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Constraining the Kähler Moduli in the Heterotic Standard Model

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Abstract: Phenomenological implications of the volume of the Calabi-Yau threefolds on the hidden and observable M-theory boundaries, together with slope stability of their corresponding vector bundles, constrain the set of Kähler moduli which give rise to realistic compactifications of the strongly coupled heterotic string. When vector bundles are constructed using extensions, we provide simple rules to determine lower and upper bounds to the region of the Kähler moduli space where such compactifications can exist. We show how small these regions can be, working out in full detail the case of the recently proposed Heterotic Standard Model. More explicitly, we exhibit Kähler classes in these regions for which the visible vector bundle is stable. On the other hand, there is no polarization for which the hidden bundle is stable.

1. Introduction

Our understanding of Calabi-Yau compactifications of string/M-theory has been increased considerably during the last years. On the one hand, distributions of vacua for type IIB, IIA and type I string theory are much better understood. On the other hand, promising compactifications of the heterotic string have been found at special points of the moduli space.

Although a systematic study of distributions of vacua for compactifications of the heterotic string is much harder, because our primitive understanding of their moduli stabilization and the huge amount of vector bundle moduli, we can still find systematic criteria to constrain the regions of the moduli space where realistic vacua should be located.

Recently, phenomenologically interesting Calabi-Yau compactifications of the heterotic string have appeared in the literature [2, 5]. Using certain elliptically fibered threefold with fundamental group $\mathbb{Z}_3 \times \mathbb{Z}_3$, and an $SU(4) \times \mathbb{Z}_3 \times \mathbb{Z}_3$ instanton living on the visible E_8-bundle, give rise to an effective field theory on $\mathbb{R}^4$ which has the particle spectrum of the Minimal Supersymmetric Standard Model (MSSM), with no exotic matter but an additional pair of Higgs-Higgs conjugate superfields. In these models, vector bundles are constructed using vector bundle extensions, which correspond to Hermitian Yang-Mills connections when they are slope-stable. We use this specific construction to exemplify how a systematic selection of realistic Kähler moduli can be done.[1]

The organization of the paper is as follows: Sect. 2 contains an outline of the natural criteria for selecting Kähler moduli in realistic Calabi-Yau compactifications of the heterotic string. In Sect. 3, we analyze the case of the Heterotic Standard Model, describe the geometry of the elliptic Calabi-Yau and construct its Kähler cone. Section 4 provides lower and upper bounds to the region of the Kähler cone that makes stable the observable vector bundle of the HSM. In such construction we find a destabilizing sub-line bundle for the hidden vector bundle, and exhibit Kähler classes that make stable the visible one.

2. Picking Kähler Moduli

The spacetime in a Calabi-Yau compactification of the strongly coupled heterotic string [15], defined through the direct product eleven-dimensional manifold $Y = \mathbb{R}^4 \times X \times [0,\ 1]$, with X a Calabi-Yau threefold. $\mathcal{N} = 1$ supersymmetry on the four dimensional Effective Field Theory, requires to fix a G_2-holonomy metric on $X \times [0,\ 1]$ plus gauge connections at the hidden and visible vector bundles, which satisfy the Hermitian Yang-Mills equations. In order to define a barely G_2-holonomy metric on $X \times [0,\ 1]$ we introduce a calibration 3-form, according to D. Joyce [16],

$$\Phi = (At + B)\omega \wedge \mathrm{d}t + \mathrm{Re}(\Omega), \tag{2.1}$$

which depends on the differential dt along the interval and the holomorphic 3-form Ω and Kähler class ω of the threefold. Such a calibration defines a barely G_2-holonomy metric on $X \times [0,\ 1]$, where the Kähler class is linearly dilated along the interval; therefore at the visible and hidden boundaries the Kähler classes are $\omega_0 = B\omega$ and $\omega_1 = (A + B)\omega$ ($i = 1$ stands for the 'hidden' boundary and $i = 0$ for the 'visible' one). The set of Kähler classes on X is usually known as the Kähler cone and denoted by $\mathcal{K}(X) \subset H^2(X,\ \mathbb{Z})$.

One approach to model building is to attach a $SU(n) \times G$ Hermitian Yang-Mills gauge connection at the boundary, to obtain an Effective Field Theory with the commutant of $SU(n) \times G \subset E_8$ as gauge group while the $\mathcal{N} = 1$ supersymmetry of the EFT is preserved. Here G is the non-trivial holonomy group associated to a certain flat line bundle. By the theorem of Donaldson and Uhlenbeck-Yau [9], we know that $SU(n)$-connections that satisfy the Hermitian Yang-Mills equations and slope-stable rank-n holomorphic vector bundles with vanishing first Chern class are in one-to-one correspondence.

2.1. Constraining angular degrees of freedom. Thus, the holomorphic vector bundles $V_i \rightarrow X$ that we fix at the hidden and visible sectors, have to be slope stable in order to get a sensible vacuum. Slope stability can impose severe constraints on the Kähler moduli.

[1] Recently, Donagi and Bouchard [8] have also proposed an independent CY compactification of the heterotic string with the spectrum of the MSSM and no exotic matter, using a different Calabi-Yau with an explicitly slope-stable vector bundle in the observable sector. It would be also interesting to study in detail these questions with the vector bundle which has just appeared in [4], on the same CY [5].

If $W_i \hookrightarrow V_i$ is a rank-m (with $m < n$) holomorphic torsion free subsheaf[2], then only the $\omega_i \in \mathcal{K}(X)$ that verify

$$\frac{1}{m}\int_X \omega_i^2 \wedge c_1(W_i) < \frac{1}{n}\int_X \omega_i^2 \wedge c_1(V_i) = 0, \tag{2.2}$$

can make V_i stable. At this point we realize that if ω_i is a stablemaker for V_i, then $N\omega_i$ with $N \in \mathbb{Z}^+$ is also a stablemaker. The stablemakers form a subcone $\mathcal{K}_i^s(X) \subseteq \mathcal{K}(X)$ within the Kähler moduli, [20].

The physical importance of slope stability is clear, [9]: Non-stablemaker classes at the boundary of $\mathcal{K}_i^s(X)$ make the vector bundle V_i semistable, i.e. we can only find correspondences to connections with reduced gauge group $H \subset SU(n)$, thus the gauge dynamics of the Effective Field Theory would be governed by the commutant of $H \times G \subset E_8$ instead of $SU(n) \times G$.

Usually, a detailed computation of $\mathcal{K}_i^s(X)$ is difficult because we need to identify every subsheaf W_i of V_i. Note that if $h^0(W_i^\vee \otimes V_i) = 0$, then W_i cannot be a subsheaf of V_i, but the converse is not necessarily true. If the vector bundle V_i is constructed through a non-trivial extension, defined by a short exact sequence

$$0 \longrightarrow V_L \longrightarrow V_i \longrightarrow V_R \longrightarrow 0, \tag{2.3}$$

with $\mathrm{Ext}^1(V_R, V_L) \neq 0$, we can give upper and lower bounds to $\mathcal{K}_i^s(X)$ in a simple way, looking at subsheaves of V_L and V_R.

On the one hand, the set $\mathcal{UL}_i$ of subsheaves of V_L is a subset of the set of subsheaves of V_i, since $V_L \to V_i$ is injective. This provides an upper bound for cone $\mathcal{K}_i^s(X)$ of Kähler classes for which V_i is stable:

$$\mathcal{K}_i^s(X)^> = \left\{\omega_i \in \mathcal{K}(X) \,:\, \int_X \omega_i^2 \wedge c_1(L_i) < 0,\ \forall\, L_i \in \mathcal{UL}_i\right\}. \tag{2.4}$$

On the other hand, a subsheaf of V_i gives an element of $\mathcal{UL}_i \times \mathcal{UR}_i$, where $\mathcal{UR}_i$ is the subset of subsheaves of V_R. Indeed, if W_i is a subsheaf of V_i, there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_L & \longrightarrow & V_i & \longrightarrow & V_R & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & W_L & \longrightarrow & W_i & \longrightarrow & W_R & \longrightarrow & 0 \end{array} \tag{2.5}$$

where the vertical arrows are injective, hence we obtain subsheaves W_L and W_R of V_L and V_R. This gives a lower bound

$$\mathcal{K}_i^s(X)^< = \left\{\omega_i \in \mathcal{K}(X) : \int_X \omega_i^2 \wedge (c_1(W_L) + c_1(W_R)) < 0,\ \forall\, W_L \in \mathcal{UL}_i\,,\ W_R \in \mathcal{UR}_i\right\}. \tag{2.6}$$

Note that the ones belonging to $\mathcal{UL}_i$ are true subsheaves of V_i, and the ones in $\mathcal{UR}_i$ are possible subsheaves of V_i. Therefore, we can construct two bounds to the stablemaker Kähler subcone $\mathcal{K}_i^s(X)$,

$$\mathcal{K}_i^s(X)^< \subseteq \mathcal{K}_i^s(X) \subseteq \mathcal{K}_i^s(X)^>. \tag{2.7}$$

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[2] It is enough to consider reflexive sheaves, i.e., sheaves with $W_i = W_i^{\vee\vee}$. Furthermore, we can assume that W_i is semistable.

Sometimes we can use further information to discard some pairs (W_L, W_R) which do not come from subsheaves W_i of V_i, hence obtaining a better lower bound. For instance, the pair $(0, V_R)$ can be discarded, because it would give a splitting of the defining exact sequence (2.3), but we have assumed that the extension is not trivial, hence has no splitting. Other cases that can be discarded are pairs of the form $(0, W_R)$ when $h^0(W_R^\vee \otimes V_i) = 0$. We shall apply these ideas in the next sections to the vector bundles constructed in the Heterotic Standard Model, [2].

2.2. Constraining radial degrees of freedom. In the last subsection we have seen how to choose rays in the Kähler cone that preserve the slope stability of a given vector bundle, and thus define a consistent gauge group in the effective field theory. On the other hand, radial degrees of freedom in $\mathcal{K}_i^s(X)$ are related with variations of the volume of X, [11]. We are not free to choose arbitrary volumes for the threefolds at the hidden and observable sector, if we want to preserve sensible values for Newton's constant and the E_8 gauge coupling, [22].

