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A Complete Renormalization Group Trajectory Between Two Fixed Points

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Abstract: We give a rigorous nonperturbative construction of a massless discrete trajectory for Wilson's exact renormalization group. The model is a three dimensional Euclidean field theory with a modified free propagator. The trajectory realizes the mean field to critical crossover from the ultraviolet Gaussian fixed point to an analog recently constructed by Brydges, Mitter and Scoppola of the Wilson-Fisher nontrivial fixed point.

1. Introduction

In recent years, the mathematical community has shown an increasing interest for the important but difficult topic of quantum field theory [24]. The most comprehensive and insightful, albeit largely conjectural, mathematical framework to address this subject is Wilson's renormalization group: a grand dynamical system in the space of all imaginable observation scale dependent effective field theories [76, 77]. As emphasized by Wilson himself [77], his approach can be construed as a mathematical theory of scaling symmetry which has yet to be fully unveiled. It generalizes in a very deep way ordinary calculus which matured in the hands of 19th century mathematicians and gave the first rigorous meaning to the notion of 'continuum'. Formulating precise conjectures about the *phase portrait* of the RG dynamical system and proving them, an endeavor one could perhaps call the 'Wilson Program' [57], is one of the greatest challenges in mathematical analysis and probability theory, and will likely remain so for years to come.

When studying a phase portrait, the first features to examine are *fixed points*, which here mean scale invariant theories. If D is the dimension of space, one expects that for $D \geq 4$ there are only two fixed points: the high temperature one and the massless Gaussian one. As one lowers the dimension to the range $3 \leq D < 4$, only one new fixed point should appear: the Wilson-Fisher fixed point [75]. Its existence as well as the construction of its local stable manifold, in the hierarchical approximation, was first rigorously established in [9]; see also [20, 21, 36]. The uniqueness, in the local potential approximation, was shown in [54]. As one continues lowering the dimension to the range

$2 < D < 3$, past every threshold $D_n = 2 + \frac{2}{n-1}$, $n = 3, 4, \ldots$, a new fixed point appears corresponding to an n-well potential, as was proved in the local potential approximation by Felder [31]. For $D = 2$, the situation becomes extremely complicated: even a conjectural classification of fixed points corresponding to conformal field theories is not yet complete. Nevertheless, there have been tremendous advances in this area; see e.g. [26] and Gawędzki's lectures in [24] for an introduction.

The next stage in the investigation concerns the various *local invariant manifolds* around these fixed points and the associated *critical exponents*. The first such rigorous result, for the Gaussian fixed point, is the work of Bleher and Sinai [8]. For further developments, with emphasis on these dynamical systems aspects, see for instance [20, 36, 37, 49, 62, 74, 44].

Then, in the third stage, one would like to know more *global* features like how all these local invariant manifolds meet to form separatrices between domains exhibiting qualitatively different behaviours. This question pertains to the active field of the renormalization group theory of crossover phenomena (see e.g. [60, 55] for recent reviews). Our work falls within this third class of problems. The control of a massless RG trajectory between fixed points announced in [1] and for which details are provided here is our contribution to the grand scheme of the Wilson Program. Note that there is extensive physics literature, following the seminal work of Zamolodchikov [78, 79], on such massless RG flows in particular in two dimensions, see e.g. [80, 23, 28, 29] and references therein. However, nonperturbative results substantiated by rigorous mathematical estimates are scarce. To borrow the terminology of the French school of constructive field theory, this is the 'problème de la soudure' or the welding problem. One has to control the junction between the ultraviolet and the infrared regimes. For instance, for the two dimensional Gross-Neveu model, the UV regime has been given a rigorous mathematical treatment a long time ago [38, 32]. Likewise, the IR regime with spontaneous mass generation for a *UV-cutoff* theory is also under control [52]; see also [51] for a similar result on the sigma model. However, the junction, although probably not out of reach of present methods, has proved to be more technically demanding than expected [53]. Note that the model we consider here is simpler in that regard. It does not involve a drastic change of scenery, for instance, from a purely Fermionic theory at the ultraviolet end, to a Bosonic one at the infrared end.

Given a small positive bifurcation parameter ϵ, we consider a three dimensional ϕ^4 theory with a modified propagator: the $(\Phi^4)_{3,\epsilon}$ model of [15], which was also studied in the hierarchical approximation in [36]. Namely, we consider functional integrals of the form

$$\int d\mu_{\tilde{C}}(\phi) \ldots e^{-V(\phi)}, \tag{1}$$

where $d\mu_{\tilde{C}}$ is the Gaussian measure with covariance $\tilde{C} \stackrel{\text{def}}{=} (-\Delta)^{-\left(\frac{3+\epsilon}{4}\right)}$ and Wick ordered interaction potential

$$V(\phi) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} d^3x \, \left\{ g : \phi(x)^4 :_{\tilde{C}} + \mu : \phi(x)^2 :_{\tilde{C}} \right\} . \tag{2}$$

Over the last two decades, Brydges and his collaborators have devised a general mathematical framework, going beyond the hierarchical and local potential approximations, in order to give a rigorous nonperturbative meaning to the renormalization group dynamical system [17, 11, 12, 15]. The approach, actually involving no approximation whatsoever, is in the spirit of Wilson's exact renormalization group scheme [56]. Our article which can be viewed as a direct continuation of [15] takes place in this setting. In very rough terms, the renormalization group map, rather than flow, represents the evolution of the integrand $I(\phi)$ of functional integrals such as (1) under convolution and rescaling. The convolution is with respect to the Gaussian measure corresponding to Fourier modes p of the field ϕ which are restricted to a range of the form $L^n \le |p| \le L^{n+1}$, where the integer $L \ge 2$ is the scale ratio for one RG step. By rescaling, one can keep the integer n constant, and make the RG transformation autonomous. The latter acts on the integrand $I(\phi)$ and produces a new one $I'(\phi)$. However, the problem with expressing the renormalization group in terms of its action on $I(\phi)$ is that $I(\phi)$ does not exist in the infinite volume limit. It is essential to express $I(\phi)$ in terms of coordinates that (i) are well defined in the infinite volume limit and (ii) carry the exact action of the renormalization group in a tractable form. The key feature that these coordinates have to express is that $I(\phi)$ is approximately a product of local functionals of the field and the action of the renormalization group is also approximately local. The first step towards these coordinates is to write $I(\phi)$ via the polymer representation:

$$I(\phi) = \sum_{\{X_i\}} e^{-V(\Lambda\backslash X,\phi)} \prod_i K(X_i, \phi), \tag{3}$$

where Λ is the volume cut-off needed to perform the thermodynamic limit, and $\{X_i\}$ is a collection of disjoint polymers X_i in Λ. By polymer we mean a connected finite union of cubes cut by a fixed $\mathbb{Z}^3$ lattice inside $\mathbb{R}^3$. The union of the X_i has been denoted by X, and the functional $V(\Lambda\backslash X, \phi)$ is given by (2) except that the integration domain is the complement $\Lambda\backslash X$ instead of $\mathbb{R}^3$. Therefore, the functionals V are determined by the two variables or couplings g and μ. Now the K's are local functionals of the field, which means that $K(Y, \phi)$ only depends on the restriction of ϕ to the set Y. The knowledge of the integrand $I(\phi)$ amounts to that of the couplings g, μ together with the collection K of all the functionals $K(Y, \phi)$ corresponding to all possible polymers Y. One also needs a splitting $K = Qe^{-V} + R$ of these functionals where the Qe^{-V} part is given explicitly in terms of g, μ only. In sum, the integrand is encoded by a triple (g, μ, R). The renormalization group map in [15] is implemented as a mathematically precise transformation $(g, \mu, R) \mapsto (g', \mu', R')$. The evolution for the $:\phi^4:$ coupling g has the form

$$g' = L^{\epsilon} g - L^{2\epsilon} a(L, \epsilon) g^2 + \xi_g(g, \mu, R)\ . \tag{4}$$

The evolution of the mass term or $:\phi^2:$ coupling μ has the form

$$\mu' = L^{\frac{3+\epsilon}{2}} \mu + \xi_\mu(g, \mu, R)\ . \tag{5}$$

Finally the collection R of 'irrelevant terms', living in a suitable infinite dimensional space, evolves according to

$$R' = \mathcal{L}^{(g,\mu)}(R) + \xi_R(g, \mu, R), \tag{6}$$

where $\mathcal{L}^{(g,\mu)}$ is a (g, μ)-dependent contractive linear map in the R direction. The ξ remainder terms are higher order small nonlinearities. An important feature of this formalism is that the polymer representation (3) is not unique. As a result, one has enough freedom when defining the RG map, in order to secure the contractive property of the $\mathcal{L}^{(g,\mu)}$. This is the so-called 'extraction step' which encapsulates the renormalization subtractions familiar in quantum field theory. The transformation in [15] also carried an extra dynamical variable $\mathbf{w}$ with very simple evolution which is independent of the

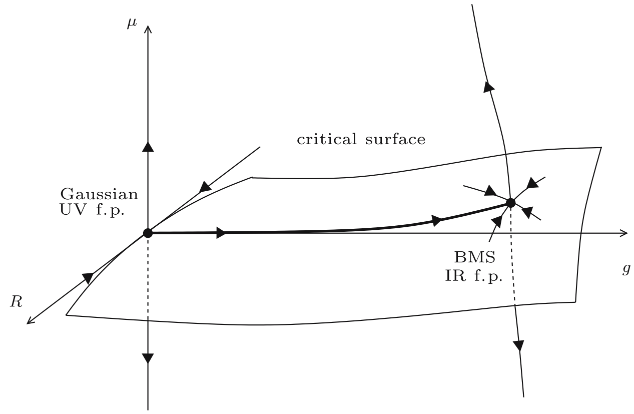

Fig. 1. The RG dynamical system

other variables, and converging exponentially fast to a fixed point $\mathbf{w}_*$. This was introduced in order to make the RG map autonomous. Throughout this article however, we take $\mathbf{w} = \mathbf{w}_*$ and incorporate $\mathbf{w}$ in the very definition of the RG map. In [15], it was shown that for small $\epsilon > 0$ there exists an infrared fixed point (g_*, μ_*, R_*) which is an analog of the Wilson-Fisher fixed point [75], and which is nontrivial, i.e., distinct from the Gaussian ultraviolet fixed point $(g, \mu, R) = (0, 0, 0)$. The local stable manifold of the infrared fixed point was also constructed. Note that if one neglects the ξ remainders, one gets an approximate fixed point $(\bar{g}_*, 0, 0)$, where

$$\bar{g}_* \stackrel{\text{def}}{=} \frac{L^\epsilon - 1}{L^{2\epsilon} a(L, \epsilon)} = \mathcal{O}(\epsilon)\ . \tag{7}$$

A schematic rendition of the phase portrait of the RG map considered in [15] is provided by Fig. 1. The precise statements of our main results, Theorem 2 and Corollary 1 below, require a substantial amount of machinery to be provided in the next sections. We can nevertheless already give an informal statement.

Main Result. *In the regime where $\epsilon > 0$ is small enough, for any $\omega_0 \in]0, \frac{1}{2}[$, there exists a complete trajectory $(g_n, \mu_n, R_n)_{n\in\mathbb{Z}}$ for the RG map given by Eqs. (4), (5), and (6), such that $\lim_{n\to-\infty} (g_n, \mu_n, R_n) = (0, 0, 0)$ the Gaussian ultraviolet fixed point, and $\lim_{n\to+\infty} (g_n, \mu_n, R_n) = (g_*, \mu_*, R_*)$ the BMS nontrivial infrared fixed point, and determined by the 'initial condition' at unit scale*

$$g_0 = \omega_0 \bar{g}_*\ . \tag{8}$$

See Fig. 2 for a sketch of such discrete RG orbits $P_n = (g_n, \mu_n, R_n)$, $n \in \mathbb{Z}$, which are parametrized by the projection of P_0 on the g axis. To the best of our knowledge, the only previous similar result is the construction of the massless connecting heteroclinic orbit going from a UV nontrivial fixed point to the Gaussian IR fixed point for a modified Gross-Neveu model in [39] (see also [22, 30] for related work in the massive case). Our work which essentially amounts to the construction of a nontrivial massless three dimensional Euclidean field theory in the continuum, is probably the first such result in the Bosonic case. This field theory is superrenormalizable in the ultraviolet sector but only barely. Namely, one needs to renormalize divergent Feynman diagrams only

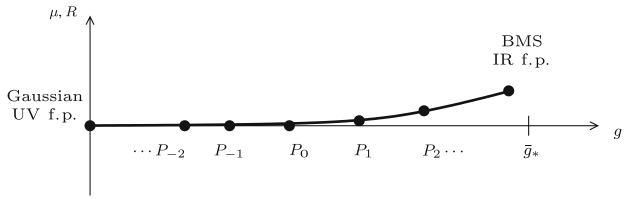

Fig. 2. The discrete trajectories

up to a finite order in perturbation theory; however this order goes to infinity when the parameter ϵ goes to zero. As shown in [15, Sect. 1.1], a proof for the difficult axiom of Osterwalder-Schrader positivity seems feasible on this model, which makes it interesting from the point of view of traditional constructive field theory [43]. Due to the lack of a nonperturbative definition of dimensional regularization, this model is the best available for the mathematically rigorous study of the Wilson-Fisher fixed point [75] which is believed to govern the infrared behavior of the tridimensional Ising model (when $\epsilon = 1$). On the technical side, as far as the construction of a global RG trajectory is concerned, one should note that the situation in [39] is facilitated by the availability of a convergent series representation in a whole neighborhood of the Gaussian fixed point which is only possible for a Fermionic theory. In the present situation, the 'trivial' fixed point around which the analysis takes place is not so trivial and in fact is highly singular from the point of view of the estimates we use. This is a manifestation of the so-called 'large field problem' and the need for the 'domination procedure' (see e.g. [64]). The norms needed for the control of R which implement a measurement of the typical size of the field $\phi \sim g^{-\frac{1}{4}}$ through a parameter h appearing in the definition of these norms, create one of the main difficulties we had to overcome: the 'fibered norm problem'. Namely, the norm for R involves the *dynamical variable* g. The approach we used is to construct the trajectory $s = (g_n, \mu_n, R_n)_{n\in\mathbb{Z}}$ via its deviation δs with respect to an approximate trajectory $(\bar{g}_n, 0, 0)_{n\in\mathbb{Z}}$ which solves the RG recursion when the ξ terms are thrown out. This is done thanks to a contraction mapping argument in a big Banach space of sequences δs. This approach, in the spirit of Irwin's proof of the stable manifold theorem [47, 67], was suggested to us by D. C. Brydges. We then realized that one can resolve the vicious circle entailed by the 'fibered norm problem' by using the approximate values $\bar{g}_n$ in the definition of the norms.

In principle, Wilson's RG picture reduces deep questions in quantum field theory and statistical mechanics to a chapter in the theory of bifurcations and dynamical systems. In practice, it has proved hard to get away with the application of a ready-made theorem from the corresponding literature, as emphasized in [20, p. 70] from the beginning of the subject and even for the simpler hierarchical models. Most of the works on the rigorous renormalization group use an ad hoc method developed in [9]. An innovation was introduced in [12], by the construction of the stable manifold of the nontrivial fixed point using an iteration in a space of sequences, along the lines of Irwin's proof. The latter method seems more robust and easier to adapt to our present setting than the more standard Hadamard graph transform method [46, 67]. Formally, the RG map given by (4), (5), and (6), with bifurcation parameter ϵ corresponds to a *transcritical bifurcation*, according to the classification given e.g. in [19, p. 177]. The moving nontrivial fixed point goes through the Gaussian one as one increases the ϵ parameter. The negative ϵ region is forbidden however, since it would put the nontrivial fixed point in the undefined $g < 0$

region. Most pertinent to the construction of a connecting heteroclinic orbit between RG fixed points, in the dynamical systems literature, is the article [50], which is based on Kelley's center manifold theorem [48, 18, 68]. However, we have so far been unable to apply these methods in the present situation.

In the same way [39] is based on the hard analysis estimates of [38], our proof is based on Theorem 1 below which summarizes a slight adaptation of the estimates in [15, Sect. 5] built on the techniques of [17, 11]. With the exception of the proof of this theorem which needs a working knowledge of [15, Sect. 5], our article can be read with only modest prerequisites in functional analysis as covered e.g. in [3, 7, 25], and in the theory of Gaussian probability measures in Hilbert spaces [10, 70]. We provided a completely self-contained definition of the renormalization group map $(g, \mu, R) \mapsto (g', \mu', R')$ in Sects. 2, 3, and 4. Apart from making the so-called extraction step explicit, this gives us the opportunity to correct some minor sign and numerical factor errors, but also one serious error, namely that in [15] the Banach fixed point theorem was used for a normed space that is not complete. Fortunately, we obtained, through discussions with D. C. Brydges and P. K. Mitter, an amendment which is provided in Sect. 3. It has the advantage that all the estimates in [15, Sect. 5] hold in this new setting without the need for a touch up. For more efficiency, in the sections defining the RG map, we adopted a rather terse style of presentation. We refer the newcomer seeking a proper motivation for this formalism to [56] and the introductory sections of [17, 11, 15]. Note that these definitions are quite involved and by no means the first that would come to one's mind. Nevertheless, they are about the simplest which give a rigorous nonperturbative meaning to Wilson's exact renormalization group, and at the same time navigate around the pitfalls of more naïve approaches. These pitfalls have been mapped by the pioneering work of Balaban, Federbush, Feldman, Gallavotti, Gawędzki, Glimm, Jaffe, Kupiainen, Magnen, Rivasseau, Seiler, Sénéor, Spencer, and many others we apologize for not citing [43, 34]. For more ample introduction to the rigorous renormalization group than we provide here, the reader from other areas of mathematics may most profitably read [71, 66, 35, 5] and Gawędzki's lecture in [24] for a first contact. More technical or specialized material is covered in [6, 43, 64].

