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Canonical Structure and Symmetries of the Schlesinger Equations

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Abstract: The Schlesinger equations $S_{(n,m)}$ describe monodromy preserving deformations of order m Fuchsian systems with $n+1$ poles. They can be considered as a family of commuting time-dependent Hamiltonian systems on the direct product of n copies of $m \times m$ matrix algebras equipped with the standard linear Poisson bracket. In this paper we present a new canonical Hamiltonian formulation of the general Schlesinger equations $S_{(n,m)}$ for all n, m and we compute the action of the symmetries of the Schlesinger equations in these coordinates.

Contents

1. Introduction

The *Schlesinger equations* $S_{(n,m)}$ [52] is the following system of nonlinear differential equations

$$\frac{\partial}{\partial u_j} A_i = \frac{[A_i, A_j]}{u_i - u_j}, \qquad i \neq j,$$
$$\frac{\partial}{\partial u_i} A_i = -\sum_{j\neq i} \frac{[A_i, A_j]}{u_i - u_j}, \tag{1.1}$$

for $m \times m$ matrix valued functions $A_1 = A_1(u), \ldots, A_n = A_n(u)$, $u = (u_1, \ldots, u_n)$, the independent variables $u_1, \ldots, u_n$ must be pairwise distinct. The first non-trivial case $S_{(3,2)}$ of the Schlesinger equations corresponds to the famous sixth Painlevé equation [17, 52, 18], the most general of all Painlevé equations. In the case of any number $n > 3$ of 2×2 matrices A_j, the Schlesinger equations reduce to the Garnier systems $\mathcal{G}_n$ (see [18, 19, 47]).

The Schlesinger equations $S_{(n,m)}$ appeared in the theory of *isomonodromic deformations* of Fuchsian systems. Namely, the monodromy matrices of the Fuchsian system

$$\frac{d\Phi}{dz} = \sum_{k=1}^{n} \frac{A_k(u)}{z - u_k} \Phi, \qquad z \in \mathbb{C}\backslash\{u_1, \ldots, u_n\} \tag{1.2}$$

do not depend on $u = (u_1, \ldots, u_n)$ if the matrices $A_i(u)$ satisfy (1.1). Conversely, under certain assumptions for the matrices $A_1, \ldots, A_n$ and for the matrix

$$A_\infty := -(A_1 + \cdots + A_n), \tag{1.3}$$

all isomonodromic deformations of the Fuchsian system (1.2) are given by solutions to the Schlesinger equations (see, e.g., [54])[1].

The solutions to the Schlesinger equations can be parameterized by the *monodromy data* of the Fuchsian system (1.2) (see the precise definition below in Sect. 2). To reconstruct the solution starting from given monodromy data one is to solve the classical *Riemann - Hilbert problem* of reconstruction of the Fuchsian system from its monodromy data. The main outcome of this approach says that the solutions $A_i(u)$ can be continued analytically to meromorphic functions on the universal covering of

$$\left\{(u_1, \ldots, u_n) \in \mathbb{C}^n \mid u_i \neq u_j \text{ for } i \neq j\right\}$$

[38, 44]. This is a generalization of the celebrated *Painlevé property* of absence of movable critical singularities (see details in [26, 27]). In certain cases the technique based on the theory of Riemann–Hilbert problem gives a possibility to compute the

[1] Bolibruch constructed non-Schlesinger isomonodromic deformations in [8]. These can occur when the matrices A_i are resonant, i.e. admit pairs of eigenvalues with positive integer differences.

asymptotic behavior of the solutions to Schlesinger equations near the critical locus $u_i = u_j$ for some $i \neq j$, although there are still interesting open problems [29, 14, 22, 10].

It is the Painlevé property that was used by Painlevé and Gambier as the basis for their classification scheme of nonlinear differential equations. Of the list of some 50 second order nonlinear differential equations possessing Painlevé property the six (nowadays known as *Painlevé equations*) are selected due to the following crucial property: the general solutions to these six equations cannot be expressed in terms of *classical functions*, i.e., elementary functions, elliptic and other classical transcendental functions (see [57] for a modern approach to this theory based on a nonlinear version of the differential Galois theory). In particular, according to these results the general solution to the Schlesinger system $S_{(3,2)}$ corresponding to the Painlevé-VI equation cannot be expressed in terms of classical functions.

A closely related question is the problem of construction and classification of *classical solutions* to the Painlevé equations and their generalizations. This problem remains open even for the case of Painlevé-VI although there are interesting results based on the theory of symmetries of the Painlevé equations [50, 49, 2] and on the geometric approach to studying the space of monodromy data [14, 25, 42, 43].

The above methods do not give any clue to the solution of the following general problems: are solutions of $S_{(n+1,m)}$ or of $S_{(n,m+1)}$ more complicated than those of $S_{(n,m)}$? Which solutions to $S_{(n+1,m)}$ or $S_{(n,m+1)}$ can be expressed via solutions to $S_{(n,m)}$? Furthermore, which of them can ultimately be expressed via classical functions?

Interest in these problems was one of the starting points for our work. We began to look at the theory of *symmetries* of Schlesinger equations, i.e., of birational transformations acting in the space of Fuchsian systems that map solutions to solutions. One class of symmetries is well known [30, 31]: they are the so-called Schlesinger transformations

$$A(z)=\sum_{i=1}^{n}\frac{A_i}{z-u_i}\mapsto \tilde{A}(z)=\frac{dG(z)}{dz}G^{-1}(z)+G(z)A(z)G^{-1}(z)=\sum_{i=1}^{n}\frac{\tilde{A}_i}{z-u_i} \qquad (1.4)$$

with a rational invertible $m \times m$ matrix valued function $G(z)$ preserving the class of Fuchsian systems. Clearly such transformations preserve the monodromy of the Fuchsian system.

More general symmetries of the $S_{(3,2)}$ Schlesinger equations *do not* preserve the monodromy. They can be derived from the theory, due to K.Okamoto [48], of canonical transformations of the Painlevé-VI equation considered as a time-dependent Hamiltonian system (see also [39, 4] regarding an algebro-geometric approach to some of the Okamoto symmetries). Some of the Okamoto symmetries were later generalized to the Schlesinger systems $S_{(n,2)}$ with arbitrary $n > 3$ [50, 56] (see also [28]) using the Hamiltonian formulation of the related Garnier equations. The generalization of these symmetries to $S_{(n,m)}$ with arbitrary n, m was one of the motivations for our work.

With this problem in mind, in this paper we present a canonical Hamiltonian formulation of Schlesinger equations $S_{(n,m)}$ for all n, m.

Recall [32, 40] that Schlesinger equations can be written as Hamiltonian systems on the Lie algebra

$$\mathfrak{g} := \oplus_{i=1}^{n}\mathfrak{gl}(m) \ni (A_1, \ldots, A_n)$$

with respect to the standard linear Lie - Poisson bracket on $\mathfrak{g}^\star \sim \mathfrak{g}$ with quadratic time-dependent Hamiltonians of the form

$$H_k := \sum_{l\neq k} \frac{\mathrm{tr}\,(A_k A_l)}{u_k - u_l}. \tag{1.5}$$

Because of isomonodromicity they can be restricted onto the symplectic leaves

$$\mathcal{O}_1 \times \cdots \times \mathcal{O}_n \in \mathfrak{g}$$

obtained by fixation of the conjugacy classes $\mathcal{O}_1,\dots, \mathcal{O}_n$ of the matrices $A_1, \dots, A_n$. The matrix A_∞ given in (1.3) is a common integral of the Schlesinger equations. Applying the procedure of symplectic reduction [41] we obtain the reduced symplectic space

$$\begin{gathered}\{A_1 \in \mathcal{O}_1, \dots, A_n \in \mathcal{O}_n,\ A_\infty = \text{given diagonal matrix}\} \\ \text{modulo simultaneous diagonal conjugations.}\end{gathered} \tag{1.6}$$

The dimension of this reduced symplectic leaf in the generic situation is equal to $2g$ where

$$g = \frac{m(m-1)(n-1)}{2} - (m-1).$$

Our aim is to introduce "good" canonical Darboux coordinates on the generic reduced symplectic leaf (1.6).

Actually, there is a natural system of canonical coordinates on (1.6): it is obtained in [53, 20] within the general framework of algebro-geometric Darboux coordinates introduced by S.Novikov and A.Veselov [58] (see also [1, 13]). They are given by the z- and w-projections of the points of the divisor of a suitably normalized eigenvector of the matrix $A(z)$ considered as a section of a line bundle on the spectral curve

$$\det(A(z) - w\mathbb{1}) = 0. \tag{1.7}$$

However, the symplectic reduction (1.6) is time–dependent. This produces a shift in the Hamiltonian functions that can only be computed by knowing the explicit parameterization of the matrices $A_1, \dots, A_n$ by the spectral coordinates. The same difficulty makes the computation of the action of the symmetries on the spectral coordinates for $m > 2$ essentially impossible.

Instead, we construct a new system of the so-called *isomonodromic Darboux coordinates* $q_1, \dots, q_g, p_1, \dots, p_g$ on generic symplectic manifolds (1.6) and we give the new Hamiltonians in these coordinates. Let us explain our construction.

The Fuchsian system (1.2) can be reduced to a scalar differential equation of the form

$$y^{(m)} = \sum_{l=1}^{m-1} d_l(z) y^{(l)}. \tag{1.8}$$

For example, one can eliminate the last $m-1$ components of Φ to obtain a m^{th} order equation for the first component $y := \Phi_1$. (Observe that the reduction procedure depends on the choice of the component of Φ.) The resulting Fuchsian equation will have regular singularities at the same points $z = u_1, \dots, z = u_n, z = \infty$. It will also have other singularities produced by the reduction procedure. However, they will be *apparent* singularities, i.e., the solutions to (1.8) will be analytic in these points. Generically there will be exactly g apparent singularities (cf. [46]; a more precise result about the number of apparent singularities working also in the nongeneric situation was obtained in [7]); they will be the first part $q_1, \ldots, q_g$ of the canonical coordinates. The conjugated momenta are defined by

$$p_i = \mathrm{Res}_{z=q_i}\left(d_{m-2}(z) + \frac{1}{2}d_{m-1}(z)^2\right), \quad i = 1, \ldots, g.$$

Our first result is

Theorem 1.1. *Let the eigenvalues of the matrices A_1, ..., A_n, A_∞ be pairwise distinct. Then the map*

$$\left\{\begin{matrix}\text{Fuchsian systems with given poles}\\ \text{and given eigenvalues of } A_1, \ldots, A_n, A_\infty\\ \text{modulo diagonal conjugations}\end{matrix}\right\} \to (q_1, \ldots, q_g, p_1, \ldots, p_g) \quad (1.9)$$

gives a system of rational Darboux coordinates on a large Zariski open set[2] *in the generic reduced symplectic leaf* (1.6). *The Schlesinger equations $S_{(n,m)}$ in these coordinates are written in the canonical Hamiltonian form*

$$\frac{\partial q_i}{\partial u_k} = \frac{\partial \mathcal{H}_k}{\partial p_i},$$
$$\frac{\partial p_i}{\partial u_k} = -\frac{\partial \mathcal{H}_k}{\partial q_i},$$

with the Hamiltonians

$$\mathcal{H}_k = \mathcal{H}_k(q, p; u) = -\mathrm{Res}_{z=u_k}\left(d_{m-2}(z) + \frac{1}{2}d_{m-1}(z)^2\right), \quad k = 1, \ldots, n.$$

Here *rational Darboux coordinates* means that the elementary symmetric functions $\sigma_1(q), \ldots, \sigma_g(q)$ and $\sigma_1(p), \ldots, \sigma_g(p)$ are rational functions of the coefficients of the system and of the poles $u_1, \ldots, u_n$. Moreover, there exists a section of the map (1.9) given by rational functions

$$A_i = A_i(q, p), \quad i = 1, \ldots, n, \quad (1.10)$$

symmetric in (q_1, p_1), ..., (q_g, p_g) with coefficients depending on $u_1, \ldots, u_n$ and on the eigenvalues of the matrices A_i, $i = 1, \ldots, n, \infty$. All other Fuchsian systems with the same poles $u_1, \ldots, u_n$, the same exponents and the same $(p_1, \ldots, p_g, q_1, \ldots, q_g)$ are obtained by simultaneous diagonal conjugation

$$A_i(q, p) \mapsto C^{-1}A_i(q, p)C, \quad i = 1, \ldots, n, \quad C = \mathrm{diag}\,(c_1, \ldots, c_m).$$

In the course of the proof of Theorem 1.1, we establish that the same parameters $(p_1, \ldots, p_g, q_1, \ldots, q_g)$ are rational coordinates in the space of what we call *special Fuchsian equations*, i.e., m^{th} order Fuchsian equations with $n + 1$ regular singularities

[2] A precise characterization of this large open set is given in Theorem 4.14 and Remark 4.21.

with given exponents and g apparent singularities with the exponents[3] 0, 1, ..., $m-2$, m. We then prove that there is a birational map between special Fuchsian equations and Fuchsian systems. This allows us to conclude that $(p_1,\dots,p_g,q_1,\dots,q_g)$ are rational coordinates in the space of Fuchsian systems.

The natural action of the symmetric group S_n on the Schlesinger equations is described in the following

Theorem 1.2. *The Schlesinger equations $S_{(n,m)}$ written in the canonical form of Theorem 1.1 admit a group of birational canonical transformations $\langle S_2,\dots,S_m,S_\infty\rangle$*

$$S_k:\quad\begin{cases}\tilde q_i=u_1+u_k-q_i, & i=1,\dots,g,\\ \tilde p_i=-p_i, & i=1,\dots,g,\\ \tilde u_l=u_1+u_k-u_l, & l=1,\dots,n,\\ \tilde{\mathcal{H}}_l=-\mathcal{H}_l, & l=1,\dots,n,\end{cases}\tag{1.11}$$

$$S_\infty:\quad\begin{cases}\tilde q_i=\frac{1}{q_i-u_1}, \quad i=1,\dots,g,\\ \tilde p_i=-p_iq_i^2-\frac{2m^2-1}{m}q_i, \quad i=1,\dots,g,\\ \tilde u_l=\frac{1}{u_l-u_1}, \quad l=2,\dots,n,\\ u_1\mapsto\infty,\\ \infty\mapsto u_1,\\ \tilde{\mathcal{H}}_1=\mathcal{H}_1,\\ \tilde{\mathcal{H}}_l=-\mathcal{H}_l(u_l-u_1)^2+(u_l-u_1)(d^0_{m-1}(u_l-u_1))^2-\\ \qquad -(u_l-u_1)\frac{(m-1)(m^2-m-1)}{m}d^0_{m-1}(u_l-u_1), \quad l=2,\dots,n,\end{cases}\tag{1.12}$$

where

$$d^0_{m-1}(u_k)=\sum_{s=1}^{g}\frac{1}{u_k-q_s}-\frac{m\,(m-1)}{2}\sum_{l\neq k}\frac{1}{u_k-u_l}.$$

The transformation S_k acts on the monodromy matrices as follows

$$\begin{aligned}\tilde M_1&=M_1^{-1}\dots M_{k-1}^{-1}M_kM_{k-1}\dots M_1,\\ \tilde M_j&=M_j, \quad j\neq 1,k,\\ \tilde M_k&=M_{k-1}\dots M_2M_1M_2^{-1}\dots M_{k-1}^{-1}, \quad i=k+1,\dots,n.\end{aligned}$$

The transformation S_∞ acts on the monodromy matrices as follows

$$\begin{aligned}\tilde M_\infty&=e^{-\frac{2\pi i}{m}}C_1M_\infty^{-1}M_1M_\infty C_1^{-1},\\ \tilde M_1&=e^{\frac{2\pi i}{m}}C_1M_\infty C_1^{-1},\\ \tilde M_j&=C_1^{-1}M_jC_1, \quad j\neq 1,\infty,\end{aligned}$$

where C_1 is the connection matrix defined in Section 2.

[3] As it was discovered by H.Kimura and K.Okamoto [34] these are the exponents at the generic apparent singularities of the scalar reduction of a Fuchsian system.

Our approach to the construction of Darboux coordinates seems not to work for nongeneric reduced symplectic leaves. The problem is that, for a nongeneric orbit the number of apparent singularities of the scalar reduction is bigger than half of the dimension of the symplectic leaf. The most striking is the example of rigid Fuchsian systems. They correspond to the extreme nongeneric case where the reduced symplectic leaf is zero dimensional. One could expect to have no apparent singularities of the scalar reduction; by no means is this the case (see [33]). The study of reductions of Schlesinger systems on nongeneric reduced symplectic leaves can reveal new interesting systems of nonlinear differential equations. Just one example of a nongeneric situation is basic for the analytic theory of semisimple Frobenius manifolds. In this case $m = n$, the monodromy group of the Fuchsian system is generated by n reflections [12]. The dimension of the symplectic leaves equals

$$\frac{n(n-1)}{2} - \left[\frac{n}{2}\right].$$

We do not know yet how to construct isomonodromic Darboux coordinates for this case if the dimension of the Frobenius manifold is greater than 3.

Our next result is the comparison of the isomonodromic Darboux coordinates with those obtained in the framework of the theory of algebro-geometrically integrable systems (dubbed here *spectral Darboux coordinates*). The spectral Darboux coordinates are constructed as follows. In the generic situation under consideration the genus of the spectral curve (1.7) equals g. The affine part of the divisor of the eigenvector

$$A(z)\psi = w\psi, \quad \psi = (\psi_1, \ldots, \psi_m)^T$$

of the matrix $A(z)$ has degree g. Denote $\gamma_1, \ldots, \gamma_g$ the z-projections of the points of the divisor and $\mu_1, \ldots, \mu_g$ their w-projections. These are the spectral Darboux coordinates in the case under consideration.

Theorem 1.3. *Let us consider a family of Fuchsian systems*

$$\epsilon \frac{d\Phi}{dz} = A(z)\Phi, \quad A(z) = \sum_{i=1}^{n} \frac{A_i}{z - u_i}$$

depending on a small parameter ϵ*. The matrices* $A_1, \ldots, A_n$*,* A_∞ *are assumed to be independent of* ϵ*. Then the isomonodromic Darboux coordinates of this Fuchsian system have the following expansion as* $\epsilon \to 0$*:*

$$q_k = \gamma_k + O(\epsilon), \quad p_k = \epsilon^{-1}\mu_k + O(1), \quad k = 1, \ldots, g.$$

Here γ_k*,* μ_k *are the spectral Darboux coordinates of the matrix* $A(z)$ *defined above.*

We do not even attempt in this paper to discuss physical applications of our results. However one of them looks particularly attractive. According to an idea of N.Reshetikhin [51] (see also [23]) the well known Knizhnik–Zamolodchikov equations in conformal field theory can be considered as a quantization of Schlesinger equations. We believe that our isomonodromic canonical coordinates could play an important role in the analysis of the quantization procedure, somewhat similar to the role played by the spectral canonical coordinates in the Sklyanin scheme of quantization of integrable systems [55]. We plan to address this problem in subsequent publications.

The paper is organized as follows. In Sect. 2 we recall the relationship between Schlesinger equations and isomonodromic deformations of Fuchsian systems. In Sect.3 we discuss the Hamiltonian formulation of Schlesinger equations. A formula for the symplectic structure of Schlesinger equations recently found by I.Krichever [35] proved to be useful for subsequent calculations with isomonodromic coordinates; we prove that this formula is equivalent to the standard one. In Sect. 4 we construct the isomonodromic Darboux coordinates and establish a birational isomorphism between the space of Fuchsian systems considered modulo conjugations and the space of special Fuchsian differential equations. In Sect. 5 we express the semiclassical asymptotics of the isomonodromic Darboux coordinates via spectral Darboux coordinates. The necessary facts from the theory of spectral Darboux coordinates are collected in the Appendix. Finally we apply the above results to constructing the nontrivial symmetries of Schlesinger equations.

2. Schlesinger Equations as Monodromy Preserving Deformations of Fuchsian Systems

In this section we establish our notations, recall a few basic definitions and prove some technical lemmata that will be useful throughout this paper.

The Schlesinger equations $\mathcal{S}_{(n,m)}$ describe monodromy preserving deformations of Fuchsian systems (1.2) with $n+1$ regular singularities at $u_1, \ldots, u_n, u_{n+1} = \infty$:

$$\frac{\mathrm{d}}{\mathrm{d}z}\Phi = \sum_{k=1}^{n} \frac{A_k}{z-u_k}\Phi, \qquad z \in \mathbb{C}\backslash\{u_1, \ldots, u_n\}, \tag{2.1}$$

A_k being $m \times m$ matrices independent on z, and $u_k \neq u_l$ for $k \neq l, k, l = 1, \ldots, n+1$. Let us explain the precise meaning of this claim.

2.1. Levelt basis near a logarithmic singularity and local monodromy data. A system

$$\frac{d\Phi}{dz} = \frac{A(z)}{z-z_0}\Phi \tag{2.2}$$

is said to have a *logarithmic*, or *Fuchsian* singularity at $z = z_0$ if the $m \times m$ matrix valued function $A(z)$ is analytic in some neighborhood of $z = z_0$. By definition the *local monodromy data* of the system is the class of equivalence of such systems w.r.t. local gauge transformations

$$A(z) \mapsto G^{-1}(z)A(z)\,G(z) + (z-z_0)G^{-1}(z)\partial_z G(z) \tag{2.3}$$

analytic near $z = z_0$ satisfying

$$\det G(z_0) \neq 0.$$

The parameters of the local monodromy can be obtained by choosing a suitable fundamental matrix solution of the system (2.2). The most general construction of such a fundamental matrix was given by Levelt [37]. We will briefly recall this construction in the form suggested in [12].

Without loss of generality one can assume that $z_0 = 0$. Expanding the system near $z = 0$ one obtains

$$\frac{d\Phi}{dz} = \left(\frac{A_0}{z} + A_1 + z\, A_2 + \dots \right) \Phi. \tag{2.4}$$

Let us now describe the structure of local monodromy data.

Two linear operators Λ, R acting in the complex m-dimensional space V,

$$\Lambda,\ R : V \to V,$$

are said to form an *admissible pair* if the following conditions are fulfilled:

1. The operator Λ is semisimple and the operator R is nilpotent.
2. R commutes with $e^{2\pi i \Lambda}$,

$$e^{2\pi i \Lambda} R = R\, e^{2\pi i \Lambda}. \tag{2.5}$$

Observe that, due to the last condition the operator R satisfies

$$R(V_\lambda) \subset \oplus_{k\in\mathbb{Z}} V_{\lambda+k} \quad \text{for} \quad \text{any} \quad \lambda \in \operatorname{Spec}\Lambda, \tag{2.6}$$

where $V_\lambda \subset V$ is the subspace of all eigenvectors of Λ with the eigenvalue λ. The last condition says that

3. The sum in the r.h.s. of (2.6) contains only non-negative values of k.

A decomposition

$$R = R_0 + R_1 + R_2 + \dots \tag{2.7}$$

is defined where

$$R_k(V_\lambda) \subset V_{\lambda+k} \quad \text{for} \quad \text{any} \quad \lambda \in \operatorname{Spec}\Lambda. \tag{2.8}$$

Clearly this decomposition contains only a finite number of terms. Observe the useful identity

$$z^\Lambda R\, z^{-\Lambda} = R_0 + z\, R_1 + z^2 R_2 + \dots. \tag{2.9}$$

Theorem 2.1. *For a system* (2.4) *with a logarithmic singularity at* $z = 0$ *there exists a fundamental matrix solution of the form*

$$\Phi(z) = \Psi(z) z^\Lambda z^R, \tag{2.10}$$

where $\Psi(z)$ *is a matrix valued function analytic near* $z = 0$ *satisfying*

$$\det \Psi(0) \neq 0$$

and $\Lambda,\ R$ *is an admissible pair.*

The formula (2.10) makes sense after fixing a branch of logarithm $\log z$ near $z = 0$. Note that z^R is a polynomial in $\log z$ due to nilpotency of R.

The proof can be found in [37] (cf. [12]). Clearly Λ is the semisimple part of the matrix A_0; R_0 coincides with its nilpotent part. The remaining terms of the expansion appear only in the *resonant case*, i.e., if the difference between some eigenvalues of Λ is a positive integer. In the important particular case of a diagonalizable matrix A_0,

$$T^{-1}A_0T = \Lambda = \mathrm{diag}\,(\lambda_1, \dots, \lambda_m)$$

with some nondegenerate matrix T, the matrix function $\Psi(z)$ in the fundamental matrix solution (2.10) can be obtained in the form

$$\Psi(z) = T\left(\mathbb{1} + z\,\Psi_1 + z^2\Psi_2 + \dots\right).$$

The matrix coefficients Ψ_1, Ψ_2, …of the expansion as well as the matrix components R_1, R_2, …of the matrix R (see (2.7)) can be found recursively from the equations

$$[\Lambda, \Psi_k] - k\,\Psi_k = B_k - R_k + \sum_{i=1}^{k-1} \Psi_{k-i}B_i - R_i\Psi_{k-i}, \quad k \geq 1.$$

Here

$$B_k := T^{-1}A_kT, \quad k \geq 1.$$

If $k_{\max}$ is the maximal integer among the differences $\lambda_i - \lambda_j$ then

$$R_k = 0 \quad \text{for } k > k_{\max}.$$

Observe that vanishing of the logarithmic terms in the fundamental matrix solution (2.10) is a constraint imposed only on the first $k_{\max}$ coefficients A_1, …, $A_{k_{\max}}$ of the expansion (2.4).

It is not difficult to describe the ambiguity in the choice of the admissible pair of matrices Λ, R describing the local monodromy data of the system (2.4). Namely, the diagonal matrix Λ is defined up to permutations of diagonal entries. Assuming the order fixed, the ambiguity in the choice of R can be described as follows [12]. Denote $\mathcal{C}_0(\Lambda) \subset GL(V)$ the subgroup consisting of invertible linear operators $G : V \to V$ satisfying

$$z^{\Lambda}G\,z^{-\Lambda} = G_0 + z\,G_1 + z^2G_2 + \dots. \tag{2.11}$$

The definition of the subgroup can be reformulated [12] in terms of invariance of a certain flag in V naturally associated with the semisimple operator Λ. The matrix $\tilde{R}$ obtained from R by the conjugation of the form

$$\tilde{R} = G^{-1}R\,G \tag{2.12}$$

will be called *equivalent* to R. Multiplying (2.10) on the right by G one obtains another fundamental matrix solution to the same system of the same structure

$$\tilde{\Phi}(z) := \Psi(z)z^{\Lambda}z^{R}G = \tilde{\Psi}(z)z^{\Lambda}z^{\tilde{R}},$$

i.e., $\tilde{\Psi}(z)$ is analytic at $z = 0$ with $\det\tilde{\Psi}(0) \neq 0$.

The columns of the fundamental matrix (2.10) form a distinguished basis in the space of solutions to (2.4).

Definition 2.2. *The basis given by the columns of the matrix* (2.10) *is called the* Levelt basis *in the space of solutions to* (2.4). *The fundamental matrix* (2.10) *is called the* Levelt fundamental matrix solution.

The monodromy transformation of the Levelt fundamental matrix solution reads

$$\Phi\left(z\, e^{2\pi i}\right) = \Phi(z) M, \quad M = e^{2\pi i \Lambda} e^{2\pi i R}. \tag{2.13}$$

To conclude this section let us denote $\mathcal{C}(\Lambda, R)$ the subgroup of invertible transformations of the form

$$\mathcal{C}(\Lambda, R) = \{\, G \in GL(V) \,|\, z^{\Lambda} G\, z^{-\Lambda} = \sum_{k\in\mathbb{Z}} G_k z^k \text{ and } [G, R] = 0\}. \tag{2.14}$$

The subgroups $\mathcal{C}(\Lambda, R)$ and $\mathcal{C}(\Lambda, \tilde{R})$ associated with equivalent matrices R and $\tilde{R}$ are conjugated. It is easy to see that this subgroup coincides with the centralizer of the monodromy matrix (2.13),

$$G \in \mathcal{C}(\Lambda, R) \quad \text{iff} \quad G\, e^{2\pi i \Lambda} e^{2\pi i R} = e^{2\pi i \Lambda} e^{2\pi i R} G, \quad \det G \neq 0. \tag{2.15}$$

Denote

$$\mathcal{C}_0(\Lambda, R) \subset \mathcal{C}(\Lambda, R) \tag{2.16}$$

the subgroup consisting of matrices G such that the expansion (2.14) contains only non-negative powers of z. Multiplying the Levelt fundamental matrix (2.10) by a matrix $G \in \mathcal{C}_0(\Lambda, R)$ one obtains another Levelt solution to (2.4),

$$\Psi(z) z^{\Lambda} z^{R} G = \tilde{\Psi}(z) z^{\Lambda} z^{R}. \tag{2.17}$$

In the next section we will see that the quotient $\mathcal{C}(\Lambda, R)/\mathcal{C}_0(\Lambda, R)$ plays an important role in the theory of monodromy preserving deformations.

Example 2.3. For

$$\Lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ c & b & 0 \end{pmatrix}$$

the quotient $\mathcal{C}(\Lambda, R)/\mathcal{C}_0(\Lambda, R)$ is trivial iff $a\, b \neq 0$.

2.2. Monodromy data and isomonodromic deformations of a Fuchsian system. Denote $\lambda_j^{(k)}$, $j = 1, \ldots, m$, the eigenvalues of the matrix A_k, $k = 1, \ldots, n, \infty$, where the matrix A_∞, is defined as

$$A_\infty := -\sum_{k=1}^{n} A_k.$$

For the sake of technical simplicity let us assume that

$$\lambda_i^{(k)} \neq \lambda_j^{(k)} \text{ for } \; i \neq j, \qquad k = 1, \ldots, n, \infty. \tag{2.18}$$

Moreover, it will be assumed that A_∞ is a constant diagonal $m \times m$ matrix with eigenvalues $\lambda_j^{(\infty)}$, $j = 1, \dots, m$.

Denote $\Lambda^{(k)}$, $R^{(k)}$ the local monodromy data of the Fuchsian system near the points $z = u_k$, $k = 1, \dots, n, \infty$. The matrices $\Lambda^{(k)}$ are all diagonal

$$\Lambda^{(k)} = \mathrm{diag}\,(\lambda_1^{(k)}, \dots, \lambda_m^{(k)}), \quad k = 1, \dots, n, \infty, \tag{2.19}$$

and under our assumptions

$$\Lambda^{(\infty)} = A_\infty.$$

Recall that the matrix $G \in GL(m, \mathbb{C})$ belongs to the group $\mathcal{C}_0(\Lambda^{(\infty)})$ *iff*

$$z^{-\Lambda^{(\infty)}} G\, z^{\Lambda^{(\infty)}} = G_0 + \frac{G_1}{z} + \frac{G_2}{z^2} + \dots. \tag{2.20}$$

It is easy to see that our assumptions about eigenvalues of A_∞ imply diagonality of the matrix G_0.

Let us also recall that the matrices $\Lambda^{(k)}$ satisfy

$$\mathrm{tr}\,\Lambda^{(1)} + \dots + \mathrm{tr}\,\Lambda^{(\infty)} = 0. \tag{2.21}$$

The numbers $\lambda_1^{(k)}, \dots, \lambda_m^{(k)}$ are called the *exponents* of the system (1.2) at the singular point u_k.

