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Poisson Quasi-Nijenhuis Manifolds

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Abstract: We introduce the notion of Poisson quasi-Nijenhuis manifolds generalizing Poisson-Nijenhuis manifolds of Magri-Morosi. We also investigate the integration problem of Poisson quasi-Nijenhuis manifolds. In particular, we prove that, under some topological assumption, Poisson (quasi)-Nijenhuis manifolds are in one-one correspondence with symplectic (quasi)-Nijenhuis groupoids. As an application, we study generalized complex structures in terms of Poisson quasi-Nijenhuis manifolds. We prove that a generalized complex manifold corresponds to a special class of Poisson quasi-Nijenhuis structures. As a consequence, we show that a generalized complex structure integrates to a symplectic quasi-Nijenhuis groupoid, recovering a theorem of Crainic.

Contents

1. Introduction

Poisson Nijenhuis structures were introduced by Magri and Morosi [16, 18] in their study of bi-Hamiltonian systems, and intensively studied by many authors [12, 21]. Recall that a Poisson Nijenhuis manifold consists of a triple (M, π, N), where M is a

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manifold endowed with a Poisson bivector field π, and a $(1, 1)$-tensor N whose Nijenhuis torsion vanishes, i.e.

$$[NX, NY] - N([NX, Y] + [X, NY] - N[X, Y]) = 0, \qquad \forall X, Y \in \mathfrak{X}(M),$$

together with some compatibility condition between π and N. Poisson Nijenhuis structures are very important in the study of integrable systems since they produce bi-Hamiltonian systems [16, 12].

As observed by Kosmann-Schwarzbach [11], given a Poisson Nijenhuis manifold (M, π, N), $((T^*M)_\pi, (TM)_N)$ constitutes a Lie bialgebroid, where $(T^*M)_\pi$ is equipped with the standard cotangent Lie algebroid structure induced by the Poisson tensor π while $(TM)_N$ is the deformed Lie algebroid on TM induced by the Nijenhuis endomorphism N. Indeed it is proved in [11] that the Lie bialgebroid condition on $((T^*M)_\pi, (TM)_N)$ is equivalent to the triple (M, π, N) being Poisson Nijenhuis.

The main goal of the present paper is to introduce the notion of Poisson quasi-Nijenhuis structures. By definition, a Poisson quasi-Nijenhuis manifold is a quadruple (M, π, N, ϕ), where M is a manifold endowed with a Poisson bivector field π, a $(1, 1)$-tensor N and a closed 3-form ϕ such that π and N are compatible (in the usual Poisson-Nijenhuis sense) and

$$[NX, NY] - N([NX, Y] + [X, NY] - N[X, Y]) = \pi^\sharp(i_{X\wedge Y}\phi), \qquad \forall X, Y \in \mathfrak{X}(M).$$

Recall that Lie bialgebroids are pairs of transverse Dirac structures in a Courant algebroid [13]. When one of the two maximal isotropic direct summands fails to be Courant involutive, this becomes a quasi-Lie bialgebroid [20, 19]. Alternatively, a quasi-Lie bialgebroid is equivalent to the following data: a Lie algebroid A together with a degree 1 derivation δ of the associated Gerstenhaber algebra $\big(\Gamma(\wedge^\bullet A), \wedge, [\cdot, \cdot]\big)$ such that $\delta^2 = [\phi, \cdot]$ and $\delta\phi = 0$ for some $\phi \in \Gamma(\wedge^3 A)$ [9]. We prove

Theorem A. *Given (M, π, N, ϕ), the following are equivalent*

- *(M, π, N, ϕ) is a Poisson quasi-Nijenhuis manifold;*
- *$\big((T^*M)_\pi, (TM)_N, \phi\big)$ is a quasi-Lie bialgebroid.*

It is well known that the global object corresponding to a Poisson manifold is a symplectic groupoid [2, 22]. It is natural to ask what is the global object integrating a Poisson Nijenhuis manifold. We prove

Theorem B. *The base manifold of a symplectic Nijenhuis groupoid is a Poisson Nijenhuis manifold. Moreover, there is a one-one correspondence between t-connected and t-simply connected symplectic Nijenhuis groupoids $(\Gamma \rightrightarrows M, \widetilde{\omega}, \widetilde{N})$ and integrable Poisson Nijenhuis manifolds (M, π, N).*

By a symplectic Nijenhuis groupoid, we mean a symplectic groupoid $(\Gamma \rightrightarrows M, \widetilde{\omega})$ equipped with a multiplicative $(1, 1)$-tensor $\widetilde{N} : T\Gamma \to T\Gamma$ such that $(\Gamma, \widetilde{\omega}, \widetilde{N})$ is a symplectic Nijenhuis structure. The main idea of the proof of Theorem B can be outlined as follows. One proves that Poisson Nijenhuis structures on a manifold M are in one-one correspondence with Lie bialgebroids $((T^*M)_\pi, \delta)$ satisfying the condition that $[\delta, d] = 0$, where d is the de Rham differential on M. The latter are the infinitesimal of symplectic Nijenhuis groupoids, as can be shown using the universal lifting theorem [9].

The same method can be used to prove an analogous result for Poisson quasi-Nijenhuis manifolds.

Theorem C. *The base manifold of a symplectic quasi-Nijenhuis groupoid is a Poisson quasi-Nijenhuis manifold. Moreover there is a one-one correspondence between t-connected and t-simply connected symplectic quasi-Nijenhuis groupoids* $\big(\Gamma \rightrightarrows M, \widetilde{\omega}, \widetilde{N}, t^*\phi - s^*\phi\big)$ *and integrable Poisson quasi-Nijenhuis manifolds* (M, π, N, ϕ).

A symplectic quasi-Nijenhuis groupoid is a symplectic groupoid $(\Gamma \rightrightarrows M, \widetilde{\omega})$ equipped with a multiplicative $(1, 1)$-tensor $\widetilde{N} : T\Gamma \to T\Gamma$ and a closed 3-form $\phi \in \Omega^3(M)$ such that $\big(\Gamma, \widetilde{\omega}, \widetilde{N}, t^*\phi - s^*\phi\big)$ is a symplectic quasi-Nijenhuis structure.

As an application, we study generalized complex structures in terms of Poisson quasi-Nijenhuis structures. The notion of generalized complex structures was introduced by Hitchin [8] and studied by Gualtieri [7] motivated by the study of mirror symmetry. It comprises both symplectic and complex structures as extreme cases. We show that on a generalized complex manifold (M, J), where

$$J = \begin{pmatrix} N & \pi^\sharp \\ \sigma_\flat & -N^* \end{pmatrix}$$

with $N^2 + \pi^\sharp\sigma_\flat = -\mathrm{id}$, the building units π, N and σ of J do exactly determine a Poisson quasi-Nijenhuis structure. Indeed, the endomorphism N can be used to define a derivation d_N of the Gerstenhaber algebra associated to the Lie algebroid $(T^*M)_\pi$. We prove

Theorem D. *The following are equivalent*

- *J is a generalized complex structure;*
- *$(M, \pi, N, d\sigma)$ is a Poisson quasi-Nijenhuis structure such that*

$$(TM)_N \oplus (T^*M)_\pi \xrightarrow{J} TM \oplus T^*M$$

is a Courant algebroid isomorphism.

A similar result (in a different form) was already proved by Crainic using a direct argument [4].

Since a generalized complex structure corresponds to a quasi-Nijenhuis manifold according to Theorem D, as a consequence, we prove

Theorem E. *Let J be a generalized complex structure as given by Eq.* (18)*, and* $(\Gamma \rightrightarrows M, \widetilde{\omega})$ *a t-connected and t-simply connected symplectic groupoid integrating* $\widetilde{(T^*M)_\pi}$. *Then there is a multiplicative* $(1, 1)$*-tensor* $\widetilde{N}$ *on* Γ *such that* $\big(\Gamma \rightrightarrows M, \widetilde{\omega}, \widetilde{N}, t^*d\sigma - s^*d\sigma\big)$ *is a symplectic quasi-Nijenhuis groupoid.*

This result, in a disguised form, was already proved by Crainic [4] using a different method.

