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On Two-Dimensional Sonic-Subsonic Flow

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Abstract: A compensated compactness framework is established for sonic-subsonic approximate solutions to the two-dimensional Euler equations for steady irrotational flows that may contain stagnation points. Only crude estimates are required for establishing compactness. It follows that the set of subsonic irrotational solutions to the Euler equations is compact; thus flows with sonic points over an obstacle, such as an airfoil, may be realized as limits of sequences of strictly subsonic flows. Furthermore, sonic-subsonic flows may be constructed from approximate solutions. The compactness framework is then extended to self-similar solutions of the Euler equations for unsteady irrotational flows.

1. Introduction

Consider the two-dimensional Euler equations for steady irrotational flows:

$$v_x - u_y = 0, \tag{1.1}$$

$$(\rho u)_x + (\rho v)_y = 0, \tag{1.2}$$

$$(\rho u^2 + p)_x + (\rho u v)_y = 0, \tag{1.3}$$

$$(\rho u v)_x + (\rho v^2 + p)_y = 0, \tag{1.4}$$

where u and v are the two components of flow velocity and ρ is the density.

For a polytropic gas with adiabatic exponent $\gamma > 1$, $p = p(\rho) = \rho^\gamma/\gamma$ is the normalized pressure. Equations (1.1) and (1.3)–(1.4) classically yield the normalized Bernoulli's law (Courant-Friedrichs [10]):

$$\rho = \hat{\rho}(q^2) := \big(1 - \frac{\gamma - 1}{2} q^2\big)^{\frac{1}{\gamma-1}}, \tag{1.5}$$

where $q = \sqrt{u^2 + v^2}$ is the flow speed. The sound speed c is defined as

$$c^2 = p'(\rho) = 1 - \frac{\gamma - 1}{2} q^2. \tag{1.6}$$

At the sonic point $q = c$, (1.6) implies $q^2 = \frac{2}{\gamma+1}$. We define the critical speed q_{cr} as

$$q_{cr} \equiv \sqrt{\frac{2}{\gamma + 1}}$$

and rewrite Bernoulli's law (1.5) in the form

$$q^2 - q_{cr}^2 = \frac{2}{\gamma + 1}\left(q^2 - c^2\right).$$

Thus the flow is subsonic when $q < q_{cr}$, sonic when $q = q_{cr}$, and supersonic when $q > q_{cr}$.

For isothermal flow, $p = \bar{c}^2 \rho$, where $\bar{c} > 0$ is the constant sound speed. Then, in the place of (1.5), Bernoulli's law yields

$$\rho = \hat{\rho}(q^2) := \rho_0 \exp\left(-\frac{u^2 + v^2}{2\bar{c}^2}\right) \tag{1.7}$$

for some constant $\rho_0 > 0$. In this case, $q_{cr} = \bar{c}$.

Bernoulli's law plays the role of mechanical energy conservation in irrotational flow.

One may opt to determine the flow either through system (1.2)–(1.4) or through system (1.1)–(1.2) and (1.5) or (1.7). The former option secures conservation of mass and momentum, while the latter option imposes irrotationality and requires conservation of mass and mechanical energy. Both approaches are mathematically equivalent for smooth flows, because, once inserted through the initial data, irrotationality persists and yields Bernoulli's law. This is no longer the case in the presence of shocks, as vorticity may be created and mechanical energy may be converted into heat. The theory of isentropic thermodynamics favors retaining system (1.2)–(1.4) for rough flows. Nevertheless, system (1.1)–(1.2) and (1.5) has been more popular among aerodynamicists, as it is mathematically simpler and also brings out the analogy with the treatment of incompressible fluids (cf. Bers [2]). In any case, it is generally believed that, when shocks are weak, the solutions of the two systems are close. Accordingly, here we will deal with the system of potential flow, namely (1.1)–(1.2) and (1.5) or (1.7).

In weak solutions of system (1.1)–(1.2) and (1.5), the left-hand sides of (1.3) and (1.4) do not necessarily vanish, but they are equal to distributions representing the (artificial and a priori unknown) body force that would have to be imposed in order to balance momentum. These equations have been ignored in traditional treatments of the system of potential flow. The purpose of this paper is to reveal and emphasize that (1.3) and (1.4), modified by the addition of the artificial body force and interpreted as entropy (in the mathematical–not the physical sense) balance equations, play a very useful and important role in the treatment of weak solutions of the system of potential flow. Indeed, with the help of these equations, we establish, in Sect. 2, a compensated compactness framework for sonic-subsonic approximate solutions to the system of two-dimensional potential flow that may contain stagnation points. In particular, in Sect. 3, we show that sets of sonic-subsonic irrotational flows that satisfy crude bounds are precompact. Thus flows with sonic points over an obstacle, such as an airfoil, may be realized as limits of sequences of strictly subsonic flows. Furthermore, sonic-subsonic flows may be constructed from approximate solutions to the potential flow system, under very crude estimates. Finally, in Sect. 4, we extend the compensated compactness framework to the realm of self-similar solutions to the Euler equations for unsteady irrotational flow.

