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Nonexistence of Self-Similar Singularities for the 3D Incompressible Euler Equations

Dongho Chae⋆

Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Korea. E-mail: chae@skku.edu

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Abstract: We prove that there exists no self-similar finite time blowing up solution to the 3D incompressible Euler equations if the vorticity decays sufficiently fast near infinity in $\mathbb{R}^3$. By a similar method we also show nonexistence of self-similar blowing up solutions to the divergence-free transport equation in $\mathbb{R}^n$. This result has direct applications to the density dependent Euler equations, the Boussinesq system, and the quasi-geostrophic equations, for which we also show nonexistence of self-similar blowing up solutions.

1. The Incompressible Euler Equations

We are concerned here with the following Euler equations for the homogeneous incompressible fluid flows in $\mathbb{R}^3$:

$$(E)\begin{cases} \dfrac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla p, \quad (x,t) \in \mathbb{R}^3 \times (0,\infty) \\ \text{div } v = 0, \quad (x,t) \in \mathbb{R}^3 \times (0,\infty) \\ v(x,0) = v_0(x), \quad x \in \mathbb{R}^3 \end{cases},$$

where $v = (v_1, v_2, v_3), v_j = v_j(x,t), j = 1, 2, 3,$ is the velocity of the flow, $p = p(x,t)$ is the scalar pressure, and v_0 is the given initial velocity, satisfying div $v_0 = 0$. There are well-known results on the local existence of classical solutions (see e.g. [23, 18, 8] and references therein). The problem of finite time blow-up of the local classical solution is one of the most challenging open problems in mathematical fluid mechanics. On this direction there is a celebrated result on the blow-up criterion by Beale, Kato and Majda ([2]). By geometric type of consideration some of the possible scenarios to the possible singularity have been excluded (see [9, 13, 15]. One of the main purposes of this paper is to exclude the possibility of a self-similar type of singularities for the Euler system.

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The system (E) has scaling property that if (v, p) is a solution of the system (E), then for any $\lambda > 0$ and $\alpha \in \mathbb{R}$ the functions

$$v^{\lambda,\alpha}(x,t) = \lambda^{\alpha} v(\lambda x, \lambda^{\alpha+1} t), \quad p^{\lambda,\alpha}(x,t) = \lambda^{2\alpha} p(\lambda x, \lambda^{\alpha+1} t) \tag{1.1}$$

are also solutions of (E) with the initial data $v_0^{\lambda,\alpha}(x) = \lambda^{\alpha} v_0(\lambda x)$. In view of the scaling properties in (1.1), the self-similar blowing up solution $v(x,t)$ of (E) should be of the form,

$$v(x,t) = \frac{1}{(T_* - t)^{\frac{\alpha}{\alpha+1}}} V\left(\frac{x}{(T_* - t)^{\frac{1}{\alpha+1}}} \right) \tag{1.2}$$

for $\alpha \neq -1$ and t sufficiently close to T_*. Substituting (1.2) into (E), we find that V should be a solution of the system

$$(SE) \begin{cases} \dfrac{\alpha}{\alpha+1} V + \dfrac{1}{\alpha+1}(x \cdot \nabla) V + (V \cdot \nabla) V = -\nabla P, \\ \text{div } V = 0 \end{cases}$$

for some scalar function P, which could be regarded as the Euler version of the Leray equations introduced in [20]. The question of existence of a nontrivial solution to (SE) is equivalent to that of existence of a nontrivial self-similar finite time blowing up solution to the Euler system of the form (1.2). A similar question for the 3D Navier-Stokes equations was raised by J. Leray in [20], and answered negatively by the authors of [24], the result of which was refined later in [28]. Combining the energy conservation with a simple scaling argument, the author of this article showed that if there exists a nontrivial self-similar finite time blowing up solution, then its helicity should be zero ([3], see also [26] for other related discussion). To the author's knowledge, however, the possibility of self-similar blow-up of the form (1.2) has never been excluded previously. In particular, due to lack of the laplacian term in the right hand side of the first equations of (SE), we cannot apply the argument of the maximum principle, which was crucial in the works in [24] and [28] for the 3D Navier-Stokes equations. Using a completely different argument from those used in [3], or [24], we prove here that there cannot be a self-similar blowing up solution to (E) of the form (1.2), if the vorticity decays sufficiently fast near infinity. Before stating our main theorem we recall the notions of particle trajectory and the back-to-label map, which are used importantly in the recent work of [7]. Given a smooth velocity field $v(x,t)$, the particle trajectory mapping $a \mapsto X(a,t)$ is defined by the solution of the system of ordinary differential equations,

$$\frac{\partial X(a,t)}{\partial t} = v(X(a,t),t) \quad ; \quad X(a,0) = a.$$

The inverse $A(x,t) := X^{-1}(x,t)$ is called the back to label map, which satisfies $A(X(a,t),t) = a$, and $X(A(x,t),t) = x$.

