

Intrinsic Definitions of “Relative Velocity” in General Relativity

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Abstract: Given two observers, we define the “relative velocity” of one observer with respect to the other in four different ways. All four definitions are given intrinsically, i.e. independently of any coordinate system. Two of them are given in the framework of spacelike simultaneity and, analogously, the other two are given in the framework of observed (lightlike) simultaneity. Properties and physical interpretations are discussed. Finally, we study relations between them in special relativity, and we give some examples in Schwarzschild and Robertson-Walker spacetimes.

1. Introduction

The need for a strict definition of “radial velocity” was treated at the General Assembly of the International Astronomical Union (IAU), held in 2000 (see [15, 10]), due to the ambiguity of the classic concepts in general relativity. As a result, they obtained three different concepts of *radial velocity*: *kinematic* (which corresponds most closely to the line-of-sight component of space velocity), *astrometric* (which can be derived from astrometric observations) and *spectroscopic* (also called *barycentric*, which can be derived from spectroscopic measurements). The kinematic and astrometric radial velocities were defined using a particular reference system, called Barycentric Celestial Reference System (BCRS). The BCRS is suitable for accurate modelling of motions and events within the solar system, but it has not taken into account the effects produced by gravitational fields outside the solar system, since it describes an asymptotically flat metric at large distances from the Sun. Moreover, from a more theoretical point of view, these concepts can not be defined in an arbitrary space-time since they are not intrinsic, i.e. they only have sense in the framework of the BCRS. So, in this work we are going to define them intrinsically. In fact, we obtain in a natural way four intrinsic definitions of relative velocity (and consequently, radial velocity) of one observer β' with respect to another observer β , following the original ideas of the IAU.

This paper has two big parts:

- The first one is formed by Sect. 3 and 4, where all the concepts are defined, trying to make the paper as self-contained as possible. In Sect. 3, we define the *kinematic* and *Fermi* relative velocities in the framework of spacelike simultaneity (also called Fermi simultaneity), obtaining some general properties and interpretations. The kinematic relative velocity generalizes the usual concept of relative velocity when the two observers β, β' are at the same event. On the other hand, the Fermi relative velocity does not generalize this concept, but it is physically interpreted as the variation of the *relative position* of β' with respect to β along the world line of β . Analogously, in Sect. 4, we define and study the *spectroscopic* and *astrometric* relative velocities in the framework of observed (lightlike) simultaneity.
- In the second one (Sect. 5 and 6) we give some relations between these concepts in special and general relativity. In Sect. 5 we find general expressions, in special relativity, for the relation between kinematic and Fermi relative velocities, and between spectroscopic and astrometric relative velocities. Finally, in Sect. 6 we show some fundamental examples in Schwarzschild and Robertson-Walker space-times.

2. Preliminaries

We work in a 4-dimensional lorentzian space-time manifold (\mathcal{M}, g) , with $c = 1$ and ∇ the Levi-Civita connection, using the Landau-Lifshitz Spacelike Convention (LLSC). We suppose that \mathcal{M} is a convex normal neighborhood [8]. Thus, given two events p and q in \mathcal{M} , there exists a unique geodesic joining p and q and there are not caustics. The parallel transport from p to q along this geodesic will be denoted by τ_{pq} . If $\beta : I \rightarrow \mathcal{M}$ is a curve with $I \subset \mathbb{R}$ a real interval, we will identify β with the image βI (that is a subset in \mathcal{M}), in order to simplify the notation. If u is a vector, then u^\perp denotes the orthogonal space of u . The projection of a vector v onto u^\perp is the projection parallel to u . Moreover, if x is a spacelike vector, then $\|x\|$ denotes the modulus of x . Given a pair of vectors u, v , we use $g(u, v)$ instead of $u^\alpha v_\alpha$. If X is a vector field, X_p will denote the unique vector of X in $T_p\mathcal{M}$.

In general, we will say that a timelike world line β is an *observer*. Nevertheless, we will say that a future-pointing timelike unit vector u in $T_p\mathcal{M}$ is an *observer at p* , identifying it with its 4-velocity.

The relative velocity of an observer with respect to another observer is completely well defined only when these observers are at the same event: given two observers u and u' at the same event p , there exists a unique vector $v \in u^\perp$ and a unique positive real number γ such that

$$u' = \gamma (u + v). \quad (1)$$

As consequences, we have $0 \leq \|v\| < 1$ and $\gamma := -g(u', u) = \frac{1}{\sqrt{1-\|v\|^2}}$. We will say that v is the *relative velocity of u' observed by u* , and γ is the *gamma factor* corresponding to the velocity $\|v\|$. From (1), we have

$$v = \frac{1}{-g(u', u)} u' - u. \quad (2)$$

We will extend this definition of relative velocity in two different ways (*kinematic* and *spectroscopic*) for observers at different events. Moreover, we will define another two

concepts of relative velocity (*Fermi* and *astrometric*) that do not extend (2) in general, but they have clear physical sense as the variation of the *relative position*.

A *light ray* is given by a lightlike geodesic λ and a future-pointing lightlike vector field F defined in λ , tangent to λ and parallelly transported along λ (i.e. $\nabla_F F = 0$), called *frequency vector field* of λ . Given $p \in \lambda$ and u an observer at p , there exists a unique vector $w \in u^\perp$ and a unique positive real number ν such that

$$F_p = \nu (u + w). \tag{3}$$

As consequences, we have $\|w\| = 1$ and $\nu = -g(F_p, u)$. We will say that w is the *relative velocity* of λ *observed by* u , and ν is the *frequency* of λ *observed by* u . In other words, ν is the modulus of the projection of F_p onto u^\perp . A *light ray from* q *to* p is a light ray λ such that $q, p \in \lambda$ and $\exp_q^{-1} p$ is future-pointing.

3. Relative Velocity in the Framework of Spacelike Simultaneity

The spacelike simultaneity was introduced by E. Fermi (see [7]), and it was used to define the *Fermi coordinates*. So, some concepts given in this section are very related to the work of Fermi, as the *Fermi surfaces*, the *Fermi derivative* or the *Fermi distance*. The original Fermi paper and most of the modern discussions of this notion (see [11, 2]) use a coordinate language (Fermi coordinates). On the other hand, in the present work we use a coordinate-free notation that allows us to get a better understanding of the basic concepts of the Fermi work, studying them from an intrinsic point of view and, in the next section, extending them to the framework of lightlike simultaneity.

Let u be an observer at $p \in \mathcal{M}$ and $\Phi : \mathcal{M} \rightarrow \mathbb{R}$ defined by $\Phi(q) := g(\exp_p^{-1} q, u)$. Then, it is a submersion and the set $L_{p,u} := \Phi^{-1}(0)$ is a regular 3-dimensional submanifold, called *Landau submanifold* of (p, u) (see [13, 3]), also known as *Fermi surface*. In other words, $L_{p,u} = \exp_p u^\perp$. An event q is in $L_{p,u}$ if and only if q is simultaneous with p in the local inertial proper system of u .

Definition 1. Given u an observer at p , and a simultaneous event $q \in L_{p,u}$, the **relative position** of q with respect to u is $s := \exp_p^{-1} q$ (see Fig. 1).

We can generalize this definition for two observers β and β' .

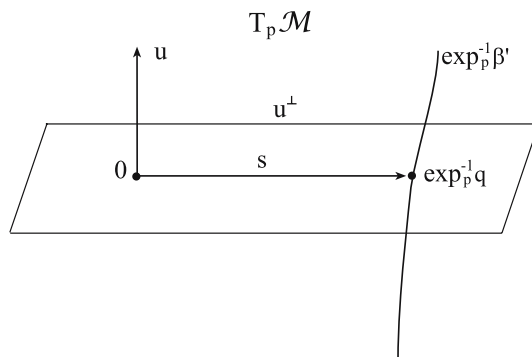


Fig. 1. Scheme in $T_p \mathcal{M}$ of the relative position s of q with respect to u

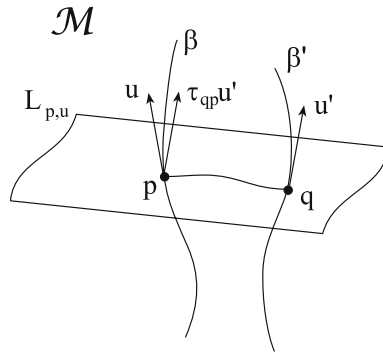


Fig. 2. Scheme in \mathcal{M} of the elements that involve the definition of the kinematic relative velocity of u' with respect to u

Definition 2. Let β, β' be two observers and let U be the 4-velocity of β . The **relative position of β' with respect to β** is the vector field S defined on β such that S_p is the relative position of q with respect to U_p , where $p \in \beta$ and q is the unique event of $\beta' \cap L_{p,U_p}$.

