3}} \Gamma(\frac{2}{3})} \left(1 + \frac{1}{3!} z^3 + \frac{1 \cdot 4}{6!} z^6 + \frac{1 \cdot 4 \cdot 7}{9!} z^9 + \dots\right) \\ &\quad - \frac{1}{Ns(3Ns)^{\frac{1}{3}} 3^{\frac{1}{3}} \Gamma(\frac{1}{3})} \left(z + \frac{2}{4!} z^4 + \frac{2 \cdot 5}{7!} z^7 + \frac{2 \cdot 5 \cdot 8}{10!} z^{10} + \dots\right) \end{aligned} \tag{42}$$

From the expansion (42), we find the intersection numbers of (36) up to a scaling factor. The first series of (42), in powers of z^{3k} , corresponds to the spin $j = 1$ case : $\langle \tau_{6,1} \rangle_{g=3} = \frac{1}{(12)^3 3! \cdot 3}$, $\langle \tau_{14,1} \rangle_{g=6} = \frac{1 \cdot 4}{(12)^6 6! \cdot 3^2}$, $\langle \tau_{22,1} \rangle_{g=9} = \frac{1 \cdot 4 \cdot 7}{(12)^9 9! \cdot 3^3}$, ... The second series in (42), in powers of z^{3k+1} corresponds to the spin $j=0$ intersection numbers, $\langle \tau_{1,0} \rangle_{g=1} = \frac{1}{12}$, $\langle \tau_{9,0} \rangle_{g=4} = \frac{2}{(12)^4 4! \cdot 3}$, $\langle \tau_{17,0} \rangle_{g=7} = \frac{2 \cdot 5}{(12)^7 7! \cdot 3^2}$, and $\langle \tau_{25,0} \rangle_{g=10} = \frac{2 \cdot 5 \cdot 8}{(12)^{10} 10! \cdot 3^3}$.

5. Conclusion

The present article, using both duality and replica, provides exact formulae for the intersection numbers of the moduli of curves on Riemann surfaces with one marked point and p -spin. The method may be extended, at least for low genera, to a higher number of marked points. It provides a relationship between Kontsevich’s Airy matrix model and Okounkov’s work of the edge of Wigner’s semi-circle for the $p = 2$ case. For the $p = 3$ case, it provides a link between a higher Airy matrix model to the critical gap closing model.

Appendix A: Gaussian Averages in the Presence of an External Matrix Source

For the sake of completeness we reproduce here the main steps of the derivation given in [7]. The probability distribution for Hermitian $n \times n$ matrices is thus

$$P_A(B) = \frac{1}{Z} e^{-\frac{\lambda}{2} \text{tr} B^2 + \text{tr} AB}. \tag{43}$$

Let us first compute the one point function

$$U_A(s) = \frac{1}{n} \langle \text{tr} e^{sB} \rangle. \tag{44}$$

Using the Itzykson-Zuber formula [11, 12] to integrate out the unitary degrees of freedom one obtains:

$$U_A(s) = \frac{1}{n} \frac{1}{Z \Delta(A)} \int \prod_1^n db_i \Delta(B) e^{\sum_1^n (-\frac{\lambda}{2} b_i^2 + a_i b_i)} \sum_1^n e^{s b_i}, \tag{45}$$

in which Z is a normalization constant fixed such as $U_A(0) = 1$ and $\Delta(B)$ is the Vander-Monde determinant of the eigenvalues $\Delta(B) = \prod_{i < j} (b_i - b_j)$. We now use repeatedly the trivial identity

$$\int \prod_1^n db_i \Delta(B) e^{\sum_1^n (-\frac{\lambda}{2} b_i^2 + a_i b_i)} = C_n e^{\sum_1^n \frac{1}{2\lambda} a_i^2} \Delta(A) \tag{46}$$

in which C_n is a simple constant, with a_i replaced by $\tilde{a}_i = a_i + s \delta_{i,\alpha}$ in which α takes the successive values 1 to n . This gives