For the fundamental ideas and early applications of compensated compactness, see the classical papers by Tartar [36] and Murat [33]. For applications to the theory of hyperbolic conservation laws, see for example [6, 7, 11, 14, 15, 17, 34]. In particular, the compensated compactness approach has been applied in [5, 9, 12, 13, 27, 28] to the one-dimensional Euler equations for unsteady isentropic flow, allowing for cavitation. Particularly, relevant to the present work are the papers by Morawetz [30, 31] for two-dimensional steady transonic flow away from stagnation and cavitation points. In particular, the basic estimate used in our proof of Theorem 2.1 is contained in Morawetz's papers (the remark following Identity I, page 505 of [30] or, equivalently, Eq. (5.5) of [31]) when applied to her case $n = 1$. However, in this work, we identify the relation of this estimate with balance of momentum, recognize its validity at stagnation points, realize that the direct use of conservation of momentum is a useful simplification, and point out how it can be applied in a simple and direct fashion.

2. Compensated Compactness Framework for Steady Flow

As noted in the Introduction, the standard presentation of steady irrotational gas dynamics is to analyze (1.1)–(1.2) subject to Bernoulli's law (1.5) or (1.7), and Eqs. (1.3)–(1.4) seem to have been lost from the story. However, (1.1)–(1.2) coupled with Bernoulli's law in fact imply the conservation of linear momentum (1.3)–(1.4). Hence, in the language of conservation laws (cf. Dafermos [11] and Lax [26]), Eqs. (1.3)–(1.4) provide "entropy-entropy flux pairs".

Let a sequence of functions $w^\varepsilon(x, y) = (u^\varepsilon, v^\varepsilon)(x, y)$, defined on open subset $\Omega \subset \mathbb{R}^2$, satisfy the following Set of Conditions (A):

(A.1) $q^\varepsilon(x, y) = |w^\varepsilon(x, y)| \le q_{cr}$ a.e. in Ω;

(A.2) $\partial_x \eta_k(w^\varepsilon) + \partial_y q_k(w^\varepsilon)$, $k = 1, 2, 3, 4$, are confined in a compact set in $H^{-1}_{loc}(\Omega)$ for the momentum entropy-entropy flux pairs:

$$(\eta_1, q_1) = (\rho u^2 + p,\ \rho uv), \qquad (\eta_2, q_2) = (\rho uv,\ \rho v^2 + p), \tag{2.1}$$

and the two natural entropy-entropy flux pairs:

$$(\eta_3, q_3) = (v,\ -u), \qquad (\eta_4, q_4) = (\rho u,\ \rho v). \tag{2.2}$$

Then, by the div-curl lemma of Tartar [36] and Murat [33] and the Young measure representation theorem for a uniformly bounded sequence of functions (cf. Tartar [36]; also Ball [1]), we have the following commutation identity:

$$\begin{aligned} &\langle \nu(w), \eta_i(w) q_j(w) - q_i(w)\eta_j(w)\rangle \\ &\quad = \langle \nu(w), \eta_i(w)\rangle\langle \nu(w), q_j(w)\rangle - \langle \nu(w), q_i(w)\rangle\langle \nu(w), \eta_j(w)\rangle, \end{aligned} \tag{2.3}$$

where $\nu = \nu_{x,y}(w)$, $w = (u, v)$, is the associated Young measure (a probability measure) for the sequence $w^\varepsilon(x, y) = (u^\varepsilon, v^\varepsilon)(x, y)$. The main point in this section for the compensated compactness framework is to prove that ν is in fact a Dirac measure by using only the above momentum entropy pairs in (2.1), besides the two natural entropy pairs in (2.2). This in turn implies the compactness of the sequence $w^\varepsilon(x, y) = (u^\varepsilon, v^\varepsilon)(x, y)$ in $L^1(\Omega)$.

Theorem 2.1 (Compensated compactness framework). *Let a sequence of functions $w^\varepsilon(x, y) = (u^\varepsilon, v^\varepsilon)(x, y)$ satisfy Framework* (A). *Then the associated Young measure ν is a Dirac mass and the sequence $w^\varepsilon(x, y)$ is compact in $L^1(\Omega)$; that is, there is a subsequence (still labeled) w^ε that converges a.e. as $\varepsilon \to 0$ to $w = (u, v)$ satisfying*

$$q(x, y) = |w(x, y)| \le q_{cr} \qquad a.e.\ (x, y) \in \Omega.$$

Proof. For simplicity of notation, we drop the subscript (x, y). Since the Young measure ν is a probability measure, Eq. (2.3) with $i = 1$, $j = 2$, for ν yields

$$\langle \nu(w_1) \otimes \nu(w_2),\ I(w_1, w_2)\rangle = 0, \tag{2.4}$$

where

$$\begin{aligned}
&I(w_1, w_2)\\
&= (\eta_1(w_1) - \eta_1(w_2))(q_2(w_1) - q_2(w_2)) - (q_1(w_1) - q_1(w_2))(\eta_2(w_1) - \eta_2(w_2))\\
&= (\rho_1 u_1^2 + p_1 - \rho_2 u_2^2 - p_2)(\rho_1 v_1^2 + p_1 - \rho_2 v_2^2 - p_2) - (\rho_1 u_1 v_1 - \rho_2 u_2 v_2)^2\\
&= -\rho_1\rho_2(u_1 v_2 - u_2 v_1)^2 + (p_1 - p_2)^2 + (p_1 - p_2)(\rho_1 q_1^2 - \rho_2 q_2^2),
\end{aligned}$$

$w_1 = (u_1, v_1)$ and $w_2 = (u_2, v_2)$ are two independent vector variables, and $\nu(w_1) \otimes \nu(w_2)$ should be understood as a product measure for (w_1, w_2), i.e., $\langle \nu(w_1) \otimes \nu(w_2), \varphi(w_1, w_2)\rangle := \langle \nu(w_1), \langle \nu(w_2), \varphi(w_1, w_2)\rangle\rangle$ for any test function $\varphi(w_1, w_2)$.

