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Approach to Equilibrium in a Microscopic Model of Friction

S. Caprino[1], **C. Marchioro**[2], **M. Pulvirenti**[2]

[1] Dipartimento di Matematica, Università di Roma 'Tor Vergata', via della Ricerca Scientifica, 00133 Roma, Italy. E-mail: caprino@mat.uniroma2.it
[2] Dipartimento di Matematica, Università di Roma 'La Sapienza', P.le Aldo Moro 2, 00185 Roma, Italy. E-mail: marchior@mat.uniroma1.it; pulvirenti@mat.uniroma1.it

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Abstract: We consider the time evolution of a disk under the action of a constant force and interacting with a free gas in the mean-field approximation. Letting $V_0 > 0$ be the initial velocity of the disk and $V_\infty > 0$ its equilibrium velocity, namely the one for which the external field is balanced by the friction force exerted by the background, we show that, if $V_\infty - V_0$ is positive and sufficiently small, then the disk reaches V_∞ with the power law $t^{-(d+2)}$, $d = 1, 2, 3$ being the dimension of the physical space. The reason for this behavior is the long tail memory due to recollisions. Any Markovian approximation (or simply neglecting the recollisions) yields an exponential approach to equilibrium.

1. Introduction

Consider a solid body moving along the x-axis, under the action of a constant force E, immersed in a homogeneous fluid. Then its time evolution is given by:

$$M\dot{V}(t) = -G(V) + E, \tag{1.1}$$

where $V = V(t)$ is the (horizontal) velocity of the body, M its total mass and G, the friction term, is usually determined on the basis of phenomenological considerations. Such a function, that for V small often takes the familiar form $G(V) = \lambda V$ for a positive λ, summarizes all the complex interactions between the body and the medium. If we suppose the body initially at rest and $G(V)$ increasing, the solution $V = V(t)$ of Eq. (1.1) is increasing in time and converges exponentially to the limiting velocity V_∞ which satisfies

$$G(V_\infty) = E. \tag{1.2}$$

In the equilibrium situation the external force is perfectly compensated by the friction force and the body moves with constant velocity.

It would be desirable, of course, to give a microscopic explanation of these facts. The most natural way to pose the problem is to model the medium as an infinite particle system, interacting with an obstacle accelerated by a given field E. Obviously the behavior of the obstacle will depend on the obstacle-background interaction. We address the reader to the classical monograph [10] for heuristic considerations. With regard to rigorous results, we are aware of Ref. [2], where the background is modeled by a vibration field. In this case the obstacle reaches its limiting velocity with an exponential rate. On the other hand in Refs. [4–6] it is shown that a test particle, immersed in a medium of identical interacting particles, accelerates indefinitely whenever the interaction is smooth or moderately diverging.

The simplest model to consider is a gas of free light particles elastically interacting with the body. This kind of interaction gives rise to a very irregular motion, with fluctuations which are very small if the ratio between the mass of the body and that of the gas particles is very large. However the averaged motion is expected to be regular and sufficient to give a correct description of the macroscopic behavior of the system. To avoid the difficulties connected with the computations of the averaged quantities, one can alternatively consider the gas in the mean-field approximation, that is the limit in which the mass of the particles constituting the free gas goes to zero, while the number of particles per unit volume diverges, in such a way that the mass density stays finite. Such a limit is well known for interacting particle systems with finite total mass (see [7, 8, 12, 13]) and one-dimensional systems with unbounded mass (see [3]). This is exactly what we do in the present paper, namely we study the time evolution of an obstacle elastically interacting with a free gas in a mean-field approximation. This model has been previously introduced in connection with the so-called piston problem (see [9] and also [11] and references quoted therein). We assume that the body has a particularly simple shape, namely we consider a cylinder with a negligible length. We prove that, if the initial velocity of the body is sufficiently close to the limiting velocity V_∞ then, for large t:

$$|V_\infty - V(t)| \approx \frac{C}{t^{(d+2)}}, \tag{1.3}$$

where C is a positive constant depending on the medium and the shape of the obstacle and $d = 1, 2, 3$ is the dimension of the physical space.

The law (1.3) is not exponential and hence the result is somehow surprising. The reason for this behavior is the appearance of recollisions between the gas particles and the obstacle. Indeed if the obstacle accelerates, it can hit a gas particle many times and this influences the friction force dramatically. In particular a gas particle which has collided quite early, can recollide after an arbitrarily large time. This creates a long tail memory which is responsible for the power law behavior. Neglecting the recollisions, namely assuming that the obstacle always hits new particles at a given thermal equilibrium, the friction force can be computed almost explicitly and the behavior is the one predicted by Eq. (1.1), that is exponential. We show that this approximation is not legitimate in our model. One can argue, however, that such a model is too poor to give realistic information: the background is schematized by a free gas while an interacting system, with good ergodic properties, could reasonably destroy the memory effects which are present in our context. Unfortunately such ergodic properties for Hamiltonian systems seem far to be proven. In any case the result of the present paper at least shows that, in the suitable time scale in which the thermalization of the medium is not yet effective, the approach to the limiting velocity is not exponential but obeys a power law. It is worth mentioning that it was already known that the recollisions can produce a power-law decay. In fact the velocity-velocity correlation of a tagged particle of a one-dimensional free gas decays as t^{-3}. (See for instance Ref. [1]).

We prove (1.3) for an obstacle of a particular shape and under the hypothesis that $V_\infty - V_0$ is sufficiently small, $V_0 < V_\infty$ being the initial velocity of the obstacle. It looks quite reasonable to conjecture that a power law holds under more general assumptions on both initial conditions and external field, however this has still to be proven (see Sect. 5).

We conclude by outlining the plan of the paper. In the next section we establish and discuss the model which is heuristically justified in the Appendix. Section 3 is devoted to preliminary technicalities, while in Sect. 4 we give the proofs of the main Theorems 2.1 and 2.2. Section 5 is devoted to concluding remarks.

2. Model and Results

The body we consider is a disk of radius R in dimension $d = 3$, a stick of length $2R$ for $d = 2$ and a point particle on the line for $d = 1$. We assume, for simplicity, its mass to be unitary. The disk (or the stick) is constrained to stay orthogonal to the x axis, with the center moving along the same axis. The thickness of the disk is assumed to be negligible, however this assumption is not essential and it is made just for notational simplicity. The system is immersed in a perfect gas in equilibrium at inverse temperature proportional to β and with constant density ρ. Moreover a constant force E is acting on the disk.

We are interested in the time asymptotics of the system, in particular we want to investigate whether and how the disk reaches a limiting velocity. We assume the perfect gas in the mean-field approximation. In other words the presence of the disk modifies the equilibrium of the gas, which starts to evolve according to the free Vlasov equation.

Let $f = f(x, v; t)$, $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$ be the mass density in the phase space of the gas particles. It evolves according to:

$$(\partial_t + v \cdot \nabla_x) f(x, v; t) = 0, \qquad \text{for} \qquad x \notin D(t). \tag{2.1}$$

Here $D(t)$ denotes the $(d-1)$-dimensional circular surface of radius R:

$$D(t) = \{y \in \Pi^\perp(X(t)) | \quad |y - X(t)|^2 < R^2\}.$$

$X(t)$ denotes the position of the center of the disk at time t and $\Pi^\perp(X(t))$ the plane orthogonal to the x-axis at the point $X(t)$.

Together with Eq. (2.1) we consider the boundary conditions. They express the continuity of f along the trajectories with elastic reflection on $D(t)$. Defining $v' = (v'_x, v'_\perp)$ as

$$v'_x = 2V(t) - v_x, \quad v'_\perp = v_\perp, \tag{2.2}$$

where $V(t) = \dot{X}(t)$ is the velocity of the disk and v_x and $v_\perp$ the velocity components of the gas particles on the x-axis and the orthogonal plane respectively, we set

$$f_+(x, v'; t) = f_-(x, v; t); \quad \text{for} \qquad x \in D(t), \tag{2.3}$$

where

$$f_\pm(x, v; t) = \lim_{\varepsilon \to 0^+} f(x \pm \varepsilon v, v; t \pm \varepsilon); \qquad \text{for} \qquad x \in D(t). \tag{2.4}$$

Equation (2.3) describes both the continuity along the collisions from the right $V(t) > v_x$ and from the left $V(t) < v_x$.

