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Continuity of Information Transport in Surjective Cellular Automata

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Abstract: We introduce a local version of the Shannon entropy in order to describe information transport in spatially extended dynamical systems, and to explore to what extent information can be viewed as a local quantity. Using an appropriately defined information current, this quantity is shown to obey a local conservation law in the case of one-dimensional reversible cellular automata with arbitrary initial measures. The result is also shown to apply to one-dimensional surjective cellular automata in the case of shift-invariant measures. Bounds on the information flow are also shown.

1. Introduction

A number of authors have suggested that information should be viewed as a fundamental physical quantity, starting with the vision of "It from Bit" of Wheeler [27] and the fundamental work on the thermodynamics of computation by Landauer [14] and Bennett [1].

Information theory also has a close relation to the foundations of statistical mechanics, e.g., through the information theoretic formulation introduced by Jaynes [11], where entropy is viewed as a measure of the ignorance of the actual microstate of the system. Information theory and computation theory can also be used to define an entropy for individual microstates in spatially extended systems [17, 28].

In a microscopic view, information or entropy quantified in terms of the Gibbs H-function is a globally conserved quantity due to Liouville's theorem. A natural question to consider is to what extent this statement has a local analogue in spatially extended dynamical systems. This article explores this question for one-dimensional reversible or surjective cellular automata. Precise statements of the notion of conservation of information, and possible extensions of this formalism to other systems, could provide a more solid foundation for the use of information based concepts in different physical systems.

We first introduce a local version of the Shannon entropy. In a one-dimensional system, the local information is defined in terms of the conditional probability of a local state given its left or right infinite context. Information can only be completely localized in a system without correlations. Thus, the measure we introduce is localized to the extent that correlations allow, and reduces to a completely local quantity when correlations vanish. However, even with correlations present, this quantity does obey a local continuity equation with an appropriately defined information current.

For cellular automata, local conservation of information was first proposed by Toffoli [23], who derived a continuity equation for information transport in the case of small perturbations around the uncorrelated equlibrium states of particle conserving reversible cellular automata, such as lattice gases. Here we investigate how these concepts can be applied to a wider class of dynamical systems and to arbitrary measures, and how they can be given a rigorous formulation. We only consider one-dimensional systems; generalizations to systems in higher dimensions will be addressed in future work.

We first consider reversible cellular automata, where the cellular automaton mapping has an inverse. Reversible cellular automata have been used to simulate physical systems, e.g., for microcanonical simulations of spin systems (e.g., [25]), and simulations of fluid dynamics [4, 7], and chemical reactions [2]. They have also been used as illustrative examples of fundamental issues in statistical mechanics [21, 22]. For one-dimensional reversible cellular automata, we show local conservation of information for any initial measure, including measures without shift-invariance.

We also consider the more complicated case of surjective cellular automata, where the global mapping is finite-to-one [8]. In this case, local conservation of information is shown for all shift-invariant measures. For the simple case of permutative rules, we are able to describe the information flow in more detail.

The aim of the article is to explore exactly to what extent information can be viewed as a local quantity in spatially extended systems. The main results show that important aspects of locality remain also in systems with correlations. We also give examples which illustrate the limits of locality in the formalism.

The rest of this article is organized as follows. Section 2 contains background material on shift spaces and cellular automata. In Sect. 3 we introduce a local measure of information and show that it is well-defined. Section 4 contains the main results of the paper. We first define the information current, and prove that information is locally conserved for one-dimensional reversible cellular automata. We then extend this result to surjective cellular automata. In Sect. 5, we give an information theoretic interpretation of the current, and provide bounds on the information flow. We also characterize the information flow in permutative cellular automata, and study some examples illustrating the limits of locality. Section 6 contains conclusions and a discussion.

2. Preliminaries

2.1. The shift space. We study dynamical systems on the space $\mathcal{A}^{\mathbb{Z}}$ of all bi-infinite symbol sequences over a finite set $\mathcal{A}$. For $x \in \mathcal{A}^{\mathbb{Z}}$ we write $x = (x_i)_{i\in\mathbb{Z}}$. The length $j-i+1$ block $(x_i, x_{i+1}, \ldots, x_j)$ of symbols from $\mathcal{A}$ will be written as x_i^j. Likewise, $x_{-\infty}^i = (\ldots, x_{i-1}, x_i)$. The *shift map* σ is defined on $\mathcal{A}^{\mathbb{Z}}$ by $\sigma(x)_i = x_{i+1}$.

A probability *measure* μ on $(\mathcal{A}^{\mathbb{Z}}, \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra, is defined by assigning a probability $\mu(\mathrm{Cyl}(a_i^{i+n}))$ to each cylinder set $\mathrm{Cyl}(a_i^{i+n}) = \{x \in \mathcal{A}^{\mathbb{Z}} : x_i^{i+n} = a_i^{i+n}\}$ in a consistent way, see [26, §0.2]. We will usually write this probability $\mu(a_i^{i+n})$, thus letting a_i^{i+n} represent both the symbol block of length $n+1$ and the cylinder set. It is often convenient to consider the measure μ as defining a discrete, stochastic process $(X_n)_{n=-\infty}^{\infty}$, $X_n \in \mathcal{A}$, with joint distributions given by $\mathrm{Prob}(X_i^j = a_i^j) = \mu(a_i^j)$. A measure is said to be Bernoulli if the coordinate random variables X_i are all independent and identically distributed. The conditional probability $\mu(a_0|a_{-n}^{-1}) = \frac{\mu(a_{-n}^0)}{\mu(a_{-n}^{-1})}$ is the probability that $X_0 = a_0$ given that $X_{-n}^{-1} = a_{-n}^{-1}$.

The measure μ is *shift-invariant* if it satisfies $\mu(\sigma^{-1}(B)) = \mu(B)$ for all cylinder sets B. When μ is shift-invariant, the expectation $E[f]$ of any measurable function f on $\mathcal{A}^{\mathbb{Z}}$ satisfies $E[f] = E[f \circ \sigma]$. The Shannon entropy $h(\mu)$ of a shift-invariant measure μ can be written as

$$h(\mu) = -\lim_{n\to\infty} \sum_{a_{-n}^0 \in \mathcal{A}^{n+1}} \mu(a_{-n}^0) \log \mu(a_0|a_{-n}^{-1}). \tag{1}$$

2.2. Cellular automata. One-dimensional cellular automata (CA) are discrete dynamical systems on $\mathcal{A}^{\mathbb{Z}}$ that commute with the shift σ.

Definition 1. *A* **cellular automaton** $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ *is a dynamical system that can be defined by non-negative integers* l, r *and a map* $f : \mathcal{A}^{l+r+1} \to \mathcal{A}$*, such that*

$$(Fx)_i = f(x_{i-l}, x_{i-l+1}, \ldots, x_{i+r}) \quad \forall i \in \mathbb{Z}. \tag{2}$$

The left and right *radii* of F are the smallest such integers l and r for which there is a block map f (CA rule) that generates F.

Example 1. Let $\mathcal{A} = \{0, 1\}$, and denote by F_1 the simple CA on $\mathcal{A}^{\mathbb{Z}}$ defined by the radii $l = 0$ and $r = 1$ and the block map $f : \mathcal{A}^2 \to \mathcal{A}$ given by $f(x_0, x_1) = x_0 + x_1 \pmod 2$. The global map F_1 can be written as $F_1(x) = x + \sigma(x)$, where addition is coordinate-wise and modulo 2.

For any $n \geq 1$, the block map f can be extended in a natural way to a map $f_n : \mathcal{A}^{l+r+n+1} \to \mathcal{A}^{n+1}$ by putting

$$f_n(x_{-l}^{r+n}) = (f(x_{-l}^r), f(x_{-l+1}^{r+1}), \ldots, f(x_{n-l}^{r+n})). \tag{3}$$

We will omit the subscript n and write f for the block map applied to a block of any length.

For *reversible* CA, F has an inverse map, so that each bi-infinite sequence $y \in \mathcal{A}^{\mathbb{Z}}$ has exactly one preimage under F. The inverse map of a reversible CA is always itself a CA [20], but the inverse CA does not necessarily have the same radii as F (see, e.g., [24]).

Example 2. Denote by F_2 the reversible CA on $\{0, 1, 2\}^{\mathbb{Z}}$ having radii $l = 0$, $r = 1$ and block map given by $f(10) = f(11) = f(12) = 0$, $f(01) = f(20) = f(22) = 1$ and $f(00) = f(02) = f(21) = 2$. The preimage x of a given $y \in \mathcal{A}^{\mathbb{Z}}$ is found by the following procedure. If $y_i = 0$ we must have $x_i = 1$. If $y_i = 1$ then $x_i = 2$ unless $y_{i+1} = 0$, in which case $x_i = 0$. Finally, if $y_i = 2$ then $x_i = 0$ unless $y_{i+1} = 0$, in which case $x_i = 2$. The inverse CA $\tilde{F}_2$ also has $l = 0$ and $r = 1$, but a different block map $\tilde{f}$.

A more general class of CA are the *surjective* ones, where all $y \in \mathcal{A}^{\mathbb{Z}}$ have at least one preimage. The class of surjective CA includes all linear CA [10] and other permutative CA [8].

It is well known that a one-dimensional CA F is surjective if and only if all finite blocks have the same number of pre-images under f [8]. That is, if for all $n \geq 1$ and $y_1^n \in \mathcal{A}^n$ there are exactly $|\mathcal{A}|^{l+r}$ blocks $z_{1-l}^{n+r} \in \mathcal{A}^{l+r+n}$ that satisfy $f(z_{1-l}^{n+r}) = y_1^n$. Furthermore, there is a constant $M(F) \leq |\mathcal{A}|^{l+r}$ such that each bi-transitive $x \in \mathcal{A}^{\mathbb{Z}}$ has exactly $M(F)$ preimages.