2. The General Setting

The ambient space for the field theory we are considering is Euclidean $\mathbb{R}^3$. Given an element $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ we will use the notation $|x|_\infty \stackrel{\text{def}}{=} \max(|x_1|, |x_2|, |x_3|)$ and $|x|_2 \stackrel{\text{def}}{=} \sqrt{x_1^2 + x_2^2 + x_3^2}$. Let ϵ be a small nonnegative number, then with a slight abuse of notation the kernel of the covariance operator $\tilde{C} = (-\Delta)^{-\left(\frac{3+\epsilon}{4}\right)}$, which is formally

$$\tilde{C}(x, y) = \tilde{C}(x - y) = \int_{\mathbb{R}^3} \frac{d^3 p}{(2\pi)^3} e^{ip(x-y)} (p^2)^{-\left(\frac{3+\epsilon}{4}\right)} , \tag{9}$$

is given (see e.g. [40, Sect. II.3.3] for a careful derivation) for noncoinciding points by the Riesz potential

$$\tilde{C}(x - y) = \frac{\varkappa_\epsilon}{|x - y|_2^{\frac{3-\epsilon}{2}}} , \tag{10}$$

with

$$\varkappa_\epsilon \stackrel{\text{def}}{=} \pi^{-\frac{3}{2}} \times 2^{-\left(\frac{3+\epsilon}{2}\right)} \times \frac{\Gamma\left(\frac{3-\epsilon}{4}\right)}{\Gamma\left(\frac{3+\epsilon}{4}\right)} . \tag{11}$$

Let $\varpi : \mathbb{R}^3 \to \mathbb{R}$ be a pointwise nonnegative C^∞ and rotationally invariant function which vanishes when $|x|_2 \geq \frac{1}{2}$ and is equal to one when $|x|_2 \leq \frac{1}{4}$. Let $\tilde{u} \stackrel{\text{def}}{=} \varpi * \varpi$ be the convolution of ϖ with itself. It is nonnegative both in direct and momentum spaces, and also rotationally invariant. Since $\varpi(0) > 0$, the integral

$$\int_{\mathbb{R}^3} d^3z\ |z|_2^{-\frac{3}{2}} \varpi(z)$$

is strictly positive. We define the function u_0 to be the unique positive multiple of ϖ such that

$$\int_{\mathbb{R}^3} d^3z\ |z|_2^{-\frac{3}{2}} u_0(z) = \varkappa_0 = (2\pi)^{-\frac{3}{2}}\ . \tag{12}$$

The u_0 function is fixed once and for all in this article. Now define

$$\lambda_\epsilon \stackrel{\text{def}}{=} \frac{\varkappa_\epsilon}{\int_{\mathbb{R}^3} d^3z\ |z|_2^{-\left(\frac{3+\epsilon}{2}\right)} u_0(z)}\ , \tag{13}$$

and let $u_\epsilon(x) = \lambda_\epsilon u_0(x)$. Now we clearly have $\lambda_\epsilon \to 1$ when $\epsilon \to 0$ and for $x \neq y$ in $\mathbb{R}^3$,

$$\int_0^{+\infty} \frac{dl}{l}\ l^{-\left(\frac{3-\epsilon}{2}\right)} u_\epsilon\left(\frac{x-y}{l}\right) = \frac{\varkappa_\epsilon}{|x-y|_2^{\frac{3-\epsilon}{2}}} = \tilde{C}(x-y)\ , \tag{14}$$

i.e., the canonically normalized noncutoff covariance. We now define the scale one UV-cutoff covariance C by

$$C(x-y) \stackrel{\text{def}}{=} \int_1^{+\infty} \frac{dl}{l}\ l^{-\left(\frac{3-\epsilon}{2}\right)} u_\epsilon\left(\frac{x-y}{l}\right)\ . \tag{15}$$

Remark 1. In [15] the u_ϵ is fixed whereas here it is a variable multiple of a fixed function u_0. Since in the regime where ϵ is small the multiplier λ_ϵ can be assumed to be say between 0.9 and 1.1; this has no effect on the estimates in [15] such as the large field stability bounds: Eq. 2.3, Lemma 5.3 and Lemma 5.4 therein.

Let $L \geq 2$ be an integer. We will also need the fluctuation covariance

$$\Gamma(x-y) \stackrel{\text{def}}{=} \int_1^{L} \frac{dl}{l}\ l^{-\left(\frac{3-\epsilon}{2}\right)} u_\epsilon\left(\frac{x-y}{l}\right)\ . \tag{16}$$

Note that in Eq. (11) the letter 'Gamma' denoted the usual Euler gamma function; however, from now on the notation will be reserved for the fluctuation covariance (16). The engineering scaling dimension of the field ϕ which is denoted by $[\phi]$ is defined by the property $\tilde{C}(lx) = l^{-2[\phi]}\tilde{C}(x)$. One can read it off Eq. (14): $[\phi] = \frac{3-\epsilon}{4}$. As in [15] we use the notation

$$C_L(x) \stackrel{\text{def}}{=} L^{2[\phi]} C(Lx) \tag{17}$$

for scaling of covariances. We define $v^{(2)}(x) \stackrel{\text{def}}{=} C_L(x)^2 - C(x)^2$ and let

$$a(L,\epsilon) \stackrel{\text{def}}{=} 36 \int_{\mathbb{R}^3} d^3x\ v^{(2)}(x)\ . \tag{18}$$

It is a simple exercise in analysis to show that, regardless of the precise shape of the initial cutoff function u_0, one has

$$\lim_{\epsilon\to 0} a(L,\epsilon) = a(L,0) = \frac{\log L}{18\pi^2} \tag{19}$$

as expected for the second order coefficient of the beta function of a marginal (at $\epsilon = 0$) coupling. As a result, the approximate fixed point

$$\bar{g}_* = \frac{L^\epsilon - 1}{L^{2\epsilon} a(L,\epsilon)} \tag{20}$$

satisfies

$$\bar{g}_* \sim 18\pi^2\epsilon \tag{21}$$

when $\epsilon \to 0$.

Now consider the lattice $\mathbb{Z}^3$ inside $\mathbb{R}^3$. A *unit box* is any closed cube of the form $[m_1, m_1+1] \times [m_2, m_2+1] \times [m_3, m_3+1]$ with $m = (m_1, m_2, m_3) \in \mathbb{Z}^3$. The set of all unit boxes is denoted by Box_0. A nonempty connected subset of $\mathbb{R}^3$ which is a finite union of unit boxes is called a *polymer*. The denumerable set of all polymers is denoted by Poly_0. We will also need the set $\mathrm{Poly}_{-1} \stackrel{\mathrm{def}}{=} \{L^{-1}X | X \in \mathrm{Poly}_0\}$ whose elements are called L^{-1}*-polymers*, as well as $\mathrm{Poly}_{+1} \stackrel{\mathrm{def}}{=} \{L.X | X \in \mathrm{Poly}_0\}$, whose elements are called L*-polymers*. Unless otherwise specified, by polymer we will always mean a unit polymer, i.e., an element of Poly_0. For a polymer X, we denote $|X| \stackrel{\mathrm{def}}{=} \mathrm{Vol}(X)$ which is also the number of unit boxes in X. We also define its L*-closure* $\bar{X}^L$ as the union of all boxes of size L, cut by the $(L\mathbb{Z})^3$ lattice, which contain a unit box in X. This is the same as the smallest L-polymer containing X, which explains the terminology. A polymer $X \in \mathrm{Poly}_0$ with $|X| \le 8$ is called a *small polymer*. A polymer $X \in \mathrm{Poly}_0$ with $|X| \le 2$ is called an *ultrasmall polymer*. A *large* polymer simply is one which is not small. We finally define the *large set regulator* which is a function $\mathcal{A} : \mathrm{Poly}_0 \to \mathbb{R}_+^*$, by $\mathcal{A}(X) \stackrel{\mathrm{def}}{=} L^{5|X|}$.

3. Functional Spaces

3.1. Sobolev spaces with gluing conditions. To each $X \in \mathrm{Poly}_0$, we associate a real separable Hilbert space $\mathrm{Fld}(X)$ where the fields $\phi : X \to \mathbb{R}$ will live. Given any *open* unit box $\mathring{\Delta}$, with $\Delta \in \mathrm{Box}_0$, we consider the standard Sobolev space $W^{4,2}(\mathring{\Delta})$ with the norm

$$||\phi||_{W^{4,2}(\mathring{\Delta})} \stackrel{\mathrm{def}}{=} \left(\sum_{|\nu|\le 4} ||\partial^\nu \phi||^2_{L^2(\mathring{\Delta})} \right)^{\frac{1}{2}} . \tag{22}$$

Since obviously $\mathring{\Delta}$ satisfies the so-called strong local Lipschitz condition, by the Sobolev embedding theorem [3, Theorem 4.12] one has a continuous injection

$$W^{4,2}(\mathring{\Delta}) \hookrightarrow C^2(\Delta),$$

where $C^2(\Delta)$ is the real Banach space of functions $\phi : \Delta \to \mathbb{R}$ which are of class C^2 in the open box $\overset{\circ}{\Delta}$ and which are continuous together with their first and second derivatives on all of the closed box Δ. The norm used on $C^2(\Delta)$ is the standard one

$$||\phi||_{C^2(\Delta)} \stackrel{\text{def}}{=} \sup_{x\in\Delta} \max_{|\nu|\leq 2} |\partial^\nu \phi(x)| \ . \tag{23}$$

Besides there is a constant C_{Sobolev} independent of the choice of Δ in Box_0, such that

$$||\phi||_{C^2(\Delta)} \leq C_{\text{Sobolev}} ||\phi||_{W^{4,2}(\overset{\circ}{\Delta})} \ . \tag{24}$$

Now define $\widetilde{\text{Fld}}(X)$ to be the finite direct sum of the Hilbert spaces $W^{4,2}(\overset{\circ}{\Delta})$ for Δ contained in X. We let $\text{Fld}(X)$ be the subspace of $\widetilde{\text{Fld}}(X)$ obtained by imposing the following *gluing conditions*. A field $\phi = (\phi_\Delta)_{\Delta\subset X}$ belongs to $\text{Fld}(X)$ if and only if, for any neighbouring boxes Δ_1, Δ_2 in X, the C^2 images by the Sobolev embedding of ϕ_{Δ_1} and ϕ_{Δ_2} coincide as well as their first and second derivatives, on the common boundary component $\Delta_1 \cap \Delta_2$. Again by the embedding theorem, this is a closed condition, and $\text{Fld}(X)$ is a real Hilbert space with the norm

$$||\phi||_{\text{Fld}(X)} \stackrel{\text{def}}{=} \left(\sum_{\Delta\subset X} \sum_{|\nu|\leq 4} ||\partial^\nu \phi_\Delta||^2_{L^2(\overset{\circ}{\Delta})} \right)^{\frac{1}{2}} \ . \tag{25}$$

Note that any polymer X is the closure of its interior. Hence, if one lets as before $C^2(X)$ be the space of functions $\phi : X \to \mathbb{R}$ which are of class C^2 in the, possibly disconnected, open set $\overset{\circ}{X}$ and which are continuous together with their first and second derivatives on the closed connected set X; and if the norm used on $C^2(X)$ is again the standard one

$$||\phi||_{C^2(X)} \stackrel{\text{def}}{=} \sup_{x\in X} \max_{|\nu|\leq 2} |\partial^\nu \phi(x)| \ ; \tag{26}$$

then it is not difficult to show that one has an embedding

$$\text{Fld}(X) \hookrightarrow C^2(X) \tag{27}$$

and an inequality

$$||\phi||_{C^2(X)} \leq C_{\text{Sobolev}} ||\phi||_{\text{Fld}(X)} \ . \tag{28}$$

The important thing here is that the constant is independent of X. We will often regard ϕ as a single function on X.

Remark 2. With this definition the lemmata [15, Lemma 5.1, Lemma 5.2] which are used for pointwise estimation of the fields, remain valid. The polygonal line arguments needed in [15, Lemma 5.1] as well as [11, Lemma 15] on which [15, Lemma 5.24] rests, are also preserved.

Now we will also need the notation

$$||\phi||_{X,1,4} \stackrel{\text{def}}{=} \left(\sum_{\Delta\subset X} \sum_{1\leq|\nu|\leq 4} ||\partial^\nu \phi_\Delta||^2_{L^2(\overset{\circ}{\Delta})} \right)^{\frac{1}{2}} \ . \tag{29}$$

This allows, given a parameter $\kappa > 0$, to define for any $\phi \in \mathrm{Fld}(X)$ the *large field regulator*

$$G_\kappa(X,\phi) \stackrel{\mathrm{def}}{=} \exp\left(\kappa||\phi||^2_{X,1,4}\right) . \tag{30}$$

An important point is that $G_\kappa(X,\cdot)$ is continuous on $\mathrm{Fld}(X)$.

3.2. Some natural maps. Note that if $X_1 \subset X_2$ are two polymers then there is an obvious linear continuous restriction map $\mathrm{Fld}(X_2) \to \mathrm{Fld}(X_1)$, $\phi \mapsto \phi|_{X_1}$. Indeed one first defines this projection from $\widetilde{\mathrm{Fld}}(X_2)$ to $\widetilde{\mathrm{Fld}}(X_1)$. Namely, it projects $\phi = (\phi_\Delta)_{\Delta\subset X_2}$ onto $(\phi_\Delta)_{\Delta\subset X_1}$. The gluing conditions for the image are automatically satisfied if they hold for the input ϕ.

Now let τ be an isometry of Euclidean $\mathbb{R}^3$ which leaves the lattice $\mathbb{Z}^3$ globally invariant, and let X be a polymer. One has a natural Hilbert space isometry $\mathrm{Fld}(X) \to \mathrm{Fld}(\tau^{-1}(X))$, $\phi \mapsto \phi\circ\tau$. Indeed one first defines this map on elements $\phi = (\phi_\Delta)_{\Delta\subset X} \in \widetilde{\mathrm{Fld}}(X)$, where each component is smooth on $\overset{\circ}{\Delta}$, by ordinary composition with τ. Then by density [3, Theorem 3.17], one extends it to a map $\widetilde{\mathrm{Fld}}(X) \to \widetilde{\mathrm{Fld}}(\tau^{-1}(X))$. Finally one takes the restriction to $\mathrm{Fld}(X)$ and corestriction to $\mathrm{Fld}(\tau^{-1}(X))$, since the gluing conditions are preserved.

We will also need an additional map. Let $X \in \mathrm{Poly}_0$. Then LX is also in Poly_0. Given $\phi \in \mathrm{Fld}(X)$ one can associate to it by a linear continuous map an element $\phi_{L^{-1}} \in \mathrm{Fld}(LX)$ as follows. First assume that $\phi = (\phi_\Delta)_{\Delta\subset X} \in \widetilde{\mathrm{Fld}}(X)$ is such that each ϕ_Δ is smooth on $\overset{\circ}{\Delta}$. Then for each $\Delta \subset X$, define $(\phi_\Delta)_{L^{-1}}(x) \stackrel{\mathrm{def}}{=} L^{-[\phi]}\phi_\Delta(L^{-1}x)$ which is smooth in the interior of $L\Delta$. Then for any unit box $\Delta' \subset L\Delta$ consider the restriction $(\phi_\Delta)_{L^{-1}}|_{\overset{\circ}{\Delta'}}$ to the interior of Δ'. The collection of all such restrictions for $\Delta' \subset L\Delta$ with $\Delta \subset X$ is by definition the image of ϕ in $\widetilde{\mathrm{Fld}}(LX)$. Then extend the map, by density, to all of $\widetilde{\mathrm{Fld}}(X)$. Finally the wanted map is obtained by restriction to $\mathrm{Fld}(X)$ and corestriction to $\mathrm{Fld}(LX)$, since the gluing conditions are easily seen to be preserved.

3.3. Gaussian measures. Now given any polymer X, and using the standard theory of Gaussian probability measures in Hilbert spaces [10, 70], it is not difficult to show that there exists a unique Borel (with respect to the $||.||_{\mathrm{Fld(X)}}$ norm topology) centered Gaussian probability measure $d\mu_{\Gamma,X}$ on $\mathrm{Fld}(X)$ such that for any $x, y \in X$, one has

$$\int d\mu_{\Gamma,X}(\zeta)\ \zeta(x)\zeta(y) = \Gamma(x-y), \tag{31}$$

where $\zeta(x)$ and $\zeta(y)$ are defined using the $C^2(X)$ realization of ζ. In other words the covariance of $d\mu_{\Gamma,X}$ is the *fluctuation covariance* Γ.

Indeed, one can define a continuous operator $\tilde{S} : \widetilde{\mathrm{Fld}}(X) \to \widetilde{\mathrm{Fld}}(X)$ as follows. If $\phi = (\phi_\Delta)_{\Delta\subset X} \in \widetilde{\mathrm{Fld}}(X)$ has smooth components, one defines its image $\tilde{S}\phi = \left((\tilde{S}\phi)_\Delta\right)_{\Delta\subset X}$ by letting for any $x \in \overset{\circ}{\Delta}$,

$$(\tilde{S}\phi)_\Delta(x) \stackrel{\mathrm{def}}{=} \sum_{\Delta'\subset X}\sum_{|\alpha|\le 4}\int_{\Delta'} dy\ (-1)^{|\alpha|}\partial^\alpha\Gamma(x-y)\ \partial^\alpha\phi_{\Delta'}(y) . \tag{32}$$

It is easy to see that $\tilde{S}$ extends on all of $\widetilde{\mathrm{Fld}}(X)$ to a continuous operator with norm bounded by $\max_{|\alpha|\le 8} ||\partial^\alpha \Gamma||_{L^\infty(\mathbb{R}^3)}$. Clearly this operator $\tilde{S}$ has its image contained in the closed subspace $\mathrm{Fld}(X)$. It is also symmetric, and positive. Now define the operator $S : \mathrm{Fld}(X) \to \mathrm{Fld}(X)$ by restriction and corestriction. It is easy to show that

$$tr\ \tilde{S} = tr\ S = |X| \cdot \sum_{|\alpha|\le 4} (-1)^{|\alpha|} \partial^{2\alpha} \Gamma(0)\ . \tag{33}$$

As a result S is a continuous symmetric positive trace class operator on $\mathrm{Fld}(X)$, i. e., a covariance operator. By the results in [70, Chap. 1], there exists a unique centered Borel Gaussian probability measure $d\mu_{\Gamma,X}$ on $\mathrm{Fld}(X)$ such that for any $\phi_1, \phi_2 \in \mathrm{Fld}(X)$,

$$\int d\mu_{\Gamma,X}(\zeta)\ (\phi_1, \zeta)(\phi_2, \zeta) = (\phi_1, S\phi_2)\ . \tag{34}$$

This equality also holds for ϕ_1, ϕ_2 more generally in $\widetilde{\mathrm{Fld}}(X)$ and with S replaced by $\tilde{S}$. It is not difficult to show that (31) follows from (34). The uniqueness of Gaussian measures satisfying (31) is also easy. Indeed one has the uniqueness of Gaussian measures satisfying (34), see [70, Chap. 1]. Besides, consider the continuous linear forms on $\widetilde{\mathrm{Fld}}(X)$ indexed by pairs (Δ, x), where $\Delta \subset X$ and $x \in \Delta$, obtained by evaluating at x the $C^2(\Delta)$ image of the component ϕ_Δ of a vector $\phi \in \widetilde{\mathrm{Fld}}(X)$. Let $\psi_{\Delta,x} \in \widetilde{\mathrm{Fld}}(X)$ be the corresponding vectors obtained by the Riesz representation theorem. By the injectivity of the Sobolev embedding, it is clear that the subspace generated by the vectors $\psi_{\Delta,x}$ is dense in $\widetilde{\mathrm{Fld}}(X)$. The uniqueness then follows easily.

Finally, note that if $X_1 \subset X_2$ are two polymers, then the direct image measure of $d\mu_{\Gamma,X_2}$, obtained by the restriction map $\phi \mapsto \phi|_{X_1}$, coincides with $d\mu_{\Gamma,X_1}$.

3.4. Polymer activities. Let $\mathbb{K}$ denote either the (algebraic) field of real numbers $\mathbb{R}$ or that of complex numbers $\mathbb{C}$. The main objects of study in this article are *polymer activities* or *polymer amplitudes*. These are functions (or functionals) $K(X, \cdot)$ from $\mathrm{Fld}(X)$ to $\mathbb{K}$. We will only consider functionals which are n_0 times continuously differentiable in the sense of Frechet between the *real* Banach spaces $\mathrm{Fld}(X)$ and $\mathbb{K}$ [7, Chap. 2], [25, Chap. VIII]. Here n_0 is a nonnegative integer constant which we will actually take to be $n_0 = 9$ as in [15].

