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Wegner Bounds for a Two-Particle Tight Binding Model

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Abstract: We consider a quantum two-particle system on a lattice $\mathbb{Z}^d$ with interaction and in presence of an IID external potential. We establish Wegner-type estimates for such a model. The main tool used is Stollmann's lemma.

1. Introduction. The Results

This paper considers a two-particle Anderson tight binding model on lattice $\mathbb{Z}^d$ with interaction. The Hamiltonian $H\left(= H^{(2)}_{U,V}(\omega)\right)$ is a lattice Schrödinger operator (LSO) of the form $H^0 + U + V_1 + V_2$ acting on functions $\phi \in \ell_2(\mathbb{Z}^d \times \mathbb{Z}^d)$:

$$\begin{aligned} H\phi(\mathbf{x}) &= H^0\phi(\mathbf{x}) + [(U + V_1 + V_2)\,\phi]\,(\mathbf{x}) \\ &= \sum_{\mathbf{y}:\,\|\mathbf{y}-\mathbf{x}\|=1} \phi(\mathbf{y}) + \left(U(\mathbf{x}) + \textstyle\sum_{j=1}^2 V(x_j;\omega)\right)\phi(\mathbf{x}), \\ &\qquad \mathbf{x} = (x_1, x_2),\ \mathbf{y} = (y_1, y_2) \in \mathbb{Z}^d \times \mathbb{Z}^d. \end{aligned} \tag{1.1}$$

Here, $x_j = \left(x_j^{(1)}, \dots, x_j^{(d)}\right)$ and $y_j = \left(y_j^{(1)}, \dots, y_j^{(d)}\right)$ stand for coordinate vectors of the j^{th} particle in $\mathbb{Z}^d$, $j = 1, 2$, and $\|\cdot\|$ is the sup-norm in $\mathbb{R}^d \times \mathbb{R}^d$:

$$\|\mathbf{x}\| = \max_{j=1,2}\ \max_{i=1,\dots,d} \left|x_j^{(i)}\right|, \quad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d.$$

The same notation, $\|\cdot\|$, is used for the sup-norm in $\mathbb{R}^d$; this should not lead to confusion.

We will use boldface notations, like $\mathbf{x}$, for points in $\mathbb{Z}^d \times \mathbb{Z}^d$ describing positions of the two-particle system. In Sect. 2, boldface notations are used for vectors in an auxiliary Euclidean space $\mathbb{R}^p$.

Throughout this paper, the random external potential $V(x;\omega)$, $x \in \mathbb{Z}^d$, is assumed to be real IID, with a common distribution function F on $\mathbb{R}$. It can be quite arbitrary, although in many applications it is assumed at least continuous. In any case, the continuity modulus of a given non-decreasing function F can be defined by

$$s(\epsilon)\ (= s(F,\epsilon)) := \sup_{a\in\mathbb{R}}\ (F(a+\epsilon) - F(a-0)),\ \ \epsilon > 0, \tag{1.2}$$

where $F(a-0) := \lim_{\delta\uparrow 0} F(a-\delta)$. Naturally, $s(\epsilon) \le 1$ for any probability distribution function F.

In physically interesting models, the interaction potential U cannot be completely arbitrary. It is usually assumed to be symmetric ($U(x_1, x_2) = U(x_2, x_1)$) or even translation invariant ($U(x_1, x_2) = \widetilde{U}(\|x_1 - x_2\|)$), and sufficiently rapidly decaying as $\|x_1 - x_2\| \to \infty$. However, in the present paper we treat only finite-volume particle systems, and such assumptions are no longer imperative for the operator $H^0 + V(x_1;\omega) + V(x_2;\omega) + U(x_1, x_2)$ to be well-defined in a finite volume. It suffices, e.g., to assume $U(\mathbf{x})$ to be locally finite, so that the operator of multiplication by the function $U(\mathbf{x})$, restricted to Hilbert space $\ell_2(\Lambda)$ with $|\Lambda| :=$ card $\Lambda < \infty$, is bounded. Even this assumption can be relaxed so as to include the case of hard-core interactions, where $U(x_1, x_2) = +\infty$ for (x_1, x_2) with $\|x_1 - x_2\| \le r_0 < \infty$, and $U(\mathbf{x})$ is (at least locally) bounded for all other $\mathbf{x} = (x_1, x_2)$. The latter case (hard-core interactions) would require certain technical modifications of notations and arguments, while our main results given in Theorems 1, 2 and 3 below would essentially remain valid.

We do assume, however, symmetry of the function $U(x_1, x_2)$, having in mind future applications to quantum systems under Bose-Einstein or Fermi-Dirac quantum statistics. Again, it is worth mentioning that such an assumption is not imperative for our results to be valid; only the general setup would need to be modified. We plan to address various possible generalisations in a separate paper.

Therefore, below we always assume that $U(\mathbf{x})$ is a fixed (non-random), symmetric, locally bounded function on $\mathbb{Z}^d \times \mathbb{Z}^d$.

The purpose of this paper is to establish the so-called Wegner-type estimates for H. More precisely, these estimates are produced for the eigen-values of a finite-volume approximation $H_\Lambda \left(= H^{(2)}_{\Lambda,U,V}(\omega)\right)$ acting in $\ell_2(\Lambda)$:

$$\begin{aligned} H_\Lambda\phi(\mathbf{x}) &= H^0_\Lambda\phi(\mathbf{x}) + \left[(U + V_1 + V_2)_\Lambda\,\phi\right](\mathbf{x}) \\ &= \sum_{\mathbf{y}\in\Lambda:\ \|\mathbf{y}-\mathbf{x}\|=1} \phi(\mathbf{y}) + \left(U(\mathbf{x}) + \textstyle\sum_{j=1}^{2} V(x_j;\omega)\right)\phi(\mathbf{x}), \\ &\qquad \mathbf{x} = (x_1, x_2),\ \ \mathbf{y} = (y_1, y_2) \in \Lambda. \end{aligned} \tag{1.3}$$

Here $\Lambda \subset \mathbb{Z}^d \times \mathbb{Z}^d$ is a finite set of cardinality $|\Lambda|$. For definiteness, we will focus on the case where Λ is specified as a $\mathbb{Z}^d \times \mathbb{Z}^d$ lattice parallelepiped written as the Cartesian product of two $\mathbb{Z}^d$ lattice cubes centred at points $u_1 = \left(u_1^{(1)}, \ldots, u_1^{(d)}\right) \in \mathbb{Z}^d$ and $u_2 = \left(u_2^{(1)}, \ldots, u_2^{(d)}\right) \in \mathbb{Z}^d$:

$$\left[\left(\mathop{\times}_{i=1}^{d}\left[-L_1 + u_1^{(i)}, u_1^{(i)} + L_1\right]\right) \times \left(\mathop{\times}_{i=1}^{d}\left[-L_2 + u_2^{(i)}, u_2^{(i)} + L_2\right]\right)\right] \cap \left(\mathbb{Z}^d \times \mathbb{Z}^d\right). \tag{1.4}$$

Here $\mathbb{L} := (L_1, L_2) \in \mathbb{N}^2$. Notice that the above lattice subset is non-empty even for $\mathbb{L} = 0$, and its diameter equals $2\|\mathbb{L}\|$.