For surjective CA one can define *Welch coefficients*. Let $x_1^n \in \mathcal{A}^n$ with $n \geq l + r$. A compatible right extension of x_1^n of length m is a collection $B \subset \mathcal{A}^m$ such that for each $z_1^m \in B$, the (n+m−l−r)-block $f(x_1^n z_1^m)$ is the same. Define the integer $R(F)$ as the maximal number of elements in any compatible right extension of any length m and of any block x_1^n. Define compatible left extensions and $L(F)$ in the same way. The coefficients $L(F)$ and $R(F)$ are finite, and satisfy the relation $L(F) \cdot M(F) \cdot R(F) = |\mathcal{A}|^{l+r}$ [8, Th. 14.9].

3. Local Information

The intent of introducing a local information quantity is to measure how much information that is located at each position of an infinite symbol sequence generated by some stochastic process. However, the correlations in such symbol sequences can in general be arbitrarily long, and it is impossible for information to be completely localized. The natural approach is therefore to define the local information as a limit which converges to a local analogue of the Shannon entropy as more and more distant neighbours are taken into account. While the Shannon entropy is limited to shift-invariant measures, we can define left local information for any measure.

Definition 2. *Let μ be a measure on $\mathcal{A}^{\mathbb{Z}}$. The* **left local information** *at coordinate i of $x \in \mathcal{A}^{\mathbb{Z}}$ with respect to μ is given by*

$$S_{\mathrm{L}}(x; i; \mu) = -\lim_{n\to\infty} \log \mu(x_i | x_{i-n}^{i-1}). \tag{4}$$

The quantity $-\log \mu(x_i | x_{i-n}^{i-1})$ is the information gained from the symbol at position i when only knowledge of the n left symbols is assumed. If, and only if, μ is Markov there is a fixed n such that $S_{\mathrm{L}}(x; i; \mu) = -\log \mu(x_i | x_{i-n}^{i-1})$. We will often use the intuitive notation $-\log \mu(x_i | x_{-\infty}^{i-1})$ for $S_{\mathrm{L}}(x; i; \mu)$.

The following theorem ensures that the left local information with respect to μ is a well-defined function on the probability space $(\mathcal{A}^{\mathbb{Z}}, \mathcal{B}, \mu)$.

Theorem 1. *For each $i \in \mathbb{Z}$, $-\log \mu(x_i | x_{i-n}^{i-1})$ converges μ-almost everywhere and in $L^1(\mu)$. Consequently, for each fixed measure μ, $S_{\mathrm{L}}(x; i; \mu) \in L^1(\mu)$.*

The validity of the theorem follows from the martingale convergence theorem, see, e.g., [12, Th. 3.1.10]. Note that from L^1-convergence and (1) it follows that $E[S_{\mathrm{L}}(x; i; \mu)] = h(\mu)$ for all i in the shift-invariant case.

The local information $S_{\mathrm{L}}(x; i; \mu)$ depends on the measure μ. However, for a shift-invariant measure μ the left local information at position i of x can be recovered with probability one from $x_{-\infty}^{i}$ only by considering the empirical measure ν_x obtained from the frequencies of finite blocks in $x_{-\infty}^{i}$,

$$\nu_x(a_1^n) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} 1_{\mathrm{Cyl}(a_1^n)}(\sigma^{i-n-k} x). \tag{5}$$

If μ is ergodic, then $\nu_x = \mu$ a.e. However, even when μ is only shift-invariant it suffices to look at the local information with respect to ν_x.

Theorem 2. *Let μ be a shift-invariant measure on $\mathcal{A}^{\mathbb{Z}}$ and ν_x the empirical measure generated by $x_{-\infty}^i$. Then $S_{\mathrm{L}}(x; i; \mu) = S_{\mathrm{L}}(x; i; \nu_x)$ μ-a.e.*

Proof. The result follows since the infinite history determines with probability one which ergodic component of μ x is generated by, see Lemma 8.6.2. in [6]. □

We can also define the *right local information* at coordinate i of x with respect to μ as

$$S_{\mathrm{R}}(x; i; \mu) = -\lim_{n\to\infty} \log \mu(x_i | x_{i+1}^{i+n}). \tag{6}$$

All results we show for the left information will have corresponding results for the right information. However, although the left and right information have the same expectation for all shift-invariant measures, they are not equal nor do they in general have the same probability distribution. This is exemplified by the Markov measure on $\{0, 1, 2\}^{\mathbb{Z}}$ defined by the following non-zero transition probabilities: $p(0|0) = p(1|0) = \frac{1}{2}$; $p(0|1) = p(2|1) = \frac{1}{2}$; $p(0|2) = p(1|2) = \frac{1}{2}$.

4. Information Transport

In this section we investigate the transport of local information in the time-evolution of a one-dimensional surjective cellular automaton. We show that the left local information satisfies a continuity equation involving an information current J_{L}, and supply an expression for this current.

Let μ^0 be a measure on $\mathcal{A}^{\mathbb{Z}}$. The measure $F(\mu^0) = \mu^0 \circ F^{-1}$ gives the joint distributions of the stochastic process $(Y_i)_{i\in\mathbb{Z}}$ with $Y_i = f(X_{i-l}^{i+r})$ when $(X_i)_{i\in\mathbb{Z}}$ has joint distributions given by μ^0. Denote $\mu^0 \circ F^{-1}$ by μ^1 and, more generally, set $\mu^t = \mu^0 \circ F^{-t}$. The block probabilities of μ^1 can be calculated from

$$\mu^1(y_0^n) = \sum_{z_{-l}^{n+r} \in f^{-1}(y_0^n)} \mu^0(z_{-l}^{n+r}). \tag{7}$$

It is well known that $h(\mu^1) = h(\mu^0)$ whenever F is surjective and μ^0 is shift-invariant (if F is non-surjective, this relation is replaced by $h(\mu^1) \le h(\mu^0)$), see, e.g., [16]. Our goal is to prove the much stronger result that the local information obeys a local continuity equation under the time evolution of the CA. This is an equation of the form

$$\Delta_t S_{\mathrm{L}} + \Delta_i J_{\mathrm{L}} = 0, \tag{8}$$

where $J_{\mathrm{L}}(x; i; \mu)$ is the information current. The operator Δ is the forward difference operator, so that

$$\begin{aligned} \Delta_t S_{\mathrm{L}}(x; i; \mu^t) &= S_{\mathrm{L}}(F(x); i; \mu^{t+1}) - S_{\mathrm{L}}(x; i; \mu^t), \\ \Delta_i J_{\mathrm{L}}(x; i; \mu^t) &= J_{\mathrm{L}}(x; i+1; \mu^t) - J_{\mathrm{L}}(x; i; \mu^t). \end{aligned} \tag{9}$$

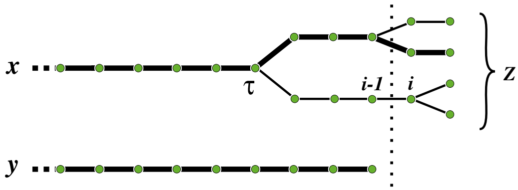

Fig. 1. An illustration of $Z = Z(x_{-\infty}^{i+r-1})$ and $\tau = \tau(x_{-\infty}^{i+r-1})$. In this case $r = 2$, $|Z| = 4$ and $\tau = i - 4$. The circles represent symbols in $\mathcal{A}$. These are connected by lines to semi-infinite sequences which all map to the same $y_{-\infty}^{i-1}$ and coincide with x at all $j \le \tau$.

With these definitions, $J_{\mathrm{L}}(x; i; \mu^t)$ can be interpreted as the information flow from position $i - 1$ to position i generated by applying the CA. Note that the local information of $F(x)$ is taken with respect to a different measure than x, unless μ^t happens to be invariant for the CA.

For a semi-infinite sequence $x_{-\infty}^i$, define $Z(x_{-\infty}^i)$ as the set of all semi-infinite sequences that have the same image and the same tail as $x_{-\infty}^i$ (see Fig. 1):

Definition 3. *For $x \in \mathcal{A}^{\mathbb{Z}}$ and a surjective CA F, define the sets $Z(x_{-\infty}^i)$ as*

$$Z(x_{-\infty}^i) = \{z_{-\infty}^i : f(z_{-\infty}^i) = f(x_{-\infty}^i) \text{ and } \exists\, j \le i \text{ such that } z_{-\infty}^j = x_{-\infty}^j\}.$$

Note that $|Z(x_{-\infty}^i)| \le R(F)$ for all x by the definition of the Welch coefficient $R(F)$. Define $\tau(x_{-\infty}^i)$ as the largest index less than $i - r$ for which all sequences in $Z(x_{-\infty}^i)$ coincide (recall that r is the right radius of F):

Definition 4. *For $x \in \mathcal{A}^{\mathbb{Z}}$, define $\tau(x_{-\infty}^i) \in \mathbb{Z}$ as*

$$\tau(x_{-\infty}^i) = \max_j \{j : j < i - r, \text{ and } z_{-\infty}^j = x_{-\infty}^j \;\forall\, z_{-\infty}^i \in Z(x_{-\infty}^i)\}.$$

We are now ready to define the information current.

Definition 5. *Let F be a surjective one-dimensional CA with right radius r, and μ a measure on $\mathcal{A}^{\mathbb{Z}}$. Put $Z = Z(x_{-\infty}^{i+r-1})$ and $\tau = \tau(x_{-\infty}^{i+r-1})$. Define the* **left information current** *at coordinate i of x with respect to μ and F as*

$$J_{\mathrm{L}}(x; i; \mu) = -\log \mu(x_{\tau+1}^{i-1} | x_{-\infty}^{\tau}) + \log \sum_{Z} \mu(z_{\tau+1}^{i+r-1} | x_{-\infty}^{\tau}). \tag{10}$$

The quantities $\mu(z_{\tau+1}^{i-1} | x_{-\infty}^{\tau})$ are defined as $\lim_{n\to\infty} \mu(z_{\tau+1}^{i-1} | x_{\tau-n}^{\tau})$. Since J_{L} is constructed entirely from conditional probabilities of this type, an analogue to Theorem 2 yields

$$J_{\mathrm{L}}(x; i; \mu) = J_{\mathrm{L}}(x; i; \nu_x) \quad \mu\text{-a.e.} \tag{11}$$

Note that $\tau < i - 1$, by the requirement that $\tau(x_{-\infty}^i) < i - r$ included in Def. 4. In the case of a reversible CA, the existence of an inverse CA ensures that τ is bounded below. Let $\tilde{r}$ be the right radius of the inverse CA. Then $\tau \ge i - 1 - \tilde{r}$ unless $\tilde{r} = 0$, in which case $\tau = i - 2$. For non-reversible CA, τ is in general unbounded but always finite.