Coupled to Eq. (2.1) we consider the evolution equation for the disk:

$$\begin{aligned} \dot{X}(t) &= V(t), \quad \dot{V}(t) = E - F(t), \\ X(0) &= 0, \quad V(0) = V_0, \end{aligned} \tag{2.5}$$

where $E > 0$ is a constant given field and

$$\begin{aligned} F(t) = 2 \int_{D(t)} dx \int_{v_x < V(t)} & dv (V(t) - v_x)^2 f_-(x, v; t) \\ -2 \int_{D(t)} dx \int_{v_x \geq V(t)} & dv (V(t) - v_x)^2 f_-(x, v; t) \end{aligned} \tag{2.6}$$

is the action of the gas on the disk.

As initial state for the gas distribution we assume the thermal equilibrium, namely

$$\lim_{\varepsilon \to 0^+} f(x + \varepsilon v, v; \varepsilon) = \rho (\frac{\beta}{\pi})^{d/2} e^{-\beta v^2}, \tag{2.7}$$

for $\beta > 0$.

We incidentally remark that the results in the present paper hold for any initial datum of the form $\rho g(v^2)$, with g integrably decreasing.

Summarizing we define a solution to the friction problem any pair (f, V) where $V = V(t)$ solves, for almost all $t \in \mathbb{R}^+$, Eqs. (2.5),(2.6) and f satisfies Eq. (2.8) below

$$\frac{d}{dt} f(x + vt, v; t) = 0, \qquad a.e.(x, v), \tag{2.8}$$

together with boundary conditions (2.3) and initial condition (2.7).

Note that similar models have been introduced in Ref [9]. Here we give a heuristic derivation of the model in the Appendix.

We first observe that Eq. (2.1) can be solved by means of characteristics. More precisely, knowing the evolution of the disk $X(t)$, $V(t)$, we can trace back the time evolution of position and velocity of the gas particle $x(s, t; x, v)$, $v(s, t; x, v)$ at time $s \leq t$, having position and velocity x, v at time t. Such backward evolution is the free motion up to the last instant $\tau < t$ in which the particle hits the disk. On the surface of the disk we impose the elastic collision, namely:

$$v_x(\tau^-) = 2V(\tau) - v_x(\tau^+), \quad v_\perp(\tau^+) = v_\perp(\tau^-).$$

Then we go backward in time, up to the one but the last collision. Impose again the reflection condition and so on. Note that if $x \in D(t)$ then v has to be interpreted as a precollisional velocity namely $v = \lim_{s \to t} v(s, t; x, v)$. At the end we obtain

$$\begin{aligned} F(t) = 2\rho(\frac{\beta}{\pi})^{d/2} \Bigg[\int_{D(t)} dx \int_{v_x < V(t)} & dv (V(t) - v_x)^2 e^{-\beta v^2(0,t;x,v)} \\ - \int_{D(t)} dx \int_{v_x \geq V(t)} & dv (V(t) - v_x)^2 e^{-\beta v^2(0,t;x,v)} \Bigg]. \end{aligned} \tag{2.9}$$

Note that to compute $F(t)$ we need to evaluate $v(0, t; x, v)$ and hence to know all the previous history $\{X(s), V(s), \quad s < t\}$. On the other hand, if the light particle goes back without undergoing any collision, then

$$v(0, t; x, v) = v.$$

In this case we say, for obvious reasons, that the gas particle has no recollisions (the very last collision, namely the one at time t, is automatically taken into consideration because the gas particle is, at time t, on the surface of the disk). In absence of the recollisions the friction term is easily computed:

$$F_0(V) = 2\rho(\frac{\beta}{\pi})^{d/2}\sigma_d\Big[\int_{v_x<V} dv(V-v_x)^2 e^{-\beta v^2} \\ - \int_{v_x \geq V} dv(V-v_x)^2 e^{-\beta v^2}\Big], \tag{2.10}$$

where σ_d is the area of the disk.

Let V_∞ be the solution of

$$F_0(V_\infty) = E. \tag{2.11}$$

We assume that $V_\infty \geq V_0 > 0$.

We will show in Lemma 2.1 that F_0 is a positive, increasing and convex function in the interval $(0, V_\infty]$.

Now we see that, neglecting recollisions, our problem becomes trivial. Indeed replacing F by F_0 in Eq. (2.5) we have:

$$\begin{gathered} \dot{X}(t) = V(t), \quad \dot{V}(t) = E - F_0(V(t)) = K(t)(V_\infty - V(t)); \\ X(0) = 0, \quad V(0) = V_0, \end{gathered} \tag{2.12}$$

where

$$K(t) = \frac{F_0(V_\infty) - F_0(V(t))}{V_\infty - V(t)}.$$

The solution to Eq. (2.12) can be almost explicitly computed. We note that V is increasing in time and converging to V_∞. Furthermore a standard comparison argument shows that

$$\gamma e^{-C_- t} \leq V_\infty - V(t) \leq \gamma e^{-C_+ t}, \tag{2.13}$$

where

$$\gamma = V_\infty - V_0 \quad \text{and} \quad C_+ = F_0'(V_0) \leq C_- = F_0'(V_\infty). \tag{2.14}$$

The Vlasov equation (2.1) is then solved by characteristics.

The full problem (including recollisions) is considerably more difficult because it is not Markovian since the friction term F at time t depends on the previous history $X(s), V(s)$ with $s < t$. As we shall see, this long range memory affects the behavior of the system in a crucial way.

For further convenience we rewrite the full friction term F as:

$$F(t) = F_0(V(t)) + r^+(t) + r^-(t), \tag{2.15}$$

where $r^+(t)$ and $r^-(t)$ are the contribution coming from right and left recollisions respectively. Explicitly:

$$r^+(t) = 2\rho \left(\frac{\beta}{\pi}\right)^{d/2} \int_{D(t)} dx \int_{v_x < V(t)} dv (V(t) - v_x)^2 \left[e^{-\beta v^2(0,t;x,v)} - e^{-\beta v^2}\right] \tag{2.16}$$

and

$$r^-(t) = 2\rho \left(\frac{\beta}{\pi}\right)^{d/2} \int_{D(t)} dx \int_{v_x \geq V(t)} dv (V(t) - v_x)^2 \left[e^{-\beta v^2} - e^{-\beta v^2(0,t;x,v)}\right]. \tag{2.17}$$

We note that $r^+(t)$ and $r^-(t)$ are both not negative as it follows by the collision law (2.2). It turns out that $r^-(t)$ slows down the disk, in spite of the fact that this term arises from the left recollisions. The reason is that, if the disk slows down, F_0 includes many kinematically impossible left collisions which must be compensated.

We consider as data of the problem the quantities ρ, β, R, V_∞ (or equivalently E) and $\gamma = V_\infty - V_0$.

We are now in the position to state the main result of the present paper.

Theorem 2.1. *There exists $\gamma_0 = \gamma_0(\rho, \beta, R, V_\infty) > 0$ sufficiently small such that, for any $\gamma \in (0, \gamma_0)$ there exists at least one solution $(V(t), f(t))$ to problem (2.1)–(2.6). Moreover any solution $(V(t), f(t))$ satisfies, for any $t \geq 0$:*

$$V_\infty - V(t) \leq e^{-C_+ t}\gamma + \frac{A_+}{(1+t)^{d+2}}\gamma^3, \tag{2.18}$$

for a suitable positive constant A_+ independent of γ and

$$V_\infty - V(t) \geq e^{-C_- t}\gamma. \tag{2.19}$$

The next theorem shows that bound (2.19) can be improved.

Theorem 2.2. *Let $\gamma \in (0, \gamma_0)$. There exists a sufficiently large $\bar{t}$, depending on γ, such that any solution (f, V) to problem (2.1)–(2.6) satisfies for any $t \geq 0$:*

$$V_\infty - V(t) \geq e^{-C_- t}\gamma + \frac{A_-}{t^{d+2}}\gamma^4 \chi(\{t > \bar{t}\}), \tag{2.20}$$

where A_- is a positive constant, independent of γ, and $\chi(\{\dots\})$ is the characteristic function of $\{\dots\}$.