$$\begin{aligned} U_A(s) &= \frac{1}{n} e^{\frac{s^2}{2\lambda}} \sum_1^n e^{s a_\alpha / \lambda} \frac{\Delta(\tilde{A}_\alpha)}{\Delta(A)} \\ &= \frac{1}{n} e^{\frac{s^2}{2\lambda}} \sum_\alpha e^{s a_\alpha / \lambda} \prod_{\beta \neq \alpha} \frac{a_\alpha - a_\beta + s}{a_\alpha - a_\beta} \end{aligned} \tag{47}$$

which may be replaced by the contour integral around the n points (a_1, \dots, a_n)

$$U_A(s) = \frac{1}{ns} e^{\frac{s^2}{2\lambda}} \oint \frac{du}{2i\pi} e^{su/\lambda} \prod_1^n \frac{u + s - a_\alpha}{u - a_\alpha}. \tag{48}$$

Note that this is an exact formula; it simplifies of course for the pure Gaussian case $a_\alpha = 0$, although the derivation does require a source matrix in the intermediate steps.

The derivation for the k-point functions follows exactly the same lines: the identity (46) requires now an \tilde{a} which carries k-indices. The result of the calculation is similar:

$$\begin{aligned}
 U_A(s_1, \dots, s_k) &= \frac{1}{n} \langle \text{tr} e^{s_1 B} \dots \text{tr} e^{s_k B} \rangle \\
 &= \frac{1}{n s_1 \dots s_k} e^{\sum_1^k \frac{s_i^2}{2\lambda}} \oint \prod_{i=1}^k \frac{du_i}{2i\pi} e^{\sum_{i=1}^k s_i u_i / \lambda} \prod_{i < j} \frac{(u_i - u_j + s_i - s_j)(u_i - u_j)}{(u_i - u_j + s_i)(u_i - u_j - s_j)} \\
 &\quad \times \prod_{i=1}^k \prod_{\alpha=1}^n \left(1 + \frac{s_i}{u_i - a_\alpha}\right). \tag{49}
 \end{aligned}$$

One can simplify the expression by noticing the Cauchy determinant identity

$$\det \frac{1}{x_i - y_j} = (-1)^{n(n-1)/2} \frac{\prod_{i < j} (x_i - x_j)(y_i - y_j)}{\prod_{i, j} (x_i - y_j)} \tag{50}$$

with $x_i = u_i + s_i$, $y_i = u_i$, namely

$$\det \frac{1}{u_i + s_i - u_j} = \frac{1}{s_1 \dots s_k} \prod_{i < j} \frac{(u_i - u_j + s_i - s_j)(u_i - u_j)}{(u_i - u_j + s_i)(u_i - u_j - s_j)}. \tag{51}$$

In the $n = 0$ limit we need to consider only the connected part of U_A (the disconnected ones vanishing with n, as is obvious from Feynman diagrams or explicit formulae) which correspond to connected permutations in the expansion of the determinant. Taking a vanishing source ($a_\alpha = 0$) the $n = 0$ limit gives immediately (19).

Appendix B: Duality

Let us consider the Gaussian average, with a matrix source A , of the product of characteristic determinants of the $N \times N$ Hermitian random matrices

$$\begin{aligned}
 F_k(\lambda_1, \dots, \lambda_k) &= \frac{1}{Z_N} \langle \prod_{\alpha=1}^k \det(\lambda_\alpha - M) \rangle_{A, M} \\
 &= \frac{1}{Z_N} \int dM \prod_{\alpha=1}^k \det(\lambda_\alpha \cdot I - M) e^{-\frac{1}{2} \text{Tr}(M - A)^2} \tag{52}
 \end{aligned}$$

in which $Z_N = \int dM \exp(-1/2 \text{Tr} M^2) = 2^{N/2} (\pi)^{N^2/2}$. The unitary invariance of the measure allows us to assume, without loss of generality, that A is a diagonal matrix with eigenvalues (a_1, \dots, a_N) . We introduce now $N \times k$ complex Grassmannian variables $(i = 1, 2, \dots, N$ and $\alpha = 1, 2, \dots, k)$ $(\bar{\psi}_i^\alpha, \psi_i^\alpha)$ ($i = 1, 2, \dots, N$ and $\alpha = 1, 2, \dots, k$) with the normalization

$$\int d\bar{\psi} d\psi \begin{pmatrix} 1 \\ \bar{\psi} \psi \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{53}$$