It remains to show that $J_{\mathrm{L}}(x; i; \mu)$ is well defined as a function on $(X, \mathcal{B}, \mu)$. This is ascertained by the following lemma, whose proof follows from the martingale convergence theorem.

Lemma 1. *For any measure μ on $\mathcal{A}^{\mathbb{Z}}$,*

$$\mu(\{x : \lim_{n\to\infty} \mu(a_{-k}^{-1}|x_{-n}^{-k-1}) \text{ exists for all } k \geq 0 \text{ and all } a_{-k}^{-1} \in \mathcal{A}^k\}) = 1.$$

It is furthermore the case that $J_{\mathrm{L}}(x; i; \mu) \in L^1(\mu)$, see Theorem 5 in Sect. 5.3.

We now proceed to present Theorems 3 and 4, which are the main results of the paper. The first theorem states that for reversible CA the continuity equation is valid for all initial measures.

Theorem 3. *Let F be a reversible one-dimensional CA, and μ a measure on $\mathcal{A}^{\mathbb{Z}}$. Then $\Delta_t S_{\mathrm{L}}(x; i; \mu) + \Delta_i J_{\mathrm{L}}(x; i; \mu) = 0$ for all $i \in \mathbb{Z}$ μ-a.e.*

For a general surjective CA, the requirement of μ being shift-invariant is necessary to ensure the validity of the continuity equation.

Theorem 4. *Let F be a surjective one-dimensional CA, and μ a shift-invariant measure on $\mathcal{A}^{\mathbb{Z}}$. Then $\Delta_t S_{\mathrm{L}}(x; i; \mu) + \Delta_i J_{\mathrm{L}}(x; i; \mu) = 0$ for all $i \in \mathbb{Z}$ μ-a.e.*

Example 3 in Sect. 5 shows that the continuity equation as defined above can fail to be valid if μ is not shift-invariant and F is surjective without being reversible.

Note that if one of the theorems is valid for a CA F together with an initial measure μ^0, then the continuity equation will be satisfied at all time steps of the iteration by F.

Proof (*of Theorem* 3). We first show that it is sufficient to consider the case $r = 0$. Here, and in the rest of the proof, we look at the initial measure μ^0 and its image μ^1.

Assume that Theorem 3 is valid for CA with $r = 0$, and let F have right radius r. There exist a CA G with $r = 0$ such that $F = \sigma^r \circ G$. We have $S_{\mathrm{L}}(Fx; i; \mu^1) = S_{\mathrm{L}}(Gx; i + r; \mu^1)$, since $F(\mu^0) = G(\mu^0) = \mu^1$. Write $\tau_1 = \tau(x_{-\infty}^{i+r-1})$, $\tau_2 = \tau(x_{-\infty}^{i+r})$, $Z_1 = Z(x_{-\infty}^{i+r-1})$ and $Z_2 = Z(x_{-\infty}^{i+r})$. Using the formula for J_{L}, we obtain

$$\begin{aligned} S_{\mathrm{L}}(Gx; i + r; \mu^1) = &- \log \mu^0(x_{i+r}|x_{-\infty}^{i+r-1}) \\ &- \log \mu^0(x_{\tau_1+1}^{i+r-1}|x_{-\infty}^{\tau_1}) + \log \sum_{Z_1} \mu^0(z_{\tau_1+1}^{i+r-1}|x_{-\infty}^{\tau_1}) \\ &+ \log \mu^0(x_{\tau_2+1}^{i+r}|x_{-\infty}^{\tau_2}) - \log \sum_{Z_2} \mu^0(z_{\tau_2+1}^{i+r}|x_{-\infty}^{\tau_2}). \end{aligned} \tag{12}$$

Since $\tau_1 \leq i - 2$ and $\tau_2 \leq i - 1$ by definition, we can write

$$\log \mu^0(x_{\tau_2+1}^{i+r}|x_{-\infty}^{\tau_2}) = \log \mu^0(x_{\tau_2+1}^{i}|x_{-\infty}^{\tau_2}) + \log \mu^0(x_{i+1}^{i+r}|x_{-\infty}^{i}), \tag{13}$$

and

$$- \log \mu^0(x_{i+r}|x_{-\infty}^{i+r-1}) - \log \mu^0(x_{\tau_1+1}^{i+r-1}|x_{-\infty}^{\tau_1})$$
$$= - \log \mu^0(x_{\tau_1+1}^{i-1}|x_{-\infty}^{\tau_1}) - \log \mu^0(x_i|x_{-\infty}^{i-1}) - \log \mu^0(x_{i+1}^{i+r}|x_{-\infty}^{i}). \tag{14}$$

Substituting (13) and (14) into (12) gives the correct continuity equation for F.

For the rest of the proof we assume that F has right radius $r = 0$, and left radius $l \geq 0$. We look only at coordinate $i = 0$. This leads to no loss of generality. Call the inverse CA $\tilde{F}$, and let $\tilde{F}$ have left radius $\tilde{l}$ and right radius $\tilde{r}$. Sequences at time $t = 0$ are denoted by x or z and sequences at time $t = 1$ by y.

We first define the joint measure ν of two consecutive time steps. Let ν be the measure on $(\mathcal{A} \times \mathcal{A})^{\mathbb{Z}}$ defined by the block probabilities

$$\nu(x_i^j, y_i^j) = \mu^0(\{z_{i-l}^j \in \mathcal{A}^{j-i+l+1} | z_i^j = x_i^j \quad \text{and} \quad f(z_{i-l}^j) = y_i^j\}). \tag{15}$$

It is easy to show that ν actually is a measure and that ν is shift-invariant if μ^0 is shift-invariant. By summing over all possible y_i^j or x_i^j we obtain from the definition that $\nu(x_i^j) = \mu^0(x_i^j)$ and $\nu(y_i^j) = \mu^1(y_i^j)$.

We will need the following lemma, which follows from the martingale convergence theorem.

Lemma 2. *Let ν be a measure on $\mathcal{A}_1^{\mathbb{Z}} \times \mathcal{A}_2^{\mathbb{Z}}$, where each $\mathcal{A}_i$ is a finite set. Let $(x, y) \in \mathcal{A}_1^{\mathbb{Z}} \times \mathcal{A}_2^{\mathbb{Z}}$. Then there is a $g \in L^1(\nu)$ such that for any $k \in \mathbb{Z}$,*

$$\lim_{n\to\infty} \nu(x_0, y_0 | x_{-n-k}^{-1}, y_{-n}^{-1}) = g \quad \nu\text{-a.e.} \tag{16}$$

Let $y = F(x)$. From the definition of local information we have

$$\begin{aligned} S_{\mathrm{L}}(x; 0; \mu^0) &= - \lim_{n\to\infty} \log \mu^0(x_0|x_{-n}^{-1}) \\ &= \lim_{n\to\infty} \left(\log \frac{\nu(x_0, y_0|x_{-n}^{-1}, y_{-n+l}^{-1})}{\mu^0(x_0|x_{-n}^{-1})} - \log \nu(x_0, y_0|x_{-n}^{-1}, y_{-n+l}^{-1}) \right) \\ &= \lim_{n\to\infty} \log \frac{\nu(y_{-n+l}^0|x_{-n}^0)}{\nu(y_{-n+l}^{-1}|x_{-n}^{-1})} - \lim_{n\to\infty} \log \nu(x_0, y_0|x_{-n}^{-1}, y_{-n+l}^{-1}) \\ &= - \log \nu(x_0, y_0|x_{-\infty}^{-1}, y_{-\infty}^{-1}) \end{aligned} \tag{17}$$

by virtue of Lemma 2 and the fact that $\nu(y_{-n+l}^0|x_{-n}^0) = \nu(y_{-n+l}^{-1}|x_{-n}^{-1}) = 1$ for all $n > l$, since y_{-n+l}^0 in this case is uniquely determined by x_{-n}^0 through the local map f and likewise for y_{-n+l}^{-1} and x_{-n}^{-1}. A similar treatment of $S_{\mathrm{L}}(y; 0; \mu^1)$ yields

$$S_{\mathrm{L}}(y; 0; \mu^1) = \lim_{n\to\infty} \log \frac{\nu(x_{-n+\tilde{l}}^0|y_{-n}^0)}{\nu(x_{-n+\tilde{l}}^{-1}|y_{-n}^{-1})} - \log \nu(x_0, y_0|x_{-\infty}^{-1}, y_{-\infty}^{-1}). \tag{18}$$

When taking the difference $\Delta_t S_{\mathrm{L}}$, the last term is canceled out, so

$$\Delta_t S_{\mathrm{L}}(x; 0; \mu^0) = \lim_{n\to\infty} \log \nu(x_{-n+\tilde{l}}^0|y_{-n}^0) - \lim_{n\to\infty} \log \nu(x_{-n+\tilde{l}}^{-1}|y_{-n}^{-1}).$$

To conclude the proof, we show that

$$- \lim_{n\to\infty} \log \nu(x_{-n+\tilde{l}}^{-1}|y_{-n}^{-1}) = J_{\mathrm{L}}(x; 0; \mu^0) \tag{19}$$

through a sequence of transformations. Firstly,

$$\log \nu(x_{-n+\tilde{l}}^{-1}|y_{-n}^{-1}) = \log \nu(x_{-n+\tilde{l}}^{-\tilde{r}-1}|y_{-n}^{-1}) + \log \nu(x_{-\tilde{r}}^{-1}|x_{-n+\tilde{l}}^{-\tilde{r}-1}, y_{-n}^{-1}). \tag{20}$$