Note that the above theorems establish the power law approach to the stationary state.

For the sake of concreteness we shall prove Thms. 2.1 and 2.2 for the three-dimensional case. The remaining cases $d = 1, 2$ follow by the same arguments with obvious modifications.

Now we prove the announced properties of F_0.

Lemma 2.1. *F_0 is a positive, increasing and convex function in $(0, V_\infty]$.*

Proof. By (2.10) it is, for a constant $C > 0$:

$$F_0(V) = C \int dv_\perp e^{-\beta v_\perp^2} \Big[\int_{-\infty}^{V} dv_x (V - v_x)^2 e^{-\beta v_x^2} - \int_{V}^{+\infty} dv_x (V - v_x)^2 e^{-\beta v_x^2} \Big]. \tag{2.21}$$

By the simple change of variables $v_x \to -v_x$ we obtain

$$\begin{aligned} F_0(V) &= C \left[\int_{-\infty}^{V} dv_x (V - v_x)^2 e^{-\beta v_x^2} - \int_{-\infty}^{-V} dv_x (V + v_x)^2 e^{-\beta v_x^2} \right] \\ &\geq C \int_{-\infty}^{-V} dv_x \left[(V - v_x)^2 - (V + v_x)^2 \right] e^{-\beta v_x^2} \\ &= -4CV \int_{-\infty}^{-V} dv_x v_x e^{-\beta v_x^2} > 0. \end{aligned} \tag{2.22}$$

Moreover

$$\begin{aligned} F_0'(V) &= 2C \left[\int_{-\infty}^{V} dv_x (V - v_x) e^{-\beta v_x^2} - \int_{-\infty}^{-V} dv_x (V + v_x) e^{-\beta v_x^2} \right] \\ &\geq -4C \int_{-\infty}^{-V} dv_x v_x e^{-\beta v_x^2} > 0. \end{aligned} \tag{2.23}$$

Finally

$$F_0''(V) = 2C \left[\int_{-\infty}^{V} dv_x e^{-\beta v_x^2} - \int_{-\infty}^{-V} dv_x e^{-\beta v_x^2} \right] > 0. \tag{2.24}$$

□

3. Preliminary Results

In what follows the symbol C will indicate any positive constant, possibly depending on β, V_∞, ρ, R, but not on γ which is our small parameter. Any such constant is explicitly computable.

For any $\gamma \in (0, \gamma_0)$ with γ_0 sufficiently small, we introduce a t- a.e. differentiable function $t \to W(t) \in [V_0, V_\infty]$, with bounded derivative, such that $W(0) = V_0$, $\lim_{t\to\infty} W(t) = V_\infty$ and satisfying the following properties:

(i) W is increasing over the interval $[0, t_0]$, with

$$t_0 = \frac{1}{2C_-} \log \frac{C_+}{\gamma}. \tag{3.1}$$

(ii) There exists a positive constant A_+ such that, for any $t \geq 0$, it is:

$$V_\infty - W(t) \leq e^{-C_+ t} \gamma + \frac{A_+}{(1+t)^5} \gamma^3 \tag{3.2}$$

and

$$V_\infty - W(t) \geq e^{-C_- t} \gamma. \tag{3.3}$$

The constant A_+, independent of γ and γ_0, will be fixed later on.

We collect in the following lemma some properties of the function W, which will be useful in the sequel. For $0 \le s < t$, we set

$$\langle W \rangle_{s,t} = \frac{1}{t-s} \int_s^t W(\tau) d\tau \tag{3.4}$$

and

$$\langle W \rangle_{0,t} = \langle W \rangle_t. \tag{3.5}$$

Lemma 3.1. *Suppose γ_0 sufficiently small. Then:*

i) For any $t > 0$ we have:

$$W(t) > \langle W \rangle_t. \tag{3.6}$$

ii) $t \to \langle W \rangle_t$ is a strictly increasing function.

iii) For any $s \in (0, t)$,

$$\langle W \rangle_{s,t} > \langle W \rangle_t. \tag{3.7}$$

iv) For any $t > 0$, the following bound holds:

$$W(t) - \langle W \rangle_t \le \frac{C}{1+t}(\gamma + A_+ \gamma^3). \tag{3.8}$$

Proof.

Proof of i). The result is true for $t \le t_0$ because in this region W is increasing. For $t \ge t_0$ we have by (3.2) and (3.3):

$$\begin{aligned} W(t) - \langle W \rangle_t &= \frac{1}{t} \int_0^t ds[(V_\infty - W(s)) - (V_\infty - W(t))] \\ &\ge \frac{\gamma}{t} \int_0^t ds \left[e^{-C_- s} - e^{-C_+ t} \right] - \gamma^3 \left[\frac{A_+}{(1+t)^5} \right] \\ &= \gamma \left[\frac{1 - e^{-C_- t}}{C_- t} - e^{-C_+ t} \right] - \gamma^3 \left[\frac{A_+}{(1+t)^5} \right] \end{aligned} \tag{3.9}$$

which is positive, by choosing γ sufficiently small and consequently t_0 sufficiently large.
Proof of ii)

$$\frac{d}{dt} \langle W \rangle_t = -\frac{1}{t^2} \int_0^t d\tau W(\tau) + \frac{1}{t} W(t) = \frac{1}{t}[W(t) - \langle W \rangle_t] > 0 \tag{3.10}$$

by the previous lemma.
Proof of iii)

$$\begin{aligned} &\frac{1}{t-s} \int_s^t W(\tau) d\tau - \frac{1}{t} \int_0^t W(\tau) d\tau \\ &\quad = \left(\frac{1}{t-s} - \frac{1}{t} \right) \int_0^t W(\tau) d\tau - \frac{1}{t-s} \int_0^s W(\tau) d\tau \\ &\quad = \frac{s}{t-s} \left[\frac{1}{t} \int_0^t W(\tau) d\tau - \frac{1}{s} \int_0^s W(\tau) d\tau \right] > 0 \end{aligned} \tag{3.11}$$

by ii).

Proof of iv) For $t \leq 1$ we have

$$W(t) - \langle W \rangle_t \leq \gamma \leq \frac{2\gamma}{1+t}. \tag{3.12}$$

On the other hand by (3.2) we have, for $t > 1$,

$$\begin{aligned} W(t) - \langle W \rangle_t &= \frac{1}{t} \int_0^t ds [V_\infty - W(s) - (V_\infty - W(t))] \\ &\leq \frac{1}{t} \int_0^t ds \big(e^{-C_+ s} \gamma + \frac{A_+}{(1+s)^5} \gamma^3 \big) \leq \frac{C}{t} (\gamma + A_+ \gamma^3) \\ &\leq \frac{C}{1+t} (\gamma + A_+ \gamma^3). \end{aligned} \tag{3.13}$$

□

4. Proofs

Proof of Theorem 2.1. The strategy in proving Theorem 2.1 is the following. For an assigned velocity W of the disk (with the properties stated in the previous section), we can solve the free Vlasov equation outside the disk moving with velocity W and compute the friction contribution due to the recollisions, namely r_W^+ and r_W^- defined below. Then we solve Eq. (2.5) for the disk with assigned r_W^+ and r_W^-, finding a new velocity V_W. Obviously the solution of our problem is the fixed point of the map $W \to V_W$ (if any), so that our main goal is to infer for V the same properties established in (3.1),(3.2) and (3.3) for W (see Proposition 4.1 below).