Notations. We denote the bracket on the sections of a Courant algebroid by $[\![\cdot, \cdot]\!]$, except for the standard Courant bracket on $TM \oplus T^*M$, which is denoted by $(\!|\cdot, \cdot|\!)$. The Lie bracket of vector fields and its extension to polyvector fields (i.e. the Schouten bracket) are denoted by $[\cdot, \cdot]$. Any bundle map $B : T^*M \to TM$ induces a bracket on the space of 1-forms (see Eq. (8)). It is denoted by $[\cdot, \cdot]_B$ as well as its extension to the space of differential forms of all degrees. Finally, if $[\![\cdot, \cdot]\!]$ is a bracket on the space of sections of a vector bundle E of which J is a bundle endomorphism, then its deformation by J is denoted by $[\![\cdot, \cdot]\!]_J$ (see Eq. (19)).

2. Preliminaries

Definition 2.1 ([13]). *A Courant algebroid is a triple consisting of*

- *a vector bundle* $E \to M$ *equipped with a non-degenerate symmetric bilinear form* $\langle\cdot,\cdot\rangle$,
- *a skew-symmetric bracket* $[\![\cdot,\cdot]\!]$ *on* $\Gamma(E)$, *and*
- *a smooth bundle map* $E \xrightarrow{\rho} M$ *called the anchor, which induces a natural differential operator* $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ *defined by*

$$\langle \mathcal{D}f, A\rangle = \tfrac{1}{2}\rho(A)f$$

for all $f \in C^\infty(M)$ *and* $A \in \Gamma(E)$.

These structures must be compatible in the following sense: $\forall A, B, C \in \Gamma(E)$ *and* $\forall f, g \in C^\infty(M)$,

- $\rho([\![A, B]\!]) = [\rho(A), \rho(B)]$,
- $[\![[\![A, B]\!], C]\!] + [\![[\![B, C]\!], A]\!] + [\![[\![C, A]\!], B]\!] = \frac{1}{3}\mathcal{D}\big(\langle[\![A, B]\!], C\rangle + \langle[\![B, C]\!], A\rangle + \langle[\![C, A]\!], B\rangle\big)$,
- $[\![A, fB]\!] = f[\![A, B]\!] + \big(\rho(A)f\big)B - \langle A, B\rangle\mathcal{D}f$,
- $\rho\circ\mathcal{D} = 0$, *i.e.* $\langle\mathcal{D}f, \mathcal{D}g\rangle = 0$,
- $\rho(A)\langle B, C\rangle = \langle[\![A, B]\!] + \mathcal{D}\langle A, B\rangle, C\rangle + \langle B, [\![A, C]\!] + \mathcal{D}\langle A, C\rangle\rangle$.

Note that a Courant algebroid is not a Lie algebroid as the Jacobi identity is not satisfied.

Example 2.2 ([3]). The generalized tangent bundle $TM \oplus T^*M$ of a manifold M is a Courant algebroid, where the anchor is the projection onto the first component and the pairing and bracket are given, respectively, by

$$\langle X + \xi, Y + \eta\rangle = \tfrac{1}{2}\big(\xi(Y) + \eta(X)\big), \tag{1}$$

$$(\!|X + \xi, Y + \eta|\!) = [X, Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi + \tfrac{1}{2}d\big(\xi(Y) - \eta(X)\big), \tag{2}$$

$\forall X, Y \in \mathfrak{X}(M), \forall \xi, \eta \in \Omega^1(M)$.

Definition 2.3. *A Dirac structure is a smooth subbundle* L *of a Courant algebroid* E*, which is maximal isotropic with respect to* $\langle\cdot,\cdot\rangle$ *and whose space of sections* $\Gamma(L)$ *is closed under* $[\![\cdot,\cdot]\!]$. *It is thus naturally a Lie algebroid.*

It is well-known [23] that a Lie algebroid $(A, [\cdot,\cdot]_A, \rho_A)$ gives rise to a Gerstenhaber algebra $(\Gamma(\wedge^\bullet A), \wedge, [\cdot,\cdot]_A)$, and a degree 1 derivation δ_A of the graded commutative algebra $(\Gamma(\wedge^\bullet A^*), \wedge)$ such that $(\delta_A)^2 = 0$. Here δ_A is given by

$$\begin{aligned}(\delta_A\alpha)(X_0, X_1, \dots, X_n) = &\sum_{i=0}^{n}(-1)^i(\rho_A X_i)\alpha(X_0, \dots, \widehat{X_i}, \dots, X_n)\\ &+ \sum_{i<j}(-1)^{i+j}\alpha([X_i, X_j]_A, X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_n).\end{aligned} \tag{3}$$

A Lie bialgebroid [15, 14] is a pair of Lie algebroid structures on A and its dual A^* such that δ_{A^*} is a derivation of the Gerstenhaber algebra $(\Gamma(\wedge^\bullet A), \wedge, [\cdot,\cdot]_A)$ or, equivalently, such that δ_A is a derivation of the Gerstenhaber algebra $(\Gamma(\wedge^\bullet A^*), \wedge, [\cdot,\cdot]_{A^*})$. Since the bracket $[\cdot,\cdot]_{A^*}$ can be recovered from the derivation δ_{A^*}, one is led to the following alternative definition.

Definition 2.4. *A Lie bialgebroid is a pair* (A, δ) *consisting of a Lie algebroid* $(A, [\cdot,\cdot]_A, \rho_A)$ *and a degree* 1 *derivation* δ *of the Gerstenhaber algebra* $(\Gamma(\wedge^\bullet A), \wedge, [\cdot,\cdot]_A)$ *such that* $\delta^2 = 0$.

More generally, we can speak about quasi-Lie bialgebroids [20, 9].

Definition 2.5 ([9]). *A quasi-Lie bialgebroid is a triple* (A, δ, ϕ) *consisting of a Lie algebroid A, a degree* 1 *derivation* δ *of the Gerstenhaber algebra* $(\Gamma(\wedge^\bullet A), \wedge, [\cdot,\cdot]_A)$ *and an element* $\phi \in \Gamma(\wedge^3 A)$ *such that* $\delta^2 = [\phi, \cdot]_A$ *and* $\delta\phi = 0$.

The link between Courant, Lie bi- and quasi-Lie bialgebroids is given by the following

Theorem 2.6 ([13, 20, 19]). (i) *There is a* 1-1 *correspondence between Lie bialgebroids and pairs of transversal Dirac structures in a Courant algebroid.*

(ii) *There is a* 1-1 *correspondence between quasi Lie bialgebroids and Dirac structures with transversal isotropic complements in a Courant algebroid.*

Proof. The proof of (i) can be found in [13], and (ii) was proved in [20, 19]. Below we give an explicit formula describing such a correspondence, which will be needed later.

Let (A, δ, ϕ) be a quasi Lie bialgebroid. Let $\rho_{A*} : A^* \to TM$ be the bundle map given by

$$\rho_{A^*}(\xi)(f) = \xi(\delta f), \quad \forall \xi \in A^*, \ \forall f \in C^\infty(M).$$

Introduce a bracket on $\Gamma(A^*)$ by

$$[\xi, \eta]_{A^*}(X) = (\rho_{A^*}\xi)(\eta X) - (\rho_{A^*}\eta)(\xi X) - (\delta X)(\xi, \eta).$$

Note that $(A^*, \rho_{A^*}, [\cdot,\cdot]_{A*})$ is in general not a Lie algebroid. Let $E = A^* \oplus A$ and $\rho : E \to TM$ be the bundle map

$$\rho(\xi + X) = \rho_{A^*}(\xi) + \rho_A(X).$$

Define a non-degenerate symmetric pairing on E by

$$\langle \xi + X, \eta + Y \rangle = \tfrac{1}{2}\big(\xi(Y) + \eta(X)\big),$$

and a bracket $[\![\cdot,\cdot]\!]$ on $\Gamma(E)$ by

$$\begin{aligned} [\![X, Y]\!] &= [X, Y]_A, \\ [\![\xi, \eta]\!] &= [\xi, \eta]_{A^*} + \phi(\xi, \eta, \cdot), \\ [\![X, \xi]\!] &= \big(i_X \delta_{A^*}\xi + \tfrac{1}{2}\delta_{A^*}(\xi X)\big) - \big(i_\xi \delta_A X + \tfrac{1}{2}\delta_A(\xi X)\big), \end{aligned} \tag{4}$$

for all $X, Y \in \Gamma(A)$ and $\xi, \eta \in \Gamma(A^*)$. Here $\delta_{A*} : \Gamma(\wedge^\bullet A^*) \to \Gamma(\wedge^{\bullet+1} A^*)$ is the derivation given by Eq. (3). Then $(E, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!], \rho)$ is a Courant algebroid.