Using Liouville's measure, we can estimate the volume of the Calabi-Yau threefold at the point $\omega_i \in \mathcal{K}(X)$ as [3]

$$\mathrm{Vol}(X)_i = \frac{1}{3!}\int_X \omega_i^3, \tag{2.8}$$

thus radial dilations in the Kähler cone $\omega_i \mapsto N\omega_i$ with $N \in \mathbb{Z}^+$, map the volume as $\mathrm{Vol}(X)_i \mapsto N^3 \mathrm{Vol}(X)_i$.

The volume of the threefolds at the boundaries of Y, are related through Witten's formula [22]

$$\mathrm{Vol}(X)_1 = \mathrm{Vol}(X)_0 + 2\pi \frac{\rho}{\ell_P}\int_X \omega_0 \wedge \left(c_2(V_0) - \frac{1}{2}c_2(TX)\right) + \mathcal{O}(\rho^2), \tag{2.9}$$

with ℓ_P the eleven dimensional Planck length and ρ the length of the M-theory interval. This formula (2.9) holds at first order in ρ, which is the limit where we work, as in (2.1). A more accurate relation between the volumes of the CYs at the boundaries, taking into account the non-linear corrections in ρ, was derived in [6] and [7]. Newton's constant in the effective supergravity theory on the observable $\mathbb{R}^4$ of Y goes as

$$G_N \sim \frac{\ell_P^9}{\rho \mathrm{Vol}(X)_0}, \tag{2.10}$$

and the E_8 gauge coupling as

$$\alpha_{GUT} \sim \frac{\ell_P^6}{\mathrm{Vol}(X)_0}. \tag{2.11}$$

Witten observed in [22], that in order to find realistic values for these physical quantities, the volume of the threefold in the visible sector has to be very large. As the integral in the right-hand side of (2.9) is negative due to Chern-Weil theory, and the identity

$$\int_X \mathrm{Tr}\left(F^2\right) \wedge \omega = -\int_X |F|^2 \omega^3, \tag{2.12}$$

[3] Being rigorous, we should work with the dimensionfull measure $(\alpha'\omega)^3$, although this will be irrelevant for our purposes because α' factorizes out in the formulae that we use. In the small volume limit this approximation can fail, and we should use conformal field theory to give a more accurate estimation.

he deduced that sensible values for G_N and α_{GUT} are only possible for very small $\mathrm{Vol}(X)_1$.

Summarizing: Let $\mathcal{K}_0^s(X)$ and $\mathcal{K}_1^s(X)$ be the set of Kähler classes that make stable $V_0 \to X$ and $V_1 \to X$, respectively. Physically interesting vacua should be located in rays of the Kähler cone lying in the intersection $\mathcal{K}_0^s(X) \cap \mathcal{K}_1^s(X) \subset \mathcal{K}(X)$, such that the relative dilating factor ω_0/ω_1 is very large, and the Witten's correlation

$$\frac{1}{3!}\int_X \omega_1^3 \sim \frac{1}{3!}\int_X \omega_0^3 + 2\pi \frac{\rho}{\ell_P}\int_X \omega_0 \wedge \left(c_2(V_0) - \frac{1}{2}c_2(TX)\right) \tag{2.13}$$

is satisfied.

Remark 1. Although the study of distributions of vacua for these models is not as developed as for Calabi-Yau compactifications of the type II string theory, the presence of vacua in these regions of the Kähler moduli space should be statistically favorable along the lines of [10], once the vector bundle, dilaton and complex moduli are stabilized.

We have shown how to identify these regions explicitly. In the rest of the paper we determine them for the recently proposed Heterotic Standard Model.

3. The Elliptic Calabi-Yau and its Kähler Cone

First, we briefly recall the construction of the Calabi-Yau threefold used the Heterotic Standard Model, following the reference [5]. Let $\widetilde{X}$ be the fiber product over $\mathbb{P}^1$ of two rational elliptic surfaces $\widetilde{X} = B_1 \times_{\mathbb{P}^1} B_2$, as in the diagram:

$$\begin{tikzcd}[column sep=small] & \widetilde{X} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, "\pi"] & \\ B_1 \arrow[dr, "\beta_1"'] & & B_2 \arrow[dl, "\beta_2"] \\ & \mathbb{P}^1 & \end{tikzcd} \tag{3.1}$$

This kind of Calabi-Yau threefolds was already studied by C. Schoen in **S**. The geometry of $\widetilde{X}$ is basically encoded in the geometry of the rational elliptic surfaces B_1 and B_2. Due to the phenomenological interest in finding threefolds which admit certain Wilson lines[4], the aim of [5] was to look for threefolds $\widetilde{X}$ such that $\mathbb{Z}_3 \times \mathbb{Z}_3 \subseteq \mathrm{Aut}(\widetilde{X})$. This search was achieved thanks to the existence of certain elliptic surfaces that admit an action of $\mathbb{Z}_3 \times \mathbb{Z}_3$ which can be characterized explicitly through a proper understanding of the Mordell-Weil group of B.

Following Kodaira's classification of singular fibers, our elliptic surfaces B_1 and B_2 are characterized by three I_1 and three I_3 singular fibers. Such rational elliptic surfaces are described by one-dimensional families that allow us to build fiber products $\widetilde{X}$, corresponding to smooth Calabi-Yau threefolds. Furthermore, $\widetilde{X}$ admits a free action of $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ and the quotient $X = \widetilde{X}/G$ is also a smooth Calabi-Yau threefold with fundamental group $\pi_1(X) = \mathbb{Z}_3 \times \mathbb{Z}_3$. Let g_1 and g_2 be generators of this group, g_2 acting as translation t_η by a section η of the fibration $\beta : B \to \mathbb{P}^1$, and g_1 acting nontrivially on the base. Let $\xi = g_1(\sigma)$ and $\alpha_B = t_{-\xi} \circ g_1$.

The threefold used in the description of the Heterotic Standard Model is $X = \widetilde{X}/G$, although we will work with G-equivariant objects on $\widetilde{X}$. In the rest of this section

[4] I.e. flat line bundles with non-trivial holonomy.

we describe the G-invariant homology rings of B and $\widetilde{X}$, and their corresponding G-invariant Kähler cones (i.e. their ample cones, or spaces of polarizations).

For the homology of a surface B, we choose a set of generators: the 0-section σ, the generic fiber F, the 6 irreducible components of the three I_3 singular fibers that do not intersect the 0-section $\Theta_{1,1}, \Theta_{1,2}, \ldots, \Theta_{3,1}, \Theta_{3,2}$, and the two sections generating the free part of the Mordell-Weil group[5] ξ and $\alpha_B \xi$. These generators are a basis for $H_2(B, \mathbb{Z}) \otimes \mathbb{Q}$, but adding the torsion generator of the Mordell-Weil group

$$\eta = \sigma + F - \frac{2}{3}\left(\Theta_{1,1} + \Theta_{2,1} + \Theta_{3,1}\right) - \frac{1}{3}\left(\Theta_{1,2} + \Theta_{2,2} + \Theta_{3,2}\right), \tag{3.2}$$

we generate all $H_2(B, \mathbb{Z})$.

The intersection matrix of the homology generators is as follows:

$$\begin{pmatrix} \sigma \\ F \\ \Theta_{1,1} \\ \Theta_{2,1} \\ \Theta_{3,1} \\ \Theta_{1,2} \\ \Theta_{2,2} \\ \Theta_{3,2} \\ \xi \\ \alpha_B\xi \\ \eta \end{pmatrix}^T \cdot \begin{pmatrix} \sigma \\ F \\ \Theta_{1,1} \\ \Theta_{2,1} \\ \Theta_{3,1} \\ \Theta_{1,2} \\ \Theta_{2,2} \\ \Theta_{3,2} \\ \xi \\ \alpha_B\xi \\ \eta \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3.3}$$

The invariant homology under the action of $G = \mathbb{Z}_3 \times \mathbb{Z}_3$, is generated by

$$H_2(B, \mathbb{Z})^G = \operatorname{span}_{\mathbb{Z}} \left\{F,\ t = -\sigma + \Theta_{2,1} + \Theta_{3,1} + \Theta_{3,2} + 2\xi + \alpha_B\xi + \eta - F\right\}, \tag{3.4}$$

where t can be also expressed as the homological sum of three sections, i.e. $t = \xi + \alpha_B\xi + \eta \boxplus \xi$.

The cohomology ring of X can be expressed as

$$H^*(X,\ \mathbb{Q}) = H^*(\widetilde{X},\ \mathbb{Q})^G, \tag{3.5}$$

using the G-invariant cohomology of $\widetilde{X}$. Hence

$$H^2(\widetilde{X},\ \mathbb{Q})^G = \left(\frac{H^2(B_1,\ \mathbb{Q}) \oplus H^2(B_2, \mathbb{Q})}{H^2(\mathbb{P}^1, \mathbb{Q})}\right)^G = \frac{H^2(B_1, \mathbb{Q})^G \oplus H^2(B_2, \mathbb{Q})^G}{H^2(\mathbb{P}^1, \mathbb{Q})}, \tag{3.6}$$

that due to (3.4), is the same as

$$\begin{aligned} H^2(X,\ \mathbb{Q}) &= H^2(\widetilde{X},\ \mathbb{Q})^G \\ &= \operatorname{span}_{\mathbb{Q}} \left\{\tau_1 = \pi_1^*(t_1),\ \tau_2 = \pi_2^*(t_2),\ \phi = \pi_1^*(F_1) = \pi_2^*(F_1)\right\}, \end{aligned} \tag{3.7}$$

where t_1 and t_2 (respectively, F_1 and F_2) are the t-classes (respectively, F-classes) defined in (3.4), corresponding to each surface B_1 and B_2. Using Poincaré duality, we know that $H^4(X,\ \mathbb{Q})$ is isomorphic to $H^2(X,\ \mathbb{Q})$, also $H_1(X,\ \mathbb{Z}) \simeq \pi_1(X) = \mathbb{Z}_3 \times \mathbb{Z}_3$ because the Hurewicz theorem, thus $H^1(X,\ \mathbb{Q}) = H^1(X,\ \mathbb{Z}) \otimes \mathbb{Q} = 0$.