For $\gamma > 1$,

$$\rho = (\gamma p)^{\frac{1}{\gamma}}, \quad q^2 = \frac{2}{\gamma - 1}\left(1 - \rho^{\gamma-1}\right),$$

and then

$$\rho q^2 = \frac{2}{\gamma - 1}\left((\gamma p)^{\frac{1}{\gamma}} - \gamma p\right).$$

Thus,

$$\begin{aligned}
&I(w_1, w_2)\\
&= -\rho_1\rho_2(u_1 v_2 - u_2 v_1)^2 + (p_1 - p_2)^2\\
&\quad + \frac{2(p_1 - p_2)}{\gamma - 1}\left((\gamma p_1)^{\frac{1}{\gamma}} - \gamma p_1 - (\gamma p_2)^{\frac{1}{\gamma}} + \gamma p_2\right)\\
&= -\rho_1\rho_2(u_1 v_2 - u_2 v_1)^2 - \frac{\gamma + 1}{\gamma - 1}(p_1 - p_2)^2 + \frac{2\gamma^{\frac{1}{\gamma}}}{\gamma - 1}(p_1 - p_2)\big(p_1^{\frac{1}{\gamma}} - p_2^{\frac{1}{\gamma}}\big)\\
&= -\rho_1\rho_2(u_1 v_2 - u_2 v_1)^2 - (p_1 - p_2)^2\big(\frac{\gamma + 1}{\gamma - 1} - \frac{2\gamma^{\frac{1}{\gamma}-1}}{\gamma - 1}\tilde{p}^{\frac{1}{\gamma}-1}\big)\\
&= -\rho_1\rho_2(u_1 v_2 - u_2 v_1)^2 - \frac{\gamma + 1}{\gamma - 1}(p_1 - p_2)^2 \frac{q_{cr}^2 - \tilde{q}^2}{\frac{2}{\gamma-1} - \tilde{q}^2},
\end{aligned}$$

where $\tilde{p} = p(\tilde{\rho})$ lies between p_1 and p_2 as determined by the mean-value theorem on $p^{\frac{1}{\gamma}}$, and $\tilde{q}$ is determined by $\tilde{\rho}$ through Bernoulli's law (1.5). We notice that setting $u = q\cos\theta$ and $v = q\sin\theta$ implies

$$u_1 v_2 - u_2 v_1 = q_1 q_2 \sin(\theta_2 - \theta_1);$$

$$I(w_1, w_2) \le 0 \quad \text{when } q_i \le q_{cr} \text{ for } i = 1, 2 \text{ (so that } \tilde{q} \le q_{cr} \text{ and } \tfrac{2}{\gamma-1} - \tilde{q}^2 > 0);$$

and

$$I(w_1, w_2) = 0$$

if and only if

$$\text{either } p_1 = p_2 > 0 \text{ and } u_1 v_2 - u_2 v_1 = 0, \text{ or } \rho_1 = \rho_2 = p_1 = p_2 = 0.$$

This implies that

$$\operatorname{supp}(\nu(w_1) \otimes \nu(w_2)) \subset \{w_2 = \pm w_1\},$$

that is,

$$\operatorname{supp} \nu \subset \{\pm P_0\} \qquad \text{for some point } P_0 = (u_0, v_0) \text{ with } q_0 \le q_{cr}.$$

When $q_0 = 0$, then $P_0 = -P_0$ and the Young measure is a Dirac mass concentrated at the stagnation point.

When $q_0 \neq 0$, the support of the Young measure ν consists of at most two points $\pm P_0$ in the (u, v)–plane, that is, there exists $\alpha \in [0, 1]$ such that

$$\nu = \alpha\, \delta_{P_0} + (1 - \alpha)\, \delta_{\{-P_0\}},$$

in the (u, v)–phase plane. Then, taking $i = 3$, $j = 4$ in (2.3) and using $\rho > 0$ as $q \le q_{cr}$, we have

$$q_0^2 = (2\alpha - 1)^2 q_0^2,$$

that is,

$$\alpha(1 - \alpha) q_0^2 = 0.$$

Since $q_0 \neq 0$, then either $\alpha = 0$ or $\alpha = 1$, which implies that the Young measure is a Dirac measure.