Conversely, assume that $(E, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!], \rho)$ is a Courant algebroid, and A is a Dirac structure with an isotropic complement B. The duality pairing

$$A \otimes B \to \mathbb{R} : X \otimes \xi \mapsto 2\langle \xi, X \rangle$$

identifies B with A^*. Let ϕ be the element in $\Gamma(\wedge^3 A)$ defined by

$$\phi(\xi, \eta, \zeta) = 2\langle [\![\xi, \eta]\!], \zeta \rangle, \quad \forall \xi, \eta, \zeta \in \Gamma(B), \tag{5}$$

$\rho_B = \rho|_B$ be the restriction of ρ to B and $[\cdot,\cdot]_B$ be the bracket on $\Gamma(B)$ such that

$$[\![\xi, \eta]\!] - [\xi, \eta]_B \in \Gamma(A), \quad \forall \xi, \eta \in \Gamma(B). \tag{6}$$

Define a derivation $\delta : \Gamma(\wedge^\bullet A)(\cong \Gamma(\wedge^\bullet B^*)) \to \Gamma(\wedge^{\bullet+1} A)(\cong \Gamma(\wedge^{\bullet+1} B^*)$ as in Eq. (3). The triple (A, δ, ϕ) becomes a quasi-Lie bialgebroid. □

3. Poisson Quasi-Nijenhuis Manifolds

Let M be a smooth manifold, π a Poisson bivector field, and $N : TM \to TM$ a $(1, 1)$-tensor.

Definition 3.1 ([11]). *The bivector field π and the tensor N are said to be compatible* [12] *if*

$$N \circ \pi^\sharp = \pi^\sharp \circ N^{\mathrm{T}} \quad \textit{and} \quad C^N_{\pi^\sharp} = 0, \tag{7}$$

where

$$C^N_{\pi^\sharp}(\alpha, \beta) := [\alpha, \beta]_{N\pi^\sharp} - \big([N^{\mathrm{T}}\alpha, \beta]_{\pi^\sharp} + [\alpha, N^{\mathrm{T}}\beta]_{\pi^\sharp} - N^{\mathrm{T}}[\alpha, \beta]_{\pi^\sharp}\big)$$

and

$$[\alpha, \beta]_B := \mathcal{L}_{B\alpha}(\beta) - \mathcal{L}_{B\beta}(\alpha) - d\big(\beta(B\alpha)\big) \tag{8}$$

for all $\alpha, \beta \in \Omega^1(M)$ and any skew-symmetric bundle map $B : T^*M \to TM$.

The $(1, 1)$-tensor N is said to have zero Nijenhuis torsion if

$$[NX, NY] - N\big([NX, Y] + [X, NY] - N[X, Y]\big) = 0, \quad \forall X, Y \in \mathfrak{X}(M).$$

In [17], Magri and Morosi defined Poisson Nijenhuis manifolds as triples (M, π, N) such that π and N are compatible and the Nijenhuis torsion of N vanishes.

This definition is motivated by the following

Fact 3.2 ([12, 21]). Assume that $\pi \in \mathfrak{X}^2(M)$ is a Poisson tensor and $N : TM \to TM$ a $(1, 1)$-tensor on M. The tensor π_N defined by

$$\pi_N(\alpha, \beta) := \beta(N\pi^\sharp\alpha), \quad \forall \alpha, \beta \in \Omega^1(M)$$

is skew-symmetric if, and only if, $N \circ \pi^\sharp = \pi \circ^\sharp N^T$. In this case, we have

(i) $[\pi, \pi_N] = 0$ if $C^N_{\pi^\sharp} = 0$;
(ii) $[\pi_N, \pi_N] = 0$ if the Nijenhuis torsion of N vanishes.

Moreover the converse is true when π is non-degenerate.

Hence, any Poisson Nijenhuis manifold (M, π, N) is endowed with a bi-Hamiltonian structure (π, π_N), i.e.

$$[\pi, \pi] = 0, \qquad [\pi, \pi_N] = 0, \qquad [\pi_N, \pi_N] = 0.$$

Similarly, one can define Poisson quasi-Nijenhuis manifolds.

Let i_N be the degree 0 derivation of $(\Omega^\bullet(M), \wedge)$ defined by

$$(i_N\alpha)(X_1, \dots, X_p) = \sum_{i=1}^{p} \alpha(X_1, \dots, NX_i, \dots, X_p), \quad \forall \alpha \in \Omega^p(M).$$

Definition 3.3. *A Poisson quasi-Nijenhuis manifold is a quadruple (M, π, N, ϕ), where $\pi \in \mathfrak{X}^2(M)$ is a Poisson bivector field, $N : TM \to TM$ is a $(1, 1)$-tensor compatible with π, and ϕ is a closed 3-form on M such that*

$$[NX, NY] - N\big([NX, Y] + [X, NY] - N[X, Y]\big) = \pi^\sharp(i_{X\wedge Y}\phi), \quad \forall X, Y \in \mathfrak{X}(M)$$

and $i_N\phi$ is closed.

It is well known that, on a Poisson manifold (M, π), the bracket on $\Omega^1(M)$ associated to the bundle map $\pi^\sharp$ through Eq. (8) makes T^*M into a Lie algebroid with anchor $\pi^\sharp : T^*M \to TM$. The usual cotangent bundle will be denoted by $(T^*M)_\pi$ when equipped with this Lie algebroid structure. More precisely, we have the following

Fact 3.4 ([2]). Let π be a bivector field on M. Then $[\pi, \pi] = 0$ if, and only if, $(T^*M)_\pi$ is a Lie algebroid.

On the other hand, defining a bracket $[\cdot, \cdot]_N$ on $\mathfrak{X}(M)$ by

$$[X, Y]_N = [NX, Y] + [X, NY] - N[X, Y], \quad \forall X, Y \in \mathfrak{X}(M)$$

as in [11], and considering $N : TM \to TM$ as an anchor map, we obtain a degree 1 derivation d_N of $(\Omega^\bullet(M), \wedge)$ inspired by Eq. (3):

$$\begin{aligned}(d_N\alpha)(X_0, X_1, \ldots, X_n) = &\sum_{i=0}^{n}(-1)^i (NX_i)\alpha(X_0, \ldots, \widehat{X_i}, \ldots, X_n)\\ &+ \sum_{i<j}(-1)^{i+j}\alpha([X_i, X_j]_N, X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_n).\end{aligned} \tag{9}$$

Moreover, as proved in [11], we have the following identity

$$d_N = [i_N, d] = i_N \circ d - d \circ i_N. \tag{10}$$

The following proposition extends a result of Kosmann-Schwarzbach [11, Prop. 3.2].

Proposition 3.5. *The quadruple (M, π, N, ϕ) is a Poisson quasi-Nijenhuis manifold if, and only if, $\big((T^*M)_\pi, d_N, \phi\big)$ is a quasi Lie bialgebroid and ϕ is a closed 3-form.*

This is an immediate consequence of Fact 3.4 and the following two lemmas.