The first term on the right hand side is zero, since $x_{-n+\tilde{l}}^{-\tilde{r}-1}$ is uniquely determined by y_{-n}^{-1} through the local map $\tilde{f}$ of the inverse CA. For the second term, a generalization of Lemma 2 gives

$$\lim_{n\to\infty} \log \nu(x_{-\tilde{r}}^{-1}|x_{-n+\tilde{l}}^{-\tilde{r}-1}, y_{-n}^{-1}) = \lim_{n\to\infty} \log \nu(x_{-\tilde{r}}^{-1}|x_{-n-l}^{-\tilde{r}-1}, y_{-n}^{-1}). \tag{21}$$

Furthermore, for any events A, B and C in a probability space it is true that $\nu(A|BC) = \frac{\nu(C|AB)\nu(A|B)}{\nu(C|B)}$. Let $A = x_{-\tilde{r}}^{-1}$, $B = x_{-n-l}^{-\tilde{r}-1}$ and $C = y_{-n}^{-1}$. Then $\nu(C|AB) = 1$. Thus,

$$\log \nu(x_{-\tilde{r}}^{-1}|x_{-n-l}^{-\tilde{r}-1}, y_{-n}^{-1}) = \log \mu^0(x_{-\tilde{r}}^{-1}|x_{-n-l}^{-\tilde{r}-1}) - \log \nu(y_{-n}^{-1}|x_{-n-l}^{-\tilde{r}-1}). \tag{22}$$

By the definition of ν, the last term can be written as

$$-\log \nu(y_{-n}^{-1}|x_{-n-l}^{-\tilde{r}-1}) = -\log \sum_{Z(x_{-\infty}^{-1})} \mu^0(z_{-\tilde{r}}^{-1}|x_{-n-l}^{-\tilde{r}-1}). \tag{23}$$

Substituting (23) into (22) and taking the limit $n \to \infty$ we arrive at the equation for $J_{\mathrm{L}}(x; 0; \mu^0)$ presented in Def. 5. $\square$

For general surjective CA, there is no inverse CA and in general several possible preimages. As a consequence, the proof of Theorem 4 requires a different approach.

Proof (*of Theorem* 4). By the same argument as in the proof of Theorem 3 it suffices to consider right radius $r = 0$. As before, we only look at coordinate $i = 0$ and consider the initial measure μ^0 and its image μ^1.

Let $y = F(x)$, and define

$$q(x) = S_{\mathrm{L}}(y; 0; \mu^1) - S_{\mathrm{L}}(x; 0; \mu^0) + J_{\mathrm{L}}(x; 1; \mu^0) - J_{\mathrm{L}}(x; 0; \mu^0). \tag{24}$$

Our goal is to prove that $q(x) = 0$ μ^0-a.e, or equivalently that $\mathrm{E}[|q|] = 0$. To prove this we will introduce a sequence q_k of approximations to q which are measurable with respect to finite parts of the history.

First, define the following equivalence relation on $\mathcal{A}^{l+n+1}$ for $n \geq 0$:

$$x_{-l}^{n} \sim z_{-l}^{n} \text{ iff } f(x_{-l}^{n}) = f(z_{-l}^{n}) \quad \text{and} \quad x_{-l}^{-1} = z_{-l}^{-1}. \tag{25}$$

That is, two blocks in $\mathcal{A}^{l+n+1}$ are equivalent if they have the same image under f and agree on the first l coordinates. Denote the equivalence class containing z_{-l}^{n} by $[z_{-l}^{n}]$. For an $x \in \mathcal{A}^{\mathbb{Z}}$ we will in particular look at the equivalence classes $[x_{-k-l}^{-1}]$ for $k \geq 1$. For each $k \geq 1$ we have the inclusion

$$Z(x_{-\infty}^{-1}) \supseteq \{x_{-\infty}^{-l-k-1} z_{-l-k}^{-1} : z_{-l-k}^{-1} \in [x_{-l-k}^{-1}]\}. \tag{26}$$

Recall that $\tau(x_{-\infty}^{-1})$ is largest index such that all sequences in $Z(x_{-\infty}^{-1})$ agree on and to the left of $\tau(x_{-\infty}^{-1})$. Therefore, for all $k \geq -\tau(x_{-\infty}^{-1}) - 1$ Eq. (26) is an equality.

Define $\tau^k(x)$ for $k \geq 1$ as the analogue of τ obtained when considering $[x_{-k-l}^{-1}]$ rather than $Z(x_{-\infty}^{-1})$,

$$\tau^k(x) = \max_j\{j \leq -2, \quad \text{and} \quad z_{-k-l}^{-1} \in [x_{-k-l}^{-1}] \Rightarrow z_{-k-l}^{j} = x_{-k-l}^{j}\}. \tag{27}$$

Note that $\tau^k(x) = \tau(x_{-\infty}^{-1})$ iff $k \geq -\tau(x_{-\infty}^{-1}) - 1$.

We now define finite versions of the current $J_{\mathrm{L}}(x; 0; \mu^0)$ and of $q(x)$. Write τ^k for $\tau^k(x)$ and put for $k \geq 1$,

$$J_{\mathrm{L}}^k(x) = -\log \mu^0(x_{\tau^k+1}^{-1} | x_{-k-l}^{\tau^k}) + \log \sum_{z_{-k-l}^{-1} \in [x_{-k-l}^{-1}]} \mu^0(z_{\tau^k+1}^{-1} | x_{-k-l}^{\tau^k}), \tag{28}$$

and

$$q_k(x) = -\log \mu^1(y_0 | y_{-k}^{-1}) + \log \mu^0(x_0 | x_{-k-l}^{-1}) + J_{\mathrm{L}}^{k+1}(\sigma x) - J_{\mathrm{L}}^k(x). \tag{29}$$

It is straightforward to check that $q_k(x) \to q(x)$ a.e. by using the properties of $[x_{-k-l}^{-1}]$ and τ^k discussed above.

Proceeding, we can write

$$\int |q| d\mu^0 \leq \int |q_k| d\mu^0 + \int |q - q_k| d\mu^0. \tag{30}$$

The theorem will follow if we can prove that both integrals on the right-hand side converge to zero. In order to do this we investigate the stochastic process $(g_n)_{n \geq 0}$ on $\mathcal{A}^{\mathbb{Z}}$ defined by

$$g_n(x) = \frac{\mu^1(y_0^n)}{\mu^0([x_{-l}^n])}, \quad \text{where } y_0^n = f(x_{-l}^n). \tag{31}$$

Here, $\mu^0([x_{-l}^n])$ means $\sum_{z_{-l}^n \in [x_{-l}^n]} \mu^0(z_{-l}^n)$. The interest in this process is due to the relationship

$$\log g_{n-1}(x) - \log g_n(x) = q_n(\sigma^n x). \tag{32}$$

We will prove that the process $(g_n)_{n \geq 0}$ is a supermartingale with respect to a filtration that we now will describe.

Let $\mathcal{P}^n$ be the partition of $\mathcal{A}^{\mathbb{Z}}$ defined by the equivalence relation (25) on $\mathcal{A}^{l+n+1}$. That is, the elements of $\mathcal{P}^n$ are the sets $P_{[u]} = \{x : x_{-l}^n \in [u]\}$ for all equivalence classes $[u]$ of $\mathcal{A}^{l+n+1}$. Let $\mathcal{F}_n = \sigma(\mathcal{P}^n)$. Then, $\mathcal{F}_n$ is the σ-algebra generated by g_n. We have to show that $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ for all n. This follows if we can show that the partition $\mathcal{P}^{n+1}$ is a refinement of $\mathcal{P}^n$. Consider a general element $P_{[w_{-l}^n]}$ of $\mathcal{P}^n$. We claim that

$$P_{[w_{-l}^n]} = \bigcup_{z_{-l}^n \in [w_{-l}^n]} \left(\bigcup_{a \in \mathcal{A}} P_{[z_{-l}^n a]} \right). \tag{33}$$

If $x \in P_{[w_{-l}^n]}$, then $P_{[x_{-l}^{n+1}]}$ clearly is a member of the double union. Conversely, if x is in some $P_{[z_{-l}^n a]}$ in the union, then $x_{-l}^{-1} = z_{-l}^{-1} = w_{-l}^{-1}$ and $f(x_{-l}^n) = f(z_{-l}^n) = f(w_{-l}^n)$. Thus, $x \in P_{[w_{-l}^n]}$. The claim follows, and $(\mathcal{F}_n \colon n \geq 0)$ is a filtration.

To prove that g_n is a supermartingale with respect to this filtration we show that $E[g_{n+1}|\mathcal{F}_n] \leq g_n$. Since each sub-$\sigma$-algebra $\mathcal{F}_n$ is finite it suffices to show $\int g_{n+1} d\mu^0 \leq \int g_n d\mu^0$ over any $P_{[u]} \in \mathcal{P}^n$. We find that

$$\int_{P_{[u]}} g_n \, d\mu^0 = \sum_{x_{-l}^n \in [u]} \mu^0(x_{-l}^n) \frac{\mu^1(y_0^n)}{\mu^0([x_{-l}^n])} = \mu^1(y_0^n). \tag{34}$$

For g_{n+1}, we split the integral into cylinder sets where g_{n+1} is constant:

$$\begin{aligned} \int_{P_{[u]}} g_{n+1} \, d\mu^0 &= \sum_{x_{-l}^n \in [u]} \sum_{a \in \mathcal{A}} \int_{\mathrm{Cyl}(x_{-l}^n a)} g_{n+1}(x) d\mu^0 \\ &= \sum_{x_{-l}^n \in [u]} \sum_{a \in \mathcal{A}} \mu^0(x_{-l}^n a) \frac{\mu^1(y_0^n f(x_{n-l+1}^n a))}{\mu^0([x_{-l}^n a])} \\ &= \sum_{b \in \mathcal{A}} \psi_b \cdot \mu^1(y_0^n b), \end{aligned} \tag{35}$$

where

$$\psi_b = \sum_{x_{-l}^n \in [u]} \sum_{a \in \mathcal{A}} \frac{\mu^0(x_{-l}^n a)}{\mu^0([x_{-l}^n a])} \cdot 1_{\{f(x_{n-l+1}^n a) = b\}}(x). \tag{36}$$

We claim that for each b, the quantity ψ_b is equal either to 0 or to 1. Fix a b. First note that all blocks $x_{-l}^n a$ in the double sum in (36) that satisfy $f(x_{n-l+1}^n a) = b$ must generate the same equivalence class $[x_{-l}^n a] = [v]$. Conversely, each block $z_{-l}^{n+1} \in [v]$ is an element of the double sum, since it will satisfy $z_{-l}^n \in [x_{-l}^n] = [u]$. Consequently, $\psi_b = 1$ by summation of the fractions and cancelation. The exception is the case where no $x_{-l}^n \in [u]$ can be extended with one symbol to the right such that the new block maps to $y_0^n b$ under f. For such b, $\psi_b = 0$. We can conclude that

$$\int_{P_{[u]}} g_{n+1} \, d\mu^0 \leq \sum_{b \in \mathcal{A}} \mu^1(y_0^n b) = \mu^1(y_0^n) = \int_{P_{[u]}} g_n \, d\mu^0. \tag{37}$$

Finally, it is easy to prove that $E[|g_0|] \leq |\mathcal{A}|^{l+1} < \infty$. This finalizes the proof that g_n is a supermartingale, so by the martingale convergence theorem g_n converges a.e. to a $g \in L^1(\mu^0)$.