[5] See Appendix A, for a complete description of the Mordell-Weil group of the elliptic surface.

The ring $H^{\mathrm{ev}}(\widetilde{X}, \mathbb{Q})^G$ is generated through the cup product of the generators (3.7), and is isomorphic to

$$H^{\mathrm{ev}}(\widetilde{X}, \mathbb{Q})^G = \mathbb{Q}[\tau_1, \tau_2, \phi]/\langle \phi^2, \phi\tau_1 = 3\tau_1^2, \phi\tau_2 = 3\tau_2^2 \rangle, \tag{3.8}$$

with the top cohomology element being $\tau_1^2\tau_2 = \tau_1\tau_2^2 = 3\{\mathrm{pt.}\}$.

3.1. The ample cone of the elliptic surface. As first step to determine the Kähler cone on the threefold, we build the G-invariant ample cone of the rational elliptic surface through Nakai's criterion. The set of ample classes is by definition the integral cohomology part of the Kähler moduli.

Using Looijenga's classification of the effective curves in a rational elliptic surface [17], we know that the cone of effective classes in $H_2(B, \mathbb{Z})$ is generated by the following classes $e \in H_2(B, \mathbb{Z})$:

1) The *exceptional curves* $e^2 := -1$, i.e. every section of the elliptic fibration.
2) The *nodal curves* $e^2 := -2$, i.e. the irreducible components of the singular fibers.
3) The *positive classes*, i.e. the classes that live in the "future" side of the cone of $e^2 > 0$.

Nakai's criterion for surfaces says that a class s is ample if and only if $s \cdot s > 0$ and $e \cdot s > 0$ for every effective curve e. We will apply this criterion to the invariant classes $s = aF + bt$.

• Intersection of s with the *exceptional curves*. Although there is an infinite amount of *exceptional curves* or sections in the elliptic surface, we can characterize them completely thanks to our understanding of the Mordell-Weil group.

As it is explained in Appendix A, the representation of the Mordell-Weil group $E(K) \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_3$ in $\mathrm{End}(H_2(B, \mathbb{Z}))$ has as generators: $(t_\xi)_*$, $(t_{\alpha_B \xi})_*$ and $(t_\eta)_*$. Thus, the homology of an arbitrary section can be expressed as

$$\left[\boxplus x\xi \boxplus y\alpha_B\xi \boxplus z\eta \right] = (t_\xi)_*^x (t_{\alpha_B\xi})_*^y (t_\eta)_*^z \sigma, \tag{3.9}$$

where $\boxplus x\xi$ (respectively $\boxplus y\alpha_B\xi$ and $\boxplus z\eta$) means $\boxplus x\xi = \underbrace{\xi \boxplus \xi \boxplus \ldots \boxplus \xi}_{x}$.

Finding the Jordan canonical forms associated to $(t_\xi)_*$, $(t_{\alpha_B\xi})_*$ and $(t_\eta)_*$, allows us to expand (3.9), explicitly. We exhibit the list of homology classes associated to the sections in Appendix A. Hence, the intersections of the *exceptional curves* with the generators of the invariant homology are

$$F \cdot [\boxplus x\xi \boxplus y\alpha_B\xi \boxplus z\eta] = 1 \tag{3.10}$$

and

$$t \cdot [\boxplus x\xi \boxplus y\alpha_B\xi \boxplus z\eta] = x^2 + y^2 - xy - x. \tag{3.11}$$

It is easy to check that $x^2 + y^2 - xy - x$ as a function $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ is non-negative and becomes zero for $(x = 0, y = 0)$, $(x = 1, y = 0)$ and $(x = 1, y = 1)$. Therefore a G-invariant ample class $s = aF + bt$ has to verify

$$s \cdot [\boxplus 0\xi \boxplus 0\alpha_B\xi \boxplus z\eta] = a > 0, \tag{3.12}$$

and

$$s \cdot [\boxplus\infty\xi \boxplus \infty\alpha_B\xi \boxplus z\eta] = a + \infty b > 0, \Rightarrow\ b > 0. \tag{3.13}$$

• Intersection of s with the *nodal curves*. The nodal curves are identified with the irreducible components $\Theta_{i,j}$ of the singular fibers, thus their intersections with the invariant class $s = aF + bt$ give us

$$s \cdot \Theta_{i,j} = b > 0, \tag{3.14}$$

an identical result to the inequality (3.13), derived above.
• Intersection of s with the *positive classes*. Let $\mathcal{K}^+(B)$ be the cone of positive classes in B, i.e. $\mathcal{K}^+(B) = \{e \in H_2(B,\mathbb{Z}) | \, e \cdot e > 0\}$. As $\mathcal{K}^+(B)$ is a convex set and we have to take intersections of elements in $\mathcal{K}^+(B)$ with invariant classes in $H_2(B,\mathbb{Z})^G$, only the intersection $\mathcal{K}^+(B) \cap H_2(B,\mathbb{Z})^G$ matters. From the intersection matrix of the homology generators, we know that the intersection matrix of the invariant homology $H_2(B,\mathbb{Z})^G$ is

$$\begin{pmatrix} F \\ t \end{pmatrix}^T \cdot \begin{pmatrix} F \\ t \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 3 & 1 \end{pmatrix} \tag{3.15}$$

hence, we find

$$\mathcal{K}^+(B) \cap H_2(B,\mathbb{Z})^G := \left\{ e = xF + yt | \, 6xy + y^2 > 0 \right\}, \tag{3.16}$$

being the edges of such a "future" cone F and $6t - F$. Furthermore, their intersections with our ample candidate $s = aF + bt$ give us the conditions

$$\begin{aligned} s \cdot F &= (aF + bt) \cdot F = 3b > 0, \\ s \cdot (6t - F) &= 18a + 6b - 3b = 18a + 3b > 0, \end{aligned} \tag{3.17}$$

that do not constrain the inequalities (3.12), and (3.13).

Finally, as the cone generated by F and t is within $\mathcal{K}^+(B) \cap H_2(B,\mathbb{Z})^G$, the last Nakai condition $s \cdot s > 0$ or positivity of the Liouville measure is verified. Therefore, the G-invariant ample cone associated to the elliptic surface B is simply

$$\mathcal{K}(B)^G = \mathrm{span}_{\mathbb{Z}^+} \{F, \, t\}. \tag{3.18}$$

3.2. Ampleness in the threefold. Once we have characterized the G-invariant ample cone on the rational surface, we can construct G-invariant ample classes on the threefold $\widetilde{X}$ as a product of ample classes on the surfaces B_1 and B_2. In fact, the following proposition shows that the ample classes on $\widetilde{X}$ constructed in this way determine explicitly its G-invariant ample cone $\mathcal{K}(\widetilde{X})^G = \mathcal{K}(X)$.

Proposition 3.1. *The G-invariant ample cone of $\widetilde{X}$ is*

$$\mathcal{K}(\widetilde{X})^G = \mathrm{span}_{\mathbb{Z}^+} \{\tau_1, \tau_2, \phi\}. \tag{3.19}$$

Proof. If L_i is an ample class in B_i, then $\pi_1^* L_1 \otimes \pi_2^* L_2$ is an ample class in $\widetilde{X}$, hence $\mathcal{K}(\widetilde{X})^G$ contains the positive linear span of τ_1, τ_2 and ϕ.

To show the opposite inclusion, we apply Nakai's criterion to some effective classes. Let $H = a\tau_1 + b\tau_2 + c\phi$ be an ample class. If C_1 is the class of a fiber of π_1,

$$0 < H \cdot C_1 = 0a + 3b + 0c = 3b.$$

Analogously, if C_2 is the class of a fiber of π_2, we obtain $a > 0$. Let $i : B_1 \times_{\mathbb{P}^1} B_2 \to B_1 \times B_2$. Let C be the class of $\sigma_1 \times_{\mathbb{P}^1} \sigma_2$, let c_1, c_2 be two integers with $c = c_1 + c_2$, and denote $[B_i]$ (respectively, [pt]) the class of B_i in $H^0(B_i, \mathbb{Z})$ (respectively, of a point in $H^4(B_i, \mathbb{Z})$),

$$\begin{aligned} 0 < H \cdot C &= i^* \left((at_1 + c_1 f_1) \otimes [B_2] + [B_1] \otimes (bt_2 + c_2 f_2)\right) \cdot i^*[\sigma_1 \otimes \sigma_2] \\ &= i^* \left((at_1 + c_1 f_1)\sigma_1 \, [\mathrm{pt}] \otimes \sigma_2 + \sigma_1 \otimes [\mathrm{pt}] \, (bt_2 + c_2 f_2)\sigma_2\right) \\ &= i^* \left(c_1 \, [\mathrm{pt}] \otimes \sigma_2 + c_2 \, \sigma_1 \otimes [\mathrm{pt}]\right) \\ &= c_1 + c_2 = c. \end{aligned} \tag{3.20}$$

□

4. Slope Stability of the Vector Bundles

The concept of (slope) stability of a vector bundle depends on the choice of a polarization $H \in \mathcal{K}(X) \subset H^2(X, \mathbb{Z})$, i.e., we say that a holomorphic vector bundle $\mathcal{E} \to X$ is stable iff

$$\mu(\mathcal{F}) < \mu(\mathcal{E}) \, ; \quad \text{with } \mu(\cdot) = \frac{H^2 \cdot \det(\cdot)}{\mathrm{rank}\,(\cdot)}, \tag{4.1}$$

for every reflexive subsheaf $\mathcal{F} \to \mathcal{E}$. By $\det(\mathcal{E})$ and $\det(\mathcal{F})$ we mean the determinant line bundles associated to $\mathcal{E}$ and $\mathcal{F}$.