Lemma 3.6 ([11, Proposition 3.2]). *Assume that $\pi \in \mathfrak{X}^2(M)$ is a Poisson tensor and $N : TM \to TM$ a $(1, 1)$-tensor on M. The differential d_N is a derivation of the graded Lie algebra $(\Omega^\bullet(M), [\cdot, \cdot]_{\pi^\sharp})$ if, and only if, π and N are compatible.*

Lemma 3.7. *Let (M, π) be a Poisson manifold and $N : TM \to TM$ a $(1, 1)$-tensor compatible with $\pi^\sharp$. Then $d_N^2 = [\phi, \cdot]_{\pi^\sharp}$ if, and only if,*

$$[NX, NY] - N([NX, Y] + [X, NY] - N[X, Y]) = \pi^\sharp(i_{X\wedge Y}\phi), \quad \forall X, Y \in \mathfrak{X}(M)$$

and $\pi^\# \circ (d\phi)_\flat = 0$, where $(d\phi)_\flat : \wedge^3 TM \to T^*M$ is the bundle map defined by $(d\phi)_\flat(u, v, w) = i_{u\wedge v\wedge w}d\phi$, $\forall u, v, w \in TM$.

Proof. It follows from an easy computation that

$$\begin{aligned}\big(d_N^2 f - [\phi, f]_{\pi^\sharp}\big)(X, Y) = (df)\big(&[NX, NY] - N([NX, Y]\\ &+ [X, NY] - N[X, Y]) - \pi^\sharp(i_{X\wedge Y}\phi)\big)\end{aligned}$$

for all $f \in C^\infty(M)$. Moreover, since $d \circ d_N + d_N \circ d = 0$, one has

$$\begin{aligned}d_N^2(df) - [\phi, df]_{\pi^\sharp} &= d(d_N^2 f) - \big(d[\phi, f]_{\pi^\sharp} - [d\phi, f]_{\pi^\sharp}\big)\\ &= d(d_N^2 f - [\phi, f]_{\pi^\sharp}) + [d\phi, f]_{\pi^\sharp}.\end{aligned}$$

Hence, $d_N^2 - [\phi, \cdot]_{\pi^\sharp}$ vanishes on 0- and exact 1-forms if, and only if,

$$[NX, NY] - N([NX, Y] + [X, NY] - N[X, Y]) = \pi^\sharp(i_{X\wedge Y}\phi), \quad \forall X, Y \in \mathfrak{X}(M)$$

and $[d\phi, f]_{\pi^\sharp} = 0$, $\forall f \in C^\infty(M)$. The latter is easily seen to be equivalent to $\pi^\#\circ(d\phi)_\flat = 0$. And in this case, since both d_N^2 and $[\phi, \cdot]_{\pi^\sharp}$ are derivations with respect to $\wedge$, we get $d_N^2 = [\phi, \cdot]_{\pi^\sharp}$. □

As an immediate consequence, we obtain the following result of Kosmann-Schwarzbach [11].

Corollary 3.8. *The triple (M, π, N) is a Poisson Nijenhuis manifold if, and only if, $((T^*M)_\pi, d_N)$ is a Lie bialgebroid.*

We now turn our attention to the particular case where the Poisson bivector field π is non-degenerate. Together with Lemma 3.6, the following two lemmas give another proof of the equivalence between the relation $[\pi, \pi_N] = 0$ and the compatibility condition (7) when π is non-degenerate (see Fact 3.2).

Lemma 3.9. *Assume that $\pi \in \mathfrak{X}^2(M)$ is a Poisson tensor and $N : TM \to TM$ a $(1, 1)$-tensor on M. Then π_N is a bivector field such that $[\pi, \pi_N] = 0$ if, and only if, all the squares in the following diagram commute.*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & C^\infty(M) & \xrightarrow{d_N} & \Omega^1(M) & \xrightarrow{d_N} & \Omega^2(M) & \xrightarrow{d_N} & \Omega^3(M) & \xrightarrow{d_N} & \cdots \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \pi^\sharp} & & \downarrow{\scriptstyle \pi^\sharp} & & \downarrow{\scriptstyle \pi^\sharp} & & \\ 0 & \longrightarrow & C^\infty(M) & \xrightarrow[{[\pi_N,\cdot]}]{} & \mathfrak{X}^1(M) & \xrightarrow[{[\pi_N,\cdot]}]{} & \mathfrak{X}^2(M) & \xrightarrow[{[\pi_N,\cdot]}]{} & \mathfrak{X}^3(M) & \xrightarrow[{[\pi_N,\cdot]}]{} & \cdots \end{array} \qquad (11)$$

Proof. We have $\pi^\sharp N^T = N\pi^\sharp$ (i.e. π_N is a bivector field) if, and only if, $\forall f \in C^\infty(M)$,

$$\begin{aligned} \pi^\sharp N^T df &= N\pi^\sharp df \\ \Leftrightarrow \pi^\sharp i_N df &= \pi_N^\sharp df \\ \Leftrightarrow \pi^\sharp d_N f &= [\pi_N, f]. \end{aligned} \qquad (12)$$

And $[\pi_N, \pi] = 0$ is equivalent to

$$\begin{aligned} & [\pi_N, \pi]^\sharp(df) = 0 \\ \Leftrightarrow\ & [[\pi_N, \pi], f] = 0 \\ \Leftrightarrow\ & [[\pi_N, f], \pi] + [\pi_N, [\pi, f]] = 0 \\ \Leftrightarrow\ & [\pi_N^\sharp df, \pi] + [\pi_N, \pi^\sharp df] = 0 \\ \Leftrightarrow\ & [\pi, \pi^\sharp N^T df] = [\pi_N, \pi^\sharp df] \\ \Leftrightarrow\ & [\pi, \pi^\sharp(i_N df)] = [\pi_N, \pi^\sharp df] \\ \Leftrightarrow\ & \pi^\sharp d(i_N df) = [\pi_N, \pi^\sharp df] \\ \Leftrightarrow\ & \pi^\sharp d_N(df) = [\pi_N, \pi^\sharp df] \end{aligned} \qquad (13)$$

for all $f \in C^\infty(M)$. Since both $\pi^\sharp\circ d_N$ and $[\pi_N, \pi^\sharp(\cdot)]$ are derivations of $(\Omega^\bullet(M), \wedge)$, the equivalence follows from Eqs. (12)–(13). □

Lemma 3.10. *Assume that* $\pi \in \mathfrak{X}^2(M)$ *is a non-degenerate Poisson tensor, and* $N : TM \to TM$ *is a* $(1,1)$*-tensor on* M*. If* π_N *is a bivector field and Diagram* (11) *commutes, then* d_N *is a derivation of* $[\cdot,\cdot]_{\pi^\sharp}$*.*

Proof. Since π is Poisson, we have

$$\pi^\sharp[\alpha,\beta]_{\pi^\sharp} = [\pi^\sharp\alpha, \pi^\sharp\beta], \quad \forall \alpha,\beta \in \Omega^\bullet(M).$$

Then, the Jacobi identity for the Schouten bracket gives

$$[\pi_N, \pi^\sharp[\alpha,\beta]_{\pi^\sharp}] = [[\pi_N, \pi^\sharp\alpha], \pi^\sharp\beta] + [\pi^\sharp\alpha, [\pi_N, \pi^\sharp\beta]],$$

which can be rewritten as

$$\pi^\sharp d_N\big([\alpha,\beta]_{\pi^\sharp}\big) = \pi^\sharp\big([d_N\alpha,\beta]_{\pi^\sharp} + [\alpha, d_N\beta]_{\pi^\sharp}\big)$$

since $\pi^\sharp \circ d_N = [\pi_N, \pi^\sharp(\cdot)]$. The conclusion follows from the invertibility of $\pi^\sharp$. □

The previous lemmas are used to prove the following

Proposition 3.11. (i) *Let* (M,π,N,ϕ) *be a Poisson quasi-Nijenhuis manifold. Then,*

$$[\pi,\pi_N] = 0, \tag{14}$$

and

$$[\pi_N,\pi_N] = 2\pi^\sharp(\phi). \tag{15}$$

(ii) *Conversely, assume that* $\pi \in \mathfrak{X}^2(M)$ *is a non-degenerate Poisson bivector field,* $N : TM \to TM$ *is a* $(1,1)$*-tensor and* ϕ *is a closed 3-form. If Eqs.* (14)–(15) *are satisfied, then* (M,π,N,ϕ) *is a Poisson quasi-Nijenhuis manifold.*

Proof. (i) Fact 3.2 implies Eq. (14). By Proposition 3.5, $\big((T^*M)_\pi, d_N, \phi\big)$ is a quasi-Lie bialgebroid. It is simple to see that its induced bivector field on M as in Proposition 4.8 of [9] is π_N. From Proposition 4.8 of [9], it follows that $[\pi_N,\pi_N] = 2\pi^\sharp(\phi)$.