Now consider for any integer n, $0 \le n \le n_0$, the $\mathbb{K}$-Banach space $\mathcal{L}_n(\mathrm{Fld}(X), \mathbb{K})$ of $\mathbb{R}$-multilinear continuous maps $W : \mathrm{Fld}(X)^n \to \mathbb{K}$ with the natural norm

$$||W||_\natural \stackrel{\mathrm{def}}{=} \sup_{\phi_1,\ldots,\phi_n \in \mathrm{Fld}(X)\setminus\{0\}} \frac{|W(\phi_1, \ldots, \phi_n)|}{||\phi_1||_{\mathrm{Fld}(X)} \cdots ||\phi_n||_{\mathrm{Fld}(X)}}\ . \tag{35}$$

Inside it sits the space $\mathcal{L}_n(\mathrm{Fld}(X), C^2(X), \mathbb{K})$ of W's for which the stronger norm

$$||W||_\sharp \stackrel{\mathrm{def}}{=} \sup_{\phi_1,\ldots,\phi_n \in \mathrm{Fld}(X)\setminus\{0\}} \frac{|W(\phi_1, \ldots, \phi_n)|}{||\phi_1||_{C^2(X)} \cdots ||\phi_n||_{C^2(X)}} \tag{36}$$

is finite. We indeed have for any $W \in \mathcal{L}_n(\mathrm{Fld}(X), C^2(X), \mathbb{K})$,

$$||W||_\natural \le C^n_{\mathrm{Sobolev}} ||W||_\sharp\ . \tag{37}$$

It is easy to see that $\mathcal{L}_n(\mathrm{Fld}(X), C^2(X), \mathbb{K})$ equipped with the sharp norm is a $\mathbb{K}$-Banach space. Let us denote by $C^{n_0}_\natural(\mathrm{Fld}(X), \mathbb{K})$ the $\mathbb{K}$-vector space of $\mathbb{K}$-valued functionals

$K(X,\cdot)$ defined on all of $\mathrm{Fld}(X)$, which are n_0 times continuously Frechet differentiable in the usual sense [7, 25] with respect to the $||\cdot||_{\mathrm{Fld}(X)}$ topology. We will also denote the n^{th} Frechet differential at the point $\phi \in \mathrm{Fld}(X)$ of a polymer activity $K(X,\cdot)$ by $D^n K(X,\phi)$. Its evaluation at the sequence of vectors $f_1,\ldots,f_n$ of $\mathrm{Fld}(X)$ is

$$D^n(X,\phi;f_1,\ldots,f_n) = \frac{\partial^n}{\partial s_1\ldots\partial s_n} K(X,\phi+s_1f_1+\cdots+s_nf_n)\Big|_{s=0}\ , \tag{38}$$

i.e., the corresponding directional or Gateau derivative. We then define the space $C_\sharp^{n_0}$ $(\mathrm{Fld}(X),\mathbb{K})$ of all $K(X,\cdot) \in C_\natural^{n_0}(\mathrm{Fld}(X),\mathbb{K})$ such that for all $\phi \in \mathrm{Fld}(X)$ and all integer n, $0 \le n \le n_0$, the differential $D^nK(X,\phi)$ belongs to $\mathcal{L}_n(\mathrm{Fld}(X), C^2(X),\mathbb{K})$, and such that the maps $\phi \mapsto D^nK(X,\phi)$ are continuous from $(\mathrm{Fld}(X),||\cdot||_{\mathrm{Fld}(X)})$ to $(\mathcal{L}_n(\mathrm{Fld}(X),C^2(X),\mathbb{K}),||\cdot||_\sharp)$. From now on the only norm we will be considering for differentials is the sharp one, therefore we will omit the symbol from the norm notation.

Given a parameter $h > 0$, a functional $K(X,\cdot) \in C_\sharp^{n_0}(\mathrm{Fld}(X),\mathbb{K})$ and a field $\phi \in \mathrm{Fld}(X)$, we define the *local norm*

$$||K(X,\phi)||_h \stackrel{\mathrm{def}}{=} \sum_{0\le n\le n_0} \frac{h^n}{n!}||D^nK(X,\phi)||\ . \tag{39}$$

This allows us to define the space $\mathcal{B}^{\mathbb{K}}_{h,G_\kappa}(X)$ of all $K(X) \in C_\sharp^{n_0}(\mathrm{Fld}(X),\mathbb{K})$ for which the norm

$$||K(X)||_{h,G_\kappa} \stackrel{\mathrm{def}}{=} \sup_{\phi\in\mathrm{Fld}(X)} G_\kappa(X,\phi)^{-1}||K(X,\phi)||_h \tag{40}$$

is finite. Now one has the following easy proposition.

Proposition 1. *For any $h,\kappa > 0$, the normed $\mathbb{K}$-vector space*

$$(\mathcal{B}^{\mathbb{K}}_{h,G_\kappa}(X), ||\cdot||_{h,G_\kappa})$$

is complete.

Now we consider an arbitrary element $K = (K(X))_{X\in\mathrm{Poly}_0}$ in the product

$$\prod_{X\in\mathrm{Poly}_0} \mathcal{B}^{\mathbb{K}}_{h,G_\kappa}(X)\ ,$$

and define the norm

$$||K||_{h,G_\kappa,\mathcal{A}} \stackrel{\mathrm{def}}{=} \sup_{\Delta\in\mathrm{Box}_0} \sum_{\substack{X\in\mathrm{Poly}_0\\ X\supset\Delta}} \mathcal{A}(X)\ ||K(X)||_{h,G_\kappa}, \tag{41}$$

where $\mathcal{A}$ is the previously defined large set regulator. Given a parameter $h_* > 0$, we also define the *kernel semi-norm*,

$$|K|_{h_*,\mathcal{A}} \stackrel{\mathrm{def}}{=} \sup_{\Delta\in\mathrm{Box}_0} \sum_{\substack{X\in\mathrm{Poly}_0\\ X\supset\Delta}} \mathcal{A}(X)\ ||K(X,0)||_{h_*}, \tag{42}$$

where the differentials are taken at the point $\phi = 0$ in each $\mathrm{Fld}(X)$. We now introduce the notion of *calibrator*: it is a new parameter $\bar{g} > 0$. We will use it to set

$$h = c\bar{g}^{-\frac{1}{4}} \tag{43}$$

for some fixed constant $c > 0$ to be adjusted later. We will take

$$h_* \stackrel{\mathrm{def}}{=} L^{\frac{3+\epsilon}{4}} . \tag{44}$$

The space of all K in the previous product space, such that $||K||_{h,G_\kappa,\mathcal{A}}$ and $|K|_{h_*,\mathcal{A}}$ are finite, is equipped with the *calibrated norm*

$$|||K|||_{\bar{g}} \stackrel{\mathrm{def}}{=} \max\left(|K|_{h_*,\mathcal{A}}, \bar{g}^2||K||_{h,G_\kappa,\mathcal{A}}\right), \tag{45}$$

and it is denoted by $\mathcal{BB}_{\bar{g}}^{\mathbb{K}}$. So as to keep notations under control we only emphasized the dependence on the calibrator $\bar{g}$ which is the most important one in what follows. One should keep in mind that the calibrated norm depends on $\bar{g}$ through the $\bar{g}^2$ factor in front of $||\cdot||_{h,G_\kappa,\mathcal{A}}$, but also through the relation (43) imposed between the h parameter and the calibrator $\bar{g}$. It is easy to see that $\mathcal{BB}_{\bar{g}}^{\mathbb{K}}$ with the norm $|||\cdot|||_{\bar{g}}$, is a $\mathbb{K}$-Banach space.

Now let τ be an isometry of Euclidean $\mathbb{R}^3$ which leaves the lattice $\mathbb{Z}^3$ globally invariant. This transformation can be made to act on an element K of $\mathcal{BB}_{\bar{g}}^{\mathbb{K}}$ by letting for any $X \in \mathrm{Poly}_0$, and any $\phi \in \mathrm{Fld}(X)$,

$$(\tau K)(X, \phi) \stackrel{\mathrm{def}}{=} K(\tau^{-1}(X), \phi \circ \tau), \tag{46}$$

where the map from $\mathrm{Fld}(X)$ to $\mathrm{Fld}(\tau^{-1}(X))$, given by $\phi \mapsto \phi \circ \tau$ is the one defined in Sect. 3.2. We will only consider $\tau \in \mathrm{Transf}$, where the set Transf is made of all translations by a vector $m = (m_1, m_2, m_3)$ in $\mathbb{Z}^3$, together with the three orthogonal reflections with respect to the coordinate planes respectively given by the equations $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$. We also define a transformation $K \mapsto K^-$ of $\mathcal{BB}_{\bar{g}}^{\mathbb{K}}$ by letting $K^-(X, \phi) \stackrel{\mathrm{def}}{=} K(X, -\phi)$.

The following lemma is an easy consequence of our previous definitions for norms.

Lemma 1. *The maps $K \mapsto \tau K$, for $\tau \in$ Transf, as well as the map $K \mapsto K^-$, are Banach space isometries of $\mathcal{BB}_{\bar{g}}^{\mathbb{K}}$.*

Thanks to this lemma we can finally define the main setting for a single RG map. It is the space $\mathcal{BBS}_{\bar{g}}^{\mathbb{K}}$ of all collections of polymer activities $K \in \mathcal{BB}_{\bar{g}}^{\mathbb{K}}$ such that $K^- = K$ and for any $\tau \in \mathrm{Transf}$, $\tau K = K$. By the previous lemma it is a closed subspace of $\mathcal{BB}_{\bar{g}}^{\mathbb{K}}$ and therefore a $\mathbb{K}$-Banach space for the norm $|||\cdot|||_{\bar{g}}$.

Remark 3. Note that all the calibrated norms, obtained for different values of $\bar{g}$, are equivalent. The underlying topological vector spaces of the $\mathcal{BB}_{\bar{g}}^{\mathbb{K}}$'s are therefore the same.

The RG map we are interested in is one from a domain in $\mathbb{R} \times \mathbb{R} \times \mathcal{BBS}_{\bar{g}}^{\mathbb{R}}$ for some values of the parameters into another analogous triple-product space with a slightly different value of $\bar{g}$. We will need complex versions of these spaces in order to obtain Lipschitz contractive estimates with the least effort. The global trajectory we construct in this article will be obtained by a contraction mapping theorem in a big Banach space of sequences $\mathcal{BBSS}^{\mathbb{K}}$ to be precisely defined in Sect. 5 below.

4. The Algebraic Definition of the RG Map

In this section we provide all the formulae which express the RG map studied in [15]. We consider an input $(g, \mu, R) \in \mathbb{C} \times \mathbb{C} \times \mathcal{BBS}^{\mathbb{C}}$; and we will give the algebraic definition for the output (g', μ', R'). Recall that the latter have the form

$$g' = L^{\epsilon} g - L^{2\epsilon} a(L, \epsilon) g^2 + \xi_g(g, \mu, R) , \tag{47}$$

$$\mu' = L^{\frac{3+\epsilon}{2}} \mu + \xi_\mu(g, \mu, R) , \tag{48}$$

$$R' = \mathcal{L}^{(g,\mu)}(R) + \xi_R(g, \mu, R), \tag{49}$$

where $a(L, \epsilon)$ has already been defined. We will therefore provide the expressions for the ξ remainders as well as for $\mathcal{L}^{(g,\mu)}(R)$.

4.1. The local potentials. For any $X \in \text{Poly}_0$, any Borel set $Z \subset \mathbb{R}^3$, and any $\phi \in \text{Fld}(X)$, we let

$$V(X, Z, \phi) \stackrel{\text{def}}{=} g \int_{Z\cap X} d^3x \ :\phi(x)^4 :_C + \mu \int_{Z\cap X} d^3x \ :\phi(x)^2 :_C \ . \tag{50}$$

We refer for instance to [43, 66] for a discussion of Wick ordering $:\bullet:_C$. Otherwise the explicit expressions

$$:\phi(x)^2:_C = \phi(x)^2 - C(0) \tag{51}$$

and

$$:\phi(x)^4:_C = \phi(x)^4 - 6C(0)\phi(x)^2 + 3C(0)^2 \tag{52}$$

may be used as definitions. Note that in [15] the notation is simplified to $V(Z, \phi)$ or even $V(Z)$ leaving the ϕ dependence implicit. Here we prefer to keep everything explicit including the first X argument which allows one to keep track of which space $\text{Fld}(\cdot)$ the field ϕ lives in. Also note that the function ϕ used in the integral formula above is of course the $C^2(X)$ realization of $\phi \in \text{Fld}(X)$ via the embedding (27). Another remark is that although we made the definition sound quite general by allowing Z to be any Borel set, we will only need such Z's which are complements of the union of some L^{-1}-polymers in X. Now define

$$g_L \stackrel{\text{def}}{=} L^{\epsilon} g , \tag{53}$$

$$\mu_L \stackrel{\text{def}}{=} L^{\frac{3+\epsilon}{2}} \mu , \tag{54}$$

$$C_{L^{-1}}(x) \stackrel{\text{def}}{=} L^{-2[\phi]} C(L^{-1}x), \tag{55}$$

and as in (50) let

$$\tilde{V}(X, Z, \phi) \stackrel{\text{def}}{=} g \int_{Z\cap X} d^3x :\phi(x)^4:_{C_{L^{-1}}} + \mu \int_{Z\cap X} d^3x :\phi(x)^2:_{C_{L^{-1}}}, \tag{56}$$

where Wick ordering is with respect to $C_{L^{-1}}$ instead of C. Also let

$$\tilde{V}_L(X, Z, \phi) \stackrel{\text{def}}{=} g_L \int_{Z\cap X} d^3x :\phi(x)^4:_C + \mu_L \int_{Z\cap X} d^3x :\phi(x)^2:_C \ . \tag{57}$$

4.2. The w kernels. We now deal with the hidden variable $\mathbf{w}$. Note that by construction the cutoff function u_ϵ satisfies $u_\epsilon(x) = 0$ if $|x|_2 \geq 1$ and a fortiori if $|x|_\infty \geq 1$. This implies that the fluctuation covariance Γ satisfies $\Gamma(x) = 0$ if $|x|_\infty \geq L$. Now we define $\mathbf{w} = \mathbf{w}_* = (w^{(1)}, w^{(2)}, w^{(3)})$ to be a triple of real functions $w^{(p)} \in \mathcal{W}_p$, where $\mathcal{W}_p$, $p = 1, 2, 3$, is the weighted L^∞ space $L^\infty(\mathbb{R}^3, |x|_\infty^{\frac{3p}{2}} d^3x)$. Namely, $f \in \mathcal{W}_p$ if and only if $f : \mathbb{R}^3 \to \mathbb{R}$ is measurable and

$$||f||_p \stackrel{\text{def}}{=} \text{ess.} \sup_{x \in \mathbb{R}^3} \left(|x|_\infty^{\frac{3p}{2}} |f(x)| \right) \tag{58}$$

is finite. The w's were constructed in [15, Lemma 5.9] by a Banach fixed point argument. We instead give them explicitly, for $x \neq 0$, by

$$w^{(p)}(x) \stackrel{\text{def}}{=} \tilde{C}(x)^p - C(x)^p \tag{59}$$

$$= \left[C(x) + \int_0^1 \frac{dl}{l}\, l^{-\left(\frac{3-\epsilon}{2}\right)} u_\epsilon\left(\frac{x}{l}\right) \right]^p - C(x)^p \,. \tag{60}$$

From the last equation it is clear that $w^{(p)}(x) = 0$ if $|x|_\infty \geq 1$. Besides, since $u_\epsilon \geq 0$, for ϵ small one has

$$|w^{(p)}(x)| \leq \tilde{C}(x)^p = \frac{\varkappa_\epsilon^p}{|x|_2^{p\left(\frac{3-\epsilon}{2}\right)}} \leq \frac{\mathcal{O}(1)}{|x|_\infty^{\frac{3p}{2}}} \,. \tag{61}$$

The fixed point property $\mathbf{w} = \mathbf{w}_*$ is embodied in the equation

$$w^{(p)}(x) = v^{(p)}(x) + w_L^{(p)}(x) \tag{62}$$

for any $x \neq 0$, where we used the notation

$$v^{(p)}(x) \stackrel{\text{def}}{=} C_L(x)^p - C(x)^p \tag{63}$$

and

$$w_L^{(p)}(x) \stackrel{\text{def}}{=} L^{2p[\phi]} w^{(p)}(Lx) \,. \tag{64}$$

Equation (62) trivially follows from the given definition.

4.3. The renormalized expanded quadratic activity Q. For $X \in \text{Poly}_0$ and $\phi \in \text{Fld}(X)$ we define the activity $Q(X, \phi)$ as follows. If X is not ultrasmall we let $Q(X, \phi) \stackrel{\text{def}}{=} 0$. If X is ultrasmall we introduce an associated integration domain $\tilde{X} \subset \mathbb{R}^3 \times \mathbb{R}^3$. If X is reduced to a single unit box Δ, we let $\tilde{X} \stackrel{\text{def}}{=} \Delta \times \Delta$. If $X = \Delta_1 \cup \Delta_2$, where the boxes Δ_1 and Δ_2 are distinct but neighbouring, we let

$$\tilde{X} \stackrel{\text{def}}{=} (\Delta_1 \times \Delta_2) \cup (\Delta_2 \times \Delta_1) \,. \tag{65}$$

We now write

$$\begin{aligned} Q(X, \phi) \stackrel{\text{def}}{=} g^2 \int_{\tilde{X}} d^3x\, d^3y \Big\{ & -24 w^{(3)}(x-y) : (\phi(x) - \phi(y))^2 :_C \\ & -18 w^{(2)}(x-y) : (\phi(x)^2 - \phi(y)^2)^2 :_C \\ & + 8 w^{(1)}(x-y) : \phi(x)^3 \phi(y)^3 :_C \Big\} \,. \end{aligned} \tag{66}$$

For reference, the Wick ordered expressions are explicitly given by

$$:(\phi(x)-\phi(y))^2:_C = (\phi(x)-\phi(y))^2 - 2C(0) + 2C(x-y)\ , \quad (67)$$

$$:(\phi(x)^2-\phi(y)^2)^2:_C = (\phi(x)^2-\phi(y)^2)^2 - 4C(0)\phi(x)^2 - 4C(0)\phi(y)^2 + 8C(x-y)\phi(x)\phi(y) + 4C(0)^2 - 4C(x-y)^2, \quad (68)$$

and

$$\begin{aligned}:\phi(x)^3\phi(y)^3:_C = {} & \phi(x)^3\phi(y)^3 - 3C(0)\phi(x)\phi(y)^3 - 3C(0)\phi(x)^3\phi(y) \\ & -9C(x-y)\phi(x)^2\phi(y)^2 + 9C(0)^2\phi(x)\phi(y) \\ & +18C(x-y)^2\phi(x)\phi(y) + 9C(0)C(x-y)\phi(x)^2 \\ & +9C(0)C(x-y)\phi(y)^2 - 9C(0)^2C(x-y) \\ & -6C(x-y)^3\ . \end{aligned} \quad (69)$$

4.4. Integration on fluctuations, reblocking and rescaling. For any unit box Δ and fields $\phi, \zeta \in \text{Fld}(\Delta)$ we define

$$P(\Delta,\phi,\zeta) \stackrel{\text{def}}{=} e^{-V(\Delta,\Delta,\phi+\zeta)} - e^{-\tilde{V}(\Delta,\Delta,\phi)}\ . \quad (70)$$

Now for any $X \in \text{Poly}_0$ and $\phi \in \text{Fld}(X)$ we let

$$K(X,\phi) \stackrel{\text{def}}{=} Q(X,\phi)e^{-V(X,X,\phi)} + R(X,\phi)\ . \quad (71)$$

We also define

$$R^\sharp(X,\phi) \stackrel{\text{def}}{=} \int d\mu_{\Gamma,X}(\zeta)\ R(X,\phi+\zeta)\ , \quad (72)$$

as well as

$$\begin{aligned}(\mathcal{S}K)^\natural(X,\phi) \stackrel{\text{def}}{=} \int d\mu_{\Gamma,LX}(\zeta) \Bigg\{ & \sum_{\substack{M,N \\ M+N\geq 1}} \frac{1}{M!N!} \sum_{(\Delta_1,\ldots,\Delta_M),(X_1,\ldots,X_N)} \\ & \exp\left[-\tilde{V}\left(LX, LX\backslash\left(\left(\cup_{i=1}^M \Delta_i\right)\cup\left(\cup_{j=1}^N X_j\right)\right), \phi_{L^{-1}}\right)\right] \\ & \times \prod_{i=1}^M P\left(\Delta_i, \phi_{L^{-1}}|_{\Delta_i}, \zeta|_{\Delta_i}\right) \times \prod_{j=1}^N K\left(X_j, \phi_{L^{-1}}|_{X_j} + \zeta|_{X_j}\right)\Bigg\}\ , \end{aligned} \quad (73)$$

where the sum over sequences $(\Delta_1,\ldots,\Delta_M)$ and $(X_1,\ldots,X_N)$ is subjected to the following conditions:

1. The Δ_i are *distinct* boxes in Box_0.
2. The X_j are *disjoint* polymers in Poly_0.
3. None of the Δ_i is contained in an X_j.
4. The L-closure of the union of all the Δ_i and the X_j is exactly the set LX.