3.1. *Kinematic relative velocity.* We are going to introduce the concept of “kinematic relative velocity” of one observer u' with respect to another observer u generalizing the concept of relative velocity given by (2), when the two observers are at different events.

Definition 3. Let u, u' be two observers at p, q respectively such that $q \in L_{p,u}$. The **kinematic relative velocity of u' with respect to u** is the unique vector $v_{\text{kin}} \in u^\perp$ such that $\tau_{qp}u' = \gamma(u + v_{\text{kin}})$, where γ is the gamma factor corresponding to the velocity $\|v_{\text{kin}}\|$ (see Fig. 2). So, it is given by

$$v_{\text{kin}} := \frac{1}{-g(\tau_{qp}u', u)} \tau_{qp}u' - u. \tag{4}$$

Let s be the relative position of q with respect to p , the **kinematic radial velocity of u' with respect to u** is the component of v_{kin} parallel to s , i.e. $v_{\text{kin}}^{\text{rad}} := g\left(v_{\text{kin}}, \frac{s}{\|s\|}\right) \frac{s}{\|s\|}$. If $s = 0$ (i.e. $p = q$) then $v_{\text{kin}}^{\text{rad}} := v_{\text{kin}}$. On the other hand, the **kinematic tangential velocity of u' with respect to u** is the component of v_{kin} orthogonal to s , i.e. $v_{\text{kin}}^{\text{tng}} := v_{\text{kin}} - v_{\text{kin}}^{\text{rad}}$.

So, the kinematic relative velocity of u' with respect to u is the relative velocity of $\tau_{qp}u'$ observed by u , in the sense of expression (2). Note that $\|v_{\text{kin}}\| < 1$, since the parallel transported observer $\tau_{qp}u'$ defines an observer at p .

We can generalize these definitions for two observers β and β' .

Definition 4. Let β, β' be two observers, and let U, U' be the 4-velocities of β, β' respectively. The **kinematic relative velocity of β' with respect to β** is the vector field V_{kin} defined on β such that $V_{\text{kin}}|_p$ is the kinematic relative velocity of U'_q observed by U_p (in the sense of Definition 3), where $p \in \beta$ and q is the unique event of $\beta' \cap L_{p,U_p}$. In the same way, we define the **kinematic radial velocity of β' with respect to β** , denoted by $V_{\text{kin}}^{\text{rad}}$, and the **kinematic tangential velocity of β' with respect to β** , denoted by $V_{\text{kin}}^{\text{tng}}$.

We will say that β is **kinematically comoving** with β' if $V_{\text{kin}} = 0$.

Let V'_{kin} be the kinematic relative velocity of β with respect to β' . Then, $V_{\text{kin}} = 0$ if and only if $V'_{\text{kin}} = 0$, i.e. the relation “to be kinematically comoving with” is symmetric and so, we can say that β and β' are kinematically comoving (each one with respect to the other). Note that it is not transitive in general.

3.2. Fermi relative velocity. We are going to define the “Fermi relative velocity” as the variation of the relative position.

Definition 5. Let β, β' be two observers, let U be the 4-velocity of β , and let S be the relative position of β' with respect to β . The **Fermi relative velocity of β' with respect to β** is the projection of $\nabla_U S$ onto U^\perp , i.e. it is the vector field

$$V_{\text{Fermi}} := \nabla_U S + g(\nabla_U S, U)U \tag{5}$$

defined on β . The right-hand side of (5) is known as the Fermi derivative. The **Fermi radial velocity of β' with respect to β** is the component of V_{Fermi} parallel to S , i.e.

$$V_{\text{Fermi}}^{\text{rad}} := g\left(V_{\text{Fermi}}, \frac{S}{\|S\|}\right) \frac{S}{\|S\|} \text{ if } S \neq 0; \text{ if } S_p = 0 \text{ (i.e. } \beta \text{ and } \beta' \text{ intersect at } p\text{) then}$$

$$V_{\text{Fermi } p}^{\text{rad}} := V_{\text{Fermi } p}. \text{ On the other hand, the } \mathbf{Fermi \ tangential velocity of } \beta' \text{ with respect to } \beta \text{ is the component of } V_{\text{Fermi}} \text{ orthogonal to } S, \text{ i.e. } V_{\text{Fermi}}^{\text{tng}} := V_{\text{Fermi}} - V_{\text{Fermi}}^{\text{rad}}.$$

We will say that β is **Fermi-comoving** with β' if $V_{\text{Fermi}} = 0$.

It is important to remark that the modulus of the vectors of V_{Fermi} is not necessarily smaller than one.

Since $g(V_{\text{Fermi}}, S) = g(\nabla_U S, S)$, if $S \neq 0$ we have

$$V_{\text{Fermi}}^{\text{rad}} = g\left(\nabla_U S, \frac{S}{\|S\|}\right) \frac{S}{\|S\|}. \tag{6}$$

The relation “to be Fermi-comoving with” is not symmetric in general.

An expression similar to (5) is given by the next proposition, that can be proved easily.

Proposition 1. Let β, β' be two observers, let U be the 4-velocity of β , let S be the relative position of β' with respect to β , and let V_{Fermi} be the Fermi relative velocity of β' with respect to β . Then $V_{\text{Fermi}} = \nabla_U S - g(S, \nabla_U U)U$. Note that if β is geodesic, then $\nabla_U U = 0$, and hence $V_{\text{Fermi}} = \nabla_U S$.

If $S_p = 0$, i.e. β and β' intersect at p , then $V_{\text{Fermi } p} = (\nabla_U S)_p$. So, it does not coincide in general with the concept of relative velocity given in expression (2).

We are going to introduce a concept of distance from the concept of relative position given in Definition 2. This concept of distance was previously introduced by Fermi.

Definition 6. Let u be an observer at an event p . Given $q, q' \in L_{p,u}$, and s, s' the relative positions of q, q' with respect to u respectively, the **Fermi distance from q to q' with respect to u** is the modulus of $s - s'$, i.e. $d_u^{\text{Fermi}}(q, q') := \|s - s'\|$.

We have that d_u^{Fermi} is symmetric, positive-definite and satisfies the triangular inequality. So, it has all the properties that must verify a topological distance defined on $L_{p,u}$. As a particular case, if $q' = p$ we have

$$d_u^{\text{Fermi}}(q, p) = \|s\| = \left(g\left(\exp_p^{-1} q, \exp_p^{-1} q\right)\right)^{1/2}. \tag{7}$$

The next proposition shows that the concept of Fermi distance is the arclength parameter of a spacelike geodesic, and it can be proved taking into account the properties of the exponential map (see [8]).

Proposition 2. *Let u be an observer at an event p . Given $q \in L_{p,u}$ and α the unique geodesic from p to q , if we parameterize α by its arclength such that $\alpha(0) = p$, then $\alpha(d_u^{\text{Fermi}}(q, p)) = q$.*

Definition 7. *Let β, β' be two observers and let S be the relative position of β' with respect to β . The **Fermi distance from β' to β with respect to β** is the scalar field $\|S\|$ defined in β .*

We are going to characterize the Fermi radial velocity in terms of the Fermi distance.

Proposition 3. *Let β, β' be two observers, let S be the relative position of β' with respect to β , and let U be the 4-velocity of β . If $S \neq 0$, the Fermi radial velocity of β' with respect to β reads $V_{\text{Fermi}}^{\text{rad}} = U(\|S\|) \frac{S}{\|S\|}$.*

By Definition 7 and Proposition 3, the Fermi radial velocity of β' with respect to β is the rate of change of the Fermi distance from β' to β with respect to β . So, if we parameterize β by its proper time τ , the Fermi radial velocity of β' with respect to β at $p = \beta(\tau_0)$ is given by $V_{\text{Fermi } p}^{\text{rad}} = \frac{d(\|S\| \circ \beta)}{d\tau}(\tau_0) \frac{S_p}{\|S_p\|}$.