Let us fix a fundamental matrix solution of the form (2.10) near all singular points $u_1, \dots, u_n, \infty$. To this end we are to fix branch cuts on the complex plane and choose the branches of logarithms $\log(z - u_1), \dots, \log(z - u_n), \log z^{-1}$. We will do it in the following way: perform parallel branch cuts π_k between ∞ and each of the u_k, $k = 1, \dots, n$ along a given (generic) direction. After this we can fix Levelt fundamental matrices analytic on

$$z \in \mathbb{C} \setminus \cup_{k=1}^n \pi_k, \tag{2.22}$$

$$\Phi_k(z) = T_k\, (\mathbb{1} + \mathcal{O}(z - u_k))\, (z - u_k)^{\Lambda^{(k)}} (z - u_k)^{R^{(k)}},\ z \to u_k, \quad k = 1, \dots, n, \tag{2.23}$$

and

$$\Phi(z) \equiv \Phi_\infty(z) = \left(\mathbb{1} + \mathcal{O}(\frac{1}{z})\right) z^{-A_\infty} z^{-R^{(\infty)}}, \quad \text{as} \quad z \to \infty. \tag{2.24}$$

Define the *connection matrices* by

$$\Phi_\infty(z) = \Phi_k(z) C_k, \tag{2.25}$$

where $\Phi_\infty(z)$ is to be analytically continued in a vicinity of the pole u_k along the positive side of the branch cut π_k.

The monodromy matrices M_k, $k = 1, \dots, n, \infty$ are defined with respect to a basis $l_1, \dots, l_n$ of loops in the fundamental group

$$\pi_1\,(\mathbb{C} \backslash \{u_1, \dots u_n\}, \infty)\,.$$

Choose the basis in the following way. The loop l_k arrives from infinity in a vicinity of u_k along one side of the branch cut π_k that will be called *positive*, then it encircles

u_k going in the anti-clock-wise direction leaving all other poles outside and, finally it returns to infinity along the opposite side of the branch cut π_k called *negative*.

Denote $l_j^* \Phi_\infty(z)$ the result of analytic continuation of the fundamental matrix $\Phi_\infty(z)$ along the loop l_j. The monodromy matrix M_j is defined by

$$l_j^* \Phi_\infty(z) = \Phi_\infty(z) M_j, \quad j = 1, \ldots, n. \tag{2.26}$$

The monodromy matrices satisfy

$$M_\infty M_n \cdots M_1 = \mathbb{1}, \qquad M_\infty = \exp\left(2\pi i A_\infty\right) \exp\left(2\pi i R^{(\infty)}\right) \tag{2.27}$$

if the branch cuts $\pi_1, \ldots, \pi_n$ enter the infinite point according to the order of their labels, i.e., the positive side of π_{k+1} looks at the negative side of π_k, $k = 1, \ldots, n-1$.

Clearly one has

$$M_k = C_k^{-1} \exp\left(2\pi i \Lambda^{(k)}\right) \exp\left(2\pi i R^{(k)}\right) C_k, \qquad k = 1, \ldots, n. \tag{2.28}$$

The collection of the local monodromy data $\Lambda^{(k)}$, $R^{(k)}$ together with the central connection matrices C_k will be used in order to uniquely fix the Fuchsian system with given poles. They will be defined up to an equivalence that we now describe. The eigenvalues of the diagonal matrices $\Lambda^{(k)}$ are defined up to permutations. Fixing the order of the eigenvalues, we define the class of equivalence of the nilpotent part $R^{(k)}$ and of the connection matrices C_k by factoring out the transformations of the form

$$R_k \mapsto G_k^{-1} R_k G_k, \quad C_k \mapsto G_k^{-1} C_k G_\infty, \quad k = 1, \ldots, n,$$

$$G_k \in \mathcal{C}_0(\Lambda^{(k)}), \quad G_\infty \in \mathcal{C}_0(\Lambda^{(\infty)}). \tag{2.29}$$

Observe that the monodromy matrices (2.28) will transform by a simultaneous conjugation

$$M_k \mapsto G_\infty^{-1} M_k G_\infty, \quad k = 1, 2, \ldots, n, \infty.$$

Definition 2.4. *The class of equivalence* (2.29) *of the collection*

$$\Lambda^{(1)}, R^{(1)}, \ldots, \Lambda^{(\infty)}, R^{(\infty)}, C_1, \ldots, C_n \tag{2.30}$$

is called monodromy data of the Fuchsian system with respect to a fixed ordering of the eigenvalues of the matrices A_1, ..., A_n and a given choice of the branch cuts.

Lemma 2.5. *Two Fuchsian systems of the form* (1.2) *with the same poles $u_1, \ldots, u_n, \infty$ and the same matrix A_∞ coincide, modulo diagonal conjugations if and only if they have the same monodromy data with respect to the same system of branch cuts $\pi_1, \ldots, \pi_n$.*

Proof. Let

$$\Phi_\infty^{(1)}(z) = \left(\mathbb{1} + O(\frac{1}{z})\right) z^{-\Lambda^{(\infty)}} z^{-R^{(\infty)}}, \quad \Phi_\infty^{(2)}(z) = \left(\mathbb{1} + O(\frac{1}{z})\right) z^{-\tilde{\Lambda}^{(\infty)}} z^{-\tilde{R}^{(\infty)}}$$

be the fundamental matrices of the form (2.24) of the two Fuchsian systems. Using the assumption about A_∞ we derive that $\tilde{\Lambda}^{(\infty)} = \Lambda^{(\infty)}$. Multiplying $\Phi_\infty^{(2)}(z)$ if necessary on the right by a matrix $G \in \mathcal{C}_0(\Lambda^{(\infty)})$, we can obtain another fundamental matrix of the second system with

$$\tilde{R}^{(\infty)} = R^{(\infty)}.$$

Consider the following matrix:

$$Y(z) := \Phi_\infty^{(2)}(z)[\Phi_\infty^{(1)}(z)]^{-1}. \tag{2.31}$$

$Y(z)$ is an analytic function around infinity:

$$Y(z) = G_0 + \mathcal{O}\left(\frac{1}{z}\right), \quad \text{as } z \to \infty, \tag{2.32}$$

where G_0 is a diagonal matrix. Since the monodromy matrices coincide, $Y(z)$ is a single valued function on the punctured Riemann sphere $\overline{\mathbb{C}}\backslash\{u_1, \dots, u_n\}$. Let us prove that $Y(z)$ is analytic also at the points u_k. Indeed, having fixed the monodromy data, we can choose the fundamental matrices $\Phi_k^{(1)}(z)$ and $\Phi_k^{(2)}(z)$ of the form (2.23) with the same connection matrices C_k and the same matrices $\Lambda^{(k)}$, $R^{(k)}$. Then near the point u_k, $Y(z)$ is analytic:

$$Y(z) = T_k^{(2)} \left(\mathbb{1} + \mathcal{O}(z - u_k)\right) \left[T_k^{(1)} \left(\mathbb{1} + \mathcal{O}(z - u_k)\right)\right]^{-1}. \tag{2.33}$$

This proves that $Y(z)$ is an analytic function on all $\overline{\mathbb{C}}$ and then, by the Liouville theorem $Y(z) = G_0$, which is constant. So the two Fuchsian systems coincide, after conjugation by the diagonal matrix G_0. □

Remark 2.6. The connection matrices are determined, within their equivalence classes by the monodromy matrices if the quotients $\mathcal{C}(\Lambda^{(k)}, R^{(k)})/\mathcal{C}_0(\Lambda^{(k)}, R^{(k)})$ are trivial for all $k = 1, \dots, n$. In particular this is the case when all the characteristic exponents at the poles $u_1, \dots, u_n$ are non-resonant.

From the above lemma the following result readily follows.

Theorem 2.7. *If the matrices $A_k(u_1, \dots, u_n)$ satisfy Schlesinger equations (1.1) and the matrix*

$$A_\infty = -(A_1 + \cdots + A_n)$$

is diagonal then all the characteristic exponents do not depend on $u_1, \dots, u_n$. The fundamental matrix $\Phi_\infty(z)$ can be chosen in such a way that the nilpotent matrix $R^{(\infty)}$ and also all the monodromy matrices are constant in $u_1, \dots, u_n$. Moreover, the Levelt fundamental matrices $\Phi_k(z)$ can be chosen in such a way that all the nilpotent matrices $R^{(k)}$ and also all the connection matrices C_k are constant. Viceversa, if the deformation $A_k = A_k(u_1, \dots, u_n)$ is such that the monodromy data do not depend on $u_1, \dots, u_n$ then the matrices $A_k(u_1, \dots, u_n)$, $k = 1, \dots, n$ satisfy Schlesinger equations.

At the end of this section we give a criterion that ensures that the “naive” definition of monodromy preserving deformations still gives rise to the Schlesinger equations.

Theorem 2.8. *Let $A_k = A_k(u_1, \dots, u_n)$, $k = 1, \dots, n$ be a deformation of the Fuchsian system* (1.2) *such that the following conditions hold true.*

1. *The matrix* $A_\infty = -A_1 - \cdots - A_n$ *is constant and diagonal.*
2. *The Fuchsian system admits a fundamental matrix solution of the form* (2.24) *with the* u*-independent matrix* $R^{(\infty)}$*. Denote* $\Phi_\infty(z; u)$ *the fundamental matrix solution of the family of Fuchsian systems of the form* (2.24).
3. *The monodromy matrices* $M_1, \ldots, M_n$ *defined as in* (2.26) *with respect to the fundamental matrix* $\Phi_\infty(z; u)$ *do not depend on* $u_1, \ldots, u_n$*. Note that this implies constancy of the diagonal matrices* $\Lambda^{(k)}$ *of exponents,* $k = 1, \ldots, n$.
4. *The (class of equivalence of) local monodromy data* $(\Lambda^{(k)}, R^{(k)})$ *does not depend on* $u_1, \ldots, u_n$.
5. *The quotients* $\mathcal{C}(\Lambda^{(k)}, R^{(k)})/\mathcal{C}_0(\Lambda^{(k)}, R^{(k)})$ *are zero dimensional for all* $k = 1, \ldots, n$.

Then the deformation satisfies the Schlesinger equations. Moreover, under the assumption of validity of 1 - 4, *if Condition* 5 *does not hold true, then there exist non-Schlesinger deformations preserving the monodromy matrices.*

Proof. The first statement of the theorem easily follows from Remark 2.6 and Theorem 2.7. To prove the second part, let us assume that the dimension of the quotient $\mathcal{C}(\Lambda^{(k)}, R^{(k)})/\mathcal{C}_0(\Lambda^{(k)}, R^{(k)})$ is positive for some k. Here $\Lambda^{(k)}, R^{(k)}$ are local monodromy data of the Fuchsian system (1.2) with some poles $u_1, \ldots, u_n, \infty$. Let us choose a nontrivial family of matrices $G(s) \in \mathcal{C}(\Lambda^{(k)}, R^{(k)})/\mathcal{C}_0(\Lambda^{(k)}, R^{(k)})$ for sufficiently small s, $G(0) = \mathbb{1}$. We will now obtain a deformation of the Fuchsian system (1.2) in the following way. Let us deform the k^{th} connection matrix C_k by putting

$$C_k(s) := G(s)C_k.$$

To reconstruct the deformation of the Fuchsian system, we are to solve the suitable Riemann - Hilbert problem. It will be solvable for sufficiently small s because of solvability for $s = 0$. At this point one can also deform the poles $u_i(s)$, $u_i(0) = u_i$. This deformation is obviously isomonodromic but not of the Schlesinger type. The theorem is proved. □

3. Hamiltonian Structure of the Schlesinger System

3.1. Lie-Poisson brackets for Schlesinger system. The Hamiltonian description of the Schlesinger system can be derived [24] from the general construction of a Poisson bracket on the space of flat connections in a principal G-bundle over a surface with boundary using Atiyah - Bott symplectic structure (see [5]). Explicitly this approach yields the following well known formalism representing the Schlesinger system $S_{(n,m)}$ in Hamiltonian form with n time variables $u_1, \ldots, u_n$ and n commuting time–dependent Hamiltonian flows on the dual space to the direct sum of n copies of the Lie algebra $\mathfrak{sl}(m)$,

$$\mathfrak{g} := \oplus_n \mathfrak{sl}(m) \ni (A_1, A_2, \ldots, A_n)\,. \tag{3.1}$$

The standard Lie-Poisson bracket on $\mathfrak{g}^*$ reads

$$\left\{ \left(A_p\right)^i_j, \left(A_q\right)^k_l \right\} = \delta_{pq} \left(\delta^i_l \left(A_p\right)^k_j - \delta^k_j \left(A_q\right)^i_l \right). \tag{3.2}$$

(We identify $\mathfrak{sl}(m)$ with its dual by using the Killing form $(A, B) = \operatorname{Tr} AB$, $\quad A, B \in \mathfrak{sl}(m)$.) The following statement is well-known (see [30, 40]) and can be checked by a straightforward computation.

Theorem 3.1. *The dependence of the solutions A_k, $k = 1, \dots, n$, of the Schlesinger system $S_{(n,m)}$ upon the variables $u_1, \dots, u_n$ is determined by Hamiltonian systems on* (3.1) *with time-dependent quadratic Hamiltonians*

$$H_k = \sum_{l \neq k} \frac{\mathrm{Tr}\,(A_k A_l)}{u_k - u_l}, \tag{3.3}$$

$$\frac{\partial}{\partial u_k} A_l = \{A_l, H_k\}. \tag{3.4}$$

To arrive from the Hamiltonian systems (3.3), (3.4) to the isomonodromic deformations one is to impose an additional constraint. Define

$$A_\infty := -(A_1 + \cdots + A_n). \tag{3.5}$$

It can be easily seen that

$$\frac{\partial}{\partial u_i} A_\infty = \{A_\infty, H_i\} = 0, \quad i = 1, \dots, n. \tag{3.6}$$

So, the matrix entries of A_∞ are integrals of the Schlesinger equations. They generate the action of the group $\mathfrak{sl}(m)$ on $\mathfrak{g}$ by the diagonal conjugations

$$A_k \mapsto G^{-1} A_k G, \quad k = 1, \dots, n, \quad G \in \mathfrak{sl}(m). \tag{3.7}$$

To identify the Hamiltonian equations (3.4) with isomonodromic deformations one is to apply the Marsden - Weinstein procedure of symplectic reduction [41]. In our setting this procedure works as follows. Let us choose a particular level surface of the moment map corresponding to the first integrals (3.6). We will mainly deal with the level surfaces of the form

$$A_\infty = \mathrm{diag}\left(\lambda_1^{(\infty)}, \dots, \lambda_m^{(\infty)}\right), \quad \lambda_i^{(\infty)} \neq \lambda_j^{(\infty)}, \quad i \neq j \tag{3.8}$$

for some pairwise distinct numbers $\lambda_1^{(\infty)}, \dots, \lambda_m^{(\infty)}$.

After restricting the Hamiltonian systems onto the level surface (3.8) there remains a residual symmetry with respect to conjugations by diagonal matrices

$$A_k \mapsto D^{-1} A_k D, \quad k = 1, \dots, n, \quad D = \mathrm{diag}\,(d_1, \dots, d_n). \tag{3.9}$$

Denote

$$Diag \simeq \left[\mathbb{C}^*\right]^{m-1} \subset SL(m)$$

the subgroup of diagonal matrices acting on $\mathfrak{g}$ by simultaneous conjugations (3.9).

Definition 3.2. *The quotient of the restriction of the Hamiltonian system (3.3), (3.4) onto the level surface of the form (3.8) w.r.t. the transformations (3.9) will be called the reduced Schlesinger system.*

From the above results it readily follows that the reduced Schlesinger system describes all nontrivial monodromy preserving deformations of the Fuchsian system (1.2).

3.2. Symplectic structure of the isomonodromic deformations. The symplectic leaves of the Poisson bracket (3.2) are products of adjoint orbits

$$(A_1, \dots, A_n) \in \mathcal{O}_1 \times \cdots \times \mathcal{O}_n \subset \mathfrak{g}, \tag{3.10}$$

where $\mathcal{O}_k$ is the adjoint orbit of A_k. The symplectic structure ω_H induced by (3.2) on the orbits can be represented in the following form [24]. Given two tangent vectors

$$\delta_1 A = \sum_k \frac{\delta_1 A_k}{z - u_k}, \quad \delta_2 A = \sum_k \frac{\delta_2 A_k}{z - u_k} \tag{3.11}$$

at the given point $A(z)$ the value of the symplectic form can be computed by

$$\omega_H(\delta_1 A, \delta_2 A) = -\sum_k \mathrm{Tr}(U_k^{(1)} \delta_2 A_k), \tag{3.12}$$

where the matrices $U_k^{(i)}$ are such that $\delta_i A_k = [U_k^{(i)}, A_k]$.

Actually, the symplectic structure (3.12) was obtained in [24] from the general Atiyah - Bott Poisson structure [3] on the moduli space of flat $SL(m)$ connections on the surface Σ. It is obtained by projecting the gauge invariant symplectic form

$$\omega_{AB}(\delta_1 A, \delta_2 A) = \frac{1}{2\pi i} \int_\Sigma \mathrm{Tr}\,(\delta_1 A \wedge \delta_2 A) \tag{3.13}$$

onto the moduli space. In our case Σ is the Riemann sphere without small discs around the poles of $A(z)$.

The eigenvalues of the matrices A_k are Casimirs of the Poisson bracket (3.2). We will mainly consider the generic case where these eigenvalues are distinct for all $k = 1, \dots, n$. Then the level sets of the Casimirs coincide with the symplectic leaves (3.10).

Denote

$$(\mathrm{Spec}\, A_1, \dots, \mathrm{Spec}\, A_n)$$

the collection of the eigenvalues of the matrices $(A_1, \dots, A_n) \in \mathfrak{g}$. Generically these are the parameters of the symplectic leaves. Fixing the level surface of the moment map (3.8) and taking the quotient over the action (3.9) of the group $Diag \subset SL(m)$ of diagonal matrices one obtains a manifold that we denote

$$M\,(\mathrm{Spec}\, A_1, \dots, \mathrm{Spec}\, A_n; A_\infty)\,. \tag{3.14}$$

The dimension of this manifold is equal to $2g$ where

$$g = \frac{m(m-1)(n-1)}{2} - (m-1). \tag{3.15}$$

Indeed, the dimension of a generic adjoint orbit $\mathcal{O}_i$ is equal to $m^2 - m$. Choosing a level surface (3.8) of the momentum map

$$(A_1, \dots, A_n) \mapsto A_\infty := -(A_1 + \cdots + A_n)$$

we impose only $m^2 - 1$ independent equations, since the trace of the matrix A_∞ is equal to the sum of traces of A_1, ..., A_n. Finally, subtracting the dimension $m - 1$ of the group $Diag$ we arrive at (3.15). The manifold still carries a symplectic structure since the action of the Abelian group *Diag* preserves the Poisson bracket (3.2). One of the aims of this paper is to construct a system of canonical Darboux coordinates on generic manifolds of the form (3.14).

For our aims the following approach to the Hamiltonian theory of monodromy preserving deformations developed recently by Krichever [36]will be useful. He has obtained a general formula for the symplectic structure on the space of isomonodromic deformations of a generic linear system of ODEs of the form

$$\frac{\mathrm{d}}{\mathrm{d}z}\Phi = A(z)\Phi,$$

where z is a variable on a punctured genus g algebraic curve and $A(z)$ is any meromorphic matrix function of z with poles of any order at $P_1, \dots, P_n$. In the case of genus $g = 0$ the formula reads

$$\omega_K = -\frac{1}{2}\sum_k \mathrm{Res}_{z=P_k}\mathrm{Tr}(\delta A \wedge \delta\Phi\,\Phi^{-1}), \tag{3.16}$$

where the variations δA and $\delta\Phi$ are independent. The corresponding Hamiltonian function describing the isomonodromic deformations in the parameter P_j is

$$H_{K_j} = -\frac{1}{2}\mathrm{Res}_{z=P_j}\mathrm{Tr}(A(z)^2). \tag{3.17}$$

Due to gauge invariants [36] the symplectic form admits a restriction onto the manifold (3.10). It defines therefore a symplectic structure ω_K on the product of adjoint orbits. Let us prove that the symplectic structures ω_H and ω_K coincide, up to a sign.

Theorem 3.3. *In the setting of isomonodromic deformations of the Fuchsian system (1.2) the two symplectic forms (3.12) and (3.16) coincide (up to a sign)*

$$\omega_K = -\omega_H. \tag{3.18}$$

Proof. The variations δA tangent to the orbit are obtained by means of an infinitesimal gauge transform with $G = \mathbb{1} + \phi$, i.e.

$$\delta A = GAG^{-1} + \frac{\mathrm{d}G}{\mathrm{d}z}G^{-1} - A = -[A, \phi] + \frac{\mathrm{d}}{\mathrm{d}z}\phi + O(\phi^2).$$

So, representing the tangent vectors (3.11) as

$$\delta_i A = -[A, \phi_i] + \frac{\mathrm{d}}{\mathrm{d}z}\phi_i, \quad i = 1, 2$$

we obtain

$$\delta_i A_k = -\left[A_k, U_k^{(i)}\right],$$

where $U_k^{(i)}$ is given by the first term of the expansion of ϕ_i at u_k,

$$\phi_i(z) = U_k^{(i)} + \mathcal{O}(z - u_k).$$

Because of this

$$\omega_H(\delta_1 A, \delta_2 A) = -\sum_k \operatorname{Tr}\left(A_k\left[U_k^{(1)}, U_k^{(2)}\right]\right).$$

In the formula (3.16) we can take

$$\delta_i \Phi\, \Phi^{-1} = \phi_i, \quad i = 1,\ 2.$$

Indeed, the matrices ϕ_i and $\delta_i \Phi\, \Phi^{-1}$ satisfy the same equation. This follows from

$$\frac{\mathrm{d}}{\mathrm{d}z}\delta\Phi = \delta A\, \Phi + A\delta\Phi.$$

Thus

$$\begin{aligned}\omega_K(\delta_1 A, \delta_2 A) &- \frac{1}{2}\sum_k \operatorname{Res}_{z=u_k}\operatorname{Tr}(\delta_1 A\, \phi_2 - \phi_1 \delta_2 A)\\ &= \sum_k \operatorname{Tr}\left(A_k\left[U_k^{(1)}, U_k^{(2)}\right]\right) - \omega_H(\delta_1 A, \delta_2 A).\end{aligned}$$

The theorem is proved. □

One can also prove that the Hamiltonians (3.17) correctly reproduce the formula (3.3), up to a minus sign (this is not a problem because also the signs of the two symplectic structures are opposite). We shall use the Krichever formula in the next section in order to compute Poisson brackets for a new set of Darboux coordinates that we call *isomonodromic coordinates.*

3.3. Multi-time dependent Hamiltonian systems and their canonical transformations. Let $\mathcal{P}$ be a manifold equipped with a Poisson bracket $\{\ ,\ \}$. A function $H = H(x;t)$ depending explicitly on time defines a time dependent Hamiltonian system of the form

$$\dot{x} = \{x, H\}. \tag{3.19}$$

The total energy

$$E(t) := H(x(t);t) \tag{3.20}$$

computed on an arbitrary solution $x = x(t)$ to (3.19) is not conserved. However, the following well known identity describes its dependence on time

$$\dot{E} = \frac{\partial H(x;t)}{\partial t}|_{x=x(t)}. \tag{3.21}$$

One can recast Eqs. (3.19), (3.21) into an (autonomous) Hamiltonian form using the following standard trick of introducing extended phase space:

$$\hat{\mathcal{P}} := \mathcal{P}\times\mathbb{R}^2_{t,E}$$

with a Poisson bracket $\{\ ,\ \hat{\}}$ such that

$$\{\ ,\ \hat{\}}_{\mathcal{P}} = \{\ ,\ \}, \quad \{x, t\hat{\}} = \{x, E\hat{\}} = 0, \quad \{E, t\hat{\}} = 1. \tag{3.22}$$

The Hamiltonian

$$\hat{H} := H(x;t) - E \tag{3.23}$$

yields the dynamics (3.19), (3.21) along with the trivial equation

$$\dot{t} = 1.$$

The new Hamiltonian $\hat{H}$ is a conserved quantity. One returns to the original setting considering dynamics on the zero level surface

$$\hat{H}(x;t,E) = 0.$$

Let us now consider n functions on $\mathcal{P}$ depending on n times $H_1 = H_1(x;\mathbf{t}), \ldots, H_n = H_n(x;\mathbf{t})$, where

$$\mathbf{t} := (t_1, \ldots, t_n).$$

Assume that the time dependent Hamiltonian systems

$$\frac{\partial x}{\partial t_i} = \{x, H_i\}, \quad i = 1, \ldots, n \tag{3.24}$$

commute pairwise, i.e.

$$\frac{\partial}{\partial t_j}\frac{\partial x}{\partial t_i} = \frac{\partial}{\partial t_i}\frac{\partial x}{\partial t_j} \quad \text{for all } i \neq j. \tag{3.25}$$

Because of commutativity there exist common solutions $x = x(\mathbf{t})$ of the family (3.24) of differential equations. We want to introduce an analogue of the extended phase space for these multi-time dependent systems. We begin with the following simple

Lemma 3.4. *The Hamiltonian systems (3.24) commute* iff *the functions*

$$c_{ij}(x;\mathbf{t}) := \frac{\partial H_i}{\partial t_j} - \frac{\partial H_j}{\partial t_i} + \{H_i, H_j\}, \quad i \neq j \tag{3.26}$$

are Casimirs of the Poisson bracket, i.e.

$$\{x, c_{ij}\} = 0.$$

The energy functions

$$E_i = E_i(\mathbf{t}) := H_i(x(\mathbf{t});\mathbf{t}), \quad i = 1, \ldots, n \tag{3.27}$$

satisfy

$$\frac{\partial E_i}{\partial t_j} = \left[\frac{\partial H_j}{\partial t_i} + c_{ij}(x;\mathbf{t})\right]_{x=x(\mathbf{t})}. \tag{3.28}$$

In these equations

$$\frac{\partial H_i}{\partial t_j} := \frac{\partial H_i(x;\mathbf{t})}{\partial t_j} \tag{3.29}$$

are partial derivatives.

Proof. Spelling out the left-hand side of (3.25) gives

$$\begin{aligned}\frac{\partial}{\partial t_j}\frac{\partial x}{\partial t_i} &= Lie_{\frac{\partial}{\partial t_j}}\{x, H_i\}\\ &= \left\{Lie_{\frac{\partial}{\partial t_j}} x, H_i\right\} + \left\{x, Lie_{\frac{\partial}{\partial t_j}} H_i\right\}\\ &= \left\{\{x, H_j\}, H_i\right\} + \left\{x, \frac{\partial H_i}{\partial t_j} + \{H_i, H_j\}\right\}.\end{aligned}$$

In this calculation we have used that the Hamiltonian vector fields are infinitesimal symmetries of the Poisson bracket. Substituting this expression into (3.25) and using the Jacobi identity we arrive at

$$\left\{x, \frac{\partial H_i}{\partial t_j} - \frac{\partial H_j}{\partial t_i} + \{H_i, H_j\}\right\} = 0.$$

This proves the first part of the lemma. The second part easily follows from (3.26):

$$\frac{\partial E_i}{\partial t_j} = \{H_i, H_j\} + \frac{\partial H_i}{\partial t_j} = \frac{\partial H_j}{\partial t_i} + c_{ij}.$$

The lemma is proved. □

Definition 3.5. *The functions $H_1(x; \mathbf{t}), \ldots, H_n(x; \mathbf{t})$ on $\mathcal{P} \times \mathbb{R}^n$ define n multi-time dependent commuting Hamiltonian systems if they satisfy equations*

$$\frac{\partial H_i}{\partial t_j} - \frac{\partial H_j}{\partial t_i} + \{H_i, H_j\} = 0, \quad i,\ j = 1, \ldots, n. \tag{3.30}$$

The energy functions $E_i = E_i(\mathbf{t})$ of a multi-time dependent commuting family satisfy

$$\frac{\partial E_i}{\partial t_j} = \frac{\partial H_j}{\partial t_i}|_{x=x(\mathbf{t})}. \tag{3.31}$$

Remark 3.6. The evolution equations for the energy functions take more "natural" form

$$\frac{\partial E_i}{\partial t_j} = \frac{\partial H_i}{\partial t_j}|_{x=x(\mathbf{t})}, \tag{3.32}$$

similar to (3.21) under an additional assumption of commutativity of the Hamiltonians

$$\{H_i, H_j\} = 0, \quad i \neq j$$

as functions on the phase space $\mathcal{P}$. Observe that the one-form

$$\varpi = H_1(x(\mathbf{t}))dt_1 + \cdots + H_n(x(\mathbf{t}))dt_n \tag{3.33}$$

is closed for any solution $x(\mathbf{t})$ if the Hamiltonians commute. The commutativity holds true in the case of Schlesinger equations (see below). The closeness of the one-form (3.33) is crucial in the definition of the isomonodromic tau-function.

Let us now define the extended phase space

$$\hat{\mathcal{P}} := \mathcal{P} \times \mathbb{R}^{2n}, \tag{3.34}$$

where the coordinates on the second factor will be denoted $t_1, \ldots, t_n, E_1, \ldots, E_n$. The Poisson bracket $\widehat{\{\ ,\ \}}$ on the extended phase space is defined in a way similar to (3.22),

$$\widehat{\{\ ,\ \}}_{\mathcal{P}} = \{\ ,\ \}, \quad \widehat{\{x, t_i\}} = \widehat{\{x, E_i\}} = 0, \quad \widehat{\{E_i, t_j\}} = \delta_{ij}. \tag{3.35}$$

The Hamiltonians on the extended phase space are given by

$$\hat{H}_i = H_i(x; \mathbf{t}) - E_i, \quad i = 1, \ldots, n. \tag{3.36}$$

On the extended phase space the multi-time dependent commuting Hamiltonian equations can be put into a form of autonomous commuting Hamiltonian systems. Namely, the following statement holds true.

Lemma 3.7. *For a multi-time dependent commuting family of Hamiltonian systems the Hamiltonians* (3.36) *commute pairwise. The corresponding Hamiltonian equations on the extended phase space* (3.34) *read*

$$\begin{aligned} \frac{\partial x}{\partial t_j} &= \widehat{\{x, \hat{H}_j\}} = \{x, H_j\}, \\ \frac{\partial E_i}{\partial t_j} &= \widehat{\{E_i, \hat{H}_j\}} = \frac{\partial H_j}{\partial t_i}, \\ \frac{\partial t_i}{\partial t_j} &= \widehat{\{t_i, \hat{H}_j\}} = \delta_{ij}. \end{aligned} \tag{3.37}$$

Proof is straightforward.

On the common level surface

$$\hat{H}_1 = 0, \ldots, \hat{H}_n = 0$$

in $\hat{\mathcal{P}}$, one recovers the original multi-time dependent dynamics.

Let us now consider the particular case of a symplectic phase space $\mathcal{P}$ of the dimension $2g$. Introduce canonical Darboux coordinates $q_1, \ldots, q_g, p_1, \ldots, p_g$, such that the symplectic form ω on $\mathcal{P}$ becomes

$$\omega = \sum_{i=1}^{g} dp_i \wedge dq_i.$$

Then the extended phase space carries a natural symplectic structure

$$\hat{\omega} = \omega - \sum_{i=1}^{n} dE_i \wedge dt_i. \tag{3.38}$$

The canonical transformations of a multi-time dependent commuting Hamiltonian family are defined as symplectomorphisms of the extended phase space $\hat{\mathcal{P}}$ equipped with the symplectic structure (3.38).