Similarly, for isothermal flows,

$$p = \bar{c}^2 \rho, \qquad q^2 = -2\bar{c}^2 \ln\Big(\frac{\rho}{\rho_0}\Big).$$

Then we have

$$\begin{aligned}
&I(w_1, w_2)\\
&= -\rho_1\rho_2(u_1v_2 - u_2v_1)^2 + \bar{c}^4(\rho_1 - \rho_2)^2 + \bar{c}^2(\rho_1 - \rho_2)(\rho_1 q_1^2 - \rho_2 q_2^2)\\
&= -\rho_1\rho_2(u_1v_2 - u_2v_1)^2 + \bar{c}^4(\rho_1 - \rho_2)^2 - 2\bar{c}^4(\rho_1 - \rho_2)\left(\rho_1 \ln\left(\frac{\rho_1}{\rho_0}\right) - \rho_2 \ln\left(\frac{\rho_2}{\rho_0}\right)\right)\\
&= -\rho_1\rho_2(u_1v_2 - u_2v_1)^2 + \bar{c}^4(\rho_1 - \rho_2)^2 - 2\bar{c}^4(\rho_1 - \rho_2)^2\left(1 + \ln\Big(\frac{\tilde{\rho}}{\rho_0}\Big)\right)\\
&= -\rho_1\rho_2(u_1v_2 - u_2v_1)^2 - \bar{c}^2(\rho_1 - \rho_2)^2\left(\bar{c}^2 - \tilde{q}^2\right) \le 0,
\end{aligned}$$

when $q_i \leq \bar{c}$ for $i = 1, 2$ (and thus $\tilde{q} \leq \bar{c}$), and $\tilde{\rho}$ lies between ρ_1 and ρ_2 as determined by the mean-value theorem on $1 + \ln\left(\frac{\rho}{\rho_0}\right)$. This implies that

$$\text{supp}(\nu(w_1) \otimes \nu(w_2)) \subset \{w_2 = \pm w_1\},$$

that is,

$$\text{supp}\,\nu \subset \{\pm P_0\} \qquad \text{for some point } P_0 = (u_0, v_0) \text{ with } q_0 \leq \bar{c}.$$

Then the same argument as for the case $\gamma > 1$ by using the other two entropy pairs (η_j, q_j), $j = 3, 4$, yields that the Young measure is a Dirac measure and the strong convergence follows immediately. □

We now consider a sequence of approximate solutions $w^\varepsilon(x, y)$ to the Euler equations (1.1)–(1.2) with Bernoulli's law (1.5) or (1.7). That is, besides Set of Conditions (A), the sequence $w^\varepsilon(x, y) = (u^\varepsilon, v^\varepsilon)(x, y)$ further satisfies

$$v_x^\varepsilon - u_y^\varepsilon = o_1^\varepsilon(1), \tag{2.5}$$

$$(\hat{\rho}(|w^\varepsilon|^2)u^\varepsilon)_x + (\hat{\rho}(|w^\varepsilon|^2)v^\varepsilon)_y = o_2^\varepsilon(1), \tag{2.6}$$

where $o_j^\varepsilon(1) \to 0$, $j = 1, 2$, in the sense of distributions as $\varepsilon \to 0$. Then, as a corollary of the compensated compactness framework (Theorem 2.1), we have

Theorem 2.2 (Convergence of approximate solutions) *. Let $w^\varepsilon(x, y) = (u^\varepsilon, v^\varepsilon)(x, y)$ be a sequence of approximate solutions to the Euler equations* (1.1)–(1.2) *with Bernoulli's law* (1.5) *or* (1.7) *in Ω. Then there exists a subsequence (still labeled) $w^\varepsilon(x, y)$ that converges a.e. as $\varepsilon \to 0$ to a weak solution $w = (u, v)$ of the Euler equations* (1.1)–(1.2) *with Bernoulli's law* (1.5) *or* (1.7) *satisfying*

$$q(x, y) = |w(x, y)| \leq q_{cr} \qquad a.e.\ (x, y) \in \Omega.$$

There are various ways to construct approximate solutions by either numerical methods such as finite difference schemes and finite element methods, or analytical methods such as vanishing viscosity and relaxation methods. Even though the flow may eventually turn out to be smooth, the point of considering here weak solutions is to demonstrate that such solutions may be constructed by merely using very crude estimates, which are available in a variety of approximating methods through basic energy-type estimates besides the L^∞ estimate.

3. Sonic Limit of Subsonic Flows

As our principal application, we consider the sonic limit of subsonic flows past an obstacle $\mathcal{P}$, such as an airfoil.

We follow the presentation of Bers [3, 4] (also cf. [16 – 25]). Write

$$z = x + iy,\ \ w = u - iv = qe^{-i\theta},\ \ q = \sqrt{u^2 + v^2};\quad u = q\cos\theta,\ \ v = -q\sin\theta.$$

We consider a fixed simple closed rectifiable curve $\mathcal{C}$ (the boundary of the obstacle $\mathcal{P}$) in the z-plane and a fixed point z_T on it (the trailing edge). This curve may possess at z_T a protruding corner or cusp, but should otherwise be a Lyapunov curve (a Lyapunov curve is a curve which possesses a tangent inclination which satisfies a Hölder condition with respect to the arc length). Let S be the length of $\mathcal{C}$, and $\varepsilon\pi$ the opening of the corner at z_T. If $\varepsilon = 0$, $\mathcal{C}$ has a cusp at z_T; if $\varepsilon = 1$, $\mathcal{C}$ possesses a tangent at z_T; otherwise, $0 < \varepsilon < 1$; see Figs. 1–3. The profile $\mathcal{C}$ admits the parametric representation:

$$z = z_T + \int_0^s e^{i\Theta(\sigma)} d\sigma, \quad 0 \le s \le S.$$

The function $\Theta(s)$ must satisfy the condition

$$\Theta(S) - \Theta(0) = (1 + \varepsilon)\pi, \qquad 0 \le \varepsilon \le 1, \tag{3.1}$$

and the Hölder condition

$$|\Theta(s_2) - \Theta(s_1)| \le k(s_2 - s_1)^\alpha, \quad 0 \le s_1 < s_2 \le S, \tag{3.2}$$

for some constants $k > 0$ and $0 < \alpha < 1$.