Remark 4. Note that since the X_j are *closed* polymers, the disjointness condition means that they cannot touch each other and have to be at least 1 apart in $|\cdot|_\infty$ distance. However, the Δ_i are allowed to touch each other or an X_j, by sharing no more than a boundary component. Also note that by hypothesis, X and therefore LX is connected. This rules out situations where for instance the (X_j) sequence would be empty, and the (Δ_i) sequence would be made of two boxes very far apart.

4.5. Preparations for the extraction. As a preparation for the crucial so called extraction step we need to introduce for any $X \in \mathrm{Poly}_0$ the quantities denoted by $\tilde{\alpha}_0(X)$, $\tilde{\alpha}_2(X)$, $\tilde{\alpha}_{2,\mu}(X)$ for $\mu = 1, 2, 3$, and $\tilde{\alpha}_4(X)$. These are by definition all set to zero if X is large. Now if X is small one lets

$$\tilde{\alpha}_0(X) \stackrel{\text{def}}{=} \frac{e^{\tilde{V}(X,X,0)}}{|X|} R^\sharp(X,0) , \tag{74}$$

$$\tilde{\alpha}_2(X) \stackrel{\text{def}}{=} \frac{e^{\tilde{V}(X,X,0)}}{2|X|} \left[D^2(R^\sharp)(X,0;1,1) + R^\sharp(X,0) D^2\tilde{V}(X,X,0;1,1) \right], \tag{75}$$

where the last two arguments of the differentials are given by the constant function equal to 1, seen as an element of $\mathrm{Fld}(X)$. We also let for $\mu = 1, 2, 3$,

$$\tilde{\alpha}_{2,\mu}(X) \stackrel{\text{def}}{=} \frac{e^{\tilde{V}(X,X,0)}}{|X|} \Big[D^2(R^\sharp)(X,0;1,\Delta_X x_\mu) + R^\sharp(X,0) D^2\tilde{V}(X,X,0;1,\Delta_X x_\mu) \Big], \tag{76}$$

where $\Delta_X x_\mu$ means the function

$$x \mapsto x_\mu - \frac{1}{|X|}\left(\int_X d^3y \; y_\mu \right),$$

the deviation from average of the coordinate function x_μ on the polymer X, again seen as an element of $\mathrm{Fld}(X)$. Finally one lets

$$\begin{aligned}\tilde{\alpha}_4(X) \stackrel{\text{def}}{=} \frac{e^{\tilde{V}(X,X,0)}}{24|X|} \Big[& D^4(R^\sharp)(X,0;1,1,1,1) \\ & + 6D^2(R^\sharp)(X,0;1,1) D^2\tilde{V}(X,X,0;1,1) \\ & + R^\sharp(X,0) D^4\tilde{V}(X,X,0;1,1,1,1) \\ & + 3R^\sharp(X,0) \left(D^2\tilde{V}(X,X,0;1,1) \right)^2 \Big] .\end{aligned} \tag{77}$$

Now given $Z \in \mathrm{Poly}_0$, and $x \in \mathbb{R}^3$ we define

$$\alpha_0(Z,x) \stackrel{\text{def}}{=} \sum_{X \text{ small},\ \bar{X}^L = LZ} \tilde{\alpha}_0(X) L^3 \mathbb{1}_{L^{-1}X}(x) , \tag{78}$$

$$\alpha_2(Z,x) \stackrel{\text{def}}{=} \sum_{X \text{ small},\ \bar{X}^L = LZ} \tilde{\alpha}_2(X) L^{\frac{3+\epsilon}{2}} \mathbb{1}_{L^{-1}X}(x) , \tag{79}$$

$$\alpha_{2,\mu}(Z,x) \stackrel{\text{def}}{=} \sum_{X \text{ small},\ \bar{X}^L = LZ} \tilde{\alpha}_{2,\mu}(X) L^{\frac{1+\epsilon}{2}} \mathbb{1}_{L^{-1}X}(x) , \tag{80}$$

$$\alpha_4(Z,x) \stackrel{\text{def}}{=} \sum_{X \text{ small},\ \bar{X}^L = LZ} \tilde{\alpha}_4(X) L^{\epsilon} \mathbb{1}_{L^{-1}X}(x), \tag{81}$$

where again $\mu = 1, 2, 3$, and $\mathbb{1}_{L^{-1}X}$ denotes the sharp characteristic function of the set $L^{-1}X$. Note that these quantities vanish if Z is not small or if $x \notin Z$.

Now choose some reference box $\Delta_0 \in \mathrm{Box}_0$. We define

$$\alpha_0 \stackrel{\text{def}}{=} L^3 \sum_{X \text{ small},\ X \supset \Delta_0} \tilde{\alpha}_0(X), \tag{82}$$

$$\alpha_2 \stackrel{\text{def}}{=} L^{\frac{3+\epsilon}{2}} \sum_{X \text{ small},\ X \supset \Delta_0} \tilde{\alpha}_2(X), \tag{83}$$

$$\alpha_4 \stackrel{\text{def}}{=} L^{\epsilon} \sum_{X \text{ small},\ X \supset \Delta_0} \tilde{\alpha}_4(X). \tag{84}$$

Note that the latter do not depend on the choice of Δ_0 because of the translational invariance imposed on polymer activities in Sect. 3. Also note that in [15, Eq. 4.44] the quantities

$$\alpha_{2,\mu} \stackrel{\text{def}}{=} L^{\frac{1+\epsilon}{2}} \sum_{X \text{ small},\ X \supset \Delta_0} \tilde{\alpha}_{2,\mu}(X) \tag{85}$$

for $\mu = 1, 2, 3$, were also defined. However, again by the conditions imposed on polymer activities in Sect. 3, it is easy to see that the latter always vanish. In other words, the RG flow does not create $\phi\partial\phi$ terms in the effective potential.

After one has defined

$$b(L, \epsilon) \stackrel{\text{def}}{=} 48 \int_{\mathbb{R}^3} d^3x\ v^{(3)}(x) \ ; \tag{86}$$

one can at last give some of the outputs of the RG map. Namely, one poses

$$\xi_g(g, \mu, R) \stackrel{\text{def}}{=} -\alpha_4 \ , \tag{87}$$

$$\xi_\mu(g, \mu, R) \stackrel{\text{def}}{=} -\left(L^{2\epsilon} b(L, \epsilon) g^2 + \alpha_2 + 6C(0)\alpha_4\right), \tag{88}$$

as definition of the first two remainder terms. At this point, the new couplings are defined via

$$g' \stackrel{\text{def}}{=} L^{\epsilon} g - L^{2\epsilon} a(L, \epsilon) g^2 + \xi_g(g, \mu, R) \ , \tag{89}$$

$$\mu' \stackrel{\text{def}}{=} L^{\frac{3+\epsilon}{2}} \mu + \xi_\mu(g, \mu, R) \ . \tag{90}$$

What remains is $\mathcal{L}^{(g,\mu)}(R)$, $\xi_R(g, \mu, R)$ and their combination R'.

4.6. The linear map for R. In order to define the linear part $\mathcal{L}^{(g,\mu)}(R)$ which was denoted by R_{linear} in [15], we need to introduce two polymer activities. For $X \in \mathrm{Poly}_0$, and $\phi \in \mathrm{Fld}(X)$, we let $\tilde{F}_R(X, \phi) \stackrel{\text{def}}{=} 0$ if X is large; otherwise we let

$$\tilde{F}_R(X, \phi) \stackrel{\text{def}}{=} \int_X d^3x \left[\tilde{\alpha}_4(X)\phi(x)^4 + \tilde{\alpha}_2(X)\phi(x)^2 + \sum_{\mu=1}^{3} \tilde{\alpha}_{2,\mu}(X)\phi(x)\partial_\mu\phi(x) + \tilde{\alpha}_0(X)\right] . \tag{91}$$

Regardless of whether X is small or not, we also let

$$J(X,\phi) \stackrel{\text{def}}{=} R^{\sharp}(X,\phi) - \tilde{F}_R(X,\phi)e^{-\tilde{V}(X,X,\phi)} . \tag{92}$$

The previous complicated definitions of the $\tilde{\alpha}_{\dots}(X)$ had no other purpose but to secure the following *normalization conditions*. For any small polymer X, and for $\mu = 1, 2, 3$, one needs

$$J(X,\phi) = 0 , \tag{93}$$

$$D^2 J(X, 0; 1, 1) = 0 , \tag{94}$$

$$D^2 J(X, 0; 1, \Delta_X x_\mu) = 0 , \tag{95}$$

$$D^4 J(X, 0; 1, 1, 1, 1) = 0 . \tag{96}$$

Note that one would have equivalent conditions if one replaced the function $\Delta_X x_\mu$ simply by the coordinate function x_μ. These normalization conditions are the analog in the present setting of the BPHZ subtraction prescription (see e.g. [64]). They are the main reason why the map $\mathcal{L}^{(g,\mu)}(\cdot)$ we are about to define is contractive.

Now given $X \in \text{Poly}_0$, and $\phi \in \text{Fld}(X)$, and using constrained sums over polymers $Y \in \text{Poly}_0$, we define

$$\begin{aligned}\mathcal{L}^{(g,\mu)}(R)(X,\phi) \stackrel{\text{def}}{=} & \sum_{Y \text{ small},\ \bar{Y}^L = LX} J(Y, \phi_{L^{-1}}|_Y) e^{-\tilde{V}_L(X, X\setminus L^{-1}Y, \phi)} \\ & + \sum_{Y \text{ large},\ \bar{Y}^L = LX} R^{\sharp}(Y, \phi_{L^{-1}}|_Y) e^{-\tilde{V}_L(X, X\setminus L^{-1}Y, \phi)} .\end{aligned} \tag{97}$$

4.7. The extraction proper. Given $X \in \text{Poly}_0$, and $x \in \mathbb{R}^3$, we define the function $f_Q^{(4)}(X,x)$ as follows.

First case. If X is given by a single box $\Delta \in \text{Box}_0$, and if x lies in the interior of Δ, we let

$$f_Q^{(4)}(X,x) \stackrel{\text{def}}{=} \int_{\Delta} d^3y\ v^{(2)}(x-y) . \tag{98}$$

Second case. If X is given by the union of two distinct neighbouring boxes Δ_1, $\Delta_2 \in \text{Box}_0$, and if x lies in the interior of say Δ_1, we let

$$f_Q^{(4)}(X,x) \stackrel{\text{def}}{=} \int_{\Delta_2} d^3y\ v^{(2)}(x-y) . \tag{99}$$

Third case. If none of the first two cases apply, we simply let $f_Q^{(4)}(X,x) \stackrel{\text{def}}{=} 0$.

One can in the same manner define a function $f_Q^{(2)}(X,x)$ using $v^{(3)}$ instead of $v^{(2)}$, as well as a function $f_Q^{(0)}(X,x)$ using $v^{(4)}$ which is given by $v^{(4)}(z) \stackrel{\text{def}}{=} C_L(z)^4 - C(z)^4$.

Now let $X \in \text{Poly}_0$, and Z be a Borel set in $\mathbb{R}^3$, and define

$$F_{0,Q}(X,Z) \stackrel{\text{def}}{=} 12 g_L^2 \int_Z d^3x f_Q^{(0)}(X,x), \tag{100}$$

as well as

$$F_{0,R}(X,Z) \stackrel{\text{def}}{=} \int_Z d^3x \left\{\alpha_0(X,x) + C(0)\alpha_2(X,x) + 3C(0)^2\alpha_4(X,x)\right\} \tag{101}$$

and

$$F_0(X,Z) \stackrel{\text{def}}{=} F_{0,Q}(X,Z) + F_{0,R}(X,Z)\ . \tag{102}$$

If in addition one has a polymer $Y \in \text{Poly}_0$, and a field $\phi \in \text{Fld}(Y)$, one can also define

$$\begin{aligned} F_{1,Q}(X,Y,Z,\phi) \stackrel{\text{def}}{=} & \ 36g_L^2 \int_{Z\cap Y} d^3x :\phi(x)^4:_C \ f_Q^{(4)}(X,x) \\ & +48g_L^2 \int_{Z\cap Y} d^3x :\phi(x)^2:_C \ f_Q^{(2)}(X,x) \end{aligned} \tag{103}$$

as well as

$$\begin{aligned} F_{1,R}(X,Y,Z,\phi) \stackrel{\text{def}}{=} \int_{Z\cap Y} d^3x \Big\{ & \alpha_4(X,x) :\phi(x)^4:_C \\ & + \sum_{\mu=1}^{3} \alpha_{2,\mu}(X,x) :\phi(x)\partial_\mu\phi(x):_C \\ & + (\alpha_2(X,x) + 6C(0)\alpha_4(X,x)) :\phi(x)^2:_C \Big\}, \end{aligned} \tag{104}$$

where $:\phi(x)\partial_\mu\phi(x):_C$ reduces to $\phi(x)\partial_\mu\phi(x)$. We finally need

$$F_1(X,Y,Z,\phi) \stackrel{\text{def}}{=} F_{1,Q}(X,Y,Z,\phi) + F_{1,R}(X,Y,Z,\phi)\ , \tag{105}$$

and

$$F(X,Y,Z,\phi) \stackrel{\text{def}}{=} F_0(X,Z) + F_1(X,Y,Z,\phi)\ . \tag{106}$$

As before the Y argument is for keeping track of which $\text{Fld}(\cdot)$ the ϕ lives in. The Z defines the domain of integration. The new argument X, is here to indicate that the F's are *local counterterms* for a polymer activity which originally lived on X.

Now given $X \in \text{Poly}_0$, and $\phi \in \text{Fld}(X)$, we let

$$\begin{aligned} \tilde{K}(X,\phi) \stackrel{\text{def}}{=} (\mathcal{S}K)^\natural(X,\phi) - e^{-\tilde{V}_L(X,X,\phi)} \times \sum_{N\ge 1} \frac{1}{N!} \sum_{(Y_1,\ldots,Y_N)} \\ \prod_{i=1}^{N} \left[\exp\left(F\left(Y_i,Y_i,Y_i,\phi|_{Y_i}\right)\right) - 1\right], \end{aligned} \tag{107}$$

where the sum is over all sequences of *distinct* polymers $Y_i \in \text{Poly}_0$ whose union is equal to X.

Again given $X \in \text{Poly}_0$, a Borel set Z, and a field $\phi \in \text{Fld}(X)$, we define

$$V_F(X,Z,\phi) \stackrel{\text{def}}{=} \sum_{\substack{\Delta\in\text{Box}_0 \\ \mathring{\Delta}\subset Z\cap X}} \left[\tilde{V}_L(\Delta,\Delta,\phi|_\Delta) - \sum_{\substack{Y\in\text{Poly}_0 \\ Y\supset\Delta}} F(Y,\Delta,\Delta,\phi|_\Delta) \right]. \tag{108}$$

Mind the inclusion condition only on the interior $\overset{\circ}{\Delta}$ of Δ.

Then for $X \in \text{Poly}_0$, and $\phi \in \text{Fld}(X)$, we let

$$\tilde{\mathcal{E}}(X,\phi) \stackrel{\text{def}}{=} \sum_{M\geq 1,\ N\geq 0} \frac{1}{M!N!} \sum_{(X_1,\ldots,X_M),\ (Z_1,\ldots,Z_N)} \exp\left(-V_F\left(X, X\backslash\left(\cup_{i=1}^{M} X_i\right), \phi\right)\right) \times \prod_{i=1}^{M} \tilde{K}\left(X_i, \phi|_{X_i}\right) \times \prod_{j=1}^{N} \left[\exp\left(-F\left(Z_j, Z_j, Z_j\backslash\left(\cup_{i=1}^{M} X_i\right), \phi|_{Z_j}\right)\right) - 1\right] \tag{109}$$

with the following conditions imposed on the X_i and Z_j:

1. The X_i and Z_j are polymers in Poly_0.
2. The X_i are *disjoint*.
3. The Z_j are *distinct*.
4. Every Z_j has a nonempty intersection, be it by an edge or a corner, with $\cup_{i=1}^{M} X_i$.
5. Every Z_j has a nonempty intersection with $X\backslash\left(\cup_{i=1}^{M} X_i\right)$.
6. The union of all the X_i and Z_j is exactly the given polymer X.

Remark 5. We emphasized the condition on the interior of Δ in (108), and the weak notion of intersection in items (4) and (5) above, since these are the notable modifications to make on the treatment of [11, Sect. 4.2] in order to account for the *closed* polymers used in [15] and here. The overlap connectedness in [11, Sect. 4.2] is automatically implied by item (6) above and the connectedness of the set X which is assumed a priori. Also note that this notion was defined in [11, Sect. 4.2] based on the idea of having a full box in common, whereas here a nonempty intersection by a boundary component already counts as an overlap. Finally note that if $M \geq 2$ then one needs to have $N \geq 1$; this is because the X_i are forced to be at least 1 apart with respect to the $|\cdot|_\infty$ distance, and they need a bridge of Z_j's joining them.