Theorem 1.1. *There exists no finite time blowing up self-similar solution $v(x,t)$ to the 3D Euler equations of the form (1.2) for $t \in (0, T_*)$ with $\alpha \neq -1$, if v and V satisfy the following conditions:*

(i) For all $t \in (0, T_*)$ the particle trajectory mapping $X(\cdot, t)$ generated by the classical solution $v \in C([0, T_*); C^1(\mathbb{R}^3; \mathbb{R}^3))$ is a C^1 diffeomorphism from $\mathbb{R}^3$ onto itself.

(ii) The vorticity satisfies Ω *=curl* $V \neq 0$*, and there exists* $p_1 > 0$ *such that* $\Omega \in L^p(\mathbb{R}^3)$ *for all* $p \in (0, p_1)$*.*

Remark 1.1. The condition (i), which is equivalent to the existence of the back-to-label map $A(\cdot, t)$ for our smooth velocity $v(x, t)$ for $t \in (0, T_*)$, is guaranteed if we assume a uniform decay of $V(x)$ near infinity, independent of the decay rate ([6]).

Remark 1.2. Regarding the condition (ii), for example, if $\Omega \in L^1_{loc}(\mathbb{R}^3; \mathbb{R}^3)$ and there exist constants R, K and $\varepsilon_1, \varepsilon_2 > 0$ such that $|\Omega(x)| \leq Ke^{-\varepsilon_1|x|^{\varepsilon_2}}$ for $|x| > R$, then we have $\Omega \in L^p(\mathbb{R}^3; \mathbb{R}^3)$ for all $p \in (0, 1)$. Indeed, for all $p \in (0, 1)$, we have

$$\int_{\mathbb{R}^3} |\Omega(x)|^p dx = \int_{|x|\leq R} |\Omega(x)|^p dx + \int_{|x|>R} |\Omega(x)|^p \, dx$$
$$\leq |B_R|^{1-p} \left(\int_{|x|\leq R} |\Omega(x)| dx \right)^p + K^p \int_{\mathbb{R}^3} e^{-p\varepsilon_1|x|^{\varepsilon_2}} dx < \infty,$$

where $|B_R|$ is the volume of the ball B_R of radius R.

Remark 1.3. In the zero vorticity case $\Omega = 0$, from div $V = 0$ and curl $V = 0$, we have $V = \nabla h$, where $h(x)$ is a harmonic function in $\mathbb{R}^3$. Hence, we have an easy example of self-similar blow-up,

$$v(x, t) = \frac{1}{(T_* - t)^{\frac{\alpha}{\alpha+1}}} \nabla h \left(\frac{x}{(T_* - t)^{\frac{1}{\alpha+1}}} \right),$$

in $\mathbb{R}^3$, which is also the case for the 3D Navier-Stokes with $\alpha = 1$. We do not consider this case in the theorem.

Remark 1.4. If we assume that initial vorticity ω_0 has compact support, then the non-existence of self-similar blow-up of the form given by (1.2) is immediate from the well-known formula, $\omega(X(a, t), t) = \nabla_a X(a, t)\omega_0(a)$(see e.g. [23]).

The proof of Theorem 1.1 will follow as a corollary of the following more general theorem.

Theorem 1.2. *Let* $v \in C([0, T); C^1(\mathbb{R}^3; \mathbb{R}^3))$ *be a classical solution to the 3D Euler equations generating the particle trajectory mapping* $X(\cdot, t)$ *which is a* C^1 *diffeomorphism from* $\mathbb{R}^3$ *onto itself for all* $t \in (0, T)$*. Suppose we have representation of the vorticity of the solution, by*

$$\omega(x, t) = \Psi(t)\Omega(\Phi(t)x) \qquad \forall t \in [0, T), \tag{1.3}$$

where $\Psi(\cdot) \in C([0, T); (0, \infty))$*,* $\Phi(\cdot) \in C([0, T); \mathbb{R}^{3\times3})$ *with* $\det(\Phi(t)) \neq 0$ *on* $[0, T)$*;* $\Omega = \text{curl}\, V$ *for some* V*, and there exists* $p_1 > 0$ *such that* $\Omega \in L^p(\mathbb{R}^3)$ *for all* $p \in (0, p_1)$*. Then, necessarily either* $\det(\Phi(t)) \equiv \det(\Phi(0))$ *on* $[0, T)$*, or* $\Omega = 0$*.*

Proof. By consistency with the initial condition, $\omega_0(x) = \Psi(0)\Omega(\Phi(0)x)$, and hence $\Omega(x) = \Psi(0)^{-1}\omega_0([\Phi(0)]^{-1}x)$ for all $x \in \mathbb{R}^3$. We can rewrite the representation (1.3) in the form,

$$\omega(x, t) = G(t)\omega_0(F(t)x) \qquad \forall t \in [0, T), \tag{1.4}$$

where $G(t) = \Psi(t)/\Psi(0)$, $F(t) = [\Phi(0)]^{-1}\Phi(t)$. In order to prove the theorem it suffices to show that either $\det(F(t)) = 1$ for all $t \in [0, T)$, or $\omega_0 = 0$, since $\det(F(t))= \det(\Phi(t))/\det(\Phi(0))$.