Let W be defined as in the previous section and $X(t) = \int_0^t W(\tau) d\tau$ be the position of the disk at time t. Consider the modified problem:

$$\frac{d}{dt}(V_\infty - V_W(t)) = -K(t)(V_\infty - V_W(t)) + r_W^+(t) + r_W^-(t), \tag{4.1}$$

where $K(t)$ is the function introduced in Eq. (2.12) with $V_W(t)$ in place of $V(t)$,

$$r_W^+(t) = 2\rho \left(\frac{\beta}{\pi} \right)^{\frac{3}{2}} \int_{D(t)} dx \int_{v_x \leq W(t)} dv (v_x - W(t))^2 (e^{-\beta v^2(0,t;x,v)} - e^{-\beta v^2}) \tag{4.2}$$

and

$$r_W^-(t) = 2\rho \left(\frac{\beta}{\pi} \right)^{\frac{3}{2}} \int_{D(t)} dx \int_{v_x \geq W(t)} dv (v_x - W(t))^2 (e^{-\beta v^2} - e^{-\beta v^2(0,t;x,v)}). \tag{4.3}$$

The velocities of the light particles $v(s, t; x, v)$, $s < t$, are computed according to the evolution $X(s)$, $W(s)$ of the disk and the law of elastic reflection (2.2). Moreover the dynamics of the system leads a fluid particle to have a finite number of collisions for almost all $t, x \in D(t)$ and v. Finally we note that the tangential collisions, namely those for which there exists a time $s < t$ such that $x \in D(t)$, $x(s, t; x, v) \in D(s)$ and $v_x(s, t; x, v) = W(s)$, constitute a zero (t, x, v) measure set. These claims are proven in Proposition A.1 in the Appendix and in the sequel the possibility of having infinitely many or tangential collisions will be neglected. As a consequence Eq. (4.1) holds a.e. $t \in \mathbb{R}^+$.

In view of the fixed point argument we shall use in the sequel, we want to show that V_W behaves like W. Preliminarily however we have to estimate $r_W^{\pm}$.

To have a recollision from the right it is necessary that $v_x < W(t)$, $x \in D(t)$, $v(0,t;x,v) \neq v$ and a time $s < t$ has to exist such that

$$v_x(t-s) = X(t) - X(s) = \int_s^t W(\tau)d\tau, \tag{4.4}$$

that is $v_x = \langle W \rangle_{s,t}$ for some $s \in (0,t)$ and

$$|x_\perp - v_\perp(t-s)| \leq 2R. \tag{4.5}$$

Thus by Lemma 3.1 iii), for a recollision to happen it is necessary that

$$v_x \geq \langle W \rangle_t \qquad \text{and} \qquad |v_\perp| \leq \frac{2R}{t-s}. \tag{4.6}$$

Lemma 4.1. *Let A_+ be the constant in (3.2). Then for any $t \geq 0$,*

$$r_W^+(t) \leq C\frac{(\gamma + A_+\gamma^3)^3}{(1+t)^5}, \tag{4.7}$$

$$r_W^-(t) \leq C\chi(\{t > t_0\})\Big(\frac{\gamma + A_+\gamma^3}{(1+t)^5}\Big)^3. \tag{4.8}$$

Proof. We start by estimating $r_W^+(t)$. Recalling that, by (2.2) $v_\perp(0,t;x,v) = v_\perp$, from (4.2) and (4.6) it follows:

$$r_W^+(t) \leq C\int_{\langle W \rangle_t}^{W(t)} dv_x(v_x - W(t))^2 \int dv_\perp e^{-\beta v_\perp^2}\chi(\{|v_\perp| < \frac{2R}{t-\bar{s}}\}), \tag{4.9}$$

where $\bar{s}$ is the maximal solution of Eq. (4.4), namely the first backward recollision time. For v_x such that $\bar{s} < \frac{t}{2}$, we have:

$$\int dv_\perp e^{-\beta v_\perp^2}\chi(\{|v_\perp| < \frac{2R}{t-\bar{s}}\}) \leq C\left(\frac{R}{1+t}\right)^2. \tag{4.10}$$

Therefore we have a first contribution to the estimate of $r_W^+(t)$ which is

$$C\left(\frac{1}{1+t}\right)^2 \int_{\langle W \rangle_t}^{W(t)} dv_x(v_x - W(t))^2 \leq C\left(\frac{1}{1+t}\right)^2 (W(t) - \langle W \rangle_t)^3. \tag{4.11}$$

On the other hand, if $\bar{s} > \frac{t}{2}$, from (4.4) and (3.2) it follows :

$$\begin{aligned} v_x &= W(t) - \frac{1}{t-\bar{s}}\int_{\bar{s}}^t d\tau(W(t) - W(\tau)) \\ &\geq W(t) - \frac{1}{t-\bar{s}}\int_{\bar{s}}^t d\tau(V_\infty - W(\tau)) \\ &\geq W(t) - \frac{1}{t-\bar{s}}\int_{\bar{s}}^t d\tau\big[e^{-C_+\tau}\gamma + \frac{A_+\gamma^3}{(1+\tau)^5}\big] \\ &\geq W(t) - \left[\frac{e^{-C_+\bar{s}} - e^{-C_+t}}{C_+(t-\bar{s})}\gamma + C\frac{A_+\gamma^3}{(1+t)^5}\right]. \end{aligned} \tag{4.12}$$

Since

$$\frac{1-e^{-C_+(t-\bar{s})}}{C_+(t-\bar{s})} < C,$$

it follows that

$$v_x \geq W(t) - C\left[\gamma e^{-C_+\frac{t}{2}} + \frac{A_+\gamma^3}{(1+t)^5}\right] \geq W(t) - C\frac{(\gamma + A_+\gamma^3)}{(1+t)^5}.$$

Hence the second contribution to the estimate of $r_W^+(t)$ is

$$C\int_0^{+\infty} dv_x(v_x - W(t))^2\chi(\{W(t) - \frac{C(\gamma + A_+\gamma^3)}{(1+t)^5} \leq v_x \leq W(t)\})$$
$$\leq C\left[\frac{\gamma + A_+\gamma^3}{(1+t)^5}\right]^3. \tag{4.13}$$

Collecting estimates (4.11), (4.13) and using Lemma 3.1 iv), we finally obtain (4.7).

For r_W^- we prove a similar estimate. First we notice that, as far as W is increasing, $r_W^-(t) = 0$ and this justifies the characteristic function in Eq. (4.8). Moreover if $v(0, t; x, v) \neq v$, $x \in D(t)$, and $v_x > W(t)$, there exists $s < t$ such that

$$v_x = 2W(s) - v_x^* \tag{4.14}$$

for some $v_x^* > W(s)$. Hence

$$v_x \leq 2W(s) - W(s) < V_\infty. \tag{4.15}$$

Thus from (4.3) we obtain:

$$r_W^-(t) \leq C\int_{W(t)}^{V_\infty}(v_x - W(t))^2 dv_x \leq C(V_\infty - W(t))^3. \tag{4.16}$$

By using (3.2) we obtain:

$$r_W^-(t) \leq C\left(\gamma e^{-C_+t} + \frac{\gamma^3 A_+}{(1+t)^5}\right)^3. \tag{4.17}$$

We obtain (4.8) by observing that $e^{-C_+t} \leq \frac{C}{(1+t)^5}$. □

Now we prove that the function $V_W(t)$ satisfying Eq. (4.1) enjoys, for γ suitably small, the same properties as the function W, with the same constant A_+.

Proposition 4.1. *Suppose γ sufficiently small. Then:*

(i) $t \to V_W(t)$ is a a.e. differentiable function, increasing over the interval $[0, t_0]$ with

$$t_0 = \frac{1}{2C_-}\log\frac{C_+}{\gamma}.$$

(ii) For any $t \geq 0$:

$$V_\infty - V_W(t) \geq e^{-C_-t}\gamma. \tag{4.18}$$

(iii) For any $t \geq 0$:

$$V_\infty - V_W(t) < e^{-C_+ t}\gamma + \frac{A_+}{(1+t)^5}\gamma^3. \tag{4.19}$$

Proof. Since $r_W^+(t)$ and $r_W^-(t)$ are bounded, V_W is a.e. differentiable with uniformly, essentially bounded derivative. Moreover from Eq. (4.1) and the Duhamel formula we have:

$$\begin{aligned} V_\infty - V_W(t) &= \gamma e^{-\int_0^t K(\tau)d\tau} \\ &\quad + \int_0^t ds \quad e^{-\int_s^t K(\tau)d\tau}(r_W^+(s) + r_W^-(s)) \end{aligned} \tag{4.20}$$

which shows, by the positivity of $r_W^+(t)$ and $r_W^-(t)$, that $V_W(t) < V_\infty$ for any t. Thus $K(t) < F_0'(V_\infty) = C_-$ and, again from (4.20) we get

$$V_\infty - V_W(t) \geq e^{-C_- t}\gamma, \tag{4.21}$$

which proves ii).