A set Λ of the form (1.4) will be called a box and denoted by $\Lambda_{\mathbb{L}}(\mathbf{u})$, $\mathbf{u} = (u_1, u_2) \in \mathbb{Z}^d \times \mathbb{Z}^d$, while the $\mathbb{Z}^d$ lattice cubes figuring in (1.4) as the Cartesian factors will be denoted by $\Pi_1 \Lambda_{\mathbb{L}}(\mathbf{u})$ and $\Pi_2 \Lambda_{\mathbb{L}}(\mathbf{u})$:

$$\Pi_j \Lambda_{\mathbb{L}}(\mathbf{u}) = \left(\mathop{\times}_{i=1}^{d} \left[-L_j + u_j^{(i)}, u_j^{(i)} + L_j \right] \right) \cap \mathbb{Z}^d, \quad j = 1, 2. \tag{1.5}$$

We will also call cubes $\Pi_1 \Lambda_{\mathbb{L}}(\mathbf{u})$ and $\Pi_2 \Lambda_{\mathbb{L}}(\mathbf{u})$ the projections of $\Lambda_{\mathbb{L}}(\mathbf{u})$ and set

$$\Pi \Lambda_{\mathbb{L}}(\mathbf{u}) = \Pi_1 \Lambda_{\mathbb{L}}(\mathbf{u}) \cup \Pi_2 \Lambda_{\mathbb{L}}(\mathbf{u}). \tag{1.6}$$

The cardinality of box $\Lambda_{\mathbb{L}}(\mathbf{u})$ is denoted by $|\Lambda_{\mathbb{L}}(\mathbf{u})|$ and the cardinality of cube $\Pi_j \Lambda_{\mathbb{L}}(\mathbf{u})$ by $\left|\Pi_j \Lambda_{\mathbb{L}}(\mathbf{u})\right|$. Symbol $\mathbb{P}$ will stand for the probability distribution generated by random variables $V(x; \omega)$, $x \in \mathbb{Z}^d$. Symbol $\mathfrak{B}\left[\Pi \Lambda_{\mathbb{L}}(\mathbf{u})\right]$ is used for the sigma-algebra generated by random variables

$$\omega \mapsto V(x; \omega), \quad x \in \Pi \Lambda_{\mathbb{L}}(\mathbf{u}). \tag{1.7}$$

Remark. Working with projections of different sizes may appear artificial. Indeed, in this paper this only allows to make assertions of Theorems 1, 2 and 3 slightly more general. However, having in mind future applications to quantum systems under Bose-Einstein or Fermi-Dirac statistics, it is preferable to allow projections of boxes to be of different sizes.

The spectrum $\Sigma\left(H_{\Lambda_{\mathbb{L}}(\mathbf{u})}\right)$ of $H_{\Lambda_{\mathbb{L}}(\mathbf{u})}$ is a random subset of $\mathbb{R}$ consisting of $|\Lambda_{\mathbb{L}}(\mathbf{u})|$ points (not necessarily distinct) $\lambda^{(k)}_{\Lambda_{\mathbb{L}}(\mathbf{u})} (= \lambda^{(k)}_{\Lambda_{\mathbb{L}}(\mathbf{u})}(\omega))$, $k = 1, \ldots, |\Lambda_{\mathbb{L}}(\mathbf{u})|$ (random eigen-values in volume $\Lambda_{\mathbb{L}}(\mathbf{u})$, measurable with respect to $\mathfrak{B}\left[\Pi \Lambda_{\mathbb{L}}(\mathbf{u})\right]$). Given a value $E \in \mathbb{R}$, we denote

$$\operatorname{dist}\left[\Sigma\left(H_{\Lambda_{\mathbb{L}}(\mathbf{u})}\right), E\right] = \min \left\{ \left| E - \lambda^{(k)}_{\Lambda_{\mathbb{L}}(\mathbf{u})} \right| : k = 1, \ldots, |\Lambda_{\mathbb{L}}(\mathbf{u})| \right\}. \tag{1.8}$$

Our first result in this paper is the so-called single-volume Wegner bound given in Theorem 1.

Theorem 1. *$\forall$ $E \in \mathbb{R}$, $\mathbb{L} \in \mathbb{N}^2$, $\mathbf{u} \in \mathbb{Z}^d \times \mathbb{Z}^d$ and $\epsilon > 0$,*

$$\mathbb{P}\left(\operatorname{dist}\left[\Sigma\left(H_{\Lambda_{\mathbb{L}}(\mathbf{u})}\right), E\right] \le \epsilon\right) \le |\Lambda_{\mathbb{L}}(\mathbf{u})| \min_{j=1,2} \left\{ \left|\Pi_j \Lambda_{\mathbb{L}}(\mathbf{u})\right| \right\} \cdot s(2\epsilon). \tag{1.9}$$

Single-volume Wegner-type bounds were often used in the (single-particle) Anderson localisation theory; see, e.g., original papers by Fröhlich, Martinelli, Scoppola and Spencer [4], and by von Dreifus and Klein [3].

In Theorem 2 below we deal with a two-volume Wegner bound. This bound assesses the probability that the random spectra $\Sigma\left(H_{\Lambda_{\mathbb{L}}(\mathbf{u})}\right)$ and $\Sigma\left(H_{\Lambda_{\mathbb{L}'}(\mathbf{u}')}\right)$ are close to each other, for a pair of boxes $\Lambda_{\mathbb{L}}(\mathbf{u})$ and $\Lambda_{\mathbb{L}'}(\mathbf{u}')$ positioned away from each other. It is worth mentioning that, in the conventional, single-particle localisation theory, such a bound can be derived from its single-volume counterpart (e.g., for IID random potentials). See [4] and [3] for details. With $N > 1$ particles, it requires additional arguments. Indeed, an important feature of two-particle operators is that the potential $W(u_1, u_2; \omega) = U(u_1, u_2) + V(u_1; \omega) + V(u_2; \omega)$ is a symmetric function of the pair $(u_1, u_2) \in \mathbb{Z}^d \times \mathbb{Z}^d$. Namely, let $S : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{Z}^d \times \mathbb{Z}^d$ be the following map:

$$S : (u_1, u_2) \mapsto (u_2, u_1).$$

Then the potential energy operator $W(\mathbf{x}) = U(\mathbf{x}) + V(x_1) + V(x_2)$ satisfies $W(S(\mathbf{x})) \equiv W(\mathbf{x})$. As a consequence, the spectra of operators H_Λ and $H_{S(\Lambda)}$ are identical, although the distance $\operatorname{dist}[\Lambda, S(\Lambda)]$ may be arbitrarily large.