Taking the curl of the first equation of (E), we obtain the vorticity evolution equation,

$$\frac{\partial \omega}{\partial t} + (v \cdot \nabla)\omega = (\omega \cdot \nabla)v.$$

This, taking the dot product with ω, leads to

$$\frac{\partial |\omega|}{\partial t} + (v \cdot \nabla)|\omega| = \alpha|\omega|, \tag{1.5}$$

where $\alpha(x, t)$ is defined as

$$\alpha(x,t) = \begin{cases} \sum_{i,j=1}^{3} S_{ij}(x,t)\xi_i(x,t)\xi_j(x,t) & \text{if} \quad \omega(x,t) \neq 0 \\ 0 & \text{if} \quad \omega(x,t) = 0 \end{cases}$$

with

$$S_{ij} = \frac{1}{2}\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right), \quad \text{and} \quad \xi(x,t) = \frac{\omega(x,t)}{|\omega(x,t)|}.$$

In terms of the particle trajectory mapping defined by $v(x, t)$, we can rewrite (1.5) as

$$\frac{\partial}{\partial t}|\omega(X(a,t),t)| = \alpha(X(a,t),t)|\omega(X(a,t),t)|. \tag{1.6}$$

Integrating (1.6) along the particle trajectories $\{X(a, t)\}$, we have

$$|\omega(X(a,t),t)| = |\omega_0(a)| \exp\left[\int_0^t \alpha(X(a,s),s)ds\right]. \tag{1.7}$$

Taking into account the simple estimates

$$-\|\nabla v(\cdot,t)\|_{L^\infty} \leq \alpha(x,t) \leq \|\nabla v(\cdot,t)\|_{L^\infty} \quad \forall x \in \mathbb{R}^3,$$

we obtain from (1.7) that

$$\begin{aligned} |\omega_0(a)| \exp\left[-\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty} ds\right] &\leq |\omega(X(a,t),t)| \\ &\leq |\omega_0(a)| \exp\left[\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty} ds\right], \end{aligned}$$

which, using the back to label map, can be rewritten as

$$\begin{aligned} |\omega_0(A(x,t))| \exp\left[-\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty} ds\right] &\leq |\omega(x,t)| \\ &\leq |\omega_0(A(x,t))| \exp\left[\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty} ds\right]. \end{aligned} \tag{1.8}$$

Combining this with the self-similar representation formula in (1.4), we have

$$|\omega_0(A(x,t))| \exp\left[-\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty} ds\right] \le G(t)|\omega_0(F(t)x)| \\ \le |\omega_0(A(x,t))| \exp\left[\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty} ds\right]. \qquad (1.9)$$

Given $p \in (0, p_1)$, computing the $L^p(\mathbb{R}^3)$ norm of each side of (1.9), we derive

$$\|\omega_0\|_{L^p} \exp\left[-\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty} ds\right] \le G(t)[\det(F(t))]^{-\frac{1}{p}} \|\omega_0\|_{L^p} \\ \le \|\omega_0\|_{L^p} \exp\left[\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty} ds\right], \qquad (1.10)$$

where we used the fact $\det(\nabla A(x,t)) \equiv 1$. Now, suppose $\Omega \ne 0$, which is equivalent to assuming that $\omega_0 \ne 0$, then we divide (1.10) by $\|\omega_0\|_{L^p}$ to obtain

$$\exp\left[-\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty} ds\right] \le G(t)[\det(F(t))]^{-\frac{1}{p}} \\ \le \exp\left[\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty} ds\right]. \qquad (1.11)$$

If there exists $t_1 \in (0, T)$ such that $\det(F(t_1)) \ne 1$, then either $\det(F(t_1)) > 1$ or $\det(F(t_1)) < 1$. In either case, setting $t = t_1$ and passing $p \searrow 0$ in (1.11), we deduce that

$$\int_0^{t_1} \|\nabla v(\cdot,s)\|_{L^\infty} ds = \infty.$$

This contradicts the assumption that the flow is smooth on $(0, T)$, i.e $v \in C([0, T); C^1(\mathbb{R}^3; \mathbb{R}^3))$. □