Moreover $V_W(t) > V_0$ for any $t > 0$. Indeed, by (4.1) it is :

$$\begin{aligned} \frac{d}{dt}(V_W(t) - V_0) &= F_0(V_\infty) - F_0(V_W(t)) - r_W^+(t) - r_W^-(t) \\ &= \gamma \frac{F_0(V_\infty) - F_0(V_0)}{V_\infty - V_0} \\ &\quad - \frac{F_0(V_W(t)) - F_0(V_0)}{V_W(t) - V_0}(V_W(t) - V_0) - r_W^+(t) - r_W^-(t). \end{aligned} \tag{4.22}$$

By the properties of F_0 and Lemma 4.1 we obtain, for γ sufficiently small:

$$\frac{d}{dt}(V_W(t) - V_0) > -\frac{F_0(V_W(t)) - F_0(V_0)}{V_W(t) - V_0}(V_W(t) - V_0). \tag{4.23}$$

This implies $V_W(t) > V_0$ for any $t > 0$ and consequently $K(t) > F_0'(V_0) = C_+$. Hence, by Eqs. (4.1), (4.21) and again Lemma 4.1 we have:

$$\begin{aligned} \frac{d}{dt}(V_\infty - V_W(t)) &\leq -C_+(V_\infty - V_W(t)) + r_W^+(t) + r_W^-(t) \\ &\leq -C_+\gamma e^{-C_- t} + C\frac{(\gamma + A_+\gamma^3)^3}{(1+t)^5} \\ &\leq -C_+\gamma e^{-C_- t} + \gamma^2 \end{aligned} \tag{4.24}$$

for γ sufficiently small, and this implies

$$\frac{d}{dt}(V_\infty - V_W(t)) < 0, \tag{4.25}$$

for $t \in [0, t_0]$, so that i) is proven.

It remains to prove (iii). From Eq. (4.20) and Lemma 4.1 it follows:

$$
\begin{aligned}
V_\infty - V_W(t) &\le e^{-C_+ t}\gamma + \int_0^t ds e^{-C_+(t-s)}(r_W^+(s) + r_W^-(s)) \\
&\le e^{-C_+ t}\gamma + C(\gamma + A_+\gamma^3)^3 \int_0^t ds \frac{e^{-C_+(t-s)}}{(1+s)^5}.
\end{aligned} \tag{4.26}
$$

Let us evaluate the integral:

$$
\begin{aligned}
\int_0^t ds \frac{e^{C_+ s}}{(1+s)^5} &= \int_0^{\frac{t}{2}} (\cdot) ds + \int_{\frac{t}{2}}^t (\cdot) ds \\
&\le \frac{e^{C_+\frac{t}{2}} - 1}{C_+} + \frac{2^5}{(2+t)^5} \frac{e^{C_+ t} - e^{C_+\frac{t}{2}}}{C_+}.
\end{aligned} \tag{4.27}
$$

Thus

$$
\begin{aligned}
\int_0^t ds \frac{e^{-C_+(t-s)}}{(1+s)^5} &\le \frac{e^{-C_+\frac{t}{2}} - e^{-C_+ t}}{C_+} + \frac{2^5}{(2+t)^5} \frac{1 - e^{-C_+\frac{t}{2}}}{C_+} \\
&\le \frac{1}{C_+}\left[e^{-C_+\frac{t}{2}} + \frac{2^5}{(2+t)^5}\right] \le \frac{C}{(1+t)^5}.
\end{aligned} \tag{4.28}
$$

To conclude, there exists a constant $\bar{C}$ such that:

$$
V_\infty - V_W(t) \le e^{-C_+ t}\gamma + \bar{C}(\gamma + A_+\gamma^3)^3 \frac{1}{(1+t)^5}. \tag{4.29}
$$

Therefore to obtain iii) it is sufficient that

$$
\bar{C}(\gamma + A_+\gamma^3)^3 < A_+\gamma^3. \tag{4.30}
$$

Inequality (4.30) is satisfied, for instance, by choosing $A_+ = 2\bar{C}$ (this fixes A_+) and γ consequently small. □

We note that inequality (4.19) is strict even assuming (3.2) (which is not strict). This improvement in passing from W to V_W will be used later on.

By Proposition 4.1 we can prove Thm 2.1.

We construct a sequence $\{V_n\}_{n=1}^\infty$ defined by

$$
V_n = V_{V_{n-1}}, \qquad n \ge 2 \tag{4.31}
$$

setting $V_1 = W$, W being any function with properties 3.1), 3.2) and 3.3). By Proposition 4.1 such properties hold for the whole sequence (for suitable values of A_+ and t_0 independent of n). By compactness (the sequence is equibounded and equicontinuous), we can extract a subsequence converging to a limit point $V = V(t)$. Let $f(t)$, $t \ge 0$ be solution to Eq. (2.1) with boundary conditions (2.3) given by $V(t)$, then the couple (f, V) solves problem (2.5), (2.6) for $t \ge 0$. We will prove this by showing that the characteristics solving (2.1) with boundary conditions given by $V_n(t)$ converge to characteristics solving (2.1) with boundary conditions given by $V(t)$. In order to avoid too heavy notation we consider the one dimensional case, the general one being an immediate transposition of it. For $t > 0$ and v given, consider the equation in $\tau < t$:

$$
X(t) - X(\tau) = v(t - \tau), \tag{4.32}
$$

where $\dot{X} = V$. This is a right recollision condition of a fluid particle with the disk in the limit dynamics and from Lemma 3.1 (which holds for the limit velocity $V(t)$ as well), we know that a necessary and sufficient condition for a solution to (4.32) to exist is:

$$\langle V \rangle_t < v < V(t) \tag{4.33}$$

since $v = \langle V \rangle_{\tau,t}$ is a continuous function of τ, such that $\langle V \rangle_{0,t} = \langle V \rangle_t$ and $\langle V \rangle_{t,t} = V(t)$. Let τ_* be the maximal time for which (4.32) is verified. It is characterized by the condition

$$X(s) < X(t) - v(t-s), \quad s \in (\tau_*, t). \tag{4.34}$$

Parallel to (4.32) and (4.34) we consider the equations

$$X_n(t) - X_n(\tau) = v(t-\tau), \tag{4.35}$$

$$X_n(s) < X_n(t) - v(t-s), \quad s \in (\tau, t). \tag{4.36}$$

Since V_n is converging to V, choosing n large enough we have

$$\langle V_n \rangle_t < v < V_n(t) \tag{4.37}$$

so that a maximal solution does exist also in this case and we denote it by τ_n. By compactness $\tau_n \to \bar{\tau}$ (extracting a subsequence if necessary). We want to show that $\bar{\tau} = \tau_*$. In fact, by (4.35) and (4.36) we get in the limit $n \to \infty$,

$$X(t) - X(\bar{\tau}) = v(t-\bar{\tau}), \tag{4.38}$$

$$X(s) \le X(t) - v(t-s), \quad s \in (\bar{\tau}, t). \tag{4.39}$$

We exclude equality in Eq. (4.39) because it would correspond to a tangential collision, which is not considered because it is negligible. Therefore $\bar{\tau}$ should be another maximal solution, in contrast with the uniqueness of τ_*. Thus, with regard to the first backward recollision from the right, we have proven that $\tau_n \to \tau_*$. Whenever the trajectory of the fluid particle $(x(s), v(s))$, $s \in (0, t)$ delivers k collisions at times $\tau^1, \ldots, \tau^k$ in the limiting dynamics induced by $V(t)$, the characteristics $(x_n(s), v_n(s))$ induced by $V_n(t)$ perform the same number of collisions, for n sufficiently large and the collision times $\tau_n^1, \ldots, \tau_n^k$ do converge to $\tau^1, \ldots, \tau^k$ (up to extraction of the subsequence when necessary). This can be easily proven by iterating the above arguments. We are not considering infinitely many collisions by Proposition A.1.