Denote by $\mathfrak{D}(\mathcal{C})$ the domain exterior to $\mathcal{C}$. A pair of functions $(u, v) \in C^1(\mathfrak{D}(\mathcal{C}))$ is called a solution of *Problem* **P**, if (u, v) satisfy (1.1)–(1.2) with Bernoulli's law (1.5) or (1.7), and the slip boundary condition

$$(u, v) \cdot \mathbf{n} = 0 \qquad \text{on} \quad \mathcal{C}, \tag{3.3}$$

where **n** denotes the normal on $\mathcal{C}$, and the limit

$$w_\infty = \lim_{z \to \infty} (u - iv)$$

exists and is finite.

A pair of functions (u, v) defined on $\mathfrak{D}(\mathcal{C})$ is said to satisfy the *Kutta-Joukowski condition* if

$$u^2 + v^2 \to 0 \qquad \text{as} \quad z \to z_T, \quad \text{if} \quad \varepsilon = 1$$

(a stagnation point at the trailing edge), or

$$u^2 + v^2 = O(1) \qquad \text{as} \quad z \to z_T, \quad \text{if} \quad 0 \le \varepsilon < 1.$$

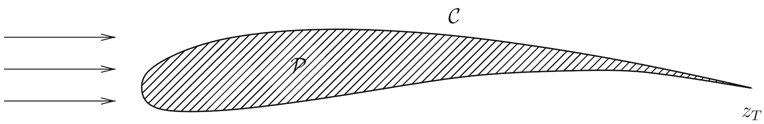

Fig. 1. Profile $\mathcal{C}$ of the obstacle $\mathcal{P}$: $\varepsilon = 0$

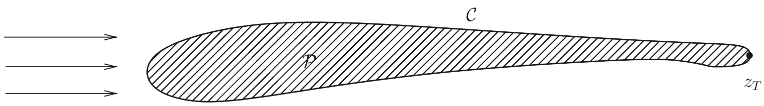

Fig. 2. Profile $\mathcal{C}$ of the obstacle $\mathcal{P}$: $\varepsilon = 1$

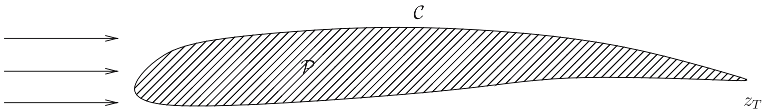

Fig. 3. Profile $\mathcal{C}$ of the obstacle $\mathcal{P}$: $0 < \varepsilon < 1$

In fact, a solution of Problem **P** automatically satisfies the Kutta-Joukowski condition if $0 \le \varepsilon < 1$; in particular, for such a function,

$$u^2 + v^2 = 0 \qquad \text{at} \quad z_T, \quad \text{if} \quad 0 < \varepsilon < 1.$$

Also, with every solution of Problem **P**, we associate the circulation Γ as

$$\Gamma = \oint_{\mathcal{C}} (u, v) \cdot \mathbf{t}\, ds,$$

where $\mathbf{t}$ is the unit tangent to $\mathcal{C}$.

Bers [3, 4] then defined the following two boundary-value problems:

Problem $\mathbf{P}_1(w_\infty)$. Find a solution of Problem **P** satisfying a Kutta-Joukowski condition and a prescribed limit w_∞ as $z \to \infty$.

Problem $\mathbf{P}_2(w_\infty, \Gamma)$. Find a solution of Problem **P** for which w_∞ and Γ are prescribed.

Problem $\mathbf{P}_2$ is only considered in the case of a smooth profile $\varepsilon = 1$.

We use Bers's fundamental existence-uniqueness theorems whose proof simplified an earlier result of Shiffman [35] on the existence of solutions.

Theorem 3.1 (Bers [4]). *For a given w_∞, there exists a number $\hat{q} < q_{cr}$, depending on the profile and the equation of state, such that Problem $\mathbf{P}_1(w_\infty)$ has a unique solution (u, v) for $q_\infty := |w_\infty| < \hat{q}$. The velocity (u, v) is Hölder continuous on the profile and depends continuously on w_∞. The maximum speed q_m of $|w|$ takes on all values between* 0 *and some critical value q_{cr}, and $q_m \to 0$ as $q_\infty \to 0$, $q_m \to q_{cr}$ as $q_\infty \to \hat{q}$. A similar result holds for Problem $\mathbf{P}_2(w_\infty, \Gamma)$.*

We note that Bers's Theorem does not apply to the critical flows, that is, those flows for which $q_\infty = \hat{q}$ and which hence must be sonic at some point in $\mathfrak{D}(\mathcal{C}) \cup \mathcal{C}$. In fact, the Gilbarg-Shiffman maximum principle [25] asserts that the sonic point should occur on $\mathcal{C}$, which presupposed the existence of critical flows. In this regard, Gilbarg and Shiffman [25] remarked in footnote 8: "The actual existence of critical flows past finite profiles of bounded curvature has been proved by M. Shiffman (unpublished)". Bers in [4] made a similar though less precise statement: "Shiffman also proved (unpublished) that for a smooth profile the solution of $\mathbf{P}_2(w_\infty, \Gamma)$ converges to a critical flow for $q_\infty \to \hat{q}$". Here, we establish a more general result.