We define the distance between two spectra in a usual way:

$$\begin{aligned} &\operatorname{dist}\left[\Sigma\left(H_{\Lambda_{\mathbb{L}}(\mathbf{u})}\right), \Sigma\left(H_{\Lambda_{\mathbb{L}'}(\mathbf{u}')}\right)\right] \\ &\quad = \min\left\{ \left|\lambda^{(k)}_{\Lambda_{\mathbb{L}}(\mathbf{u})} - \lambda^{(k')}_{\Lambda_{\mathbb{L}'}(\mathbf{u}')}\right|, 1 \le k \le |\Lambda_{\mathbb{L}}(\mathbf{u})|, 1 \le k' \le \left|\Lambda_{\mathbb{L}'}(\mathbf{u}')\right| \right\}. \end{aligned} \tag{1.10}$$

Theorem 2. $\forall\, \mathbb{L} = (L_1, L_2), \mathbb{L}' = (L'_1, L'_2) \in \mathbb{N}^2$, $\mathbf{u}, \mathbf{u}' \in \mathbb{Z}^d \times \mathbb{Z}^d$ *with*

$$\min\left\{ \|\mathbf{u} - \mathbf{u}'\|, \|\mathbf{u} - S(\mathbf{u}')\| \right\} > 8 \max\{L_1, L_2, L'_1, L'_2\} \tag{1.11}$$

and $\forall\, \epsilon > 0$, *the following inequality holds:*

$$\begin{aligned} &\mathbb{P}\left(\operatorname{dist}\left[\Sigma\left(H_{\Lambda_{\mathbb{L}}(\mathbf{u})}\right), \Sigma\left(H_{\Lambda_{\mathbb{L}'}(\mathbf{u}')}\right)\right] \le \epsilon\right) \\ &\quad \le |\Lambda_{\mathbb{L}}(\mathbf{u})| \left|\Lambda_{\mathbb{L}'}(\mathbf{u}')\right| \max_{j=1,2} \max_{\mathbf{u}'' \in \{\mathbf{u}, \mathbf{u}'\}} \left|\Pi_j \Lambda_{\mathbb{L}}(\mathbf{u}'')\right| s(2\epsilon). \end{aligned} \tag{1.12}$$

The assertions of Theorems 1 and 2 are proved in the next section of the paper, with the help of the so-called Stollmann's lemma. They are useful in the spectral analysis of H and $H_{\Lambda_{\mathbb{L}}(\mathbf{u})}$.

Throughout the paper, the symbol □ is used to mark the end of a proof.

2. Stollmann's Lemma. Proof of Theorems 1 and 2

2.1. Stollmann's lemma and its use. For the reader's convenience, we provide here the statement of Stollmann's lemma; see Lemma 2.1 below, cf. [6] and [7], Lemma 2.3.1. Let Γ be a non-empty finite set of cardinality $|\Gamma| = p$. We assume that Γ is ordered and identify it with the set $\{1, 2, \ldots, p\}$. Consider the Euclidean space $\mathbb{R}^\Gamma \cong \mathbb{R}^p$ with standard basis $(\mathbf{e}_1, \ldots, \mathbf{e}_p)$, and its positive orthant

$$\mathbb{R}^\Gamma_+ = \left\{ \mathbf{q} = (q_1, \ldots, q_p) \in \mathbb{R}^\Gamma : \, q_j \ge 0, \;\; j = 1, \ldots, p \right\}.$$

We believe that the use of boldface notations for vectors $\mathbf{q} \in \mathbb{R}^\Gamma$, in this section, should not lead to confusion.

For a given probability measure μ on $\mathbb{R}$, denote by μ^Γ the product measure $\mu \times \cdots \times \mu$ on $\mathbb{R}^\Gamma$ and by $\mu^{\Gamma \setminus \{1\}}$ be the marginal product measure induced by μ^Γ on $\mathbb{R}^{\Gamma \setminus \{1\}}$. Next, $\forall\, \epsilon > 0$ set

$$s(\mu, \epsilon) = \sup_{a \in \mathbb{R}} \; \mu\left([a, a + \epsilon]\right). \tag{2.1}$$

Definition 2.1. *A function* $\Phi : \mathbb{R}^\Gamma \to \mathbb{R}$ *is called diagonally-monotone (DM) if it satisfies the following conditions:*

(i) $\forall\, \mathbf{r} \in \mathbb{R}^\Gamma_+$ *and any* $\mathbf{v} \in \mathbb{R}^\Gamma$,

$$\Phi(\mathbf{v} + \mathbf{r}) \ge \Phi(\mathbf{v}); \tag{2.2}$$

(ii) *moreover, with vector* $\mathbf{e} = \mathbf{e}_1 + \ldots + \mathbf{e}_p \in \mathbb{R}^\Gamma$, $\forall\, \mathbf{v} \in \mathbb{R}^\Gamma$ *and* $t > 0$,

$$\Phi(\mathbf{v} + t\mathbf{e}) - \Phi(\mathbf{v}) \ge t. \tag{2.3}$$

Lemma 2.1. *Suppose function $\Phi : \mathbb{R}^\Gamma \to \mathbb{R}$ is DM. Then $\forall\, \epsilon > 0$ and any open interval $I \subset \mathbb{R}$ of length ϵ,*

$$\mu^\Gamma\{\mathbf{v} : \Phi(\mathbf{v}) \in I\} \le |\Gamma| \cdot s(\mu, \epsilon). \tag{2.4}$$

The proof of this lemma can be found in the book by P. Stollmann [7], as well as in his original paper [6].

In our situation, it is also convenient to introduce the notion of a DM operator family. As before, Γ is a finite set, $|\Gamma| = p < \infty$, identified with $\{1, \dots, p\}$, so that $\mathbb{R}^\Gamma \cong \mathbb{R}^p$.

Definition 2.2. *Let $\mathcal{H}$ be a Hilbert space of a finite dimension m. A family of Hermitian operators $B(\mathbf{v}) : \mathcal{H} \to \mathcal{H}$, $\mathbf{v} \in \mathbb{R}^\Gamma$, $|\Gamma| = p < \infty$, is called DM if*

(i) $\forall\, \mathbf{r} \in \mathbb{R}_+^\Gamma \ \forall\, \mathbf{v} \in \mathbb{R}^\Gamma$,

$$B(\mathbf{v} + \mathbf{r}) \ge B(\mathbf{v}) \tag{2.5A}$$

(*in the sense of quadratic forms*).