Example 3.8. A multi-time dependent generating function $S = S(\mathbf{q}, \tilde{\mathbf{q}}, \mathbf{t})$ satisfying

$$\det \frac{\partial^2 S}{\partial q_i \partial \tilde{q}_j} \neq 0$$

defines a canonical transformation of the form

$$\tilde{p}_i = \frac{\partial S}{\partial \tilde{q}_i}, \quad p_i = -\frac{\partial S}{\partial q_i}, \quad \tilde{E}_k = E_k - \frac{\partial S}{\partial t_k}.$$

Usually in textbooks the last equation is written as the transformation law of the Hamiltonians, i.e., the new Hamiltonians $\tilde{H}_k$ are given by

$$\tilde{H}_k = H_k - \frac{\partial S}{\partial t_k}, \quad k = 1, \ldots, n.$$

We will stick to this tradition.

Let us come back to Schlesinger equations. In this case the position of the poles $u_1, \ldots, u_n$ play the role of the (complexified) time variables. It is straightforward to prove that the Schlesinger equations on $\mathfrak{g}^*$ can be considered as a multi-time dependent commuting Hamiltonian family.

Theorem 3.9. *The Hamiltonians H_k of the form* (3.3) *on $\mathfrak{g}^*$ Poisson commute*

$$\{H_k, H_l\} = 0, \quad \forall k, l = 1, \ldots n.$$

They also satisfy

$$\frac{\partial H_k}{\partial u_l} = \frac{\partial H_l}{\partial u_k}.$$

We end this section with the following simple observations about the Hamiltonians (3.3). First, these Hamiltonians are not independent. Indeed,

$$\sum_{i=1}^{n} H_i = 0. \tag{3.39}$$

Therefore the solutions to the Schlesinger equations depend only on the differences $u_i - u_j$. Moreover,

$$\sum_{i=1}^{n} u_i H_i = \sum_{i<j} \operatorname{tr} A_i A_j = \frac{1}{2} \operatorname{tr} \left(A_\infty^2 - \sum_{i=1}^{n} A_i^2 \right). \tag{3.40}$$

So, for a solution to the Schlesinger equations

$$\sum_{i=1}^{n} u_i \frac{\partial}{\partial u_i} A_k = [A_k, A_\infty]. \tag{3.41}$$

Thus the Hamiltonian (3.40) generates trivial dynamics on the reduced symplectic leaves. This implies that the solutions to the Schlesinger equations depend only on $n-2$ combinations of the variables $u_1, \ldots, u_n$ invariant w.r.t. the action of one-dimensional affine group

$$u_i \mapsto a\, u_i + b, \quad i = 1, \ldots, n, \quad a \neq 0.$$

Due to this invariance it is sometimes convenient to normalize the position of the poles of the Fuchsian systems by

$$u_1 = 0, \quad u_2 = 1. \tag{3.42}$$

4. Scalar Reductions of Fuchsian Systems

In this section we establish a birational transformation, that we call *scalar reduction*, between the space of all $m \times m$ Fuchsian systems of the form (1.2) considered modulo diagonal conjugations and the space of *special Fuchsian differential equations*, that we describe in the next sub-section.

4.1. Special Fuchsian differential equations. Recall [9] that a scalar linear differential equation of order m with rational coefficients is called Fuchsian if it has only regular singularities. Writing the differential equation in the form

$$y^{(m)} = a_1(z)y^{(m-1)} + \cdots + a_m(z)y, \tag{4.1}$$

one spells out the condition of regularity of a point $z = z_0$ in the form of existence of the limits

$$b_k(z_0) := -\lim_{z\to z_0}(z - z_0)^k a_k(z), \quad k = 1, \ldots, m.$$

The infinite point $z = \infty$ is regular if there exist the limits

$$b_k(\infty) := -\lim_{z\to\infty} z^k a_k(z), \quad k = 1, \ldots, m.$$

All the solutions to Eq. (4.1) are analytic at the points of analyticity of the coefficients. Let $z = z_0$ be a pole of the coefficients of the Fuchsian equation. The *indicial equation* at the point $z = z_0$ reads

$$\begin{aligned}&\lambda(\lambda-1)\cdots(\lambda-m+1) + b_1(z_0)\lambda(\lambda-1)\cdots(\lambda-m+2)\\ &\quad+\cdots+b_{m-1}(z_0)\lambda + b_m(z_0) = 0.\end{aligned} \tag{4.2}$$

If the roots $\lambda_1 = \lambda_1(z_0), \ldots, \lambda_m = \lambda_m(z_0)$ of this equation are non-resonant, i.e. none of the differences $\lambda_i - \lambda_j$ is a positive integer, then there exists a fundamental system of solutions of the form

$$y_j(z) = (z - z_0)^{\lambda_j}\left(1 + \sum_{l>0} a_{jl}(z - z_0)^l\right), \quad j = 1, \ldots, m, \quad z \to z_0. \tag{4.3}$$

Therefore the roots of the indicial equation coincide with the exponents at the regular singularity $z = z_0$. Similarly, the indicial equation at $z = \infty$ reads

$$\begin{aligned}&\lambda(-\lambda-1)\cdots(-\lambda-m+1) + b_1(\infty)\lambda(-\lambda-1)\cdots(-\lambda-m+2)\\ &\quad+\cdots+b_{m-1}(\infty)\lambda - b_m(\infty) = 0.\end{aligned} \tag{4.4}$$

The roots of this equation give the exponents $\lambda_i = \lambda_i(\infty)$ at $z = \infty$. The corresponding fundamental system of solutions, in the non-resonant case is given by

$$y_j(z) = z^{-\lambda_j}\left(1 + \sum_{l>0} a_{jl} z^{-l}\right), \quad j = 1, \ldots, m. \quad, z \to \infty. \tag{4.5}$$

Remark 4.1. For a Fuchsian differential equation with non-resonant roots of (4.4) there exists a unique canonical, up to a permutation of the roots, basis of solutions (4.5). Therefore there exists a unique canonical normalization of the monodromy matrices, for a given choice of a basis in the fundamental group of the punctured plane.

If the Fuchsian equation has $N+1$ poles including infinity then the exponents satisfy the following *Fuchs relation*:

$$\sum_{\text{all poles } z_0} \sum_{i=1}^{m} \lambda_i(z_0)(N-1)\frac{m(m-1)}{2}. \tag{4.6}$$

In the resonant case logarithmic terms are to be added. They can be obtained by a method similar to the one described above for the case of Fuchsian systems. See [9] for the details.

Remark 4.2. If λ is a root of the indicial equation (4.2) and $\lambda + n$ is not a root for any positive integer n then there exists a solution

$$y(z) = (z-z_0)^{\lambda}\left(1 + \sum_{l>0} c_l (z-z_0)^l\right), \quad z \to z_0.$$

Definition 4.3. *A pole $z = z_0$ of the coefficients of the Fuchsian equation is called* **apparent singularity** *if all solutions $y(z)$ are analytic at $z = z_0$.*

A necessary condition for the pole $z = z_0$ to be an apparent singularity is that all the roots $\lambda_1, \dots, \lambda_m$ of the indicial equation (4.2) must be non-negative integers. Absence of logarithmic terms impose additional constraints onto the coefficients of the Fuchsian equation. Let M be the maximum of the roots. Then the full set of constraints can be obtained by plugging into the equation the expansions (4.3) truncated at the term of order M and requiring compatibility of the resulting linear system for the coefficients a_{jl}, $j = 1, \dots, m$, $l = 1, \dots, M$.

Definition 4.4. *A Fuchsian differential equation of order m is called* **special** *if it has $n+1$ regular singularities at the points $z = u_1, \dots, z = u_n$, $z = \infty$ and also g apparent singularities $z = q_1, \dots, z = q_g$, where g is given by the formula* (3.15) *with the indices $0, 1, \dots, m-2, m$.*

Observe that, due to Fuchs relation [3] for a special Fuchsian equation the sum of the indices at the points $z = u_1, \dots, z = u_m$, $z = \infty$ is equal to $m-1$. For this reason we will denote, as above

$$\lambda_1^{(i)}, \dots, \lambda_m^{(i)}$$

the indices at $z = u_i$ and

$$\lambda_1^{(\infty)}, \lambda_2^{(\infty)} + 1, \dots, \lambda_m^{(\infty)} + 1$$

the indices at infinity. These numbers have zero sum (cf. (2.21)

$$\sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_j^{(i)} + \sum_{j=1}^{m} \lambda_j^{(\infty)} = 0. \tag{4.7}$$

We now describe in more details the behavior of the coefficients of the special Fuchsian equation near an apparent singularity.

Lemma 4.5. *Near an apparent singularity $z = q_i$ there exist m linear independent solutions $y_1(z), \dots, y_m(z)$ to the special Fuchsian equation* (4.1) *having expansions at $z = q_i$ of the following form:*

$$\begin{aligned} y_1 &= 1 + \frac{\alpha_1^{(i)}}{(m-1)!}(z-q_i)^{m-1} + \mathcal{O}(z-q_i)^{m+1}, \\ y_2 &= (z-q_i) + \frac{\alpha_2^{(i)}}{(m-1)!}(z-q_i)^{m-1} + \mathcal{O}(z-q_i)^{m+1}, \\ &\dots\dots\dots \\ y_{m-1} &= \frac{1}{(m-2)!}(z-q_i)^{m-2} + \frac{\alpha_{m-1}^{(i)}}{(m-1)!}(z-q_i)^{m-1} + \mathcal{O}(z-q_i)^{m+1}, \\ y_m &= \frac{1}{m!}(z-q_i)^m + \frac{\alpha_m^{(i)}}{(m+1)!}(z-q_i)^{m+1} + \mathcal{O}(z-q_i)^{m+2}, \end{aligned} \tag{4.8}$$

where $\alpha_1^{(i)}, \dots, \alpha_m^{(i)}$ are some constant coefficients.

The proof is obvious.

Denote

$$\mathcal{L} := -\frac{d^m}{dz^m} + a_1(z)\frac{d^{m-1}}{dz^{m-1}} + \dots + a_m(z)$$

the differential operator in the l.h.s. of (4.1).

Lemma 4.6. *The point $z = q_i$ is an apparent singularity of the special Fuchsian equation $\mathcal{L}y = 0$ if and only if the following conditions are satisfied.*

1. *The coefficients $a_1(z)$, $a_2(z), \dots, a_n(z)$ have at most simple poles at $z = q_i$ and*

$$\mathrm{Res}_{z=q_i} a_1(z) = -1. \tag{4.9}$$

2. *There exist coefficients $\alpha_1^{(i)}, \dots, \alpha_{m-1}^{(i)}$ such that, after the substitution of the expansions (4.8) into the differential equation one obtains*

$$\mathcal{L}y_j = O(z-q_i), \quad j = 1, \dots, m-1, \quad z \to q_i. \tag{4.10}$$

Proof. Suppose that $z = q_i$ is an apparent singularity. The indicial equation (4.2) for $z_0 = q_i$ by assumption must have the roots $0, 1, \dots, m-2, m$. Because of it $b_1(z_0) = -1$ and $b_k(z_0) = 0$ for $k > 1$. So

$$a_k(z) = \frac{c_k}{(z-q_i)^{k-1}}\left(1 + O(z-q_i)\right), \quad z \to q_i, \quad k = 2, \dots, m,$$

for some constants $c_1, \dots, c_m$. Let us prove that $c_3 = \dots = c_m = 0$ (only the non-trivial case $m \geq 3$ is to be studied). Indeed substituting the solution $y_1(z)$ from (4.8) into the equation one obtains that the l.h.s. behaves as

$$\mathcal{L}y_1 \sim \frac{c_m}{(z-q_i)^{m-1}}, \quad z \to q_i.$$

Hence $c_m = 0$. Similarly, substituting $y_2(z)$ one proves that, for $m \geq 4$ $c_{m-1} = 0$. Continuing this procedure we prove that all the poles are at most simple.

Validity of (4.10) means that the solutions $y_1(z), \dots, y_{m-1}(z)$ corresponding to the roots $0, 1, \dots, m-2$ of the indicial equation at $z = q_i$ contain no logarithmic terms up to the order $O(z-q_i)^m$. As it was explained in the previous page, this implies absence of logarithmic terms also in the higher orders since the order of resonance by assumption is equal to m. It remains to observe that the holomorphic solution $y_m(z)$ corresponding to the maximal root m always exists (see Remark 4.2). The vice–versa is obvious. □

The main result of this section is a coordinate description of the space of all special Fuchsian equations with given indices. Denote

$$\begin{aligned} p_s &:= -\alpha_{m-1}^{(s)} + \delta_1^{(s)}, \quad s = 1, \dots g., \\ \delta_1^{(s)} &= \sum_{t\neq s} \frac{1}{q_s - q_t} + \sum_{i=1}^{n} \frac{1}{q_s - u_i}\left[\sigma_1^{(i)} - \frac{m(m-1)}{2}\right], \\ \sigma_1^{(i)} &= \sum_{j=1}^{m} \lambda_j^{(i)}. \end{aligned} \tag{4.11}$$

Theorem 4.7. *Any special Fuchsian equation of order m with given indices $\lambda_j^{(i)}$, $j = 1, \dots, m$, $i = 1, \dots, n, \infty$ satisfying* (4.7) *must have the form*

$$y^{(m)} = a_1(z)y^{(m-1)} + \dots + a_m(z)y, \tag{4.12}$$

where the coefficients are given by

$$\begin{aligned} a_1(z) &= \sum_{s=1}^{g} \frac{1}{z-q_s} + \sum_{i=1}^{n} \frac{1}{z-u_i}\left[\sum_{j=1}^{m} \lambda_j^{(i)} - \frac{m\,(m-1)}{2}\right], \\ a_k(z) &= \Bigg[-\sum_{s=1}^{g} \frac{c_{m-k+1}^{(s)} R(q_s)^{k-1}}{z-q_s} + (-1)^{k-1} \sum_{i=1}^{n} \frac{\beta_k^{(i)}}{z-u_i}[R'(u_i)]^{k-1} \\ &\qquad + \beta_k^{(\infty)} z^{k\,n-n-k} + P_{k\,n-n-k-1}(z)\Bigg]\frac{1}{R(z)^{k-1}}, \\ k &= 2, \dots, m. \end{aligned} \tag{4.13}$$

Here $R(z) = \prod_{k=1}^{n}(z-u_k)$, $P_{n-3}(z)$, $P_{2n-4}(z), \dots, P_{m\,n-m-n-1}(z)$ are some polynomials labeled by their degrees, $c_1^{(s)}, \dots, c_{m-1}^{(s)}$ are some numbers. The coefficients $\beta_1^{(k)}, \dots, \beta_m^{(k)}$, $k = 1, \dots, n, \infty$ depend only on the indices. They are determined from *the identities*

$$\lambda(\lambda-1)\dots(\lambda-m+1)-\beta_1^{(i)}\lambda(\lambda-1)\dots(\lambda-m+2)+$$
$$+\beta_2^{(i)}\lambda(\lambda-1)\dots(\lambda-m+3)-\dots-\beta_m^{(i)}=\prod_{j=1}^m\left(\lambda-\lambda_j^{(i)}\right), \tag{4.14}$$

$$\lambda(\lambda+1)\dots(\lambda+m-1)-\left[\frac{m(m+1)}{2}-1+\sum_{j=1}^m\lambda_j^{(\infty)}\right]\lambda(\lambda+1)\dots(\lambda+m-2)-$$
$$-\beta_2^{(\infty)}\lambda(\lambda+1)\dots(\lambda+m-3)+\dots+(-1)^{m-1}\beta_m^{(\infty)}$$
$$=(\lambda-\lambda_1^{(\infty)})\prod_{j=2}^m(\lambda-\lambda_j^{(\infty)}-1). \tag{4.15}$$

The coefficients of the polynomials $P_{n-3}(z)$, $P_{2n-4}(z),\dots,P_{mn-m-n-1}(z)$ *and the parameters* $c_1^{(s)},\dots,c_{m-1}^{(s)}$ *are rational functions of* $q_1,\dots,q_g$, $p_1,\dots,p_g$ *and* $u_1,\dots,u_n$.

Proof of the Theorem. The ansatz (4.13) follows from the definition of a Fuchsian equation and from the first of the claims of Lemma 4.6. The expressions (4.14), (4.15) via indices is nothing but the spelling of the indicial equations (4.2), (4.4). Let us now use the second statement of Lemma 4.6 in order to show that all the remaining coefficients are uniquely determined by q_s and p_s.

Denote $\delta_k^{(s)}$ the constant term in the Laurent expansion of $a_k(z)$ near the apparent singularity $z=q_s$, $s=1,\dots,g$,

$$a_1(z)=\frac{1}{z-q_s}+\delta_1^{(s)}+O(z-q_s),$$
$$a_k(z)=-\frac{c_{m-k+1}^{(s)}}{z-q_s}+\delta_k^{(s)}+O(z-q_s),\quad k=2,\dots,m. \tag{4.16}$$

□

Lemma 4.8. 1. *The coefficients* $c_j^{(s)}$ *in* (4.13) *coincide with* $\alpha_j^{(s)}$ *in* (4.8).
2. *Equation* (4.1) *with coefficients given by* (4.13) *is special Fuchsian* iff *the following equations hold valid for all* $s=1,\dots,g$*:*

$$\begin{aligned}
&p_s\alpha_1^{(s)}+\delta_m^{(s)}=0,\\
&p_s\alpha_2^{(s)}+\delta_{m-1}^{(s)}-\alpha_1^{(s)}=0,\\
&\dots\quad\dots\quad\dots\\
&p_s\alpha_{m-2}^{(s)}+\delta_3^{(s)}-\alpha_{m-3}^{(s)}=0,\\
&p_s\alpha_{m-1}^{(s)}+\delta_2^{(s)}-\alpha_{m-2}^{(s)}=0,
\end{aligned} \tag{4.17}$$

where

$$p_s:=-\alpha_{m-1}^{(s)}+\delta_1^{(s)},\quad s=1,\dots,g. \tag{4.18}$$

Proof. Substituting the solution $y_l(z)$ for $1 \le l \le m-2$ into (4.1), by using (4.13), one obtains, modulo terms of order $O(z-q_s)$ the nontrivial contributions only from the terms

$$a_1(z)y_l^{(m-1)} + a_2(z)y_l^{(m-2)} + a_{m-l+1}(z)y_l^{(l-1)} + a_{m-l+2}(z)y_l^{(l-2)}.$$

According to Lemma 4.6 this expression must be of the order $O(z-q_s)$ for $z \to q_s$. Spelling this out gives

$$\left[\frac{1}{z-q_s} + \delta_1^{(s)}\right]\alpha_l^{(s)} + \left[-\frac{c_{m-1}^{(s)}}{z-q_s} + \delta_2^{(s)}\right]\alpha_l^{(s)}(z-q_s)$$
$$+\left[-\frac{c_l^{(s)}}{z-q_s} + \delta_{m-l+1}^{(s)}\right]\left[1+\alpha_l^{(s)}\frac{(z-q_s)^{m-l}}{(m-l)!}\right] + \left[-\frac{c_{l-1}^{(s)}}{z-q_s} + \delta_{m-l+2}^{(s)}\right](z-q_s)$$
$$= O(z-q_s). \tag{4.19}$$

Expanding these equations for $l = 1, \dots, m-2$ yields

$$c_l^{(s)} = \alpha_l^{(s)}$$

and also the first $m-2$ equations of (4.17). Analogously for $l = m-1$ the nontrivial contributions in (4.13) arise only in the terms

$$a_1(z)y_{m-1}^{(m-1)} + a_2(z)y_{m-1}^{(m-2)} + a_3(z)y_{m-1}^{(m-3)}.$$

Again, imposing that this expression must be of the order $O(z-q_s)$, one obtains

$$c_{m-1}^{(s)} = \alpha_{m-1}^{(s)}$$

and also the last equation of (4.17).

Due to Lemma 4.6 the equations (4.19) for $l = 1, \dots, m-1$ are necessary and sufficient for the points $z = q_s$ to be apparent singularities of a special Fuchsian equation. The lemma is proved. □

End of the proof of Theorem 4.7. We have derived a system of linear equations (4.17) for the parameters $\alpha_1^{(s)}, \dots, \alpha_{m-1}^{(s)}$, $s = 1, \dots, g$ and for the coefficients of the polynomials $P_{n-3}(z), P_{2n-4}(z), \dots, P_{mn-m-n-1}(z)$. It is easy to see that the number of equations is equal to the number of unknowns. It remains to prove that the determinant of this linear system is not an identical zero.

Let us first eliminate the α's. To this end we need to spell out the terms $\delta_2^{(s)}, \dots, \delta_m^{(s)}$ of order zero in the expansions of $a_2, \dots, a_m$ at $z = q_s$:

$$\delta_k^{(s)} = (k-1)\frac{R'(q_s)}{R(q_s)}\alpha_{m-k+1}^{(s)} - \sum_{t\neq s}\frac{\alpha_{m-k+1}^{(t)}R(q_t)^{k-1}}{(q_s-q_t)R(q_s)^{k-1}} + f_k(q_s) + \frac{P_{kn-n-k-1}(q_s)}{R(q_s)^{k-1}},$$

where

$$f_k(z) = \frac{(-1)^{k-1}\sum_{i=1}^{n}\frac{\beta_k^{(i)}}{z-u_i}[R'(u_i)]^{k-1} + \beta_k^{(\infty)}z^{kn-n-k}}{R(z)^{k-1}}.$$

The rational functions $f_1(z), \ldots, f_m(z)$ depend only on the positions of the poles u_i and the indices. Using these notations we rewrite Equations (4.17) as follows:

$$
\begin{aligned}
& p_s\alpha_1^{(s)} + \frac{(m-1)R'(q_s)}{R(q_s)}\alpha_1^{(s)} - \sum_{t\neq s}\frac{\alpha_1^{(t)}R(q_t)^{m-1}}{(q_s-q_t)R(q_s)^{m-1}} + f_m(q_s) \\
& \quad + \frac{P_{m\,n-m-n-1}(q_s)}{R(q_s)^{m-1}} = 0, \\
& p_s\alpha_2^{(s)} + \frac{(m-2)R'(q_s)}{R(q_s)}\alpha_2^{(s)} - \sum_{t\neq s}\frac{\alpha_2^{(t)}R(q_t)^{m-2}}{(q_s-q_t)R(q_s)^{m-2}} + f_{m-1}(q_s) \\
& \quad + \frac{P_{mn-m-2n}(q_s)}{R(q_s)^{m-2}} = \alpha_1^{(s)} \\
& \ldots \quad \ldots \quad \ldots \\
& p_s\alpha_{m-1}^{(s)} + \frac{R'(q_s)}{R(q_s)}\alpha_{m-1}^{(s)} - \sum_{t\neq s}\frac{\alpha_{m-1}^{(t)}R(q_t)}{(q_s-q_t)R(q_s)} + f_2(q_s) + \frac{P_{n-3}(q_s)}{R(q_s)}\alpha_{m-2}^{(s)}.
\end{aligned} \tag{4.20}
$$

Let us introduce the following vector notations. Denote

$$\mathbf{q} = (q_1, \ldots, q_g), \quad \mathbf{p} = (p_1, \ldots, p_g).$$

For any function $f(z)$ introduce vector

$$f(\mathbf{q}) := (f(q_1), \ldots, f(q_g)).$$

Similar notations will be used for functions of $\mathbf{p}$. For example,

$$\mathbf{p}^2 = (p_1^2, \ldots, p_g^2).$$

We also introduce g-component vectors α_j with the coordinates $\alpha_j^{(s)}$ and δ_1 with the coordinates $\delta_1^{(s)}$. The last ingredient will be the $g \times g$ matrices $M^{(l)} = (M_{ij}^{(l)})$, $l = 1, \ldots, m-1$ with the matrix entries

$$M_{ij}^{(l)} = \left(p_i + (m-l)\frac{R'(q_i)}{R(q_i)}\right)\delta_{ij} - \frac{R(q_j)^{m-l}(1-\delta_{ij})}{R(q_i)^{m-l}(q_i-q_j)}. \tag{4.21}$$

Using these notations we can rewrite Eqs. (4.20) as follows:

$$
\begin{aligned}
\alpha_{m-2} &= M^{(m-1)}(\delta_1 - \mathbf{p}) + f_2(\mathbf{q}) + \frac{P_{n-3}(\mathbf{q})}{R(\mathbf{q})}, \\
\alpha_{m-3} &= M^{(m-2)}\alpha_{m-2} + f_3(\mathbf{q}) + \frac{P_{2n-4}(\mathbf{q})}{R(\mathbf{q})^2}, \\
& \ldots \quad \ldots \quad \ldots \\
\alpha_1 &= M^{(2)}\alpha_2 + f_{m-1}(\mathbf{q}) + \frac{P_{(m-1)(n-1)-n-1}(\mathbf{q})}{R(\mathbf{q})^{m-2}}, \\
0 &= M^{(1)}\alpha_1 + f_m(\mathbf{q}) + \frac{P_{m(n-1)-n-1}(\mathbf{q})}{R(\mathbf{q})^{m-1}}.
\end{aligned} \tag{4.22}
$$

Substituting the first equation into the second equation we obtain

$$\alpha_{m-3} = M^{(m-2)}\left(M^{(m-1)}\alpha_{m-1} + f_2(\mathbf{q}) + \frac{P_{n-3}(\mathbf{q})}{R(\mathbf{q})}\right) + f_3(\mathbf{q}) + \frac{P_{2n-4}(\mathbf{q})}{R(\mathbf{q})^2}.$$

Continuing this process we express all α's via the known functions and the coefficients of the polynomials $P_{n-3}(z)$, $P_{2n-4}(z), \ldots, P_{m\,n-m-n-1}(z)$. On the last step we arrive at a linear equation for these coefficients:

$$\begin{aligned}\hat{P}_{m(n-1)-n-1}(\mathbf{q}) + M^{(1)}\hat{P}_{(m-1)(n-1)-n-1}(\mathbf{q}) + \cdots + M^{(1)}M^{(2)}\cdots M^{(m-2)}\hat{P}_{n-3}(\mathbf{q})\\ = M^{(1)}M^{(2)}\cdots M^{(m-1)}[\mathbf{p}-\delta_1] - M^{(1)}M^{(2)}\cdots M^{(m-2)}f_2(\mathbf{q})\\ -\cdots - M^{(1)}f_{m-1}(\mathbf{q}) - f_m(\mathbf{q}),\end{aligned} \tag{4.23}$$

where we denote

$$\hat{P}_{k\,n-k-n-1}(z) := \frac{P_{k\,n-k-n-1}(z)}{R(z)^{k-1}}, \quad k = 2, \ldots, m.$$

It remains to prove that the determinant of the linear operator in the left-hand side of this system does not identically vanish.

Indeed, this determinant is a polynomial in $p_1, \ldots, p_g$. Let us compute the terms of highest degree in these variables. It is easy to see that those terms can be written down explicitly

$$\frac{W_{m,n}(q_1, \ldots, q_g, \hat{p}_1, \ldots, \hat{p}_g)}{[R(q_1)\ldots R(q_g)]^{m-1}},$$

where the polynomial $W_{m,n}$ in $2g$ variables is defined in (A.21),

$$\hat{p}_s := p_s R(q_s).$$

Clearly it is not an identical zero. This proves the theorem. □

Corollary 4.9. *The positions $q_1, \ldots, q_g$ of the apparent singularities along with the auxiliary parameters $p_1, \ldots, p_g$ are coordinates on a Zariski open subset in the space of all special Fuchsian equations with given indices and given Fuchsian singularities $u_1, \ldots, u_n, \infty$.*

Observe that, in terms of the special Fuchsian equation the auxiliary parameters p_i are defined by

$$p_i = \mathrm{Res}_{z=q_i}\left[a_2(z) + \frac{1}{2}a_1(z)^2\right], \quad i = 1, \ldots, g. \tag{4.24}$$

They are related to $\rho_s := \alpha_{m-1}^{(s)}$ by the shift $\delta_1^{(s)}$ (see (4.11)) of the form

$$\rho_s = -p_s + \frac{\partial S(\mathbf{q}, \mathbf{u})}{\partial q_s}, \tag{4.25}$$

$$S(\mathbf{q}, \mathbf{u}) = \frac{1}{2}\log\prod_{s\neq t} + \log\prod_{i=1}^{n}\prod_{s=1}^{g}(q_s - u_i)^{\sigma_1^{(i)} - \frac{m(m-1)}{2}}.$$

We will see below that the coordinates q_s and p_s are canonically conjugated variables for the Schlesinger equations. The shift (4.25) is a canonical transformation. So, the variables q_s and $-\rho_s$ are also canonically conjugated.

Remark 4.10. The linear system (4.23) to be solved in order to reconstruct the special Fuchsian equation with given indices and poles and g given pairs (q_i, p_i) is very similar to the linear equation (A.20) used in the Appendix below in order to reconstruct the spectral curve (A.4) starting from its behavior over $z = u_1, \dots, z = u_n, z = \infty$ and a given divisor $(\gamma_1, \mu_1) + \cdots + (\gamma_g, \mu_g)$ of the degree g. The essential difference is that, in matrix notations in (A.20)the powers of the diagonal matrix $(\mu_1, \dots, \mu_g)$ enter while in (4.23) this is to be replaced by the matrix M of the form (4.21). It is a surprise that the matrix $M^{(l)}$ for $l = m$ coincides with the Lax matrix for the Calogero - Moser system [45]! At the moment we do not have an explanation of this coincidence.

4.2. Transformation of Fuchsian systems into special Fuchsian differential equations. In this section we will assign to a Fuchsian system (1.2) a special Fuchsian differential equation of the form

$$y^{(m)} = \sum_{l=0}^{m-1} d_l(z) y^{(l)}. \tag{4.26}$$

Note a change of notations with respect to (4.1):

$$d_l(z) = a_{m-l}(z), \quad l = 1, \dots, m.$$

The reduction of a system of differential equations to a scalar equation is given by the following well known classical construction. Denote by $\phi_1, \phi_2, \dots, \phi_m$ the components of the vector function Φ. The m^{th} order linear differential equation for the scalar function $y := \phi_1$ can be written in the determinant form

$$\det \begin{pmatrix} y & y_1 & \dots & y_m \\ y' & y_1' & \dots & y_m' \\ \dots & \dots & \dots & \dots \\ y^{(m)} & y_1^{(m)} & \dots & y_m^{(m)} \end{pmatrix} = 0.$$

Here $y_1, \dots, y_m$ is the first row of a fundamental matrix solution for the system (1.2). Expanding the determinant one obtains the needed differential equation in the form

$$W(z) y^{(m)} = W'(z) y^{(m-1)} + W_{m-2}(z) y^{(m-2)} + \cdots + W_0(z) y, \tag{4.27}$$

where

$$W(z) = \det \begin{pmatrix} y_1 & \dots & y_m \\ y_1' & \dots & y_m' \\ \dots & \dots & \dots \\ y_1^{(m-1)} & \dots & y_m^{(m-1)} \end{pmatrix} \tag{4.28}$$

is the Wronskian of the functions $y_1(z), \dots, y_m(z)$, the functions W_l are certain determinants with the rows constructed from $(y_1, \dots, y_m)$ and their derivatives. Therefore

$$d_{m-1}(z) = \frac{W'(z)}{W(z)}, \quad d_l(z) = \frac{W_l(z)}{W(z)}, \quad l \leq m-2. \tag{4.29}$$

It readily follows that the coefficients of the scalar differential equation can have poles only at zeroes of the Wronskian and at the points $u_1, \dots, u_n$. Let us call the Fuchsian equation (4.27) the *1-associated* with the Fuchsian system (1.2). In a similar way one can obtain for any $j = 1, \dots, m$ the j-associated Fuchsian equation for the j^{th} component of Φ.