Theorem 3.2 *Let $q_\infty^\varepsilon < \hat{q}$ be a sequence of speeds at ∞, and let $(u^\varepsilon, v^\varepsilon)$ be the corresponding solutions to either Problem $\mathbf{P}_1(w_\infty)$ or $\mathbf{P}_2(w_\infty, \Gamma)$. Then, as $q_\infty^\varepsilon \nearrow \hat{q}$, the solution sequence $(u^\varepsilon, v^\varepsilon)$ possesses a subsequence (still denoted by) $(u^\varepsilon, v^\varepsilon)$ that converges strongly a.e. in $\mathfrak{D}(\mathcal{C})$ to a pair of functions (u, v) which is a weak solution of Eqs.* (1.1)–(1.2) *with Bernoulli's law* (1.5) *or* (1.7)*. Furthermore, the limit velocity (u, v) satisfies the boundary conditions* (3.3) *as the normal trace of the divergence-measure field $(\rho u, \rho v)$ on the boundary (see* [8]*).*

Proof. The strong solutions $(u^\varepsilon, v^\varepsilon)$ satisfy (1.1)–(1.4) and are subsonic. Hence Theorem 2.1 immediately implies that the Young measure is a Dirac mass and the convergence is strong a.e. in $\mathfrak{D}(\mathcal{C})$. The fact the boundary conditions (3.3) are satisfied for (u, v) in the sense of distributions is standard by multiplying (1.2) by a test function and applying the divergence theorem and the fact that the sequence of subsonic solutions does satisfy (3.3), which implies (u, v) satisfies the boundary conditions (3.3) actually as the normal trace of the divergence-measure field $(\rho u, \rho v)$ on the boundary in the sense of Chen-Frid [8]. □

4. Extension to Self-Similar Solutions for Unsteady Flow

Here we are concerned with self-similar solutions depending on the variables $(\xi, \eta) = (x/t, y/t)$ for the Euler equations for unsteady flows. Then the equations of mass and momentum conservation imply

$$\begin{cases} (\rho(u-\xi))_\xi + (\rho(v-\eta))_\eta + 2\rho = 0, \\ (\rho u(u-\xi) + p)_\xi + (\rho u(v-\eta))_\eta + 2\rho u = 0, \\ (\rho v(u-\xi))_\xi + (\rho v(v-\eta) + p)_\eta + 2\rho v = 0. \end{cases} \tag{4.1}$$

The last two equations are of course easily rewritten as

$$\begin{cases} (\rho(u-\xi)^2 + p)_\xi + (\rho(u-\xi)(v-\eta))_\eta + 3\rho(u-\xi) = 0, \\ (\rho(u-\xi)(v-\eta))_\xi + (\rho(v-\eta)^2 + p)_\eta + 3\rho(v-\eta) = 0. \end{cases}$$

If we introduce the velocity potential Φ:

$$(U, V) \equiv (u-\xi, v-\eta) = \nabla\Phi,$$

we again find the normalized Bernoulli relation for polytropic gas with $\gamma > 1$:

$$\frac{\rho^{\gamma-1}}{\gamma-1} + \Phi + \frac{Q^2}{2} = \frac{1}{\gamma-1}, \tag{4.2}$$

where $Q^2 = U^2 + V^2 = |\nabla\Phi|^2$. We write the Bernoulli equation (4.2) in the same form as (1.5):

$$\rho = \hat{\rho}(Q^2; \Phi) := \Big(1 - \frac{\gamma-1}{2}Q^2 - (\gamma-1)\Phi\Big)^{\frac{1}{\gamma-1}}, \tag{4.3}$$

write the two equations of momentum conservation as

$$\begin{cases} (\rho U^2 + p)_\xi + (\rho UV)_\eta + 3\rho U = 0, \\ (\rho UV)_\xi + (\rho V^2 + p)_\eta + 3\rho V = 0, \end{cases} \tag{4.4}$$

and write the continuity equation as

$$(\rho U)_\xi + (\rho V)_\eta + 2\rho = 0. \tag{4.5}$$

The Bernoulli relation (4.3) yields the sound speed c as

$$c^2 := p'(\rho) = 1 - \frac{\gamma-1}{2}Q^2 - (\gamma-1)\Phi.$$

Thus, (4.4) is elliptic when $Q^2 < c^2$, i.e.,

$$\frac{\gamma+1}{2}Q^2 + (\gamma-1)\Phi < 1; \tag{4.6}$$

and (4.4) is hyperbolic when the above inequality (4.6) is reversed.