(ii) $\forall\, f \in \mathcal{H}$

$$(B(\mathbf{v} + t \cdot \mathbf{e})f, f) - (B(\mathbf{v})f, f) \ge t \cdot \|f\|^2. \tag{2.5B}$$

That is, $\forall\, f \in \mathcal{H}$ with $\|f\| = 1$, the function $\Phi_f : \mathbb{R}^\Gamma \to \mathbb{R}$ defined by $\Phi_f(\mathbf{v}) = (B(\mathbf{v})f, f)$ is DM.

The importance of Stollmann's Lemma 2.1 in spectral theory of random operators is illustrated by the following two elementary observations.

Remark 2.1. Suppose that $B(\mathbf{v})$, $\mathbf{v} \in \mathbb{R}^\Gamma$, is a DM operator family in $\mathcal{H}$. Let $E^{(1)}_{B(\mathbf{v})} \le \dots \le E^{(m)}_{B(\mathbf{v})}$ be the eigen-values of $B(\mathbf{v})$. Then, by virtue of the variational principle, $\forall\, k = 1, \dots, m$, $\mathbf{v} \mapsto E^{(k)}_{B(\mathbf{v})}$ is a DM function.

Remark 2.2. If $B(\mathbf{v})$, $\mathbf{v} \in \mathbb{R}^\Gamma$, is a DM operator family in $\mathcal{H}$, and $K : \mathcal{H} \to \mathcal{H}$ is an arbitrary Hermitian operator, then the family $K + B(\mathbf{v})$ is also DM.

The arbitrariness of operator K in Remark 2.2 illustrates the power of Stollmann's lemma. In the context of multi-particle lattice quantum systems, it allows to consider fairly general kinetic energy operators H^0 and non-random interactions U.

For a single-particle tight binding model with non-IID random potential, similar results are presented in [1].

2.2. Proof of Theorem 1. The proof is a straightforward application of Lemma 2.1 and Remarks 2.1 and 2.2, cf. the proof of Theorems 2.3.2 and 2.3.3 in [2]. In our situation, the set Γ is identified as the set of smallest cardinality among $\Pi_1 \Lambda_{\mathbb{L}}(\mathbf{u})$ and $\Pi_2 \Lambda_{\mathbb{L}}(\mathbf{u})$, with $p = |\Gamma| = \min\{|\Pi_1 \Lambda_{\mathbb{L}}(\mathbf{u})|, |\Pi_2 \Lambda_{\mathbb{L}}(\mathbf{u})|\}$. (If both projections have equal cardinality, we can pick $\Gamma = \Pi_1 \Lambda_{\mathbb{L}}(\mathbf{u})$, for the sake of definiteness.) Vector $\mathbf{v}$ is identified with a collection $\{V(x;\omega),\ x \in \Gamma\}$ of sample values of the external potential; to stress this fact we will write

$$\mathbf{v} \sim \{V(x;\omega),\ x \in \Gamma\}. \tag{2.6}$$

Next, probability measure μ represents the distribution of a single value, say $V(0;\ \cdot\ )$, and product-measure μ^{Γ} is identified as $\mathbb{P}_{\Pi_i \Lambda_{\mathbb{L}}(\mathbf{u})}$, where the value $i \in \{1, 2\}$ is chosen so that $\Pi_i \Lambda_{\mathbb{L}}(\mathbf{u})$ has smallest cardinality. Further, the Hilbert space $\mathcal{H}$ in Remarks 2.1 and 2.2 is $\ell_2(\Lambda_{\mathbb{L}}(\mathbf{u}))$, of dimension $m = |\Lambda_{\mathbb{L}}(\mathbf{u})|$, in which the action of operator $H_{\Lambda_{\mathbb{L}}(\mathbf{u})}$ is considered. Given $\mathbf{x} = (x_1, x_2) \in \Lambda_{\mathbb{L}}(\mathbf{u})$, we can write

$$\begin{aligned} V(x_1; \omega) + V(x_2; \omega) &= \sum_{y \in \Pi \Lambda_{\mathbb{L}}(\mathbf{u})} c(\mathbf{x}, y) V(y; \omega) \\ &= \sum_{y \in \Gamma} c(\mathbf{x}, y) V(y; \omega) + \sum_{y \in \Pi \Lambda_{\mathbb{L}}(\mathbf{u}) \setminus \Gamma} c(\mathbf{x}, y) V(y; \omega), \end{aligned}$$

where $c(\mathbf{x}, y)$ is defined as a function of $y \in \Pi \Lambda_{\mathbb{L}}(\mathbf{u})$, for every $\mathbf{x} \in \Lambda_{\mathbb{L}}(\mathbf{u})$, by

$$c(\mathbf{x}, y) = \delta_{y,x_1} + \delta_{y,x_2},$$

so that, obviously, $\forall\, \mathbf{x} \in \Lambda_{\mathbb{L}}(\mathbf{u})\, \forall\, y \in \Gamma$,

$$c(\mathbf{x}, y) \geq 1.$$

Now we can re-write the external random potential $V(x_1; \omega) + V(x_2; \omega)$ as follows:

$$V(x_1; \omega) + V(x_2; \omega) = V_{\Gamma}(\mathbf{x}; \omega) + \widetilde{V}(\mathbf{x}; \omega),$$

where

$$V_{\Gamma}(\mathbf{x}; \omega) = \sum_{y \in \Gamma} c(\mathbf{x}, y) V(y; \omega)$$

and, respectively,

$$\widetilde{V}(\mathbf{x}; \omega) = \sum_{y \in \Pi \Lambda_{\mathbb{L}}(\mathbf{u}) \setminus \Gamma} c(\mathbf{x}, y) V(y; \omega)$$

so that only the term $V_{\Gamma}(\mathbf{x}; \omega)$ is measurable with respect to $\mathfrak{B}\,[\Pi_i \Lambda_{\mathbb{L}}(\mathbf{u})]$, while $\widetilde{V}(\mathbf{x}; \omega)$ is measurable with respect to $\{V(y; \cdot),\ y \in \Pi \Lambda_{\mathbb{L}}(\mathbf{u}) \setminus \Gamma\}$, and, therefore, independent of $\{V(y; \cdot),\ y \in \Pi_i \Lambda_{\mathbb{L}}(\mathbf{u})\}$.