We will now give a more precise description of the poles of the scalar Fuchsian equation and the corresponding exponents. Let us begin with the poles that come from zeroes of the Wronskian. They are so-called *apparent singularities* of the Fuchsian differential equation (4.27). That is, the coefficients of the differential equation have poles at zeroes of $W(z)$ but all solutions are analytic at these poles. Let us first compute exponents at the apparent singularities.

Lemma 4.11. *Let $z = q$ be a zero of the Wronskian $W(z)$ of the multiplicity k. Then, if $k \le m - 1$ the exponents of solutions to the Fuchsian equation are*

$$0, 1, \dots, m-k-1,\ m-k+j_1, m-k+1+j_1+j_2, \dots, m-1+j_1+\dots+j_k, \quad (4.30)$$

where $j_1, \dots, j_k$ are nonnegative integers satisfying

$$k\, j_1 + (k-1) j_2 + \dots + j_k = k; \quad (4.31)$$

if $k \ge m$, then the exponents are

$$0,\ 1 + j_1,\ 2 + j_1 + j_2, \dots, m - 1 + j_1 + \dots + j_{m-1}, \quad (4.32)$$

where $j_1, \dots, j_{m-1}$ are nonnegative integers satisfying

$$(m-1) j_1 + (m-2) j_2 + \dots + j_{m-1} = k. \quad (4.33)$$

Proof. All the exponents at an apparent singularity must be nonnegative integers $0 \le n_1 \le \dots \le n_m$. The corresponding basis of solutions must have the form

$$y_1(z) = (z-q)^{n_1}(1 + O(z-q)), \dots, y_m(z) = (z-q)^{n_m}(1 + O(z-q)).$$

Therefore all the exponents are pairwise distinct: in the opposite case the difference of two basic solutions would have a higher exponent. Besides, necessarily $n_1 = 0$. Indeed, otherwise all elements of the first line of the fundamental matrix Φ would vanish at $z = q$. This contradicts non-degeneracy of the determinant $\det \Phi(z)$ of the fundamental matrix solution to the Fuchsian system.

Let us now spell out the indicial equation (4.2) at $z = q$,

$$\lambda(\lambda-1)\dots(\lambda-m+1) - b_1\lambda(\lambda-1)\dots(\lambda-m+2) - \dots - b_m = 0,$$

where

$$b_{m-i} = \lim_{z\to q}(z-q)^{m-i} d_i(z), \quad i = 0, 1, \dots, m-1.$$

Because of (4.29) we always have

$$b_1 = k,$$

where k is the multiplicity of $z = q$ as a zero of the Wronskian. Besides, if $k \le m - 1$ then

$$b_{k+1} = \dots = b_m = 0.$$

So the indicial equation factorizes

$$\lambda(\lambda-1)\dots(\lambda-m+k+1)$$
$$\times\left[(\lambda-m+k)\dots(\lambda-m+1)-k\,(\lambda-m+k)\dots(\lambda-m+2)-\dots-b_k\right]=0.$$

The sum of the k roots of the second factor is equal to

$$(m-k)+(m-k+1)+\dots+(m-1)+k.$$

As these roots must be pairwise distinct positive integers different from zeroes of $\lambda(\lambda-1)\dots(\lambda-m+k+1)$, they can be represented in the form (4.30), (4.31).

Let us now consider the case of multiplicity $k\geq m$. Since $\lambda=0$ must be a root of the indicial equation, one has $b_0=0$. Hence the indicial equation reads

$$\lambda\left[(\lambda-1)\dots(\lambda-m+1)-k\,(\lambda-1)\dots(\lambda-m+2)-\dots-b_{m-1}\right]=0.$$

Again, the $(m-1)$ roots of the second factor are pairwise distinct positive integers with the sum equal to

$$1+2+\dots+(m-1)+k.$$

So, they can be represented in the form (4.32), (4.33). The lemma is proved. □

Corollary 4.12. *If $z=q$ is a simple root of the Wronskian $W(z)$, then the exponents of the Fuchsian differential equation (4.27) are*

$$0,1,\dots,m-2,\ m.$$

Remark 4.13. In [34] Kimura and Okamoto claimed that, for an apparent singularity of multiplicity k the exponents are

$$0,\ 1,\dots,m-2,\ m+k-1.$$

We were unable to reproduce the proof of this statement for $k>1$.

Theorem 4.14. *Take a Fuchsian system of the form* (1.2) *with pairwise distinct nonresonant exponents at ∞ $\lambda_i^{(\infty)}$, $i=1,\dots,m$. Denote $\lambda_1^{(k)},\dots,\lambda_m^{(k)}$ the exponents at u_k, $k=1,\dots,n$. Suppose that for some $j=1,\dots,m$,*

$$\sum_{k=1}^{n}A_{k_{ji}}u_k\neq 0,\quad\forall\, i=1,\dots,m,\ i\neq j.\tag{4.34}$$

Then the scalar differential equation for the j^{th} component

$$y:=\phi_j$$

of the solution Φ of the deformed system (1.2) *possesses the following properties.*

1. *The coefficients of the equation depend rationally on the matrix elements of* A_k, $k = 1, \dots, n$. *They can be represented in the form*

$$d_l(z) = \frac{f_l(z)}{\prod_{k=1}^{n}(z-u_k)^{m-l}\prod_{i=1}^{g}(z-q_i)},$$

where

$$g = \frac{(n-1)m(m-1)}{2} - (m-1),$$

the functions $f_l(z)$ *are polynomials of degree*

$$\deg f_l(z) = (n-1)(m-l) + g,$$

and $q_1, \dots, q_g$ *are zeroes of the Wronskian* (4.28). *They are apparent singularities of the Fuchsian equation.*
2. *This differential equation has regular singularities at* $u_1, \dots, u_n$ *with the exponents* $\lambda_i^{(k)}$, $i = 1, \dots, m$, *at* u_k *and also at* ∞ *with the exponents* $\lambda_1^{(\infty)}, \lambda_2^{(\infty)} + 1, \dots, \lambda_m^{(\infty)} + 1$.
3. *If the numbers* $q_1, \dots, q_g$ *are pairwise distinct then the exponents at each apparent singularity are* $0, 1, \dots, m-2, m$.

Proof. Without loss of generality we may assume $j = 1$. The above construction gives a Fuchsian equation with regular singularities at $u_1, \dots, u_n, \infty$ and apparent singularities at the zeroes of the Wronskian. Denote $\tilde{\lambda}_1^{(i)}, \dots, \tilde{\lambda}_m^{(i)}$ the exponents of the solutions to the scalar equation at the point $z = u_i$ for every $i = 1, \dots, n$. From the construction it immediately follows that for $j = 1, \dots, m$,

$$\tilde{\lambda}_j^{(i)} = \lambda_j^{(i)} + n_j^{(i)}, \quad \text{for some} \quad n_j^{(i)} \in \mathbb{N}, \tag{4.35}$$

after a suitable labelling of the exponents. Let us now consider the exponents $\tilde{\lambda}_1^{(\infty)}, \dots, \tilde{\lambda}_m^{(\infty)}$. They satisfy modified equalities

$$\tilde{\lambda}_1^{(\infty)} = \lambda_1^{(\infty)}, \quad \tilde{\lambda}_j^{(\infty)} = \lambda_j^{(\infty)} + 1 + n_j^{(\infty)}, \quad \text{for some} \quad n_j^{(\infty)} \in \mathbb{N}, j = 2, \dots, m. \tag{4.36}$$

Indeed, because of diagonality of the matrix A_∞ there exists a fundamental matrix solution of the Fuchsian system of the form

$$\Phi = \left(\mathbb{1} + O\left(\frac{1}{z}\right)\right) z^{-A_\infty} z^{-R^{(\infty)}}.$$

Looking at the first row of this matrix yields the above estimates.

We will prove below that, doing if necessary a small monodromy preserving deformation the above equalities are satisfied with $n_j^{(i)} = 0$ for all $i = 1, \dots, n, \infty$, $j = 1, \dots, m$. To this end we now proceed to considering the apparent singularities.

We already know from Corollary 4.12 that, if $z = q$ is a simple zero of the Wronskian then the exponents at the apparent singularity are $0, 1, \dots, m-2, m$. Let us now prove that, under the assumption (4.34) the Wronskian has exactly g zeroes (counted with their multiplicities).

Denote

$$R(z) := \prod_{k=1}^{n} (z - u_k).$$

Differentiating the linear system

$$\phi_i'(z) = \sum_{j=1}^{m} A_{ij}(z)\phi_j(z), \tag{4.37}$$

where

$$A_{ij}(z) = \sum_{k=1}^{n} \frac{A_{k_{ij}}}{z - u_k}$$

and using Leibnitz rule we have for each $l = 1, \ldots, m$,

$$\phi_i^{(l)}(z) = \sum_{j=1}^{m} \sum_{k=0}^{l-1} \binom{l-1}{k} A_{ij}^{(l-1-k)}(z)\phi_j^{(k)}(z). \tag{4.38}$$

Denoting $y = \phi_1$, we rewrite the system in the form

$$y^{(l)}(z) = \sum_{j=2}^{m} \mathcal{P}_{l+1,j}(z)\phi_j(z) + \sum_{k=0}^{l-1} \mathcal{Q}_{l+1,l-k}(z) y^{(l-1-k)}(z) \tag{4.39}$$

for $l = 1, \ldots, m$. The coefficients $\mathcal{P}_{l+1,j}(z)$ and $\mathcal{Q}_{l+1,l-k}(z)$ are rational functions of z such that $R(z)^l \mathcal{P}_{l+1,j}(z)$ is a polynomial in z of degree $nl - l - 1$ and $R(z)^{k+1} \mathcal{Q}_{l+1,l-k}(z)$ is a polynomial in z of degree $(k+1)(n-1)$. They can be computed from the following recursion relations starting with

$$\mathcal{Q}_{2,1}(z) = A_{11}(z), \quad \mathcal{P}_{2,j}(z) = A_{1j}(z).$$

The recursion reads

$$\begin{aligned}
\mathcal{P}_{l+2,j}(z) &= \mathcal{P}_{l+1,j}'(z) + \sum_{s=2}^{m} \mathcal{P}_{l+1,s}(z) A_{sj}(z) \qquad j = 2, \ldots, m, \\
\mathcal{Q}_{l+2,l+1-k}(z) &= \mathcal{Q}_{l+1,l-(k-1)}'(z) + \mathcal{Q}_{l+1,l-k}(z), \qquad k = 1, \ldots, l-1, \\
\mathcal{Q}_{l+2,l+1}(z) &= \mathcal{Q}_{l+1,l}(z), \\
\mathcal{Q}_{l+2,1}(z) &= \mathcal{Q}_{l+1,1}'(z) + \sum_{s=2}^{m} \mathcal{P}_{l+1,s}(z) A_{s1}(z).
\end{aligned}$$

Observe that $\mathcal{Q}_{l+1,l}(z) = A_{11}(z)$ for all l.

The system (4.39) can be written in the form

$$\mathcal{P} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \cdots \\ \phi_m \end{pmatrix} = \mathcal{Q} \begin{pmatrix} y \\ y' \\ \cdots \\ y^{(m-1)} \end{pmatrix}, \tag{4.40}$$

where

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \mathcal{P}_{22}(z) & \dots & \mathcal{P}_{2m}(z) \\ \dots & \dots & \dots & \\ 0 & \mathcal{P}_{m,2}(z) & \dots & \mathcal{P}_{m,m}(z) \end{pmatrix},$$

$$\mathcal{Q} = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ -\mathcal{Q}_{21}(z) & 1 & 0 & \dots & 0 \\ -\mathcal{Q}_{31}(z) & -\mathcal{Q}_{32}(z) & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -\mathcal{Q}_{m,1}(z) & \dots & \dots & -\mathcal{Q}_{m,m-1}(z) & 1 \end{pmatrix}. \tag{4.41}$$

Let us prove invertibility of the $m \times m$ matrix $\mathcal{P}$.

Lemma 4.15.

$$\det(\mathcal{P})\ \frac{\Delta(z)}{R(z)^{\frac{m(m-1)}{2}}}, \tag{4.42}$$

where $\Delta(z)$ is a polynomial of degree $g = \frac{(n-1)m(m-1)}{2} - (m-1)$ with leading coefficient

$$-\prod_{2\le i<j\le m}\left(\lambda_i^{(\infty)} - \lambda_j^{(\infty)}\right)\left[\prod_{j=2}^{m}\sum_{k=1}^{n} u_k A_{k_{1j}}\right]. \tag{4.43}$$

Proof. First of all we prove that, if $p_{l+1,j}z^{nl-l-1}$ is the leading term in $R(z)^l\mathcal{P}_{l+1,j}$, then the leading term in $R(z)^{l+1}\mathcal{P}_{l+2,j}$ is $p_{l+2,j}z^{n(l+1)-(l+1)-1}$ with $p_{l+2,j} = (-\lambda_j^{(\infty)} - (l+1))p_{l+1,j}$. In fact, from the above recursion relations one obtains

$$\begin{aligned} R(z)^{l+1}\mathcal{P}_{l+2,j}(z) &= R(z)^{l+1}\mathcal{P}'_{l+1,j}(z) + R(z)^{l+1}\sum_{s=2}^{m}\mathcal{P}_{l+1,s}(z)A_{sj}(z) \\ &\sim ((nl-l-1)p_{l+1,j} - nlp_{l+1,j})z^{nl-l+n} + R(z)^{l+1}\mathcal{P}_{l+1,j}(z)A_{jj}(z) \\ &\sim \left(\sum_{k=1}^{n} A_{k_{jj}}(z) - (l+1)\right)p_{l+1,j}z^{n(l+1)-(l+1)-1}. \end{aligned}$$

Thus we have

$$p_{l+1,j} = (-\lambda_j^{(\infty)} - l)p_{l,j}\cdots = (-\lambda_j^{(\infty)} - l)\cdots(-\lambda_j^{(\infty)} - 2)p_{2,j},$$

with $p_{2,j} = \sum_{k=1}^{n} A_{k_{1j}}u_k$ because $\mathcal{P}_{2,j}(z) = A_{1j}(z)$. Substituting these leading terms in the entries $\mathcal{P}_{ij}(z)$ we obtain that the leading term of $\mathcal{P}$ is **RMP** where

$$\mathbf{P} = \text{diagonal}\left(1, p_{2,2}, \dots, p_{2,m}\right),$$

$$\mathbf{R} = \text{diagonal}\left(1, \frac{z^{n-2}}{R(z)}, \dots, \frac{z^{(n-1)(i-1)-1}}{R(z)^{i-1}}, \dots, \frac{z^{(n-1)(m-1)-1}}{R(z)^{m-1}}\right),$$

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 1 \\ 0 & (-\lambda_2^{(\infty)} - 2) & \dots & (-\lambda_m^{(\infty)} - 2) \\ \dots & \dots & \dots & \dots \\ 0 & \prod_{j=2}^{m-1}(-\lambda_2^{(\infty)} - j) & \dots & \prod_{j=2}^{m-1}(-\lambda_m^{(\infty)} - j) \end{pmatrix},$$

and computing the determinant we obtain (4.42), (4.43). The lemma is proved. □

Observe that the leading coefficient of the polynomial $\Delta(z)$ cannot be zero, thanks to our hypothesis (4.34).

Continuing the procedure used in the proof of the lemma it is easy to obtain explicit formulae for the coefficients of the scalar equation. Let $\mathcal{I}$ be the inverse matrix of $\mathcal{P}$. Its leading term equals

$$\mathcal{I}_{ij}(z) \sim \left(\mathbf{M}^{-1}\right)_{ij} \frac{R(z)^{j-1}}{p_{2,i} z^{(n-1)(j-1)-1}}$$

that is,

$$\mathcal{I}_{ij}(z) = \frac{D_{ij}(z) R(z)^{j-1}}{\Delta(z)},$$

where $D_{ij}(z)$ is a polynomial in z of degree $\frac{(n-1)m(m-1)}{2} - (m-1) - ((j-1)(n-1)-1)$ with coefficients depending on i, j. Solving the system (4.40) we obtain for $i > 1$,

$$\phi_i(z) = \sum_{j=1}^{m} \left(- \sum_{2\le s<j} \mathcal{I}_{is}(z) \mathcal{Q}_{sj}(z) + \mathcal{I}_{ij}(z) \right) y^{(j-1)}. \tag{4.44}$$

Substituting (4.44) in (4.38) with $l = m$, we obtain

$$y^{(m)}(z) = \sum_{j=1}^{m} \left[\sum_{i=2}^{m} \mathcal{P}_{m+1,i}(z) \left(\mathcal{I}_{ij}(z) - \sum_{s=2}^{j-1} \mathcal{I}_{is}(z) \mathcal{Q}_{sj}(z) \right) + \mathcal{Q}_{m+1,j}(z) \right] y^{(j-1)}(z)$$

that is the requested differential equation

$$y^{(m)}(z) = \sum_{l=0}^{m-1} d_l(z) y^{(l)}(z),$$

with

$$d_l = \sum_{i=2}^{m} \mathcal{P}_{m+1,i}(z) \left(\mathcal{I}_{i,l+1}(z) - \sum_{s=2}^{l+1} \mathcal{I}_{is}(z) \mathcal{Q}_{s,l+1}(z) \right) + \mathcal{Q}_{m+1,l+1}(z) \frac{f_{1,l}(z)}{\Delta(z) R(z)^{m-l}},$$

where $f_{1,l}(z)$ are polynomials of degree $(n-1)(m-l) + g$.

It remains to prove the statement about the exponents of the scalar Fuchsian equation at the poles $z = u_i$, $z = \infty$. Let us assume for simplicity that all the apparent singularities are pairwise distinct (the general case can be considered in a similar way). Taking the sum of the equalities (4.35), (4.36) we obtain the following estimate:

$$\sum_{j=1}^{m} \left(\sum_{i=1}^{n} \tilde{\lambda}_j^{(i)} + \tilde{\lambda}_j^{(\infty)} \right) = m - 1 + \sum_{i,j} n_j^{(i)}.$$

The sum of exponents at the apparent singularity $z = q_s$ is equal to

$$1 + 2 + \cdots + (m-2) + m = \frac{m(m-1)}{2} + m.$$

So the total sum of exponents of the Fuchsian equation satisfies

$$\sum \text{exponents} = m - 1 + g\left(\frac{m(m-1)}{2} + m\right) + \sum_{i,j} n_j^{(i)}(g+n-1)\frac{m(m-1)}{2} + \sum_{i,j} n_j^{(i)}.$$

But, according to the Fuchs relation (4.6) the total sum of exponents over all $g + n + 1$ regular singularities must be equal just to $(g + n - 1)\frac{m(m-1)}{2}$. Therefore all non-negative integers $n_j^{(i)}$ appearing in the equalities (4.35), (4.36) must be zero. The theorem is proved.

Remark 4.16. The above construction of the Fuchsian equation (4.26) for a given Fuchsian system is clearly invariant w.r.t. simultaneous diagonal conjugations of the coefficients of the latter.

To make sure that (4.26) is a special Fuchsian equation for a generic Fuchsian system, we are to prove that, in the generic case all the roots of the polynomial $\Delta(z)$ are pairwise distinct. This will follow from Theorem 4.7 claiming that, in the space of all special Fuchsian equations, the positions of the apparent singularities are independent variables and from the result of the next section that says that under a certain genericity assumption the Fuchsian system can be reconstructed from the special Fuchsian equation uniquely up to a conjugation by constant diagonal matrices.

4.3. Inverse transformation

Theorem 4.17. *Consider an m^{th} order special Fuchsian equation of the form*

$$y^{(m)}(z) = \sum_{l=0}^{m-1} \frac{f_l(z)}{\Delta(z)R(z)^{m-l}} y^{(l)}(z), \tag{4.45}$$

where $R(z) = \prod_{k=1}^{n}(z-u_k)$, $\Delta(z) = \prod_{i=1}^{g}(z-q_i)$, $g = \frac{(n-1)m(m-1)}{2} - (m-1)$ and $f_l(z)$ are polynomials of degree $(n-1)(m-l) + g$. Let the exponents of the pole u_k, $k = 1, \ldots, n$, be $\lambda_i^{(k)}$, $i = 1, \ldots, m$, and the ones of ∞ be $\lambda_1^{(\infty)}, \lambda_2^{(\infty)}+1, \ldots, \lambda_m^{(\infty)}+1$ and let $q_1, \ldots, q_g$ be pairwise distinct apparent singularities of exponents $0, 1, \ldots, m-2, m$. If the monodromy group of the Fuchsian equation (4.45) is irreducible, then there exists a $m \times m$ Fuchsian system of the form

$$\frac{\mathrm{d}}{\mathrm{d}z}\Phi = \sum_{k=1}^{n} \frac{A_k}{z-u_k}\Phi$$

with exponents $\lambda_1^{(\infty)}, \ldots, \lambda_m^{(\infty)}$ at ∞ and $\lambda_1^{(k)}, \ldots, \lambda_m^{(k)}$ at u_k, and no apparent singularities, such that the first row of its fundamental matrix satisfies the given m^{th} order Fuchsian equation. The matrix entries of the matrices A_k, $k = 1, \ldots, n$, depend rationally on the coefficients of the polynomials f_l and on $q_1, \ldots, q_g$, $u_1, \ldots, u_n$. Moreover, if

$$\frac{\mathrm{d}}{\mathrm{d}z}\tilde{\Phi} = \sum_{k=1}^{n} \frac{\tilde{A}_k}{z-u_k}\tilde{\Phi}$$

is another Fuchsian system corresponding to the given special Fuchsian equation, then there exists a diagonal matrix D such that

$$\tilde{A}_k = D^{-1} A_k\, D, \quad k = 1, \dots, n.$$

Proof. This proof follows essentially the proof due to Bolibruch of reconstruction of a Fuchsian system from a given Fuchsian equation [3, 7]. We need some extra machinery in our case to eliminate the apparent singularities $q_1, \dots, q_g$.

Lemma 4.18. *The system*

$$\frac{\mathrm{d}Y}{\mathrm{d}z} = F(z)Y$$

constructed from (4.45) *by the transformation*

$$Y^j = [\Delta(z)R(z)]^{j-1}\frac{\mathrm{d}^{j-1}y}{\mathrm{d}z^{j-1}}, \qquad j = 1, \dots, m,$$

is Fuchsian at $u_1, \dots, u_n, q_1, \dots, q_g$ with the same exponents of (4.45) *and has a regular singularity at ∞.*

The proof is straightforward and can be found in [3, 7].

First of all we want to eliminate the apparent singularities $q_1, \dots, q_g$. By Lemma 4.5, near the point $z = q_i$ we can choose a basis of solutions $y_1, \dots, y_m$ such that

$$\frac{\mathrm{d}^{l-1}y_k}{\mathrm{d}z^{l-1}} = \frac{1}{(k-l)!}(z-q_i)^{k-l} + \mathcal{O}(z-q_i)^{m-l}, \qquad l \le k < m,$$
$$\frac{\mathrm{d}^{l-1}y_k}{\mathrm{d}z^{l-1}} = \frac{\alpha_k^{(i)}}{(m-l)!}(z-q_i)^{m-l} + \mathcal{O}(z-q_i)^{m+2-l}, \qquad l > k,\ k < m,$$
$$\frac{\mathrm{d}^{l-1}y_m}{\mathrm{d}z^{l-1}} = \frac{1}{(m-l+1)!}(z-q_i)^{m-l+1} + \mathcal{O}(z-q_i)^{m+2-l}, \qquad k = m, \quad l \le m.$$

To eliminate all apparent singularities $q_1, \dots, q_g$, we apply the following gauge transformation:

$$\hat{Y} = \Gamma(z)\Delta(z)^{-M}Y,$$

where $M = \mathrm{diag}\,(0, 1, \dots, m-2, m)$ and $\Gamma(z)$ is a lower triangular matrix with all diagonal elements equal to 1 and all off-diagonal elements equal to zero apart from the last row which is given by

$$\Gamma(z)_{ml} = -\frac{R(z)^{m-1}}{\Delta(z)}g_l(z), \quad l = 1, \dots, m-1.$$

where $g_l(z)$ is a degree g polynomial in $\frac{1}{z}$ such that $g_l(q_i) = \alpha_l^{(i)}$ and $g_l(z) \sim z^{-g}$ as $z \to \infty$. Let us show that the new matrix $\hat{Y}$ is holomorphic and invertible at $z = q_i$.

In fact near $z = q_i$, we have

$$\Gamma_{ml}^{(i)} = -\frac{\alpha_l^{(i)}R(q_i)^{m-1}}{(z-q_i)\Delta'(q_i)} + \mathcal{O}(1),$$

and $\Delta(z)^{-M}Y = \mathrm{diag}\left(1,\ldots,1,\frac{1}{\Delta(z)}\right)\mathrm{diag}\left(1, R(z),\ldots,R(z)^{m-1}\right)G(z)$, where

$$\begin{aligned} G(z)_{lk} &= \mathcal{O}(z-q_i), \quad l \neq k, m, \\ G(z)_{ll} &= 1+\mathcal{O}(z-q_i), \quad l \neq m \\ G(z)_{mk} &= \alpha_k^{(i)} + \mathcal{O}(z-q_i), \quad k \neq m, \\ G(z)_{mm} &= (z-q_i) + \mathcal{O}(z-q_i)^2. \end{aligned}$$

This gives

$$\hat{Y}(z) = \hat{Y}_0 + \mathcal{O}(z-q_i),$$

where $\det(\hat{Y}_0) = \frac{R(q_i)^{\frac{m(m-1)}{2}}}{\Delta'(q_i)} \neq 0$, as we wanted to prove.

We now need to study infinity. First, in the non-resonant case, we can choose a basis $y_1,\ldots,y_m$ of solutions for the differential equation of the form

$$\begin{aligned} y_1(z) &= a_1 z^{-\lambda_1^\infty}\left(1+\mathcal{O}\left(\frac{1}{z}\right)\right), \\ y_k(z) &= a_k z^{-\lambda_k^\infty - 1}\left(1+\mathcal{O}\left(\frac{1}{z}\right)\right), \qquad k = 2,\ldots,m, \end{aligned}$$

for some arbitrary non-zero coefficients $a_1,\ldots,a_m$. As a consequence we obtain that

$$\begin{aligned} Y_{l1} &= (-1)^{l-1} a_1 \lambda_1^{(\infty)}(\lambda_1^{(\infty)}+1)\cdots(\lambda_1^{(\infty)}+l-2) z^{(g+n)(l-1)} z^{-\lambda_1^\infty - l+1}\left(1+\mathcal{O}\left(\frac{1}{z}\right)\right), \\ Y_{lk} &= (-1)^{l-1} a_k (\lambda_k^{(\infty)}+1)(\lambda_k^{(\infty)}+2)\cdots(\lambda_k^{(\infty)}+l-1) z^{(g+n)(l-1)} z^{-\lambda_k^\infty - l}\left(1+\mathcal{O}\left(\frac{1}{z}\right)\right), \\ &\quad k = 2,\ldots,m, \end{aligned}$$

that gives

$$Y(z) = z^C G^{(\infty)}(z) z^{-\Theta^{(\infty)}},$$

where $\Theta^{(\infty)} = \mathrm{diagonal}(\lambda_1^\infty, \lambda_2^\infty+1,\ldots,\lambda_m^\infty+1)$, $C = \mathrm{diagonal}(0, (g+n-1),\ldots,(g+n-1)(m-1))$ and $G^{(\infty)}(z)$ is holomorphicly invertible at infinity such that $G^{(\infty)}(\infty)$ has all minors not equal to zero. In particular the arbitrary choice of the parameters $a_1,\ldots,a_m$ implies freedom of multiplication of Y by a diagonal matrix, diagonal($a_1,\ldots,a_m$), from the right.

In the resonant case, in a similar way one obtains

$$Y(z) = z^C G^{(\infty)}(z) z^{-\Theta^{(\infty)}} z^{-R^{(\infty)}}.$$

To estimate the indices at infinity of our $\hat{Y}(z) = \Gamma(z)\Delta(z)^{-M} z^C G^{(\infty)}(z) z^{-\Theta^{(\infty)}} z^{-R^{(\infty)}}$ we want to use the following lemma proved in [3, 7] (see Lemma 4.1.2 in both references).

Lemma 4.19. *Let $U(z)$ be a matrix holomorphicly invertible at ∞ and let all the principal minors of $U(\infty)$ be non-zero. Then for any integers $k_1 \leq k_2 \leq \cdots \leq k_m$ there exists a lower triangular matrix $\Gamma^{(\infty)}(z)$ with elements on the principal diagonal equal to* 1, *$\Gamma^{(\infty)}(z)$ polynomial in z, and a matrix $V^{(\infty)}(z)$ holomorphicly invertible in a neighborhood of ∞ such that*

$$\Gamma^{(\infty)}(z)z^K U(z) = V^{(\infty)}(z)z^K,$$

where $K = \mathrm{diag}\,(k_1, k_2, \dots, k_m)$.

We add that $V^{(\infty)}(z)$ and $\Gamma^{(i)}(z)$ depend only on the first $S := k_m - k_1$ terms of the series expansion of $U(z) = \sum_{s=1}^{\infty} U_s z^{-s}$ near ∞.