Now given $X \in \text{Poly}_0$, and $\phi \in \text{Fld}(X)$, we let

$$\mathcal{E}(X,\phi) \stackrel{\text{def}}{=} \tilde{\mathcal{E}}(X,\phi) \times \exp\left[-\sum_{\substack{\Delta\in\text{Box}_0\\ \Delta\subset X}} \sum_{\substack{Y\in\text{Poly}_0\\ Y\supset\Delta}} F_0(Y,\Delta)\right]. \tag{110}$$

Finally we define $Q'(X,\phi)$ in exactly the same way as $Q(X,\phi)$ in Sect. 4.3 but using the new coupling g' obtained in Sect. 4.5 instead of the old one g. Likewise we need a potential $V'(X,Z,\phi)$ defined in the exact same manner as $V(X,Z,\phi)$ in Sect. 4.1 using the new couplings g', μ' instead of g, μ. At last one can give the output R' of the RG map defined for any $X \in \text{Poly}_0$, and $\phi \in \text{Fld}(X)$ by

$$R'(X,\phi) \stackrel{\text{def}}{=} \mathcal{E}(X,\phi) - Q'(X,\phi)e^{-V'(X,X,\phi)}. \tag{111}$$

In somewhat of a roundabout manner, the definition of the ξ_R remainder is then

$$\xi_R(g,\mu,R)(X,\phi) \stackrel{\text{def}}{=} R'(X,\phi) - \mathcal{L}^{(g,\mu)}(R)(X,\phi). \tag{112}$$

The algebraic definition of the RG map is now complete. Note that the Frechet differentiability of the output polymer activities, the justification of the measurability of the integrations over ζ, follow once the proper estimates are established because of the algebraic nature of the operations used in this section. These estimates have been provided in [15, Sect. 5], and their result is summarized in Theorem 1 below.

5. The Dynamical System Construction

The RG map for which the defining formulae were given in the previous section is $(g, \mu, R) \mapsto (g', \mu', R')$, where

$$\begin{cases} g' &= L^{\epsilon} g - L^{2\epsilon} a(L, \epsilon) g^2 + \xi_g(g, \mu, R) \ , \\ \mu' &= L^{\frac{3+\epsilon}{2}} \mu + \xi_\mu(g, \mu, R) \ , \\ R' &= \mathcal{L}^{(g,\mu)}(R) + \xi_R(g, \mu, R) \ . \end{cases} \tag{113}$$

Our aim is to construct a double-sided sequence $s = (g_n, \mu_n, R_n)_{n\in\mathbb{Z}}$ which solves this recursion and such that $\lim_{n\to-\infty} (g_n, \mu_n, R_n) = (0, 0, 0)$ the Gaussian ultraviolet fixed point, and $\lim_{n\to+\infty} (g_n, \mu_n, R_n) = (g_*, \mu_*, R_*)$ the BMS nontrivial infrared fixed point [15]. We proceed as follows. We will simply write a for the coefficient $a(L, \epsilon) > 0$. We also take $\epsilon > 0$ small enough so that $L^{\epsilon} \in]1, 2[$. Recall that $\bar{g}_* = \frac{L^{\epsilon}-1}{L^{2\epsilon} a} > 0$ and consider the function

$$\begin{array}{l} f : [0, \bar{g}_*] \to [0, \bar{g}_*] \\ \quad x \mapsto f(x) = L^{\epsilon} x - L^{2\epsilon} a x^2 \ . \end{array} \tag{114}$$

It is trivial to see that f is a strictly increasing diffeomorphism of $[0, \bar{g}_*]$; it is also strictly concave. The only fixed points are 0 and $\bar{g}_*$, and $f(x) > x$ in the interval $]0, \bar{g}_*[$. Given $\omega_0 \in]0, 1[$, there is a unique double-sided sequence $(\bar{g}_n)_{n\in\mathbb{Z}}$ in $]0, \bar{g}_*[^{\mathbb{Z}}$ such that $\bar{g}_0 = \omega_0 \bar{g}_*$, and for any $n \in \mathbb{Z}$, $\bar{g}_{n+1} = f(\bar{g}_n)$. This sequence is strictly increasing from 0 when $n \to -\infty$, to $\bar{g}_*$ when $n \to +\infty$. We call $\bar{g}_0$ the coupling at unit scale. Once it is chosen it defines the sequence $(\bar{g}_n)_{n\in\mathbb{Z}}$ completely. Moreover, if one ignores the remainder terms ξ in (113) then the renormalization group recursion is solved by the approximate sequence $\bar{s} \stackrel{\text{def}}{=} (\bar{g}_n, 0, 0)_{n\in\mathbb{Z}}$. The true trajectory will be constructed in such a way that $g_0 = \bar{g}_0$, and via the construction of the deviation sequence $\delta s = (\delta g_n, \mu_n, R_n)_{n\in\mathbb{Z}}$ with respect to the approximate sequence $\bar{s}$. Using the notation $\delta g_n = g_n - \bar{g}_n$, the new recursion which is equivalent to (113) that we have to solve is

$$\begin{cases} \delta g_{n+1} &= f'(\bar{g}_n)\delta g_n + \left[-L^{2\epsilon} a \ \delta g_n^2 \ + \xi_g(\bar{g}_n + \delta g_n, \mu_n, R_n)\right] \ , \\ \mu_{n+1} &= L^{\frac{3+\epsilon}{2}} \mu_n + \xi_\mu(\bar{g}_n + \delta g_n, \mu_n, R_n) \ , \\ R_{n+1} &= \mathcal{L}^{(\bar{g}_n+\delta g_n, \mu_n)}(R_n) + \xi_R(\bar{g}_n + \delta g_n, \mu_n, R_n) \ . \end{cases} \tag{115}$$

The boundary conditions we will need can roughly be stated as:

- $\delta g_0 = 0$.
- μ_n does not blow up when $n \to +\infty$.
- R_n does not blow up when $n \to -\infty$.

Also note the behavior of the linear parts of (115) :

- When $n \to +\infty$, $f'(\bar{g}_n) \to 2 - L^{\epsilon} < 1$, i.e., one has a deamplification.
- When $n \to -\infty$, $f'(\bar{g}_n) \to L^{\epsilon} > 1$, i.e., one has an amplification.

– One always has an amplification in the 'relevant' μ or mass direction.
– Once the RG map has been properly defined, one can arrange to always have a deamplification in the 'irrelevant' R direction.

Based on these observations, it is natural using the standard method of associated 'discrete integral equations', used for instance in [47], to rewrite the system (115) as

$$\forall n > 0,\quad \delta g_n = f'(\bar{g}_{n-1})\delta g_{n-1} + \left[-L^{2\epsilon} a\ \delta g_{n-1}^2\ + \xi_g(\bar{g}_{n-1} + \delta g_{n-1}, \mu_{n-1}, R_{n-1})\right], \tag{116}$$

$$\forall n < 0,\quad \delta g_n = \frac{1}{f'(\bar{g}_n)}\delta g_{n+1} - \frac{1}{f'(\bar{g}_n)}\left[-L^{2\epsilon} a\ \delta g_n^2\ + \xi_g(\bar{g}_n + \delta g_n, \mu_n, R_n)\right], \tag{117}$$

$$\forall n \in \mathbb{Z},\quad \mu_n = L^{-\left(\frac{3+\epsilon}{2}\right)}\mu_{n+1} - L^{-\left(\frac{3+\epsilon}{2}\right)}\xi_\mu(\bar{g}_n + \delta g_n, \mu_n, R_n), \tag{118}$$

$$\forall n \in \mathbb{Z},\quad R_n = \mathcal{L}^{(\bar{g}_{n-1}+\delta g_{n-1}, \mu_{n-1})}(R_{n-1}) + \xi_R(\bar{g}_{n-1} + \delta g_{n-1}, \mu_{n-1}, R_{n-1}), \tag{119}$$

and iterate, i.e., replace the linear term occurrences of the dynamical variables δg, μ, R, in terms of the analogous equations for $n-1$ or $n+1$, and repeat ad nauseam until one hits a boundary condition. In sum, the true sequence we are seeking will be constructed as a fixed point of a map $s \mapsto s'$ or rather $\delta s \mapsto \mathfrak{m}(\delta s)$, which to a sequence $\delta s = (\delta g_n, \mu_n, R_n)_{n\in\mathbb{Z}}$ associates the new sequence $\mathfrak{m}(\delta s) = (\delta g_n', \mu_n', R_n')_{n\in\mathbb{Z}}$ which is given as follows.

Definition 1. The map on sequences. *Leaving the issue of convergence for later, the defining formulae for the map* $\mathfrak{m}$ *are :*

$$\delta g_0' \stackrel{\text{def}}{=} 0, \tag{120}$$

$$\forall n > 0,\quad \delta g_n' \stackrel{\text{def}}{=} \sum_{0\le p<n}\left(\prod_{p<j<n} f'(\bar{g}_j)\right)\left[-L^{2\epsilon} a\ \delta g_p^2\ + \xi_g(\bar{g}_p + \delta g_p, \mu_p, R_p)\right], \tag{121}$$

$$\forall n < 0,\quad \delta g_n' \stackrel{\text{def}}{=} -\sum_{n\le p<0}\left(\prod_{n\le j\le p} \frac{1}{f'(\bar{g}_j)}\right)\left[-L^{2\epsilon} a\ \delta g_p^2\ + \xi_g(\bar{g}_p + \delta g_p, \mu_p, R_p)\right], \tag{122}$$

$$\forall n \in \mathbb{Z},\quad \mu_n' \stackrel{\text{def}}{=} -\sum_{p\ge n} L^{-\left(\frac{3+\epsilon}{2}\right)(p-n+1)}\ \xi_\mu(\bar{g}_p + \delta g_p, \mu_p, R_p), \tag{123}$$

and finally

$$\forall n \in \mathbb{Z},$$
$$R'_n \stackrel{\text{def}}{=} \sum_{p<n} \mathcal{L}^{(\bar{g}_{n-1}+\delta g_{n-1},\mu_{n-1})} \circ \mathcal{L}^{(\bar{g}_{n-2}+\delta g_{n-2},\mu_{n-2})} \circ \cdots$$
$$\cdots \circ \mathcal{L}^{(\bar{g}_{p+1}+\delta g_{p+1},\mu_{p+1})} \left(\xi_R(\bar{g}_p + \delta g_p, \mu_p, R_p)\right) , \tag{124}$$

where the composition $\circ$ *is of course with respect to the* R *argument.*

We now come to the definition of the space in which the deviation sequences δs will live. Let us introduce as in [15] the exponent drops $\delta \in [0, \frac{1}{6}]$ and $\eta \in [0, \frac{3}{16}]$ which will be fixed later. We will also define for $n \in \mathbb{Z}$,

$$e_n \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } n \leq 0 \, , \\ \frac{3}{2} & \text{if } n \geq 1 \, . \end{cases} \tag{125}$$

Now we define the big Banach space of sequences

$$\mathcal{BBSS}^{\mathbb{K}} \subset \prod_{n\in\mathbb{Z}} \left(\mathbb{K} \times \mathbb{K} \times \mathcal{BBS}^{\mathbb{K}}_{\bar{g}_n}\right) \tag{126}$$

whose elements are all deviation sequences $\delta s = (\delta g_n, \mu_n, R_n)_{n\in\mathbb{Z}}$ for which the quadruple norm

$$||||\delta s|||| \stackrel{\text{def}}{=} \sup_{n\in\mathbb{Z}} \left(\max\left\{|\delta g_n|\bar{g}_n^{-e_n}, |\mu_n|\bar{g}_n^{-(2-\delta)}, |||R_n|||_{\bar{g}_n}\bar{g}_n^{-(\frac{11}{4}-\eta)}\right\}\right) \tag{127}$$

is bounded and such that $\delta g_0 = 0$. Note that the approximate sequence $\bar{s}$ itself does not belong to $\mathcal{BBSS}^{\mathbb{K}}$ which somewhat plays the role of a tangent space around it. As an easy consequence of our definitions one has the following proposition.

Proposition 2. *The space*

$$\left(\mathcal{BBSS}^{\mathbb{K}}, ||||\cdot||||\right)$$

is complete.

6. The BMS Estimates on a Single RG Step

The estimates in [15, Sect. 5], slightly modified for the needs of the present construction, can be summarized by Theorem 1 below. Before stating the theorem one can give a brief description of the main ideas behind the estimates of [15, Sect. 5]. Given some a priori hypotheses on the size of the input g, μ, R of the RG map, the goal is to prove estimates on the output g', μ', R'. The size of these variables is typically measured in powers of the ϕ^4 coupling g. However the latter is a dynamical variable of the problem, and in order to avoid a vicious circle one uses instead powers of a predetermined approximation $\bar{g}$ which we have called *the calibrator*. The true value of g is allowed to float in a small complex ball centred on $\bar{g}$. In [15, Eq. 5.1] this calibrator is taken equal to the approximate fixed point value which we denoted here by $\bar{g}_*$ and which is of order ϵ. Grosso modo the main purpose of [15, Sect. 5] is to show that provided μ is of order $\bar{g}^2$, and R is of order $\bar{g}^3$, then the linear map $\mathcal{L}^{(g,\mu)}$ is contractive in the R direction, and the remainder ξ_R remains of order $\bar{g}^3$. In fact, for technical reasons, the exponents are slightly altered and a more precise statement would be: provided μ is of order $\bar{g}^{2-\delta}$, and R is of order $\bar{g}^{\frac{11}{4}-\eta}$, then the linear map $\mathcal{L}^{(g,\mu)}$ is contractive in the R direction, and the remainder ξ_R remains of order $\bar{g}^{\frac{11}{4}}$. Here δ and η are small nonnegative discrepancies. A nice feature of the estimates [15, Eq. 5.1] is that they allow a bound on the output ξ_R which is strictly better than the one on the input R, when $\eta > 0$. This is required in the subsequent dynamical system construction, for an effective use of the splitting $R' = \mathcal{L}^{(g,\mu)}(R) + \xi_R$.

Two norms are required to measure the R coordinate. The first is the *kernel* semi-norm $|\cdot|_{h_*,\mathcal{A}}$ defined in (42). This norm detects the true power g^3 of the coupling constant inside R. On its own this norm does not carry enough information to control the action of the renormalization group because it only depends on the size of ϕ derivatives of R at $\phi = 0$. The renormalization group involves convolution by the Gaussian measure μ_Γ. The role of the second norm $\|\cdot\|_{h,G_\kappa,\mathcal{A}}$ is to control R when it is tested on the large fields in the tail of μ_Γ.

A typical polymer amplitude generated by the expansion of [15, Sect. 3.1] (see also our Sect. 4.4) is of the form $\phi_1 \cdots \phi_k e^{-V(\phi)}$, where $\phi_1, \ldots, \phi_k$ refer to the evaluations of the background field ϕ at various locations $x_1, \ldots, x_k$. The latter eventually are integrated over against a kernel $\mathcal{K}(x_1, \ldots, x_k)$. Such ϕ_i factors usually need to be estimated pointwise. This requires a two-step argument (see [15, Lemma 5.1]). One bounds the difference between ϕ_i and the average of ϕ over some polymer using the large field regulator G_κ which only involves L^2 norms of derivatives of ϕ but not ϕ itself. Then the average value of ϕ is controlled, via Hölder's inequality, thanks to a fraction of the $e^{-g\int \phi^4}$ which is extracted from $e^{-V(\phi)}$ by [15, Lemma 5.5]. The cost of the operation is a large $\bar{g}^{-\frac{1}{4}}$ factor per ϕ_i.

Note that by the choice of Q in Sect. 4.3 the action of the renormalization group keeps $K = e^{-V} Q$ fixed up to a trivial rescaling of the coupling constant g, in the second order in perturbation theory approximation. This ensures that the RG map contribution to R is entirely due to third and higher orders of perturbation theory. Now the expansion in [15, Sect. 3.1] typically produces a collection of vertices $g[(\phi + \zeta)^4 - \phi^4]$ which involve at least one fluctuation field ζ. Therefore, in the worst case scenario, the contribution of such a vertex to a $||\cdot||_h$ norm bound is $\bar{g} \times (\bar{g}^{-\frac{1}{4}})^3 = \bar{g}^{\frac{1}{4}}$. The R activities which correspond to remainders beyond second order perturbation theory essentially contain at least three vertices and satisfy a $\bar{g}^{\frac{3}{4}}$ bound. The last considerations impose the $\bar{g}^2 = \bar{g}^{\frac{11}{4}} \times \bar{g}^{-\frac{3}{4}}$ multiplicative shift of the $||\cdot||_h$ norm in the definition of the calibrated triple norm (45). This in turn affects the number n_0 of functional derivatives to be accounted for in the norms. This number has to be at least equal to 9 for the needs of [15, Lemma 5.15] which transforms a $||R||_h$ decay into a bound on $|R^\sharp|_{h*}$, using a Taylor expansion of the polymer activities in the field variable around $\phi = 0$.

Once the proper definitions for the polymer activity norms have been made available, the sequence of estimates in [15, Sect. 5] is for the most part reasonably straightforward. It successively provides bounds for activities such as P of (70) and $(\mathcal{S}K)^\natural$ of (73) which are intermediates on the way to the final RG product R'. Contour integrations are used for conceptual economy when breaking R' into pieces to be estimated separately. They are also used for bounds on the $\tilde{\mathcal{E}}$, $\mathcal{E}$ defined in the extraction step where one would otherwise need more cumbersome estimates on derivatives of polymer activities with respect to interpolation parameters.

The crucial estimates of [15, Sect. 5] are [15, Corollary 5.25] and [15, Lemma 5.27] which pertain to the linear part of the $R \to R'$ map, here denoted by $\mathcal{L}^{(g,\mu)}(\cdot)$. There lies the heart of the renormalization problem in quantum field theory: the action of the renormalization group has expanding (relevant) directions. In the present context these are manifested in (97) which contains a sum over Y small satisfying a constraint. Consider for example the case where Y is a single cube. Then the constraint amounts to summing over all small cubes contained in a fixed cube at the next scale, see the same phenomenon discussed in [64]. The renormalization group inevitably has expanding directions because of the L^3 factor resulting from this summation. In (97) there are two sums and one of them refers to Y large. Typically, for rather intuitive geometrical reasons, the number of cubes in a polymer strictly decreases when it is coarse grained to become the smallest covering by cubes on the next scale. This geometrical effect is exploited in [15, Inequality 2.7] followed by a pin and sum argument [17, Lemma 5.1] to prove that these so-called large polymers are harmless: they are not part of the expanding direction problem. However this purely geometrical effect breaks down in the case of small polymers (see also [2, Lemma 11]). A compensating good factor $L^{-\frac{7-\epsilon}{2}}$ then has to be provided by the scaling behavior of the activity J. The latter corresponds to the R-linear part of what the perturbation expansion produces, when both terms $R^\sharp$ and counterterms $\tilde{F}_R e^{-\tilde{V}}$ are accounted for. The proper scaling bound on J proceeds by the clever double Taylor expansion argument of [11, Lemma 15] and [15, Corollary 5.25]. Roughly, one expands J in the field variable ϕ around zero; then one expands the fields or test functions appearing in the low order functional derivative terms, with respect to the space variable x. The normalization conditions [15, Eq. 4.37] eliminate the low order terms in the bigrading given by the degree in ϕ and the number of spacial derivatives ∂. The surviving terms have enough $L^{-\frac{3-\epsilon}{4}}$ factors provided by the ϕ's and L^{-1} factors given by the ∂'s not only to beat the L^3 volume sum but also to leave an extra $L^{-\frac{1-\epsilon}{2}}$ which secures the contractivity of $\mathcal{L}^{(g,\mu)}(\cdot)$ for L large, uniformly in ϵ.

We may now proceed to the statement of the BMS estimates theorem. Mind the order of quantifiers which is important.