We now proceed to show L^1-convergence of $\log g_n$. A family $(f_n)_{n \geq 0}$ of measurable functions is uniformly integrable if ([5, Sect. 1.14])

$$\lim_{M \to \infty} \sup_n \int (|f_n| - M)^+ d\mu = 0. \tag{38}$$

We claim that $(\log g_n)_{n \geq 0}$ is an uniformly integrable family. Note first that

$$\log g_n(x) > t \quad \Leftrightarrow \quad \mu^0([x_{-l}^n]) < 2^{-t} \cdot \mu^1(y_0^n). \tag{39}$$

Define $A_{n,t} \subset \mathcal{A}^{l+n+1}$ as $A_{n,t} = \{x_{-l}^n : \mu^0([x_{-l}^n]) < 2^{-t} \cdot \mu^1(f(x_{-l}^n))\}$. We obtain

$$\mu^0(\{\log g_n > t\}) = \sum_{A_{n,t}} \mu^0(x_{-l}^n) \leq \sum_{A_{n,t}} 2^{-t} \mu^1(f(x_{-l}^n))$$
$$\leq \sum_{\mathcal{A}^{l+n+1}} 2^{-t} \mu^1(f(x_{-l}^n)) = 2^{-t} \cdot |\mathcal{A}|^l, \tag{40}$$

for all n. A simple application of Fubini's theorem yields

$$\sup_n \int (|\log g_n| - M)^+ d\mu^0 = \sup_n \int_M^\infty \mu^0(\{\log g_n > t\})\, dt$$
$$\leq |\mathcal{A}|^l \int_M^\infty 2^{-t}\, dt = 2^{-M} \cdot \frac{|\mathcal{A}|^l}{\ln 2}. \tag{41}$$

Thus, uniform integrability is satisfied and $\lim_{n\to\infty} \mathrm{E}[|\log g - \log g_n|] = 0$. From (32) and shift-invariance,

$$\lim_{n\to\infty} \mathrm{E}[|q_n|] = \lim_{n\to\infty} \mathrm{E}[|q_n| \circ \sigma^n] = \lim_{n\to\infty} \mathrm{E}[|\log g_{n-1} - \log g_n|] = 0. \tag{42}$$

Hence, the first integral on the right-hand side in (30) converges to zero. Regarding the second integral on the right-hand side, we know that $\lim_{n\to\infty} q_n = q$ a.e. Thus, if we can prove that $(q_n)_{n\geq 0}$ is itself a uniformly integrable family then we are done. From (32) and the fact that $\log g_n \geq 0$ for all n it follows that

$$\{|q_n| \circ \sigma^n > M\} \subseteq \{\log g_{n-1} > M\} \bigcup \{\log g_n > M\}. \tag{43}$$

We can conclude that

$$\sup_n \int (|q_n| - M)^+ d\mu^0 = \sup_n \int (|q_n \circ \sigma^n| - M)^+ d\mu^0 \leq 2^{-M+1} \cdot \frac{|\mathcal{A}|^l}{\ln 2}, \tag{44}$$

and the result follows. □

A corresponding continuity equation can also be written for right local information. The right variants of the set Z and variable τ are defined by

$$Z(x_i^\infty) = \{z_i^\infty : f(z_i^\infty) = f(x_i^\infty) \quad \text{and} \quad \exists\ j \geq i \text{ such that } z_j^\infty = x_j^\infty\},$$
$$\tau(x_i^\infty) = \min_j\{j : j > i - l, \quad \text{and} \quad z_j^\infty = x_j^\infty\ \forall\ z_i^\infty \in Z(x_i^\infty)\}.$$

Put $Z = Z(x_{i-l}^\infty)$ and $\tau = \tau(x_{i-l}^\infty)$, and define the right information current at coordinate i of x with respect to μ and F by

$$J_{\mathrm{R}}(x; i; \mu) = \log \mu(x_i^{\tau-1} | x_\tau^\infty) - \log \sum_Z \mu(z_{i-l}^{\tau-1} | x_\tau^\infty). \tag{45}$$

Then, $J_{\mathrm{R}}(x; i; \mu)$ satisfies the continuity equation $\Delta_t S_{\mathrm{R}}(x; i; \mu) + \Delta_i J_{\mathrm{R}}(x; i; \mu) = 0$ at all $i \in \mathbb{Z}\mu$-a.e., under the same conditions as in Theorem 3 or Theorem 4.

5. Further Aspects of Information Transport

5.1. Information Theoretic Interpretation. We now describe a way of decomposing $J_{\mathrm{L}}(x; i)$ into

$$J_{\mathrm{L}}(x; i) = J_{\mathrm{L}}^{+}(x; i) - J_{\mathrm{L}}^{-}(x; i), \tag{46}$$

with $J_{\mathrm{L}}^{+}, J_{\mathrm{L}}^{-} \geq 0$, such that J_{L}^{+} has a natural interpretation in terms of information flowing to the right between coordinates $i-1$ and i, and J_{L}^{-} in terms of information flowing to the left. Here, and in the rest of the section, we omit μ from the notation in J_{L} and S_{L} when considering some fixed measure μ.

First recall the definition of $Z(x_{-\infty}^{i})$ and define

$$Z_0(x_{-\infty}^{i}) = \left\{ z_{-\infty}^{i} \in Z(x_{-\infty}^{i}) : z_{-\infty}^{i-r} = x_{-\infty}^{i-r} \right\}. \tag{47}$$

We will consider the set $Z_0(x_{-\infty}^{i+r-1})$, which consists of the semi-infinite sequences which have the same image as $x_{-\infty}^{i+r-1}$ and coincide with $x_{-\infty}^{i+r-1}$ up to index $i-1$. Define J_{L}^{+} and J_{L}^{-} at coordinate $i=0$, with $\tau = \tau(x_{-\infty}^{r-1})$, $Z = Z(x_{-\infty}^{r-1})$ and $Z_0 = Z_0(x_{-\infty}^{r-1})$, as

$$J_{\mathrm{L}}^{-}(x; 0) = -\log \sum_{Z_0} \mu(z_0^{r-1} | x_{-\infty}^{-1}), \tag{48}$$

$$J_{\mathrm{L}}^{+}(x; 0) = -\log \frac{\sum_{Z_0} \mu(z_{\tau+1}^{r-1} | x_{-\infty}^{\tau})}{\sum_{Z} \mu(z_{\tau+1}^{r-1} | x_{-\infty}^{\tau})}. \tag{49}$$

It is straightforward to confirm that $J_{\mathrm{L}}^{-}(x; 0)$ and $J_{\mathrm{L}}^{+}(x; 0)$ are non-negative and satisfy (46).

We first examine J_{L}^{-}. Using the joint measure ν defined in (15) we can write

$$J_{\mathrm{L}}^{-}(x; 0) = -\lim_{n\to\infty} \log \frac{\nu(x_{-n}^{-1}, y_{-r}^{-1})}{\nu(x_{-n}^{-1})} = -\log \nu(y_{-r}^{-1} | x_{-\infty}^{-1}). \tag{50}$$

The equation states that $J_{\mathrm{L}}^{-}(x; 0)$ is the information gained by observing y_{-r}^{-1} when having knowledge of $x_{-\infty}^{-1}$. This is what one should expect. Indeed, since $x_{-\infty}^{-1}$ is known, the semi-infinite sequence $y_{-\infty}^{-r-1}$ is uniquely determined by the CA map. Hence, all uncertainty about $y_{-\infty}^{-1}$ is with respect to y_{-r}^{-1}, and this uncertainty comes from lack of knowledge about the continuation x_0^{∞} of $x_{-\infty}^{-1}$. The quantity $-\log \nu(y_{-r}^{-1} | x_{-\infty}^{-1})$ is thus the further information about the continuation x_0^{∞} found in $y_{-\infty}^{-1}$ but not in $x_{-\infty}^{-1}$. This information has been transported from x_0^{∞}, and adds a negative contribution J_{L}^{-} to the information current.

Considering J_{L}^{+}, we can write

$$\begin{aligned} J_{\mathrm{L}}^{+}(x; 0) &= -\lim_{n\to\infty} \log \frac{\sum_{Z_0} \mu(z_{-n}^{r-1})}{\sum_{Z} \mu(z_{-n}^{r-1})} \\ &= -\lim_{n\to\infty} \log \frac{\nu(x_{-n}^{-1}, y_{-n+l}^{-1})}{\nu(x_{-n}^{\tau}, y_{-n+l}^{-1})} = -\log \nu(x_{\tau+1}^{-1} | x_{-\infty}^{\tau}, y_{-\infty}^{-1}). \end{aligned} \tag{51}$$

Thus, $J_{\mathrm{L}}^{+}(x; 0)$ is the information gained from observing $x_{\tau+1}^{-1}$ when $x_{-\infty}^{\tau}$ as well as $y_{-\infty}^{-1}$ is known. Since $y_{-\infty}^{-1}$ is known, the preimage x_{∞}^{r-1} is determined up to the set $Z(x_{-\infty}^{r-1})$. This is illustrated by Fig. 1, where $x_{\tau+1}^{r-1}$ must be one of the "branches" to the right, but it is not decidable from $y_{-\infty}^{-1}$ which one. However, which member of $Z(x_{-\infty}^{r-1})$ that $x_{-\infty}^{r-1}$ actually is will, with probability one, be determined by the continuation y_0^{∞} of $y_{-\infty}^{-1}$. Therefore, the information $-\log \nu(x_{\tau+1}^{-1}|x_{-\infty}^{\tau}, y_{-\infty}^{-1})$ flows to the right and is found to the right of coordinate -1 in y.