Proof of Theorem 1.1. We apply Theorem 1.2 with

$$\Phi(t) = (T_* - t)^{-\frac{1}{\alpha+1}} I, \quad \text{and} \quad \Psi(t) = (T_* - t)^{-1},$$

where I is the unit matrix in $\mathbb{R}^{3\times 3}$. If $\alpha \ne -1$ and $t \ne 0$, then

$$\det(\Phi(t)) = (T_* - t)^{-\frac{3}{\alpha+1}} \ne T_*^{-\frac{3}{\alpha+1}} = \det(\Phi(0)).$$

Hence, we conclude that $\Omega = 0$ by Theorem 1.2. In this case, there is no finite time blow-up for $v(x, t)$, since the vorticity is zero. □

2. Divergence-Free Transport Equation

The previous argument in the proof of Theorem 1.1 can also be applied to the following transport equations by a divergence-free vector field in $\mathbb{R}^n$, $n \geq 2$:

$$(TE)\begin{cases} \dfrac{\partial \theta}{\partial t} + (v \cdot \nabla)\theta = 0, \\ \operatorname{div} v = 0, \\ \theta(x, 0) = \theta_0(x), \end{cases}$$

where $v = (v_1, \cdots, v_n) = v(x, t)$, and $\theta = \theta(x, t)$. In view of the invariance of the transport equation under the scaling transform,

$$v(x,t) \mapsto v^{\lambda,\alpha}(x,t) = \lambda^\alpha v(\lambda x, \lambda^{\alpha+1} t),$$
$$\theta(x,t) \mapsto \theta^{\lambda,\alpha,\beta}(x,t) = \lambda^\beta \theta(\lambda x, \lambda^{\alpha+1} t)$$

for all $\alpha, \beta \in \mathbb{R}$ and $\lambda > 0$, the self-similar blowing up solution is of the form,

$$v(x,t) = \frac{1}{(T_* - t)^{\frac{\alpha}{\alpha+1}}} V\left(\frac{x}{(T_* - t)^{\frac{1}{\alpha+1}}} \right), \tag{2.1}$$

$$\theta(x,t) = \frac{1}{(T_* - t)^{\beta}} \Theta\left(\frac{x}{(T_* - t)^{\frac{1}{\alpha+1}}} \right) \tag{2.2}$$

for $\alpha \neq -1$ and t sufficiently close to T_*. Substituting (2.1) and (2.2) into the above transport equation, we obtain

$$(ST)\begin{cases} \beta\Theta + \dfrac{1}{\alpha+1}(x \cdot \nabla)\Theta + (V \cdot \nabla)\Theta = 0, \\ \operatorname{div} V = 0. \end{cases}$$

The question of existence of a suitable nontrivial solution to (ST) is equivalent to that of a existence of nontrivial self-similar finite time blowing up solution to the transport equation. We will establish the following theorem.

Theorem 2.1. *Let $v \in C([0, T_*); C^1(\mathbb{R}^n; \mathbb{R}^n))$ generate a C^1 diffeomorphism from $\mathbb{R}^n$ onto itself. Suppose there exist $\alpha \neq -1$, $\beta \in \mathbb{R}$ and solution (V, Θ) to the system (ST) with $\Theta \in L^{p_1}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n)$ for some p_1, p_2 such that $0 < p_1 < p_2 \leq \infty$. Then, $\Theta = 0$.*

This theorem is a corollary of the following one.

Theorem 2.2. *Suppose there exists $T > 0$ such that there exists a representation of the solution $\theta(x, t)$ to the system (TE) by*

$$\theta(x,t) = \Psi(t)\Theta(\Phi(t)x) \qquad \forall t \in [0, T), \tag{2.3}$$

where $\Psi(\cdot) \in C([0, T); (0, \infty))$, $\Phi(\cdot) \in C([0, T); \mathbb{R}^{n \times n})$ with $\det(\Phi(t)) \neq 0$ on $[0, T)$; there exists $p_1 < p_2$ with $p_1, p_2 \in (0, \infty]$ such that $\Theta \in L^{p_1}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n)$. Then, necessarily either $\det(\Phi(t)) \equiv \det(\Phi(0))$ and $\Psi(t) \equiv \Psi(0)$ on $[0, T)$, or $\Theta = 0$.

Proof. Similarly to the proof of Theorem 1.2 the representation (2.3) reduces to the form,

$$\theta(x,t) = G(t)\theta_0(F(t)x), \tag{2.4}$$

where $G(t) = \Psi(t)/\Psi(0)$, $F(t) = \Phi(t)[\Phi(0)]^{-1}$. By standard L^p-interpolation and the relation between θ_0 and Θ by $\theta_0(x) = \Psi(0)\Theta(\Phi(0)x)$, we have that $\Theta \in L^{p_1}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n)$ implies $\theta_0 \in L^p(\mathbb{R}^n)$ for all $p \in [p_1, p_2]$. As in the proof of Theorem 1.2 we denote by $\{X(a,t)\}$ and $\{A(x,t)\}$ the particle trajectory map and the back to label map respectively, each one of which is defined by $v(x,t)$. As the solution of the first equation of (TE) we have $\theta(X(a,t),t) = \theta_0(a)$, which can be rewritten as $\theta(x,t) = \theta_0(A(x,t))$ in terms of the back to label map. This, combined with (2.4), provides us with the relation