(ii) Since $[\pi,\pi_N] = 0$, Lemma 3.9 implies that $\pi^\sharp \circ d_N = [\pi_N, \pi^\sharp(\cdot)]$ and Lemma 3.10 implies that d_N is a derivation of $[\cdot,\cdot]_{\pi^\sharp}$. Hence π and N are compatible by Lemma 3.6. Since π is non-degenerate, we may apply $(\pi^\sharp)^{-1}$ to Eq. (15). Then, making use of Lemma 3.9, we get back to $d_N^2 = [\phi,\cdot]_{\pi^\sharp}$. Equation (15) and the graded Jacobi identity yield $[\pi_N, \pi^\sharp(\phi)] = 0$. Applying $(\pi^\sharp)^{-1}$, we get $d_N\phi = 0$. □

Corollary 3.12. *Let* ω *be a symplectic* 2*-form and* ϕ *a closed* 3*-form on* M*. Then* (M,ω,N,ϕ) *is a symplectic quasi-Nijenhuis manifold if and only if*

$$[\omega_N,\omega_N] = 2\phi \quad \textit{and} \quad d\omega_N = 0,$$

where $[\cdot,\cdot]$ *stands for the Schouten bracket on* $\Omega^\bullet(M)$ *induced from the Lie algebroid* $(T^*M)_\pi$*, and* ω_N *is the* 2*-form on* M *defined by*

$$\omega_N(X,Y) = \omega(NX,Y), \quad \forall X,Y \in \mathfrak{X}(M).$$

Proof. It is well known that when π is non-degenerate, $\pi^\sharp$ is an isomorphism of differential Gerstenhaber algebras from $(\Omega^\bullet(M), d, [\cdot,\cdot])$ to $(\mathfrak{X}^\bullet(M), [\pi,\cdot], [\cdot,\cdot])$ [23, 10]. The conclusion thus follows immediately from Proposition 3.11 since $\pi^\sharp\omega_N = \pi_N$. □

Remark 3.13. Poisson Nijenhuis structures arise naturally in the study of integrable systems. It would be interesting to find applications of Poisson quasi-Nijenhuis structures in integrable systems as well.

4. Universal Lifting Theorem

In this section, we recall the universal lifting theorem and its basic ingredients, as it plays a crucial role in the following sections. For details, see [9].

Let $\Gamma \rightrightarrows M$ be a Lie groupoid, $A \to M$ its Lie algebroid and $\Pi \in \mathfrak{X}^k(\Gamma)$ a k-vector field on Γ. Define $F_\Pi \in C^\infty(T^*\Gamma \times_\Gamma \overset{(k)}{\cdots} \times_\Gamma T^*\Gamma)$ by

$$F_\Pi(\mu^1, \ldots, \mu^k) = \Pi(\mu^1, \ldots, \mu^k).$$

Definition 4.1. *A k-vector field $\Pi \in \mathfrak{X}^k(\Gamma)$ is multiplicative if, and only if, F_Π is a 1-cocycle with respect to the groupoid $T^*\Gamma \times_\Gamma \overset{(k)}{\cdots} \times_\Gamma T^*\Gamma \rightrightarrows A^* \times_M \overset{(k)}{\cdots} \times_M A^*$.*

Remark 4.2. It is simple to see that a bivector field Π is multiplicative if, and only if, the graph of the multiplication $\Lambda \subset \Gamma \times \Gamma \times \Gamma$ is coisotropic with respect to $\Pi \oplus \Pi \oplus \bar{\Pi}$, where $\bar{\Pi}$ denotes the opposite bivector field to Π.

Example 4.3. If $P \in \Gamma(\wedge^k A)$, then $\overrightarrow{P} - \overleftarrow{P}$ is multiplicative, where $\overrightarrow{P}$ and $\overleftarrow{P}$ denote, respectively, the right and left invariant k-vector fields on Γ corresponding to P.

By $\mathfrak{X}^k_{\mathrm{mult}}(\Gamma)$ we denote the space of all multiplicative k-vector fields on Γ. And $\mathfrak{X}_{\mathrm{mult}}(\Gamma) = \bigoplus_k \mathfrak{X}^k_{\mathrm{mult}}(\Gamma)$.

Proposition 4.4 ([9]). *The vector space $\mathfrak{X}_{\mathrm{mult}}(\Gamma)$ is closed under the Schouten bracket, and therefore is a graded Lie algebra.*

It is simple to show that for any given $\Pi \in \mathfrak{X}^k_{\mathrm{mult}}(\Gamma)$ and any $X \in \Gamma(\wedge^i A)$, the $(k+i-1)$-vector field $[\overleftarrow{X}, \Pi]$ is always left invariant. Define $\delta_\Pi X \in \Gamma(\wedge^{(k+i-1)} A)$ by

$$\overleftarrow{\delta_\Pi X} = [\overleftarrow{X}, \Pi].$$

Thus one obtains a linear operator $\delta_\Pi : \Gamma(\wedge^i A) \to \Gamma(\wedge^{(k+i-1)} A)$. Here we use the following convention: $\Gamma(\wedge^0 A) \cong C^\infty(M)$ and for any $f \in C^\infty(M)$, $\overleftarrow{f} = \beta^* f$. One easily checks that the following identities are satisfied:

$$\delta_\Pi(P \wedge Q) = (\delta_\Pi P) \wedge Q + (-1)^{p(k-1)} P \wedge \delta_\Pi Q,$$
$$\delta_\Pi[P, Q] = [\delta_\Pi P, Q] + (-1)^{(p-1)(k-1)}[P, \delta_\Pi Q],$$

for all $P \in \Gamma(\wedge^p A)$ and $Q \in \Gamma(\wedge^q A)$. This leads to the following definition of k-differentials. Recall that for any Lie algebroid $A \to M$, $(\Gamma(\wedge^\bullet A), \wedge, [\cdot, \cdot])$ is a Gerstenhaber algebra [23].

Definition 4.5. *A k-differential on a Lie algebroid A is a degree $(k-1)$ derivation of the Gerstenhaber algebra $(\Gamma(\wedge^\bullet A), \wedge, [\cdot, \cdot])$; i.e. a linear operator*

$$\delta : \Gamma(\wedge^\bullet A) \to \Gamma(\wedge^{\bullet+(k-1)} A)$$

satisfying

$$\delta(P \wedge Q) = (\delta P) \wedge Q + (-1)^{p(k-1)} P \wedge \delta Q,$$
$$\delta[P, Q] = [\delta P, Q] + (-1)^{(p-1)(k-1)}[P, \delta Q],$$

for all $P \in \Gamma(\wedge^p A)$ and $Q \in \Gamma(\wedge^q A)$. The set of k-differentials on A is denoted by $\mathcal{A}^k(A)$.

The space of all multi-differentials $\mathcal{A}(A) = \bigoplus_k \mathcal{A}^k(A)$ becomes a graded Lie algebra when endowed with the graded commutator:

$$[\delta_1, \delta_2] = \delta_1 \circ \delta_2 - (-1)^{(k-1)(l-1)} \delta_2 \circ \delta_1, \qquad \text{where } \delta_1 \in \mathcal{A}^k(A) \text{ and } \delta_2 \in \mathcal{A}^l(A).$$

Below is a list of basic examples.

Examples 4.6. (i) When A is a Lie algebra $\mathfrak{g}$, then k-differentials are in one-one correspondence with Lie algebra 1-cocycles $\delta : \mathfrak{g} \to \wedge^k \mathfrak{g}$ with respect to the adjoint action.

(ii) The 0-differentials correspond to sections $\phi \in \Gamma(A^*)$ such that $d_A \phi = 0$, i.e. Lie algebroid 1-cocycles with trivial coefficients.