There is a natural bijection between vector bundles on X and G-equivariant vector bundles on $\widetilde{X}$. We will recall a few general remarks on G-invariance and G-equivariance, which will be useful in the rest of this section.

Let X be a complex projective variety and G a complex algebraic group acting on it. A subvariety X' of X is said to be invariant if $gX' = X'$ for all g in G. A divisor $\mathcal{D}$ is said to be invariant if $g\mathcal{D} = \mathcal{D}$ for all g in G. A divisor class is said invariant to be if for any divisor $\mathcal{D}$ in the class and g and in G, the divisor $g\mathcal{D}$ is linearly equivalent to $\mathcal{D}$.

An equivariant structure on a vector bundle E on X is a lifting by linear maps $E(x) \longrightarrow E(gx)$ (for all $g \in G$) between fibers, of the action of G on X. We will widely use this notion, and sometimes also the notion of equivariant coherent sheaf (we will talk about some equivariant ideal sheaf) so it is convenient to generalize it defining an equivariant structure on a coherent sheaf F on X as a family of isomorphisms φ_g^F : $F \cong g^* F$, for each $g \in G$, so that $\varphi_{g'g}^F = \varphi_g^F \varphi_{g'}^F$. Equivariant morphisms

$$f : F \longrightarrow F', \tag{4.2}$$

between equivariant sheaves are those such that

$$\begin{array}{ccc} F & \xrightarrow{\varphi_g^F} & g^* F \\ f \downarrow & & g^* f \downarrow \\ F' & \xrightarrow[\varphi_g^{F'}]{} & g^* F' \end{array} \tag{4.3}$$

for all $g \in G$.

If two vector bundles have an equivariant structure, obviously their tensor products inherit an equivariant structure. If a vector bundle E has an equivariant structure, all of its exterior powers, and in particular its determinant line bundle, det (E), inherit an equivariant structure, and also its dual E^* (pointwise, take the inverse of the transposed action). The trivial bundle $L = X \times \mathbb{C}$, or $\mathcal{O}_X$ as an associated sheaf, admits a trivial equivariant structure.

A vector bundle with equivariant structure is always *invariant*, which means, by definition, that g^*E is isomorphic to E for any g in G . In the case E is a vector bundle L of rank 1, this definition means that both g^*L and L define the same point of $\mathrm{Pic}(X)$, i.e. that the point corresponding to E in $\mathrm{Pic}(X)$ is fixed by the action of the group, or still, in terms of associated divisors, that the corresponding divisor *class* is invariant.

A vector subbundle $E' \subset E$ of an equivariantly structured bundle E is called an equivariant vector subbundle when $g(E'(x)) \subset E'(x)$ for all x in X and g in G. This is equivalent to say that, for all $g \in G$, the isomorphism $E \cong g^*E$ given by the equivariant structure applies E' into g^*E', so this notion still has a meaning when E' is just a coherent subsheaf. An equivariant coherent subsheaf E' obviously inherits a structure of the equivariant coherent sheaf, as well as its quotient $E'' = E/E'$, and we just say that the extension

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0, \tag{4.4}$$

is equivariant.

An equivariant vector bundle is said to be equivariantly stable if all its equivariant coherent sheaves (enough to check with reflexive) have smaller slope. A section s of equivariantly structured E is called equivariant when, for all x in X and g in G, it is $g(s(x)) = s(gx)$. When viewing the section, as usual, as a subbundle $\mathcal{O}_X \to E$, this amounts to say that the subbundle is equivariant and the inherited equivariant structure on the trivial bundle is the trivial equivariant structure. Clearly, the vanishing locus $V(s)$ of an equivariant section is invariant. If the vector bundle E is a line bundle L of rank one, and s is a meromorphic equivariant section of it, i.e. equivariant section defined on a Zariski open set, the divisor it defines is an invariant divisor (not only a divisor of invariant class). We say L is equivariantly effective if it has a nonzero equivariant global (i.e. holomorphic) section.

In a surface X, a line bundle $L = \mathcal{O}_X(D)$ is equivariantly ample when it is equivariant and has positive selfintersecion, and its itersection number with all equivariantly effective equivariant line bundles is positive. Therefore, ample and equivariant implies equivariantly ample.

4.1. Conditions on the effective divisors. This is an analysis previous to the solution of both problems. We show now that if there exists an effective divisor in the invariant class $\mathcal{O}_B(at + bF)$ on the elliptic surface, then $a \geq -3b$. We start with the following:

Remark 2. Denote a' the defect quotient

$$a' = \left[\frac{a}{3}\right]. \tag{4.5}$$

Recall that t is the homology sum of three sections, namely ξ, $\alpha_B\xi$ and $\eta \boxplus \xi$, which we denote, respectively, s_1, s_2 and s_3. The $3a$ summands in

$$at = as_1 + as_2 + as_3 \tag{4.6}$$

can be ordered

$$at = s'_1 + \cdots + s'_{3a}, \tag{4.7}$$

so to fullfill the following three conditions:

- For all index i such that $s'_i = s_1$,

$$\sharp\{s'_j \mid j \leq i \quad \text{and} \quad s'_j = s_2\} - \{j \mid j \leq i \quad \text{and} \quad s'_j = s_1\} \leq a'. \tag{4.8}$$

- For all index i such that $s'_i = s_3$,

$$\sharp\{s'_j \mid j \leq i \quad \text{and} \quad s'_j = s_2\} - \{j \mid j \leq i \quad \text{and} \quad s'_j = s_3\} \leq a'. \tag{4.9}$$

- For all index i such that $s'_i = s_2$,

$$\sharp\{s'_j \mid j \leq i \quad \text{and} \quad s'_j = s_1 \text{ or } s_3\} - \{j \mid j \leq i \quad \text{and} \quad s'_j = s_2\} \leq a'. \tag{4.10}$$

Indeed, the following ordering of the $3a$ summands satisfies the three conditions: take its first $3a'$ summands to be

$$(s_1 + s_2 + s_3) + \cdots + (s_1 + s_2 + s_3). \tag{4.11}$$

Next, add summands of the alternating form

$$(s_1 + s_2) + (s_3 + s_2) + (s_1 + s_2) + (s_3 + s_2) + \cdots \tag{4.12}$$

(so s has already ocurred a times) and add finally summands s_1, s_3, in no matter which order, until completing a ocurrences of each.

The consequence of this remark is the following

Lemma 1. *For any direct factor $\mathcal{O}_{\mathbb{P}^1}(l)$ occurring in the splitting of $\beta_*\mathcal{O}_B(at)$ it is $l \leq a' := \left[\frac{a}{3}\right]$, i.e. $h^0(\beta_*\mathcal{O}_B(at)(-a' - 1)) = 0$.*

Proof. Recall $\beta_*\mathcal{O}_B = \mathcal{O}_{\mathbb{P}^1}$. Order the $3a$ summands in

$$at = s'_1 + \cdots + s'_{3a}, \tag{4.13}$$

as in the former remark. For some index $1 \leq i < 3a$, assume it is already proved that

$$h^0(\beta_*\mathcal{O}_B(s'_1 + \cdots + s'_{i-1})(-a' - 1)) = 0. \tag{4.14}$$

It is then enough to prove that

$$h^0(\beta_*\mathcal{O}_B(s'_1 + \cdots + s'_i)(-a' - 1)) = 0. \tag{4.15}$$

From

$$0 \longrightarrow \mathcal{O}_B(s'_1 + \cdots + s'_{i-1}) \longrightarrow \mathcal{O}_B(s'_1 + \cdots + s'_i) \longrightarrow \mathcal{O}_{s_1}(s'_1 + \cdots + s'_i) \longrightarrow 0, \tag{4.16}$$

we obtain

$$0 \longrightarrow \beta_*\mathcal{O}_B(s'_1 + \cdots + s'_{i-1}) \longrightarrow \beta_*\mathcal{O}_B(s'_1 + \cdots + s'_i) \longrightarrow \mathcal{O}_{\mathbb{P}^1}((s'_1 + \cdots + s'_i)s_1) \longrightarrow 0. \tag{4.17}$$

Assume first that $s'_i = s_1$. Recalling that $s_1^2 = -1$, $s_1 s_3 = 0$, $s_1 s_2 = 1$, we have

$$(s'_1 + \cdots + s'_i)s_1 = \sharp\{j \mid j \le i \text{ and } s'_j = s_2\} - \sharp\{j \mid j \le i \text{ and } s'_j = s_1\} \le a', \tag{4.18}$$

proving, by consulting the former exact sequence, the wanted vanishing. The vanishing is analogously proved in the case $s'_i = s_3$.

Assume now that $s'_i = s_2$. Recalling that $s_2^2 = -1$, $s_2 s_1 = s_2 s_3 = 1$, we have

$$(s'_1 + \cdots + s'_i)s_2 = \sharp\{j \mid j \le i \text{ and } s'_j = s_1 \text{ or } s_3\} - \sharp\{j \mid j \le i \quad \text{and} \quad s'_j = s_2\} \le a', \tag{4.19}$$

thus proving also the wanted vanishing. □

Corollary. *If*

$$H^0(B, \mathcal{O}_B(at + bF)) \ne 0, \tag{4.20}$$

then $a \ge -3b$.

Proof. We assume

$$\begin{aligned} 0 \ne H^0(B, \mathcal{O}_B(at + bF)) &= H^0(\mathbb{P}^1, \beta_* \mathcal{O}_B(at + bF)) \\ &= H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b) \otimes \beta_* \mathcal{O}_B(at)), \end{aligned} \tag{4.21}$$

where $\beta_* \mathcal{O}_B(at)$ is a direct sum of factors $\mathcal{O}_{\mathbb{P}^1}(l)$ with $l \le a/3$, by the lemma. Therefore, for some of these factors, we obtain

$$0 \le b + l \le b + \frac{a}{3}. \tag{4.22}$$

□

Let Θ be the inverse image of Z under the second projection.