$$\theta_0(A(x,t)) = G(t)\theta_0(F(t)x). \tag{2.5}$$

Using the fact $\det(\nabla A(x,t)) = 1$, we compute the $L^p(\mathbb{R}^n)$ norm of (2.5) to have

$$\begin{aligned}\|\theta_0\|_{L^p} &= |G(t)||\det(F(t))|^{-\frac{1}{p}}\left(\int_{\mathbb{R}^n} |\theta(F(t)x)|^p |\det(F(t))|dx\right)^{\frac{1}{p}} \\ &= |G(t)||\det(F(t))|^{-\frac{1}{p}}\|\theta_0\|_{L^p} \end{aligned}\tag{2.6}$$

for all $t \in [0,T)$ and $p \in [p_1, p_2]$. Suppose $\theta_0 \neq 0$, which is equivalent to $\Theta \neq 0$, then we divide (2.6) by $\|\theta_0\|_{L^p}$ to obtain $|G(t)|^p = \det(F(t))$ for all $t \in [0,T)$ and $p \in [p_1, p_2]$, which is possible only if $G(t) = \det(F(t)) = 1$ for all $t \in [0,T)$. Hence, $\Psi(t) \equiv \Psi(0)$, and $\det(\Phi(t)) \equiv \det(\Phi(0))$. □

Proof of Theorem 2.1. We apply Theorem 2.2 with

$$\Phi(t) = (T_* - t)^{-\frac{1}{\alpha+1}} I \quad \text{and} \quad \Psi(t) = (T_* - t)^{-\beta},$$

where I is the unit matrix in $\mathbb{R}^{n\times n}$. Then,

$$\det(\Phi(t)) = (T_* - t)^{-\frac{n}{(\alpha+1)}} \neq \det(\Phi(0)) = T_*^{-\frac{n}{(\alpha+1)}} \quad \text{if} \quad \alpha \neq -1, t \neq 0.$$

Hence, by Theorem 2.2 we have $\Theta = 0$. □

Below we present some examples of fluid mechanics, where we can apply a similar argument to the above to prove nonexistence of nontrivial self-similar blowing up solutions.

A. The density-dependent Euler equations. The density-dependent Euler equations in $\mathbb{R}^n$, $n \geq 2$, are the following system:

$$(\mathrm{E}_1)\begin{cases} \dfrac{\partial \rho v}{\partial t} + \operatorname{div}(\rho v \otimes v) = -\nabla p, \\ \dfrac{\partial \rho}{\partial t} + v \cdot \nabla \rho = 0, \\ \operatorname{div} v = 0, \\ v(x,0) = v_0(x), \quad \rho(x,0) = \rho_0(x), \end{cases}$$

where $v = (v_1, \cdots, v_n) = v(x,t)$ is the velocity, $\rho = \rho(x,t) \geq 0$ is the scalar density of the fluid, and $p = p(x,t)$ is the pressure. We refer to Sect. 4.5 in [21] for a more detailed introduction of this system. Here we just note that this system reduces to the homogeneous Euler system of the previous section when $\rho \equiv 1$. The question of finite time blow-up for the system is wide open even in the case of $n = 2$, although we have local in time existence result of the classical solution and its finite time blow-up criterion (see e.g. [1, 4]). The system (E_1) has the scaling property that if (v, ρ, p) is a solution of the system (E_1), then for any $\lambda > 0$ and $\alpha \in \mathbb{R}$ the functions

$$v^{\lambda,\alpha}(x,t) = \lambda^{\alpha} v(\lambda x, \lambda^{\alpha+1} t), \quad \rho^{\lambda,\alpha,\beta}(x,t) = \lambda^{\beta} \rho(\lambda x, \lambda^{\alpha+1} t), \tag{2.7}$$

$$p^{\lambda,\alpha,\beta}(x,t) = \lambda^{2\alpha+\beta} p(\lambda x, \lambda^{\alpha+1} t) \tag{2.8}$$

are also solutions of (E_1) with the initial data

$$v_0^{\lambda,\alpha}(x) = \lambda^{\alpha} v_0(\lambda x), \quad \rho_0^{\lambda,\alpha,\beta}(x) = \lambda^{\beta} \rho_0(\lambda x).$$

In view of the scaling properties in (2.7), we should check if there exists a nontrivial solution $(v(x,t), \rho(x,t))$ of (E_1) of the form,

$$v(x,t) = \frac{1}{(T_* - t)^{\frac{\alpha}{\alpha+1}}} V\left(\frac{x}{(T_* - t)^{\frac{1}{\alpha+1}}} \right), \tag{2.9}$$

$$\rho(x,t) = \frac{1}{(T_* - t)^{\beta}} R\left(\frac{x}{(T_* - t)^{\frac{1}{\alpha+1}}} \right) \tag{2.10}$$

for $\alpha \neq -1$ and t sufficiently close to T_*. The solution (v, ρ) of the form (2.9)–(2.10) is called the self-similar blowing up solution of the system (E_1). The following theorem establishes the nonexistence of a nontrivial self-similar blowing up solution of the system (E_1), which is immediate from Theorem 2.2.