To apply the above lemma, we first observe that

$$\Gamma(z)z^{-Mg+C} = z^{-Mg+C}\tilde{\Gamma}(z),$$

where $\tilde{\Gamma}(z)$ is a lower triangular matrix with all diagonal elements equal to 1 and all off-diagonal elements equal to zero apart from the last row which is given by

$$\tilde{\Gamma}(z)_{ml} \sim z^{(n-1)(l-m)-g}, \qquad l = 1, \dots, m-1.$$

To apply Lemma 4.19 we need to introduce a permutation P such that $Pz^{-Mg+C}P^{-1} = z^K$ where, in the case $m \geq 3$,

$$K = \mathrm{diagonal}\left((m-1)(n-1) - g, 0, (n-1), 2(n-1), \dots, (m-2)(n-1)\right),$$

and in the case $m = 2$, $K = \mathrm{diagonal}(0, 1)$ and $P = \mathbb{1}$. Moreover, in the case $m > 2$, we need to show that $P\tilde{\Gamma}(\infty)G^{(\infty)}(\infty)P^{-1}$ has all principal minors different from zero. This is a straightforward consequence of the fact that $\tilde{\Gamma}(\infty) = \mathbb{1}$ and $\tilde{G} = PG^{(\infty)}(\infty)P^{-1}$ is given by

$$\begin{aligned}
\tilde{G}_{11} &= (-1)^{m-1}a_1\lambda_1^{(\infty)}(\lambda_1^{(\infty)}+1)\dots(\lambda_1^{(\infty)}+m-2),\\
\tilde{G}_{1k} &= (-1)^{m-1}a_k(\lambda_k^{(\infty)}+1)\dots(\lambda_k^{(\infty)}+m-1), \quad k \neq 1,\\
\tilde{G}_{l1} &= (-1)^{l-1}a_1\lambda_1^{(\infty)}(\lambda_1^{(\infty)}+1)\dots(\lambda_1^{(\infty)}+l-2), \quad l \neq 1, m,\\
\tilde{G}_{lk} &= (-1)^{l-1}a_k(\lambda_k^{(\infty)}+1)\dots(\lambda_k^{(\infty)}+l-1), \quad l \neq 1, m, \quad k \neq 1,\\
\tilde{G}_{mk} &= a_k, \quad k = 1, \dots, m.
\end{aligned}$$

We can then apply Lemma 4.19 to

$$P\hat{Y} = Pz^{-Mg+C}\tilde{\Gamma}(z)G^{(\infty)}(z)z^{-\Theta^{(\infty)}}z^{-R^{(\infty)}} = z^K P\tilde{\Gamma}(z)G^{(\infty)}(z)P^{-1}Pz^{-\Theta^{(\infty)}}z^{-R^{(\infty)}}.$$

We obtain a gauge transformation with the matrix $\Gamma^{(\infty)}(z)$ polynomial in z, such that the new fundamental matrix

$$\tilde{Y} = \Gamma^{(\infty)}(z)P\hat{Y} = \Gamma^{(\infty)}(z)z^K P\tilde{\Gamma}(z)G^{(\infty)}(z)P^{-1}Pz^{-\Theta^{(\infty)}}z^{-R^{(\infty)}},$$

factors as

$$\tilde{Y}(z) = V^{(\infty)}(z)z^K Pz^{-\Theta^{(\infty)}}z^{-R^{(\infty)}} = V^{(\infty)}(z)P^{-1}z^{-Mg+C}z^{-\Theta^{(\infty)}}z^{-R^{(\infty)}},$$

with the matrix $V^{(\infty)}(z)$ holomorphicly invertible in a neighborhood of ∞. The new exponents at ∞ are $\lambda_1^{(\infty)}$, $\hat{\lambda}_m^{(\infty)} = \lambda_m^{(\infty)} + 1 + mg - (m-1)(n+g-1)$, and, for

$j = 2, \dots, m-1$, $\hat{\lambda}_j^{(\infty)} = \lambda_j^{(\infty)} + 1 - (j-1)(n-1)$. Their sum is zero, therefore ∞ is a Fuchsian singularity (see [AB]).

So we have constructed a $m \times m$ Fuchsian system of the form

$$\frac{\mathrm{d}}{\mathrm{d}z}\tilde{Y} = \sum_{k=1}^{n} \frac{\tilde{A}_k}{z - u_k}\tilde{Y}$$

with exponents $\lambda_1^{(\infty)}, \hat{\lambda}_2^{(\infty)}, \dots, \hat{\lambda}_m^{(\infty)}$ at ∞ and $\lambda_1^{(k)}, \dots, \lambda_m^{(k)}$ at u_k, and no apparent singularities. We now want to map this system to a $m \times m$ Fuchsian system with exponents $\lambda_1^{(\infty)}, \dots, \lambda_m^{(\infty)}$ at ∞ and $\lambda_1^{(k)}, \dots, \lambda_m^{(k)}$ at u_k. We need the following:

Lemma 4.20. *Given a Fuchsian system of the form* (1.2)*, let* $\lambda_1^{(k)}, \dots \lambda_m^{(k)}$ *be the eigenvalues of the matrix* A_k *for* $k = 1, \dots, n, \infty$ *and let* $\mathcal{G}_k$ *be its diagonalizing matrix,*

$$\mathcal{G}_k^{-1} A_k \mathcal{G}_k = \operatorname{diag}\left(\lambda_1^{(k)}, \dots \lambda_m^{(k)}\right).$$

Assume that there are two eigenvalues, say $\lambda_1^{(k)}$ *and* $\lambda_m^{(k)}$ *such that* $\lambda_m^{(k)} \neq \lambda_1^{(k)} + 1$, $\lambda_m^{(k)} \neq \lambda_1^{(k)} + 2$, *and for all* $l \neq 1, m$, $\lambda_1^{(k)} \neq \lambda_l^{(k)} - 1$ *and* $\lambda_m^{(k)} \neq \lambda_l^{(k)} + 1$. *If not all entries in position* $m1$ *of the matrices* $\mathcal{G}_k^{-1} A_l \mathcal{G}_k$, $l = 1, \dots, n$, *are zero, then there exists a gauge transformation* $G_k(z; A_1 \dots, A_n, u_1, \dots, u_n)$*, rational in all arguments, such that the new matrices* $\tilde{A}_l$, $l = 1, \dots, n$, $l \neq k$ *have the same eigenvalues as the old ones* A_l *and the new matrix* $\tilde{A}_k$ *has eigenvalues* $\lambda_1^{(k)} + 1, \lambda_2^{(k)} \dots, \lambda_{m-1}^{(k)}, \lambda_m^{(k)} - 1$. *Moreover the gauge transformation* $G_k(z; A_1 \dots, A_n, u_1, \dots, u_n)$ *preserves the Schlesinger equations.*

Proof. We give here the gauge transformation $G_\infty(z)$ giving rise to the change $\lambda_1^{(\infty)} \to \lambda_1^{(\infty)} + 1$, $\lambda_m^{(\infty)} \to \lambda_m^{(\infty)} - 1$. So we assume that $\lambda_1^{(\infty)}$ and $\lambda_m^{(\infty)}$ are such that $\lambda_m^{(\infty)} \neq \lambda_1^{(\infty)} + 1$, $\lambda_m^{(\infty)} \neq \lambda_1^{(\infty)} + 2$, and for all $l \neq 1, m$, $\lambda_1^{(\infty)} \neq \lambda_l^{(\infty)} - 1$ and $\lambda_m^{(\infty)} \neq \lambda_l^{(\infty)} + 1$, and if not all entries in position $m1$ of the matrices A_l, $l = 1, \dots, n$, are zero.

Let us fix a fundamental matrix Φ normalized at infinity

$$\Phi_\infty = \left(\mathbb{1} + \frac{\Psi_1}{z} + \frac{\Psi_2}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right)\right) z^{-A_\infty} z^{-R^{(\infty)}},$$

where

$$\Lambda = A_\infty, \quad R^{(\infty)} = R_1 + R_2 + \dots,$$

$$
\begin{aligned}
(R_1)_{ij} &= \begin{cases} (B_1)_{ij}\,, & \lambda_i = \lambda_j + 1 \\ 0, & \text{otherwise} \end{cases}\,, \\
B_1 &= -\sum_k A_k u_k, \\
(\Psi_1)_{ij} &= \begin{cases} -\frac{(B_1)_{ij}}{\lambda_i - \lambda_j - 1}, & \lambda_i \neq \lambda_j + 1 \\ \\ \text{arbitrary}, & \text{otherwise} \end{cases}\,, \\
(R_2)_{ij} &= \begin{cases} (B_2 - \Psi_1 R_1 + B_1 \Psi_1)_{ij}\,, & \lambda_i = \lambda_j + 2, \\ 0, & \text{otherwise} \end{cases} \\
B_2 &= -\sum_k A_k u_k^2, \\
(\Psi_2)_{ij} &= \begin{cases} \frac{(-B_2 + \Psi_1 R_1 - B_1 \Psi_1)_{ij}}{\lambda_i - \lambda_j - 2}, & \lambda_i \neq \lambda_j + 2 \\ \\ \text{arbitrary}, & \text{otherwise} \end{cases}\,.
\end{aligned}
\tag{4.46}
$$

Consider the following gauge transformation $\Phi(z) = (I(z) + G)\tilde{\Phi}(z)$ where

$$
I(z) := \text{Diagonal}\,(z, 0, \ldots, 0)\,,
$$

and

$$
\begin{aligned}
& G_{m1} = \Psi_{1_{m1}}, \qquad G_{1m} = -\frac{1}{G_{m1}}, \\
& \text{if} \quad p \neq 1, m, \quad G_{pp} = 1, \quad G_{1p} = \Psi_{1_{mp}} G_{1m}, \qquad G_{p1} = \Psi_{1_{p1}}, \\
& \text{if} \quad p, q \neq 1,\ p \neq q, \quad G_{pq} = 0, \\
& G_{11} = G_{1m}\Psi_{2_{m1}} + \Psi_{1_{11}}, \quad \text{and} \qquad G_{mm} = 0.
\end{aligned}
\tag{4.47}
$$

In order to see that this gauge transformation is always well defined it is enough to observe that $\Psi_{1_{m1}}(u)$ is never identically equal to zero if at least one of the $(m, 1)$ matrix entries of the matrices $A_1(u), \ldots, A_n(u)$ is different from identical zero. Indeed, this follows from the linear equations

$$
\partial_i \Psi_1 = -A_i, \tag{4.48}
$$

$$
\partial_i \Psi_2 = -A_i \Psi_1 - u_i A_i, \tag{4.49}
$$

which are a straightforward consequence of the equation

$$
\partial_i \Phi_\infty(z; u) = -\frac{A_i}{z - u_i}\, \Phi_\infty(z; u), \quad i = 1, \ldots, n,
$$

describing the u-dependence of the fundamental matrix $\Phi_\infty(z; u)$.

Let us prove that this transformation maps the matrices $A_1, \ldots, A_n$ to new matrices $\tilde{A}_1, \ldots, \tilde{A}_n$ given by

$$
\tilde{A}_k := (I(u_k) + G)^{-1} A_k (I(u_k) + G),
$$

such that

$$
\tilde{A}_\infty = -\sum_{k=1}^n \tilde{A}_k = \text{diagonal}\left(\lambda_1^{(\infty)} + 1, \lambda_2^{(\infty)}, \ldots, \lambda_{m-1}^{(\infty)}, \lambda_m^{(\infty)} - 1\right). \tag{4.50}
$$

In fact $(I(z)+G)^{-1} = J(z)+G^{-1}$, where

$$J(z) := \text{Diagonal}\,(0,\dots,0,z)\,,$$

therefore

$$\tilde{A}_k := G^{-1}A_kI(u_k) + G^{-1}A_kG + J(u_k)A_kI(u_k) + J(u_k)A_kG.$$

Multiplying by G from the left and summing on all k we get that the condition (4.50) is satisfied if and only if

$$\begin{pmatrix} -g_{11} & \left(\lambda_1^{(\infty)}-\lambda_2^{(\infty)}\right)g_{12} & \dots & \dots & \left(\lambda_1^{(\infty)}-\lambda_{m-1}^{(\infty)}\right)g_{1\,m-1} & \left(\lambda_1^{(\infty)}-\lambda_m^{(\infty)}+1\right)g_{1m} \\ \left(\lambda_2^{(\infty)}-\lambda_1^{(\infty)}-1\right)g_{21} & 0 & \dots & \dots & \dots & 0 \\ \dots & 0 & \dots & \dots & \dots & 0 \\ \left(\lambda_m^{(\infty)}-\lambda_1^{(\infty)}-1\right)g_{m1} & 0 & \dots & \dots & \dots & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} \sum_k A_{k_{11}}u_k & 0\dots & 0 \\ \dots & 0\dots & 0 \\ \sum_k A_{k_{m1}}u_k & 0\dots & 0 \end{pmatrix} + \begin{pmatrix} g_{1m}\sum_k A_{k_{m1}}u_k^2 & 0\dots & 0 \\ 0 & 0\dots & 0 \\ \dots & 0\dots & 0 \end{pmatrix} +$$

$$+\begin{pmatrix} g_{1m}\sum_s\sum_k A_{k_{ms}}u_kg_{s1} & \dots & g_{1m}\sum_s\sum_k A_{k_{ms}}u_kg_{sm} \\ 0 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & 0 \end{pmatrix}. \tag{4.51}$$

Observe that in our assumptions on the eigenvalues $\lambda_1^{(\infty)}$ and $\lambda_m^{(\infty)}$, these formulae are clearly satisfied thanks to the fact that Ψ_1, Ψ_2 and $R^{(\infty)}$ are given by formulae (4.46).

Let us prove that this gauge transformation preserves the Schlesinger equations. Differentiating $\tilde{A}_k$ w.r.t. u_j, with $j\neq k$ and using the Schlesinger equations for $A_1,\dots,A_n$ we get:

$$\frac{\partial\tilde{A}_k}{\partial u_j} = \left[\tilde{A}_k,\,(I(u_k)+G)^{-1}\frac{\partial G}{\partial u_j} + \frac{(I(u_k)+G)^{-1}A_j(I(u_k)+G)}{u_k-u_j}\right] =$$

$$= \frac{\left[\tilde{A}_k,\tilde{A}_j\right]}{u_k-u_j}$$

$$+\left[\tilde{A}_k,\,(I(u_k)+G)^{-1}\left(\frac{\partial G}{\partial u_j} + \frac{A_j(I(u_k)-I(u_j)) - B_{kj}A_j(I(u_j)+G)}{u_k-u_j}\right)\right],$$

where

$$B_{kj} = \begin{pmatrix} 0 & \dots & 0 & \frac{u_k-u_j}{g_{m1}} \\ 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 \end{pmatrix}.$$

Given the formulae (4.47), it is straightforward to prove that the equation

$$\frac{\partial G}{\partial u_j} + \frac{A_j(I(u_k) - I(u_j)) - B_{kj}A_j(I(u_j) + G)}{u_k - u_j} = 0,$$

is equivalent to Eqs. (4.48), (4.49). This proves that also $\tilde{A}_1, \dots, \tilde{A}_n$ satisfy the Schlesinger equations.

Analogous formulae can be derived for the transformation $\lambda_1^{(k)} \to \lambda_1^{(k)} + 1$, $\lambda_m^{(k)} \to \lambda_m^{(k)} - 1$, for $k = 1, \dots, n$. In fact, suppose $u_1 = 0$ and $k \neq 1$. We can simply apply the conformal transformation $\tilde{z} = \frac{1}{u_k} - \frac{1}{z}$. The new residue matrices are $\tilde{A}_l = A_l$ for $l \neq 1, \infty$, $\tilde{A}_1 = -\sum_l A_l$, $\tilde{A}_\infty = A_k$. We then need to diagonalize $\tilde{A}_\infty$ and apply the above gauge transformation to the new system. □

We show that it is possible to make a finite sequence of gauge transformations described in Lemma 4.20 in such a way that the final Fuchsian system has exponents $\lambda_1^{(\infty)}, \dots, \lambda_m^{(\infty)}$ at infinity and $\lambda_k^{(\infty)}, \dots, \lambda_k^{(\infty)}$ at u_k, $k = 1, \dots n$.

By means of a permutation, we choose the following ordering of the parameters $\lambda_k^{(\infty)}$:

$$\Re\left(\lambda_m^{(\infty)}\right) \geq \Re\left(\lambda_1^{(\infty)}\right) \geq \Re\left(\lambda_2^{(\infty)}\right) \cdots \geq \Re\left(\lambda_{m-1}^{(\infty)}\right).$$

We start with $\hat{\lambda}_2^{(\infty)} \to \hat{\lambda}_2^{(\infty)} + 1$ and $\hat{\lambda}_m^{(\infty)} \to \hat{\lambda}_m^{(\infty)} - 1$. We want to apply such gauge $s = \lambda_2^{(\infty)} - \hat{\lambda}_2^{(\infty)}$ times. To do this we need to check that for all $p = 0, 1, \dots, s-1$ and for all $l = 1, \dots, m$ the following conditions are satisfied:

$$\hat{\lambda}_m^{(\infty)} - \hat{\lambda}_2^{(\infty)} \neq 2p+1, 2p+2, \tag{4.52}$$

$$\hat{\lambda}_l^{(\infty)} - \hat{\lambda}_2^{(\infty)} \neq p+1, \qquad \forall\, l \neq 2, m, \tag{4.53}$$

$$\hat{\lambda}_m^{(\infty)} - \hat{\lambda}_l^{(\infty)} \neq p+1, \qquad \forall\, l \neq 2, m. \tag{4.54}$$

To prove (4.52) we observe that since $\Re\left(\lambda_m^{(\infty)}\right) > \Re\left(\lambda_2^{(\infty)}\right)$, $\Re\left(\hat{\lambda}_m^{(\infty)} - \hat{\lambda}_2^{(\infty)}\right) > \max(2p+2) = 2s$. To prove (4.53) we observe that $\Re\left(\hat{\lambda}_l^{(\infty)} - \hat{\lambda}_2^{(\infty)}\right)$ is a negative number. To prove (4.54) we observe that $\Re\left(\hat{\lambda}_m^{(\infty)} - \hat{\lambda}_1^{(\infty)}\right) > s$. Therefore all conditions (4.52), (4.53), (4.54) are satisfied and thanks to the hypothesis that the monodromy group of the Fuchsian equation (4.45) is irreducible (which implies that at each step at least one residue matrix has $m1$ entry non-identically 0) there exists a gauge transformation $G_2(z)$ such that the new Fuchsian system has exponents $\hat{\lambda}_2^{(\infty)} + s$, $\hat{\lambda}_m^{(\infty)} - s$, and $\hat{\lambda}_j^{(\infty)}$ for all $j = 1, 3, \dots, m-1$.

At the j^{th} step of this procedure the parameters are $\lambda_1^{(\infty)}, \dots, \lambda_j^{(\infty)}, \hat{\lambda}_{j+1}^{(\infty)}, \dots, \hat{\lambda}_{m-1}^{(\infty)}$, $\tilde{\lambda}_m^{(\infty)} = \lambda_m^{(\infty)} - \frac{m-j-1}{2}[(m+j-2)(n-1)-2]$. We want to apply a gauge transformation $G_{j+1}(z)$ that maps $\hat{\lambda}_{j+1}^{(\infty)} \to \hat{\lambda}_{j+1}^{(\infty)} + 1$, $\tilde{\lambda}_m^{(\infty)} \to \tilde{\lambda}_m^{(\infty)} - 1$, a number $j(n-1) - 1 = \hat{\lambda}_{j+1}^{(\infty)} - \lambda_{j+1}^{(\infty)}$ of times. As above we need to verify that for all $p = 0, 1, \dots, j(n-1) - 2$

for all $l = 1, \dots, j, j+2, \dots, m-1$ the following conditions are satisfied:

$$\tilde{\lambda}_m^{(\infty)} - \hat{\lambda}_{j+1}^{(\infty)} \neq 2p+2, 2p+1, \tag{4.55}$$

$$\lambda_l^{(\infty)} - \hat{\lambda}_{j+1}^{(\infty)} \neq p+1, \tag{4.56}$$

$$\tilde{\lambda}_m^{(\infty)} - \lambda_l^{(\infty)} \neq p+1, \tag{4.57}$$

$$\tilde{\lambda}_m^{(\infty)} - \hat{\lambda}_s^{(\infty)} \neq p+1. \tag{4.58}$$

The proof that these conditions are fulfilled at each step is straightforward. Again we can use the hypothesis that the monodromy group of the Fuchsian equation (4.45) is irreducible to prove that at each step at least one residue matrix has mj entry non-identically 0.

Therefore we have obtained a gauge transformation $G_m(z)G_{m-1}(z)\cdots G_2(z)$ such that the new Fuchsian system has exponents $\lambda_1^{(\infty)}, \dots, \lambda_m^{(\infty)}$ at infinity and $\lambda_k^{(\infty)}, \dots, \lambda_k^{(\infty)}$ at u_k, $k = 1, \dots n$. The new fundamental matrix at infinity is

$$\tilde{\Phi}_\infty := \prod_{j=1}^{m-1} G_{j+1}(z) V^{(\infty)}(z) z^{-Mg+C} z^{\Theta^{(\infty)}} z^R.$$

In order to normalize it at infinity we need to perform one last gauge transform:

$$\Phi = \left(\prod_{j=1}^{m-1} G_{j+1}(\infty) V^{(\infty)}(\infty)\right)^{-1} \tilde{\Phi}.$$

The final new Fuchsian system

$$\frac{\mathrm{d}}{\mathrm{d}z}\Phi = \sum_{k=1}^{n} \frac{A_k}{z-u_k}\Phi$$

has exponents $\lambda_1^{(\infty)}, \dots, \lambda_m^{(\infty)}$ at ∞ and $\lambda_1^{(k)}, \dots, \lambda_m^{(k)}$ at u_k, and no apparent singularities. The matrix entries of the matrices A_k, $k = 1, \dots, n$, depend rationally on the coefficients of the polynomials f_l and on $q_1, \dots, q_g$. This concludes the proof of existence. Due to the ambiguity $\prod_{j=1}^{m-1} G_{j+1}(z) \to D \prod_{j=1}^{m-1} G_{j+1}(z)$, where D is any constant diagonal matrix with non-zero entries, we have that from the differential equation (4.45) we have constructed not one Fuchsian system, but a family of them, all related by diagonal conjugation.

We now prove the last statement of the theorem. Let us start from another Fuchsian system

$$\frac{\mathrm{d}}{\mathrm{d}z}\check{\Phi} = \sum_{k=1}^{n} \frac{\check{A}_k}{z-u_k}\check{\Phi} \tag{4.59}$$

with exponents $\lambda_1^{(\infty)}, \dots, \lambda_m^{(\infty)}$ at ∞ and $\lambda_1^{(k)}, \dots, \lambda_m^{(k)}$ at u_k, and no apparent singularities. Let us normalize its fundamental matrix $\check{\Psi}$ as usual,

$$\check{\Phi}^{(\infty)} := \left(\mathbb{1} + \mathcal{O}\left(\frac{1}{z}\right)\right) z^{-A^{(\infty)}} z^{-R^{(\infty)}}.$$

Let us apply the reduction procedure described in Theorem 4.14. This means that we construct a gauge transformation $G(z) = \mathcal{Q}^{-1}\mathcal{P}$,

$$\mathcal{Q}^{-1}\mathcal{P}\check{\Phi} = Y,$$

where Y is the Wronskian matrix of the differential equations (4.45). Now from such an equation we constructed a Fuchsian system

$$\frac{d}{dz}\Phi = \sum_{k=1}^{n} \frac{A_k}{z-u_k}\Phi$$

with the same exponents as (4.59). By the above construction, Φ is also related to Y by a gauge, $\Phi = \check{G}(z)Y$. Therefore $\Phi = \check{G}(z)G(z)\check{\Phi}$. This gauge transformation preserves the normalization at infinity by construction. All monodromy data are preserved and by the uniqueness Lemma 2.5 we conclude that $\check{G}(z)G(z)$ must be diagonal and constant in z.

In particular this proves that the first row of the fundamental matrix Φ satisfies the given m^{th} order Fuchsian equation. □

4.4. Darboux coordinates for Schlesinger system. According to Corollary 4.9, the parameters q_i, p_i, $i = 1, \dots, g$ are coordinates on a Zariski open subset in the space of all special Fuchsian equations with given indices and given $u_1, \dots, u_n$.

Due to Theorems 4.14 and 4.45, the parameters q_i, p_i, $i = 1, \dots, g$ can be used as coordinates on a Zariski open subset in the space of all Fuchsian systems with given $u_1, \dots, u_n$ and given exponents, considered modulo diagonal conjugations. Indeed, for fixed $u_1, \dots, u_n$, the condition (4.34) defines a Zariski open set in the space of all Fuchsian systems with given exponents.

In this section we will prove that these coordinates are canonically conjugated with respect to the isomonodromic symplectic structure ω_K (see (3.16)) on (3.14).

Remark 4.21. In order to apply our coordinates to the description of solutions to the Schlesinger equations, one has to make sure that the Zariski closed subset where the map

$$\left\{\begin{matrix}\text{Fuchsian systems with given poles}\\ \text{and given eigenvalues of } A_1, \dots, A_n, A_\infty\\ \text{modulo diagonal conjugations}\end{matrix}\right\} \to (q_1, \dots, q_g, p_1, \dots, p_g)$$

becomes singular, or equivalently condition (4.34) is violated, is never invariant under the monodromy preserving deformation.[4] This can be proved under the following two assumptions:

i) If A_∞ has a resonance of order one then the corresponding logarithmic correction R_1 is not zero (see (2.9)).
ii) For at least one j, the entries of the j^{th} row of the matrices $A_1, \dots, A_n$ satisfy the following condition:

$$\text{for every } i \neq j \text{ there exists } k \text{ such that } A_{k\,ji} \neq 0. \tag{4.60}$$

[4] In the theory of *iso-spectral* deformations an analogous problem arises. In this case one needs to check that the dynamics on the Jacobian of the spectral curve is never tangent to the Theta-divisor.

Under these assumptions, by performing a small monodromy preserving deformation, condition (4.34) is satisfied.

In fact suppose by contradiction that $\sum_{l=1}^{n} u_l A_{l_{ji}}(u) \equiv 0$ for some i, j. A simple differentiation using the Schlesinger equations gives

$$\frac{\partial}{\partial u_k} \sum_{l=1}^{n} u_l A_{l_{ji}} = -\left(1 + \lambda_j^{(\infty)} - \lambda_i^{(\infty)}\right) A_{k_{ji}} = 0. \tag{4.61}$$

Now if $1 + \lambda_j^{(\infty)} - \lambda_i^{(\infty)} = 0$ then A_∞ has a resonance of order one and since $R_{1_{ji}} = \sum_{l=1}^{n} u_l A_{l\,ji}$ must be zero, assumption i) is contradicted. Therefore $A_{k\,ji} = 0$, but this contradicts assumption ii).

Clearly the two assumptions are satisfied in a large Zariski open set in the space of solutions of the Schlesinger equations.

Let us rewrite Eq. (4.45) in the matrix form

$$\frac{\mathrm{d}}{\mathrm{d}z}\Psi = \boldsymbol{\mathcal{B}}(z)\Psi, \tag{4.62}$$

where

$$\Psi = \begin{pmatrix} y \\ y' \\ \dots \\ y^{(m-1)} \end{pmatrix}, \qquad \boldsymbol{\mathcal{B}}(z) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots \\ 0 & \dots & 0 & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ d_0 & d_1 & \dots & \dots & d_{m-1} \end{pmatrix}, \tag{4.63}$$

with $d_l(z) := \frac{f_l(z)}{\Delta(z) R(z)^{m-l}}$, $R(z) = \prod_{k=1}^{n}(z - u_k)$, $\Delta(z) = \prod_{i=1}^{g}(z - q_i)$ and $\deg f_l(z) = (n-1)(m-l) + g$. Recall that the system (4.63) is obtained from the original Fuchsian system by a gauge transformation.

Lemma 4.22. *If the apparent singularities q_i, $i = 1, \dots, g$ in Eq.* (4.45) *are distinct, for each $i = 1, \dots, g$, the matrix $\mathcal{B}(z)$ has one and only one eigenvalue $\rho_i(z)$ with a simple pole at q_i. For each $i = 1, \dots, g$, we define*

$$p_i = \begin{cases} \mathrm{Res}_{z=q_i} \frac{\rho_i(z)}{z - q_i}, & \textit{for} \quad q_i \neq u_k,\ \forall k = 1, \dots, n, \\ \mathrm{Res}_{z=q_i} \rho_i(z), & \textit{for} \quad q_i = u_k. \end{cases}$$

Then

$$\begin{aligned} p_i &= \frac{\tilde{f}_{m-2}(q_i)}{\tilde{f}_{m-1}(q_i)} \frac{\tilde{f}'_{m-1}(q_i)}{\Delta'(q_i)} - \tilde{f}_{m-1}(q_i) \frac{\Delta''(q_i)}{\Delta'(q_i)} \\ &= \mathrm{Res}_{z=q_i}\left(d_{m-2}(z) + \frac{1}{2} d_{m-1}(z)^2\right), \end{aligned} \tag{4.64}$$

where for $l = 0, \dots, m-1$,

$$\tilde{f}_l(z) = \begin{cases} \frac{f_l(z)}{R(z)^{m-l}}, & \text{for} \quad q_i \neq u_k,\ \forall k = 1, \dots, n, \\ \frac{f_l(z)}{R'(z)^{m-l}}, & \text{for} \quad q_i = u_k. \end{cases}$$

Observe that as an immediate consequence of the second part of Eq. (4.64), one obtains that the momenta p_i coincide with those defined in (4.11).

Proof. The characteristic equation of $\mathcal{B}$ is

$$\Delta(z)\rho(z)^m = \sum_{l=0}^{m-1} \frac{f_l(z)}{R(z)^{m-l}}\rho(z)^l.$$

Let us define $\tilde\rho(z) = R(z)\rho(z)$. Since the polynomials $f_l(z)$ are regular at $z = q_i$, there is only one eigenvalue $\tilde\rho_i(z)$ that has a pole at q_i. If $q_i \neq u_k$ for all $k = 1, \dots, n$, this pole is simple. Let us expand $\tilde\rho_i$ at $z = q_i$ and compare the left and right-hand sides of the characteristic equation. We obtain

$$\tilde\rho_i(z) = \frac{f_{m-1}(q_i)}{\Delta'(q_i)(z-q_i)} + \frac{f_{m-2}(q_i)}{f_{m-1}(q_i)} + \frac{f'_{m-1}(q_i)}{\Delta'(q_i)} - f_{m-1}(q_i)\frac{\Delta''(q_i)}{\Delta'(q_i)} + \mathcal{O}(z-q_i).$$

Therefore $\rho_i(z)\frac{\tilde f_{m-1}(q_i)}{\Delta'(q_i)(z-q_i)} + p_i + \mathcal{O}(z-q_i)$, where $\tilde f_l = \frac{f_l}{R(z)^{m-l}}$. This proves (4.64) for $q_i \neq u_k$ for all $k = 1, \dots, n$. Analogously if $q_i = u_k$ for one value of $k = 0, \dots, \infty$, then $\rho_i(z)$ has a double pole at q_i and $\rho_i(z)\frac{\tilde f_{m-1}(q_i)}{\Delta'(q_i)(z-q_i)^2} + \frac{p_i}{(z-q_i)} + \mathcal{O}(1)$ and again we obtain (4.64) as we wanted to prove. The second part of formula (4.64) is immediately obtained from the formula (4.13) for $a_1(z) = d_{m-1}(z)$. □

Definition 4.23. *We call the set* $(q_1, \dots, q_g, p_1, \dots, p_g)$ *the* **isomonodromic coordinates** *of the Schlesinger equations.*

Theorem 4.24. *On a generic reduced symplectic leaf* $\mathcal{O}_1 \times \dots \times \mathcal{O}_n/Diag$ *the quantities* $(q_1, \dots, q_g, p_1, \dots, p_g)$ *are canonical coordinates. The Schlesinger equations in these coordinates are written in the canonical form*

$$\frac{\partial q_i}{\partial u_k} = \frac{\partial \mathcal{H}_k}{\partial p_i},$$
$$\frac{\partial p_i}{\partial u_k} = -\frac{\partial \mathcal{H}_k}{\partial q_i},$$

where the Hamiltonians in canonical coordinates are given by the formula

$$\mathcal{H}_k = -\mathrm{Res}_{z=u_k}\left(d_{m-2}(z) + \frac{1}{2}d_{m-1}(z)^2\right), \tag{4.65}$$

where $d_{m-2}(z)$ *and* $d_{m-1}(z)$ *are defined in Theorem* 4.14.