Let a sequence of functions $W^\varepsilon(\xi, \eta) := (U^\varepsilon, V^\varepsilon)(\xi, \eta)$, defined on an open subset $\Omega \subset \mathbb{R}^2$, satisfy the following Set of Conditions (B):

(B.1) There exist $\Phi^\varepsilon(\xi,\eta) \geq -M$ for some $M \in (0,\infty)$ independent of ε such that

$$\Phi^\varepsilon \to \Phi \qquad \text{a.e. in } \Omega \text{ as } \varepsilon \to 0, \tag{4.7}$$

$$\frac{\gamma+1}{2}(Q^\varepsilon(x,y))^2 + (\gamma-1)\Phi^\varepsilon(x,y) \leq 1 \qquad \text{a.e. } (x,y) \in \Omega; \tag{4.8}$$

(B.2) The sequences $\partial_\xi \eta_k(W^\varepsilon;\Phi^\varepsilon) + \partial_\eta q_k(W^\varepsilon;\Phi^\varepsilon)$, $k = 1,2,3,4$, are confined in a compact set in $H^{-1}_{loc}(\Omega)$ with (η_k, q_k), $k = 1,2,3,4$, defined as

$$(\eta_1,q_1)(W;\Phi) = (\hat\rho(Q^2;\Phi)U^2 + p(\hat\rho(Q^2;\Phi)),\ \hat\rho(Q^2;\Phi)UV),$$
$$(\eta_2,q_2)(W;\Phi) = (\hat\rho(Q^2;\Phi)UV,\ \hat\rho(Q^2;\Phi)V^2 + p(\hat\rho(Q^2;\Phi))),$$

and

$$(\eta_3,q_3)(W) = (V,-U),$$
$$(\eta_4,q_4)(W;\Phi) = (\hat\rho(Q^2;\Phi)U,\ \hat\rho(Q^2;\Phi)V).$$

Then we have the following compensated compactness framework.

Theorem 4.1 (Compensated compactness framework). *Let a sequence of functions $W^\varepsilon(\xi,\eta) := (U^\varepsilon,V^\varepsilon)(\xi,\eta)$ satisfy the Set of Conditions* (B) *with Φ^ε. Then the sequence W^ε is compact in $L^1(\Omega)$; that is, there is a subsequence (still labeled) W^ε that converges, as $\varepsilon \to 0$, to $W = (U,V)$ a.e. satisfying* (4.6) *with Φ.*

Proof. We proceed as in the proof of Theorem 2.1. First, Condition (B.1) implies

$$-M \leq \Phi^\varepsilon(\xi,\eta) \leq \frac{1}{\gamma-1}, \qquad (Q^\varepsilon(\xi,\eta))^2 \leq \frac{2}{\gamma+1}\left(1+(\gamma-1)M\right),$$

which indicates that the sequence $(W^\varepsilon;\Phi^\varepsilon)$ is uniformly bounded in $L^\infty(\mathbb{R}^2)^3$. Then, for any continuous function g on $\mathbb{R}^3$, we write

$$\begin{aligned} g(W^\varepsilon;\Phi^\varepsilon) &= g(W^\varepsilon;\Phi^\varepsilon) - g(W^\varepsilon;\Phi) + g(W^\varepsilon;\Phi) \\ &= \frac{\partial g}{\partial \Phi}(W^\varepsilon;\bar\Phi)(\Phi^\varepsilon - \Phi) + g(W^\varepsilon;\Phi), \end{aligned} \tag{4.9}$$

where $\bar\Phi$ lies between Φ^ε and Φ as determined by the mean-value theorem for each ε. Here, if $\frac{\partial g}{\partial \Phi}$ is uniformly bounded on the range of $(W^\varepsilon;\Phi^\varepsilon)$, then the first term on the right-hand side of the last equality in (4.9) goes to zero strongly as $\varepsilon \to 0$, and the weak* limit of $g(W^\varepsilon;\Phi^\varepsilon)$ in $L^\infty(\Omega)$ is

$$w^*\text{-}\lim g(W^\varepsilon;\Phi^\varepsilon) = \langle \nu(W), g(W;\Phi(\xi,\eta))\rangle,$$

where $\nu = \nu_{\xi,\eta}(W)$, $W = (U,V)$, is the associated Young measure with $W^\varepsilon(\xi,\eta)$, and we have used the strong convergence of Φ^ε in (4.7).

On the other hand, Condition (B.2) indicates that

$$\begin{aligned} &\left(\hat\rho((Q^\varepsilon)^2;\Phi^\varepsilon)(U^\varepsilon)^2 + p(\hat\rho((Q^\varepsilon)^2;\Phi^\varepsilon))\right)_\xi + \Big(\hat\rho((Q^\varepsilon)^2;\Phi^\varepsilon)U^\varepsilon V^\varepsilon\Big)_\eta \\ &\qquad \text{is confined in a compact set in } H^{-1}_{loc}(\Omega), \end{aligned} \tag{4.10}$$

and

$$\begin{aligned}&\big(\hat{\rho}((Q^\varepsilon)^2;\Phi^\varepsilon)U^\varepsilon V^\varepsilon\big)_\xi+\big(\hat{\rho}((Q^\varepsilon)^2;\Phi^\varepsilon)(V^\varepsilon)^2+p(\hat{\rho}((Q^\varepsilon)^2;\Phi^\varepsilon))\big)_\eta\\&\qquad\text{is confined in a compact set in } H^{-1}_{loc}(\Omega).\end{aligned} \tag{4.11}$$

Then we see that the nonlinear term g in the expressions above represents the terms of the form

$$\hat{\rho}(Q^2;\Phi)U^2,$$

etc., and $\frac{\partial g}{\partial \Phi}$ is uniformly bounded as long as $(W;\Phi)$ stay in the elliptic region. We then apply the commutation identity as in the proof of Theorem 2.1 and write it as

$$\langle \nu(W_1)\otimes\nu(W_2), I(W_1,W_2;\Phi(\xi,\eta))\rangle=0,$$

where $W_i=(U_i,V_i)$, $i=1,2$, and

$$I=-\rho_1\rho_2(U_1V_2-U_2V_1)^2+(p_1-p_2)^2+(p_1-p_2)(\rho_1Q_1^2-\rho_2Q_2^2).$$