4. Relative Velocity in the Framework of Lightlike Simultaneity

The lightlike (or observed) simultaneity is based on “what an observer is really observing” and it provides an appropriate framework to study optical phenomena and observational cosmology (see [6]).

Let $p \in \mathcal{M}$ and $\varphi : \mathcal{M} \rightarrow \mathbb{R}$ defined by $\varphi(q) := g(\exp_p^{-1} q, \exp_p^{-1} q)$. Then, it is a submersion and the set

$$E_p := \varphi^{-1}(0) - \{p\} \tag{8}$$

is a regular 3-dimensional submanifold, called *horismos submanifold of p* (see [3, 1]). An event q is in E_p if and only if $q \neq p$ and there exists a lightlike geodesic joining p and q . E_p has two connected components, E_p^- and E_p^+ [14]; E_p^- (respectively E_p^+) is the *past-pointing* (respectively *future-pointing*) *horismos submanifold of p* , and it is the connected component of (8) in which, for each event $q \in E_p^-$ (respectively $q \in E_p^+$), the preimage $\exp_p^{-1} q$ is a past-pointing (respectively future-pointing) lightlike vector. In other words, $E_p^- = \exp_p C_p^-$, and $E_p^+ = \exp_p C_p^+$, where C_p^- and C_p^+ are the past-pointing and the future-pointing light cones of $T_p \mathcal{M}$ respectively.

This section is analogous to Sect. 3, but using E_p^- instead of $L_{p,u}$.

Definition 8. *Given u an observer at p , and an observed event $q \in E_p^- \cup \{p\}$, the **relative position of q observed by u** (or the observed relative position of q with respect to u) is the projection of $\exp_p^{-1} q$ onto u^\perp (see Fig. 3), i.e. $s_{\text{obs}} := \exp_p^{-1} q + g(\exp_p^{-1} q, u)u$.*

We can generalize this definition for two observers β and β' .

Definition 9. *Let β, β' be two observers and let U be the 4-velocity of β . The **relative position of β' observed by β** is the vector field S_{obs} defined in β such that $S_{\text{obs } p}$ is the relative position of q observed by U_p , where $p \in \beta$ and q is the unique event of $\beta' \cap E_p^-$.*

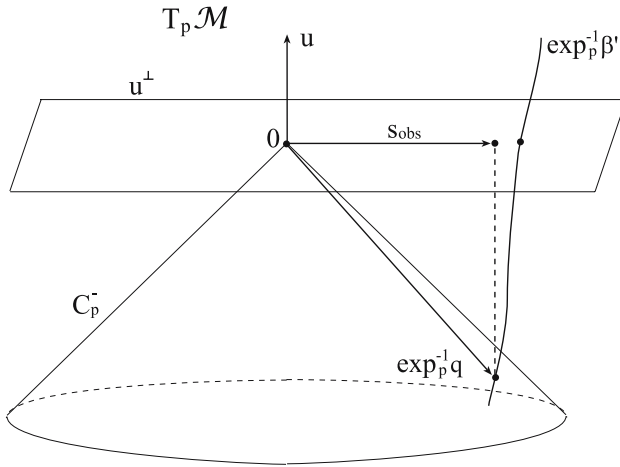


Fig. 3. Scheme in $T_p \mathcal{M}$ of the relative position s_{obs} of q observed by u

4.1. *Spectroscopic relative velocity.* In a previous work (see [4]), we defined a concept of relative velocity of an observer observed by another observer in the framework of lightlike simultaneity. We are going to rename this concept as “spectroscopic relative velocity”, and to review its properties in the context of this work.

Definition 10. Let u, u' be two observers at p, q respectively such that $q \in E_p^-$ and let λ be a light ray from q to p . The **spectroscopic relative velocity of u' observed by u** is the unique vector $v_{\text{spec}} \in u^\perp$ such that $\tau_{qp}u' = \gamma(u + v_{\text{spec}})$, where γ is the gamma factor corresponding to the velocity $\|v_{\text{spec}}\|$ (see Fig. 4). So, it is given by

$$v_{\text{spec}} := \frac{1}{-g(\tau_{qp}u', u)} \tau_{qp}u' - u. \tag{9}$$

We define the **spectroscopic radial and tangential velocity of u' observed by u** analogously to Definition 3, using s_{obs} (see Definition 8) instead of s .

So, the spectroscopic relative velocity of u' observed by u is the relative velocity of $\tau_{qp}u'$ observed by u , in the sense of expression (2), and $\|v_{\text{spec}}\| < 1$.

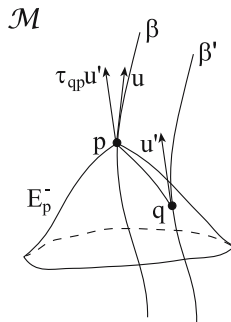


Fig. 4. Scheme in \mathcal{M} of the elements that involve the definition of the spectroscopic relative velocity of u' observed by u

Note that if w is the relative velocity of λ observed by u (see (3)), then $w = -\frac{s_{\text{obs}}}{\|s_{\text{obs}}\|}$, and so

$$v_{\text{spec}}^{\text{rad}} = g(v_{\text{spec}}, w) w. \tag{10}$$

We can generalize these definitions for two observers β and β' .

Definition 11. Let β, β' be two observers, we define V_{spec} (the **spectroscopic relative velocity of β' observed by β**) and its **radial and tangential** components analogously to Definition 4, using $E_{\bar{p}}$ instead of L_p, U_p .

We will say that β is **spectroscopically comoving** with β' if $V_{\text{spec}} = 0$.

The relation “to be spectroscopically comoving with” is not symmetric in general. The following result can be found in [4].

Proposition 4. Let λ be a light ray from q to p and let u, u' be two observers at p, q respectively. Then

$$v' = \gamma (1 - g(v_{\text{spec}}, w)) v, \tag{11}$$

where v, v' are the frequencies of λ observed by u, u' respectively, v_{spec} is the spectroscopic relative velocity of u' observed by u , w is the relative velocity of λ observed by u , and γ is the gamma factor corresponding to the velocity $\|v_{\text{spec}}\|$.

Expression (11) is the general expression for Doppler effect (that includes gravitational redshift, see [4]). Therefore, if β is spectroscopically comoving with β' , and λ is a light ray from β' to β , then, by (11), we have that β and β' observe λ with the same frequency. So, if β' emits n light rays in a unit of its proper time, then β observes also n light rays in a unit of its proper time. Hence, β observes that β' uses the “same clock” as it does.

Taking into account (10), expression (11) can be written in the form

$$v' = \frac{1 \pm \|v_{\text{spec}}^{\text{rad}}\|}{\sqrt{1 - \|v_{\text{spec}}\|^2}} v, \tag{12}$$

where we choose “+” if $g(v_{\text{spec}}, w) < 0$ (i.e. if u' is moving away from u), and we choose “-” if $g(v_{\text{spec}}, w) > 0$ (i.e. if u' is getting closer to u).

Remark 1. We can not deduce v_{spec} from the shift, v'/v , unless we make some assumptions (like considering negligible the tangential component of v_{spec} , as we will see in Remark 2). For instance, if $v'/v = 1$ then v_{spec} is not necessarily zero. Let us study this particular case: by (11) we have

$$1 = \frac{v'}{v} = \frac{1 - g(v_{\text{spec}}, w)}{\sqrt{1 - \|v_{\text{spec}}\|^2}} \longrightarrow g(v_{\text{spec}}, w) = 1 - \sqrt{1 - \|v_{\text{spec}}\|^2}.$$

Since $(1 - \sqrt{1 - \|v_{\text{spec}}\|^2}) \geq 0$, it is necessary that $g(v_{\text{spec}}, w) \geq 0$, i.e. the observed object has to be getting closer to the observer. In this case, by (12) we have $\|v_{\text{spec}}^{\text{rad}}\| = 1 - \sqrt{1 - \|v_{\text{spec}}\|^2}$. So, it is possible that $v'/v = 1$ and $v_{\text{spec}} \neq 0$ if the observed object

is getting closer to the observer. On the other hand, if the observed object is moving away from the observer then $v'/v = 1$ if and only if $v_{\text{spec}} = 0$. That is, for objects moving away, the shift is always redshift; and for objects getting closer, the shift can be blueshift, 1, or redshift.