Next, we write

$$H_{\Lambda_{\mathbb{L}}(\mathbf{u})} = H^0 + U + \widetilde{V} + V_{\Gamma} = \widetilde{K} + V_{\Gamma},\ \ \widetilde{K} = H^0 + U + \widetilde{V},$$

so that V_{Γ} is $\mathfrak{B}\,[\Pi_i \Lambda_{\mathbb{L}}(\mathbf{u})]$-measurable, and $\widetilde{K}$ is $\mathfrak{B}\,[\Pi_i \Lambda_{\mathbb{L}}(\mathbf{u})]$-independent. In other words, relative to the measure μ^{Γ}, operator $\widetilde{K}$ is non-random. So, with $\mathbf{v} \sim \{V(y; \omega),\ y \in \Gamma\}$, we identify $\widetilde{K}$ with operator K of Remark 2.2, while the role of operator family $B(\mathbf{v})$ is played by multiplication operators $V_{\Gamma}(\cdot; \omega)$:

$$B(\mathbf{v})\phi(\mathbf{x}) = V_{\Gamma}(\mathbf{x}; \omega)\phi(\mathbf{x}),\ \ \ \mathbf{x} \in \Lambda_{\mathbb{L}}(\mathbf{u}),\ \ \phi \in \ell_2(\Lambda_{\mathbb{L}}(\mathbf{u})). \tag{2.7}$$

The above lower bound $c(\mathbf{x}, y) \geq 1$, valid for any $y \in \Gamma$, implies that, with identification (2.6), Hermitian operators $B(\mathbf{v})$ form a DM family.

Then we use Remark 2.2 (cf. (1.3)), and obtain that $H_{\Lambda_{\mathbb{L}}(\mathbf{u})} = \widetilde{K} + V_{\Gamma}$ is a DM family. Next, owing to Remark 2.1, each eigen-value $\lambda^{(k)}_{\Lambda_{\mathbb{L}}(\mathbf{u})}$, $k = 1, \ldots, |\Lambda_{\mathbb{L}}(\mathbf{u})|$, is a

DM function of the sample collection $\mathbf{v} \sim \{V(x;\omega),\ x \in \Gamma\}$. Hence, by Lemma 2.1, $\forall$ $k = 1, \dots, |\Lambda_{\mathbb{L}}(\mathbf{u})|$, and owing to our choice of the projection of smallest cardinality,

$$\mathbb{P}\left(\left|E - \lambda^{(k)}_{\Lambda_{\mathbb{L}}(\mathbf{u})}\right| \leq \epsilon\right) \leq \min_{j=1,2}\left\{\left|\Pi_j\Lambda_{\mathbb{L}}(\mathbf{u})\right|\right\} s(F, 2\epsilon). \tag{2.8}$$

The final remark is that the probability in the LHS of Eq. (1.9) is $\leq$ the RHS of Eq. (2.8) times $|\Lambda_{\mathbb{L}}(\mathbf{u})|$. ◻

We will need the following elementary geometrical statement which we prove later.

Lemma 2.2. *Consider two boxes $\Lambda_{\mathbb{L}}(\mathbf{u})$ and $\Lambda_{\mathbb{L}'}(\mathbf{u}')$ and suppose that*

$$\min(\|\mathbf{u} - \mathbf{u}'\|, \|\mathbf{u} - S(\mathbf{u}')\|) > 8\max\{L_1, L_2, L_1', L_2'\}. \tag{2.9}$$

Then there are two possibilities (which in general do not exclude each other):

(i) *$\Lambda_{\mathbb{L}}(\mathbf{u})$ and $\Lambda_{\mathbb{L}'}(\mathbf{u}')$ are 'completely separated', when*

$$\Pi\Lambda_{\mathbb{L}}(\mathbf{u}) \cap \Pi\Lambda_{\mathbb{L}'}(\mathbf{u}') = \emptyset. \tag{2.10}$$

(ii) *$\Lambda_{\mathbb{L}}(\mathbf{u})$ and $\Lambda_{\mathbb{L}'}(\mathbf{u}')$ are 'partially separated'. In this case one (or more) of the four possibilities can occur:*

$$\begin{aligned}
&\text{(A)}\ \Pi_1\Lambda_{\mathbb{L}}(\mathbf{u}) \cap \left[\Pi_2\Lambda_{\mathbb{L}}(\mathbf{u}) \cup \Pi\Lambda_{\mathbb{L}'}(\mathbf{u}')\right] = \emptyset,\\
&\text{(B)}\ \Pi_2\Lambda_{\mathbb{L}}(\mathbf{u}) \cap \left[\Pi_1\Lambda_{\mathbb{L}}(\mathbf{u}) \cup \Pi\Lambda_{\mathbb{L}'}(\mathbf{u}')\right] = \emptyset,\\
&\text{(C)}\ \Pi_1\Lambda_{\mathbb{L}'}(\mathbf{u}') \cap \left[\Pi\Lambda_{\mathbb{L}}(\mathbf{u}) \cup \Pi_2\Lambda_{\mathbb{L}'}(\mathbf{u}')\right] = \emptyset,\\
&\text{(D)}\ \Pi_2\Lambda_{\mathbb{L}'}(\mathbf{u}') \cap \left[\Pi\Lambda_{\mathbb{L}}(\mathbf{u}) \cup \Pi_1\Lambda_{\mathbb{L}'}(\mathbf{u}')\right] = \emptyset.
\end{aligned} \tag{2.11}$$

Pictorially, case (ii) is where one of the cubes $\Pi_j\Lambda_{\mathbb{L}}(\mathbf{u})$, $\Pi_j\Lambda_{\mathbb{L}'}(\mathbf{u}')$, $j = 1, 2$, is disjoint from the union of the rest of the projections of $\Lambda_{\mathbb{L}}(\mathbf{u})$ and $\Lambda_{\mathbb{L}'}(\mathbf{u}')$.

We note that the use of the max-norm $\|\cdot\|$ is convenient here: it leads to the constant 8 (two times the number of projections $\Pi_j\Lambda_{\mathbb{L}}(\mathbf{u})$ and $\Pi_j\Lambda_{\mathbb{L}'}(\mathbf{u}')$, $j = 1, 2$) which does not depend on dimension d.

2.3. Proof of Theorem 2. Owing to Lemma 2.2, boxes $\Lambda_{\mathbb{L}}(\mathbf{u})$ and $\Lambda_{\mathbb{L}'}(\mathbf{u}')$ satisfy either (i) or (ii), i.e. they are either completely or partially separated. Passing to the proof of Theorem 2 proper, consider separately cases where boxes $\Lambda_{\mathbb{L}}(\mathbf{u})$ and $\Lambda_{\mathbb{L}'}(\mathbf{u}')$ satisfy (i) or (ii).