(iii) The 1-differentials correspond to the infinitesimals of Lie algebroid automorphisms.

(iv) If $P \in \Gamma(\wedge^k A)$, then $\mathrm{ad}_P = [P, \cdot]$ is clearly a k-differential, which is called the *coboundary* k-differential associated to P.

(v) A Lie bialgebroid can be seen as a Lie algebroid together with a 2-differential of square zero. The converse is also true.

From the previous discussion, we know that there exists a linear map

$$\mathfrak{X}^\bullet_{\mathrm{mult}}(\Gamma) \to \mathcal{A}^\bullet(A) : \Pi \mapsto \delta_\Pi,$$

which is a Lie algebra homomorphism since the graded Jacobi identity satisfied by the Schouten bracket implies that

$$[\delta_\Pi, \delta_{\Pi'}] = \delta_{[\Pi, \Pi']}. \tag{16}$$

Moreover, one has the following

Universal Lifting Theorem ([9]). *Assume that $\Gamma \rightrightarrows M$ is a target-connected and target-simply connected Lie groupoid with Lie algebroid A. Then*

$$\mathfrak{X}^\bullet_{\mathrm{mult}}(\Gamma) \to \mathcal{A}^\bullet(A) : \Pi \mapsto \delta_\Pi$$

is an isomorphism *of graded Lie algebras.*

5. Symplectic Nijenhuis Groupoids

Definition 5.1. *A symplectic Nijenhuis groupoid is a symplectic groupoid $(\Gamma \rightrightarrows M, \widetilde{\omega})$ equipped with a multiplicative $(1,1)$-tensor $\widetilde{N} : T\Gamma \to T\Gamma$ such that $(\Gamma, \widetilde{\omega}, \widetilde{N})$ is a symplectic Nijenhuis structure.*

The main result of this section is the following

Theorem 5.2. (i) *The unit space of a symplectic Nijenhuis groupoid is a Poisson Nijenhuis manifold.*

(ii) *Every integrable Poisson Nijenhuis manifold is the unit space of a unique target-connected, target-simply connected symplectic Nijenhuis groupoid.*

Here, by an integrable Poisson Nijenhuis manifold, we mean the corresponding Poisson structure is integrable, i.e. it admits an associated symplectic groupoid. See [5, 6] for the solution of the integrability problem for Poisson manifolds and, more generally, Lie algebroids.

Recall that a Poisson Nijenhuis manifold (M, π, N) gives rise to a Lie bialgebroid $((T^*M)_\pi, d_N)$ according to Corollary 3.8. The following lemma gives a useful characterization of those Lie bialgebroids arising from Poisson Nijenhuis structures.

Lemma 5.3. *Let (M, π) be a Poisson manifold. A Lie bialgebroid $((T^*M)_\pi, \delta)$ is induced by a Poisson Nijenhuis structure if and only if $[\delta, d] = 0$, where d stands for the de Rham differential.*

Proof. If (M, π, N) is a Poisson Nijenhuis manifold, then $d_N = i_N \circ d - d \circ i_N$. Thus

$$[d_N, d] = d_N \circ d + d \circ d_N = (i_N \circ d - d \circ i_N) \circ d + d \circ (i_N \circ d - d \circ i_N) = 0.$$

Conversely, given a Lie bialgebroid $((T^*M)_\pi, \delta)$ such that $[\delta, d] = 0$, one obtains a Lie algebroid structure on TM. Let $N : TM \to TM$ be its anchor map. Thus $\delta = d_N : C^\infty(M) \to \Omega^1(M)$. Since $[\delta, d] = 0$, we have $\forall f \in C^\infty(M)$, $\delta(df) = -d\delta f = -dd_N f = d_N(df)$. It thus follows that $\delta = d_N$ on any differential forms since both δ and d_N are derivations and they agree on 0- and exact 1-forms. According to Corollary 3.8, it follows that (M, π, N) is a Poisson Nijenhuis manifold. □

Proof of Theorem 5.2. (i) FROM SYMPLECTIC NIJENHUIS GROUPOIDS TO POISSON NIJENHUIS MANIFOLDS. Assume that $(\Gamma, \widetilde{\omega}, \widetilde{N})$ is a symplectic Nijenhuis groupoid. Let $\widetilde{\pi}$ be the bivector field on Γ which is the inverse of $\widetilde{\omega}$ and let $\widetilde{\pi}_{\widetilde{N}} \in \mathfrak{X}^2(\Gamma)$ be the bivector field defined by $\widetilde{\pi}^\sharp_{\widetilde{N}} = \widetilde{N} \circ \widetilde{\pi}^\sharp$.

- Since $[\widetilde{\pi}, \widetilde{\pi}] = 0$, the induced bivector field $\pi = t_* \widetilde{\pi}$ on the base manifold of the symplectic groupoid $\Gamma \rightrightarrows M$ is Poisson [22]. The Lie algebroid of $\Gamma \to M$ is isomorphic to $(T^*M)_\pi$ [2]. And the multiplicative bivector field $\widetilde{\pi}$ corresponds to a 2-differential on $(T^*M)_\pi$, which is the de Rham differential d. That is, $((T^*M)_\pi, d)$ is the Lie bialgebroid corresponding to the symplectic groupoid $(\Gamma, \widetilde{\omega})$.
- As pointed out in Fact 3.2, $\widetilde{\pi}_{\widetilde{N}}$ is a Poisson tensor on Γ [16, 12, 21]. Moreover, $\widetilde{\pi}_{\widetilde{N}}$ is a multiplicative bivector field since $\widetilde{N}$ is a multiplicative $(1,1)$-tensor and $\widetilde{\pi}$ is a multiplicative bivector field. In other words, $(\Gamma, \widetilde{\pi}_{\widetilde{N}})$ is a Poisson groupoid [14]. Let $\delta_{\widetilde{\pi}_{\widetilde{N}}} : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ be the 2-differential on $(T^*M)_\pi$ induced by the multiplicative Poisson bivector field $\widetilde{\pi}_{\widetilde{N}}$ on Γ. Since $[\widetilde{\pi}_{\widetilde{N}}, \widetilde{\pi}_{\widetilde{N}}] = 0$, the universal lifting theorem implies that

$$0 = \delta_{[\widetilde{\pi}_{\widetilde{N}}, \widetilde{\pi}_{\widetilde{N}}]} = [\delta_{\widetilde{\pi}_{\widetilde{N}}}, \delta_{\widetilde{\pi}_{\widetilde{N}}}] = \delta_{\widetilde{\pi}_{\widetilde{N}}} \circ \delta_{\widetilde{\pi}_{\widetilde{N}}} + \delta_{\widetilde{\pi}_{\widetilde{N}}} \circ \delta_{\widetilde{\pi}_{\widetilde{N}}} = 2\delta^2_{\widetilde{\pi}_{\widetilde{N}}}.$$

  Thus, $((T^*M)_\pi, \delta_{\widetilde{\pi}_{\widetilde{N}}})$ is a Lie bialgebroid.
- Likewise, it is standard that $[\widetilde{\pi}_{\widetilde{N}}, \widetilde{\pi}] = 0$. Thus the universal lifting theorem implies that $[\delta_{\widetilde{\pi}_{\widetilde{N}}}, d] = 0$. According to Lemma 5.3, $\delta_{\widetilde{\pi}_{\widetilde{N}}} = d_N$ for some Nijenhuis tensor N on M and (M, π, N) is a Poisson Nijenhuis manifold.