Remark 2. If we suppose that $v_{\text{spec}}^{\text{tg}} = 0$, i.e. $v_{\text{spec}} = v_{\text{spec}}^{\text{rad}} = kw$ with $k \in]-1, 1[$, then we can deduce v_{spec} from the shift v'/v :

$$\frac{v'}{v} = \frac{1 - g(v_{\text{spec}}, w)}{\sqrt{1 - \|v_{\text{spec}}\|^2}} = \frac{1 - k}{\sqrt{1 - k^2}} = \frac{\sqrt{1 - k}}{\sqrt{1 + k}} \rightarrow k = \frac{1 - \left(\frac{v'}{v}\right)^2}{1 + \left(\frac{v'}{v}\right)^2},$$

and hence

$$v_{\text{spec}} = \left(\frac{1 - \left(\frac{v'}{v}\right)^2}{1 + \left(\frac{v'}{v}\right)^2} \right) w = - \left(\frac{1 - \left(\frac{v'}{v}\right)^2}{1 + \left(\frac{v'}{v}\right)^2} \right) \frac{S_{\text{obs}}}{\|S_{\text{obs}}\|}. \tag{13}$$

4.2. Astrometric relative velocity. We are going to define the “astrometric relative velocity” as the variation of the observed relative position.

Definition 12. Let β, β' be two observers, we define V_{ast} (the **astrometric relative velocity of β' observed by β**) and its **radial** and **tangential** components analogously to Definition 5, using S_{obs} (see Definition 9) instead of S . So,

$$V_{\text{ast}} := \nabla_U S_{\text{obs}} + g(\nabla_U S_{\text{obs}}, U) U, \tag{14}$$

where U is the 4-velocity of β .

We will say that β is **astrometrically comoving** with β' if $V_{\text{ast}} = 0$.

It is important to remark that the modulus of the vectors of V_{ast} is not necessarily smaller than one.

Analogously to (6), since $g(V_{\text{ast}}, S_{\text{obs}}) = g(\nabla_U S_{\text{obs}}, S_{\text{obs}})$, if $S_{\text{obs}} \neq 0$ we have

$$V_{\text{ast}}^{\text{rad}} = g\left(\nabla_U S_{\text{obs}}, \frac{S_{\text{obs}}}{\|S_{\text{obs}}\|}\right) \frac{S_{\text{obs}}}{\|S_{\text{obs}}\|}. \tag{15}$$

The relation “to be astrometrically comoving with” is not symmetric in general.

An expression similar to (14) is given by the next proposition, which proof is analogous to the proof of Proposition 1.

Proposition 5. Let β, β' be two observers, let U be the 4-velocity of β , let S_{obs} be the relative position of β' observed by β , and let V_{ast} be the astrometric relative velocity of β' observed by β . Then $V_{\text{ast}} = \nabla_U S_{\text{obs}} - g(S_{\text{obs}}, \nabla_U U) U$. Note that if β is geodesic, then $\nabla_U U = 0$, and hence $V_{\text{ast}} = \nabla_U S_{\text{obs}}$.

If $S_{\text{obs } p} = 0$, i.e. β and β' intersect at p , then $V_{\text{ast } p} = (\nabla_U S_{\text{obs}})_p$. So, it does not coincide in general with the concept of relative velocity given in (2).

We are going to introduce another concept of distance from the concept of observed relative position given in Definition 8. This distance was previously introduced in [9] and studied in [4], and it plays a basic role for the construction of *optical coordinates* whose relevance for cosmology was stressed in many articles by G. Ellis and his school (see [6]).

Definition 13. Let u be an observer at an event p . Given $q, q' \in E_p^- \cup \{p\}$, and $s_{\text{obs}}, s'_{\text{obs}}$ the relative positions of q, q' observed by u respectively, the **affine distance from q to q' observed by u** is the modulus of $s_{\text{obs}} - s'_{\text{obs}}$, i.e. $d_u^{\text{affine}}(q, q') := \|s_{\text{obs}} - s'_{\text{obs}}\|$.

We have that d_u^{affine} is symmetric, positive-definite and satisfies the triangular inequality. So, it has all the properties that must verify a topological distance defined on $E_p^- \cup \{p\}$. As a particular case, if $q' = p$ we have

$$d_u^{\text{affine}}(q, p) = \|s_{\text{obs}}\| = g\left(\exp_p^{-1} q, u\right). \tag{16}$$

The next proposition shows that the concept of affine distance is according to the concept of “length” (or “time”) parameter of a lightlike geodesic for an observer, and it is proved in [4].

Proposition 6. Let λ be a light ray from q to p , let u be an observer at p , and let w be the relative velocity of λ observed by u . If we parameterize λ affinely (i.e. the vector field tangent to λ is parallelly transported along λ) such that $\lambda(0) = p$ and $\dot{\lambda}(0) = -(u + w)$, then $\lambda(d_u^{\text{affine}}(q, p)) = q$.

Definition 14. Let β, β' be two observers and let S_{obs} be the relative position of β' observed by β . The **affine distance from β' to β observed by β** is the scalar field $\|S_{\text{obs}}\|$ defined in β .

We are going to characterize the astrometric radial velocity in terms of the affine distance. The proof of the next proposition is analogous to the proof of Proposition 3, taking into account expression (15).

Proposition 7. Let β, β' be two observers, let S_{obs} be the relative position of β' observed by β , and let U be the 4-velocity of β . If $S_{\text{obs}} \neq 0$, the astrometric radial velocity of β' observed by β reads $V_{\text{ast}}^{\text{rad}} = U(\|S_{\text{obs}}\|) \frac{S_{\text{obs}}}{\|S_{\text{obs}}\|}$.

By Definition 14 and Proposition 7, the astrometric radial velocity of β' observed by β is the rate of change of the affine distance from β' to β observed by β . So, if we parameterize β by its proper time τ , the astrometric radial velocity of β' observed by β at $p = \beta(\tau_0)$ is given by $V_{\text{ast } p}^{\text{rad}} = \frac{d(\|S_{\text{obs}}\| \circ \beta)}{d\tau}(\tau_0) \frac{S_{\text{obs } p}}{\|S_{\text{obs } p}\|}$.

5. Special Relativity

In this section, we are going to work in the Minkowski space-time, with β, β' two observers, and U the 4-velocity of β .

Proposition 8. Let S be the relative position of β' with respect to β , and let $V_{\text{kin}}, V_{\text{Fermi}}$ be the kinematic and Fermi relative velocities of β' with respect to β respectively. Then

$$V_{\text{Fermi}} = (1 + g(S, \nabla_U U)) V_{\text{kin}}. \tag{17}$$

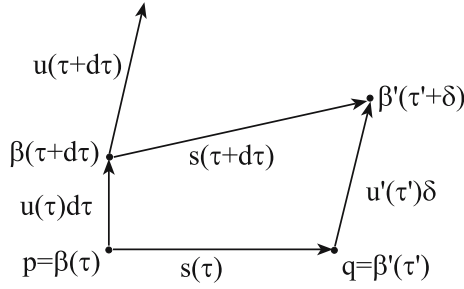


Fig. 5. Scheme of the proof of Proposition 8

Proof. We are going to consider the observers parameterized by their proper times. Let $p = \beta(\tau)$ be an event of β , let $u(\tau)$ be the 4-velocity of β at p , and let $q = \beta'(\tau')$ be the event of β' such that $g(u(\tau), q - p) = 0$ (note that the Minkowski space-time has an affine structure, and $q - p$ denotes the vector which joins p and q). So, $q - p$ is the relative position of q with respect to $u(\tau)$, denoted by $s(\tau)$. Considering the differential diagram given in Fig. 5, where $u'(\tau')$ is the 4-velocity of β' at q , it is easy to check that

$$\delta = \frac{1 + g(s(\tau), \dot{u}(\tau))}{-g(u'(\tau'), u(\tau + d\tau))} d\tau \quad (18)$$

for an infinitesimally small $d\tau$ (i.e. it holds in quadratic approximation in $d\tau$). Since $s(\tau + d\tau) = s(\tau) + u'(\tau')\delta - u(\tau)d\tau$, from (18) we have

$$\dot{s}(\tau) = \lim_{d\tau \rightarrow 0} \frac{s(\tau + d\tau) - s(\tau)}{d\tau} = \frac{1 + g(s(\tau), \dot{u}(\tau))}{-g(u'(\tau'), u(\tau))} u'(\tau') - u(\tau). \quad (19)$$