Then one may write

$$\prod_{i=1}^k \det(\lambda_i \cdot \mathbf{I} - \mathbf{M}) = \int \prod d\bar{\psi}_i^\alpha d\psi_i^\alpha e^{\sum_{i,j=1}^N \sum_{\alpha=1}^k \bar{\psi}_i^\alpha (\lambda_\alpha - M)_{ij} \psi_j^\alpha}, \quad (54)$$

and perform the Gaussian average

$$\frac{1}{Z_N} \int dM e^{-\frac{1}{2} \text{Tr} M^2 + \text{Tr} M X} = e^{\frac{1}{2} \text{Tr} X^2}. \quad (55)$$

We deal here with a matrix X given by

$$X_{pq} = a_p \delta_{pq} - \sum_{\alpha=1}^k \bar{\psi}_q^\alpha \psi_p^\alpha \quad (56)$$

and thus

$$\text{Tr} X^2 = \text{Tr} A^2 - 2 \sum_{p=1}^N \sum_{\alpha=1}^k a_p \bar{\psi}_p^\alpha \psi_p^\alpha - \sum_{\alpha, \beta=1, \dots, k} \gamma_{\alpha, \beta} \gamma_{\beta, \alpha}, \quad (57)$$

where

$$\gamma_{\alpha, \beta} = \sum_{i=1}^N \bar{\psi}_i^\alpha \psi_i^\beta. \quad (58)$$

To make the notations more transparent let us denote by ‘Tr’ the trace on the initial space of $N \times N$ matrices and by ‘tr’ the trace in the new space of $k \times k$ matrices; the last term in (57) for $\text{Tr} X^2$ is thus $-\text{tr} \gamma^2$, the minus sign being due to the anticommuting nature of the psi’s. We then introduce an auxiliary Hermitian $k \times k$ matrix β so that

$$e^{-\frac{1}{2} \text{tr} \gamma^2} = \frac{1}{Z_k} \int d\beta e^{-\frac{1}{2} \text{tr} \beta^2 + i \text{tr} \gamma \beta}. \quad (59)$$

This leads us to the representation

$$F_k(\lambda_1, \dots, \lambda_k) = \frac{1}{Z_k} \int \prod d\bar{\psi}_i^\alpha d\psi_i^\alpha \times \int d\beta e^{-\frac{1}{2} \text{tr} \beta^2 + i \text{tr} \gamma (\beta - i\Lambda)} e^{-\sum_{i=1}^N \sum_{\alpha=1}^k a_i \bar{\psi}_i^\alpha \psi_i^\alpha} \quad (60)$$

in which Λ is the $k \times k$ diagonal matrix with eigenvalues $(\lambda_1, \dots, \lambda_k)$. One can now easily perform the integration over the Grassmanian variables since the exponent is quadratic in those variables. This yields the announced dual representation

$$F_k(\lambda_1, \dots, \lambda_k) = \frac{1}{Z_k} \int d\beta e^{-\frac{1}{2} \text{tr} \beta^2} \prod_{j=1}^N \det[(\lambda_\rho - a_j) \delta_{\rho, \sigma} + i \beta_{\rho, \sigma}] \quad (61)$$

in which the matrices are now $k \times k$.

Remark. This duality takes a more symmetric form if we choose a pure imaginary diagonal matrix A , and if we shift the matrix β by $i\Lambda$ in the final formula (61). Then the duality reads

$$\begin{aligned} & \frac{1}{Z_N} \int dM \prod_{\alpha=1}^k \det(\lambda_i \cdot \mathbf{I} - M) e^{-\frac{1}{2} \text{Tr}(M-iA)^2} \\ &= (-i)^{Nk} \frac{1}{Z_k} \int d\beta e^{-\frac{1}{2} \text{tr}(\beta+i\Lambda)^2} \prod_{j=1}^N \det(a_j \delta_{\rho,\sigma} - \beta_{\rho,\sigma}). \end{aligned} \tag{62}$$

This duality may be extended to a two-matrix model when the measure is an exponential of a quadratic form in the two matrices M and M' , namely $\exp(a \text{Tr} M^2 + b \text{Tr} M'^2 + c \text{Tr} M M')$.