Theorem 2.3. *Let v generate a particle trajectory, which is a C^1 diffeomorphism from $\mathbb{R}^n$ onto itself for all $t \in (0, T_*)$. Suppose there exist $\alpha \neq -1$ and a solution (v, ρ) to the system (E_1) of the form (2.9)–(2.10), for which there exists p_1, p_2 with $0 < p_1 < p_2 \le \infty$ such that $R \in L^{p_1}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n)$. Then, $R = 0$.*

B. The 2D Boussinesq system. The Boussinesq system for the inviscid fluid flows in $\mathbb{R}^2$ is given by

$$(B) \begin{cases} \dfrac{\partial v}{\partial t} + (v \cdot \nabla) v = -\nabla p + \theta e_1, \\ \dfrac{\partial \theta}{\partial t} + (v \cdot \nabla)\theta = 0, \\ \quad \mathrm{div}\ v = 0, \\ v(x,0) = v_0(x), \qquad \theta(x,0) = \theta_0(x), \end{cases}$$

where $v = (v_1, v_2) = v(x,t)$ is the velocity, $e_1 = (1,0)$, and $p = p(x,t)$ is the pressure, while $\theta = \theta(x,t)$ is the temperature function. The local in time existence of the solution and the blow-up criterion of the Beale-Kato-Majda type has been well known (see e.g. [16, 5]). The question of finite time blow-up has been open until now. Here, we exclude the possibility of a self-similar finite time blow-up for the system. The system (B) has scaling property that if (v, θ, p) is a solution of the system (B), then for any $\lambda > 0$ and $\alpha \in \mathbb{R}$ the functions

$$v^{\lambda,\alpha}(x,t) = \lambda^{\alpha} v(\lambda x, \lambda^{\alpha+1} t), \quad \theta^{\lambda,\alpha}(x,t) = \lambda^{2\alpha+1} \theta(\lambda x, \lambda^{\alpha+1} t), \tag{2.11}$$

$$p^{\lambda,\alpha}(x,t) = \lambda^{2\alpha} p(\lambda x, \lambda^{\alpha+1} t) \tag{2.12}$$

are also solutions of (B) with the initial data

$$v_0^{\lambda,\alpha}(x) = \lambda^{\alpha} v_0(\lambda x), \quad \theta_0^{\lambda,\alpha}(x) = \lambda^{2\alpha+1}\theta_0(\lambda x).$$

In view of the scaling properties in (2.11), the self-similar blowing-up solution $(v(x,t), \theta(x,t))$ of (B) should of the form,

$$v(x,t) = \frac{1}{(T_* - t)^{\frac{\alpha}{\alpha+1}}} V\left(\frac{x}{(T_* - t)^{\frac{1}{\alpha+1}}}\right), \tag{2.13}$$

$$\theta(x,t) = \frac{1}{(T_* - t)^{2\alpha+1}} \Theta\left(\frac{x}{(T_* - t)^{\frac{1}{\alpha+1}}}\right), \tag{2.14}$$

where $\alpha \neq -1$. We have the following nonexistence result of such type of solution.

Theorem 2.4. *Let v generate a particle trajectory, which is a C^1 diffeomorphism from $\mathbb{R}^2$ onto itself for all $t \in (0, T_*)$. There exists no nontrivial solution (v, θ) of the system (B) of the form (2.13)–(2.14), if there exists $p_1, p_2 \in (0, \infty]$, $p_1 < p_2$, such that $\Theta \in L^{p_1}(\mathbb{R}^2) \cap L^{p_2}(\mathbb{R}^2)$, and $V \in H^m(\mathbb{R}^2)$, $m > 2$.*

Proof. Similarly to the proof of Theorem 2.1, we first conclude $\Theta = 0$, and hence $\theta(\cdot, t) \equiv 0$ on $[0, T_*)$. Then, the system (B) reduces to the 2D incompressible Euler equations, for which we have a global in time regular solution for $v_0 \in H^m(\mathbb{R}^2)$, $m > 2$ (see e.g. [19]). Hence, we should have $v(\cdot, t) \equiv 0$ on $[0, T_*)$. □

Note added to the proof. A similar proof to the one above leads to the nonexistence of a self-similar blowing up solution to the axisymmetric 3D Euler equations with swirl of the form, (1.2), if $\Theta = rV^{\theta}$ satisfies the condition of Theorem 2.4, and curl $V \in H^m(\mathbb{R}^3)$, $m > 5/2$, where $r = \sqrt{x_1^2 + x_2^2}$, and V^{θ} is the angular component of V. Indeed, applying Theorem 2.2 to the θ-component of the Euler equations, $\frac{D}{Dt}(rv^{\theta}) = 0$, we show that $v^{\theta} = 0$ as in the above proof, and then we use the global regularity result for the 3D axisymmetric Euler equations without swirl ([22, 27]) to conclude that (v^r, v^3) is also zero.