Let U, U' be the 4-velocities of β and β' respectively, and let S be the relative position of β' with respect to β . Then, from (19) we have

$$(\nabla_U S)_p = \frac{1 + g(S_p, (\nabla_U U)_p)}{-g(U_p, U'_q)} U'_q - U_p. \quad (20)$$

So, by Proposition 1 and expression (20), the Fermi relative velocity $V_{\text{Fermi } p}$ of β' with respect to β at p is given by

$$\begin{aligned} V_{\text{Fermi } p} &= (\nabla_U S)_p - g(S_p, (\nabla_U U)_p) U_p \\ &= (1 + g(S_p, (\nabla_U U)_p)) \left(\frac{1}{-g(U_p, U'_q)} U'_q - U_p \right). \end{aligned} \quad (21)$$

On the other hand, the kinematic relative velocity $V_{\text{kin } p}$ of β' with respect to β at p reads

$$V_{\text{kin } p} = \frac{1}{-g(U'_q, U_p)} U'_q - U_p. \quad (22)$$

Hence, from (21) and (22) we obtain (17), concluding the proof.

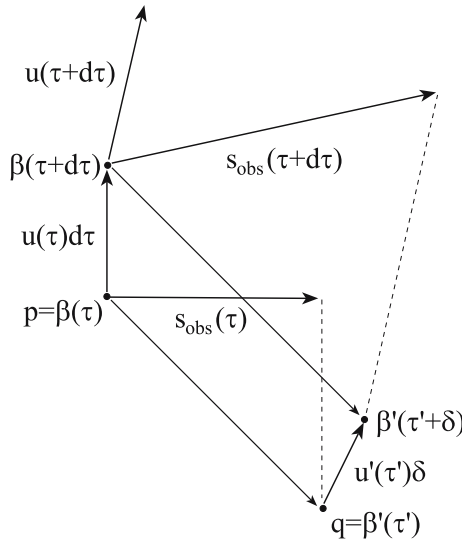


Fig. 6. Scheme of the proof of Proposition 9

So, V_{kin} and V_{Fermi} are proportional. Moreover, if β is geodesic, then $V_{\text{Fermi}} = V_{\text{kin}}$.

The proof of the next proposition is similar to the proof of Proposition 8 (but a bit more complicated), considering the differential diagram given in Fig. 6.

Proposition 9. *Let S_{obs} be the relative position of β' observed by β , and let V_{spec} , V_{ast} be the spectroscopic and astrometric relative velocities of β' observed by β respectively. If $S_{\text{obs}} \neq 0$ then*

$$V_{\text{ast}} = \|S_{\text{obs}}\| \nabla_U U + \frac{1}{1 + g\left(V_{\text{spec}}, \frac{S_{\text{obs}}}{\|S_{\text{obs}}\|}\right)} V_{\text{spec}}. \tag{23}$$

So, V_{spec} and V_{ast} are not proportional unless β is geodesic.

If β' is geodesic then it is clear that $V_{\text{spec}} = V_{\text{kin}}$. Moreover, if β is also geodesic then $V_{\text{spec}} = V_{\text{kin}} = V_{\text{Fermi}}$.

Remark 3. Let us suppose that β and β' intersect at p , let u, u' be the 4-velocities of β, β' at p respectively, and let v be the relative velocity of u' observed by u , in the sense of expression (2). Let us study the relations between $v, V_{\text{kin } p}, V_{\text{Fermi } p}, V_{\text{spec } p}$ and $V_{\text{ast } p}$.

It is clear that $V_{\text{kin } p} = V_{\text{spec } p} = v$, even in general relativity. Moreover, since $S_p = 0$, by (17) we have $V_{\text{Fermi } p} = v$. On the other hand, since $S_{\text{obs } p} = 0$, it is easy to prove that $V_{\text{ast } p} = \frac{1}{1 \pm \|v\|} v$, where we choose “+” if we consider that β' is leaving from β , and we choose “-” if we consider that β' is arriving at β . Therefore, if β and β' intersect at p , then it is not possible to write $V_{\text{ast } p}$ in a unique way in terms of v .

Example 1. Using rectangular coordinates (t, x, y, z) , let us consider the following observers parameterized by their proper times: $\beta(\tau) := (\tau, 0, 0, 0)$, and $\beta'(\tau') :=$

$$\begin{cases} (\gamma\tau', v\gamma\tau', 0, 0) & \text{if } \tau' \in \left[0, \frac{1}{\gamma v}\right] \\ (\gamma\tau', 2 - v\gamma\tau', 0, 0) & \text{if } \tau' \in \left]\frac{1}{\gamma v}, \frac{2}{\gamma v}\right] \end{cases}, \text{ where } v \in]0, 1[\text{ and } \gamma := \frac{1}{\sqrt{1-v^2}}. \text{ That is,}$$

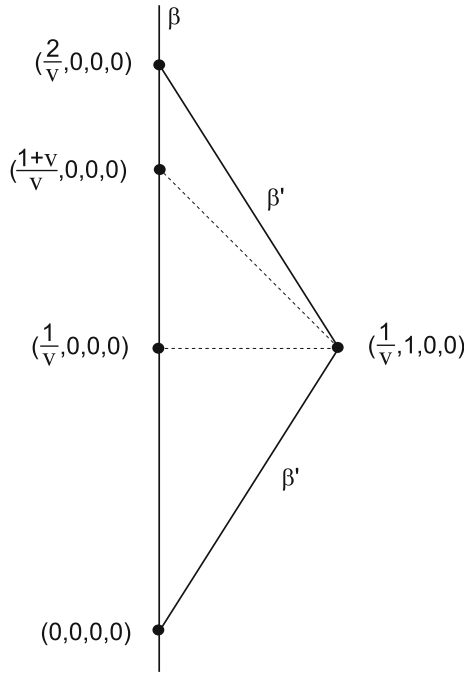


Fig. 7. Scheme of the observers of Example 1

β is a stationary observer with $x = 0, y = 0, z = 0$ and β' is an observer moving from $x = 0, y = 0, z = 0$ to $x = 1, y = 0, z = 0$ with velocity of modulus v and returning (see Fig. 7). It is satisfied that

$$V_{\text{kin}} \beta(\tau) = \begin{cases} v \frac{\partial}{\partial x} \Big|_{\beta(\tau)} & \text{if } \tau \in [0, \frac{1}{v}] \\ -v \frac{\partial}{\partial x} \Big|_{\beta(\tau)} & \text{if } \tau \in]\frac{1}{v}, \frac{2}{v}] \end{cases},$$

$$V_{\text{spec}} \beta(\tau) = \begin{cases} v \frac{\partial}{\partial x} \Big|_{\beta(\tau)} & \text{if } \tau \in [0, \frac{1+v}{v}] \\ -v \frac{\partial}{\partial x} \Big|_{\beta(\tau)} & \text{if } \tau \in]\frac{1+v}{v}, \frac{2}{v}] \end{cases}.$$

Applying (17), we obtain $V_{\text{Fermi}} \beta(\tau) = V_{\text{kin}} \beta(\tau)$. Moreover

$$S_{\text{obs}} \beta(\tau) = \begin{cases} \frac{v\tau}{1+v} \frac{\partial}{\partial x} \Big|_{\beta(\tau)} & \text{if } \tau \in [0, \frac{1+v}{v}] \\ \frac{2-v\tau}{1-v} \frac{\partial}{\partial x} \Big|_{\beta(\tau)} & \text{if } \tau \in]\frac{1+v}{v}, \frac{2}{v}] \end{cases}.$$

Hence, by (23) we have

$$V_{\text{ast}} \beta(\tau) = \begin{cases} \frac{v}{1+v} \frac{\partial}{\partial x} \Big|_{\beta(\tau)} & \text{if } \tau \in [0, \frac{1+v}{v}] \\ -\frac{v}{1-v} \frac{\partial}{\partial x} \Big|_{\beta(\tau)} & \text{if } \tau \in]\frac{1+v}{v}, \frac{2}{v}] \end{cases}.$$

Consequently, $\|V_{\text{ast}} \beta(\tau)\| \in]0, 1/2[$ if $\tau \in [0, \frac{1+v}{v}]$, i.e. if β' is moving away radially. On the other hand, $\|V_{\text{ast}} \beta(\tau)\| \in]0, +\infty[$ if $\tau \in]\frac{1+v}{v}, \frac{2}{v}]$, i.e. if β' is getting closer radially. This corresponds to what β observes.