The recollisions due to fluid particles coming from the left can be treated in the same way. We remark that, in two or three dimensions we have to exclude also the null measure set of initial conditions (x, v) for which $x(t_k) \in \partial D(t_k)$. Finally the convergence of the characteristics shows that $r^{\pm}_{V_n} \to r^{\pm}_{V}$, so that (f, V) is a solution to our friction problem.

To conclude the proof of Theorem 2.1, let us consider any solution (f, V) to problem (2.5) (2.6). By the continuity of V, there exists a time interval for which

$$V_\infty - V(t) < e^{-C_+ t}\gamma + \frac{A_+}{(1+t)^5}\gamma^3, \tag{4.40}$$

because it is obviously verified at time zero. Let T be the first time for which (4.40) is violated. The same arguments used to prove Proposition 4.1 (i) (replacing W by V) apply here to show that

$$V_\infty - V(t) \ge e^{-C_- t}\gamma \tag{4.41}$$

and for $t \in [0, \min(t_0, T))$:

$$\frac{d}{dt}(V_\infty - V(t)) \le 0. \tag{4.42}$$

Proceeding as in the proof of Proposition 4.1 (iii), since V enjoys the same properties as W for $t \in [0, T)$, we infer that (4.40) is still valid for $t = T$. Hence (4.40) holds globally in time.

This concludes the proof of Theorem 2.1.

Proof of Theorem 2.2. Consider now a solution (f, V) to problem (2.1), (2.6). The lower bound (2.20) will be obtained by considering the integration over the velocities v producing a single recollision in the past. This allows us to estimate explicitly $v(0, t; x, v)$ in Eq. (4.2). To this end we introduce $s_0 > 0$ defined as:

$$s_0 = \min\left\{s \in (0, t) : V(s) \ge \frac{V_0 + \langle V\rangle_{s,t}}{2}\right\}. \tag{4.43}$$

Such s_0 does exist by continuity, since by Lemma 3.2 we have at time 0:

$$V(0) = V_0 < \frac{V_0 + \langle V\rangle_t}{2} = \frac{V_0 + \langle V\rangle_{0,t}}{2}, \tag{4.44}$$

while at time $s = t$,

$$V(t) > \frac{V_0 + V(t)}{2} = \frac{V_0 + \langle V\rangle_{t,t}}{2}. \tag{4.45}$$

The set $\{(x, v)|x \in D(t), \langle V\rangle_t \le v_x \le \langle V\rangle_{s_0,t}\}$ generates a subfamily of characteristics which had at most one recollision with the disk in the past. Indeed, consider a light particle which is to collide at x and let it go back up to the time $s < t$ of the first recollision in the past. Then $v_x = \langle V\rangle_{s,t}$ for some $s \le s_0$. Hence, denoting by $v_x(s^-)$ the $x-$ component of the precollisional velocity, by (4.43) we have:

$$v_x(s^-) = -v_x + 2V(s) = -\langle V\rangle_{s,t} + 2V(s) \le V_0 \tag{4.46}$$

so that

$$v_x(0, t; x, v) = 2V(s) - v_x. \tag{4.47}$$

We now prove that s_0 is bounded from above and from below, independently of γ , in the following way:

$$\frac{1}{C_-}\log\frac{3}{2} \le s_0 \le \frac{\log 4}{C_+}, \tag{4.48}$$

provided that t is sufficiently large independently of γ.

Indeed s_0 is the minimal solution of the equation:

$$V(s_0) = \frac{V_0 + \langle V\rangle_{s_0,t}}{2} \tag{4.49}$$

which gives

$$V_\infty - V(s_0) = \frac{\gamma}{2} + \frac{\langle V_\infty - V\rangle_{s_0,t}}{2}. \tag{4.50}$$

By the use of property (2.18) for V we obtain:

$$e^{-C_+ s_0}\gamma \geq \frac{\gamma}{2} - \frac{A_+}{(1+s_0)^5}\gamma^3 + \frac{\langle V_\infty - V\rangle_{s_0,t}}{2} \geq \frac{\gamma}{2} - \frac{A_+}{(1+s_0)^5}\gamma^3. \tag{4.51}$$

For γ small, $A_+\gamma^2 \leq \frac{1}{4}$, so that

$$e^{-C_+ s_0} \geq \frac{1}{4} \tag{4.52}$$

and we have proved the right bound in (4.48).

To prove the left bound, again by (2.18) we have:

$$\begin{aligned}\langle V_\infty - V\rangle_{s_0,t} &\leq \frac{1}{t-s_0}\int_{s_0}^{t} d\tau \left(\gamma e^{-C_+\tau} + \frac{A_+\gamma^3}{(1+\tau)^5}\right)\\ &\leq \frac{1}{t-s_0}\left[\gamma \frac{e^{-C_+ s_0} - e^{-C_+ t}}{C_+} + \gamma^3\frac{A_+}{4}\right].\end{aligned} \tag{4.53}$$

Hence, for t sufficiently large independent of γ:

$$\langle V_\infty - V\rangle_{s_0,t} \leq \frac{2\gamma}{t}\left(\frac{1}{C_+} + \frac{\gamma^2 A_+}{4}\right) \leq \frac{\gamma}{3}. \tag{4.54}$$

Therefore (4.50) and (2.19) yield:

$$e^{-C_- s_0}\gamma \leq \frac{2}{3}\gamma, \tag{4.55}$$

proving also the left bound in (4.48).

Now we set

$$I(t) = \int_{D(t)} dx \int dv_\perp \int_{\langle V\rangle_t}^{\langle V\rangle_{s_0,t}} dv_x (v_x - V(t))^2 [e^{-\beta v^2(0,t;x,v)} - e^{-\beta v^2}]. \tag{4.56}$$

Note that, on the basis of the same arguments leading to inequality (3.6), we can show that $V(t) > \langle V\rangle_{s_0,t}$. Hence

$$r^+(t) \geq C I(t). \tag{4.57}$$

For $s \leq s_0$ by (4.43) we get:

$$\begin{aligned}v_x^2 - v_x^2(0,t;x,v) &= v_x^2 - (2V(s) - v_x)^2 = 4V(s)(\langle V\rangle_{s,t} - V(s))\\ &\geq 2V(s)(\langle V\rangle_{s,t} - V_0).\end{aligned} \tag{4.58}$$

The quantity in the right-hand side of (4.58) can be estimated from below in the following way:

$$2V(s)(\langle V\rangle_{s,t} - V_0) \geq V_0\gamma \tag{4.59}$$

for t sufficiently large independent of γ. Indeed, by Lemma 3.1 (iii) and Eq. (2.18):

$$\begin{aligned}\inf_{s\le t}(\langle V\rangle_{s,t} - V_0) &\ge \langle V\rangle_t - V_0 = \frac{1}{t}\int_0^t ds V(s) - V_0\\ &\ge \frac{1}{t}\int_0^t ds[\gamma(1-e^{-C_+s}) - \frac{A_+}{(1+s)^5}\gamma^3]\\ &\ge \gamma(1-\frac{1-e^{-C_+t}}{C_+t}) - \frac{A_+}{4t}\gamma^3 \ge \frac{\gamma}{2}. \end{aligned} \tag{4.60}$$

By these considerations, we have:

$$\begin{aligned} I(t) &\ge \beta\int_{D(t)} dx \int dv_\perp \int_{\langle V\rangle_t}^{\langle V\rangle_{s_0,t}} dv_x (v_x - V(t))^2 e^{-\beta v^2}[v_x^2 - v_x^2(0,t;x.v)]\\ &\ge C\gamma \int_{\langle V\rangle_t}^{\langle V\rangle_{s_0,t}} dv_x (v_x - V(t))^2 e^{-\beta v_x^2}\int_{|v_\perp|<\frac{C}{t}} dv_\perp e^{-\beta v_\perp^2}\\ &\ge \frac{C\gamma}{t^2}[(V(t)-\langle V\rangle_t)^3 - (V(t)-\langle V\rangle_{s_0,t})^3]\\ &= \frac{C\gamma}{t^2}[(V(t)-\langle V\rangle_{s,t})^2(\langle V\rangle_{s_0,t} - \langle V\rangle_t)], \end{aligned} \tag{4.61}$$

for some $s \in (0, s_0)$. We now estimate both differences appearing in (4.61) showing that they are both $O(\frac{1}{t})$.