Corollary 4.25. *The Hamiltonians* (4.65) *are given by*

$$\mathcal{H}_k = \left[\sum_{s=1}^{g}\frac{\delta_1^{(s)} - p_s}{u_k - q_s} + \sum_{i\neq k}\frac{\beta_2^{(i)}R'(u_i)}{u_k - u_i} - \frac{\beta_2^{(k)}R''(u_k)}{2} - \beta_2^{(\infty)}u_k^{n-2} - P_{n-3}(u_k)\right]$$
$$\times \frac{1}{R'(u_k)} - \left\{\sum_{s=1}^{g}\frac{1}{u_k - q_s} + \sum_{i\neq k}\frac{\sum_{j=1}^m \lambda_j^{(i)} - \frac{m\,(m-1)}{2}}{u_k - u_i}\right\}$$
$$\times \left[\sum_{j=1}^{m}\lambda_j^{(i)} - \frac{m\,(m-1)}{2}\right],$$

where the coefficients of the polynomial $P_{n-3}(z)$ *are rational functions of* $p_1, \dots, p_g$, $q_1, \dots, q_g$ *uniquely determined by* (4.23).

Example 4.26. In the 2×2 case the polynomial $\Delta(z)$ coincides with the $(1,2)$-matrix entry of $A(z) = \sum_k A_k/(z - u_k)$,

$$\Delta(z) = R(z) A_{12}(z).$$

So the isomonodromic coordinates q_i coincide with the spectral coordinates (see below). Our $p_1, \dots, p_g$ are slightly different from the usual momenta $\hat{p}_1, \dots, \hat{p}_g$ defined for Garnier systems (see [28]). In fact in our case we imposed the trace of all matrices A_k to be zero, while in [28], the determinant is zero. The relation between our coordinates and [28] is given by

$$p_i = -\hat{p}_i + \sum_{k=1}^{n} \frac{\lambda_1^{(k)}}{q_i - u_k} + \frac{q_i (2 \sum_{k=1}^{n} u_k - n q_i) + \sum_{1 \le k < l \le n} u_k u_l}{\prod_{k=1}^{n} (q_i - u_k)}.$$

Keeping track of this time-dependent canonical transformation, it is not difficult to verify that our Hamiltonian functions (4.65) coincide with the one given in [28].

Proof of the theorem. Since the system (4.63) is gauge equivalent to the original Fuchsian system (1.2), it suffices to perform all computations with (4.63). First of all formula (4.65) is obtained by straightforward computation applying formula (3.17) to the matrix (4.63). We want to show that q_i, p_i are canonical coordinates on the reduced symplectic leaf (3.14) and that in those coordinates the Hamiltonian is indeed (4.65). To this aim observe that we can always put equations of the form (4.45) in the matrix form (4.62), (4.63). The apparent singularities are poles of the eigenvectors of the matrix $\mathcal{B}(z)$.

In this way the proof reduces to proving the following

Lemma 4.27. *If* $q_i \neq u_1, \dots, u_n, \infty$ *for all* $i = 1, \dots, g$*, the symplectic structure* (3.16) *on the space of monodromy data of the linear system of ODEs of the form* (4.62)*,* (4.63) *is*

$$\omega_K = \sum_{i=1}^{g} \mathrm{d}p_i \wedge \mathrm{d}q_i.$$

Proof. Due to gauge invariance of the form ω_K we have to compute

$$\omega_K = -\frac{1}{2} \sum_{k=1}^{n} \mathrm{Res}_{z=u_k} \mathrm{Tr}(\delta \mathcal{B} \wedge \delta \Psi\, \Psi^{-1}) - \frac{1}{2} \sum_{i=1}^{g} \mathrm{Res}_{z=q_i} \mathrm{Tr}(\delta \mathcal{B} \wedge \delta \Psi\, \Psi^{-1}).$$

Observe that $\delta \mathcal{B}_{il}$ is zero for all $i \neq m$ and $\delta \mathcal{B}_{ml} = \delta d_{l-1}$ so that ω_K depends only on the m^{th} column of the matrix Ψ^{-1}, i.e.

$$\omega_K = -\frac{1}{2} \sum_{k=1}^{n} \mathrm{Res}_{z=u_k} \sum_{l,j=0}^{m} \delta d_{l-1} \wedge \delta \Psi_{lj} \left(\Psi^{-1}\right)_{jm} -$$
$$-\frac{1}{2} \sum_{i=1}^{g} \mathrm{Res}_{z=q_i} \sum_{l,j=0}^{m} \delta d_{l-1} \wedge \delta \Psi_{lj} \left(\Psi^{-1}\right)_{jm}.$$

Let us deal with the apparent singularities first. We have to compute the expansion of $\boldsymbol{\mathcal{B}}(z)$ and $\Psi(z)$ at q_i.

Choose m linear independent solutions $y_1, \ldots, y_m$ having expansions at $z = q_i$ of the form described in Lemma 4.5 where the constants α_l are determined by the differential equation (4.62) and are

$$\begin{aligned} \alpha_{l+1}^{(i)} &= -\mathrm{Res}_{z=q_i} d_l(z), \qquad l = 0, \ldots, m-2, \\ \alpha_m^{(i)} &= \mathrm{Res}_{z=q_i} \left(2\frac{d_{m-1}(z)}{z-q_i} + d_{m-2}(z) \right). \end{aligned} \tag{4.66}$$

Comparing these with (4.64) we have $p_i = \frac{\alpha_m^{(i)} - \alpha_{m-1}^{(i)}}{2} = \mathrm{Res}_{z=q_i}\left(\frac{d_{m-1}(z)}{z-q_i} + d_{m-2}(z)\right)$.

Observe that the fundamental matrix Ψ of the scalar equation (4.45) has matrix elements given by

$$\Psi_{lj} = \frac{\mathrm{d}^{l-1} y_j}{\mathrm{d}z^{l-1}}, \qquad l, j = 1, \ldots, m.$$

We can show that in the computation of the residue in (3.16) at q_i, one can neglect $\mathcal{O}(z-q_i)^m$ in $y_1, \ldots, y_m$ and in the coefficient d_{m-1} and $\mathcal{O}(1)$ in $d_0, \ldots, d_{m-2}$. This follows by straightforward computations based on a list of observations:

1. $\left(\Psi^{-1}\right)_{jm} = \mathcal{O}(z-q_i)^{m-j}$ for $j = 1, \ldots, m-1$ and $\left(\Psi^{-1}\right)_{mm} \mathcal{O}(z-q_i)^{-1}$.
2. From (4.11) and (4.66) we have that

$$\frac{\alpha_m^{(i)} + \alpha_{m-1}^{(i)}}{2} = \sum_{j \neq i} \frac{1}{q_i - q_j} + \sum_{k=1}^{n} \frac{\sum_{j=1}^m \lambda_j^{(k)} - \frac{m(m-1)}{2}}{q_i - u_k}.$$

3. For $l = 1, \ldots, m-1$,

$$\delta d_{l-1} = \frac{-\alpha_l^{(i)} \delta q_i}{(z-q_i)^2} - \frac{\delta \alpha_l^{(i)}}{(z-q_i)} + \mathcal{O}(1),$$

and

$$\delta d_{m-1} = \frac{\delta q_i}{(z-q_i)^2} + \frac{1}{2}\left(\delta\alpha_m^{(i)} + \delta\alpha_{m-1}^{(i)}\right) + \mathcal{O}(1)\delta q_i + \mathcal{O}(z-q_i).$$

4. For $1 \le j \le m-1$ and $l \le j-1$:

$$\begin{aligned} \delta\Psi_{lj} = &-\frac{\delta q_i}{(j-l-1)!}(z-q_i)^{j-l-1} + \frac{\delta\alpha_j^{(i)}}{(m-l)!}(z-q_i)^{m-l} - \\ &-\frac{\alpha_j^{(i)}\delta q_i}{(m-l-1)!}(z-q_i)^{m-l-1} + \delta q_i \mathcal{O}(z-q_i)^{m-l+1} + \mathcal{O}(z-q_i)^{m-l+2}, \end{aligned}$$

for $1 \le j \le m-1$ and $l \ge j$:

$$\begin{aligned} \delta\Psi_{lj} = &\frac{\delta\alpha_j^{(i)}}{(m-l)!}(z-q_i)^{m-l} - \frac{\alpha_j^{(i)}\delta q_i}{(m-l-1)!}(z-q_i)^{m-l-1} + \\ &+ \delta q_i \mathcal{O}(z-q_i)^{m-l+1} + \mathcal{O}(z-q_i)^{m-l+2}, \end{aligned}$$

and for $j = m$:

$$\delta\Psi_{lm} = -\frac{\delta q_i}{(m-l)!}(z-q_i)^{m-l} + \frac{\delta\alpha_m^{(i)}}{(m-l+2)!}(z-q_i)^{m-l+2} - \\ -\frac{\alpha_m^{(i)}\delta q_i}{(m-l-1)!}(z-q_i)^{m-l-1} + \delta q_i \mathcal{O}(z-q_i)^{m-l+2} + \mathcal{O}(z-q_i)^{m-l+3}.$$

From 1) and 3) we immediately see that only the terms with $j = m-1, m$ can contribute to the residue. From 1), 2), 3) and 4) we have

$$\sum_{i=1}^{g} \mathrm{Res}_{z=q_i} \sum_l \delta d_{l-1} \wedge \delta\Psi_{l,m-1} \left(\Psi^{-1}\right)_{m-1,m} = \\ = \sum_{i=1}^{g} \mathrm{Res}_{z=q_i} \delta d_{m-1} \wedge \delta\Psi_{m,m-1} \left(\Psi^{-1}\right)_{m-1,m} \frac{1}{2}\delta\alpha_{m-1}^{(i)} \wedge \delta q_i,$$

and

$$\sum_{i=1}^{g} \mathrm{Res}_{z=q_i} \sum_l \delta d_{l-1} \wedge \delta\Psi_{lm} \left(\Psi^{-1}\right)_{mm} = \\ = \sum_{i=1}^{g} \mathrm{Res}_{z=q_i} \left(\delta d_{m-1} \wedge \delta\Psi_{mm} \left(\Psi^{-1}\right)_{mm} + \delta d_{m-2} \wedge \delta\Psi_{m-1,m} \left(\Psi^{-1}\right)_{mm}\right) = \\ = -\delta\alpha_m^{(i)} \wedge \delta q_i + \frac{1}{2}\delta\alpha_{m-1}^{(i)} \wedge \delta q_i,$$

thus we obtain

$$-\frac{1}{2}\sum_{i=1}^{g} \mathrm{Res}_{z=q_i} \sum_{l,j} \delta d_{l-1} \wedge \delta\Psi_{lj} \left(\Psi^{-1}\right)_{jm} = -\frac{1}{2}\sum_{i=1}^{g} \delta\left(\alpha_{m-1}^{(i)} - \alpha_m^{(i)}\right) \wedge \delta q_i = \\ = \sum_{i=1}^{g} \delta p_i \wedge \delta q_i.$$

Let us now show that

$$\mathrm{Res}_{z=u_k} \sum_{l,j} \delta d_{l-1} \wedge \delta\Psi_{lj} \left(\Psi^{-1}\right)_{jm} = 0, \qquad \forall k = 1, \dots, n.$$

Let us expand d_{l-1} at u_k. We have $d_{l-1} = \frac{c_{l-1}}{(z-u_k)^{m-l+1}} + \mathcal{O}(z-u_k)^{l-m}$, where c_{l-1} are uniquely determined by the indicial equation, thus by the exponents. As a consequence $\delta d_{l-1} = \mathcal{O}(z-u_k)^{l-m}$. Analogously to estimate $\Psi_{lj} = \frac{\mathrm{d}^{l-1} y_j}{\mathrm{d}z^{l-1}}$, we can again normalize the solutions y_j at u_k in such a way that $y_j = (z-u_k)^{\lambda_j^{(k)}} + \mathcal{O}(z-u_k)^{\lambda_j^{(k)}+1}$ so that $\delta y_j = \mathcal{O}(z-u_k)^{\lambda_j^{(k)}-1}$. Thus $\delta d_{l-1} \wedge \delta\Psi_{lj} = \mathcal{O}(z-u_k)^{\lambda_j^{(k)}-m}$. Now $\left(\Psi^{-1}\right)_{jm} = \mathcal{O}(z-u_k)^{m-\lambda_j^{(k)}}$ so that near u_k we have $\sum_{l,j} \delta d_{l-1} \wedge \delta\Psi_{lj} \left(\Psi^{-1}\right)_{jm} = \mathcal{O}(1)$ and the pole u_k does not contribute to the residue. □

The above lemma proves that q_i, p_i are canonical coordinates on the reduced symplectic leaf (3.14). We now want to prove that in those coordinates the Hamiltonians are indeed given by formula (4.65). To this aim we need to extend the phase space.

Let us consider the space of all matrices $\mathcal{B}$ of the form (4.63) with coefficients $d_0, \ldots, d_{m-1}$,

$$
\begin{aligned}
d_{m-1}(z) &= \sum_{s=1}^{g} \frac{1}{z-q_s} + \sum_{i=1}^{n} \frac{1}{z-u_i} \left[\sum_{j=1}^{m} \lambda_j^{(i)} - \frac{m\,(m-1)}{2} \right] \\
d_k(z) &= \Bigg[-\sum_{s=1}^{g} \frac{c_{k+1}^{(s)} R(q_s)^{m-k-1}}{z-q_s} + (-1)^{m-k-1} \sum_{i=1}^{n} \frac{\beta_{m-k}^{(i)}}{z-u_i} [R'(u_i)]^{m-k-1} + \\
&\quad + \beta_{m-k}^{(\infty)} z^{(m-k)(n-1)-n} + P_{(m-k)(n-1)-n-1}(z) \Bigg] \frac{1}{R(z)^{m-k-1}}, \\
&\quad k = 0, \ldots, m-2.
\end{aligned}
$$

We recall that this means that Eq. (4.45) has $n+1$ Fuchsian poles at $u_1, \ldots, u_n, \infty$ with indices $\lambda_1^{(k)}, \ldots, \lambda_m^{(k)}$ for $k = 1, \ldots, n, \infty$ given by Eqs. (4.14) and (4.15) and it has simple poles at the points $q_1, \ldots, q_g$.

Near each simple pole q_s, the matrix $\mathcal{B}$ can be expanded as

$$
\mathcal{B} = \frac{\mathcal{B}_0^{(s)}}{z-q_s} + \mathcal{B}_1^{(s)} + \mathcal{O}(z-q_s),
$$

where

$$
\mathcal{B}_0^{(s)} = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \\ -c_1^{(s)} & \ldots & -c_{m-1}^{(s)} & 1 \end{pmatrix},
$$

$$
\mathcal{B}_1^{(s)} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots \\ 0 & \ldots & 0 & 1 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \delta_m^{(s)} & \delta_{m-1}^{(s)} & \ldots & \ldots & \delta_1^{(s)} \end{pmatrix},
$$

where $\delta_1^{(s)}, \ldots, \delta_m^{(s)}$ are given in Eqs. (4.11). If we do not impose Eqs. (4.17), that is if we do not assume the singularities $q_1, \ldots, q_g$ to be apparent, there exists a fundamental solution of the form

$$
\Psi = G^{(s)}(z)(z-q_s)^{\Lambda}(z-q_s)^{R^{(s)}}, \tag{4.67}
$$

where $G^{(s)}(z) = G_0^{(s)} + G_1^{(s)}(z-q_s) + \mathcal{O}(z-q_s)^2$, $\Lambda = \text{diagonal}(0, \ldots, 0, 1)$, $R^{(s)}$ is an off-diagonal matrix with all entries equal to zero, apart from the last row. The matrices $R^{(s)}$, $G_0^{(s)}$ and $G_1^{(s)}$ are determined by the following equations:

$$
\mathcal{B}_0^{(s)} G_0^{(s)} = G_0^{(s)} \Lambda, \qquad \mathcal{B}_0^{(s)} G_1^{(s)} + \mathcal{B}_1^{(s)} G_0^{(s)} = G_1^{(s)} + G_0^{(s)} R^{(s)} + G_1^{(s)} \Lambda.
$$

This fact is a simple consequence of the gauge formula applied to the gauge $\Psi = G^{(s)}(z)\tilde{\Psi}$ that maps $\mathcal{B}$ to $\frac{\Lambda}{z-q_s} + R^{(s)}$.

Observe that the $m-1$ equations (4.17) imposing that the simple pole q_s is apparent, coincide with the $m-1$ equations $R^{(s)} \equiv 0$.

We consider now the symplectic structure on the extended space of isomonodromic deformations of systems of the form (4.62), (4.63) where the entries of the matrices $R^{(s)}$, $s = 1, \ldots, g$ are not necessarily null.

The following lemma concludes the proof of Theorem 4.24, by proving that the Hamiltonians in the isomonodromic coordinates are indeed given by (4.65).

Lemma 4.28. *On the extended space $S \times X_n$, where S are the symplectic leaves and $X_n = \mathbb{C}^n \setminus \{diagonals\}$ is the configuration space of n points, the symplectic structure* (3.16) *becomes*

$$\omega_K = \sum_{i=1}^{g} \mathrm{d}p_i \wedge \mathrm{d}q_i - \sum_{k=1}^{n} \mathrm{d}\mathcal{H}_k \wedge \mathrm{d}u_k.$$

Proof. There are two main differences with the previous proof. The first one is that now we have to take into account the variations δu_k, $k = 1, \ldots, n$, the second one is that now the entries of the matrices $R^{(s)}$ are not necessarily zero. Let's first look at the term $\delta\Psi \wedge \Psi^{-1}$ near the point q_s. Using formula (4.67) we obtain

$$\delta\Psi\Psi^{-1} = \delta G^{(s)}(z)\left(G^{(s)}(z)\right)^{-1} - G^{(s)}(z)\frac{\Lambda}{z-q_s}\left(G^{(s)}(z)\right)^{-1}\delta q_s - \\ - G^{(s)}(z)\frac{R^{(s)}}{z-q_s}\left(G(z)^{(s)}\right)^{-1}\delta q_s$$

because all resonances are of order one and $(z-q_s)^{\Lambda}R^{(s)}(z-q_s)^{-\Lambda} = R^{(s)}$.

Therefore, when computing the residue at q_s of $\mathrm{Tr}(\delta\mathcal{B} \wedge \delta\Psi\,\Psi^{-1})$ we just need to add the contribution of the term $-\mathrm{Tr}\left(\delta\mathcal{B} \wedge G(z)^{(s)}\frac{R^{(s)}}{z-q_s}\left(G(z)^{(s)}\right)^{-1}\right)\delta q_s$ to $\mathrm{d}p_s \wedge \mathrm{d}q_s$. We are now going to prove that this extra contribution is zero.

In fact

$$G(z)\frac{R^{(s)}}{z-q_s}\left(G(z)^{(s)}\right)^{-1} = \frac{G_0^{(s)}R^{(s)}(G_0^{(s)})^{-1}}{z-q_s} - \left[G_0^{(s)}R^{(s)}(G_0^{(s)})^{-1}, G_1^{(s)}(G_0^{(s)})^{-1}\right] + \\ + \mathcal{O}(z-q_s).$$

Only the last column contributes to the trace. It is not difficult to see that the last column of $G_0^{(s)}R^{(s)}G_0^{(s)}$ is zero and that the only non–zero element of the last column of $\left[G_0^{(s)}R^{(s)}(G_0^{(s)})^{-1}, G_1^{(s)}(G_0^{(s)})^{-1}\right]$ is the last one. Since $\delta d_m = \frac{\delta q_s}{z-q_s} + \mathcal{O}(1)$, the residue is zero.

Let us now show that

$$\mathrm{Res}_{z=u_k}\sum_{l,j}\delta d_{l-1} \wedge \delta\Psi_{lj}\left(\Psi^{-1}\right)_{jm} = 2\delta\mathcal{H}_k \wedge \delta u_k, \qquad \forall k = 1, \ldots, n.$$

The contribution of the matrices $R^{(s)}$ does not play any role here because we are expanding at u_k. On the other side, this time we need to take into account the variations $\delta u_1, \ldots, \delta u_n$.

Let us expand d_{l-1} at u_k. We have

$$d_{m-1} = \frac{\beta_1^{(k)} - \frac{m(m-1)}{2}}{z-u_k} + D_{m-1}^{(k)} + \mathcal{O}(z-u_k),$$
$$d_{l-1} = \frac{(-1)^{m-l}\beta_{m-l+1}^{(k)}}{(z-u_k)^{m-l+1}} + \frac{D_{l-1}^{(k)}}{(z-u_k)^{m-l}} + \mathcal{O}(z-u_k)^{l+1-m}, \qquad (4.68)$$

where (cfr. (4.13) with $d_{l-1}(z) = a_{m-l+1}(z)$)

$$D_{m-1}^{(k)} = \sum_{s=1}^{g} \frac{1}{u_k - q_s} + \sum_{i\neq k} \frac{1}{u_k-u_i}\left[\sum_{j=1}^{m}\lambda_j^{(i)} - \frac{m\,(m-1)}{2}\right],$$
$$D_{l-1}^{(k)} = \left[(-1)^{m-l}\sum_{i\neq k}\frac{\beta_{m-l+1}^{(i)}}{u_k-u_i}[R'(u_i)]^{m-l} + \beta_{m-l+1}^{(\infty)}u_k^{(n-1)(m-l+1)-n}\right.$$
$$\left. + P_{(m-l+1)(n-1)-n-1}(u_k)\right]\frac{1}{R'(u_k)^{m-l}}, \qquad l=0,\ldots,m-2.$$

We need to introduce some notation:

$$[\lambda]_0 := 1, \qquad [\lambda]_1 := \lambda, \qquad [\lambda]_n := \lambda(\lambda-1)\ldots(\lambda-n+1),\ \forall n = 2,3,\ldots.$$

The indicial equations (4.14) read

$$[\lambda^{(k)}]_m = \left(\beta_1^{(k)} - \frac{m(m-1)}{2}\right)[\lambda^{(k)}]_{m-1} + \sum_{l=1}^{m-1}(-1)^{m-l}\beta_{m-l+1}^{(k)}[\lambda^{(k)}]_{l-1}. \quad (4.69)$$

To start with, we perform our computation in the case when the exponents $\lambda_1^{(k)}\ldots,\lambda_m^{(k)}$ of the pole u_k are non-resonant. Thanks to (4.3), there exists a basis of solutions $y_1,\ldots,y_m$ of the form

$$y_i(z) = (z-u_k)^{\lambda_i^{(k)}}(1+\eta_i^{(k)}(z-u_k) + \mathcal{O}(z-u_k)^2), \qquad i=1,\ldots,m.$$

Therefore we have

$$\Psi_{li}(z) = \frac{\mathrm{d}^{l-1}y_i}{\mathrm{d}z^{l-1}} = [\lambda_i^{(k)}]_{l-1}(z-u_k)^{\lambda_i^{(k)}-l} + [\lambda_i^{(k)}+1]_{l-1}\eta_i^{(k)}(z-u_k)^{\lambda_i^{(k)}-l+1} +$$
$$+\mathcal{O}(z-u_k)^{\lambda_i^{(k)}-l+2},$$

where the constants $\eta_1^{(k)},\ldots,\eta_m^{(k)}$ are determined by the following equations:

$$\left\{[\lambda_i^{(k)}+1]_m - \left(\beta_1^{(k)} - \frac{m(m-1)}{2}\right)[\lambda_i^{(k)}+1]_{m-1} - \right.$$
$$\left. - \sum_{l=1}^{m-1}(-1)^{m-l}\beta_{m-l+1}^{(k)}[\lambda_i^{(k)}+1]_{l-1}\right\}\eta_i^{(k)} = \sum_{l=1}^{m-1}D_{l-1}^{(k)}[\lambda_i^{(k)}]_{l-1}. \quad (4.70)$$

As above, only the last column of the inverse matrix Ψ^{-1} enters in the computation of the symplectic structure ω,

$$\left(\Psi^{-1}\right)_{im} = \frac{(z-u_k)^{m-1-\lambda_i^{(k)}}}{\prod_{j\neq i}(\lambda_i^{(k)}-\lambda_j^{(k)})} + \mathcal{O}(z-u_k)^{m-\lambda_i^{(k)}}.$$

Proceeding in a similar way as in the first part of this proof, we arrive at the formula

$$\begin{aligned}\mathrm{Res}_{z=u_k}\sum_{l,i}\delta d_{l-1}\wedge\delta\Psi_{li}\left(\Psi^{-1}\right)_{im} &= \sum_{i=1}^{m}\frac{-1}{\prod_{j\neq i}(\lambda_i^{(k)}-\lambda_j^{(k)})}\left\{\sum_{l=1}^{m-1}\left[\lambda_i^{(k)}\right]_l\,\delta D_{l-1}^{(k)}\wedge\delta u_k+\right.\\ &+\left(\beta_1^{(k)}-\frac{m(m-1)}{2}\right)\left[\lambda_i^{(k)}+1\right]_{m-1}\,\delta\eta_i^{(k)}\wedge\delta u_k+\\ &\left.+\left(\sum_{l=1}^{m-1}(-1)^{m-l}(m-l+1)\beta_{m-l+1}^{(k)}[\lambda^{(k)}+1]_{l-1}\right)\delta\eta_i^{(k)}\wedge\delta u_k\right\}. \qquad (4.71)\end{aligned}$$

Now using the indicial equation (4.69), we get

$$\begin{aligned}&\left(\beta_1^{(k)}-\frac{m(m-1)}{2}\right)[\lambda_i^{(k)}+1]_{m-1}+\sum_{l=1}^{m-1}(-1)^{m-l}(m-l+1)\beta_{m-l+1}^{(k)}[\lambda^{(k)}+1]_{l-1}=\\ &(\lambda_i^{(k)}-m+1)\left\{[\lambda^{(k)}+1]_m-\left(\beta_1^{(k)}-\frac{m(m-1)}{2}\right)[\lambda^{(k)}+1]_{m-1}-\right.\\ &\left.-\sum_{l=1}^{m-1}(-1)^{m-l}\beta_{m-l+1}^{(k)}[\lambda^{(k)}+1]_{l-1}\right\}. \qquad (4.72)\end{aligned}$$

Observe that the right-hand-side of Eq. (4.72) is $(\lambda_i^{(k)}-m+1)$ times the coefficient of $\eta_i^{(k)}$ in (4.70). Using this in Eq. (4.71), we get

$$\begin{aligned}\mathrm{Res}_{z=u_k}\sum_{l,i}\delta d_{l-1}\wedge\delta\Psi_{li}\left(\Psi^{-1}\right)_{im} =&\sum_{i=1}^{m}\frac{-1}{\prod_{j\neq i}(\lambda_i^{(k)}-\lambda_j^{(k)})}\left\{2[\lambda_i^{(k)}]_m\,\delta D_{m-1}^{(k)}\wedge\delta u_k+\right.\\ &\left.+\sum_{l=1}^{m-1}[\lambda_i^{(k)}]_{l-1}(2\lambda_i^{(k)}+2-m-l)\delta D_{l-1}^{(k)}\wedge\delta u_k\right\}. \qquad (4.73)\end{aligned}$$

To conclude we observe that

$$\begin{aligned}\sum_{i=1}^{m}[\lambda_i]_l\frac{2\lambda_i-m+2-l}{\prod_{j\neq i}(\lambda_i^{(k)}-\lambda_j^{(k)})} &= \sum_{i=1}^{n}\mathrm{res}_{\lambda=\lambda_i}[\lambda]_l\,\frac{2\lambda-m-l+2}{\prod_{j=1}^{n}(\lambda-\lambda_j)}\\ &= -\mathrm{res}_{\lambda=\infty}[\lambda]_l\,\frac{2\lambda-m-l+2}{\prod_{j=1}^{n}(\lambda-\lambda_j)}\\ &= \begin{cases}0, & \text{for } l=0,1,\ldots,m-3,\\ 2, & \text{for } l=m-2.\end{cases}\end{aligned}$$

Analogously

$$\sum_{i=1}^{m} \frac{[\lambda_i^{(k)}]_m}{\prod_{j\neq i}(\lambda_i^{(k)}-\lambda_j^{(k)})} = \beta_1^{(k)} - \frac{m(m-1)}{2}.$$

Using these in (4.73), we get finally

$$\mathrm{Res}_{z=u_k} \sum_{l,i} \delta d_{l-1} \wedge \delta\Psi_{li} \left(\Psi^{-1}\right)_{im} = -2\left(\beta_1^{(k)} - \frac{m(m-1)}{2}\right) \delta D_{m-1}^{(k)} \wedge \delta u_k - \\ - 2\,\delta D_{m-2}^{(k)} \wedge \delta u_k = 2\,\mathrm{d}\mathcal{H}_k \wedge \mathrm{d}u_k,$$

as we wanted to prove.

To conclude, we observe that if some of the exponents $\lambda_1^{(k)} \dots, \lambda_m^{(k)}$ of the pole u_k are resonant, then by Theorem 2.1 there exists a fundamental solution

$$\Psi = G^{(k)}(z)(z-u_k)^{\Lambda^{(k)}}(z-u_k)^{R^{(k)}},$$

where $G^{(k)}(z) = \sum_{j=0}^{\infty} G_j^{(k)}(z-u_k)$, and $R^{(k)} = \sum R_j^{(k)}$ is a finite sum of off-diagonal matrices such that

$$(z-u_k)^{\Lambda^{(k)}} R^{(k)} (z-u_k)^{-\Lambda^{(k)}} = R_0^{(k)} + R_1^{(k)}(z-u_k) + \cdots .$$

By applying enough iterates of Lemma 4.20 we can increase the order of the resonances arbitrarily, i.e. we can always assume that

$$(z-u_k)^{\Lambda^{(k)}} R^{(k)} (z-u_k)^{-\Lambda^{(k)}} = R_p^{(k)}(z-u_k)^p + R_{p+1}^{(k)}(z-u_k)^{p+1} + \cdots .,$$

with p large enough. Then the extra term in $\delta\Psi\Psi^{-1}$ due to $R^{(k)}$ is given by

$$-G^{(k)}(z)(R_p^{(k)}(z-u_k)^{p-1} + \mathcal{O}(z-u_k)^p)\left(G(z)^{(k)}\right)^{-1} \delta u_k$$

which does not contribute to the residue. □

4.5. An example. As we already know, for $m = 2$ the isomonodromic coordinates coincide with the spectral ones. Starting from $m = 3$ they are different.

In this subsection we give an explicit parametrization of special Fuchsian equations (4.26) in the first non-trivial case $m = 3$ and $n = 3$, in terms of our isomonodromic coordinates $q_1, \dots, q_g, p_1, \dots, p_g$ and compute the Hamiltonians of the Schlesinger equations in the canonical coordinates. Note that $g = 4$ for $m = 3$ and $n = 3$.