6. Examples in General Relativity

6.1. Stationary observers in Schwarzschild. In the Schwarzschild metric with spherical coordinates

$$ds^2 = -a^2(r) dt^2 + \frac{1}{a^2(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2),$$

where $a(r) = \sqrt{1 - \frac{2m}{r}}$ and $r > 2m$, let us consider two equatorial stationary observers, $\beta_1(\tau) = \left(\frac{1}{a_1}\tau, r_1, \pi/2, 0\right)$ and $\beta_2(\tau) = \left(\frac{1}{a_2}\tau, r_2, \pi/2, 0\right)$ with $\tau \in \mathbb{R}$, $r_2 > r_1 > 2m$, $a_1 := a(r_1)$ and $a_2 := a(r_2)$, and let U be the 4-velocity of β_2 , i.e. $U := \frac{1}{a_2} \frac{\partial}{\partial t}$. We are going to study the relative velocities of β_1 with respect to and observed by β_2 .

6.1.1. Kinematic and Fermi relative velocities Let us consider the vector field $X := a(r) \frac{\partial}{\partial r}$. This vector field is spacelike, unit, geodesic, and orthogonal to U . Since $\nabla_X \left(\frac{1}{a(r)} \frac{\partial}{\partial t}\right) = 0$, we have that the kinematic relative velocity V_{kin} of β_1 with respect to β_2 is given by $V_{\text{kin}} = 0$.

Let $\alpha(\sigma) = (t_0, \alpha^r(\sigma), \pi/2, 0)$ be an integral curve of X such that $q := \alpha(\sigma_1) \in \beta_1$ and $p := \alpha(\sigma_2) \in \beta_2$, with $\sigma_2 > \sigma_1$ (i.e. $\alpha(\sigma)$ is a spacelike geodesic from q to p , parameterized by its arclength, and its tangent vector at p is X_p). Then, by Proposition 2, the Fermi distance $d_{U_p}^{\text{Fermi}}(q, p)$ from q to p with respect to U_p is $\sigma_2 - \sigma_1$. Since α is an integral curve of X , we have $\dot{\alpha}^r(\sigma) = \sqrt{1 - \frac{2m}{\alpha^r(\sigma)}}$. So, $\int_{r_1}^{r_2} \left(1 - \frac{2m}{\alpha^r(\sigma)}\right)^{-1/2} \dot{\alpha}^r(\sigma) d\sigma = \sigma_2 - \sigma_1$, and then

$$d_{U_p}^{\text{Fermi}}(q, p) = 2m \ln \left(\frac{(1 - a_1) \sqrt{r_1}}{(1 - a_2) \sqrt{r_2}} \right) + r_2 a_2 - r_1 a_1. \quad (24)$$

Since (24) does not depend on t_0 , the Fermi distance from β_1 to β_2 with respect to β_2 is also given by expression (24). Hence, by (7), the relative position S of β_1 with respect to β_2 is given by

$$S = \left(2m \ln \left(\frac{(1 - a_2) \sqrt{r_2}}{(1 - a_1) \sqrt{r_1}} \right) + r_1 a_1 - r_2 a_2 \right) a_2 \frac{\partial}{\partial r}.$$

It is easy to prove that $\nabla_U S$ is proportional to U . Therefore, the Fermi relative velocity V_{Fermi} of β_1 with respect to β_2 reads $V_{\text{Fermi}} = 0$.

6.1.2. Spectroscopic and astrometric relative velocities It is easy to prove that the spectroscopic relative velocity V_{spec} of β_1 observed by β_2 is radial. Since the gravitational redshift is given by $\frac{a_2}{a_1}$ (see [4]), by (13) we obtain

$$V_{\text{spec}} = -a_2 \frac{a_2^2 - a_1^2}{a_2^2 + a_1^2} \frac{\partial}{\partial r}. \quad (25)$$

Expression (25) is also obtained in [4]. We have $\lim_{r_1 \rightarrow 2m} \|V_{\text{spec}}\| = 1$.

On the other hand, in [4] it is also proved (by using Proposition 6) that the affine distance from β_1 to β_2 observed by β_2 is $\frac{r_2-r_1}{a_2}$. Hence, by (16), the relative position S_{obs} of β_1 observed by β_2 is given by

$$S_{\text{obs}} = (r_1 - r_2) \frac{\partial}{\partial r}. \tag{26}$$

It is easy to prove that $\nabla_U S_{\text{obs}}$ is proportional to U . Therefore, the astrometric relative velocity V_{ast} of β_1 observed by β_2 reads $V_{\text{ast}} = 0$.

6.2. Free-falling observers in Schwarzschild. Let us consider a radial free-falling observer β_1 parameterized by the coordinate time t , $\beta_1(t) = (t, \beta_1^r(t), \pi/2, 0)$. Given an event $q = (t_1, r_1, \pi/2, 0) \in \beta_1$, the 4-velocity of β_1 at q is given by

$$u_1 = \frac{E}{a_1^2} \frac{\partial}{\partial t} \Big|_q - \sqrt{E^2 - a_1^2} \frac{\partial}{\partial r} \Big|_q, \tag{27}$$

where E is a constant of motion given by $E := \left(\frac{1-2m/r_0}{1-v_0^2}\right)^{1/2}$, r_0 is the radial coordinate at which the fall begins, v_0 is the initial velocity (see [5]), and $a_1 := a(r_1)$. Moreover, let us consider an equatorial stationary observer $\beta_2(\tau) = \left(\frac{1}{a_2}\tau, r_2, \pi/2, 0\right)$ with $\tau \in \mathbb{R}$, $r_2 > r_1 > 2m$, $a_2 := a(r_2)$, and $U := \frac{1}{a_2} \frac{\partial}{\partial t}$ its 4-velocity. We are going to study the relative velocities of β_1 with respect to and observed by β_2 at p , where p will be a determined event of β_2 .

6.2.1. Kinematic and Fermi relative velocities Let $p = (t_1, r_2, \pi/2, 0)$. This is the unique event of β_2 such that $q \in L_{p,U_p}$, i.e. there exists a spacelike geodesic $\alpha(\sigma)$ from $q = \alpha(\sigma_1)$ to $p = \alpha(\sigma_2)$ such that the tangent vector $\dot{\alpha}(\sigma_2)$ is orthogonal to U_p . We can consider $\alpha(\sigma)$ parameterized by its arclength and $\sigma_2 > \sigma_1$. So, $\alpha(\sigma)$ is an integral curve of the vector field $X = a(r) \frac{\partial}{\partial r}$. If we parallelly transport u_1 from q to p along α we obtain $\tau_{qp}u_1 = \frac{E}{a_1 a_2} \frac{\partial}{\partial t} \Big|_p - \frac{a_2}{a_1} \sqrt{E^2 - a_1^2} \frac{\partial}{\partial r} \Big|_p$. By (4), the kinematic relative velocity $V_{\text{kin } p}$ of β_1 with respect to β_2 at p reads

$$V_{\text{kin } p} = -\frac{a_2}{E} \sqrt{E^2 - a_1^2} \frac{\partial}{\partial r} \Big|_p.$$

So, it is satisfied that $\lim_{r_1 \rightarrow 2m} \|V_{\text{kin } p}\| = 1$.