Using Eq. (3.11) we have:

$$\langle V\rangle_{s_0,t} - \langle V\rangle_t = \frac{s_0}{t-s_0}[\langle V\rangle_t - \langle V\rangle_{s_0}]. \tag{4.62}$$

By estimate (4.48) we know that, for γ small, t_0 is much larger than s_0 so that, by monotonicity, we have for any $\tau \in (0, s_0)$:

$$V(\tau) < V(s_0) < \frac{V_0 + V_\infty}{2}, \tag{4.63}$$

after using Eq. (4.49). This implies that

$$\begin{aligned}\langle V\rangle_t - \langle V\rangle_{s_0} &= \left[\langle V_\infty - V\rangle_{s_0} - \langle V_\infty - V\rangle_t\right]\\ &\ge \Big[\frac{\gamma}{2} - \frac{1}{t}\int_0^t (V_\infty - V(\tau))d\tau\Big]. \end{aligned} \tag{4.64}$$

By (2.18):

$$\begin{aligned}&\int_0^t (V_\infty - V(\tau))d\tau\\ &\quad\le \gamma\int_0^t d\tau e^{-C_+\tau} + \gamma^3 A_+ \int_0^\infty d\tau \frac{1}{(1+\tau)^5} \le \frac{\gamma}{C_+} + \frac{\gamma^3 A_+}{4}. \end{aligned} \tag{4.65}$$

Hence we obtain:

$$\langle V\rangle_t - \langle V\rangle_{s_0} \ge \frac{\gamma}{2} - \frac{1}{t}[\frac{\gamma}{C_+} + \frac{\gamma^3 A_+}{4}] \ge \frac{\gamma}{4}, \tag{4.66}$$

for t large independently of γ. Thus by (4.62) and (4.66) we arrive at:

$$\langle V\rangle_{s_0,t} - \langle V\rangle_t \geq C\frac{\gamma}{t}. \tag{4.67}$$

Let us now estimate the remaining term in (4.61). It is:

$$V(t) - \langle V\rangle_{s,t} = \langle V_\infty - V\rangle_{s,t} - (V_\infty - V(t)). \tag{4.68}$$

Again by properties (2.18) and (2.19) we obtain:

$$\begin{aligned} V(t) - \langle V\rangle_{s,t} &\geq \frac{\gamma}{t-s}\int_s^t d\tau e^{-C_-\tau} - \gamma e^{-C_+t} - \frac{\gamma^3 A_+}{(1+t)^5} \\ &\geq \frac{\gamma}{t-s_0}\int_{s_0}^t d\tau e^{-C_-\tau} - \gamma e^{-C_+t} - \frac{\gamma^3 A_+}{(1+t)^5}, \end{aligned} \tag{4.69}$$

because $s \leq s_0$ and $e^{-C_-\tau}$ is decreasing in τ.

Consequently, for t sufficiently large independently of γ:

$$V(t) - \langle V\rangle_{s,t} \geq \gamma\frac{C}{t}. \tag{4.70}$$

Inserting estimates (4.67) and (4.70) in (4.61), by (4.57) we conclude that, for t sufficiently large, independently of γ,

$$r^+(t) \geq C\frac{\gamma^4}{t^5}. \tag{4.71}$$

Actually Eq. (4.71) holds (a fortiori) for any $t > t_0$, provided that γ is sufficiently small, since t_0 is diverging when γ is vanishing.

For $t \geq 2t_0$, by virtue of (4.71) and the Duhamel formula, it is:

$$\begin{aligned} V_\infty - V(t) &\geq e^{-C_-t}\gamma + \int_0^t ds e^{-C_-(t-s)} r^+(s) \\ &\geq e^{-C_-t}\gamma + C\int_{t_0}^t ds e^{-C_-(t-s)}\frac{\gamma^4}{s^5}. \end{aligned} \tag{4.72}$$

Now we have:

$$\int_{t_0}^t ds \frac{e^{-C_-(t-s)}}{s^5} \geq \frac{1-e^{-C_-(t-t_0)}}{C_-t^5} \geq \frac{1-e^{-C_-t_0}}{C_-t^5} = \frac{1-(\frac{\gamma}{C_+})^{\frac{1}{2}}}{C_-t^5} \geq \frac{C}{t^5}, \tag{4.73}$$

by (3.1) since $t \geq 2t_0$. Hence:

$$V_\infty - V(t) \geq e^{-C_-t}\gamma + C\frac{\gamma^4}{t^5}. \tag{4.74}$$

The last inequality fixes A_- and the proof is finally complete. □

5. Comments

In this paper we proved some significant and somehow surprising effects of recollisions in a suitable microscopic model of friction. Our techniques are perturbative and work only when the parameter $\gamma = V_\infty - V_0$ is small.

We did not prove uniqueness of the solution to problem (2.1)–(2.6). Such a property should follow from a rather detailed analysis of the entire recollision sequence. Such a deeper analysis would also improve the upper bound (2.18) as regards to γ-dependence. On the other hand we were able to outline the asymptotic behavior of the solution taking into explicit account one recollision only.

We emphasize that a small change in the model can cause a drastic change of the time asymptotics. For instance, assuming a lower bound on the vertical component of the gas particles velocity, namely $|v_\perp| > \varepsilon > 0$, two consecutive collisions may happen in a time interval of length at most $\frac{2R}{\varepsilon}$. This means that the memory effects are bounded in time and it can be proven that this implies an exponential decay.

In the present paper we have considered the case $0 < V_0 \leq V_\infty$. Of course other cases can be studied, for instance $V_0 \geq V_\infty$ or $0 = E = V_\infty$. Another physical interesting case is when the external field depends on the position of the disk. Unfortunately it seems hard to find a unique approach to all these cases: we analyzed the easiest one.

Let us briefly discuss the case $V_0 \geq V_\infty > 0$ which, apparently, is symmetric to ours. Our techniques give the paradoxical result that the difference $V(t) - V_\infty$ becomes negative before vanishing as $t \to \infty$. Indeed, let us suppose (by absurdum) that $V(t) > V_\infty$ for all times. Then $V(t) - V_\infty$ is decreasing in time (in particular $r^+ = 0$).

By the Duhamel formula:

$$\begin{aligned} V(t) - V_\infty &= (V_0 - V_\infty)e^{-\int_0^t K(s)ds} - \int_0^t ds e^{-\int_s^t K(\tau)d\tau} r^-(s) \\ &\leq (V_0 - V_\infty)e^{-F_0'(V_\infty)t} - \int_0^t ds e^{-F_0'(V_0)(t-s)} r^-(s). \end{aligned} \tag{5.1}$$

Analogously to what we have proven in Thm. 2.2 we can find that, for small $(V_0 - V_\infty)$ and large t:

$$r^-(s) > C\frac{(V_0 - V_\infty)^4}{t^{d+2}}. \tag{5.2}$$

Therefore, for large t we find a contradiction because $V(t) - V_\infty$ becomes negative. Moreover the positivity of $r^\pm(s)$ prevents $V(t) - V_\infty$ from becoming positive later on. Since there is a change of sign in the difference $V(t) - V_\infty$, a detailed analysis of the asymptotics is delicate. Even more involved is the case in which $E = 0$. Again after some time $V(t)$ becomes negative and the quantities in the square brackets in (2.16) and (2.17) are no more positive, while in our paper the positivity of $r^\pm$ played an important role. More generally, the cases in which there is a change of sign of the velocity of the disk, for instance when $V_0 < 0$ or when E is not constant, are beyond a straightforward application of the techniques used in the present paper.

We did not make explicit the dependence on β of the constants, even if it is reasonable to believe that the long tail memory becomes irrelevant as $\beta \to \infty$. Indeed, in the limiting case, all the gas particles are initially at rest and the recollisions are absent when $|V_\infty - V_0|$ is small. This is because the first collision yields an outgoing velocity larger than $2V_0 > V_\infty$, so that the gas particles cannot be hit anymore. On the other hand, the probability of finding a post-collisional velocity between V_0 and V_∞, is vanishing as $\beta \to 0$.