Lemma 2. a) *If* $H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(a_1\tau_1 + a_2\tau_2 + b\phi)) \ne 0$, *then* $a_1, a_2 \ge 0$ *and* $b \ge -\frac{1}{3}(a_1 + a_2)$.

b) *If* $H^0(\widetilde{X}, \mathcal{I}_\Theta(a_1\tau_1 + b\phi)) \ne 0$, *then* $a_1 \ge 0$, $b \ge -\frac{1}{3}a_1 + 3$.

Proof. a) If a_i were negative, then the restriction of this section to any elliptic fibre E_i of π_2 would be

$$\mathcal{O}_{E_i} \longrightarrow \mathcal{O}_{E_i}(a_1(p_1 + p_2 + p_3)), \tag{4.23}$$

and this is impossible. On the other hand,

$$\begin{aligned} H^0\left(\mathcal{O}_{\widetilde{X}}(a_1\tau_1 + a_2\tau_2 + b\phi)\right) &= H^0\left(\mathcal{O}_{B_1}(a_1 t_1 + bF_1) \otimes \pi_{1*}\pi_2^* \mathcal{O}_{B_2}(a_2 t_2)\right) \\ &= H^0(\mathcal{O}_{B_1}(a_1 t_1 + bF_1) \otimes \beta_1^* \beta_{2*} \mathcal{O}_{B_2}(a_2 t_2)) \\ &= H^0(\beta_{1*}\mathcal{O}_{B_1}(a_1 t_1) \otimes \mathcal{O}_{\mathbb{P}^1}(b) \otimes \beta_{2*}\mathcal{O}_{B_2}(a_2 t_2)) \\ &= H^0(\bigoplus_i \mathcal{O}_{\mathbb{P}^1}(l_{1i}) \otimes \mathcal{O}_{\mathbb{P}^1}(b) \otimes \bigoplus_j \mathcal{O}_{\mathbb{P}^1}(l_{2j})). \end{aligned} \tag{4.24}$$

In these sums $l_{1i} \le a'_1 := \left[\frac{a_1}{3}\right]$ and $l_{2j} \le a'_2 := \left[\frac{a_2}{3}\right]$, because of the former lemma, so if this is nonzero, then for some direct factors $\mathcal{O}_{\mathbb{P}^1}(l_1)$ and $\mathcal{O}_{\mathbb{P}^1}(l_2)$ appearing in the decomposition it is

$$0 \le l_1 + b + l_2 \le \frac{a_1}{3} + b + \frac{a_2}{3} \tag{4.25}$$

b) Remark that

$$\pi_{1*}\pi_2^*\mathcal{I}_\Theta = \beta_1^*\beta_{2*}\mathcal{I}_\Theta = \beta_1^*\mathcal{O}_{\mathbb{P}^1}(-3) = \mathcal{O}_{\widetilde{X}}(-3\phi). \tag{4.26}$$

since $\beta_{2*}\mathcal{I}_\Theta = \mathcal{O}_{\mathbb{P}^1}(-p_1 - p_2 - p_3) \cong \mathcal{O}_{\mathbb{P}^1}(-3)$ (because Z lies in the fibers of β_2 at three different points $p_1, p_2, p_3 \in \mathbb{P}^1$). Therefore

$$\begin{aligned} 0 \neq H^0(\widetilde{X}, \mathcal{I}_\Theta(a_1\tau_1 + b\phi)) &= H^0(B_1, \pi_{1*}\pi_2^*\mathcal{I}_\Theta \otimes \mathcal{O}_{B_1}(a_1\tau_1 + b\phi)) \\ &= H^0(B_1, \mathcal{O}_{B_1}(a_1\tau_1 + (b-3)\phi)), \end{aligned} \tag{4.27}$$

and we conclude using the previous corollary. □

4.2. The hidden bundle. Let $\mathcal{H}$ be a rank-2 subbundle of the vector bundle $\mathcal{E}_{\mathrm{h}} \to X$, adjoint representation of the hidden E_8 gauge group, defined through the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\widetilde{X}}(2\tau_1 + \tau_2 - \phi) \longrightarrow \mathcal{H} \longrightarrow \mathcal{O}_{\widetilde{X}}(-2\tau_1 - \tau_2 + \phi) \longrightarrow 0. \tag{4.28}$$

By construction of the extension, the determinant line bundle associated to $\mathcal{H}$ is trivial, thus the slope of the rank-2 vector bundle is $\mu(\mathcal{H}) = 0$. On the other hand, $\mathcal{O}_{\widetilde{X}}(2\tau_1 + \tau_2 - \phi)$ admits a morphism to $\mathcal{H}$ as it is shown in the diagram (4.28), therefore given a polarization $H = \mathcal{O}_{\widetilde{X}}(x\tau_1 + y\tau_2 + z\phi)$ with $x,\ y,\ z \in \mathbb{Z}^+$, we have

$$\mu\left(\mathcal{O}_{\widetilde{X}}(2\tau_1+\tau_2 - \phi)\right) = H^2 \cdot \mathcal{O}_{\widetilde{X}}(2\tau_1+\tau_2-\phi) = 3(x^2+2y^2 + 6xz+12yz) > 0, \tag{4.29}$$

that is positive for all $H \in \mathcal{K}(X)$, thus $\mu\left(\mathcal{O}_{\widetilde{X}}(2\tau_1 + \tau_2 - \phi)\right) > \mu(\mathcal{H})$, which means that $\mathcal{O}_{\widetilde{X}}(2\tau_1 + \tau_2 - \phi)$ is a destabilizing line bundle for $\mathcal{H}$. As $\mathcal{H}$ is not stable, we cannot integrate the hermitian Yang-Mills equations in order to construct an $SU(2)$-instanton on $\mathcal{H}$. We must substitute $\mathcal{H}$ in order to find a sensible vacuum for the heterotic string.

4.3. The visible bundle. Here we recall the construction of the visible bundle, [2]. First it is defined as an equivariant rank 2 vector bundle V_2 on B of trivial determinant given as a nontrivial extension

$$0 \longrightarrow \mathcal{O}_B(-2F) \longrightarrow V_2 \longrightarrow \mathcal{I}_Z(2F) \longrightarrow 0, \tag{4.30}$$

with Z the scheme of 9 points, together with an equivariant structure on V_2 so that this extension is equivariant

$$0 \longrightarrow \mathcal{O}_{\widetilde{X}}(-2\phi) \longrightarrow \pi_2^* V_2 \longrightarrow \mathcal{I}_\Theta(2\phi) \longrightarrow 0. \tag{4.31}$$

Recall that Θ is the lifting to $\widetilde{X}$ of Z by the second projection. Then the visible rank 4 vector bundle V_4 of the trivial determinant is defined through the extension

$$0 \longrightarrow \mathcal{O}(-\tau_1 + \tau_2) \oplus \mathcal{O}(-\tau_1 + \tau_2) \longrightarrow V_4 \longrightarrow V_2(\tau_1 - \tau_2) \longrightarrow 0, \tag{4.32}$$

together with an equivariant structure making this extension equivariant, and general among such extensions.

We will show there exists some equivariant line bundle $\mathcal{O}_{\widetilde{X}}(x_1\tau_1 + x_2\tau_2 + y\phi)$, thus of corresponding class of divisors H being invariant, i.e. $H = x_1\tau_1 + x_2\tau_2 + y\phi$, such that the integers x, y, z are positive (thus $\mathcal{O}_{\widetilde{X}}(x_1\tau_1 + x_2\tau_2 + y\phi)$ equivariantly ample) and making the equivariant bundle V_4 equivariantly stable.

The degree of a line bundle $\mathcal{O}_{\widetilde{X}}(a_1\tau_1 + a_2\tau_2 + b\phi)$, with respect to the polarization H is

$$\begin{aligned} H^2(a_1\tau_1 + a_2\tau_2 + bf) &= 3(x_1 + x_2 + 6y)(a_1x_2 + a_2x_1) + x_1x_2(3a_1 + 3a_2 + 18b) \\ &= 3x_2(2x_1 + x_2 + 6y)a_1 + 3x_1(x_1 + 2x_2 + 6y)a_2 + 6x_1x_2b. \end{aligned} \tag{4.33}$$

Clearly, this degree function is strictly monotonous with respect to the obvious partial ordering among these line bundles or triples of integers (a_1, a_2, b). Now we will list all possible subsheaves.

1). *Possible line subbundles.*

For this we see the first necessary conditions on a_1, a_2, b for $\pi_2^*V_2$ to admit $\mathcal{O}_{\widetilde{X}}(a_1\tau_1 + a_2\tau_2 + b\phi)$ as an equivariant line subbundle

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\widetilde{X}}(-2\phi) & \longrightarrow & \pi_2^*V_2 & \longrightarrow & \mathcal{I}_\Theta(2\phi) & \longrightarrow & 0. \\ & & & & \uparrow & & & & \\ & & & & \mathcal{O}_{\widetilde{X}}(a_1\tau_1 + a_2\tau_2 + b\phi) & & & & \end{array} \tag{4.34}$$

If $a_1 \le 0$ and $a_2 \le 0$ and $b \le -2 - \frac{1}{3}(a_1 + a_2)$ is not fulfilled, then the intersection of this subbundle with the one on the left must be null, so $\mathcal{O}_{\widetilde{X}}(a_1\tau_1 + a_2\tau_2 + b\phi)$ becomes an equivariant subsheaf of the one on the right, thus giving an equivariant nonzero section of $\mathcal{O}_{\widetilde{X}}(-a_1\tau_1 - a_2\tau_2 + b\phi)$ vanishing at Θ. We thus get possibilities

$$\begin{aligned} &\text{i)} \quad a_1 \le 0 \text{ and } a_2 \le -1 \text{ and } b \le 2 - \frac{1}{3}(a_1 + a_2), \\ &\text{ii)} \quad a_1 \le 0 \text{ and } a_2 = 0 \text{ and } b \le -1 - \frac{1}{3}a_1, \\ &\text{iii)} \quad a_1 \le 0 \text{ and } a_2 \le 0 \text{ and } b \le -2 - \frac{1}{3}(a_1 + a_2). \end{aligned} \tag{4.35}$$