Starting from a Fuchsian system

$$\frac{dY}{dz} = A(z)Y, \quad A(z) = \frac{A_1}{z-u_1} + \frac{A_2}{z-u_2} + \frac{A_3}{z-u_3},$$

where A_i are 3×3 matrices with the eigenvalues $\lambda_1^{(i)}, \lambda_2^{(i)}, \lambda_3^{(i)}, i = 1, 2, 3$ satisfying

$$-(A_1 + A_2 + A_3) = A_\infty = \mathrm{diag}(\lambda_1^{(\infty)}, \lambda_2^{(\infty)}, \lambda_3^{(\infty)})$$

we arrive at a third order Fuchsian equation with eight regular singularities with the following Riemann scheme:

$$\mathcal{P}\left\{\begin{array}{cccccccc} \infty & u_1 & u_2 & u_3 & q_1 & q_2 & q_3 & q_4 \\ \lambda_1^{(\infty)} & \lambda_1^{(1)} & \lambda_1^{(2)} & \lambda_1^{(3)} & 0 & 0 & 0 & 0 \\ \lambda_2^{(\infty)}+1 & \lambda_2^{(1)} & \lambda_2^{(2)} & \lambda_2^{(3)} & 1 & 1 & 1 & 1 \\ \lambda_3^{(\infty)}+1 & \lambda_3^{(1)} & \lambda_3^{(2)} & \lambda_3^{(3)} & 3 & 3 & 3 & 3 \end{array}\right\}$$

satisfying the additional constraint of absence of logarithmic terms at the points $q_1, \ldots, q_4$. From the previous considerations it follows that the Fuchsian equation must have the form

$$\begin{aligned} y''' = &\left[\sum_{s=1}^{4} \frac{1}{z-q_s} - 3\sum_{i=1}^{3} \frac{1}{z-u_i}\right] y'' + \\ &+ \left[\sum_{s=1}^{4} \frac{c_2^{(s)} R(q_s)}{z-q_s} + \sum_{i=1}^{3} \frac{\beta_2^{(i)} R'(u_i)}{z-u_i} + \beta_2^{(\infty)} z + h\right] \frac{y'}{R(z)} + \\ &+ \left[-\sum_{s=1}^{4} \frac{c_1^{(s)} R^2(q_s)}{z-q_s} + \sum_{i=1}^{3} \frac{\beta_3^{(i)} R'^2(u_i)}{z-u_i} + \beta_3^{(\infty)} z^3 + a\, z^2 + b\, z + c\right] \frac{y}{R^2(z)}. \end{aligned} \tag{4.74}$$

Let us spell out the notations. The polynomial $R(z)$ is given by

$$R(z) = (z-u_1)(z-u_2)(z-u_3).$$

The coefficients $\beta_k^{(i)}$, $\beta_k^{(\infty)}$ are given by the following formulae:

$$\beta_2^{(i)} = \lambda_1^{(i)}\lambda_2^{(i)} + \lambda_1^{(i)}\lambda_3^{(i)} + \lambda_2^{(i)}\lambda_3^{(i)} - 5, \quad \beta_3^{(i)} = \lambda_1^{(i)}\lambda_2^{(i)}\lambda_3^{(i)}, \quad i = 1,\ 2,\ 3, \tag{4.75}$$

$$\beta_2^{(\infty)} = [\lambda_1^{(\infty)}]^2 + [\lambda_2^{(\infty)}]^2 + [\lambda_3^{(\infty)}]^2 - 2\lambda_1^{(\infty)} - 5, \quad \beta_3^{(\infty)} = -\lambda_1^{(\infty)}(\lambda_2^{(\infty)}+1)(\lambda_3^{(\infty)}+1).$$

In this example we assume all the matrices A_i to be traceless:

$$\mathrm{tr} A_i = 0, \quad i = 1,\ 2,\ 3. \tag{4.76}$$

We also put

$$c_2^{(s)} = -p_s - 3\,\frac{R'(q_s)}{R(q_s)} + \frac{1}{2}\frac{\Delta''(q_s)}{\Delta'(q_s)}$$

as in Eq. (4.11). We introduce the following quantities:

$$p_s^{[k]} = p_s + k\,\frac{R'(q_s)}{R(q_s)}, \quad k = 0,\ 1,\ 2,\ 3, \quad s = 1, \ldots, 4 \tag{4.77}$$

and

$$\tilde{p}_s^{[k]} = p_s^{[k]} - \frac{1}{2}\frac{\Delta''(q_s)}{\Delta'(q_s)} \tag{4.78}$$

where, as above the monic polynomial $\Delta(z)$ is defined by

$$\Delta(z) = (z - q_1) \dots (z - q_4) \equiv z^4 - \sigma_1 z^3 + \sigma_2 z^2 - \sigma_3 z + \sigma_4. \tag{4.79}$$

Here σ_1, ..., σ_4 are just the elementary symmetric functions of $q_1, \dots, q_4$. In these notations, $c_2^{(s)} = -\tilde{p}_s^{[3]}$.

The coefficients h, a, b, c and $c_1^{(s)} = \alpha_1^{(s)}$, $s = 1, \dots, 4$ are to be expressed in terms of the canonical coordinates q_1, ..., q_4, p_1, ..., p_4 and u_1, u_2, u_3 from the assumption of absence of logarithmic terms at the apparent singularities. This assumption yields a linear 8×8 system for the above unknowns. Eliminating the unknowns $c_1^{(s)}$ one arrives at the following system:

$$a\, q_s^2 + b\, q_s + c + h \cdot \sum_t M_{st}(p^{[2]}, q) R(q_t) = w_s, \quad s = 1, \dots, 4, \tag{4.80}$$

where the 4×4 matrix $M(p, q) = (M_{st}(p, q))$ is defined by

$$M_{st}(p, q) = \begin{cases} p_s, & t = s \\ \frac{1}{q_t - q_s}, & t \neq s \end{cases}$$

and

$$\begin{aligned} w_s = & \sum_{t,r} M_{st}(p^{[2]}, q) R(q_t) M_{tr}(p^{[1]}, q) R(q_r) \tilde{p}_r^{[0]} - \\ & - \sum_t M_{st}(p^{[2]}, q) R(q_t) f_2(q_t) - f_3(q_s), \quad s = 1, \dots, 4 \end{aligned} \tag{4.81}$$

and

$$f_2(z) = -\left[\sum_{i=1}^{3} \beta_2^{(i)} \frac{R'(u_i)}{z - u_i} + \beta_2^{(\infty)} z \right],$$

$$f_3(z) = \sum_{i=1}^{3} \beta_3^{(i)} \frac{R'^2(u_i)}{z - u_i} + \beta_3^{(\infty)} z^3.$$

Denote

$$D = D(p, q, u) = \sum_{s=1}^{4} \tilde{p}_s^{[2]} \frac{R(q_s)}{\Delta'(q_s)} \tag{4.82}$$

the determinant of the linear system (4.80). Then

$$\begin{aligned} a = -\sum_{s=1}^{4} \frac{w_s}{D} \Bigg[& \frac{1}{2} \sum_{j,k,l} \operatorname{sign}(s, j, k, l) p_j^{[2]} R(q_j) \frac{q_k - q_l}{W(q)} + \frac{R(q_s)}{\Delta'^2(q_s)} + \\ & + \frac{1}{\Delta'(q_s)} \sum_{j=1}^{4} (q_s - q_j)^2 \frac{R(q_j)}{\Delta'^2(q_j)} (\sigma_1 - q_s - 3 q_j) \Bigg], \end{aligned}$$

$$b = \sum_{s=1}^{4} \frac{w_s}{D} \left[\frac{1}{2} \sum_{j,k,l} \mathrm{sign}(s,\, j,\, k,\, l) p_j^{[2]} R(q_j) \frac{q_k^2 - q_l^2}{W(q)} + \frac{R(q_s)(\sigma_1 - 2q_s)}{\Delta'^2(q_s)} + \right.$$
$$\left. + \frac{1}{\Delta'(q_s)} \sum_{j=1}^{4} (q_s - q_j)^2 \frac{R(q_j)}{\Delta'(q_j)^2} (\sigma_1^2 - \sigma_2 - \sigma_1 q_s - 2\sigma_1 q_j + 2 q_s q_j) \right],$$

$$c = -\sum_{s=1}^{4} \frac{w_s}{D} \left[\frac{1}{2} \sum_{j,k,l} \mathrm{sign}(s,\, j,\, k,\, l) p_j^{[2]} R(q_j) q_k q_l \frac{q_k - q_l}{W(q)} + \right.$$
$$+ \frac{1}{\Delta'(q_s)} \sum_{j=1}^{4} (q_s - q_j)^2 \frac{R(q_j)}{\Delta'^2(q_j)} q_j \left(\sigma_1^2 - 4\sigma_2 - 3q_s^2 + 2\sigma_1 q_s + 3\frac{\sigma_4}{q_s q_j} \right) +$$
$$\left. + \frac{R(q_s)}{\Delta'^2(q_s)} (3q_s^2 - 2\sigma_1 + 2\sigma_2) \right],$$

$$h = \frac{1}{D} \sum_{s=1}^{4} \frac{w_s}{\Delta'(q_s)}, \tag{4.83}$$

where

$$W(q) = \prod_{i<j} (q_i - q_j), \tag{4.84}$$

$\mathrm{sign}(s,\, j,\, k,\, l)$ is the sign of the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ s & j & k & l \end{pmatrix}$.

Using (4.65) one obtains the following expression for the Hamiltonians of Schlesinger equations $S_{(3,3)}$:

$$\mathcal{H}_i = -\frac{1}{R'(u_i)} \left[\beta_2^{(\infty)} u_i + h - \sum_{s=1}^{4} \tilde{p}_s^{[3]} \frac{R(q_s)}{u_i - q_s} + \sum_{j \neq i} \frac{\beta_2^{(j)}}{u_i - u_j} R'(u_j) \right]$$
$$+ \frac{1}{2} \beta_2^{(i)} \frac{R''(u_i)}{R'(u_i)} + 3 \frac{\Delta'(u_i)}{\Delta(u_i)} - \frac{9}{2} \frac{R''(u_i)}{R'(u_i)}, \quad i = 1,\, 2,\, 3, \tag{4.85}$$

where the rational function $h = h(p, q, u)$ was defined in (4.83). Clearly of these three Hamiltonians only one is independent: the solutions depend only on the combination $(u_3 - u_1)/(u_2 - u_1)$.

5. Comparison of Spectral and Isomonodromic Coordinates

5.1. Spectral coordinates. We recall the construction of the algebro-geometric Darboux coordinates on the generic reduced symplectic leaves (3.14) following the scheme of [58, 1, 13, 20]. We call these algebro-geometric Darboux coordinates *spectral coordinates*.

The spectral coordinates are defined as follows. Let us assume that all the matrices A_i have pairwise distinct nonzero eigenvalues $\lambda_1^{(i)}, \ldots, \lambda_m^{(i)}$, and that the diagonal matrix

$$A_\infty := -(A_1 + \cdots + A_n) = \mathrm{diag}\,(\lambda_1^{(\infty)}, \ldots, \lambda_m^{(\infty)})$$

has distinct nonzero diagonal entries. Consider the characteristic polynomial of the matrix $A(z)$ of the form

$$A(z) = \sum_{i=1}^{n} \frac{A_i}{z - u_i} \tag{5.1}$$

with constant matrices $A_1, \ldots, A_n$ satisfying the following properties. Denote

$$\mathcal{R}(z, w) = \det(w - A(z)) = w^m + \alpha_1(z) w^{m-1} + \cdots + \alpha_m(z) \tag{5.2}$$

the characteristic polynomial of the matrix $A(z)$. Denote by

$$D(z) := \left[\prod_{i=1}^{n}(z - u_i)\right]^{m(m-1)} \prod_{i \neq j}(w_i(z) - w_j(z)) \tag{5.3}$$

the discriminant of the polynomial $R(z, w)$. In this formula $w_1(z),\ldots,w_m(z)$ are roots of the equation $R(z, w) = 0$. The resulting expression is a polynomial in the coefficients $\alpha_1(z)$, ..., $\alpha_m(z)$. Under the assumption that the matrix A_∞ has simple spectrum, the degree of the discriminant is equal to

$$N = m(m - 1)(n - 1). \tag{5.4}$$

Assumption 1. The N roots of the discriminant are simple and pairwise distinct. Also we require that

$$D(u_i) \neq 0, \quad i = 1, \ldots, n. \tag{5.5}$$

Due to this assumption the *spectral curve*

$$\mathcal{R}(z, w) = 0 \tag{5.6}$$

of the matrix $A(z)$ is smooth outside the lines $z = u_1, \ldots, z = u_n$, $z = \infty$. These lines intersect the spectral curve in singular points of multiplicity m.

Let us introduce the row vectors b_0, $b_1(z)$, $b_2(z)$, ...by

$$b_0 = (1, 0, \ldots, 0), \quad b_k(z) = b_0 A^{k-1}(z), \; k > 0. \tag{5.7}$$

Denote $B(z)$ the $m \times m$ matrix with the rows $b_0, b_1(z), \ldots, b_{m-1}(z)$,

$$B(z) = \begin{pmatrix} b_0 \\ b_1(z) \\ \cdot \\ \cdot \\ \cdot \\ b_{m-1}(z) \end{pmatrix}. \tag{5.8}$$

Put

$$\Delta_0(z) = \left[\prod_{i=1}^{n}(z - u_i)\right]^{\frac{m(m-1)}{2}} \det B(z). \tag{5.9}$$

Assumption 2. All the roots $\gamma_1, \ldots, \gamma_g$ of the polynomial $\Delta_0(z)$ are pairwise distinct and the degree of $\Delta_0(z)$ is equal to the maximal value

$$g = \deg \Delta_0(z) = \frac{1}{2} m\,(m\,n - m - n - 1) + 1. \tag{5.10}$$

We also assume that the roots $\gamma_1, \ldots, \gamma_g$ do not coincide with the poles $z = u_i$ of the Fuchsian system neither with the zeroes of the discriminant $D(z)$. Under this assumption there exists, for any $i = 1, \ldots, g$, a unique, up to normalization, eigenvector ψ^i of the matrix $B(\gamma_i)$ with zero eigenvalue. The first component of the eigenvector vanishes. It is also an eigenvector of the matrix $A(\gamma_i)$ with some eigenvalue μ_i,

$$B(\gamma_i)\psi^i = 0, \;\; A(\gamma_i)\psi^i = \mu_i \psi^i, \;\; \psi^i(\psi^i_1, \psi^i_2, \ldots, \psi^i_m)^T, \;\; \psi^i_1 = 0, \;\; i = 1, \ldots, g.$$

Observe that the genus of the Riemann surface (5.6) is equal to g. The spectral curve (5.6) together with the divisor

$$\mathcal{D} = \sum_{i=1}^{g} (\gamma_i, \mu_i) \tag{5.11}$$

determines the matrix $A(z)$ uniquely up to a conjugation by a constant diagonal matrix. Moreover, the matrices $A(z)$ satisfying the above assumptions form a Zariski open subset in the space of all matrices of the form (5.1). All these facts are rather standard for the theory of algebraically completely integrable systems. We give a sketch of proofs of these statements in the Appendix.

Definition 5.1. *We call the set* $(\gamma_1, \ldots, \gamma_g, \mu_1, \ldots, \mu_g)$ *the* **spectral coordinates** *on* (3.14).

Example 5.2. For the $m = 3$ case the polynomial $\Delta(z)$ determining the isomonodromic coordinates reads

$$\Delta(z) = R(z) \left[A_{12}A_{13}\,(A_{22} - A_{33}) - A_{12}^2 A_{23} + A_{13}^2 A_{32} - A_{12}A'_{13} + A'_{12}A_{13} \right],$$

where, as usual $R(z) = \prod_i (z - u_i)$ and $A_{ij} = A_{ij}(z)$ are the entries of the 3×3 matrix $A(z) = \sum_i A_i/(z - u_i)$. The positions of the spectral coordinates are determined by the polynomial

$$\Delta_0(z) = R(z) \left[A_{12}A_{13}\,(A_{22} - A_{33}) - A_{12}^2 A_{23} + A_{13}^2 A_{32} \right].$$

We see that, unlike the case $m = 2$ (see above Eg. 4.26) starting from $m = 3$ the spectral and isomonodromic coordinates do not coincide.

5.2. Spectral coordinates versus isomonodromic coordinates. In this subsection we prove that in a certain semi–classical limit the isomonodromic coordinates $q_1, \ldots, q_g$, $p_1, \ldots, p_g$ converge to the algebro–geometric Darboux coordinates.

Let us consider the following family of Fuchsian systems depending on a small parameter ϵ.

$$\epsilon \frac{d\Phi}{dz} = A(z)\,\Phi, \;\; \Phi = (\phi_1(z), \ldots, \phi_m(z))^T. \tag{5.12}$$

Theorem 5.3. *Under the assumptions of Theorem 4.14 the apparent singularities of the scalar reduction of the Fuchsian system admit the following expansion*

$$q_k = \gamma_k + O(\epsilon), \quad \epsilon \to 0, \quad k = 1, \dots, g. \tag{5.13}$$

Moreover, the scalar reduction can be written in the following form

$$\epsilon^m y^{(m)} + \epsilon^{m-1} a_1(z,\epsilon) y^{(m-1)} + \dots + a_m(z,\epsilon) y = 0, \tag{5.14}$$

where the functions $a_1(z,\epsilon), \dots, a_m(z,\epsilon)$ *are analytic in* (z,ϵ) *for* $z \neq u_i$, $z \neq \gamma_j$, $|\epsilon| << 1$ *and*

$$a_k(z,\epsilon) = \alpha_k(z) - \epsilon\, \beta_k(z) + O(\epsilon^2), \quad k = 1, \dots, m, \tag{5.15}$$

where

$$\beta_k(z) = \frac{1}{z-\gamma_j}\left[\mu_j^{k-1} + \alpha_1(\gamma_j)\mu_j^{k-2} + \dots + \alpha_{k-1}(\gamma_j)\right] + O(1), \quad z \to \gamma_j. \tag{5.16}$$

In particular, the parameters p_j *defined in* (4.64) *have the following expansion*

$$p_j = \epsilon^{-1}[\mu_j + \alpha_1(\gamma_j)] + O(1), \quad \epsilon \to 0, \quad j = 1, \dots, g. \tag{5.17}$$

Here γ_k, μ_k are the spectral coordinates of the matrix $A(z)$.

Lemma 5.4. *The following formula holds true for any* $k \geq 0$,

$$\epsilon^k y^{(k)} = [b_k(z) + \epsilon\, \tilde{b}_k(z) + O(\epsilon^2)]\, \Phi, \tag{5.18}$$

where the row vectors $b_k(z)$ *were defined in* (5.7) *and the row vectors* $\tilde{b}_k(z)$ *are defined by the following recursive procedure:*

$$\tilde{b}_0 = 0, \quad \tilde{b}_{k+1}(z) = \tilde{b}_k(z) A(z) + b_k'(z), \; k \geq 0. \tag{5.19}$$

Here and below it will be understood that the product of the row vector by the column vector is a scalar.

Proof. For $k = 0$ (5.18) is obvious. Since the first row of $A(z)$ is $b_1(z)$, the first equation of the Fuchsian system can be recast into the form

$$\epsilon\, y' = b_1(z)\, \Phi.$$

This proves (5.18) for $k = 1$. Let us now assume (5.18) for k and prove it for $k+1$. Differentiating both sides of (5.18) in z and multiplying by ϵ yields

$$\epsilon^{k+1} y^{(k+1)} = \epsilon[b_k'(z) + \epsilon \tilde{b}_k'(z) + O(\epsilon^2)]\, \Phi(z) + [b_k(z) + \epsilon\, \tilde{b}_k(z) + O(\epsilon^2)] \\ A(z)\Phi(z) = [b_{k+1}(z) + \epsilon\, (\tilde{b}_k(z) A(z) + b_k'(z)) + O(\epsilon^2)]\, \Phi(z).$$

The proof of the lemma is completed by induction. □

Corollary 5.5. *Define an $m \times m$ matrix valued function $\tilde{B}(z)$ with the rows $\tilde{b}_0(z)$, $\tilde{b}_1(z)$, ..., $\tilde{b}_{m-1}(z)$. Then the scalar reduction of the Fuchsian system reads*

$$\epsilon^m y^{(m)} = [b_m(z) + \epsilon \tilde{b}_m(z) + O(\epsilon^2)] \left[B(z) + \epsilon \tilde{B}(z) + O(\epsilon^2)\right]^{-1} \hat{y}, \tag{5.20}$$

where

$$\hat{y} := (y, \epsilon\, y', \ldots, \epsilon^{m-1} y^{(m-1)})^T. \tag{5.21}$$

Proof. From (5.18) we obtain

$$\begin{pmatrix} y \\ \epsilon\, y' \\ \ldots \\ \epsilon^{m-1} y^{(m-1)} \end{pmatrix} = (B + \epsilon \tilde{B} + O(\epsilon^2))\, \Phi.$$

This proves (5.20). □

From the corollary the claim of the theorem about expansions (5.13) of the apparent singularities, readily follows. Analyticity of the coefficients

$$\begin{aligned} &(a_m(z,\epsilon), a_{m-1}(z,\epsilon), \ldots, a_1(z,\epsilon)) = \\ &= [b_m(z) + \epsilon \tilde{b}_m(z) + O(\epsilon^2)] \left[B(z) + \epsilon \tilde{B}(z) + O(\epsilon^2)\right]^{-1} = \\ &= b_m(z) B^{-1}(z) + \epsilon \left[\tilde{b}_m(z) B^{-1}(z) - b_m(z) B^{-1}(z) \tilde{B}(z)\, B^{-1}(z)\right] + O(\epsilon^2) \end{aligned} \tag{5.22}$$

also follows for small ϵ and for z away from the poles $z = u_i$ and $z = \gamma_j$.

Let us now simplify the r.h.s. of the formula (5.22).

Lemma 5.6. *The leading term in the r.h.s. of (5.22) reads*

$$b_m(z)\, B^{-1}(z) = -(\alpha_m(z), \alpha_{m-1}(z), \ldots, \alpha_1(z)). \tag{5.23}$$

Proof. Using the Cayley - Hamilton theorem we obtain

$$\begin{aligned} b_m(z)\, B^{-1}(z) &= b_0 A^m(z) B^{-1}(z) \\ &= -b_0 \left(\alpha_m(z) + \alpha_{m-1}(z) A(z) + \cdots + \alpha_1(z) A^{m-1}(z)\right) B^{-1}(z) \\ &= -\left[\alpha_m(z) b_0 + \alpha_{m-1}(z) b_1(z) + \cdots + \alpha_1(z) b_{m-1}(z)\right] B^{-1}(z) \\ &= -(\alpha_m(z), \alpha_{m-1}(z), \ldots, \alpha_1(z)). \end{aligned}$$

In the last line we use the definition of the inverse matrix $B^{-1}(z)$. The lemma is proved. □

We will now simplify the linear in ϵ term. We need the following simple

Lemma 5.7. *Let us introduce matrix $T(z)$ by*

$$T(z) = \begin{pmatrix} 0 & 1 & 0 \ldots & 0 \\ 0 & 0 & 1 \ldots & 0 \\ & \ldots & \ldots & \\ 0 & 0 & \ldots & 1 \\ -\alpha_m(z) & -\alpha_{m-1}(z) & \ldots & -\alpha_1(z) \end{pmatrix}. \tag{5.24}$$

Then

$$T(z) B(z) = B(z) A(z). \tag{5.25}$$

Proof. Use the definition of the matrix $B(z)$ and the Cayley - Hamilton theorem. □

Lemma 5.8. *The following identity holds true*

$$\tilde{b}_m(z)B^{-1}(z) - b_m(z)B^{-1}(z)\tilde{B}(z)B^{-1}(z)$$
$$= \sum_{k=1}^{m} e_k B'(z)B^{-1}(z)\left[T^{m-k}(z) + \alpha_1(z)T^{m-k-1}(z) + \cdots + \alpha_{m-k}(z)\right]. \quad (5.26)$$

Here $e_1, \dots, e_m$ are row vectors of the standard basis in $\mathbb{C}^{m*}$, $(e_k)_i = \delta_{i\,k}$.

Proof. Using arguments of Lemma 5.6 we replace $b_m(z)B^{-1}(z)$ by $-(\alpha_m(z), \dots, \alpha_1(z))$. Next, using induction we derive the following formula for the row vectors (5.19):

$$\tilde{b}_k(z) = e_2 B'(z)A^{k-2}(z) + e_3 B'(z)A^{k-3}(z) + \cdots + e_k B'(z), \quad k \geq 2.$$

Using identity (5.25) the last formula can be recast into the form

$$\tilde{b}_k(z) = \Big[e_2 B'(z)B^{-1}(z)T^{k-2}(z) + e_3 B'(z)B^{-1}(z)T^{k-3}(z)$$
$$+ \cdots + e_k B'(z)B^{-1}(z)\Big] B(z), \quad k \geq 2.$$

This implies formula (5.26) with the summation in the r.h.s. starting from $k = 2$. Since the first row of the matrix $B'(z)$ identically vanishes, adding the term with $k = 1$ does not change the sum. □

We now want to compute the Laurent expansion of the coefficients of the scalar reduction (5.14) at the points $z = \gamma_j = q_j + O(\epsilon)$. Let γ be one of the zeroes of $\Delta_0(z)$, μ the eigenvalue of the matrix $A(\gamma)$ such that the corresponding eigenvector $\psi = (\psi_1, \psi_2, \dots, \psi_m)^T$,

$$A(\gamma)\psi = \mu\,\psi$$

satisfies

$$\psi_1 = 0.$$

According to our assumptions the eigenvector is defined uniquely up to a scalar factor. As we know, ψ is also an eigenvector of the matrix $B(\gamma)$ with zero eigenvalue,

$$B(\gamma)\psi = 0.$$

Moreover, there exists an analytic function $\lambda(z)$ defined for $|z - \gamma| << 1$ being an eigenvalue of $B(z)$ s.t. $\lambda(z)$ has a simple zero at $z = \gamma$, and $\lambda(z)$ does not coincide with other eigenvalues of $B(z)$. Denote $\psi(z) = (\psi_1(z), \dots, \psi_m(z))^T$ the analytic vector valued function s.t.

$$B(z)\psi(z) = \lambda(z)\psi(z), \quad |z - \gamma| << 1,$$
$$\lambda(\gamma) = 0, \quad \psi(\gamma) = \psi.$$

We also introduce a left eigenvector $\psi^*(z) = (\psi_1^*(z), \dots, \psi_m^*(z))$,

$$\psi^*(z)B(z) = \lambda(z)\psi^*(z), \quad |z - \gamma| << 1. \quad (5.27)$$

Denote

$$\psi^* := \psi^*(\gamma).$$

Lemma 5.9. *ψ^* is a left eigenvector of $T(\gamma)$ with the eigenvalue μ,*

$$\psi^* T(\gamma) = \mu \psi^*. \tag{5.28}$$

In particular, it can be chosen in the form

$$\psi_k^* = \mu^{m-k} + \alpha_1(\gamma)\mu^{m-k-1} + \cdots + \alpha_{m-k}(\gamma), \quad k = 1, \ldots, m. \tag{5.29}$$

Proof. From (5.27) for $z = \gamma$ it follows

$$\psi^* B(\gamma) = 0. \tag{5.30}$$

Let us prove that $\psi^* T(\gamma)$ is again a left eigenvector of $B(\gamma)$ with zero eigenvalue. Indeed, using (5.25) we obtain

$$\psi^* T(\gamma) B(\gamma) = \psi^* B(\gamma) A(\gamma) = 0.$$

The eigenvector of $T(\gamma)$ with the eigenvalue μ can be written in the form (5.29). Let us prove that this eigenvector satisfies (5.30).

According to our assumptions all the eigenvalues of the matrix $A(\gamma)$ are pairwise distinct. Of course, they coincide with the eigenvalues of the matrix $T(\gamma)$. Therefore it suffices to prove that

$$\psi^* B(\gamma) \psi' = 0$$

for an arbitrary eigenvector ψ' of the matrix $A(\gamma)$,

$$A(\gamma)\psi' = \mu' \psi'.$$

Indeed, from (5.29) we obtain

$$\psi^* B(\gamma) \psi' = b_0 \psi' \, \hat{R}(\mu, \mu', \gamma),$$

where

$$\hat{R}(\mu, \mu', \gamma) = \frac{\mathcal{R}(z, w) - \mathcal{R}(z, w')}{w - w'}|_{z=\gamma,\ w=\mu,\ w'=\mu'}$$

for $\mu' \neq \mu$ and

$$\hat{\mathcal{R}}(\mu, \mu, \gamma) = \frac{\partial \mathcal{R}(z, w)}{\partial w}|_{z=\gamma,\ w=\mu}.$$

It is clear that $\hat{R}(\mu, \mu', \gamma) = 0$ for $\mu' \neq \mu$. So $\psi^* B(\gamma)\psi' = 0$. For $\mu' = \mu$ we have $\hat{R}(\mu, \mu, \gamma) \neq 0$ (since γ is not a zero of the discriminant $D(z)$) but $b_0 \, \psi = 0$ since $\psi_1 = 0$. The lemma is proved. □

We will now compute the leading term of the Laurent expansion of the logarithmic derivative $B'(z) B^{-1}(z)$ at $z \to \gamma$.

Lemma 5.10. *For $z \to \gamma$,*

$$B'(z) B^{-1}(z) = \frac{1}{z - q} \frac{B'(\gamma)\psi \otimes \psi^*}{\psi^* B'(\gamma)\psi} + O(1). \tag{5.31}$$

Proof. Let

$$\Pi_1(z) = \frac{\psi(z)\otimes\psi^*(z)}{\psi^*(z)\psi(z)} \tag{5.32}$$

be the projector of $\mathbb{C}^m$ onto the direction of the eigenvector $\psi(z)$ parallel to the $(m-1)$-dimensional subspace spanned by other eigenvectors. Denote $\Pi_2(z) = \mathrm{id} - \Pi_1(z)$ the complementary projector and put

$$B_2(z) := B(z)\Pi_2(z).$$

We have

$$B(z) = \lambda(z)\Pi_1(z) + B_2(z).$$

All these matrix valued functions are analytic for z sufficiently close to γ and since $B(z)$ has a unique zero eigenvalue,

$$\mathrm{rank}\,(B_2(\gamma)) = m-1,$$

and the image of $B_2(z)$ is transverse to that of Π_1. So

$$B'(z)B^{-1}(z) = (\log\lambda(z))'\Pi_1(z) + \lambda^{-1}(z)B_2'(z)\Pi_1(z) + \text{regular terms}.$$

Since $B_2(z)\Pi_1(z) \equiv 0$, we obtain

$$B_2'(z)\Pi_1(z) = -B_2(z)\Pi_1'(z) = -\frac{B_2(z)\psi'(z)\otimes\psi^*(z)}{\psi^*(z)\psi(z)} + \cdots$$
$$= -\frac{B(z)\psi'(z)\otimes\psi^*(z)}{\psi^*(z)\psi(z)} + \cdots .$$

In the last equation dots denote terms analytic at $z=\gamma$. We obtain

$$B'(z)B^{-1}(z) = \frac{1}{z-\gamma}\left[\frac{\psi\otimes\psi^*}{\psi^*\psi} - \frac{1}{\lambda'(\gamma)}\frac{B(\gamma)\psi'(\gamma)\otimes\psi^*}{\psi^*\psi}\right] + O(1).$$

Using the well-known formula of the "perturbation theory"

$$\lambda'(\gamma) = \frac{\psi^*B'(\gamma)\psi}{\psi^*\psi}$$

(observe that the formula implies $\psi^*B'(\gamma)\psi \neq 0$) and the identity

$$B(\gamma)\psi'(\gamma) + B'(\gamma)\psi = \lambda'(\gamma)\psi,$$

we complete the proof of the lemma. □

End of the proof of the Theorem 5.3. We are to compute the sum

$$\sum_{k=1}^m e_k B'(z)B^{-1}(z)\left[T^{m-k}(z) + \alpha_1(z)T^{m-k-1}(z) + \cdots + \alpha_{m-k}(z)\right]$$
$$= \frac{1}{z-\gamma}\sum_{k=1}^m e_k\frac{B'(\gamma)\psi\otimes\psi^*}{\psi^*B'(\gamma)\psi}\left[T^{m-k}(\gamma) + \alpha_1(\gamma)T^{m-k-1}(\gamma) + \cdots + \alpha_{m-k}(\gamma)\right] + O(1)$$
$$= \frac{1}{z-\gamma}\sum_{k=1}^m e_k\frac{B'(\gamma)\psi\otimes\psi^*}{\psi^*B'(\gamma)\psi}\left[\mu^{m-k} + \alpha_1(\gamma)\mu^{m-k-1} + \cdots + \alpha_{m-k}(\gamma)\right] + O(1)$$
$$= \frac{1}{z-\gamma}\sum_{k=1}^m \frac{\psi_k^*\left(B'(\gamma)\psi\right)_k}{\psi^*B'(\gamma)\psi}\,\psi^* + O(1) = \frac{1}{z-\gamma}\psi^* + O(1),$$

where the left eigenvector ψ^* is chosen in the form (5.29). The theorem is proved. □

5.3. Canonical transformations. Let us consider the isomonodromic coordinates $(q_1, \dots, q_g, p_1, \dots, p_g)$ obtained from the scalar reduction w.r.t. the first row of (1.2).