(i) 'Complete separation'. Then we can write

$$\begin{aligned}
&\mathbb{P}\left(\mathrm{dist}\left[\Sigma\left(H_{\Lambda_{\mathbb{L}}(\mathbf{u})}\right), \Sigma\left(H_{\Lambda_{\mathbb{L}'}(\mathbf{u}')}\right)\right] \leq \epsilon\right)\\
&\quad = \mathbb{E}\left[\mathbb{P}\left(\mathrm{dist}\left[\Sigma\left(H_{\Lambda_{\mathbb{L}}(\mathbf{u})}\right), \Sigma\left(H_{\Lambda_{\mathbb{L}'}(\mathbf{u}')}\right)\right] \leq \epsilon \mid \mathfrak{B}\left[\Pi\Lambda_{\mathbb{L}'}(\mathbf{u}')\right]\right)\right].
\end{aligned} \tag{2.12}$$

Note first that, under conditioning in Eq. (2.12), the eigen-values $\lambda^{(k')}_{\Lambda_{\mathbb{L}}(\mathbf{u}')}$, $k' = 1, \dots,$ $\left|\Lambda_{\mathbb{L}}(\mathbf{u}')\right|$, forming the set $\Sigma\left(H_{\Lambda_{\mathbb{L}'}(\mathbf{u}')}\right)$ are non-random. Therefore, it makes sense to use the following inequality:

$$\begin{aligned}
&\mathbb{P}\left(\mathrm{dist}\left[\Sigma\left(H_{\Lambda_{\mathbb{L}}(\mathbf{u})}\right), \Sigma\left(H_{\Lambda_{\mathbb{L}'}(\mathbf{u}')}\right)\right] \leq \epsilon \mid \mathfrak{B}\left[\Pi\Lambda_{\mathbb{L}'}(\mathbf{u}')\right]\right)\\
&\quad \leq |\Lambda_{\mathbb{L}'}(\mathbf{u}')| \sup_{E\in\mathbb{R}} \mathbb{P}\left(\mathrm{dist}\left[\Sigma\left(H_{\Lambda_{\mathbb{L}}(\mathbf{u})}\right), E\right] \leq \epsilon\right),
\end{aligned} \tag{2.13}$$

since there are $|\Lambda_{\mathbb{L}'}(\mathbf{u}')|$ eigen-values $\lambda^{(k')}_{\Lambda_{\mathbb{L}}(\mathbf{u}')}$ (counting multiplicities). Then, by virtue of Theorem 1,

$$\mathbb{P}\left(\operatorname{dist}\left[\Sigma\left(H_{\Lambda_{\mathbb{L}}(\mathbf{u})}\right), E\right] \leq \epsilon\right) \leq |\Lambda_{\mathbb{L}}(\mathbf{u})| \min_{j=1,2}\left\{\left|\Pi_j \Lambda_{\mathbb{L}}(\mathbf{u})\right|\right\} \cdot s(2\epsilon), \quad (2.14)$$

implying bound (1.12).

(ii) 'Partial separation'. For example, assume case (A) where $\Pi_1\Lambda_{\mathbb{L}}(\mathbf{u})$, is disjoint from the union of the rest of the projections of $\Lambda_{\mathbb{L}}(\mathbf{u})$ and $\Lambda_{\mathbb{L}'}(\mathbf{u}')$:

$$\Pi_1\Lambda_{\mathbb{L}}(\mathbf{u}) \cap \left[\Pi_2\Lambda_{\mathbb{L}}(\mathbf{u}) \cup \Pi\Lambda_{\mathbb{L}'}(\mathbf{u}'))\right] = \emptyset. \quad (2.15)$$

We then estimate the probability in the LHS of (2.13) with the help of the conditional expectation

$$\begin{aligned}&\mathbb{P}\left(\left|\lambda^{(k)}_{\Lambda_{\mathbb{L}}(\mathbf{u})} - \lambda^{(k')}_{\Lambda_{\mathbb{L}}(\mathbf{u}')}\right| \leq \epsilon \,\left|\mathfrak{B}\left[\Pi\Lambda_{\mathbb{L}'}(\mathbf{u}')\right]\right.\right)\\ &\quad = \mathbb{E}\left[\mathbb{P}\left(\left|\lambda^{(k)}_{\Lambda_{\mathbb{L}}(\mathbf{u})} - \lambda^{(k')}_{\Lambda_{\mathbb{L}}(\mathbf{u}')}\right| \leq \epsilon\left|\mathfrak{B}\left[\Pi_2\Lambda_{\mathbb{L}}(\mathbf{u}) \cup \Pi\Lambda_{\mathbb{L}'}(\mathbf{u}')\right]\right.\right) \left|\mathfrak{B}\left[\Pi\Lambda_{\mathbb{L}'}(\mathbf{u}')\right]\right.\right].\end{aligned} \quad (2.16)$$

Here $\mathfrak{B}\left[\Pi_2\Lambda_{\mathbb{L}}(\mathbf{u}) \cup \Pi\Lambda_{\mathbb{L}'}(\mathbf{u}')\right]$ is the sigma-algebra generated by the random variables

$$\omega \mapsto V(x;\omega), \quad x \in \Pi_2\Lambda_{\mathbb{L}}(\mathbf{u}) \cup \Pi\Lambda_{\mathbb{L}'}(\mathbf{u}');$$

owing to (2.15) it is independent of the sigma-algebra $\mathfrak{B}\left[\Pi_1\Lambda_{\mathbb{L}}(\mathbf{u})\right]$ generated by the random variables

$$\omega \mapsto V(x;\omega), \quad x \in \Pi_1\Lambda_{\mathbb{L}}(\mathbf{u}).$$

We see that the argument used in the proof of Theorem 1 is still applicable; here, we take the product-measure $\mathbb{P}_{\Pi_1\Lambda_{\mathbb{L}}(\mathbf{u})}$ (which again is identified with the product-measure μ^{Γ} from Lemma 2.1, with $|\Gamma| = p = |\Pi_1\Lambda_{\mathbb{L}}(\mathbf{u})|$). This allows us to write

$$\mathbb{P}\left(\left|\lambda^{(k)}_{\Lambda_{\mathbb{L}}(\mathbf{u}')} - \lambda^{(k')}_{\Lambda_{\mathbb{L}}(\mathbf{u}')}\right| \leq \epsilon \,\left|\mathfrak{B}\left[\Pi_2\Lambda_{\mathbb{L}}(\mathbf{u}) \cup \Pi\Lambda_{\mathbb{L}'}(\mathbf{u}')\right]\right.\right) \leq |\Pi_1\Lambda_{\mathbb{L}}(\mathbf{u})|\; s(F, 2\epsilon) \quad (2.17)$$

and deduce the required bound for the conditional probability in the LHS of (2.16).