For ii) we have used Lemma 2 b). Let us find now necessary conditions for the existence of an equivariant rank 1 reflexive sheaf, i.e. equivariant subbundle $\mathcal{O}_{\widetilde{X}}(a_1\tau_1 + a_2\tau_2 + b\phi)$, of V_4:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\widetilde{X}}(-\tau_1 + \tau_2) \oplus \mathcal{O}_{\widetilde{X}}(-\tau_1 + \tau_2) & \longrightarrow & V_4 & \longrightarrow & \pi_2^*V_2(\tau_1 - \tau_2) & \longrightarrow & 0. \\ & & & & \uparrow & & & & \\ & & & & \mathcal{O}_{\widetilde{X}}(a_1\tau_1 + a_2\tau_2 + b\phi) & & & & \end{array} \tag{4.36}$$

By the same argument as above, combined with our former discusion on equivariant line subbundles of $\pi_2^*V_2$, we obtain these possibilities:

$$\begin{aligned} &\text{i.1)} \quad a_1 \le -1 \text{ and } a_2 \le 1 \text{ and } b \le -\frac{1}{3}(a_1 + a_2), \\ &\text{i.2)} \quad a_1 \le 1 \text{ and } a_2 \le -2 \text{ and } b \le 2 - \frac{1}{3}(a_1 + a_2), \\ &\text{i.3)} \quad a_1 \le 1 \text{ and } a_2 = -1 \text{ and } b \le -\frac{2}{3} - \frac{1}{3}a_1, \\ &\text{i.4)} \quad a_1 \le 1 \text{ and } a_2 \le -1 \text{ and } b \le -2 - \frac{1}{3}(a_1 + a_2). \end{aligned} \tag{4.37}$$

2). *Possible reflexive sheaves of rank 2.*

Let us consider now an equivariant reflexive subsheaf of rank 2

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\widetilde{X}}(-\tau_1+\tau_2)\oplus\mathcal{O}_{\widetilde{X}}(-\tau_1+\tau_2) & \longrightarrow & V_4 & \longrightarrow & \pi_2^* V_2(\tau_1-\tau_2) & \longrightarrow & 0 \\ & & & & \uparrow & & & & \\ & & & & R_2 & & & & \end{array} \tag{4.38}$$

having nonnegative degree. Since all of its equivariant line subbundles, as equivariant subbundles of V_4, must have, as seen, negative degree, the reflexive sheaf R_2 is equivariantly semistable. If its intersection with the subbundle of V_4 in its above presentation were not zero, then there would be a nonzero equivariant morphism

$$R_2 \longrightarrow \mathcal{O}_{\widetilde{X}}(-\tau_1+\tau_2)\oplus\mathcal{O}_{\widetilde{X}}(-\tau_1+\tau_2), \tag{4.39}$$

between both equivariantly semistable sheaves, so that the first should have slope not bigger than the slope of the second, i.e. R_2 should have degree not bigger than the degree of the direct sum, which is negative (as seen in the former step). We thus obtain an injection

$$0 \longrightarrow R_2 \longrightarrow V_2(\tau_1-\tau_2) \longrightarrow Q \longrightarrow 0, \tag{4.40}$$

between these equivariant reflexive sheaves of rank 2, thus its quotient Q is a torsion sheaf. We thus obtain a nonzero equivariant morphism

$$\mathcal{O}_{\widetilde{X}}(a_1\tau_1+a_2\tau_2+b\phi) = \bigwedge^2 R_2 \longrightarrow \bigwedge^2 V_2(\tau_1-\tau_2) = \mathcal{O}_{\widetilde{X}}(2\tau_1-2\tau_2). \tag{4.41}$$

Therefore, necessarily

$$\text{ii.1) } a_1 \le 2 \text{ and } a_2 \le -2 \text{ and } b \le -\frac{1}{3}(a_1+a_2). \tag{4.42}$$

The top case $a_1 = 2$ and $a_2 = -2$ and $b = 0$, would give a contradiction to what we want to prove, if it occurred, since no polarization of $\widetilde{X}$ giving negative degree to $\mathcal{O}(-\tau_1+\tau_2)\oplus\mathcal{O}(-\tau_1+\tau_2)$ would give negative degree to $\mathcal{O}_{\widetilde{X}}(2\tau_1-2\tau_2)$, but fortunately it does not occur. Indeed, if this were the case, then the quotient Q would be supported in codimension at least two, but this is incompatible with the kernel R_2 of such a quotient being reflexive, unless $Q = 0$, i.e. $R_2 \cong V_2(\tau_1-\tau_2)$, thus splitting the sequence presenting V_4. This would contradict the genericity of the extension taken in its presentation. Therefore, we get three subcases:

$$\begin{aligned} &\text{ii.1.a)} \quad a_1 \le 1 \text{ and } a_2 \le -2 \text{ and } b \le -\frac{1}{3}(a_1+a_2), \\ &\text{ii.1.b)} \quad a_1 \le 2 \text{ and } a_2 \le -3 \text{ and } b \le -\frac{1}{3}(a_1+a_2), \\ &\text{ii.1.c)} \quad a_1 \le 2 \text{ and } a_2 \le -2 \text{ and } b \le -1-\frac{1}{3}(a_1+a_2). \end{aligned} \tag{4.43}$$

3). *Possible rank 3 equivariant reflexive sheaves.* We can consider these equivariant subsheaves saturated, i.e. having as a quotient a rank 1 torsion free sheaf, with a line bundle $\mathcal{O}_{\widetilde{X}}(a_1\tau_1+a_2\tau_2+b\phi)$ as dual. In other words, giving such a subsheaf is equivalent to giving an equivariant line subbundle as in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_2^* V_2(-\tau_1+\tau_2) & \longrightarrow & V_4^\vee & \longrightarrow & \mathcal{O}_{\widetilde{X}}(\tau_1-\tau_2)\oplus\mathcal{O}_{\widetilde{X}}(\tau_1-\tau_2) & \longrightarrow & 0. \\ & & & & \uparrow & & & & \\ & & & & \mathcal{O}_{\widetilde{X}}(a_1\tau_1+a_2\tau_2+b\phi) & & & & \end{array} \tag{4.44}$$

Here we have used that $V_2^\vee \cong V_2$, since it is a rank two bundle of trivial determinant. Since $V_4^\vee$ has zero degree for any polarization, all we must show is that the equivariant line subbundle $\mathcal{O}_{\widetilde{X}}(a_1\tau_1 + a_2\tau_2 + b\phi)$ has negative degree for the polarization we are considering. If the compositions

$$\mathcal{O}_{\widetilde{X}}(a_1\tau_1 + a_2\tau_2 + b\phi) \longrightarrow \mathcal{O}_{\widetilde{X}}(\tau_1 - \tau_2), \tag{4.45}$$

with each of the two direct factors on the right-hand side were both null, then we would have a nonzero equivariant morphism

$$\mathcal{O}_{\widetilde{X}}(a_1\tau_1 + a_2\tau_2 + b\phi) \longrightarrow \pi_2^* V_2(-\tau_1 + \tau_2), \tag{4.46}$$

and these morphisms have already been analyzed in step one. Therefore, in our situation we are necessarily in one of the following cases:

$$\begin{aligned}
&\text{iii.1)} \quad a_1 \le 1 \text{ and } a_2 \le -1 \text{ and } b \le -\frac{1}{3}(a_1 + a_2),\\
&\text{iii.2)} \quad a_1 \le -1 \text{ and } a_2 \le 0 \text{ and } b \le 2 - \frac{1}{3}(a_1 + a_2),\\
&\text{iii.3)} \quad a_1 \le -1 \text{ and } a_2 = 1 \text{ and } b \le -\frac{4}{3} - \frac{1}{3}a_1,\\
&\text{iii.4)} \quad a_1 \le -1 \text{ and } a_2 \le 1 \text{ and } b \le -2 - \frac{1}{3}(a_1 + a_2).
\end{aligned} \tag{4.47}$$

In case iii.1), the top instance ($a_1 = 1$ and $a_2 = -1$ and $b = 0$) would provide an essential contradiction to what we want, if it ocurred, since no polarization giving $\mathcal{O}_{\widetilde{X}}(\tau_1 - \tau_2)$ negative degree could also give negative degree to the bundle $\mathcal{O}_{\widetilde{X}}(-\tau_1 + \tau_2) \oplus \mathcal{O}_{\widetilde{X}}(-\tau_1 + \tau_2)$ in the presentation of V_4. Fortunately, this instance does not occur. Indeed, in such a case the above morphism $\mathcal{O}_{\widetilde{X}}(a_1\tau_1 + a_2\tau_2 + b\phi) \to \mathcal{O}_{\widetilde{X}}(\tau_1 - \tau_2)$ would be isomorphic, thus splitting the bottom sequence presenting V_3 in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{\widetilde{X}}(-\tau_1 + \tau_2) \oplus \mathcal{O}_{\widetilde{X}}(-\tau_1 + \tau_2) & \longrightarrow & V_4 & \longrightarrow & V_2(\tau_1 - \tau_2) & \longrightarrow & 0,\\
 & & \text{inclusion } \uparrow \text{ of one summand} & & \uparrow & & \uparrow \text{ id.} & & \\
0 & \longrightarrow & \mathcal{O}_{\widetilde{X}}(-\tau_1 + \tau_2) & \longrightarrow & V_3 & \longrightarrow & V_2(\tau_1 - \tau_2) & \longrightarrow & 0,
\end{array} \tag{4.48}$$

in contradiction with the fact that the extension presenting V_4 has been taken to be general, with both of its components in the decomposition

$$\begin{aligned}
&\mathrm{Ext}^1(V_2(\tau_1 - \tau_2), \mathcal{O}_{\widetilde{X}}(-\tau_1 + \tau_2) \oplus \mathcal{O}_{\widetilde{X}}(-\tau_1 + \tau_2))\\
&\quad = \mathrm{Ext}^1(V_2(\tau_1 - \tau_2), \mathcal{O}_{\widetilde{X}}(-\tau_1 + \tau_2)) \oplus \mathrm{Ext}^1(V_2(\tau_1 - \tau_2), \mathcal{O}_{\widetilde{X}}(-\tau_1 + \tau_2))
\end{aligned} \tag{4.49}$$

being nonzero. Therefore, the first case splits into three subcases:

$$\begin{aligned}
&\text{iii.1.a)} \quad a_1 \le 0 \text{ and } a_2 \le -1 \text{ and } b \le -\frac{1}{3}(a_1 + a_2),\\
&\text{iii.1.b)} \quad a_1 \le 1 \text{ and } a_2 \le -2 \text{ and } b \le -\frac{1}{3}(a_1 + a_2),\\
&\text{iii.1.c)} \quad a_1 \le 1 \text{ and } a_2 \le -1 \text{ and } b \le -1 - \frac{1}{3}(a_1 + a_2).
\end{aligned} \tag{4.50}$$