Theorem 1. $\exists \kappa_0 > 0$, $\exists L_0 \in \mathbb{N}$,
$\forall \kappa \in]0, \kappa_0]$, $\forall \delta \in [0, \frac{1}{6}]$, $\forall \eta \in [0, \frac{3}{16}]$,
$\forall A_g \in]0, \frac{1}{2}]$, $\forall A_\mu > 0$, $\forall A_R > 0$, $\forall A_{\bar{g}} > 0$,
$\exists c_0 > 0$, $\forall c \in]0, c_0]$,
$\exists B_g > 0$, $\exists B_{R\mathcal{L}} > 0$,
$\forall L \in \mathbb{N}$ *such that* $L \geq L_0$,
$\exists B_\mu > 0$, $\exists B_{R\xi} > 0$,
$\exists \epsilon_0 > 0$,
$\forall \epsilon \in]0, \epsilon_0]$, $\forall \bar{g} \in]0, A_{\bar{g}}\epsilon]$,
if one uses the notations

$$D_g \stackrel{\text{def}}{=} \left\{ g \in \mathbb{C} |\ |g - \bar{g}| < A_g \bar{g} \right\} , \tag{128}$$

$$D_\mu \stackrel{\text{def}}{=} \left\{ \mu \in \mathbb{C} |\ |\mu| < A_\mu \bar{g}^{2-\delta} \right\} , \tag{129}$$

$$D_R \stackrel{\text{def}}{=} \left\{ R \in \mathcal{BBS}^{\mathbb{C}} |\ |||R|||_{\bar{g}} < A_R \bar{g}^{\frac{11}{4}-\eta} \right\} ; \tag{130}$$

then

1. *The maps* ξ_g*,* ξ_μ*,* ξ_R*, are well defined and analytic on the open set* $D_g \times D_\mu \times D_R$ *with values in* $\mathbb{C}$*,* $\mathbb{C}$*, and* $\mathcal{BBS}^{\mathbb{C}}$ *respectively.*
2. *The map* $(g, \mu, R) \mapsto \mathcal{L}^{(g,\mu)}(R)$ *is well defined and analytic from* $D_g \times D_\mu \times \mathcal{BBS}^{\mathbb{C}}$ *to* $\mathcal{BBS}^{\mathbb{C}}$*. Besides, for any* $(g, \mu) \in D_g \times D_\mu$*, the map* $R \mapsto \mathcal{L}^{(g,\mu)}(R)$ *is linear continuous from* $\mathcal{BBS}^{\mathbb{C}}$ *to itself.*
3. *The maps* ξ_g*,* ξ_μ*,* ξ_R *send the real cross-section*

$$(D_g \cap \mathbb{R}) \times (D_\mu \cap \mathbb{R}) \times (D_R \cap \mathcal{BBS}^{\mathbb{R}})$$

into $\mathbb{R}$*,* $\mathbb{R}$*, and* $\mathcal{BBS}^{\mathbb{R}}$ *respectively.*

4. *The map* $(g, \mu, R) \mapsto \mathcal{L}^{(g,\mu)}(R)$ *sends* $(D_g \cap \mathbb{R}) \times (D_\mu \cap \mathbb{R}) \times \mathcal{BBS}^{\mathbb{R}}$ *into* $\mathcal{BBS}^{\mathbb{R}}$*.*
5. *For any* $(g, \mu, R) \in D_g \times D_\mu \times D_R$ *one has the estimates*

$$|\xi_g(g, \mu, R)| \le B_g \bar{g}^{\frac{11}{4}-\eta} , \tag{131}$$

$$|\xi_\mu(g, \mu, R)| \le B_\mu \bar{g}^2 , \tag{132}$$

$$|||\xi_R(g, \mu, R)|||_{\bar{g}} \le B_{R\xi} \bar{g}^{\frac{11}{4}} . \tag{133}$$

6. *For any* $(g, \mu, R) \in D_g \times D_\mu \times \mathcal{BBS}^{\mathbb{C}}$ *one has the estimate*

$$|||\mathcal{L}^{(g,\mu)}(R)|||_{\bar{g}} \le B_{R\mathcal{L}} L^{-\left(\frac{1-\epsilon}{2}\right)} |||R|||_{\bar{g}} . \tag{134}$$

Remark 6. We suppressed the reference to a calibrator $\bar{g}$ when mentioning the spaces $\mathcal{BBS}^{\mathbb{K}}$. This is because the corresponding statements do not really depend on the choice of one of the equivalent norms $|||\cdot|||_{\bar{g}}$. Also note that the notion of analyticity we used is the standard one in the Banach space context (see for instance [7, Sect. 2.3]). Finally remember that the c quantity is the one involved in the relation (43).

For the proof of the theorem we refer to [15, Sect. 5]. The statements about the maps being well defined and analytic will follow from the algebraic nature of the formulae in Sect. 4, once the estimates are established. The statements about the map taking real values are obvious from the formulae in Sect. 4. Now for the estimates, one should say that it is the $\bar{g} \sim \epsilon$ special case of Theorem 1 which is proven in [15]. This is because the analysis takes place in the vicinity of the infrared fixed point where one can assume that the g coupling is almost constant equal to $\bar{g}_* = \mathcal{O}(\epsilon)$. In other words, the small ϵ parameter is attributed two roles at once: bifurcation parameter and calibrator. However, by carefully following [15, Sect. 5], one can see that the arguments still apply if one dissociates the two functions. Therefore all one needs is to go over and redo the series of Lemmata from [15, Sect. 5], except that one has to replace the hypothesis in Eqs. (5.1–5.3) of [15] by the new conditions given by the domains D_g, D_μ and D_R, namely,

$$|g - \bar{g}| < A_g \bar{g} , \tag{135}$$

$$|\mu| < A_\mu \bar{g}^{2-\delta} , \tag{136}$$

$$|||R|||_{\bar{g}} < A_R \bar{g}^{\frac{11}{4}-\eta} , \tag{137}$$

to which one adds

$$0 < \bar{g} \le A_{\bar{g}} \epsilon , \tag{138}$$

knowing that in the end ϵ will be taken to be small, after having fixed L. Then instead of using powers of ϵ in the bounds, one has to use powers of the calibrator $\bar{g}$ instead. In Lemmata 5.26 and 5.27 of [15], one has to use bounds in terms of the norms $||R||_{h,G_\kappa,\mathcal{A}}$ and $|R|_{h_*,\mathcal{A}}$. Note that for [15, Lemma 5.5], one needs $(\Re g)^{\frac{1}{4}}h$ to be small, which can be achieved by taking c small provided $\frac{\Re g}{\bar{g}}$ is bounded from above. This is guaranteed by our assumption (135). Rather than [15, Lemma 5.5], the reader might find it more convenient to use instead specializations of [11, Theorem 1]. The latter needs the ratio $\frac{\Im g}{\Re g}$ to be bounded, which again is guaranteed by (135) and the condition $A_g \le \frac{1}{2}$. Note that the important [15, Eq. 5.58] on the other hand cannot allow $(\Re g)^{\frac{1}{4}}h$ to be too small either. This is why it seems hard to avoid the fibered norm problem, and we need to keep g rather close to the calibrator $\bar{g}$ as in (135). Note that a stronger hypothesis was used in [15, Eq. 5.1]. However, as far as [15, Sect. 5] alone is concerned, this hypothesis only serves to show that it reproduces itself, in [15, Corollary 5.18]. We relaxed this conclusion in Theorem 1, and therefore we can drop this hypothesis.

Remark that in [15, Sect. 5] the exponents δ, η were taken equal to $\frac{1}{64}$. The reader who prefers this choice, can simply make the corresponding modifications in our Sect. 8. The ranges $[0, \frac{1}{6}]$ for δ and $[0, \frac{3}{16}]$ for η which we have given come from the following considerations. First note that the hypothesis $\delta, \eta > 0$ in [15, Sect. 5] is only used in order to absorb some constant factors in the bounds provided in [15, Theorem 1]. We do not need this, since we allow the B factors above. Then note that each time in [15, Sect. 5] one has a bound with a sum of terms with different powers of ϵ, or rather here $\bar{g}$, one has to pick the dominant term in the $\delta, \eta \to 0$ limit. Collecting the inequalities on δ, η which ensure that the term picked is indeed dominant, one can see that $\delta \le \frac{1}{6}$ and $\eta \le \frac{3}{16}$ are sufficient for these inequalities to hold. Finally, the modifications introduced in our Sect. 3 for the functional analytic setting, do not affect the bounds. One may simply mention that [15, Lemma 5.15] uses the Taylor formula with integral remainder. Of course one first has to apply it in the textbook setting of the space we denoted by $C^{n_0}_{\natural}(\mathrm{Fld}(X), \mathbb{K})$; and only then, one can use the sharp norm for the differentials and the $||\cdot||_{C^2(X)}$ norms for the fields when performing the bounds.

Armed with the previous remarks, the precise statement of Theorem 1 to aim for, and some patience, the reader with expertise on the techniques from [17, 11, 15] will have no difficulty adapting the arguments of [15, Sect. 5].

7. Elementary Estimates on the Approximate Sequence

This section collects the elementary but crucial estimates on the sequence $(\bar{g}_n)_{n\in\mathbb{Z}}$.

7.1. The discrete step function lemma. We firstly need some basic bounds on the sequence.

Lemma 2. The step function behaviour. *1) For any nonnegative integer n,*

$$\bar{g}_* \left(1 - (1-\omega_0)(1+\omega_0 - L^\epsilon \omega_0)^n\right) \le \bar{g}_n \le \bar{g}_* \left(1 - (1-\omega_0)(2 - L^\epsilon)^n\right) . \quad (139)$$

2) For any nonpositive integer n,

$$\bar{g}_* \omega_0 L^{\epsilon n} \le \bar{g}_n \le \bar{g}_* \omega_0 \left(\frac{2}{L^\epsilon + \sqrt{L^{2\epsilon} - 4\omega_0(L^\epsilon - 1)}} \right)^{-n} . \quad (140)$$

Remark 7. This simply says that, for $n \to +\infty$, $\bar{g}_n$ goes exponentially fast to $\bar{g}_*$ and that, for $n \to -\infty$, $\bar{g}_n$ goes exponentially fast to 0, with a transition or 'step' in between. These exponential rates are very weak in the $\epsilon \to 0$ limit. We need as precise estimates on these rates as we can, to be used as input for the following analysis. Indeed, based on these estimates, we will have to determine the winner between close competing effects, as one can see in the next subsections. This is why we included this otherwise trivial lemma.

Proof. On the interval $[\bar{g}_0, \bar{g}_*]$ we define the two functions f_{+h} and f_{+l} by

$$f_{+h}(x) \stackrel{\text{def}}{=} f(\bar{g}_*) + (x - \bar{g}_*) f'(\bar{g}_*) \,, \tag{141}$$

$$f_{+l}(x) \stackrel{\text{def}}{=} f(\bar{g}_0) + (x - \bar{g}_0) \times \frac{f(\bar{g}_*) - f(\bar{g}_0)}{\bar{g}_* - \bar{g}_0} \,. \tag{142}$$

Since f is increasing and concave, one has for any $x \in [\bar{g}_0, \bar{g}_*]$,

$$\bar{g}_0 \le f_{+l}(x) \le f(x) \le f_{+h}(x) \le \bar{g}_* \,. \tag{143}$$

A trivial iteration then implies

$$\forall n \in \mathbb{N}, \forall x \in [\bar{g}_0, \bar{g}_*], \quad \bar{g}_0 \le (f_{+l})^n(x) \le f^n(x) \le (f_{+h})^n(x) \le \bar{g}_* \,. \tag{144}$$

Now note that

$$(f_{+h})^n(x) = \bar{g}_* + (x - \bar{g}_*)[f'(\bar{g}_*)]^n \tag{145}$$

$$= \bar{g}_* + (x - \bar{g}_*)(2 - L^\epsilon)^n \,. \tag{146}$$

Likewise

$$f_{+l}(x) = \bar{g}_* + (x - \bar{g}_*) \left(\frac{\bar{g}_* - \bar{g}_1}{\bar{g}_* - \bar{g}_0} \right)^n \,. \tag{147}$$

Let $\bar{g}_1 = \omega_1 \bar{g}_*$, for $\omega_1 \in]0, 1[$, then

$$\bar{g}_1 = f(\bar{g}_0) = \omega_0 \bar{g}_* (L^\epsilon - L^{2\epsilon} a \omega_0 \bar{g}_*) \,, \tag{148}$$

or

$$\omega_1 = \omega_0 (L^\epsilon - \omega_0 (L^\epsilon - 1)) = L^\epsilon \omega_0 - L^\epsilon \omega_0^2 + \omega_0^2 \,, \tag{149}$$

so

$$\frac{\bar{g}_* - \bar{g}_1}{\bar{g}_* - \bar{g}_0} = \frac{1 - \omega_1}{1 - \omega_0} \tag{150}$$

$$= \frac{1 - L^\epsilon \omega_0 + L^\epsilon \omega_0^2 - \omega_0^2}{1 - \omega_0} \tag{151}$$

$$= \frac{(1 - \omega_0)(1 + \omega_0) - L^\epsilon \omega_0 (1 - \omega_0)}{1 - \omega_0} \tag{152}$$

$$= 1 + \omega_0 - L^\epsilon \omega_0 \,. \tag{153}$$

Thus,

$$(f_{+l})^n(x) = \bar{g}_* + (x - \bar{g}_*) \left(1 + \omega_0 - L^\epsilon \omega_0\right)^n \,. \tag{154}$$

Now on the interval $[0, \bar{g}_0]$ we also define, using the inverse f^{-1}, the two functions f_{-h} and f_{-l} by

$$f_{-h}(x) \stackrel{\text{def}}{=} x \times \frac{f^{-1}(\bar{g}_0)}{\bar{g}_0} , \tag{155}$$

$$f_{-l}(x) \stackrel{\text{def}}{=} x \times (f^{-1})'(0) . \tag{156}$$

One has for any $x \in [0, \bar{g}_0]$,

$$0 \le f_{-l}(x) \le f^{-1}(x) \le f_{-h}(x) \le \bar{g}_0, \tag{157}$$

which trivially iterates into

$$\forall n \in \mathbb{N}, \forall x \in [0, \bar{g}_0], \quad 0 \le (f_{-l})^n(x) \le (f^{-1})^n(x) \le (f_{-h})^n(x) \le \bar{g}_0. \tag{158}$$

Now

$$(f_{-l})^n(x) = L^{-\epsilon n} x \tag{159}$$

and

$$(f_{-h})^n(x) = \left(\frac{\bar{g}_{-1}}{\bar{g}_0}\right)^n x . \tag{160}$$

Let $\bar{g}_{-1} = \omega_{-1}\bar{g}_*$ for $\omega_{-1} \in]0, 1[$. The latter is the smallest of the two solutions of the quadratic equation

$$L^\epsilon(\omega_{-1}\bar{g}_*) - L^{2\epsilon} a(\omega_{-1}\bar{g}_*)^2 = \omega_0\bar{g}_* , \tag{161}$$

i.e.,

$$(L^\epsilon - 1)\omega_{-1}^2 - L^\epsilon\omega_{-1} + \omega_0 = 0 ; \tag{162}$$

therefore

$$\omega_{-1} = \frac{L^\epsilon - \sqrt{L^{2\epsilon} - 4\omega_0(L^\epsilon - 1)}}{2(L^\epsilon - 1)} . \tag{163}$$

As a result

$$(f_{-h})^n(x) = \left(\frac{L^\epsilon - \sqrt{L^{2\epsilon} - 4\omega_0(L^\epsilon - 1)}}{2\omega_0(L^\epsilon - 1)}\right)^n x \tag{164}$$

$$= \left(\frac{2}{L^\epsilon + \sqrt{L^{2\epsilon} - 4\omega_0(L^\epsilon - 1)}}\right)^n x . \tag{165}$$

From the previous considerations, applied to the sequence $(\bar{g}_n)_{n\in\mathbb{Z}}$, the lemma follows. □

This taken care of, we now proceed to the key lemmata for the construction of a global RG trajectory.

Firstly, the forward 'integral equation' (122) for δg, or the deviation of the running coupling constant with respect to the reference sequence $(\bar{g}_n)_{n\in\mathbb{Z}}$, requires an *explicit* bound on

$$\Sigma_{\delta g - f}(\epsilon, \gamma, \nu) \stackrel{\text{def}}{=} \sup_{n<0} \left\{ \frac{1}{\bar{g}_n^\gamma} \sum_{n\le p<0} \bar{g}_p^\nu \prod_{n\le j\le p} \frac{1}{f'(\bar{g}_j)} \right\} , \tag{166}$$

where γ, ν are some nonnegative real exponents.

Secondly, the backward 'integral equation' (121) for δg, requires an analogous bound on

$$\Sigma_{\delta g-b}(\epsilon,\gamma,\nu) \stackrel{\text{def}}{=} \sup_{n>0}\left\{\frac{1}{\bar{g}_n^{\gamma}}\sum_{0\le p<n}\bar{g}_p^{\nu}\prod_{p<j<n}f'(\bar{g}_j)\right\} . \tag{167}$$

Thirdly, the forward 'integral equation' (123) for μ, or the squared mass, requires a bound on

$$\Sigma_{\mu-f}(\epsilon,\gamma,\nu) \stackrel{\text{def}}{=} \sup_{n\in\mathbb{Z}}\left\{\frac{1}{\bar{g}_n^{\gamma}}\sum_{p\ge n}L^{-\left(\frac{3+\epsilon}{2}\right)(p-n+1)}\bar{g}_p^{\nu}\right\} . \tag{168}$$

Fourthly, the backward 'integral equation' (124) for R, or the irrelevant terms generated by the RG transformation, requires a bound on

$$\Sigma_{R-b}(\epsilon,\gamma,\nu) \stackrel{\text{def}}{=} \sup_{n\in\mathbb{Z}}\left\{\frac{1}{\bar{g}_n^{\gamma}}\sum_{p<n}c_R^{n-p-1}\bar{g}_p^{\nu}\right\} , \tag{169}$$

where $c_R \in]0,1[$ is an upper bound on the operator norms of the linearized RG maps $\mathcal{L}^{(\cdot,\cdot)}$ in the R direction. We will provide the necessary estimates in reverse order, i.e., from simple to more involved.

7.2. The backward bound for R. Assuming the already mentioned hypotheses on L, ϵ, a, c_R, ω_0 we have the following result.