Appendix C: Contour integrals in the $n = 0$ limit

We return to the contour integral (19) which gives the $n = 0$ limit of the k -point function. Let us consider for simplicity the case of the two-point function. We are then dealing with the sum of two integrals:

$$\begin{aligned} \lim_{n \rightarrow 0} U(s_1, s_2) &= -e^{\frac{s_1^2+s_2^2}{2\lambda}} \oint \prod_1^2 \frac{du_i}{2i\pi} e^{\sum_1^2 (u_i s_i / \lambda)} [\log(1 + \frac{s_1}{u_1}) + \log(1 + \frac{s_2}{u_2})] \\ &\quad \times \det \frac{1}{u_i + s_i - u_j}. \end{aligned} \tag{63}$$

The continuation to $n = 0$ requires to take contours in the u_1 and u_2 planes which circle around the respective cuts $[-s_1, 0]$ and $[-s_2, 0]$. We have to choose some well-defined contours on the two variables before we can write the integral (63) as a sum of two integrals. For instance we choose an integral over a large contour in the u_2 plane, and close to the cut in the u_1 plane. The disconnected term $1/s_1 s_2$ of the determinant gives a vanishing contribution. Thus

$$\begin{aligned} U(s_1, s_2) &= e^{\frac{s_1^2+s_2^2}{2\lambda}} \oint \prod_1^2 \frac{du_i}{2i\pi} e^{\sum_1^2 (u_i s_i / \lambda)} [\log(1 + \frac{s_1}{u_1}) + \log(1 + \frac{s_2}{u_2})] \\ &\quad \times \frac{1}{(u_1 + s_1 - u_2)(u_2 + s_2 - u_1)}. \end{aligned} \tag{64}$$

The second integral, the one which involves $\log(1 + \frac{s_2}{u_2})$, vanishes for the choice of contours that we have made, since we can integrate over u_1 first and there are no poles inside the contour. For the first part, that with $\log(1 + \frac{s_1}{u_1})$, we integrate over u_2 first, pick up the two poles and find

$$U(s_1, s_2) = e^{\frac{s_1^2+s_2^2}{2\lambda}} \frac{1}{\sigma} (\epsilon^{-s_2^2/\lambda} - e^{s_1 s_2/\lambda}) \oint \frac{du_1}{2i\pi} e^{u_1 \sigma/\lambda} \log(1 + \frac{s_1}{u_1}) \tag{65}$$

with $\sigma = s_1 + s_2$. Since

$$\oint \frac{du_1}{2i\pi} e^{u_1 \sigma/\lambda} \log(1 + \frac{s_1}{u_1}) = - \int_{s_1}^0 dx e^{x\sigma/\lambda} = -\frac{\lambda}{\sigma} (1 - e^{-s_1 \sigma/\lambda}), \tag{66}$$

we end up with

$$U(s_1, s_2) = 4 \frac{\lambda}{\sigma^2} \sinh \frac{\sigma s_1}{2\lambda} \sinh \frac{\sigma s_2}{2\lambda}. \quad (67)$$

This calculation may be repeated for the k -point function, although the combinatorics becomes heavy. This yields the final result (theorem in (20)),

$$\lim_{n \rightarrow 0} U(s_1, s_2, \dots, s_k) = \frac{\lambda}{\sigma^2} \prod_1^k 2 \sinh \frac{s_i \sigma}{2\lambda}, \quad (68)$$

where $\sigma = s_1 + s_2 + \dots + s_k$.

It is straightforward to generalize this formula to the case of a non-zero external matrix source A with eigenvalues (a_1, \dots, a_n) . We assume that the density of eigenvalues

$$\rho(x) = \frac{1}{n} \sum_1^n \delta(x - a_i) \quad (69)$$

has a finite limit $\rho_0(x)$ when n goes to zero. Up to now we have dealt with $\rho_0(x) = \delta(x)$, but we could take other examples such as $n/2$ eigenvalues equal to $+a$ and $n/2$ equal to $-a$. Then we would deal with $\rho_0(x) = \frac{1}{2}[\delta(x - a) + \delta(x + a)]$. The derivation goes through now in the same way: we write

$$\prod_1^n \left(1 + \frac{s}{u - a_\alpha}\right) = e^{n \int dx \rho(x) \log \left(1 + \frac{s}{u-x}\right)} \quad (70)$$

which may be expanded when n goes to zero provided, as we have assumed, that $\rho(x)$ has a limit. The result is then simply

$$\lim_{n \rightarrow 0} U_A(s_1, s_2, \dots, s_k) = \lim_{n \rightarrow 0} U(s_1, s_2, \dots, s_k) \int dx \rho_0(x) e^{x\sigma/\lambda}, \quad (71)$$

in which the first factor U is given again by (68).