Summing up, the vector bundle V_4 will then be stable if all the subsheaves that we have listed have negative degree. Recall that the degree $d(x_1, x_2, y, a_1, a_2, b)$ is monotonous in a_1, a_2 and b, so in each case it is enough to check that it is negative when these numbers take the maximum possible value. Therefore, we get the following *sufficient* conditions for a polarization to make V_4 stable:

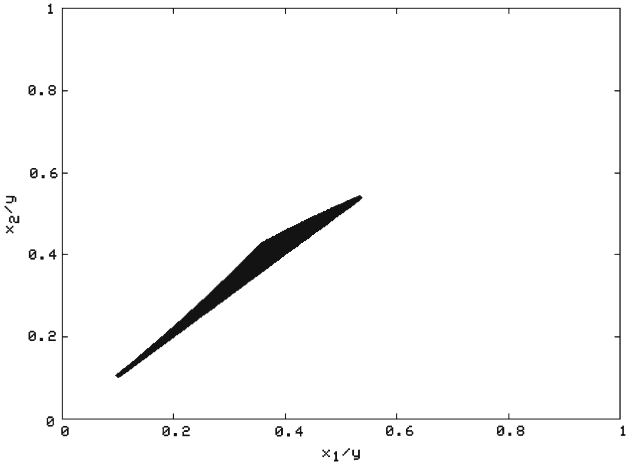

Fig. 1. Polarizations which make V_4 stable

Proposition 1. *The vector bundle V_4 is equivariantly stable for any polarization $\mathcal{O}_{\tilde{X}}(x_1, x_2, y)$ admitting equivariant structure (for instance, x_1, x_2 multiple of 3) and making the number*

$$d(x_1, x_2, y, a_1, a_2, b) := 3(x_1 + x_2 + 6y)(a_1 x_2 + a_2 x_1) + x_1 x_2 (3a_1 + 3a_2 + 18b) \quad (4.51)$$

negative for the following triples (a_1, a_2, b) of integers

$$\begin{aligned} &\text{i.1)} \quad (-1, 1, 0),\\ &\text{i.2)} \quad (1, -2, 7/3),\\ &\text{i.3)} \quad (1, -1, -1),\\ &\text{ii.1.b)} \quad (2, -3, 0),\\ &\text{ii.1.c)} \quad (2, -2, -1),\\ &\text{iii.1.a)} \quad (0, -1, 0),\\ &\text{iii.2)} \quad (-1, 0, 5/3). \end{aligned} \quad (4.52)$$

Remark. We have removed some cases which are redundant. For instance, case i.4) corresponds to the point $(1, -1, -2)$, but this case is automatic once case i.3), corresponding to $(1, -1, -1)$, has been checked, since the degree function is monotonous in a_1, a_2 and b.

Using the proposition, it is easy to find examples of ample sheaves which make V_4 stable. For instance, $\mathcal{O}_{\tilde{X}}(18\tau_1 + 21\tau_2 + 49\phi)$. In Figure 1 we have ploted the region of ample bundles which satisfy the conditions of Proposition 1, and hence make stable the vector bundle V_4.

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Appendix A. Action of the Mordell-Weil Group on the Homology

The Mordell-Weil group $E(K)$, is defined adding sections fiberwise thanks to the group structure of an elliptic curve, once the zero section is fixed. More rigorously, we define $E(K)$ in terms of the short exact sequence

$$0 \longrightarrow T \longrightarrow H_2(B,\ \mathbb{Z}) \longrightarrow E(K) \longrightarrow 0 \tag{A.1}$$

for a certain subgroup T in $H_2(B,\ \mathbb{Z})$.

For our elliptic surface, we know that the Mordell-Weil group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_3$ and is generated by the sections ξ, $\alpha_B \xi$ and η, thus we can express every section as

$$\boxplus x\xi \boxplus y\alpha_B\xi \boxplus z\eta \quad \text{for } x,\ y \in \mathbb{Z} \text{ and } z \in \mathbb{Z}_3 \tag{A.2}$$

with $\boxplus x\xi$ (respectively $\boxplus y\alpha_B\xi$ and $\boxplus z\eta$) meaning $\boxplus x\xi = \underbrace{\xi \boxplus \xi \boxplus \ldots \boxplus \xi}_{x}$.

Therefore, if $t_a : B \to B$ is the Mordell-Weil action of translating by the section a, we have to determine the push forwards $(t_\xi)_*$, $(t_{\alpha_B\xi})_*$, $(t_\eta)_*$ as maps $H_2(B) \to H_2(B)$, in order to express the homology class of an arbitrary section as

$$[\boxplus x\xi \boxplus y\alpha_B\xi \boxplus z\eta] = (t_\xi)_*^x \cdot (t_{\alpha_B\xi})_*^y \cdot (t_\eta)_*^z \sigma \tag{A.3}$$

with σ the zero section.

The push forwards $(t_\xi)_*$, $(t_\eta)_*$ and $(\alpha_B)_*$ were already determined in [5], using the quotient structure of the Mordell-Weil group on $H_2(B,\ \mathbb{Z})$ and computing intersection numbers with sections. Here, we state their result, and derive $(t_{\alpha_B\xi})_*$ as $(\alpha_B)_*(t_\xi)_*(\alpha_B)_*^{-1}$, hence we have

$$(t_\xi)_* \cdot \begin{pmatrix} \sigma \\ F \\ \Theta_{1,1} \\ \Theta_{2,1} \\ \Theta_{3,1} \\ \Theta_{1,2} \\ \Theta_{2,2} \\ \Theta_{3,2} \\ \xi \\ \alpha_B\xi \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \sigma \\ F \\ \Theta_{1,1} \\ \Theta_{2,1} \\ \Theta_{3,1} \\ \Theta_{1,2} \\ \Theta_{2,2} \\ \Theta_{3,2} \\ \xi \\ \alpha_B\xi \end{pmatrix}, \tag{A.4}$$

$$(t_{\alpha_B\xi})_* \cdot \begin{pmatrix} \sigma \\ F \\ \Theta_{1,1} \\ \Theta_{2,1} \\ \Theta_{3,1} \\ \Theta_{1,2} \\ \Theta_{2,2} \\ \Theta_{3,2} \\ \xi \\ \alpha_B\xi \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} \sigma \\ F \\ \Theta_{1,1} \\ \Theta_{2,1} \\ \Theta_{3,1} \\ \Theta_{1,2} \\ \Theta_{2,2} \\ \Theta_{3,2} \\ \xi \\ \alpha_B\xi \end{pmatrix}, \tag{A.5}$$

$$(t_\eta)_* \cdot \begin{pmatrix} \sigma \\ F \\ \Theta_{1,1} \\ \Theta_{2,1} \\ \Theta_{3,1} \\ \Theta_{1,2} \\ \Theta_{2,2} \\ \Theta_{3,2} \\ \xi \\ \alpha_B\xi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ -2/3 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -2/3 & 1/3 \\ -2/3 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1/3 & -2/3 \\ -2/3 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1/3 & 1/3 \\ -1/3 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1/3 & 2/3 \\ -1/3 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1/3 & -1/3 \\ -1/3 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 2/3 & -1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \sigma \\ F \\ \Theta_{1,1} \\ \Theta_{2,1} \\ \Theta_{3,1} \\ \Theta_{1,2} \\ \Theta_{2,2} \\ \Theta_{3,2} \\ \xi \\ \alpha_B\xi \end{pmatrix}. \tag{A.6}$$

Another way of looking at these three matrices is as generators of the representation of the Mordell-Weil group in End $\big(H_2(B, \mathbb{Z})\big)$. The commutation relations $[(t_\xi)_*, (t_{\alpha_B\xi})_*] = 0$, $[(t_\xi)_*, (t_\eta)_*] = 0$, $[(t_\eta)_*, (t_{\alpha_B\xi})_*] = 0$ are obeyed and the torsion generator $(t_\eta)_*$, verifies $(t_\eta)_*^3 = \mathbf{1}$ as expected.