By virtue of the mean-value theorem and similar calculations to those in the proof of Theorem 2.1, we obtain

$$\begin{aligned}I&=-\rho_1\rho_2(U_1V_2-U_2V_1)^2-(p_1-p_2)^2\left(\frac{\gamma+1}{\gamma-1}-\frac{2(1-(\gamma-1)\Phi)}{\gamma-1}(\gamma\tilde{p})^{\frac{1}{\gamma}-1}\right)\\&=-\rho_1\rho_2(U_1V_2-U_2V_1)^2-(p_1-p_2)^2\frac{1-(\gamma-1)\Phi-\frac{\gamma+1}{2}\tilde{Q}^2}{1-(\gamma-1)\Phi-\frac{\gamma-1}{2}\tilde{Q}^2},\end{aligned}$$

where $\tilde{p}=p(\tilde{\rho})$ lies between $p_1=p(\rho_1)$ and $p_2=p(\rho_2)$ as determined by the mean-value theorem on $p^{\frac{1}{\gamma}}$, and $\tilde{Q}$ is determined by $\tilde{\rho}$ through the relation (4.3). The ellipticity condition (4.6) ensures $I\le 0$. Then, by using the other two natural entropy pairs (η_k,q_k), $k=3,4$, the proof is completed as in the proof of Theorem 2.1. □

We now consider a sequence of approximate solutions $(W^\varepsilon;\Phi^\varepsilon)(\xi,\eta)$ to

$$\begin{cases}V_\xi-U_\eta=0,\\(\rho U)_\xi+(\rho V)_\eta+2\rho=0,\end{cases} \tag{4.12}$$

defined on an open subset $\Omega\subset\mathbb{R}^2$, satisfying the Bernoulli relation (4.3) and the ellipticity constraint (4.6). That is, besides the Set of Conditions (B) with Φ^ε, the sequence $(W^\varepsilon;\Phi^\varepsilon)(\xi,\eta)=(U^\varepsilon,V^\varepsilon;\Phi^\varepsilon)(\xi,\eta)$ further satisfies

$$(U^\varepsilon,V^\varepsilon)=\nabla\Phi^\varepsilon+o_1^\varepsilon(1), \tag{4.13}$$

$$(\hat{\rho}(|W^\varepsilon|^2;\Phi^\varepsilon)U^\varepsilon)_\xi+(\hat{\rho}(|W^\varepsilon|^2;\Phi^\varepsilon)V^\varepsilon)_\eta+2\hat{\rho}(|W^\varepsilon|^2;\Phi^\varepsilon)=o_2^\varepsilon(1), \tag{4.14}$$

where $o_j^\varepsilon(1)\to 0$, $j=1,2$, in the sense of distributions as $\varepsilon\to 0$.

We note that (4.13) implies

$$V_\xi^\varepsilon-U_\eta^\varepsilon=o_3^\varepsilon(1),$$

with $o_3^\varepsilon(1)\to 0$ in the sense of distributions as $\varepsilon\to 0$.

Then, as a corollary of the compensated compactness framework (Theorem 4.1), we have

Theorem 4.2 (Convergence of approximate solutions). *Let* $(W^\varepsilon; \Phi^\varepsilon)(\xi, \eta) = (U^\varepsilon, V^\varepsilon; \Phi^\varepsilon)(\xi, \eta)$ *be a sequence of approximate solutions to the Euler equations* (4.12) *with the Bernoulli relation* (4.3) *in* Ω. *Then there exists a subsequence (still labeled)* $W^\varepsilon(\xi, \eta)$ *that converges a.e. as* $\varepsilon \to 0$ *to a weak solution* $W = (U, V) = \nabla\Phi$ *of the Euler equations* (4.12) *together with the Bernoulli relation* (4.3) *which satisfies the elliptic constraint* (4.6) *a.e. in* Ω.

In particular, when

$$(W^\varepsilon; \Phi^\varepsilon)(\xi, \eta) = (\nabla\Phi^\varepsilon; \Phi^\varepsilon)(\xi, \eta)$$

are exact subsonic solutions to the Euler equations (4.12) with the Bernoulli relation (4.3) in Ω and

$$\Phi^\varepsilon(\xi, \eta) \geq -M \quad \text{for some } M \in (0, \infty) \text{ independent of } \varepsilon,$$

Φ^ε and $\nabla\Phi^\varepsilon$ are uniformly bounded and thus

$$\Phi^\varepsilon(\xi, \eta) \to \Phi(\xi, \eta) \qquad \text{a.e. as } \varepsilon \to 0.$$

Therefore, the sequence of the exact, subsonic solutions $(W^\varepsilon, \Phi^\varepsilon)$ satisfies the Set of Conditions (B) with Φ^ε. Then Theorems 4.1–4.2 imply that the sequence W^ε possesses a subsequence (still denoted by) W^ε, which converges a.e. in Ω to a function W such that $W(\xi, \eta) = \nabla\Phi(\xi, \eta)$ is a weak solution of Eqs. (4.12) with the Bernoulli relation (4.3).

Finally, we remark that, in the isothermal case $\gamma = 1$, we have the same results just as we did in Theorems 4.1–4.2.

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