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References

1. Witten, E.: Two dimensional gravity and intersection theory on moduli space. *Surv. Differ. Geom.* **1**, 243 (1991)
2. Witten, E.: Algebraic geometry associated with matrix models of two dimensional gravity. In: *Topological methods in modern mathematics (Stony Brook, NY, 1991)*, Houston, TX: Publish or Perish, 1993, p.235
3. Kontsevich, M.: Intersection Theory on the moduli Space of Curves and Matrix Airy Function. *Commun. Math. Phys.* **147**, 1 (1992)
4. Brézin, E., Hikami, S.: Vertices from replica in a random matrix theory. *J. Phys. A* **40**, 13545 (2007)
5. Brézin, E., Hikami, S.: Correlations of nearby levels induced by a random potential. *Nucl. Phys. B* **479**, 697 (1996)
6. Brézin, E., Hikami, S.: Extension of level-spacing universality. *Phys. Rev.* **E56**, 264 (1997)
7. Brézin, E., Hikami, S.: Spectral form factor in a random matrix theory. *Phys. Rev.* **E55**, 4067 (1997)
8. Okounkov, A., Pandharipande, R.: *Gromov-Witten theory, Hurwitz numbers, and Matrix models*, I. <http://arxiv.org/list/math.AG/0101147>, 2001

9. Okounkov, A.: Generating functions for the intersection numbers on moduli spaces of curves. Intern. Math. Research. Notices **18**, 933 (2002)
10. Okounkov, A.: Random matrices and random permutations. Intern. Math. Research. Notices **20**, 1043 (2000)
11. Harish-Chandra.: Proc. Nat. Acad. Sci. **42**, 252 (1956)
12. Itzykson, C., Zuber, J.-B.: The planar approximation II. J. Math. Phys. **21**, 411 (1980)
13. Brézin, E., Hikami, S.: Characteristic polynomials of random matrices. Commun. Math. Phys. **214**, 111–135 (2000)
14. Hashimoto, A., Min-xin Huang, Klemm, A., Shih, D.: Open/closed string duality for topological gravity with matter. JHEP **05**, 007 (2005)
15. Kazakov, V.A.: External matrix field problem and new multicriticalities in (-2)-dimensional random surfaces. Nucl. Phys. B **354**, 614 (1991)
16. Itzykson, C., Zuber, J.-B.: Combinatorics of the Modular Group II. The Kontsevich Integrals. Int. J. Mod. Phys. **A7**, 5661 (1992)
17. Liu, K., Xu, H.: New properties of the intersection numbers on moduli spaces of curves. Math. Res. Lett. **14**, 1041–1054 (2007)
18. Brézin, E., Hikami, S.: Universal singularity at the closure of a gap in a random matrix theory. Phys. Rev. **E57**, 4140 (1998)
19. Brézin, E., Hikami, S.: Level spacing of random matrices in an external source. Phys. Rev. **E58**, 7176 (1998)
20. Jarvis, T.J., Kimura, T., Vaintrob, A.: Moduli Spaces of Higher Spin Curves and Integrable Hierarchies. Comp. Math. **126**, 157–212 (2001)
21. Adler, M., van Moerbeke, P.: A matrix integral solution to two-dimensional Wp-gravity. Commun. Math. Phys. **147**, 25–56 (1992)
22. Shadrin, S.: Geometry of meromorphic functions and intersections on moduli spaces of curves. Int. Math. Res. Not. **38**, 2051 (2003); Faber, C., Shadrin, S., Zvonkine, D.: *Tautological relations and the r-spin Witten conjecture*. <http://arxiv.org/list/math/0612510>, 2006

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