On the other hand, by (24), the relative position S of β_1 with respect to β_2 is given by

$$S = \left(2m \ln \left(\frac{(1 - a_2) \sqrt{r_2}}{(1 - a(\beta_1^r(t))) \sqrt{\beta_1^r(t)}} \right) + \beta_1^r(t) a(\beta_1^r(t)) - r_2 a_2 \right) a_2 \frac{\partial}{\partial r}.$$

By (5), the Fermi relative velocity V_{Fermi} of β_1 with respect to β_2 reads

$$V_{\text{Fermi}} = (\nabla_U S)^r \frac{\partial}{\partial r} = \frac{1}{a_2} \frac{\partial S^r}{\partial t} \frac{\partial}{\partial r} = \frac{1}{a_2} \frac{\dot{\beta}_1^r(t)}{a(\beta_1^r(t))} \frac{\partial}{\partial r}.$$

Taking into account (27), we have $\dot{\beta}_1^r(t_1) = -\frac{a_1^2}{E}\sqrt{E^2 - a_1^2}$. Hence

$$V_{\text{Fermi } p} = -\frac{a_1}{a_2 E} \sqrt{E^2 - a_1^2} \frac{\partial}{\partial r} \Big|_p.$$

So, it is satisfied that $\lim_{r_1 \rightarrow 2m} \|V_{\text{Fermi } p}\| = 0$.

6.2.2. Spectroscopic and astrometric relative velocities Let p be the unique event of β_2 such that there exists a light ray λ from q to p , and let us suppose that $p = (t_2, r_2, \pi/2, 0)$. In [4] it is shown that the spectroscopic relative velocity $V_{\text{spec } p}$ of β_1 observed by β_2 at p is given by

$$V_{\text{spec } p} = -a_2 \frac{(a_2^2 + a_1^2) \sqrt{E^2 - a_1^2} + E(a_2^2 - a_1^2)}{(a_2^2 - a_1^2) \sqrt{E^2 - a_1^2} + E(a_2^2 + a_1^2)} \frac{\partial}{\partial r} \Big|_p. \tag{28}$$

So, it follows that $\lim_{r_1 \rightarrow 2m} \|V_{\text{spec } p}\| = 1$.

On the other hand, it can be checked that

$$\lambda(r) := \left(t_1 + r - r_1 + 2m \ln \left(\frac{r - 2m}{r_1 - 2m} \right), r, \pi/2, 0 \right), \quad r \in [r_1, r_2]$$

is a light ray from $q = \lambda(r_1)$ to $p = \lambda(r_2)$. So,

$$t_2 = \lambda^t(r_2) = t_1 + r_2 - r_1 + 2m \ln \left(\frac{r_2 - 2m}{r_1 - 2m} \right). \tag{29}$$

Let us define implicitly the function $f(t)$ by the expression

$$f(t) := t - \left(r_2 - \beta_1^r(f(t)) + 2m \ln \left(\frac{r_2 - 2m}{\beta_1^r(f(t)) - 2m} \right) \right). \tag{30}$$

Taking into account (29), $f(t)$ is the coordinate time at which β_1 emits a light ray that arrives at β_2 at coordinate time t . Applying (26), the relative position S_{obs} of β_1 observed by β_2 reads

$$S_{\text{obs}} = (\beta_1^r(f(t)) - r_2) \frac{\partial}{\partial r}.$$

By (14), the astrometric relative velocity V_{ast} of β_1 observed by β_2 is given by

$$V_{\text{ast}} = (\nabla_U S_{\text{obs}})^r \frac{\partial}{\partial r} = \frac{1}{a_2} \frac{\partial S_{\text{obs}}^r}{\partial t} \frac{\partial}{\partial r} = \frac{1}{a_2} \dot{\beta}_1^r(f(t)) \dot{f}(t) \frac{\partial}{\partial r}.$$

From (30), we have $\dot{f}(t_2) = \frac{a_1^2}{a_1^2 - (a_1^2 - 1)\dot{\beta}_1^r(t_1)}$. Moreover, taking into account (27), we have $\dot{\beta}_1^r(t_1) = -\frac{a_1^2}{E}\sqrt{E^2 - a_1^2}$. Hence

$$V_{\text{ast } p} = -\frac{a_1^2}{a_2} \frac{\sqrt{E^2 - a_1^2}}{E + (a_1^2 - 1)\sqrt{E^2 - a_1^2}} \frac{\partial}{\partial r} \Big|_p, \tag{31}$$

and, in consequence, $\lim_{r_1 \rightarrow 2m} \|V_{\text{ast } p}\| = \frac{1}{a_2^2} \frac{2E^2}{1+2E^2} \in]0, +\infty[$.

6.3. *Comoving observers in Robertson-Walker.* In a Robertson-Walker metric with cartesian coordinates

$$ds^2 = -dt^2 + \frac{a^2(t)}{\left(1 + \frac{1}{4}kr^2\right)^2} \left(dx^2 + dy^2 + dz^2\right),$$

where $a(t)$ is the scale factor, $k = -1, 0, 1$ and $r := \sqrt{x^2 + y^2 + z^2}$, we consider two comoving (in the classical sense, see [14]) observers $\beta_0(\tau) = (\tau, 0, 0, 0)$ and $\beta_1(\tau) = (\tau, x_1, 0, 0)$ with $\tau \in \mathbb{R}$ and $x_1 > 0$. Let $t_0 \in \mathbb{R}$, $p := \beta_0(t_0)$ and $u := \dot{\beta}_0(t_0) = \frac{\partial}{\partial t}\Big|_p$ (i.e. the 4-velocity of β_0 at p). We are going to study the relative velocities of β_1 with respect to and observed by β_0 at p .

6.3.1. *Kinematic and Fermi relative velocities.* The vector field

$$X := -\sqrt{\frac{a_0^2}{a^2(t)} - 1} \frac{\partial}{\partial t} + \frac{a_0}{a^2(t)} \left(1 + \frac{1}{4}kx^2\right) \frac{\partial}{\partial x}$$

is geodesic, spacelike, unit, and X_p is orthogonal to u , i.e. it is tangent to the Landau submanifold $L_{p,u}$. Let $\beta_1(t_1) =: q$ be the unique event of $\beta_1 \cap L_{p,u}$. We can find t_1 for concrete scale factors $a(t)$ taking into account the expression of X , but we can not find an explicit expression in the general case. If $u' := \dot{\beta}_1(t_1) = \frac{\partial}{\partial t}\Big|_q$, then $\tau_{qp}u' = \frac{a_0}{a_1} \frac{\partial}{\partial t}\Big|_p + \sqrt{\frac{1}{a_1^2} - \frac{1}{a_0^2}} \frac{\partial}{\partial x}\Big|_p$, where $a_1 := a(t_1)$ (it is well defined because $a_0 \geq a_1 > 0$). So, by (4), the kinematic relative velocity $V_{\text{kin } p}$ of β_1 with respect to β_0 at p is given by

$$V_{\text{kin } p} = \frac{1}{a_0^2} \sqrt{a_0^2 - a_1^2} \frac{\partial}{\partial x}\Big|_p.$$

Given a concrete scale factor $a(t)$, the Fermi distance d^{Fermi} from β_1 to β_0 with respect to β_0 can be also found, taking into account the expression of X . So, the relative position S of β_1 with respect to β_0 reads

$$S = d^{\text{Fermi}} \frac{\left(1 + \frac{1}{4}kr^2\right)}{a(t)} \frac{\partial}{\partial x},$$

because $d^{\text{Fermi}} = \|S\|$. Hence, the Fermi relative velocity $V_{\text{Fermi } p}$ of β_1 with respect to β_0 at p is given by

$$V_{\text{Fermi } p} = \left(\frac{d}{dt} \left(\frac{d^{\text{Fermi}}}{a(t)} \right) \Big|_{t=t_0} + \frac{d^{\text{Fermi}} \dot{a}(t_0)}{a_0^2} \right) \frac{\partial}{\partial x}\Big|_p.$$

6.3.2. *Spectroscopic and astrometric relative velocities* Let λ be a light ray received by β_0 at p and emitted from β_1 at $\beta_1(t_1)$. Note that t_1 can be found from x_1 and t_0 taking into account that $\int_0^{x_1} \frac{dx}{1+\frac{1}{4}kx^2} = \int_{t_1}^{t_0} \frac{dt}{a(t)}$. It can be easily proved that the spectroscopic relative velocity $V_{\text{spec } p}$ of β_1 observed by β_0 at p is radial (by isotropy). So, by (13) taking into account that the cosmological shift is given by $\frac{a_0}{a_1}$ (see [4]), where $a_0 := a(t_0)$ and $a_1 := a(t_1)$, we have

$$V_{\text{spec } p} = \frac{1}{a_0} \frac{a_0^2 - a_1^2}{a_0^2 + a_1^2} \left. \frac{\partial}{\partial x} \right|_p. \tag{32}$$