We incidentally observe that for $E = 0$ and $\frac{1}{\beta} = 0$, the asymptotic behavior is not exponential, even neglecting recollisions. In fact in this case $F_0(V) = CV|V|$ and hence

$$V(t) = \frac{V_0}{1 + CV_0 t}. \tag{5.3}$$

We finally remark that in the present paper we have essentially studied the asymptotic behavior of the motion of the solid body. It would also be interesting to understand the behavior of the Vlasov fluid. In particular one may ask whether the velocity distribution at a given point (say the origin) converges to the Maxwellian when $t \to \infty$. This is not true in one dimension. Indeed a light particle with velocity $v < -V_\infty$, at a large time, has surely collided with the disk in the past, while in higher dimension the transversal velocity makes this event exceptional.

Appendix

We give an heuristic derivation of our model in the one-dimensional case. The case of a d-dimensional disk follows with minor modifications.

Denoting by V and M velocity and mass of the heavy particle and by v and m velocity and mass of a gas particle, the law of elastic collision says that:

$$V' = V + \frac{2m}{M+m}(v - V); \quad v' = V - \frac{M-m}{M+m}(v - V), \tag{A.1}$$

where V' and v' are the outgoing velocities.

As usual in the mean field limit, we assume the mass of any light particle to be $m = \frac{1}{N} << M$, N being the total number of the gas particles, so that, by (A.1), we have:

$$V' \approx V + \frac{2}{NM}(v - V); \quad v' \approx 2V - v. \tag{A.2}$$

We now evaluate the variation of the velocity ΔV of the heavy particle in the time interval $[t, t + \Delta t]$. It is:

$$\Delta V = E\Delta t - \frac{1}{N} \sum_{j \in I^+(\Delta t)} \frac{2}{M} |v_j - V| + \frac{1}{N} \sum_{j \in I^-(\Delta t)} \frac{2}{M} |v_j - V| + h, \tag{A.3}$$

where h denotes a term $o(\Delta t)$ and $I^\pm(\Delta t)$ denote the indices of the light particles which are colliding from the right ($v_j < V$) and from the left ($v_j \geq V$) respectively.

We finally apply our mean-field hypothesis by setting:

$$\frac{1}{N} \sum_{j \in I^\pm(\Delta t)} \frac{2}{M} |v_j - V| = \Delta t \frac{2}{M} \int dv |v - V|^2 f^\pm(X, v, t). \tag{A.4}$$

Taking the limit $\Delta t \to 0$, we obtain Eqs. (2.1)–(2.6). We also set $M = 1$, M being an irrelevant constant.

Proposition A. 1. *Consider the dynamics of the disk with given velocity $W = W(t)$ and the fluid trajectories $x(s,t;x,v)$, $v(s,t;x,v)$ computed according to the evolution of the disk and the law of the elastic reflection (2.2). Assume W differentiable for almost all t and such that*

$$ess\ sup_{t\in\mathbb{R}^+}(|W(t)| + |\dot{W}(t)|) = L < +\infty. \tag{A.5}$$

Then the set of all $t \in \mathbb{R}^+, x \in D(t), v \in \mathbb{R}^d$ for which $x(s,t;x,v)$, $v(s,t;x,v)$, $0 \le s < t$, delivers infinitely many backward collisions, or has a tangential collision, has vanishing Lebesgue measure.

Proof. We give the proof for the one-dimensional case, for notational simplicity.

For a given $T > 0$, we shall prove that the set of $(x,v) \in \mathbb{R}^2$ for which $x(s,T;x,v)$, $v(s,T;x,v)$, $s < T$ yields infinitely many collisions or has a tangential collision, has null measure. Then Proposition A.1 follows easily.

We consider a partition $I_1, \dots I_N$ of the time interval $[0,T)$ into intervals of the same length δ. Obviously $N = T/\delta$. We denote by t_k the middle point of I_k.

We shall not consider the case in which t_k is a collision time because it is a (x,v) measure zero event.

Consider the set

$$\begin{aligned} A_k^\delta = \{(x,v)|x(s_1,T;x,v) = X(s_1), \quad x(s_2,T;x,v) = X(s_2) \\ \text{for some} \quad s_1, s_2 \in I_k\}. \end{aligned} \tag{A.6}$$

Denote also by R_T the set of all configurations at time T leading (backward) to infinitely many collisions. Then, to have an accumulation point of the collision times, we necessarily have two consecutive collisions falling in the same time interval I_k for some k. Hence

$$R_T \subset \bigcup_k A_k^\delta \tag{A.7}$$

for all $\delta > 0$.

We finally set, for $s < T$:

$$\begin{aligned} D^\delta(s) = \{(x,v)||x(s,T;x,v) - X(s)| < 2L\delta, \\ |v(s,T;x,v) - W(s)| < 2L\delta\}. \end{aligned} \tag{A.8}$$

We shall prove that

$$A_k^\delta \subset D^\delta(t_k). \tag{A.9}$$

By Eq. (A.9) we easily conclude the proof. Indeed, by the time invariance (with respect to the time evolution of the fluid particle flow) of the Lebesgue measure, the map $(x,v) \to (x(s,T;x,v), v(s,T;x,v))$ has unitary Jacobian and hence

$$|D^\delta(t_k)| \le C\delta^2, \tag{A.10}$$

where $|A|$ denotes the Lebesgue measure of the set A. Therefore

$$|R_T| \le \sum_{k=1}^N |A_k^\delta| \le \sum_{k=1}^N |D^\delta(t_k)| \le CN\delta^2 = C\delta. \tag{A.11}$$

By the arbitrariness of δ we conclude that $|R_T| = 0$.

Moreover the set Z_T of all (x, v) leading to a tangential collision, i.e.

$$x(s, T; x, v) = X(s); \quad v(s, T; x, v) = W(s)$$

for some $s \in I_k$, $k = 1 \dots N$, trivially satisfies:

$$Z_T \subset \bigcup_k D^\delta(t_k)$$

and hence has vanishing measure.

To prove (A.9) let $(x, v) \in A_k^\delta$ and s_1 and s_2, with $s_1 < s_2$, be two consecutive collision instants in I_k. Then, if $t_k \in (s_1, s_2)$,

$$v(t_k)(s_2 - s_1) = \int_{s_1}^{s_2} W(s)ds, \tag{A.12}$$

and hence

$$v(t_k) = W(\bar{s}), \tag{A.13}$$

for some $\bar{s} \in (s_1, s_2)$. Here and in the sequel we shall use the shorthand notation $v(s) = v(s, T; x, v)$, $x(s) = x(s, T; x, v)$. Therefore

$$|v(t_k) - W(t_k)| = |\int_{\bar{s}}^{t_k} \dot{W}(s)ds| \le L\delta. \tag{A.14}$$

On the other hand if $t_k \in I_k/(s_1, s_2)$, say for instance $t_k < s_1 < s_2$, with no collision in (t_k, s_1), then we have simultaneously:

$$v(s_1^+) = W(\bar{s}) \tag{A.15}$$

for some $\bar{s} \in (s_1, s_2)$ and

$$v(s_1^+) = 2W(s_1) - v(t_k). \tag{A.16}$$

Hence

$$|v(t_k) - W(t_k)| = |2W(s_1) - W(\bar{s}) - W(t_k)| \le 2L\delta. \tag{A.17}$$

Finally, let s_1 be the closest collision time to t_k, say for instance, $s_1 < t_k$. Then

$$|X(t_k) - x(t_k)| \le \int_{s_1}^{t_k} ds|v(s_1^+) - W(s)| \le 2L\delta. \tag{A.18}$$

In fact if (x, v) develops at least two collisions in I_k, then $\sup_{s \in I_k} |v(s)| < 2L$. □

We finally remark that it is possible to prove that infinitely many collisions in a finite time interval cannot occur, for all initial data (x, v), just for geometrical reasons, provided that W has bounded second derivative. However such an extra regularity property does not follow easily from our arguments.

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