Lemma 3. *Provided the exponents γ, μ satisfy $\nu \ge \gamma \ge 0$, the following inequality holds:*

$$\Sigma_{R-b}(\epsilon,\gamma,\nu) \le \bar{\Sigma}_{R-b}(\epsilon,\gamma,\nu) \stackrel{\text{def}}{=} \frac{\bar{g}_*^{\nu-\gamma}}{1-c_R} . \tag{170}$$

Proof. Let $n \in \mathbb{Z}$ and denote

$$\Delta_n \stackrel{\text{def}}{=} \frac{1}{\bar{g}_n^{\gamma}}\sum_{p<n}c_R^{n-p-1}\bar{g}_p^{\nu} . \tag{171}$$

Since the sequence $(\bar{g}_n)_{n\in\mathbb{Z}}$ contained in $]0,\bar{g}_*[$ is increasing, and $\nu \ge \gamma \ge 0$, we trivially have

$$\Delta_n \le \frac{1}{\bar{g}_n^{\gamma}}\sum_{p<n}c_R^{n-p-1}\bar{g}_n^{\nu} \tag{172}$$

$$\le \frac{\bar{g}_n^{\nu-\gamma}}{1-c_R} \tag{173}$$

$$\le \frac{\bar{g}_*^{\nu-\gamma}}{1-c_R} . \tag{174}$$

□

7.3. The forward bound for μ. Again with the assumptions of Sect. 5, we have the following result.

Lemma 4. *Provided the exponents γ, ν satisfy $\nu \geq \gamma \geq 0$, and $\epsilon\nu < \frac{3+\epsilon}{2}$, we have*

$$\Sigma_{\mu-f}(\epsilon,\gamma,\nu) \leq \bar{\Sigma}_{\mu-f}(\epsilon,\gamma,\nu) \stackrel{\text{def}}{=} \frac{\bar{g}_*^{\nu-\gamma}}{L^{\frac{3+\epsilon}{2}} - L^{\epsilon\nu}} . \tag{175}$$

Proof. Let $n \in \mathbb{Z}$ and write

$$\Delta_n \stackrel{\text{def}}{=} \frac{1}{\bar{g}_n^{\gamma}} \sum_{p \geq n} L^{-\left(\frac{3+\epsilon}{2}\right)(p-n+1)} \bar{g}_p^{\nu} \tag{176}$$

$$= \bar{g}_n^{\nu-\gamma} \sum_{p \geq n} L^{-\left(\frac{3+\epsilon}{2}\right)(p-n+1)} \left(\prod_{n<j\leq p} \frac{\bar{g}_j}{\bar{g}_{j-1}} \right)^{\nu} . \tag{177}$$

Now

$$\frac{\bar{g}_j}{\bar{g}_{j-1}} = \frac{f(\bar{g}_{j-1}) - f(0)}{\bar{g}_{j-1} - 0} = f'(\xi) > 0 \tag{178}$$

for some $\xi \in]0, \bar{g}_{j-1}[$. Since f is concave $f'(\xi) \leq f'(0) = L^{\epsilon}$, and therefore

$$\Delta_n \leq \bar{g}_n^{\nu-\gamma} \sum_{p \geq n} L^{-\left(\frac{3+\epsilon}{2}\right)(p-n+1)} L^{\epsilon\nu(p-n)} \tag{179}$$

$$\leq \bar{g}_n^{\nu-\gamma} L^{-\left(\frac{3+\epsilon}{2}\right)} \times \frac{1}{1 - L^{\epsilon\nu - \left(\frac{3+\epsilon}{2}\right)}} . \tag{180}$$

Since $\nu - \gamma \geq 0$, $\bar{g}_n^{\nu-\gamma} \leq \bar{g}_*^{\nu-\gamma}$, and we are done. □

7.4. The backward bound for δg. Again with the assumptions of Sect. 5, we have the following result.

Lemma 5. *For any $\gamma, \nu \geq 0$ we have*

$$\Sigma_{\delta g-b}(\epsilon,\gamma,\nu) \leq \bar{\Sigma}_{\delta g-b}(\epsilon,\gamma,\nu), \tag{181}$$

where

$$\bar{\Sigma}_{\delta g-b}(\epsilon,\gamma,\nu) \stackrel{\text{def}}{=} \frac{\omega_0^{-\gamma} \bar{g}_*^{\nu-\gamma}}{L^{\epsilon} - 1} \exp\left[\frac{2(1-\omega_0)(1+\omega_0 - L^{\epsilon}\omega_0)}{\omega_0(2-L^{\epsilon})} \right] . \tag{182}$$

Proof. Let n be a strictly positive integer, and denote

$$\Delta_n \stackrel{\text{def}}{=} \frac{1}{\bar{g}_n^{\gamma}} \sum_{0 \leq p < n} \bar{g}_p^{\nu} \prod_{p<j<n} f'(\bar{g}_j) . \tag{183}$$

Lemma 2 shows that $\bar{g}_n \to \bar{g}_*$ when $n \to +\infty$. We therefore expect most of the $f'(\bar{g}_j)$ to be very close to $f'(\bar{g}_*) = 2 - L^{\epsilon}$. This motivates the rewriting

$$\Delta_n = \frac{1}{\bar{g}_n^{\gamma}} \sum_{0 \leq p < n} \left\{ \prod_{p<j<n} \frac{f'(\bar{g}_j)}{2 - L^{\epsilon}} \right\} (2 - L^{\epsilon})^{n-p-1} \bar{g}_p^{\nu} . \tag{184}$$

Since f' is decreasing, for any $j \geq 1$,

$$\frac{f'(\bar{g}_j)}{2-L^\epsilon} = \frac{L^\epsilon - 2L^{2\epsilon} a \bar{g}_j}{2-L^\epsilon} > 1 \,, \tag{185}$$

and thus

$$\prod_{p<j<n} \frac{f'(\bar{g}_j)}{2-L^\epsilon} \leq \prod_{j\geq 1} \frac{L^\epsilon - 2L^{2\epsilon} a \bar{g}_j}{2-L^\epsilon} \tag{186}$$

$$\leq \exp\left[\sum_{j\geq 1}\left(\frac{L^\epsilon - 2L^{2\epsilon} a \bar{g}_j}{2-L^\epsilon} - 1\right)\right] . \tag{187}$$

Now

$$\frac{L^\epsilon - 2L^{2\epsilon} a \bar{g}_j}{2-L^\epsilon} - 1 = \frac{2L^{2\epsilon} a}{2-L^\epsilon} \times (\bar{g}_* - \bar{g}_j) \tag{188}$$

and Lemma 2 implies

$$\bar{g}_j \geq \bar{g}_* - \bar{g}_*(1-\omega_0)(1+\omega_0 - L^\epsilon \omega_0)^j \,, \tag{189}$$

i.e.,

$$\frac{L^\epsilon - 2L^{2\epsilon} \bar{g}_j}{2-L^\epsilon} - 1 \leq \frac{2L^{2\epsilon} a}{2-L^\epsilon} \times \bar{g}_*(1-\omega_0)(1+\omega_0 - L^\epsilon \omega_0)^j, \tag{190}$$

where $1+\omega_0 - L^\epsilon \omega_0$ belongs to $]0, 1[$. Hence

$$\prod_{p<j<n} \frac{f'(\bar{g}_j)}{2-L^\epsilon} \leq \exp\left[\frac{2L^{2\epsilon} a \bar{g}_*(1-\omega_0)}{2-L^\epsilon} \times \frac{(1+\omega_0 - L^\epsilon \omega_0)}{1-(1+\omega_0 - L^\epsilon \omega_0)}\right] \tag{191}$$

$$\leq \exp\left[\frac{2(1-\omega_0)(1+\omega_0 - L^\epsilon \omega_0)}{\omega_0(2-L^\epsilon)}\right]. \tag{192}$$

So we are left with bounding

$$\varDelta'_n \stackrel{\text{def}}{=} \frac{1}{\bar{g}_n^\gamma} \sum_{0\leq p<n} (2-L^\epsilon)^{n-p-1} \bar{g}_p^\nu \,. \tag{193}$$

To this effect we use the very coarse estimates $\bar{g}_n \geq \bar{g}_0 = \omega_0 \bar{g}_*$ and $\bar{g}_p \leq \bar{g}_*$ with the result that

$$\varDelta'_n \leq (\omega_0 \bar{g}_*)^{-\gamma} \sum_{0\leq p<n} (2-L^\epsilon)^{n-p-1} \bar{g}_*^\nu \tag{194}$$

$$\leq \omega_0^{-\gamma} \bar{g}_*^{\nu-\gamma} \times \frac{1}{1-(2-L^\epsilon)} \,. \tag{195}$$

Inequalities (192) and (195) now imply

$$\varDelta_n \leq \frac{\omega_0^{-\gamma} \bar{g}_*^{\nu-\gamma}}{L^\epsilon - 1} \exp\left[\frac{2(1-\omega_0)(1+\omega_0 - L^\epsilon \omega_0)}{\omega_0(2-L^\epsilon)}\right] . \tag{196}$$

□

7.5. The forward bound for δg. Once more, with the assumptions of Sect. 5, we have the following result.

Lemma 6. *For any exponents* γ, ν *such that* $0 \le \gamma \le 1$, $\nu > 0$ *and*

$$\Upsilon \stackrel{\text{def}}{=} \frac{2L^{\frac{\epsilon}{\nu}}}{L^\epsilon + \sqrt{L^{2\epsilon} - 4\omega_0(L^\epsilon - 1)}} \in]0, 1[\ , \tag{197}$$

we have

$$\Sigma_{\delta g - f}(\epsilon, \gamma, \nu) \le \bar{\Sigma}_{\delta g - f}(\epsilon, \gamma, \nu), \tag{198}$$

where

$$\bar{\Sigma}_{\delta g - f}(\epsilon, \gamma, \nu) \stackrel{\text{def}}{=} \frac{(\omega_0 \bar{g}_*)^{\nu - \gamma}}{1 - \Upsilon^\nu} \times \exp\left[\frac{\omega_0 \left(2 - L^\epsilon + \sqrt{L^{2\epsilon} - 4\omega_0(L^\epsilon - 1)}\right)}{(1 - \omega_0)\,(L^\epsilon - 2\omega_0(L^\epsilon - 1))}\right]. \tag{199}$$

Proof. Let n be a strictly negative integer, and define

$$\Delta_n \stackrel{\text{def}}{=} \frac{1}{\bar{g}_n^\gamma} \sum_{n \le p < 0} \bar{g}_p^\nu \prod_{n \le j \le p} \frac{1}{f'(\bar{g}_j)}\ . \tag{200}$$

Lemma 2 shows that $\bar{g}_n \to 0$ when $n \to -\infty$. We therefore expect most of the $f'(\bar{g}_j)$ to be very close to $f'(0) = L^\epsilon$. Therefore write

$$\Delta_n = \frac{1}{\bar{g}_n^\gamma} \sum_{n \le p < 0} \left(\prod_{n \le j \le p} \frac{L^\epsilon}{f'(\bar{g}_j)}\right) \left(L^{-\epsilon}\right)^{p - n + 1} \bar{g}_p^\nu\ . \tag{201}$$

Now

$$\frac{L^\epsilon}{f'(\bar{g}_j)} = \frac{1}{1 - 2L^\epsilon a \bar{g}_j} > 1\ . \tag{202}$$

We use

$$\prod_{n \le j \le p} \frac{L^\epsilon}{f'(\bar{g}_j)} \le \prod_{j \le -1} \frac{1}{1 - 2L^\epsilon a \bar{g}_j} \tag{203}$$

$$\le \exp\left[\sum_{j \le -1} \left(\frac{1}{1 - 2L^\epsilon a \bar{g}_j} - 1\right)\right] \tag{204}$$

$$\le \exp\left[\sum_{j \le -1} \frac{2L^\epsilon a \bar{g}_j}{1 - 2L^\epsilon a \bar{g}_j}\right]\ . \tag{205}$$

Now for $j \le -1$, $\bar{g}_j \le \bar{g}_0 = \omega_0 \bar{g}_*$; hence

$$\frac{2L^\epsilon a \bar{g}_j}{1 - 2L^\epsilon a \bar{g}_j} \le \frac{2L^\epsilon a \bar{g}_j}{1 - 2L^\epsilon a \bar{g}_0} = \frac{2L^{2\epsilon} a \bar{g}_j}{L^\epsilon - 2\omega_0(L^\epsilon - 1)}\ , \tag{206}$$

and by Lemma 2

$$\frac{2L^{\epsilon}a\bar{g}_j}{1-2L^{\epsilon}a\bar{g}_j} \le \frac{2L^{2\epsilon}a\omega_0\bar{g}_*}{L^{\epsilon}-2\omega_0(L^{\epsilon}-1)}\left(\frac{2}{L^{\epsilon}+\sqrt{L^{2\epsilon}-4\omega_0(L^{\epsilon}-1)}}\right)^{-j} . \tag{207}$$

As a result

$$\prod_{n\le j\le p}\frac{L^{\epsilon}}{f'(\bar{g}_j)} \le \exp\left[\frac{2\omega_0(L^{\epsilon}-1)}{L^{\epsilon}-2\omega_0(L^{\epsilon}-1)}\times\frac{\left(\frac{2}{L^{\epsilon}+\sqrt{L^{2\epsilon}-4\omega_0(L^{\epsilon}-1)}}\right)}{1-\left(\frac{2}{L^{\epsilon}+\sqrt{L^{2\epsilon}-4\omega_0(L^{\epsilon}-1)}}\right)}\right]. \tag{208}$$

Note that

$$0<\frac{2}{L^{\epsilon}+\sqrt{L^{2\epsilon}-4\omega_0(L^{\epsilon}-1)}}<1\,, \tag{209}$$

because of the global assumptions $1 < L^{\epsilon} < 2$ and $0 < \omega_0 < 1$. A straightforward simplification of the argument of the exponential leads to

$$\prod_{n\le j\le p}\frac{L^{\epsilon}}{f'(\bar{g}_j)} \le \exp\left[\frac{\omega_0\left(2-L^{\epsilon}+\sqrt{L^{2\epsilon}-4\omega_0(L^{\epsilon}-1)}\right)}{(1-\omega_0)\left(L^{\epsilon}-2\omega_0(L^{\epsilon}-1)\right)}\right]. \tag{210}$$

Now we are left with bounding

$$\Delta'_n \stackrel{\text{def}}{=} \frac{1}{\bar{g}_n^{\gamma}}\sum_{n\le p<0}\left(L^{-\epsilon}\right)^{p-n+1}\bar{g}_p^{\nu}\,. \tag{211}$$

We now use Lemma 2 to obtain

$$\Delta'_n \le (\omega_0\bar{g}_*)^{-\gamma}L^{-\gamma\epsilon n} \times\sum_{n\le p<0}\left(L^{-\epsilon}\right)^{p-n+1}(\omega_0\bar{g}_*)^{\nu}\left(\frac{2}{L^{\epsilon}+\sqrt{L^{2\epsilon}-4\omega_0(L^{\epsilon}-1)}}\right)^{-\nu p}, \tag{212}$$

i.e.,

$$\Delta'_n \le (\omega_0\bar{g}_*)^{\nu-\gamma}L^{\epsilon(\gamma-1)|n|}\times L^{-\epsilon}\times\frac{\Upsilon^{\nu}}{1-\Upsilon^{\nu}}, \tag{213}$$

where Υ is the one defined in the statement of the lemma. We now need a bound which is n-independent; this requires the hypothesis $\gamma \le 1$. Inequalities (210) and (213) now clearly imply

$$\forall n\le -1,\quad \Delta_n \le \bar{\Sigma}_{\delta g-f}(\epsilon,\gamma,\nu)\,, \tag{214}$$

and the lemma is proved. □

7.6. The $\epsilon \to 0$ limit. Leaving L, c_R, ω_0 and the exponents γ, ν fixed, we now analyze the $\epsilon \to 0$ asymptotics of the previous bounds. Note that in this limit we will have $a = a(L, \epsilon) \to \frac{\log L}{18\pi^2}$. The crux of our construction lies in the following result.

Lemma 7. *For $\epsilon \to 0^+$ we have*
1)

$$\bar{\Sigma}_{R-b}(\epsilon, \gamma, \nu) = \epsilon^{\nu-\gamma} \left(K_{R-b} + \mathcal{O}(\epsilon)\right), \tag{215}$$

where

$$K_{R-b} = \frac{1}{1-c_R}\left(18\pi^2\right)^{\nu-\gamma}, \tag{216}$$

provided $\nu \geq \gamma \geq 0$;
2)

$$\bar{\Sigma}_{\mu-f}(\epsilon, \gamma, \nu) = \epsilon^{\nu-\gamma} \left(K_{\mu-f} + \mathcal{O}(\epsilon)\right), \tag{217}$$

where

$$K_{\mu-f} = \frac{1}{L^{\frac{3}{2}}-1}\left(18\pi^2\right)^{\nu-\gamma}, \tag{218}$$

provided $\nu \geq \gamma \geq 0$;
3)

$$\bar{\Sigma}_{\delta g-b}(\epsilon, \gamma, \nu) = \epsilon^{\nu-\gamma-1} \left(K_{\delta g-b} + \mathcal{O}(\epsilon)\right), \tag{219}$$

where

$$K_{\delta g-b} = \frac{\left(18\pi^2\right)^{\nu-\gamma}}{\omega_0^{\gamma}(\log L)} \exp\left[\frac{2(1-\omega_0)}{\omega_0}\right], \tag{220}$$

provided $\nu \geq 0$ and $\gamma \geq 0$;
4)

$$\bar{\Sigma}_{\delta g-f}(\epsilon, 1, \nu) = \epsilon^{\nu-2} \left(K_{\delta g-f} + \mathcal{O}(\epsilon)\right), \tag{221}$$

where

$$K_{\delta g-f} = \frac{\omega_0^{\nu-1}\left(18\pi^2\right)^{\nu-1}}{(\log L)\left[\nu(1-\omega_0)-1\right]} \exp\left[\frac{2\omega_0}{1-\omega_0}\right], \tag{222}$$

provided $\nu > \frac{1}{1-\omega_0}$.

Proof. Straightforward first year calculus; the only delicate point is in checking condition (197). Simply note the asymptotics

$$\Upsilon = 1 - \left(1 - \omega_0 - \frac{1}{\nu}\right)\epsilon \log L + o(\epsilon), \tag{223}$$

in order to check that Lemma 6 applies, with the above hypothesis on ν. □

8. Fixed Point in the Space of Sequences

We start by applying Theorem 1. So we choose some $\kappa_0 > 0$ and $L_0 \in \mathbb{N}$ whose existence is guaranteed by the theorem. We set $\kappa = \kappa_0$, and we take

$$A_g = \frac{1}{2}\ , \tag{224}$$

$$A_\mu = 1\ , \tag{225}$$

$$A_R = 1\ , \tag{226}$$

$$A_{\bar{g}} = 19\pi^2\ , \tag{227}$$

$$\delta = \frac{1}{6}\ , \tag{228}$$

$$\eta = \frac{3}{16}\ . \tag{229}$$

Now take c to be equal to a c_0 provided by the theorem, which also produces some B_g and $B_{R\mathcal{L}}$ only depending on the quantities which have been fixed so far. Now choose $L \geq L_0$ large enough so that

$$B_{R\mathcal{L}} L^{-\frac{1}{4}} \leq \frac{1}{3}\ . \tag{230}$$

This will guarantee that for any $\epsilon \in]0, \frac{1}{2}]$,

$$B_{R\mathcal{L}} L^{-\left(\frac{1-\epsilon}{2}\right)} \leq \frac{1}{3}\ . \tag{231}$$

Now the theorem provides us with B_μ, $B_{R\xi}$, and ϵ_0. We will choose some ϵ_1 such that $0 < \epsilon_1 < \min(\frac{1}{2}, \epsilon_0)$, and such that for all $\epsilon \in]0, \epsilon_1]$ one has $\frac{\bar{g}_*}{\epsilon} < A_{\bar{g}}$. This is possible thanks to (21) and (227).

We now have the following specialization of Theorem 1.