Thus, expanding the equation

$$[\boxplus x\xi \boxplus y\alpha_B\xi \boxplus z\eta] = (t_\xi)_*^x \cdot (t_{\alpha_B\xi})_*^y \cdot (t_\eta)_*^z \sigma \tag{A.7}$$

for the homology classes of the sections, gives us the following list[6]: If

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} (\mathrm{mod}\, 3) \quad \text{or} \quad \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} (\mathrm{mod}\, 3) \quad \text{or} \quad \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} (\mathrm{mod}\, 3), \tag{A.8}$$

then
$\big[\boxplus x\xi \boxplus y\alpha_B\xi \boxplus z\eta\big] = (1-x-y)\sigma + (1/3x^2 + 1/3y^2 - 1/3xy - x - y)F + 1/3y\Theta_{1,1} + 2/3x\Theta_{2,1} + (1/3x + 2/3y)\Theta_{3,1} + 2/3y\Theta_{1,2} + 1/3x\Theta_{2,2} + (2/3x + 1/3y)\Theta_{3,2} + x\xi + y\alpha_B\xi$.
If

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} (\mathrm{mod}\, 3) \quad \text{or} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (\mathrm{mod}\, 3) \quad \text{or} \quad \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} (\mathrm{mod}\, 3) \tag{A.9}$$

then
$\big[\boxplus x\xi \boxplus y\alpha_B\xi \boxplus z\eta\big] = (1-x-y)\sigma + (1/3x^2 + 1/3y^2 - 1/3xy - x - y + 1)F + (1/3y - 2/3)\Theta_{1,1} + (2/3x - 2/3)\Theta_{2,1} + (1/3x + 2/3y - 2/3)\Theta_{3,1} + (2/3y - 1/3)\Theta_{1,2} + (1/3x - 1/3)\Theta_{2,2} + (2/3x + 1/3y - 1/3)\Theta_{3,2} + x\xi + y\alpha_B\xi$. If

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} (\mathrm{mod}\, 3) \quad \text{or} \quad \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} (\mathrm{mod}\, 3) \quad \text{or} \quad \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} (\mathrm{mod}\, 3) \tag{A.10}$$

then
$\big[\boxplus x\xi \boxplus y\alpha_B\xi \boxplus z\eta\big] = (1-x-y)\sigma + (1/3x^2 + 1/3y^2 - 1/3xy - x - y + 1)F + (1/3y - 1/3)\Theta_{1,1} + (2/3x - 1/3)\Theta_{2,1} + (1/3x + 2/3y - 1/3)\Theta_{3,1} + (2/3y - 2/3)\Theta_{1,2} + (1/3x - 2/3)\Theta_{2,2} + (2/3x + 1/3y - 2/3)\Theta_{3,2} + x\xi + y\alpha_B\xi$. If

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} (\mathrm{mod}\, 3) \quad \text{or} \quad \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} (\mathrm{mod}\, 3) \quad \text{or} \quad \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} (\mathrm{mod}\, 3) \tag{A.11}$$

[6] It can be proven to hold by using induction.

then
$\left[\boxplus x\xi \boxplus y\alpha_B\xi \boxplus z\eta\right] = (1-x-y)\sigma + (1/3x^2+1/3y^2-1/3xy-x-y+2/3)F + (1/3y-1/3)\Theta_{1,1} + 2/3x\Theta_{2,1} + (1/3x+2/3y-2/3)\Theta_{3,1} + (2/3y-2/3)\Theta_{1,2} + 1/3x\Theta_{2,2} + (2/3x+1/3y-1/3)\Theta_{3,2} + x\xi + y\alpha_B\xi$.If

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (\operatorname{mod} 3) \quad \text{or} \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} (\operatorname{mod} 3) \quad \text{or} \quad \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} (\operatorname{mod} 3) \tag{A.12}$$

then
$\left[\boxplus x\xi \boxplus y\alpha_B\xi \boxplus z\eta\right] = (1-x-y)\sigma + (1/3x^2+1/3y^2-1/3xy-x-y+2/3)F + 1/3y\Theta_{1,1} + (2/3x-2/3)\Theta_{2,1} + (1/3x+2/3y-1/3)\Theta_{3,1} + 2/3y\Theta_{1,2} + (1/3x-1/3)\Theta_{2,2} + (2/3x+1/3y-2/3)\Theta_{3,2} + x\xi + y\alpha_B\xi$. If

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} (\operatorname{mod} 3) \quad \text{or} \quad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} (\operatorname{mod} 3) \quad \text{or} \quad \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} (\operatorname{mod} 3) \tag{A.13}$$

then
$\left[\boxplus x\xi \boxplus y\alpha_B\xi \boxplus z\eta\right] = (1-x-y)\sigma + (1/3x^2+1/3y^2-1/3xy-x-y+2/3)F + (1/3y-2/3)\Theta_{1,1} + (2/3x-1/3)\Theta_{2,1} + (1/3x+2/3y)\Theta_{3,1} + (2/3y-1/3)\Theta_{1,2} + (1/3x-2/3)\Theta_{2,2} + (2/3x+1/3y)\Theta_{3,2} + x\xi + y\alpha_B\xi$. If

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} (\operatorname{mod} 3) \quad \text{or} \quad \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} (\operatorname{mod} 3) \quad \text{or} \quad \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} (\operatorname{mod} 3) \tag{A.14}$$

then
$\left[\boxplus x\xi \boxplus y\alpha_B\xi \boxplus z\eta\right] = (1-x-y)\sigma + (1/3x^2+1/3y^2-1/3xy-x-y+2/3)F + (1/3y-2/3)\Theta_{1,1} + 2/3x\Theta_{2,1} + (1/3x+2/3y-1/3)\Theta_{3,1} + (2/3y-1/3)\Theta_{1,2} + 1/3x\Theta_{2,2} + (2/3x+1/3y-2/3)\Theta_{3,2} + x\xi + y\alpha_B\xi$. If

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} (\operatorname{mod} 3) \quad \text{or} \quad \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} (\operatorname{mod} 3) \quad \text{or} \quad \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} (\operatorname{mod} 3) \tag{A.15}$$

then
$\left[\boxplus x\xi \boxplus y\alpha_B\xi \boxplus z\eta\right] = (1-x-y)\sigma + (1/3x^2+1/3y^2-1/3xy-x-y+2/3)F + (1/3y-1/3)\Theta_{1,1} + (2/3x-2/3)\Theta_{2,1} + (1/3x+2/3y)\Theta_{3,1} + (2/3y-2/3)\Theta_{1,2} + (1/3x-1/3)\Theta_{2,2} + (2/3x+1/3y)\Theta_{3,2} + x\xi + y\alpha_B\xi$. If

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} (\operatorname{mod} 3) \quad \text{or} \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} (\operatorname{mod} 3) \quad \text{or} \quad \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} (\operatorname{mod} 3) \tag{A.16}$$

then
$\left[\boxplus x\xi \boxplus y\alpha_B\xi \boxplus z\eta\right] = (1-x-y)\sigma + (1/3x^2+1/3y^2-1/3xy-x-y+2/3)F + 1/3y\Theta_{1,1} + (2/3x-1/3)\Theta_{2,1} + (1/3x+2/3y-2/3)\Theta_{3,1} + 2/3y\Theta_{1,2} + (1/3x-2/3)\Theta_{2,2} + (2/3x+1/3y-1/3)\Theta_{3,2} + x\xi + y\alpha_B\xi$.

References

1. Banks, T., Dine, M.: Couplings and Scales in Strongly Coupled Heterotic String Theory. Nucl. Phys. B **479**, 173–196 (1996)
2. Braun, V., He, Y-H., Ovrut, B.A., Pantev, T.: Vector Bundle Extensions, Sheaf Cohomology, and the Heterotic Standard Model. Adv. Theor. Math. Phys. **10**, 4 (2006)
3. Braun, V., He, Y-H., Ovrut, B.A., Pantev, T.: Heterotic Standard Model Moduli. JHEP **0601**, 025 (2006)
4. Braun, V., He, Y-H., Ovrut, B.A., Pantev, T.: The Exact MSSM Spectrum from String Theory. JHEP **0605**, 043 (2006)
5. Braun, V., Ovrut, B.A., Pantev, T., Reinbacher, R.: Elliptic Calabi-Yau Threefolds with $\mathbb{Z}_3 \times \mathbb{Z}_3$ Wilson Lines. JHEP **0412**, 062 (2004)
6. Curio, G., Krause, A.: Nucl.Phys. B **602**, 172–200 (2001)
7. Curio, G., Krause, A.: Nucl.Phys. B **693**, 195–222 (2004)
8. Donagi, R., Bouchard, V.: An SU(5) Heterotic Standard Model. Phys. Lett. B **633**, 483–791 (2006)
9. Donaldson, S.K., Kronheimer, P.B.: *The Geometry of Four-Manifolds*. Oxford: Oxford University Press, 1990
10. Douglas, M.R.: The Statistics of String/M-Theory Vacua. JHEP **0305**, 046 (2003)
11. Douglas, M.R., Fiol, B., Römelsberger, C.: Stability and BPS branes. JHEP **0509**, 006 (2005)
12. Grassi, A., Morrison, D.R.: Automorphisms and the Kähler cone of certain Calabi-Yau manifolds. Duke Math. J. **71**, 831–838 (1993)
13. Gukov, S., Kachru, S., Liu, X., McAllister, L.: Heterotic Moduli Stabilization with Fractional Chern-Simons Invariants. Phys. Rev. D **69**, 086008 (2004)
14. Hartshorne, R.: *Algebraic Geometry*. Graduate Texts in Mathematics, No. **52**, New York: Springer-Verlag, 1977
15. Hořava, P., Witten, E.: Eleven-dimensional supergravity on a manifold with boundary. Nucl. Phys. B **475**, 94–114 (1996)
16. Joyce, D.D.: *Compact Manifolds with Special Holonomy*. Oxford: Oxford University Press, 2000
17. Looijenga, E.: Rational surfaces with an anti-canonical cycle. Ann. of Math. (2) **114**, 267–322 (1981)
18. Namikawa, Yo.: On the birational structure of certain Calabi-Yau threefolds. J. Math. Kyoto Univ. **31**, 151–164 (1991)
19. Schoen, C.: On the fiber products of rational elliptic surfaces with sections. Math. Ann. **197**, 177–199 (1988)
20. Sharpe, E.: Kähler Cone Substructure. Adv. Theor. Math. Phys. **2**, 1441 (1998)
21. Wilson, P.M.H.: The Kähler cone on Calabi-Yau threefolds. Invent. Math. **107**, 561–583 (1992)
22. Witten, E.: Strong coupling expansion of Calabi-Yau compactification. Nucl. Phys. B **471**, 135–158 (1996)

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