(ii) FROM POISSON NIJENHUIS MANIFOLDS TO SYMPLECTIC NIJENHUIS GROUPOIDS. Given a Poisson Nijenhuis manifold (M, π, N), then $((T^*M)_\pi, d_N)$ is a Lie bialgebroid by Corollary 3.8. Assume that $(T^*M)_\pi$ is integrable (see [5, 6] for the integrability condition) and $(\Gamma \rightrightarrows M, \widetilde{\omega})$ is a target-connected and target simply-connected symplectic groupoid of M. Since $d_N^2 = 0$ and $[d_N, d] = 0$, the universal lifting theorem implies that d_N corresponds to a multiplicative Poisson bivector field $\widetilde{\pi}_{\widetilde{N}}$ on Γ such that $[\widetilde{\pi}_{\widetilde{N}}, \widetilde{\pi}] = 0$, where $\widetilde{\pi}$ is the Poisson tensor on Γ inverse to $\widetilde{\omega}$. Let $\widetilde{N} = \widetilde{\pi}^{\sharp}_{\widetilde{N}} \circ \widetilde{\omega}_{\flat} : T\Gamma \to T\Gamma$. Then it is clear that $\widetilde{N}$ is a multiplicative $(1, 1)$-tensor, and $(\Gamma, \widetilde{\omega}, \widetilde{N})$ is a symplectic Nijenhuis groupoid.

Since these two constructions are inverse to each other, the theorem is proved. □

6. Symplectic Quasi-Nijenhuis Groupoids

The goal of this section is to generalize Theorem 5.2 to the quasi-setting. More precisely, we will give an integration theorem for Poisson quasi-Nijenhuis manifolds.

Definition 6.1. *A symplectic quasi-Nijenhuis groupoid is a symplectic groupoid* $(\Gamma \rightrightarrows M, \widetilde{\omega})$ *equipped with a multiplicative* $(1, 1)$*-tensor* $\widetilde{N} : T\Gamma \to T\Gamma$ *and a closed* 3*-form* $\phi \in \Omega^3(M)$ *such that* $(\Gamma, \widetilde{\omega}, \widetilde{N}, t^*\phi - s^*\phi)$ *is a symplectic quasi-Nijenhuis structure.*

The following result is a generalization of Theorem 5.2.

Theorem 6.2. (i) *The unit space of a symplectic quasi-Nijenhuis groupoid is a Poisson quasi-Nijenhuis manifold.*

(ii) *Every integrable Poisson quasi-Nijenhuis manifold* (M, π, N, ϕ) *is the unit space of a unique target-connected and target-simply connected symplectic quasi-Nijenhuis groupoid* $(\Gamma \rightrightarrows M, \widetilde{\omega}, \widetilde{N}, t^*\phi - s^*\phi)$.

Proof. The proof is similar to that of Theorem 5.2, so we will merely sketch it.

Assume that (M, π, N, ϕ) is an integrable Poisson quasi-Nijenhuis manifold. Let $\Gamma \rightrightarrows M$ be a target-connected and target-simply connected groupoid integrating the Lie algebroid $(T^*M)_\pi$. By Proposition 3.5, $((T^*M)_\pi, d_N, \phi)$ is a quasi-Lie bialgebroid, which integrates to a quasi-Poisson groupoid by the universal lifting theorem. Let $\widetilde{\pi}_{\widetilde{N}} \in \mathfrak{X}(\Gamma)$ be the bivector field on Γ corresponding to d_N. Then we have

$$\tfrac{1}{2}[\widetilde{\pi}_{\widetilde{N}}, \widetilde{\pi}_{\widetilde{N}}] = \overrightarrow{\phi} - \overleftarrow{\phi}.$$

On the other hand, we know that $\Gamma \rightrightarrows M$ is a symplectic groupoid, whose corresponding Lie bialgebroid is $((T^*M)_\pi, d)$. The symplectic form on Γ is denoted by $\widetilde{\omega}$. Let $\widetilde{\pi} \in \mathfrak{X}^2(\Gamma)$ be its corresponding Poisson tensor. Since $[d_N, d] = 0$, we have $[\widetilde{\pi}_{\widetilde{N}}, \widetilde{\pi}] = 0$ according to the universal lifting theorem. Let $\widetilde{N} = \widetilde{\pi}^{\sharp}_{\widetilde{N}} \circ \widetilde{\omega}_{\flat} : T\Gamma \to T\Gamma$. Then it is clear that $\widetilde{N}$ is a multiplicative $(1, 1)$-tensor. Since $\overrightarrow{\phi} - \overleftarrow{\phi} = \widetilde{\pi}^{\sharp}(t^*\phi - s^*\phi)$, from Proposition 3.11, it follows that $(\Gamma, \widetilde{\omega}, \widetilde{N}, t^*\phi - s^*\phi)$ is a symplectic quasi-Nijenhuis groupoid.

The other direction can be proved by going backwards. □

Remark 6.3. Note that $\widetilde{\omega}_{\flat}(\widetilde{\pi}_{\widetilde{N}})$ is a multiplicative 2-form on $\Gamma \rightrightarrows M$. It would be interesting to see what is the corresponding Dirac structure on M and how the integration result in [1] can be applied to this situation.

7. Generalized Complex Structures

This section is devoted to the investigation of the relationship between generalized complex structures and Poisson quasi-Nijenhuis structures. Let us first recall the definition of generalized complex structures [8, 7].

Definition 7.1. *A generalized complex structure on a manifold M is a bundle map*

$$J : TM \oplus T^*M \to TM \oplus T^*M$$

satisfying the algebraic properties

$$J^2 = -I \qquad \text{and} \qquad \langle Jv, Jw \rangle = \langle v, w \rangle \tag{17}$$

and the integrability condition

$$(\!| Jv, Jw |\!) - (\!| v, w |\!) - J((\!| Jv, w |\!) + (\!| v, Jw |\!)) = 0$$

$\forall v, w \in \Gamma(TM \oplus T^*M)$*. Here $\langle \cdot, \cdot \rangle$ and $(\!| \cdot, \cdot |\!)$ are the pairing and bracket on the standard Courant algebroid $TM \oplus T^*M$ as in Example 2.2.*

The first two algebraic conditions (17) imply that J must be of the form

$$J = \begin{pmatrix} N & \pi^\sharp \\ \sigma_\flat & -N^* \end{pmatrix}, \tag{18}$$

where $\pi \in \mathfrak{X}^2(M)$ is a bivector field, $\sigma \in \Omega^2(M)$ is a 2-form and $N : TM \to TM$ is a $(1, 1)$-tensor. Here $\sigma_\flat : TM \to T^*M$ is the map given by $(\sigma_\flat X)(Y) = \sigma(X, Y)$, $\forall X, Y \in \mathfrak{X}(M)$.

On the other hand, a Courant algebroid can be deformed using a bundle map J. More precisely, let $(E, \langle \cdot, \cdot \rangle, [\![\cdot, \cdot]\!], \rho)$ be a Courant algebroid over M and let

$$\begin{array}{ccc} E & \xrightarrow{J} & E \\ \downarrow & & \downarrow \\ M & \xrightarrow[\text{id}]{} & M \end{array}$$

be a vector bundle automorphism of $E \to M$. Consider

- the inner product
$$\langle A, B \rangle_J = \langle JA, JB \rangle,$$
- the bracket
$$[\![A, B]\!]_J = [\![JA, B]\!] + [\![A, JB]\!] - J[\![A, B]\!], \tag{19}$$
- and the bundle map
$$\rho_J = \rho \circ J$$

induced by J.

A natural question is

Question 7.2. When is the quadruple $(E, \langle \cdot, \cdot \rangle_J, [\![\cdot, \cdot]\!]_J, \rho_J)$ still a Courant algebroid?

The next proposition gives a trivial sufficient condition.

Proposition 7.3. *The quadruple* $(E, \langle\cdot,\cdot\rangle_J, [\![\cdot,\cdot]\!]_J, \rho_J)$ *is a Courant algebroid if*

$$[\![JA, JB]\!] + J^2[\![A, B]\!] - J\big([\![JA, B]\!] + [\![A, JB]\!]\big) = 0, \quad \forall A, B \in \Gamma(E).$$

Moreover, in this case, J is a Courant algebroid isomorphism from $(E, \langle\cdot,\cdot\rangle_J, [\![\cdot,\cdot]\!]_J, \rho_J)$ *to* $(E, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!], \rho)$.

We now give an answer to Question 7.2 in the special case of the standard Courant algebroid $TM \oplus T^*M$, where J satisfies Eqs. (17), and is given by Eq. (18) .