Proposition 3. *There exists an $\epsilon_2 \in]0, \epsilon_1]$ such that for any $\epsilon \in]0, \epsilon_2]$, and for any calibrator $\bar{g} \in]0, \bar{g}_*[$, the conclusions (1)–(6) of Theorem 1 are valid with the inequality in (134) replaced by*

$$|||\mathcal{L}^{(g,\mu)}(R)|||_{f(\bar{g})} \leq \frac{1}{2}|||R|||_{\bar{g}}\ . \tag{232}$$

The proof is an immediate corollary of the following lemma.

Lemma 8. *Provided*

$$\max\left(L^{2\epsilon}, (2 - L^{\epsilon})^{-\frac{1}{4}}\right) \leq \frac{3}{2}, \tag{233}$$

which will hold true when $\epsilon \to 0$, one has for any $\bar{g} \in]0, \bar{g}_*[$, and any $R \in \mathcal{BBS}^{\mathbb{K}}$,

$$|||R|||_{f(\bar{g})} \leq \frac{3}{2}|||R|||_{\bar{g}}\ . \tag{234}$$

Proof. Let $\bar{g}' = f(\bar{g})$. Since $\bar{g}' > \bar{g}$, and from the definition of the triple norms it is immediate that for any R one has

$$|||R|||_{\bar{g}'} \leq \max\left[\left(\frac{\bar{g}'}{\bar{g}}\right)^2, \left(\frac{\bar{g}'}{\bar{g}}\right)^{\frac{7}{4}}, \ldots, 1, \left(\frac{\bar{g}'}{\bar{g}}\right)^{-\frac{1}{4}}\right] \times |||R|||_{\bar{g}}\ . \tag{235}$$

However, by the mean value theorem,

$$\frac{\bar{g}'}{\bar{g}} = \frac{f(\bar{g}) - f(0)}{\bar{g} - 0} = f'(\varsigma) \tag{236}$$

for some $\varsigma \in]0, \bar{g}_*[$. As a result

$$2 - L^{\epsilon} < \frac{\bar{g}'}{\bar{g}} < L^{\epsilon} , \tag{237}$$

and the lemma follows. □

Now given $\omega_0 \in]0, \frac{1}{2}[$, we construct the sequence $(\bar{g}_n)_{n\in\mathbb{Z}}$ as in Sect. 5, as well as the associated spaces $(\mathcal{BBSS}^{\mathbb{K}}, ||||\cdot||||)$. Given an element $\delta s \in \mathcal{BBSS}^{\mathbb{K}}$, and a positive number β we use the notation $B_{\mathbb{K}}(\delta s, \beta)$ for the open ball of radius β around δs in $\mathcal{BBSS}^{\mathbb{K}}$. We also use $\bar{B}_{\mathbb{K}}(\delta s, \beta)$ for the analogous closed ball. We can now state our main theorem.

Theorem 2. The Main Theorem.
$\exists \beta_0, \forall \beta \in]0, \beta_0]$,
$\exists \epsilon_3 > 0, \forall \epsilon \in]0, \epsilon_3]$,
one has

1. *The* $\mathcal{BBSS}^{\mathbb{C}}$ *valued map* $\mathfrak{m}$ *from Sect. 5 is well defined and analytic on* $B_{\mathbb{C}}(0, \beta)$.
2. *The image by* $\mathfrak{m}$ *of* $B_{\mathbb{C}}(0, \beta)$ *is contained in* $\bar{B}_{\mathbb{C}}(0, \frac{\beta}{6})$.
3. *The restriction of* $\mathfrak{m}$ *to the closed ball* $\bar{B}_{\mathbb{R}}(0, \frac{\beta}{6})$ *is a contraction from that ball to itself.*
4. *There exists a unique fixed point for the map* $\mathfrak{m}$ *inside the ball* $\bar{B}_{\mathbb{R}}(0, \frac{\beta}{6})$.

Proof. Let $\beta > 0$ be such that the condition $\beta \le \frac{1}{2} = A_g$ is realized. Then by construction, for any $n \in \mathbb{Z}$, $\bar{g}_n \in]0, A_{\bar{g}}\epsilon[$. Therefore, as a consequence of Proposition 3, for any

$$\delta s = (\delta g_n, \mu_n, R_n)_{n\in\mathbb{Z}} \in B_{\mathbb{C}}(0, \beta) ,$$

all the summands in (121), (122), (123), and (124) are well defined and analytic with respect to δs. The analyticity property required in statement *(1)* will therefore follow from the uniform absolute convergence of the series. The latter will in turn result from the estimates, required for the statement *(2)*, which we now proceed to establish. Using the notations of Definition 1, we assume that δs is in $B_{\mathbb{C}}(0, \beta)$, and we apply the estimates of Sect. 7, in order to obtain the following results.
The backward δg bound . Let $n > 0$, then

$$\frac{1}{\beta}|\delta g_n'|\bar{g}_n^{-\frac{3}{2}} \le \frac{1}{\beta \bar{g}_n^{\frac{3}{2}}} \sum_{0\le p<n} \left(\prod_{p<j<n} f'(\bar{g}_j) \right)$$
$$\times \left[L^{2\epsilon} a(L,\epsilon)|\delta g_p|^2 + |\xi_g(\bar{g}_p + \delta g_p, \mu_p, R_p)| \right]$$
$$\le \frac{1}{\beta \bar{g}_n^{\frac{3}{2}}} \sum_{0\le p<n} \left(\prod_{p<j<n} f'(\bar{g}_j) \right)$$

$$\times \left[L^{2\epsilon} a(L,\epsilon)\beta^2 \bar{g}_p^3 + B_g \bar{g}_p^{\left(\frac{11}{4}-\frac{3}{16}\right)} \right] \tag{238}$$

$$\leq \beta L^{2\epsilon} a(L,\epsilon) \bar{\Sigma}_{\delta g-b}\left(\epsilon, \frac{3}{2}, 3\right) + \frac{1}{\beta} B_g \bar{\Sigma}_{\delta g-b}\left(\epsilon, \frac{3}{2}, \frac{11}{4} - \frac{3}{16}\right) . \tag{239}$$

Now by part *3)* of Lemma 7 and for any fixed β, the last upper bound goes to zero when $\epsilon \to 0$. Therefore, by choosing ϵ small enough, one will have

$$\forall n > 0\ ,\ \frac{1}{\beta}|\delta g_n'|\bar{g}_n^{-\frac{3}{2}} \leq \frac{1}{6} . \tag{240}$$

The forward δg bound . Let $n > 0$, then in the same vein one will have

$$\frac{1}{\beta}|\delta g_n'|\bar{g}_n^{-1} \leq \beta L^{2\epsilon} a(L,\epsilon)\bar{\Sigma}_{\delta g-f}(\epsilon,1,2) + \frac{1}{\beta} B_g \bar{\Sigma}_{\delta g-b}\left(\epsilon, 1, \frac{11}{4} - \frac{3}{16}\right) . \tag{241}$$

Now here comes the narrowest passage in the proof. Provided that $\omega_0 \in]0, \frac{1}{2}[$, the limiting case of part *4)* in Lemma 7 shows that

$$L^{2\epsilon} a(L,\epsilon)\bar{\Sigma}_{\delta g-f}(\epsilon,1,2) \to \frac{\omega_0}{1-2\omega_0}\exp\left[\frac{2\omega_0}{1-\omega_0}\right] \tag{242}$$

when $\epsilon \to 0$. Therefore we need to take

$$\beta < \frac{1-2\omega_0}{6\omega_0}\exp\left[-\frac{2\omega_0}{1-\omega_0}\right] . \tag{243}$$

Then after β is fixed accordingly, the first term in (241) will be strictly less than $\frac{1}{6}$ in the $\epsilon \to 0$ limit while the second term will go to zero, again by *4)* of Lemma 7. We will then have

$$\forall n < 0\ ,\ \frac{1}{\beta}|\delta g_n'|\bar{g}_n^{-1} \leq \frac{1}{6} . \tag{244}$$

The forward μ bound . Let $n \in \mathbb{Z}$, then by the same reasoning one will have

$$\frac{1}{\beta}|\mu_n'|\bar{g}_n^{-(2-\frac{1}{6})} \leq \frac{1}{\beta} B_\mu \bar{\Sigma}_{\mu-f}\left(\epsilon, 2-\frac{1}{6}, 2\right) , \tag{245}$$

which will go to zero when $\epsilon \to 0$, as results from *case 2)* of Lemma 7. We will then have

$$\forall n \in \mathbb{Z}\ ,\ \frac{1}{\beta}|\mu_n'|\bar{g}_n^{-(2-\frac{1}{6})} \leq \frac{1}{6} . \tag{246}$$

The backward *R* bound . Let $n \in \mathbb{Z}$, then proceed in the same manner except that the varying norms require a little care. We have

$$\frac{1}{\beta}|||R'_n|||_{\bar{g}_n} \times \bar{g}_n^{-\left(\frac{11}{4}-\frac{3}{16}\right)}$$
$$\leq \frac{1}{\beta \bar{g}_n^{\left(\frac{11}{4}-\frac{3}{16}\right)}} \times \sum_{p<n} |||\mathcal{L}^{(\bar{g}_{n-1}+\delta g_{n-1},\mu_{n-1})} \circ \mathcal{L}^{(\bar{g}_{n-2}+\delta g_{n-2},\mu_{n-2})} \circ \cdots \tag{247}$$
$$\cdots \circ \mathcal{L}^{(\bar{g}_{p+1}+\delta g_{p+1},\mu_{p+1})} \left(\xi_R(\bar{g}_p+\delta g_p,\mu_p,R_p)\right) |||_{\bar{g}_n}$$
$$\leq \frac{1}{\beta \bar{g}_n^{\left(\frac{11}{4}-\frac{3}{16}\right)}} \times \sum_{p<n} \left(\frac{1}{2}\right)^{n-p-1} \times \frac{3}{2} \times B_{R\xi} \times \bar{g}_p^{\frac{11}{4}}, \tag{248}$$

where we repeatedly used the inequality (232), as well as (234), and the ξ_R estimate in item (5) of Theorem 1. In sum one has

$$\frac{1}{\beta}|||R'_n|||_{\bar{g}_n} \times \bar{g}_n^{-\left(\frac{11}{4}-\frac{3}{16}\right)} \leq \frac{3B_{R\xi}}{2\beta} \times \bar{\Sigma}_{R-b}\left(\epsilon, \frac{11}{4}-\frac{3}{16}, \frac{11}{4}\right), \tag{249}$$

and this goes to zero when $\epsilon \to 0$, as shown in part *1)* of Lemma 7, with $c_R = \frac{1}{2}$.

At this point, statements *(1)* and *(2)* of the theorem are proved.

The contraction property . Let $\delta s_1 \neq \delta s_2$ be two elements of the open ball $B_{\mathbb{C}}(0, \frac{\beta}{6})$. Let

$$r = \frac{2\beta}{3||||\delta s_1 - \delta s_2||||} \ . \tag{250}$$

Then

$$||||\delta s_1 - \delta s_2|||| \leq ||||\delta s_1|||| + ||||\delta s_2|||| \leq \frac{\beta}{3} \tag{251}$$

implies that $r \geq 2$. Therefore, if one defines the contour γ as the counterclockwise oriented circle or radius r around the origin in the complex plane; one has by the Cauchy theorem

$$\mathfrak{m}(\delta s_1) - \mathfrak{m}(\delta s_2) = \frac{1}{2\pi i}\oint_\gamma dz \left(\frac{1}{z-1} - \frac{1}{z}\right) \mathfrak{m}\left(\delta s_2 + z(\delta s_1 - \delta s_2)\right) \ . \tag{252}$$

Now for $z \in \gamma$ we have

$$||||\delta s_2 + z(\delta s_1 - \delta s_2)|||| \leq ||||\delta s_2|||| + r||||\delta s_1 - \delta s_2|||| \tag{253}$$
$$\leq \frac{\beta}{6} + \frac{2\beta}{3} \tag{254}$$
$$< \beta \ . \tag{255}$$

As a result of the already established statement *(2)*, one has

$$||||\mathfrak{m}(\delta s_1) - \mathfrak{m}(\delta s_2)|||| \leq \frac{1}{r-1} \times \max_{0\leq\theta\leq 2\pi} ||||\mathfrak{m}(\delta s_2 + re^{i\theta}(\delta s_1 - \delta s_2))|||| \tag{256}$$
$$\leq \frac{\beta}{6(r-1)} \tag{257}$$
$$\leq \frac{\beta}{3r} \tag{258}$$

because $r \geq 2$. Inserting the definition of r shows that

$$|||\mathfrak{m}(\delta s_1) - \mathfrak{m}(\delta s_2)||| \leq \frac{1}{2} \times |||\delta s_1 - \delta s_2||| \, , \tag{259}$$

i.e., the contraction property.

The real ball stability follows from statements *(3)* and *(4)* in Theorem 1/Proposition 3 and Definition 1. Now statement *(3)* is proved, and *(4)* follows from the Banach fixed point theorem. This concludes the proof of the main theorem. □

Corollary 1. *The constructed two-sided trajectory* $(g_n, \mu_n, R_n)_{n\in\mathbb{Z}}$ *is the unique such sequence inside the ball* $\bar{B}_{\mathbb{R}}(0, \frac{\beta}{6})$ *of* $\mathcal{BBSS}^{\mathbb{R}}$ *which solves the recursion (113). One has*

$$\lim_{n\to-\infty} (g_n, \mu_n, R_n) = (0, 0, 0) \, , \tag{260}$$

the trivial Gaussian ultraviolet fixed point, and

$$\lim_{n\to+\infty} (g_n, \mu_n, R_n) = (g_*, \mu_*, R_*) \, , \tag{261}$$

the BMS nontrivial infrared fixed point [15].

Proof. The proof of the first statement is easy and left to the reader. Note that the statements concerning the limits for R_n are topological and do not depend on a particular choice of a calibrated norm $||| \cdot |||_{\bar{g}}$. The last statement follows from the possibility of making β as small as we want, provided ϵ is small enough. Indeed, because of the choice of exponent $\frac{3}{2}$ in (125) at the positive end for n, the convergence of $\bar{g}_n$ to $\bar{g}_*$ when $n \to +\infty$ will ensure that for large positive values of n, (g_n, μ_n, R_n) will fall within the small domain around the approximate IR fixed point where the stable manifold has been constructed, and where the convergence of all one-sided sequences which remain bounded in the future, towards the IR fixed point, has been been established [15, Sect. 6]. □

9. Suggestions for Future Work

The following is a list of problems which are natural continuations of the present work.
1) The continuous connecting orbit between the two fixed points should be the graph of a function $g \mapsto (\mu(g), R(g))$ with g in the range $0 < g < g_*$. In principle, when considering one of the sequences $(g_n, \mu_n, R_n)_{n\in\mathbb{Z}}$ we constructed as a function of g_0 only, this map should correspond to the one giving μ_0 and R_0 in terms of g_0 (which is here provided in the range $0 < g < \frac{\bar{g}_*}{2}$). One could even say that it is also the map giving μ_n and R_n in terms of g_n, for any n, provided one could do the proper inversions. Although we did not yet explore this, it seems likely that by a more refined analysis, one can construct the full invariant curve connecting the two fixed points. This would open the door to the investigation, in a constructive setting, of the old 'reparametrization' renormalization group [72, 41]. This has so far remained inaccessible in Bosonic constructive field theory. In contrast, a continuous RG for Fermions has been developed through work initiated in [65] and completed in [27].
2) If one could answer the first question, then the immediate one that follows is: what would be the regularity of this curve? It seems reasonable to conjecture real analyticity in the range $0 < g < g_*$. An interesting question in this regard raised by K. Gawędzki,

concerns the C^∞ behavior, or not, of this curve at $g = 0^+$. A similar question was mentioned in [39], related to a possible explanation of the breakdown of the traditional perturbative argument ruling out nonrenormalizable theories as consistent [63, 73, 61]. To gain insight on this issue, consider the following simplified flow which mimics the behavior of the RG map considered here:

$$\begin{cases} \frac{dg}{dt} = \alpha g - \beta g^2 \ , \\ \frac{d\mu}{dt} = \gamma \mu - \delta g^2 \ . \end{cases}$$

If one eliminates the time variable then the connecting orbit can be expressed exactly in terms of an incomplete beta function, which admits a convergent hypergeometric series representation near $g = 0$. If one rescales g writing $s = \frac{\beta g}{\alpha}$ and letting $\nu = \frac{\gamma}{\alpha}$, then the smoothness of the orbit at 0^+ is reduced to that of the function

$$s \mapsto s^\nu \frac{\pi(\nu-1)}{\sin[\pi(\nu-1)]} + \frac{s^2}{\nu-2} \times {}_2F_1\left[\begin{matrix} 1-\nu\ ,\ 2-\nu \\ 3-\nu \end{matrix} ; s\right]$$

at $s = 0$, when ν is not an integer. In this case C^∞ behavior is ruled out. In our setting ν is roughly given by

$$\frac{L^{\left(\frac{3+\epsilon}{2}\right)} - 1}{L^\epsilon - 1} ,$$

which is very large.

3) The RG map considered in [15] and also here is in the so-called ‘formal infinite volume limit’. With more work one can probably perform the true scaling limit of the theory, using an appropriate bare ansatz as in [39] for instance. One should also try to develop a streamlined rigorous RG framework for the handling of correlation functions, including those of more general observables, like composite operators. An important step in this direction was taken in [14]. One should then prove or disprove the existence of anomalous scaling dimensions not only for the field $\phi(x)$, but also for composite operators. In the hierarchical model there is no anomalous dimension for the field $\phi(x)$ as shown in [37]. A similar result for the full model was recently obtained [57, 58], together with a preliminary perturbative calculation which supports the hypothesis of a nonzero anomalous dimension of order ϵ for the composite field $\phi(x)^2$. Justifying this last statement by a rigorous nonperturbative proof, however is a tantalizing open problem. Finally if one can go as far, the investigation by analytical means of Wilson’s operator product expansion would make a nice crowning achievement.

4) Orthogonal to the RG approach by Brydges and collaborators, where one tries to know as little as possible about the irrelevant terms R, there is also the phase space expansion method [42] which has become the trademark of the French school of constructive field theory [32, 33, 64] (see also [5]). In this other approach one, on the contrary, tries to know as much as possible about the explicit structure of these terms [2]. We therefore hope to have the future opportunity of investigating the same model as considered here, with this alternative approach. The lessons learnt with the methods of Brydges and collaborators will be useful in this regard. For instance, in [2] the hypothesis of large L was not used and polymers were allowed which have large gaps in the vertical direction. Albeit esthetically pleasing, these features lead to additional technical difficulties which drive one away from maximal simplicity. The use of strictly short ranged fluctuation

covariances introduced in [59], exploited in [15] as well as the present article, and systematically developed in [13, 16], should allow major simplifications in the multiscale phase space expansions framework.
5) Important new methods for dealing with ϕ^4-type lattice models, based on Witten Laplacian techniques, have been developed recently [45, 69, 4]. It would be desirable to extend their reach to the case of critical theories. Although one should bear in mind that according to RG wisdom, rather than the weakly convex case (no ϕ^2 in the bare potential), it is the double well case (properly adjusted strictly negative ϕ^2 coupling) which should entail a power law behaviour of correlations. The result in the present article adds a new confirmation to this picture. Indeed, our trajectory which lies on the critical manifold essentially has $\mu = \mathcal{O}(g^{2-\delta})$. Undoing the Wick ordering, this means that the ϕ^2 coupling is $\mu - 6C(0)g < 0$.

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