C. The 2D quasi-geostrophic equation. The following 2D quasi-geostrophic equation (QG) models the dynamics of the mixture of cold and hot air, and the fronts between them,

$$(QG)\begin{cases} \dfrac{\partial \theta}{\partial t} + (v \cdot \nabla)\theta = 0, \\ v = -\nabla^{\perp}(-\Delta)^{-\frac{1}{2}}\theta \left(= \nabla^{\perp} \displaystyle\int_{\mathbb{R}^2} \frac{\theta(y,t)}{|x-y|} dy \right), \\ \theta(x,0) = \theta_0(x), \end{cases}$$

where $\nabla^{\perp} = (-\partial_2, \partial_1)$. Besides its physical significance, mainly due to its similar structure to the 3D Euler equations, there have been many recent studies on this system (see e.g. [10–12] and references therein). Although the question of finite time singularities is still open, some type of scenarios of singularities have been excluded ([11, 12, 14]).

Here we exclude the scenario of self-similar singularity. The system (QG) has the scaling property that if θ is a solution of the system, then for any $\lambda > 0$ and $\alpha \in \mathbb{R}$ the functions

$$\theta^{\lambda,\alpha}(x,t) = \lambda^{\alpha}\theta(\lambda x, \lambda^{\alpha+1}t) \tag{2.15}$$

are also solutions of (QG) with the initial data $\theta_0^{\lambda,\alpha}(x) = \lambda^{\alpha}\theta_0(\lambda x)$. Hence, the self-similar blowing up solution should be of the form,

$$\theta(x,t) = \frac{1}{(T_* - t)^{\frac{\alpha}{\alpha+1}}}\Theta\left(\frac{x}{(T_* - t)^{\frac{1}{\alpha+1}}}\right) \tag{2.16}$$

for t sufficiently close to T_* and $\alpha \neq -1$. Applying the same argument as in the proof of Theorem 2.1, we have the following theorem.

Theorem 2.5. *Let v generate a particle trajectory, which is a C^1 diffeomorphism from $\mathbb{R}^2$ onto itself for all $t \in (0, T_*)$. There exists no nontrivial solution θ to the system (QG) of the form (2.16), if there exists $p_1, p_2 \in (0, \infty]$, $p_1 < p_2$, such that $\Theta \in L^{p_1}(\mathbb{R}^2) \cap L^{p_2}(\mathbb{R}^2)$.*

3. Remarks on the Locally Self-Similar Blow-up

The notion of self-similar solutions considered in the previous sections are apparently 'global' in the sense that the self-similar representation of the solution in (1.2) should hold for all space points in $\mathbb{R}^3$. For convenience we call the self-similar solutions considered above global self-similar solutions. On the other hand, many physicists have been trying to seek a 'locally self-similar' solution of the 3D Euler equations (see e.g. [17, 25] and the references therein). Our aim in this section is to show that the nonexistence of the global self-similar solution as proved in the previous sections implies the nonexistence of the locally self-similar solutions. Thus we exclude the most popular scenario (at least among the physicists) leading to the singularities of the 3D Euler equations. We first formulate the precise definition of the locally self-similar solution of the 3D Euler equations. A similar definition applies obviously to the locally self-similar solutions to the other equations.

Definition 1. *A solution $v(x,t)$ of the solution to (E) is called a locally self-similar blowing up solution near a space-time point $(x_*, T_*) \in \mathbb{R}^3 \times (-\infty, +\infty)$ if there exist $r > 0$, $\alpha > -1$ and a solenoidal vector field V defined on $\mathbb{R}^3$ such that the representation*

$$v(x,t) = \frac{1}{(T_* - t)^{\frac{\alpha}{\alpha+1}}}V\left(\frac{x - x_*}{(T_* - t)^{\frac{1}{\alpha+1}}}\right) \ \forall (x,t) \in B(x_*, r) \times (T_* - r^{\alpha+1}, T_*) \tag{3.1}$$

holds true, where $B(x_*, r) = \{x \in \mathbb{R}^3 \mid |x - x_*| < r\}$.

The following is our main result in this section.

Theorem 3.1. *The nonexistence of the globally self-similar solution of the 3D Euler equations implies the nonexistence of the locally self-similar solution.*

Combining Theorem 3.1 with Theorem 1.1, we have the following corollary.