If, instead of (2.15), we have one of the other disjointedness relations (B)-(D) in Eq. (2.11), then the argument is conducted in a similar fashion. Specifically, in case (B) we exchange projections $\Pi_1\Lambda_{\mathbb{L}}(\mathbf{u})$ and $\Pi_2\Lambda_{\mathbb{L}}(\mathbf{u})$ in the above argument. In cases (C) and (D), we should exchange $\mathbf{u}$ and $\mathbf{u}'$ as compared to arguments in cases (A) and (B).

This concludes the proof of Theorem 2. □

2.4. Proof of Lemma 2.2. Recall that we have two boxes, $\Lambda_{\mathbb{L}}(\mathbf{u})$ and $\Lambda_{\mathbb{L}'}(\mathbf{u}')$, satisfying the condition (2.9):

$$\min\left\{\|\mathbf{u}-\mathbf{u}'\|, \|\mathbf{u}-S(\mathbf{u}')\|\right\} > 8\max\{L_1, L_2, L_1', L_2'\}.$$

Notice that this can be viewed as a lower bound for the distance in the factor space $\mathbb{Z}^d \times \mathbb{Z}^d/S$, where $S(u_1, u_2) = (u_2, u_1)$.

Since $\operatorname{diam} \Lambda_{\mathbb{L}}(\mathbf{u}) = 2\|\mathbb{L}\|$, $\operatorname{diam} \Lambda_{\mathbb{L}'}(\mathbf{u}') = 2\|\mathbb{L}'\|$, this implies that the union of the four coordinate projections,

$$\Pi_1\Lambda_{\mathbb{L}}(\mathbf{u}), \Pi_2\Lambda_{\mathbb{L}}(\mathbf{u}), \Pi_1\Lambda_{\mathbb{L}'}(\mathbf{u}'), \Pi_2\Lambda_{\mathbb{L}'}(\mathbf{u}')$$

cannot be connected. Therefore, it can be decomposed into two or more connected components. Cases (A), (B), (C) and (D) in the statement of Lemma 2.2 correspond to the situation where one of these coordinate projections is disjoint with the three remaining projections. So, it suffices to analyse the case where each connected component of the union

$$\Pi\Lambda_{\mathbb{L}}(\mathbf{u}) \cup \Pi\Lambda_{\mathbb{L}'}(\mathbf{u}') \tag{2.18}$$

contains exactly two coordinate projections. Furthermore, it suffices to show that the only possible case is (2.10). To do so, we have to exclude two remaining cases, namely,

$$\begin{cases} \left(\Pi_1\Lambda_{\mathbb{L}}(\mathbf{u}) \cup \Pi_1\Lambda_{\mathbb{L}'}(\mathbf{u}')\right) \cap \left(\Pi_2\Lambda_{\mathbb{L}}(\mathbf{u}) \cup \Pi_2\Lambda_{\mathbb{L}'}(\mathbf{u}')\right) = \emptyset, \\ \Pi_1\Lambda_{\mathbb{L}}(\mathbf{u}) \cap \Pi_1\Lambda_{\mathbb{L}'}(\mathbf{u}') \neq \emptyset, \\ \Pi_2\Lambda_{\mathbb{L}}(\mathbf{u}) \cap \Pi_2\Lambda_{\mathbb{L}'}(\mathbf{u}') \neq \emptyset, \end{cases} \tag{2.19}$$

and

$$\begin{cases} \left(\Pi_1\Lambda_{\mathbb{L}}(\mathbf{u}) \cup \Pi_2\Lambda_{\mathbb{L}'}(\mathbf{u}')\right) \cap \left(\Pi_1\Lambda_{\mathbb{L}'}(\mathbf{u}') \cup \Pi_2\Lambda_{\mathbb{L}}(\mathbf{u})\right) = \emptyset, \\ \Pi_1\Lambda_{\mathbb{L}}(\mathbf{u}) \cap \Pi_2\Lambda_{\mathbb{L}'}(\mathbf{u}') \neq \emptyset, \\ \Pi_1\Lambda_{\mathbb{L}'}(\mathbf{u}') \cap \Pi_2\Lambda_{\mathbb{L}}(\mathbf{u}) \neq \emptyset. \end{cases} \tag{2.20}$$

First, observe that (2.19) contradicts the assumption that $\Lambda_{\mathbb{L}}(\mathbf{u})$ and $\Lambda_{\mathbb{L}'}(\mathbf{u}')$ are disjoint. Indeed, in such a case, there exist lattice points

$$v_1 \in \Pi_1\Lambda_{\mathbb{L}}(\mathbf{u}) \cap \Pi_1\Lambda_{\mathbb{L}'}(\mathbf{u}'),\ \ v_2 \in \Pi_2\Lambda_{\mathbb{L}}(\mathbf{u}) \cap \Pi_2\Lambda_{\mathbb{L}'}(\mathbf{u}'),$$

so that

$$\begin{aligned} \exists\,(v_1, v_2) \in [\Pi_1\Lambda_{\mathbb{L}}(\mathbf{u}) \times \Pi_2\Lambda_{\mathbb{L}}(\mathbf{u})] \cap \left[\Pi_1\Lambda_{\mathbb{L}'}(\mathbf{u}') \times \Pi_2\Lambda_{\mathbb{L}'}(\mathbf{u}')\right] \\ = \Lambda_{\mathbb{L}}(\mathbf{u}) \cap \Lambda_{\mathbb{L}'}(\mathbf{u}') = \emptyset, \end{aligned}$$

which is impossible.

The case (2.20) can be reduced to (2.19), by the symmetry S. Namely, let $\mathbf{u}'' = S(\mathbf{u}')$, then

$$\Pi_1\Lambda_{\mathbb{L}'}(\mathbf{u}'') = \Pi_2\Lambda_{\mathbb{L}'}(\mathbf{u}'),\ \ \Pi_2\Lambda_{\mathbb{L}'}(\mathbf{u}'') = \Pi_1\Lambda_{\mathbb{L}'}(\mathbf{u}').$$

Now (2.20) reads as follows in terms of boxes $\Lambda_{\mathbb{L}}(\mathbf{u})$ and $\Lambda_{\mathbb{L}'}(\mathbf{u}'')$:

$$\begin{cases} \left(\Pi_1\Lambda_{\mathbb{L}}(\mathbf{u}) \cup \Pi_1\Lambda_{\mathbb{L}'}(\mathbf{u}'')\right) \cap \left(\Pi_2\Lambda_{\mathbb{L}'}(\mathbf{u}'') \cup \Pi_2\Lambda_{\mathbb{L}}(\mathbf{u})\right) = \emptyset, \\ \Pi_1\Lambda_{\mathbb{L}}(\mathbf{u}) \cap \Pi_1\Lambda_{\mathbb{L}'}(\mathbf{u}'') \neq \emptyset, \\ \Pi_2\Lambda_{\mathbb{L}'}(\mathbf{u}'') \cap \Pi_2\Lambda_{\mathbb{L}}(\mathbf{u}) \neq \emptyset. \end{cases} \tag{2.21}$$

The same argument as above shows then that $\Lambda_{\mathbb{L}}(\mathbf{u}) \cap \Lambda_{\mathbb{L}'}(\mathbf{u}'') \neq \emptyset$, which is impossible, since, by virtue of (2.9),

$$\mathrm{dist}(\mathbf{u}, S(\mathbf{u}')) > 8 \max \left\{L_1, L_2, L_1', L_2'\right\}.$$

This completes the proof. □

3. Concluding Remarks

Remarks made by the referees of this paper allowed us to establish a sharper version of Eq. (1.9) and, consequently, a sharper version of Eq. 1.12 which include a factor $s(\epsilon)$. The reader may compare the current assertion of Theorem 1 with its counterpart in the preliminary version of this paper on [arXiv:0708.2056].