5.2. Permutative Cellular Automata. For the class of permutative CA the information dynamics has a particularly simple form. A CA F is called right permutative if $R(F) = 1$ or, equivalently, if $|Z(x_{-\infty}^{i})| = 1$ for all pairs x and i. Thus, for right permutative CA (48) gives $J_{\mathrm{L}}^{-}(x; i; \mu) = -\log \mu(x_i^{i+r-1}|x_{-\infty}^{i-1})$ and (49) gives $J_{\mathrm{L}}^{+} \equiv 0$. This gives the following corollary to Theorem 4.

Corollary 1. *Let μ be any shift-invariant measure on $\mathcal{A}^{\mathbb{Z}}$ and $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ a right permutative CA with right radius r. Then, μ-almost everywhere,*

$$S_{\mathrm{L}}(Fx; i; \mu \circ F^{-1}) = S_{\mathrm{L}}(x; i + r; \mu). \tag{52}$$

In particular, if $r = 0$, then $S_{\mathrm{L}}(F^t x; i; \mu^t) = S_{\mathrm{L}}(x; i; \mu^0)$ for all $t \geq 0$ so that the local information is locally constant. As a result of Corollary 1, the distribution of local information will also remain unchanged.

Corollary 2. *Let μ^0 be any ergodic measure on $\mathcal{A}^{\mathbb{Z}}$ and $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ a right permutative CA. Then, for all measures ν being a weighted sum of the measures μ^t, $t \geq 0$, the random variables $S_{\mathrm{L}}(x; i; \nu)$ have the same distribution.*

Note that even though the behaviour of the local information is very simple in the case of permutative CA, the sequence μ^t of measures generated by a linear CA under iteration is quite complicated. Block probabilities and the structure of correlations in the system varies widely with t [16]. On the other hand, for many bipermutative CA large classes of initial measures weak* converge in Cesàro mean to the uniform Bernoulli measure. That is,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=0}^{n-1} \mu^t(a_1^k) = \frac{1}{|\mathcal{A}|^k} \tag{53}$$

for all $k \geq 1$ and finite blocks $a_1^k \in \mathcal{A}^k$. This was first proved for the linear CA $F = \sigma + \sigma^{-1}$ on $\{0, 1\}^{\mathbb{Z}}$ with μ^0 a Bernoulli measure by Lind [15]. It has later been extended to a larger subclass of the permutative CA and classes of measures [3, 9, 18, 19].

For the uniform Bernoulli measure $\bar{\mu}$ the local information has a uniform distribution, i.e., $S_{\mathrm{L}}(x; i; \bar{\mu}) = \log |\mathcal{A}|$ for all x and i. We can use the result on Cesàro convergence to demonstrate that convergence of a sequence $(\mu_n)_{n\geq 0}$ of measures to a limit measure μ in the weak*-topology does not, in any sense, mean that $S_{\mathrm{L}}(x; i; \mu_n)$ converges to $S_{\mathrm{L}}(x; i; \mu)$. Indeed, let F and μ^0 be any combination of a CA and an ergodic measure such that (53) is valid, and put $\mu_n = \frac{1}{n} \sum_{t=0}^{n-1} \mu^t$. Then μ_n converges to the uniform Bernoulli measure, but by Corollary 2 all $S_{\mathrm{L}}(x; 0; \mu_n)$ have the same non-uniform probability distribution on $\mathbb{R}$. The reason that the distribution of $S_{\mathrm{L}}(x; i; \mu_n)$ can remain unchanged even though $\mu_n \to \bar{\mu}$, with $\bar{\mu}$ uniform Bernoulli, is that local information takes *all* correlations in the system into account while the weak* topology only considers finite blocks. In this sense local information yields a different, more microscopic, view of the system than the weak* topology does.

We now use Corollary 1 to demonstrate the necessity of the shift invariance condition on the measure in Theorem 4.

Example 3. Let F be the CA $\sigma^{-1} \circ F_1$, with F_1 from Example 1. Let μ^0 be the uniform Bernoulli measure on $\{0, 1\}^{\mathbb{Z}}$, except that x_0 always is zero. Then, for all $x \in \mathcal{A}^{\mathbb{Z}}$ we have $S_{\mathrm{L}}(x; i, \mu^0) = 1$ for $i \neq 0$ and $S_{\mathrm{L}}(x; 0, \mu^0) = 0$. However, we claim that $S_{\mathrm{L}}(y; i, \mu^1) = 1$ for *all* i.

Each sequence $y \in \mathcal{A}^{\mathbb{Z}}$ has two preimages under f, call them z and w. These have the property that z_i and w_i always are different, $z_i = 1 - w_i$. Assume that $z_0 = 0$. We obtain

$$\mu^0(z_j^k) = \begin{cases} 2^{j-k} & \text{if } j \leq 0 \leq k \\ 2^{j-k-1} & \text{otherwise} \end{cases}, \qquad \mu^0(w_j^k) = \begin{cases} 0 & \text{if } j \leq 0 \leq k \\ 2^{j-k-1} & \text{otherwise} \end{cases}. \tag{54}$$

The local information $S_{\mathrm{L}}(y; i, \mu^1)$ is the limit $n \to \infty$ of

$$-\log \mu^1(y_i | y_{i-n}^{i-1}) = -\log \frac{\mu^0(z_{i-n-1}^i) + \mu^0(w_{i-n-1}^i)}{\mu^0(z_{i-n-1}^{i-1}) + \mu^0(w_{i-n-1}^{i-1})}. \tag{55}$$

In all three cases: $i < 0$, $i = 0$ and $i > 0$, inserting the probabilities from (54) gives $-\log \mu^1(y_i | y_{i-n}^{i-1}) = 1$ for all n.

Now assume that $\Delta_t S_{\mathrm{L}} + \Delta_i J_{\mathrm{L}} = 0$ at all i μ^0-a.e. Since F is right permutative and has $r = 0$, Corollary 1 states that $J_{\mathrm{L}} = 0$ μ^0-a.e. However, this makes $S_{\mathrm{L}}(y; 0, \mu^1) = 0$, which is not satisfied for the image of any $x \in \mathcal{A}^{\mathbb{Z}}$.

An alternative way to appreciate that $S_{\mathrm{L}}(y; 0, \mu^1) = 1$ for the system in the example is to realize that $y_{-\infty}^{-1}$ does not give any information about x_{-1}. Therefore, even though $x_0 = 0$ with certainty, $\mu^1(y_0 | y_{-\infty}^{-1}) = \frac{1}{2}$. Note that once y_0 is observed, we will have perfect knowledge of $x_{-\infty}^{-1}$. Thus, information about the preimage that is not contained in the tail of y is made available at some position in the sequence. In similar constructs the information need not appear at a single position as it did in Example 3. A case illustrating this would be to let $\mu^0(X_i = 1) = \frac{1}{4}$ for all $i \geq 0$ and $\mu^0(X_i = 1) = \frac{1}{2}$ for $i \leq 0$. Then the correct preimage will be learnt gradually from observing $y_0, y_1, y_2, \ldots$, since the fraction of 1's in the preimage block x_0^n will converge either to $\frac{1}{4}$ or $\frac{3}{4}$. In this case, the continuity equation will in general not be satisfied at any $i \geq 0$, but will be an increasingly better approximation as i increases.

Finally in this section, we use a left and right permutative CA to illustrate the difference between S_{L} and S_{R}, and the effect of the choice of a frame of reference on the distribution of information in the system. We consider the CA on $\{0, 1\}^{\mathbb{Z}}$ defined by the radii $l = r = 1$ and local rule

$$f(x_{-1}, x_0, x_1) = x_{-1} + x_0 + x_1 \pmod 2. \tag{56}$$

Let the initial measure μ^0 be Bernoulli with a very small probability for a 1, say $\mu(1) = 2^{-10}$. Assume that $i = 0$ is the only coordinate in the interval $-100 \leq i \leq 100$ initially having $x_i = 1$. Figure 2 shows the configurations of the interval $-50 \leq i \leq 50$ for all iteration up to time $t = 50$. Coordinate $i = 0$ initially has 10 bits of left and right

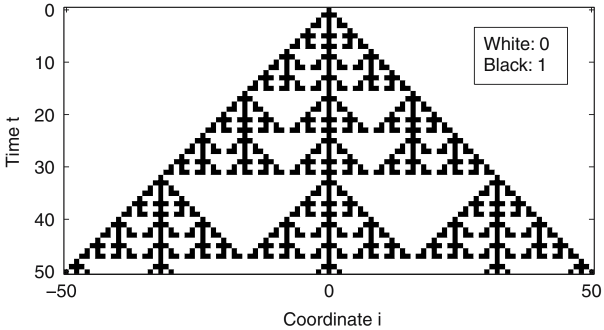

Fig. 2. The evolution of the symbol sequence with a single 1 located at $i = 0$ under the CA rule defined in (56). The left local information from the initial 1 is located at the left boundary of the expanding pattern while the right local information is located at the right boundary.

local information, $S_{\mathrm{L}}(x; 0; \mu^0) = S_{\mathrm{R}}(x; 0; \mu^0) = 10$. Let $y = F^t(x)$. An observer which knows the left history will by observing $y_{-t} = 1$ learn that $x_0 = 1$ and gain 10 bits of information. However, from each of the subsequent symbols the observer will gain only $-\log(1-2^{-10})$ bits of information. This is in agreement with Corollary 1. On the other hand, an observer knowing the right history will gain the 10 bits of information by observing that $y_t = 1$. Thus, the question about where in the pattern the information generated by the unlikely event $\{x_0 = 1\}$ is located at time t cannot be answered without also taking into account which frame of reference an observer has.