Lemma 7.4. *Assume that* $J : TM\oplus T^*M \to TM\oplus T^*M$ *is given by Eq.* (18). *Let* $(\!|\cdot,\cdot|\!)_J$ *be the deformed bracket on* $\mathfrak{X}(M) \oplus \Omega^1(M)$ *as in Eq.* (19). *Then, for all* $X, Y \in \mathfrak{X}(M)$ *and* $\xi, \eta \in \Omega^1(M)$*, we have*

$$(\!|\xi, \eta|\!)_J = [\xi, \eta]_{\pi^\sharp}, \tag{20}$$

$$(\!|X, Y|\!)_J = [X, Y]_N + (d\sigma)(X, Y, \cdot), \tag{21}$$

$$\begin{aligned}(\!|X, \xi|\!)_J = {} & \big([X, \pi^\sharp\xi] - \pi^\sharp(\mathcal{L}_X\xi - \tfrac{1}{2}d(\xi X))\big) + \big(\mathcal{L}_{NX}\xi - \mathcal{L}_X(N^T\xi) \\ & + N^T(\mathcal{L}_X\xi - \tfrac{1}{2}d(\xi X))\big).\end{aligned} \tag{22}$$

Proof. This follows from a straightforward computation using Eqs. (2) and (19), and is left for the reader. □

Proposition 7.5. *Let* $J : TM \oplus T^*M \to TM \oplus T^*M$ *be a bundle map which satisfies Eqs.* (17)*, and is given by Eq.* (18). *Then* $(TM \oplus T^*M, \langle\cdot,\cdot\rangle_J, (\!|\cdot,\cdot|\!)_J, \rho_J)$ *is a Courant algebroid if, and only if,* $(M, \pi, N, d\sigma)$ *is a Poisson quasi-Nijenhuis manifold. And in this case,* $(TM \oplus T^*M, \langle\cdot,\cdot\rangle_J, (\!|\cdot,\cdot|\!)_J, \rho_J)$ *is naturally identified with the double of the quasi-Lie bialgebroid* $\big((T^*M)_\pi, d_N, d\sigma\big)$.

Proof. Assume that $(TM \oplus T^*M, \langle\cdot,\cdot\rangle_J, (\!|\cdot,\cdot|\!)_J, \rho_J)$ is a Courant algebroid. It is clear that $A := T^*M$ and $B := TM$ are transversal, maximal isotropic subbundles. By Eq. (20), $A = T^*M$ is a Dirac structure with the induced bracket $[\cdot,\cdot]_{\pi^\sharp}$. Thus, according to Theorem 2.6, we obtain a quasi-Lie bialgebroid. The construction of the corresponding derivation δ of $(\Omega^\bullet(M), \wedge, [\cdot,\cdot]_{\pi^\sharp})$ and the twisting 3-form ϕ was outlined in the proof of Theorem 2.6. In the present situation, we have

$$\rho_B(X) = \rho_J(X) = \rho(JX) = \rho(NX + \sigma_\flat X) = NX, \quad \forall X \in TM$$

and, combining Eqs. (21) and (6),

$$[X, Y]_B = [X, Y]_N, \quad \forall X, Y \in \mathfrak{X}(M).$$

Therefore, comparing Eqs. (3) and (9), we conclude that $\delta = d_N$. And, combining Eqs. (5) and (21), we get

$$\begin{aligned}\phi(X, Y, Z) &= 2\langle(\!|X, Y|\!)_J, Z\rangle_J = 2\langle J(\!|X, Y|\!)_J, JZ\rangle = 2\langle(\!|X, Y|\!)_J, Z\rangle \\ &= 2\langle[X, Y]_N + d\sigma(X, Y, \cdot), Z\rangle = d\sigma(X, Y, Z), \quad \forall X, Y, Z \in \mathfrak{X}(M).\end{aligned}$$

Hence $\big((T^*M)_\pi, d_N, d\sigma\big)$ is a quasi-Lie bialgebroid or, equivalently according to Proposition 3.5, $(M, \pi, N, d\sigma)$ is a Poisson quasi-Nijenhuis manifold.

Conversely, assume that $(M, \pi, N, d\sigma)$ is a Poisson quasi-Nijenhuis manifold. By Proposition 3.5, $((T^*M)_\pi, d_N, d\sigma)$ is a quasi-Lie bialgebroid. Its double E is a Courant algebroid. We will show that E is indeed isomorphic to $(TM\oplus T^*M, \langle\cdot,\cdot\rangle_J, (\!|\cdot,\cdot|\!)_J, \rho_J)$.

First, it is simple to check that their anchors and non-degenerate symmetric pairings coincide. It remains to check that their brackets coincide. According to Eq. (4), the bracket $[\![\cdot,\cdot]\!]$ on $\Gamma(E)$ is given by

$$[\![\xi,\eta]\!] = [\xi,\eta]_\pi, \tag{23}$$

$$[\![X,Y]\!] = [X,Y]_N + (d\sigma)(X,Y,\cdot), \tag{24}$$

$$[\![X,\xi]\!] = \left(i_X\delta_{TM}\xi + \tfrac{1}{2}\delta_{TM}(\xi X)\right) - \left(i_\xi\delta_{T^*M}X + \tfrac{1}{2}\delta_{T^*M}(\xi X)\right) \tag{25}$$

for all $X, Y \in \mathfrak{X}(M)$ and $\xi, \eta \in \Omega^1(M)$. In our case, we have

$$\delta_{T^*M} = [\pi,\cdot] \qquad \text{and} \qquad \delta_{TM} = d_N.$$

It follows from a straightforward verification that the right hand sides of Eqs. (20)–(22) and (23)–(25) coincide. Therefore, $(TM \oplus T^*M, \langle\cdot,\cdot\rangle_J, (\!|\cdot,\cdot|\!)_J, \rho_J)$ is indeed a Courant algebroid. □

We are now ready to state the main result of this section.

Theorem 7.6. *Assume that* $J : TM \oplus T^*M \to TM \oplus T^*M$ *as given by Eq.* (18) *satisfies Eqs.* (17). *Then the following are equivalent*

- J *is a generalized complex structure;*
- $(M, \pi, N, d\sigma)$ *is a Poisson quasi-Nijenhuis manifold such that*

$$(TM)_N \oplus (T^*M)_\pi \xrightarrow{J} TM \oplus T^*M$$

  is a Courant algebroid isomorphism.

Here $(TM)_N \oplus (T^*M)_\pi$ *denotes the Courant algebroid corresponding to the quasi-Lie bialgebroid* $((T^*M)_\pi, d_N, d\sigma)$.

Proof. By Proposition 7.3, J is a generalized complex structure if, and only if, $(TM \oplus T^*M, \langle\cdot,\cdot\rangle_J, (\!|\cdot,\cdot|\!)_J, \rho_J)$ is a Courant algebroid and $(TM \oplus T^*M, \langle\cdot,\cdot\rangle_J, (\!|,|\!)_J, \rho_J) \xrightarrow{J} (TM \oplus T^*M, \langle\cdot,\cdot\rangle, (\!|\cdot,\cdot|\!), \rho)$ is a Courant algebroid isomorphism. The result follows immediately from Proposition 7.5. □

Since any generalized complex structure naturally gives rise to a Poisson quasi-Nijenhuis manifold, as an immediate consequence of Theorem 6.2, we have the following

Theorem 7.7. *Let* J *be a generalized complex structure as given by Eq.* (18)*, and* $(\Gamma \rightrightarrows M, \widetilde{\omega})$ *a target-connected and target-simply connected symplectic groupoid integrating* $(T^*M)_\pi$. *Then there is a multiplicative* $(1,1)$*-tensor* $\widetilde{N}$ *on* Γ *such that* $(\Gamma \rightrightarrows M, \widetilde{\omega}, \widetilde{N}, t^*d\sigma - s^*d\sigma)$ *is a symplectic quasi-Nijenhuis groupoid.*

Remark 7.8. Note that Theorem 3.3–3.4 in [4] essentially imply our Theorem 7.7. Our proof is conceptual, while Crainic used a direct argument. It would be interesting to see how Theorem 3.4 (ii) in [4] can be proved conceptually.

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