In an earlier manuscript [2], we proved Wegner-type bounds for two-particle lattice systems under a much more restrictive assumption of *analyticity* of the distribution function F of the random external potential V. The proofs, which were more involved than in this paper, also required the amplitude of the potential V to be sufficiently big. However, both approaches revealed an interesting fact. Speaking informally, having more than one particle can only make Wegner type bounds *stronger*, not weaker, as one might suppose. Assertions of Theorems 1 and 2 in [arXiv:0708.2056] make this particularly clear.

A Wegner-type bound for multi-particle systems, with an arbitrary number of particles, close to Theorem 1 (but not Theorem 2) has been independently obtained by W. Kirsch [5] using arguments which are closer in spirit to the original argument given by F. Wegner [8] than to Stollmann's method.

In fact, our Theorem 1 can be extended without difficulty to the general case of $N \geq 2$ particles. Denoting as before N-particle configurations by $\mathbf{x}$, $\mathbf{y}$, etc., the Hamiltonian H $(= H^{(N)}_{U,V})$ reads

$$\begin{aligned} H\phi(\mathbf{x}) &= H^0\phi(\mathbf{x}) + [(U + V_1 + \cdots + V_N)\,\phi]\,(\mathbf{x}) \\ &= \sum_{\mathbf{y}:\, \|\mathbf{y}-\mathbf{x}\|=1} \phi(\mathbf{y}) + \Big(U(\mathbf{x}) + \textstyle\sum_{j=1}^N V(x_j;\omega)\Big)\,\phi(\mathbf{x}), \\ &\qquad \mathbf{x} = (x_1, \ldots, x_N),\ \mathbf{y} = (y_1, \ldots, y_N) \in \mathbb{Z}^d \times \cdots \times \mathbb{Z}^d, \end{aligned} \tag{3.1}$$

where $x_j = \left(x_j^{(1)}, \ldots, x_j^{(d)}\right)$, $y_j = \left(y_j^{(1)}, \ldots, y_j^{(d)}\right) \in \mathbb{Z}^d$, $j = 1, \ldots, N$, and

$$\|\mathbf{x}\| = \max_{j=1,\ldots,N}\ \max_{i=1,\ldots,d} \left|x_j^{(i)}\right|, \quad \mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}^d \times \cdots \times \mathbb{R}^d.$$

A similar formula defines $H_{\Lambda_{\mathbb{L}}(\mathbf{u})}$; cf. (1.3). We again assume that values $V(x;\omega)$, $x \in \mathbb{Z}^d$, are IID with a common distribution function F. Function $U(\mathbf{x})$ is assumed to be locally bounded and symmetric on $\mathbb{Z}^d \times \cdots \times \mathbb{Z}^d$. (As before, value $+\infty$ can also be incorporated.)

The statement of Theorem 1 does not change: given $\mathbf{u} = (u_1, \ldots, u_N)$, with $u_j = (u_j^{(1)}, \ldots, u_j^{(d)}) \in \mathbb{Z}^d$, and $\mathbb{L} = (L_1, \ldots, L_N) \in \mathbb{N}^N$, define $\Pi_j \Lambda_{\mathbb{L}}(\mathbf{u})$ as

$$\Pi_j \Lambda_{\mathbb{L}}(\mathbf{u}) = \left(\mathop{\times}_{i=1}^{d} \left[-L_j + u_j^{(i)}, u_j^{(i)} + L_j\right]\right) \cap \mathbb{Z}^d, \quad j = 1, \ldots, N. \tag{3.2}$$

Then set

$$\Lambda_{\mathbb{L}}(\mathbf{u}) = \Pi_1 \Lambda_{\mathbb{L}}(\mathbf{u}) \times \cdots \times \Pi_N \Lambda_{\mathbb{L}}(\mathbf{u}). \tag{3.3}$$

Theorem 3. *$\forall\, E \in \mathbb{R}$, $\mathbb{L} \in \mathbb{N}^N$, $\mathbf{u} \in \mathbb{Z}^d \times \cdots \times \mathbb{Z}^d$ and $\epsilon > 0$,*

$$\mathbb{P}\left(\mathrm{dist}\left[\Sigma\left(H_{\Lambda_{\mathbb{L}}(\mathbf{u})}\right), E\right] \le \epsilon\right) \le |\Lambda_{\mathbb{L}}(\mathbf{u})| \min_{j=1,\dots,N} \left\{\left|\Pi_j \Lambda_{\mathbb{L}}(\mathbf{u})\right|\right\} \cdot s(2\epsilon). \tag{3.4}$$

The proof is completely analogous to that of Theorem 1, based on the representation

$$\sum_{j=1}^{N} V(x_j;\omega) = \sum_{y\in\Gamma} c(\mathbf{x}, y) V(y;\omega).$$

Here Γ is as before the set of smallest cardinality among $\Pi_1\Lambda_{\mathbb{L}}(\mathbf{u}), \dots, \Pi_N\Lambda_{\mathbb{L}}(\mathbf{u})$ and $c(\mathbf{x}, y)$ is given as a function of $y \in \Gamma$, for every $\mathbf{x} \in \Lambda_{\mathbb{L}}(\mathbf{u})$, by

$$c(\mathbf{x}, y) = \sum_{j=1}^{N} \delta_{y,x_j} \ge 1.$$

The statement of an analog of Theorem 2 for a general N-particle case will be a subject of a forthcoming paper.

We want to conclude by noticing that Theorem 1 can be further extended when $\Lambda_{\mathbb{L}}(\mathbf{u})$ is replaced by a general lattice domain (in $\mathbb{Z}^d \times \cdots \times \mathbb{Z}^d$). We decided to focus on lattice parallelepipeds because it suffices for traditional applications (Anderson localisation).

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