5.3. Bounds on the Current. We first consider bounds for the local information flow. Let $\tau = \tau(x_{-\infty}^{i+r-1})$. Then, from (46), (48), and monotonicity of $\log x$,

$$-\sum_{k=i}^{i+r-1} S_{\mathrm{L}}(x; k) \leq J_{\mathrm{L}}(x; i) \leq \sum_{k=\tau+1}^{i-1} S_{\mathrm{L}}(x; k). \tag{57}$$

Thus, the amount of information that flows from coordinate $i-1$ to i is limited by the amount of information available in the intervals $[\tau+1, i-1]$ and $[i, i+r-1]$. The appearance of $\tau+1$ rather than $i-l$ in the first interval warrants a closer examination, because a perturbation of one symbol in the initial configuration only can propagate a distance l per time step.

For reversible CA the existence of an inverse CA ensures that $\tau + 1 \geq i - \tilde{r}$, where $\tilde{r}$ is the right radius of the inverse CA. Therefore, the distance over which information can flow in a single iteration is uniformly bounded for a given reversible CA.

For surjective non-reversible CA, τ is in general unbounded. In the following discussion, assume that $r = 0$, since this case gives the maximal flow of left local information to the right. We look at coordinate $i = 0$. The appearance of $\tau+1$ in (57) is related to the interpretation of J_{L}^{+} as the information gained by observing $x_{\tau+1}^{-1}$ when the image $y_{-\infty}^{-1}$ as well as the history $x_{-\infty}^{\tau}$ is known. The second inequality in (57) is an equality if and only if the additional knowledge of $y_{-\infty}^{-1}$ leads to no reduction in information gain compared to knowledge of only $x_{-\infty}^{\tau}$. From (49) this is equivalent to having $\sum_Z \mu(z_{\tau+1}^{-1}|x_{-\infty}^{\tau}) = 1$. However, since $|Z| \leq R(F)$ the sum is rarely close to this magnitude, particularly when

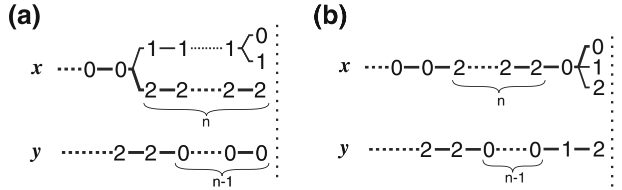

Fig. 3. An illustration of the sets (a) $Z(x_{-\infty}^{-1})$ and (b) $Z(x_{-\infty}^{1})$ for the CA in Example 4 when $x_{-n}^{-1} = 22\ldots2$ and $x_{-n-2} = x_{-n-1} = x_0 = x_1 = 0$.

$|\tau|$ is large. A further argument that large information flows are improbable is the observation that $J_{\mathrm{L}}(x; i; \mu) > s$ requires $\mu(x_{\tau+1}^{-1}|x_{-\infty}^{\tau}) / \sum_Z \mu(z_{\tau+1}^{-1}|x_{-\infty}^{\tau}) < 2^{-s}$, so $x_{\tau+1}^{-1}$ must be a very unlikely continuation of $x_{-\infty}^{\tau}$ to generate a large current s. We illustrate these considerations with the following example.

Example 4. Let the surjective CA F on $\{0, 1, 2\}^{\mathbb{Z}}$ be defined by the radii $l = 1, r = 0$ and local function f given by $f(10) = f(11) = f(22) = 0$, $f(12) = f(20) = f(21) = 1$ and $f(00) = f(01) = f(02) = 2$. Unlikely events in this system can generate information flows over large distances in a single iteration.

Let μ be Bernoulli with a low probability $p = \mu(2)$ for the symbol 2 occurring, and $q = \mu(0) = \mu(1) = \frac{1-p}{2}$. Although p is small, long blocks of successive 2's will occur at some points. Assume that $x_{-n}^{-1} = 22\ldots2$ while $x_{-n-1}^{-n-2} = 00$ and $x_0^3 = 0000$. The set $Z(x_{-\infty}^{-1})$ is illustrated to the left in Fig. 3 using the representation introduced in Fig. 1. For p small, the quantity $\sum_Z \mu(z_{\tau+1}^{-1}|x_{-\infty}^{\tau})$ is much larger than $\mu(x_{\tau+1}^{-1}|x_{-\infty}^{\tau})$, since the two other elements in $Z(x_{-\infty}^{-1})$ consist entirely of 0's and 1's. It follows from (49) that J_{L}^{+} is large and an increasing function of the number n of 2's. Equation (10) yields for $1 \le k \le n$,

$$J_{\mathrm{L}}(x; -n+k; \mu) = \log\left(1 + 2\left(\frac{q}{p}\right)^k\right) \approx 1 + k\log\left(\frac{q}{p}\right). \tag{58}$$

Only approximately $-\log q$ bits of information remain at each coordinate $-n \le i \le -1$, while the surplus information is transported to the right of $i = -1$, see Fig. 4. Most of the information is accumulated at position $i = 1$. The reason is that observing the value $y_1 = 2$ while knowing $y_{-\infty}^{0}$ establishes that the actual preimage was the one containing the large block of 2's, see the right part of Fig. 3. This preimage was highly improbable, and a high local information results.

Finally, note that since F is left permutative, the transport of right local information is simply given by $J_{\mathrm{R}}(x; i; \mu) = S_{\mathrm{L}}(x; i-1; \mu)$.

The possibility that $J_{\mathrm{L}}(x; i) > -\log\mu(x_{i-l}^{i-1}|x_{-\infty}^{i-l-1})$ can be better appreciated by recalling that local information S_{L} is defined with respect to an infinite frame of reference. Therefore, a permutation arbitrary far to the left of i can alter the conditional probability $\mu(x_i|x_{-\infty}^{i-1})$ and hence $S_{\mathrm{L}}(x; i)$. Contrary to this, the propagation of a perturbation in the initial configuration consists of the symbols at an increasing number of coordinates deviating from some reference symbols. Clearly, no frame of reference is needed to detect these deviations.

We can compare the results above to a situation that involves communication between two parts of the lattice. Consider an observer A who knows the initial configuration $x_{-\infty}^{0}$

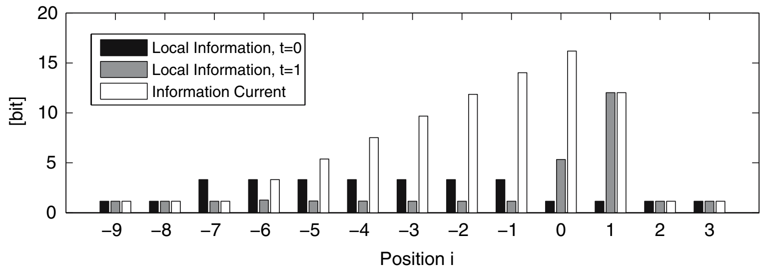

Fig. 4. The distribution of local information S_L and the corresponding information currents $J_L(x; i; \mu)$ for the situation in Example 4 with $n = 7$ and $p = 0.1$. The information is accumulated at coordinates $i = 0$ and $i = 1$ at time $t = 1$.

of the negative part of the lattice. How much information about the continuation x_1^∞ can A gain by observing the configurations $(F^k x)_{-\infty}^0$ for $0 < k \leq t$? This question can be answered by using the concept of relative entropy, or Kullback Liebler distance [13]. The relative entropy of a posterior measure μ with respect to a prior measure μ_0 satisfying $\mu \ll \mu_0$ is defined as

$$D(\mu||\mu_0) = \int_X \log \frac{d\mu}{d\mu_0} d\mu, \tag{59}$$

where $\frac{d\mu}{d\mu_0}$ is the Radon-Nikodym derivative. The quantity $D(\mu||\mu_0)$ is interpreted as the Shannon information gained by going to the posterior.

The posterior is in our case expressed in terms of the joint measure ν^t of all times $0 \leq k \leq t$ obtained as a straightforward generalization of ν defined in (15). For $x \in \mathcal{A}^{\mathbb{Z}}$, define the measure μ_x^t on $\sigma(X_i : i > 0)$ as

$$\mu_x^t(z_1^n) = \nu^t(z_1^n | x_{-\infty}^0, (Fx)_{-\infty}^0, \ldots, (F^t x)_{-\infty}^0), \quad n \geq 1. \tag{60}$$

The information the observer A gains by time t is given by the relative entropy $D(\mu_x^t||\mu_x^0)$. The following relations are valid

Proposition 1. *The measures defined in Eq.* (60) *satisfy*

$$D(\mu_x^t||\mu_x^0) \leq \sum_{i=1}^{rt} S_L(x; i; \mu), \tag{61}$$

$$D(\mu_x^1||\mu_x^0) = J_L^-(x; 1; \mu). \tag{62}$$

This means that A during the first t iterations of F cannot gain more information about $x_{-\infty}^0$ than the left local information initially located within the interval $[1, rt]$. Similarly, if B is an observer knowing x_0^∞ and observing the symbols at $i \geq 0$ for times $1 \leq k \leq t$, his information gain about $x_{-\infty}^{-1}$ would be bounded by $\sum_{i=-lt}^{-1} S_R(x; i; \mu^0)$.

Proof. Define $B_x \subseteq \mathcal{A}^{rt}$ as $B_x = \{z_1^{rt} : f^k(x_{-kr}^0 z_1^{kr}) = f(x_{-kr}^{kr}) \text{ for } 1 \le k \le t\}$. Both results follow from

$$\begin{aligned}
D(\mu_x^t||\mu_x^0) &= \lim_{n\to\infty} \sum_{z_1^n} \mu_x^t(z_1^n) \log \frac{\mu_x^t(z_1^n)}{\mu_x^0(z_1^n)} \\
&= \lim_{n\to\infty} \sum_{z_1^n} \mu(z_1^n | x_{-\infty}^0, z_1^{rt} \in B_x) \log \frac{\mu(z_1^n|x_{-\infty}^0, z_1^{rt} \in B_x)}{\mu(z_1^n|x_{-\infty}^0)} \\
&= -\log \mu(\{z_1^{rt} \in B_x\}|x_{-\infty}^0) \le -\log \mu(x_1^{rt}|x_{-\infty}^0).
\end{aligned}$$

□

We now move on to determine bounds on the *average* information flow generated by a surjective one-dimensional CA.