Given a concrete scale factor $a(t)$, the affine distance d^{affine} from β_1 to β_0 observed by β_0 can be found. So, the relative position S_{obs} of β_1 observed by β_0 is given by

$$S_{\text{obs}} = d^{\text{affine}} \frac{(1 + \frac{1}{4}kr^2)}{a(t)} \left. \frac{\partial}{\partial x} \right|_p,$$

because $d^{\text{affine}} = \|S_{\text{obs}}\|$. Hence, the astrometric relative velocity $V_{\text{ast } p}$ of β_1 observed by β_0 at p reads

$$V_{\text{ast } p} = \left(\left. \frac{d}{dt} \left(\frac{d^{\text{affine}}}{a(t)} \right) \right|_{t=t_0} + \frac{d_p^{\text{affine}} \dot{a}(t_0)}{a_0^2} \right) \left. \frac{\partial}{\partial x} \right|_p. \tag{33}$$

Let us study these relative velocities in more detail. In cosmology it is usual to consider the scale factor in the form

$$a(t) = a_0 \left(1 + H_0(t - t_0) - \frac{1}{2}q_0 H_0^2(t - t_0)^2 \right) + \mathcal{O} \left(H_0^3(t - t_0)^3 \right),$$

where $t_0 \in \mathbb{R}$, $a_0 = a(t_0) > 0$, $H(t) = \dot{a}(t)/a(t)$ is the Hubble ‘‘constant’’, $H_0 = H(t_0) > 0$, $q(t) = -a(t)\ddot{a}(t)/\dot{a}(t)^2$ is the deceleration coefficient, and $q_0 = q(t_0)$, with $|H_0(t - t_0)| \ll 1$ (see [12]). This corresponds to a universe in decelerated expansion and the time scales that we are going to use are relatively small. Let us define $p := \beta_0(t_0)$ and $u := \dot{\beta}_0(t_0) = \left. \frac{\partial}{\partial t} \right|_p$.

We are going to express the spectroscopic and the astrometric relative velocity of β_1 observed by β_0 at p in terms of the redshift parameter at $t = t_0$, defined as $z_0 := \frac{a_0}{a_1} - 1$, where $a_1 := a(t_1)$. This parameter is very usual in cosmology since it can be measured by spectroscopic observations. By (32), the spectroscopic relative velocity $V_{\text{spec } p}$ of β_1 observed by β_0 at p is given by

$$V_{\text{spec } p} = \frac{1}{a_0} \frac{a_0^4 - (z_0 + 1)^2}{a_0^4 + (z_0 + 1)^2} \left. \frac{\partial}{\partial x} \right|_p. \tag{34}$$

In [4] it is shown that the affine distance d^{affine} from β_1 to β_0 observed by β_0 reads

$$d^{\text{affine}}(t) = \frac{z(t)}{H(t)} \left(1 - \frac{1}{2}(3 + q(t))z(t) \right) + \mathcal{O} \left(z^3(t) \right),$$

where $z(t)$ is the redshift function. So, by (33), the astrometric relative velocity $V_{\text{ast } p}$ of β_1 observed by β_0 at p is given by

$$V_{\text{ast } p} = \left(\frac{\dot{z}(t_0)}{a_0 H_0} + \frac{z_0}{a_0} \left(q_0 + 1 - \frac{\dot{z}(t_0)}{H_0} (3 + q_0) \right) + \mathcal{O}(z_0^2) \right) \frac{\partial}{\partial x} \Big|_p.$$

Hence, if we suppose that $\dot{z}(t_0) \approx 0$ (i.e., the redshift is constant in our time scale), then

$$V_{\text{ast } p} \approx \left(\frac{z_0}{a_0} (q_0 + 1) + \mathcal{O}(z_0^2) \right) \frac{\partial}{\partial x} \Big|_p. \tag{35}$$

7. Discussion and Comments

It is usual to consider the spectroscopic relative velocity as a non-acceptable “physical velocity”. However, in this paper we have defined it in a geometric way, showing that it is, in fact, a very plausible physical velocity.

- Firstly, in other works (see [3, 4]), we have discussed pros and cons of spacelike and lightlike simultaneities, coming to the conclusion that lightlike simultaneity is physically and mathematically more suitable. Since the spectroscopic relative velocity is the natural generalization (in the framework of lightlike simultaneity) of the usual concept of relative velocity (given by (2)), it might have a lot of importance.
- Secondly, there are some good properties suggesting that the spectroscopic relative velocity has a lot of physical sense. For instance, if we work with the spectroscopic relative velocity, it is shown in [4] that gravitational redshift is just a particular case of a generalized Doppler effect.

Nevertheless, all four concepts of relative velocity have full physical sense and they must be studied equally.

Finally, one can wonder whether the discussed concepts of relative velocity can be actually determined experimentally. A priori, only the spectroscopic and astrometric relative velocities can be measured by direct observation. The shift allows us to find relations between the modulus of the spectroscopic relative velocity and its tangential component, as we show in (12). But, in general, it is not enough information to determine it completely (as we discuss in Remark 1), unless we make some assumptions (see Remark 2) or we use a model for the space-time and apply some expressions like (25), (28), or (34). Finding the astrometric relative velocity is basically the same problem as finding the optical coordinates. It is non-trivial and it has been widely treated, for instance, in [6]. Nevertheless, expressions like (31) or (35) could be very useful in particular situations. Since the measure of these velocities is rather difficult, any expression relating them can be very helpful in order to determine them, as, for example, expression (23) in special relativity.

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References

1. Beem, J.K., Ehrlich, P.E.: *Global Lorentzian Geometry*. Marcel Dekker, New York, 1981
2. Bini, D., Lusanna, L., Mashhoon, B.: Limitations of radar coordinates. *Int. J. Mod. Phys. D* **14**, 1413–1429 (2005)

3. Bolós, V.J., Liern, V., Olivert, J.: Relativistic simultaneity and causality. *Int. J. Theor. Phys.* **41**, 1007–1018 (2002)
4. Bolós, V.J.: Lightlike simultaneity, comoving observers and distances in general relativity. *J. Geom. Phys.* **56**, 813–829 (2006)
5. Crawford, P., Tereno, I.: Generalized observers and velocity measurements in general relativity. *Gen. Rel. Grav.* **34**, 2075–2088 (2002)
6. Ellis, G.F.R., Nel, S.D., Maartens, R., Stoeger, W.R., Whitman, A.P.: Ideal observational cosmology. *Phys. Rep.* **124**, 315–417 (1985)
7. Fermi, E.: Sopra i fenomeni che avvengono in vicinanza di una linea oraria. *Atti R. Accad. Naz. Lincei, Rendiconti, Cl. sci. fis. mat & nat.* **31**(1), 21–23, 51–52, and 101–103 (1922)
8. Helgason, S.: *Differential Geometry and Symmetric Spaces*. Academic Press, London, 1962
9. Kermack, W.O., McCrea, W.H., Whittaker, E.T.: On properties of null geodesics and their application to the theory of radiation. *Proc. R. Soc. Edinburgh* **53**, 31–47 (1932)
10. Lindegren, L., Dravins, D.: The fundamental definition of “radial velocity”. *Astronomy & Astrophysics* **401**, 1185–1201 (2003)
11. Marzlin, K.P.: The physical meaning of Fermi coordinates. *Gen. Rel. Grav.* **26**, 619–636 (1994)
12. Misner, W., Thorne, K., Wheeler, J.: *Gravitation*. Freeman, New York, 1973
13. Olivert, J.: On the local simultaneity in general relativity. *J. Math. Phys.* **21**, 1783–1785 (1980)
14. Sachs, R.K., Wu, H.: *Relativity for Mathematicians*. Springer Verlag, Berlin, 1977
15. Soffel, M., *et al.*: The IAU 2000 resolutions for astrometry, celestial mechanics and metrology in the relativistic framework: explanatory supplement. *The Astronomical Journal* **126**, 2687–2706 (2003)

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