Corollary 3.1. *Suppose there exists a locally self-similar blowing up solution v of the 3D Euler equations in the form (3.1), which generates a C^1 diffeomorphism on $\mathbb{R}^3$ for all time before the blow-up. If there exists $p_1 > 0$ such that $\Omega = \text{curl } V \in L^p(\mathbb{R}^3)$ for all $p \in (0, p_1)$, then necessarily $\Omega = 0$. In other words, there exists no nontrivial locally self-similar solution to the 3D Euler equation if the vorticity $\Omega \neq 0$ satisfies such an integrability condition.*

Proof of Theorem 3.1. We assume there exists a locally self-similar solution $v(x,t)$ in the sense of Definition 1. The proof of Theorem 3.1 follows if we prove the existence of the global self-similar solution. By translation in space-time variables, we can rewrite the velocity in (3.1) as

$$v(x,t) = \frac{1}{t^{\frac{\alpha}{\alpha+1}}} V\left(\frac{x}{t^{\frac{1}{\alpha+1}}}\right) \quad \text{for} \quad (x,-t) \in B(0,r) \times (-r^{\alpha+1}, 0). \tag{3.2}$$

We observe that, under the scaling transform (1.1), we have the invariance of the representation,

$$v(x,t) \mapsto v^{\lambda,\alpha}(x,t) = \lambda^\alpha v(\lambda x, \lambda^{\alpha+1} t) = \frac{1}{t^{\frac{\alpha}{\alpha+1}}} V\left(\frac{x}{t^{\frac{1}{\alpha+1}}}\right) (= v(x,t)),$$

while the region of space-time, where the self-similar form of solution is valid, transforms according to

$$B(0,r) \times (-r^{\alpha+1}, 0) \mapsto B(0, r/\lambda) \times \left(-(r/\lambda)^{\alpha+1}, 0\right).$$

We set $\lambda = 1/n$, and define the sequence of locally self-similar solutions $\{v^n(x,t)\}$ by $v^n(x,t) := v^{\frac{1}{n},\alpha}(x,t)$ with $v^1(x,t) := v(x,t)$. In the above we find that

$$v^n(x,t) = \frac{1}{t^{\frac{\alpha}{\alpha+1}}} V\left(\frac{x}{t^{\frac{1}{\alpha+1}}}\right) \quad \text{for} \quad (x,-t) \in B(0,nr) \times (-(nr)^{\alpha+1}, 0),$$

and each $v^n(x,t)$ is a solution of the Euler equations for all $(x,t) \in \mathbb{R}^3 \times (-\infty, 0)$. Let us define $v^\infty(x,t)$ by

$$v^\infty(x,t) = \frac{1}{t^{\frac{\alpha}{\alpha+1}}} V\left(\frac{x}{t^{\frac{1}{\alpha+1}}}\right) \quad \text{for} \quad (x,-t) \in \mathbb{R}^3 \times (-\infty, 0).$$

Given a compact set $K \subset \mathbb{R}^3 \times (-\infty, 0)$, we observe that $v^n \to v^\infty$ as $n \to \infty$ on K in any strongest possible topology of convergence. Indeed, for sufficiently large $N = N(K)$, $v^n(x,t) \equiv v^\infty(x,t)$ for all $(x,t) \in K$, if $n > N$. Hence, we find that $v^\infty(x,t)$ is a solution of the Euler equations for all $(x,t) \in \mathbb{R}^3 \times (-\infty, 0)$, which is a global self-similar blowing up solution, after translation in space and time. □

We note that the above proof does not depend on the specific form of the Euler equations, and hence obviously works also for the self-similar solutions of the other equations, e.g. for the Leray type of self-similar solutions of the Navier-Stokes equations. We state the result more precisely below.

Corollary 3.2. *Let v be a weak solution of the 3D Navier-Stokes equations. Suppose there exist $r > 0$, $(x_*, T_*) \in \mathbb{R}^3 \times (-\infty, \infty)$, and $V \in L^p(\mathbb{R}^3) \cap L^2_{loc}(\mathbb{R}^3)$ for some $p \in [3, \infty)$ such that*

$$v(x,t) = \frac{1}{\sqrt{T_* - t}} V\left(\frac{x - x_*}{\sqrt{T_* - t}}\right) \ \forall (x,t) \in B(x_*, r) \times (T_* - r^2, T_*)$$

holds true, then $V = 0$.

The proof is similar to the previous one, where we use the result in [24] for the nonexistence of a weak solution to the Leray system in $L^3(\mathbb{R}^3)$, while we use the corresponding result in [28] for the case of a weak solution of the Leray system in $L^p(\mathbb{R}^3)$, $p \in (3, \infty)$. A similar type of nonexistence theorems hold true also for the other equations considered in Sect. 2 with the appropriate integrability conditions.

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