Theorem 5. *Let $J_{\mathrm{L}}(x; i; \mu)$ be the information current with respect to a surjective CA F and a measure μ. Then, for each $i \in \mathbb{Z}$, $J_{\mathrm{L}}(x; i; \mu) \in L^1(\mu)$. Furthermore, if μ is shift-invariant, then $E[J_{\mathrm{L}}]$ satisfies the relationship*

$$-rh(\mu) \le E[J_{\mathrm{L}}] \le \log R(F) - rh(\mu). \tag{63}$$

The term $\log R(F)$ on the right hand-side in (63) is related to the interpretation of J_{L}^+ as the information about which member of $Z(x_{-\infty}^{r-1})$ that $x_{-\infty}^{r-1}$ is. Since $|Z(x_{-\infty}^{r-1})| \le R(F)$, the average of this information cannot exceed $\log R(F)$. The term $-rh(\mu)$ is related to J_{L}^-.

Proof. We look at coordinate $i = 0$. The current can be written as

$$J_{\mathrm{L}}(x; 0; \mu) = -\log \frac{\mu(x_{\tau+1}^{r-1}|x_{-\infty}^{\tau})}{\sum_Z \mu(x_{\tau+1}^{r-1}|x_{-\infty}^{\tau})} + \log \mu(x_0^{r-1}|x_{-\infty}^{-1}), \tag{64}$$

where the first term is non-negative and the second term is non-positive. To bound the integral of the first term we divide $\mathcal{A}^{\mathbb{Z}}$ into the sets $T_k = \{x : \tau(x_{-\infty}^{r-1}) = k\}$ for $k \le -2$. Furthermore, we wish to subdivide each T_k through an equivalence relation similar to that defined in (25). Define the following relation on $\mathcal{A}^{|k|+2r+l-1}$:

$$x_{k+1-l-r}^{r-1} \sim z_{k+1-l-r}^{r-1} \text{ iff } f(_{k+1-l-r}^{r-1}) = f(_{k+1-l-r}^{r-1}) \text{ and } x_{k+1-l-r}^{k} = z_{k+1-l-r}^{k}.$$

Then, transfer the equivalence relation to T_k through

$$x \sim z \text{ iff } x_{k+1-l-r}^{r-1} \sim z_{k+1-l-r}^{r-1}.$$

We denote the equivalence classes of T_k by $P_{k,j}$ with j in some finite index set. Furthermore, for each $P_{k,j}$ denote the corresponding equivalence class of $\mathcal{A}^{|k|+2r+l-1}$ by $\bar{P}_{k,j}$. Each $\bar{P}_{k,j}$ has at most $R(F)$ members.

For each j there is a set $P_{k,j}^- \in \sigma(X_i : i \le k - l - r)$ of histories $x_{-\infty}^{k-l-r}$ such that

$$P_{k,j} = P_{k,j}^- \bigcap \bigcup_{\bar{P}_{k,j}} \mathrm{Cyl}(z_{k+1-l-r}^{k}). \tag{65}$$

If $|\bar{P}_{k,j}| = R(F)$ then $P^-_{k,j} = \mathcal{A}^{\mathbb{Z}}$, but otherwise $P^-_{k,j}$ can be a subset of $\mathcal{A}^{\mathbb{Z}}$. For instance, let F be the CA from Example 4 and x have $x_i = 1$ for all i. Then $\tau(x^{-1}_{-\infty}) = -2$, $|\bar{P}_{-2,j}| = 2$, and $P^-_{-2,j} = \{z : z_i \neq 0 \text{ for } i \leq -3\}$.

Using the subdivision, we can write

$$-\int_{\mathcal{A}^{\mathbb{Z}}} \log \frac{\mu(x^{r-1}_{\tau+1}|x^{\tau}_{-\infty})}{\sum_Z \mu(x^{r-1}_{\tau+1}|x^{\tau}_{-\infty})} d\mu$$

$$= \sum_{k=-\infty}^{-2} \sum_{P_{k,j}\in T_k} \int_{P^-_{k,j}} \Psi_{k,j}(x^{k-l-r}_{-\infty}) \left(\sum_{\bar{P}_{k,j}} \mu(z^{r-1}_{k+1-l-r}|x^{k-l-r}_{-\infty}) \right) d\mu(x^{k-l-r}_{-\infty}), \tag{66}$$

where

$$\Psi_{k,j}(x^{k-l-r}_{-\infty}) = -\sum_{\bar{P}_{k,j}} \frac{\mu(z^{r-1}_{k+1}|x^{k}_{-\infty})}{\sum_{\bar{P}_{k,j}} \mu(z^{r-1}_{k+1}|x^{k}_{-\infty})} \log \frac{\mu(z^{r-1}_{k+1}|x^{k}_{-\infty})}{\sum_{\bar{P}_{k,j}} \mu(z^{r-1}_{k+1}|x^{k}_{-\infty})}. \tag{67}$$

The function $\Psi_{k,j}(x^{k-l-r}_{-\infty})$ is for each $x^{k-l-r}_{-\infty}$ the entropy of a discrete random variable with at most $R(F)$ outcomes. Therefore, $\Psi_{k,j}(x^{\tau-l-r}_{-\infty}) \leq \log R(F)$. By using this inequality, we obtain from (66) that

$$-\int_{\mathcal{A}^{\mathbb{Z}}} \log \frac{\mu(x^{r-1}_{\tau+1}|x^{\tau}_{-\infty})}{\sum_Z \mu(x^{r-1}_{\tau+1}|x^{\tau}_{-\infty})} d\mu \leq \log R(F) \sum_{k=-\infty}^{-2} \sum_{T_k} \mu(P_{k,j}) = \log R(F). \tag{68}$$

Considering the second term in (64),

$$\int_{\mathcal{A}^{\mathbb{Z}}} \log \mu(x^{r-1}_0|x^{-1}_{-\infty}) d\mu \geq -r \log |\mathcal{A}|.$$

Therefore,

$$E[|J_{\mathrm{L}}(x; 0; \mu)|] \leq r \log |\mathcal{A}| + \log R(F) < \infty,$$

so $J_{\mathrm{L}}(x; 0; \mu) \in L^1(\mu)$. The second statement follows since for μ shift-invariant,

$$\int_{\mathcal{A}^{\mathbb{Z}}} \log \mu(x^{r-1}_0|x^{-1}_{-\infty}) d\mu = -r h(\mu).$$

□

From Theorem 5 it follows that for any surjective CA and any measure μ, the following uniform bound is valid:

$$\frac{|E[J_{\mathrm{L}}(x; i; \mu)]|}{(l + r) \log |\mathcal{A}|} \leq 1. \tag{69}$$

When μ is the uniform Bernoulli measure $\bar{\mu}$, the inequality is sharp for all CA with $r = 0$ and maximal R. We close this section by looking briefly at information transport for $\bar{\mu}$, which is invariant for all surjective CA.

With $\bar{\mu}$ there are no correlations in the system, so $S_{\mathrm{L}}(x; i; \bar{\mu}) \equiv S_{\mathrm{R}}(x; i; \bar{\mu}) \equiv \log |\mathcal{A}|$. The information currents J_{L} and J_{R} are consequently also constant, but these depend on the radii and the Welch coefficients $L(F)$, $M(F)$ and $R(F)$. Using (10) and (45) we obtain

$$J_{\mathrm{L}}(x; i; \bar{\mu}) \equiv \log R - r \log |\mathcal{A}|, \tag{70}$$

$$J_{\mathrm{R}}(x; i; \bar{\mu}) \equiv -\log L + l \log |\mathcal{A}|. \tag{71}$$

By using the relation $L \cdot M \cdot R = |\mathcal{A}|^{l+r}$, we obtain

$$J_{\mathrm{R}} - J_{\mathrm{L}} \equiv \log M. \tag{72}$$

Thus, the sum of the velocity of left information to the left and right information to the right only depends on $M(F)$, the number of preimages that almost all bi-infinite sequences possess. The higher M is, the higher the potential for information transport. The choice of radii decides how the potential is allocated to transport of information to the right and to the left.

6. Conclusions and Discussion

The main concern of the paper has been to investigate transport of local information in the time evolution of a cellular automaton F. In particular, we have introduced an information current $J_{\mathrm{L}}(x; i; \mu)$ such that the continuity equation $\Delta_t S_{\mathrm{L}} + \Delta_i J_{\mathrm{L}} = 0$ holds under very general conditions. This is expressed in our main results, Theorems 3 and 4 in Sect. 4. We have also given an information theoretic interpretation of the current, and shown bounds for the information flow.

The fact that the local information is a locally conserved quantity for *all* measures under iteration of any reversible CA is a clear indication that the function S_{L} is an appropriate local information measure in a spatially extended system. However, we still need to consider the fact that information is not a strictly local quantity when correlations are present, and that it depends on the choice of context. In one dimension, this is illustrated by the fact that both the left and right local information are locally conserved, and in general different (as seen, e.g., for bipermutative CA). We have also given other examples which illustrate the limits of locality when correlations are present.

We are currently investigating how the continuity equation can be extended to other classes of cellular automata. In particular this includes non-surjective and probabilistic CA in dimension one as well as CA in dimension two and higher. For non-surjective or non-deterministic systems a continuity equation must take loss and production of information into account. One can also look at local information and transport of local information for other types of spatially extended dynamical systems, in particular coupled map lattices. Extensions of the formalism to other systems will also bring us closer to addressing fundamental issues relating to information transport and conservation in physical systems.

The continuity equation is a fundamental property of information transport in reversible systems. But we expect local information in cellular automata to have further interesting properties. In particular, a continuity equation is a constraint, rather than an equation that determines the dynamics of the system. One may expect information flow to have different dynamic characteristics in different cellular automata. In particular it should be investigated whether some systems allow a description of the dynamics of information separate from the underlying dynamical system, which would provide an additional argument for viewing information as a fundamental physical quantity.

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