Proposition 5.11. *For every $k = 2, \dots, n$, the following transformation*

$$S_k : \begin{cases} \tilde{q}_i = u_1 + u_k - q_i, & i = 1, \dots, g, \\ \tilde{p}_i = -p_i, & i = 1, \dots, g, \\ \tilde{u}_l = u_1 + u_k - u_l, & l = 1, \dots, n, \\ \tilde{\lambda}_j^{(k)} = \lambda_j^{(1)}, & j = 1, \dots, m, \\ \tilde{\lambda}_j^{(1)} = \lambda_j^{(k)}, & j = 1, \dots, m, \\ \tilde{\mathcal{H}}_l = -\mathcal{H}_l, & l = 1, \dots, n, \end{cases} \tag{5.33}$$

is a birational canonical transformation of the Schlesinger systems. This transformation acts on the monodromy matrices as follows:

$$\begin{aligned} \tilde{M}_1 &= M_1^{-1} \dots M_{k-1}^{-1} M_k M_{k-1} \dots M_1, \\ \tilde{M}_j &= M_j, \quad j \neq 1, k, \\ \tilde{M}_k &= M_{k-1} \dots M_2 M_1 M_2^{-1} \dots M_{k-1}^{-1}, \quad i = k+1, \dots, n. \end{aligned} \tag{5.34}$$

Proof. The transformation (5.33) is obviously birational. To show that (5.33) is a canonical transformation of the Schlesinger systems, we just observe that it is obtained by the conformal transformation $\zeta = u_1 + u_k - z$ of the scalar reduction (4.45). In fact (4.45) is transformed to

$$\frac{\mathrm{d}^m y}{\mathrm{d}\zeta^m} = \sum_{l=0}^{m-1} \tilde{d}_l(\zeta) \frac{\mathrm{d}^l y}{\mathrm{d}\zeta^l},$$

where $\tilde{d}_l(\zeta) = (-1)^{l-m} d_l(u_1 + u_k - \zeta) \frac{\tilde{f}_l(\zeta)}{\tilde{\Delta}(\zeta)\tilde{R}(\zeta)^{m-l}}$ with $\tilde{R}(\zeta) = \prod_{k=1}^n (\zeta - \tilde{u}_k)$, $\Delta(\zeta) = \prod_{i=1}^g (\zeta - \tilde{q}_i)$ and $\tilde{f}_l(\zeta) = (-1)^{(m-l)(n-1)+g} f_l(u_1 + u_k - \zeta)$. To obtain $\tilde{p}_i$ we use formula (4.64):

$$\begin{aligned} \tilde{p}_i &= \mathrm{Res}_{\zeta=\tilde{q}_i} \left(\tilde{d}_{m-2}(\zeta) + \frac{\tilde{d}_{m-1}(\zeta)}{\zeta - \tilde{q}_i} \right) = \\ &= \mathrm{Res}_{\zeta=\tilde{q}_i} \left(d_{m-2}(u_1 + u_k - \zeta) - \frac{d_{m-1}(u_1 + u_k - \zeta)}{\zeta - \tilde{q}_i} \right) = -p_i. \end{aligned}$$

To obtain the formulae for the exponents, just observe that the conformal transformation $\zeta = u_1 + u_k - z$ permutes u_1 with u_k.

Let us now prove the formula (5.34). The involution $i_k : u_1 \leftrightarrow u_k$ changes the basis in the fundamental group $\pi_1(\overline{\mathbb{C}} \backslash \{u_1, \dots, u_n, \infty\})$. In fact, as explained in Sect. 2.2, the cuts $\pi_1, \dots, \pi_n$ along which we take our basis $l_1, \dots, l_n$, are ordered according to the order of the poles. Applying the transformation i_k we then arrive at the new basis of loops

$$\begin{aligned} l_1' &= l_1 l_2 \cdots l_{k-1} l_k l_{k-1}^{-1} \cdots l_2^{-1} l_1^{-1}, \\ l_j' &= l_j, \quad j \neq 1, k, \\ l_k' &= l_{k-1}^{-1} \cdots l_2^{-1} l_1 l_2 \cdots l_{k-1}. \end{aligned}$$

from these formulae we immediately obtain (5.34). □

Proposition 5.12. *The following transformation:*

$$S_\infty : \begin{cases} \tilde q_i = \frac{1}{q_i - u_1}, \quad i = 1, \dots, g, \\ \tilde p_i = -p_i q_i^2 - \frac{2m^2-1}{m} q_i, \quad i = 1, \dots, g, \\ \tilde u_l = \frac{1}{u_l - u_1}, \quad l = 2, \dots, n, \\ u_1 \mapsto \infty, \\ \infty \mapsto u_1, \\ \tilde\lambda_1^{(\infty)} = \lambda_1^{(1)} + \frac{m-1}{m}, \\ \tilde\lambda_j^{(\infty)} = \lambda_j^{(1)} - \frac{1}{m}, \quad j = 2, \dots, m, \\ \tilde\lambda_1^{(1)} = \lambda_1^{(\infty)} - \frac{m-1}{m}, \\ \tilde\lambda_j^{(1)} = \lambda_j^{(\infty)} + \frac{1}{m}, \quad j = 2, \dots, m, \\ \tilde{\mathcal{H}}_1 = \mathcal{H}_1, \\ \tilde{\mathcal{H}}_l = -\mathcal{H}_l (u_l - u_1)^2 + (u_l - u_1)(d_{m-1}^0 (u_l - u_1))^2 - \\ \qquad -(u_l - u_1)\frac{(m-1)(m^2-m-1)}{m} d_{m-1}^0 (u_l - u_1), \quad l = 2, \dots, n \end{cases} \tag{5.35}$$

where

$$d_{m-1}^0(u_k) = \sum_{s=1}^g \frac{1}{u_k - q_s} - \frac{m(m-1)}{2} \sum_{l \neq k} \frac{1}{u_k - u_l}$$

is a birational canonical transformation of the Schlesinger systems. This transformation acts on the monodromy matrices M_1 and M_∞ as follows:

$$\begin{aligned} \tilde M_\infty &= e^{-\frac{2\pi i}{m}} C_1 M_\infty^{-1} M_1 M_\infty C_1^{-1}, \\ \tilde M_1 &= e^{\frac{2\pi i}{m}} C_1 M_\infty C_1^{-1}, \\ \tilde M_j &= C_1^{-1} M_j C_1, \quad j \neq 1, \infty, \end{aligned} \tag{5.36}$$

Proof. The fact that the above transformation is birational is trivial. To show that it is a canonical transformation of the Schlesinger systems, we just observe that it is obtained by a conformal transformation $\zeta = \frac{1}{z - u_1}$ and a gauge transformation $y = g(\zeta)\tilde y$, $g(\zeta) = \zeta^{\frac{m-1}{m}}$ of the scalar reduction (4.45). In fact (4.45) is transformed to

$$\frac{\mathrm{d}^m \tilde y}{\mathrm{d}\zeta^m} = -\sum_{p=1}^{m-1} \binom{m}{p} \frac{1}{g(\zeta)} \frac{\mathrm{d}^p g}{\mathrm{d}\zeta^p} \frac{\mathrm{d}^{m-p} \tilde y}{\mathrm{d}\zeta^{m-p}} + \sum_{s=0}^{m-1} \hat d_s \sum_{p=0}^{s} \binom{s}{p} \frac{1}{g(\zeta)} \frac{\mathrm{d}^p g}{\mathrm{d}\zeta^p} \frac{\mathrm{d}^{s-p} \tilde y}{\mathrm{d}\zeta^{s-p}},$$

where

$$\begin{aligned} \hat d_0 &= (-1)^m \zeta^{-2m} d_0(\tfrac{1}{\zeta} + u_1), \\ \hat d_s &= (-1)^{m+1} c_{s+1}^{m+1} \zeta^{s-m} + (-1)^m \textstyle\sum_{l=s}^{m-1} \zeta^{l+s-2m} c_{s+1}^{l+1} d_l(\tfrac{1}{\zeta} + u_1), \end{aligned}$$

and $c_i^j := (-1)^{j-1}(j-i)! \binom{j-2}{i-2} \binom{j-1}{i-1}$. Using the above formula and (4.64) it is a straightforward computation to obtain the formulae for $\tilde q_i$, $\tilde p_i$ and $\tilde{\mathcal{H}}_l$ in (5.35). The transformation law of the exponents is obtained in two stages: first the conformal transformation $\zeta = \frac{1}{z} + u_1$ maps

$$\begin{aligned} &\lambda_1^{(\infty)} \to \lambda_1^{(1)}, \quad \lambda_i^{(\infty)} + 1 \to \lambda_i^{(1)}, \quad i = 2, \dots, m, \\ &\lambda_1^{(1)} \to \lambda_1^{(\infty)}, \quad \lambda_i^{(1)} \to \lambda_i^{(\infty)} + 1, \quad i = 2, \dots, m, \end{aligned}$$

then the gauge transformation $y = g(\zeta)\tilde{y}$, $g(\zeta) = \zeta^{\frac{m-1}{m}}$ adds $\frac{m-1}{m}$ to all exponents at infinity and subtracts the same quantity to all exponents at 0. To show (5.36) we proceed as in the previous proof: the involution $i_\infty: u_1 \leftrightarrow \infty$ changes the base point of the fundamental group (this is obtained by conjugating all monodromy matrices with the connection matrix C_1 of M_1), and it changes the basis of loops as in the previous proof with k replaced by ∞ and $k-1$ by n. This implies immediately (5.36).

Remark 5.13. Obviously we can obtain analogous birational canonical transformations acting on the isomonodromic coordinates obtained from the scalar reduction w.r.t. any row of (1.2).

Remark 5.14. Apart from the above symmetries, there are other birational canonical transformations. In fact let us denote by $(q_1^{(j)}, \ldots, q_g^{(j)}, p_1^{(j)}, \ldots, p_g^{(j)})$ the isomonodromic coordinates obtained from the scalar reduction w.r.t. the j^{th} row. The transformation that maps $(q_1^{(j)}, \ldots, q_g^{(j)}, p_1^{(j)}, \ldots, p_g^{(j)})$ to the isomonodromic coordinates obtained from the scalar reduction w.r.t. the i^{th} row$(q_1^{(i)}, \ldots, q_g^{(i)}, p_1^{(i)}, \ldots, p_g^{(i)})$ is by construction a birational canonical transformation. These transformations are the analogues of Okamoto's w_3 for the Painlevé sixth equation (see [48]).

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Appendix: Algebro-Geometric Darboux Coordinates

Here we outline the construction of the so-called algebro-geometric Darboux coordinates. Our presentation follows [13]. However, the idea of constructing canonical coordinates for integrable systems by using the projections of the points in the divisor of a suitable normalized line–bundle on the spectral curve appeared for the first time in a paper by H. Flaschka and D.W. McLaughlin [16] where the special cases of the Toda system and KdV equation were dealt with. Later S.P.Novikov and A.P.Veselov [58] generalized the construction to any hyperelliptic spectral curve and introduced a general class of finite-and infinite-dimensional Poisson brackets. The Flaschka–McLaughlin construction was then generalized to generic rational Lax pairs by M.R. Adams, J. Harnad and J. Hurtubise [1]. A quantum version of this method was initiated by E.Sklyanin [55].

A construction of the polynomial $\Delta_0(z)$ equivalent to ours was given in [53, 20]. Our Theorem A.2 that enables to construct *rational* Darboux coordinates on the reduced symplectic leaves seems to be new (cf. however the recent paper [6] where a similar approach to constructing the Darboux coordinates was developed).

Let us rewrite the characteristic polynomial (5.2) of the matrix

$$A(z) = \sum_{i=1}^{n} \frac{A_i}{z - u_i} = \frac{\hat{A}(z)}{\prod_{i=1}^{n}(z - u_i)}, \tag{A.1}$$

$$\hat{A}(z) = -A_\infty z^{n-1} + O(z^{n-2}), \quad z \to \infty \tag{A.2}$$

in the form

$$\hat{\mathcal{R}}(z,\hat{w}) = \det\left(\hat{w} - \det \hat{A}(z)\right) = \left(\prod_{i=1}^{n}(z-u_i)\right)^m \mathcal{R}(z,w), \quad \hat{w} = w\prod_{i=1}^{n}(z-u_i). \tag{A.3}$$

Expanding the determinant one obtains a polynomial

$$\hat{\mathcal{R}}(z,\hat{w}) = \hat{w}^m + \hat{\alpha}_1(z)\hat{w}^{m-1} + \cdots + \hat{\alpha}_m(z) = \sum_{i+(n-1)j\le m(n-1)} a_{ij} z^i \hat{w}^j, \tag{A.4}$$

where

$$a_{ij} = a_{ij}(A_1,\ldots,A_n; u_1,\ldots,u_n) \tag{A.5}$$

are some polynomials in the entries of the matrices A_k and in u_l,

$$a_{0m} = 1, \quad \hat{\alpha}_s(z) = \sum_{i=0}^{s\,(n-1)} a_{i\,,m-s} z^i, \quad s = 0,\ 1,\ldots,m.$$

It is well known that for a Zariski open subset in the projective space $\mathbf{P}^{M-1}$,

$$M = (n-1)\frac{m(m+1)}{2} + m + 1 \tag{A.6}$$

with the homogeneous coordinates

$$a_{ij}, \quad i + (n-1)j \le m(n-1),$$

the affine algebraic curve

$$\hat{\mathcal{R}}(z,\hat{w}) = \sum_{i+(n-1)j\le m(n-1)} a_{ij} z^i \hat{w}^j = 0 \tag{A.7}$$

is smooth. Indeed, it suffices to check smoothness of one of the curves of the above family, e.g. of

$$\hat{w}^m = z^{m(n-1)} - 1.$$

Under the smoothness assumption the standard compactification of (A.7) gives a compact Riemann surface Γ of the genus

$$g = (n-1)\frac{m(m-1)}{2} - m + 1 \tag{A.8}$$

(cfr. the formula (5.10)). The infinite part of Γ is a divisor D_∞ of the degree m. If the normal Jordan form of the matrix $\mathcal{A}_\infty$ contains k Jordan blocks of the multiplicities $m_1,\ldots,m_k$ then the divisor D_∞ has the form

$$D_\infty = m_1\infty_1 + \cdots + m_k\infty_k. \tag{A.9}$$

Here $\infty_1, \dots, \infty_k \in \Gamma$ are the points added at infinity. In particular, if the spectrum of A_∞ is simple then the divisor D_∞ is a sum of m distinct points $\infty_1, \dots, \infty_m$:

$$\infty_k := \{z \to \infty,\ \hat{w} \to \infty,\ \frac{\hat{w}}{z^{n-1}} \to -\lambda_k^{(\infty)}\},\quad k = 1, \dots, m. \tag{A.10}$$

Here $\lambda_1^{(\infty)}, \dots, \lambda_m^{(\infty)}$ are the eigenvalues of the matrix A_∞.

Let us give an intrinsic characterization of algebraic curves of the form (A.7).

The coordinate functions z and $\hat{w}$ have poles at the divisors D_∞ and $(n-1)D_\infty$ respectively. Conversely, the following simple statement can be proved by using standard arguments based on the Riemann - Roch theorem.

Lemma A.1. *Let* Γ *be a Riemann surface of the genus* (A.8). *Let* D_∞ *be a divisor of the degree* m *on* Γ *such that*

$$\dim H^0(\Gamma, \mathcal{O}(D_\infty)) = 2, \quad \dim H^0(\Gamma, \mathcal{O}((n-1)D_\infty)) = n+1.$$

Then the Riemann surface can be represented in the form (A.7).

Proof. The first of the assumptions implies existence of a non-constant meromorphic function z with poles at the points of the divisor D_∞. The second one yields existence of another function $\hat{w}$ with poles at $(n-1)D_\infty$ that cannot be represented as a polynomial in z. Let us now consider the space $H^0(\Gamma, \mathcal{O}\,(m(n-1)D_\infty))$. The M monomials

$$z^i \hat{w}^j, \quad i + (n-1)j \le m(n-1) \tag{A.11}$$

belong to this space. Let us prove that these monomials are linearly dependent. To this end let us compute the dimension

$$\dim H^0(\Gamma, \mathcal{O}\,(m(n-1)D_\infty)).$$

First of all, the degree of the divisor $\mathcal{D} := m(n-1)D_\infty$ equals

$$\deg \mathcal{D} = m^2(n-1) = 2g - 2 + m(n+1) > 2g - 2.$$

So the Riemann - Roch theorem gives

$$\dim H^0(\Gamma, \mathcal{O}\,(m(n-1)D_\infty)) = \deg \mathcal{D} - g + 1 = M - 1.$$

This proves linear dependence of M functions of the form (A.11). The lemma is proved. □

Observe that for $n > 1$ the eigenvalues $\lambda_1^{(\infty)}, \dots, \lambda_m^{(\infty)}$ are determined by the Riemann surface uniquely up to permutations and common affine transformations

$$\lambda_i^{(\infty)} \mapsto a\lambda_i^{(\infty)} + b, \quad i = 1, \dots, m.$$

We will say that our Riemann surface Γ is D_∞*-generic* if the eigenvalues are pairwise distinct,

$$\lambda_i^{(\infty)} \ne \lambda_j^{(\infty)}, \quad i \ne j.$$

Instead of using the coefficients a_{ij} as the homogeneous coordinates in the space of algebraic curves (A.7) we will construct another system of coordinates on a Zariski open subspace in $\mathbf{P}^{M-1}$. Let us choose $n+g$ pairwise distinct numbers $u_1, \ldots, u_n$, $\gamma_1, \ldots, \gamma_g$ and also $n+1$ m-tuples of pairwise distinct numbers $\lambda_1^{(i)}, \ldots, \lambda_m^{(i)}$ for every $i = 1, \ldots, n, \infty$,

$$\lambda_r^{(i)} \neq \lambda_s^{(i)}, \quad s \neq r, \quad i = 1, 2, \ldots, m, \infty,$$

satisfying the constraint

$$\sum_{i=1}^{n}\sum_{r=1}^{m}\lambda_r^{(i)} + \sum_{r=1}^{m}\lambda_r^{(\infty)} = 0. \tag{A.12}$$

Finally, choose arbitrary g numbers $\mu_1, \ldots, \mu_g$. Denote

$$\hat{\lambda}_r^{(i)} := \prod_{j\neq i}(u_i - u_j)\lambda_r^{(i)}, \quad r = 1, \ldots, m, \quad i = 1, \ldots, n, \tag{A.13}$$

$$\hat{\mu}_s = \prod_{i=1}^{n}(\gamma_s - u_i)\mu_s, \quad s = 1, \ldots, g. \tag{A.14}$$

Theorem A.2. *For generic values of the parameters*

$$u_1, \ldots, u_n, \quad \lambda_1^{(i)}, \ldots, \lambda_m^{(i)}, \quad i = 1, \ldots, n, \infty, \quad \gamma_1, \ldots, \gamma_g, \ \mu_1, \ldots, \mu_g \tag{A.15}$$

satisfying the constraint (A.12) there exists a unique curve $\hat{\mathcal{R}}(z, \hat{w}) = 0$ *of the form (A.7) with* $a_{0m} = 1$ *passing through the points*

$$(u_i, \hat{\lambda}_r^{(i)}), \ r = 1, \ldots, m, \ i = 1, \ldots, n, \tag{A.16}$$

$$z, \ \hat{w} \to \infty, \ \frac{\hat{w}}{z^{n-1}} \to -\lambda_r^{(\infty)}, \ r = 1, \ldots, m, \tag{A.17}$$

$$(\gamma_s, \hat{\mu}_s), \quad s = 1, \ldots, g. \tag{A.18}$$

Proof. Let us denote

$$\sigma_k^{(i)} := \sigma_k(\lambda_1^{(i)}, \ldots, \lambda_m^{(i)}), \quad k = 1, \ldots, m$$

the value of the k^{th} elementary symmetric function of $\lambda_1^{(i)}, \ldots, \lambda_m^{(i)}$, $i = 1, \ldots, n, \infty$. The equation of the algebraic curve must have the form

$$\hat{w}^m - R(z)\sum_{i=1}^{n}\frac{\sigma_1^{(i)}}{z-u_i}\hat{w}^{m-1} \tag{A.19}$$
$$+R(z)\sum_{k=2}^{m}\left[(-1)^k\sum_{i=1}^{n}\frac{\sigma_k^{(i)}}{z-u_i}[R'(u_i)]^{k-1} + \sigma_k^{(\infty)}z^{kn-n-k} + p_{kn-n-k-1}(z)\right]\hat{w}^{m-k} = 0.$$

Here, as above

$$R(z) := \prod_{i=1}^{n}(z-u_i),$$

the polynomials $p_{n-3}(z)$, $p_{2n-4}(z)$, …, $p_{m\,n-m-n-1}(z)$ labeled by their degrees are to be determined later. Such a curve will pass through the points (A.16), (A.17). To have it passing also through (A.18) the following system of equations must be satisfied:

$$\sum_{k=2}^{m} p_{k\,n-n-k-1}(\gamma_s)\hat{\mu}_s^{m-k} + Q(\gamma_s, \hat{\mu}_s) = 0, \quad s = 1, \ldots, g, \tag{A.20}$$

where

$$Q(z, \hat{w}) := \frac{\hat{w}^m}{R(z)} - \sum_{i=1}^{n} \frac{\sigma_1^{(i)}}{z - u_i}\hat{w}^{m-1} + \sum_{k=2}^{m}\left[(-1)^k \sum_{i=1}^{n} \frac{\sigma_k^{(i)}}{z - u_i}[R'(u_i)]^{k-1} + \sigma_k^{(\infty)} z^{k\,n-n-k}\right]\hat{w}^{m-k}.$$

This is a linear system for the g coefficients of the polynomials $p_{n-3}(z)$, $p_{2n-4}(z)$, …, $p_{m\,n-m-n-1}(z)$. Let us prove that the determinant of this linear system does not vanish identically. Indeed, this determinant is equal to the following polynomial in γ_1, …, γ_g, $\hat{\mu}_1$, …, $\hat{\mu}_g$,

$$\begin{aligned} &W_{m,n}(\gamma_1, \ldots, \gamma_g, \hat{\mu}_1, \ldots, \hat{\mu}_g) \\ &:= \sum_{\pi}(-1)^{|\pi|}(\hat{\mu}_{i_1} \ldots \hat{\mu}_{i_{n-3}})^{m-2}(\hat{\mu}_{j_1} \ldots \hat{\mu}_{j_{2n-4}})^{m-3}(\hat{\mu}_{k_1} \ldots \hat{\mu}_{k_{3n-5}})^{m-4} \ldots \\ &\quad \times V(\gamma_{i_1} \ldots \gamma_{i_{n-3}})V(\gamma_{j_1} \ldots \gamma_{j_{2n-4}})V(\gamma_{k_1} \ldots \gamma_{k_{3n-5}})V(\gamma_{l_1}, \ldots \gamma_{l_{m\,n-m-n-1}}). \end{aligned} \tag{A.21}$$

Here the summation is over the partitions

$$\begin{aligned} \pi : \{1, 2, \ldots, g\} = \{i_1, \ldots, i_{n-3}\} \sqcup \{j_1, \ldots, j_{2n-4}\} \sqcup \\ \times\{k_1, \ldots, k_{3n-5}\} \sqcup \cdots \sqcup \{l_1, \ldots, l_{m\,n-m-n-1}\}, \end{aligned} \tag{A.22}$$

$|\pi|$ stands for the parity of the permutation $\pi \in S_g$,

$$V(x_1, \ldots, x_k) := \prod_{1 \le i < j \le k} (x_i - x_j)$$

is the Vandermonde determinant. It is clear that this polynomial is not an identical zero. The theorem is proved. □

We want now to show that the same data used in Theorem A.2 determine the matrix valued polynomial $A(z)$ in the determinant representation (A.3). We will now prove that any generic curve of the form (A.7) can be represented in the determinant form (A.3). Actually, this can be done in many ways; we will describe the parameters of all determinant representations of Γ.

Let Γ be the spectral curve (A.4) of a matrix $\hat{A}(z)$. Assuming smoothness of the spectral curve, we will associate with the determinant representation a degree g divisor D on Γ. Let us first consider the eigenvector line bundle $\mathcal{L}$ on Γ,

$$\hat{A}(z)\psi = \hat{w}\psi, \tag{A.23}$$

$\psi = (\psi_1, \ldots, \psi_m)^T$. Define the divisor

$$\hat{D} := \{\psi_1 = 0\}. \quad \text{(A.24)}$$

In other words, $\hat{D}$ is the divisor of poles of the meromorphic functions ψ_2/ψ_1, ψ_3/ψ_1, ..., ψ_m/ψ_1.

Lemma A.3. *The degree of the divisor $\hat{D}$ is equal to*

$$\deg \hat{D} = g + m - 1.$$

See [21] for a simple proof.
Denote

$$D = \hat{D} \cap \Gamma \setminus D_\infty. \quad \text{(A.25)}$$

Lemma A.4. *The point $(z_0, \hat{w}_0) \in D$ only if*

$$\Delta_0(z_0) = 0.$$

Here the polynomial $\Delta_0(z)$ was defined in (5.9). *Conversely, for any root z_0 of the polynomial $\Delta_0(z)$ there exists a point $(z_0, \hat{w}_0) \in D$.*

Proof. For the convenience of the reader we will give here the proof. Rewriting the equation of the divisor in the form

$$< b_0, \psi >= 0$$

(here $< \ , \ >$ stands for the natural pairing between row- and column-vectors, the row-vector b_0 was defined in (5.7)) we also derive that, for $k = 1, \ldots, m-1$,

$$< b_k(z_0), \psi >=< b_0, A^k(z_0)\psi >= w_0^k < b_0, \psi >= 0, \quad w_0 = \frac{\hat{w}_0}{R(z_0)}.$$

The determinant of this linear homogeneous system must be equal to 0. This gives $\Delta_0(z_0) = 0$.

Conversely, let z_0 be a zero of Δ_0 . Using the identity (5.25) we derive that the subspace Ker $B(z_0)$ of the m-dimensional space is invariant w.r.t. the linear operator $A(z_0)$. Here the matrix $B(z)$ was defined in the line after the formula (5.7). Therefore there exists an eigenvector $\psi \in \text{Ker } B(z_0)$,

$$A(z_0)\psi = w_0\psi, \quad < b_0, \psi >= \psi_1 = 0.$$

By definition the point (z_0, w_0) belongs to the divisor $D \in \Gamma$. The lemma is proved.

Let us now compute the degree of the polynomial $\Delta_0(z)$. Let

$$\hat{A}(z) = -A_\infty z^{n-1} + C z^{n-2} + O(z^{n-3}).$$

Explicitly, the matrix $C = (C_{ij})$ reads

$$C = A_\infty \bar{u} + \sum_{i=1}^n u_i A_i, \quad \bar{u} = \sum_{i=1}^n u_i. \quad \text{(A.26)}$$

Lemma A.5. *The polynomial* $\Delta_0(z)$ *has the form*

$$\Delta_0(z) = (-1)^{m-1} C_{12} C_{13} \dots C_{1m} \prod_{2 \le i < j} (\lambda_i^{(\infty)} - \lambda_j^{(\infty)})\, z^g + O(z^{g-1}),$$

where g is given by the formula (A.8).

Proof. For the j^{th} coordinate of the row-vector

$$\hat{b}_k := b_0 \hat{A}^{k-1}(z)$$

(cf. (5.7) one obtains

$$(\hat{b}_k)_j = (-1)^k \left[\left(\lambda_1^{(\infty)}\right)^k \delta_{1\,j} z^{k(n-1)} - \frac{(\lambda_1^{(\infty)})^k - (\lambda_j^{(\infty)})^k}{\lambda_1^{(\infty)} - \lambda_j^{(\infty)}} C_{1\,j} z^{k(n-1)-1} + \cdots \right],$$

where dots stand for the terms of lower order in z. Computing the determinant of this matrix we obtain the proof of the needed formula. □

Corollary A.6. *If the eigenvalues of the matrix* A_∞ *are pairwise distinct and all the elements of the first row of the matrix (A.26) are not equal to zero then the degree of the divisor D is equal to g. The remaining points of the divisor* $\hat{D}$ *are at infinity,*

$$\hat{D} - D = \infty_2 + \infty_3 + \cdots + \infty_m.$$

The statement of the corollary is a formalization of the following asymptotic behavior of the eigenvectors $\psi = (\psi_1, \dots, \psi_m)^T$ of the matrix $\hat{A}(z)$ at $z \to \infty$:

$$\psi_k = \delta_{kj} + O\left(\frac{1}{z}\right), \quad z \to \infty, \quad \frac{\hat{w}}{z^{n-1}} \to -\lambda_j^{(\infty)}.$$

Such a normalized eigenvector will have $g + m - 1$ poles on $\Gamma \setminus D_\infty$. Under the above assumptions the first component ψ_1 has simple zeroes at the points $\infty_2, \dots, \infty_m$.

Remark A.7. We observe that, as it was shown in the proof of Lemma 4.20, the element C_{ij} of the matrix (A.26) is identically equal to zero for an isomonodromic deformation $A_k A_k(u_1, \dots, u_n)$, if and only if

$$(1 - \lambda_i^{(\infty)} + \lambda_j^{(\infty)}) A_{k_{ij}} = -A_{\infty_{ij}},$$

that is either $1 - \lambda_i^{(\infty)} + \lambda_j^{(\infty)} = 0$ or $A_{k_{ij}} = 0$ for all k.

Denote

$$\gamma_s = \gamma_s(\mathbf{A}), \quad \mu_s = \mu_s(\mathbf{A}), \quad s = 1, \dots, g, \tag{A.27}$$

the coordinates of the points of the divisor D,

$$D = (\gamma_1, \hat{\mu}_1) + \cdots + (\gamma_g, \hat{\mu}_g), \quad \hat{\mu}_s = \mu_s \prod_{i=1}^{n} (\gamma_s - u_i). \tag{A.28}$$

We want to show that, given the generic values of the functions $\gamma_s(\mathbf{A})$, $\mu_s(\mathbf{A})$, $s = 1, \ldots, g$ together with the numbers $u_1, \ldots, u_n$ and the pairwise distinct eigenvalues $\lambda_1^{(i)}, \ldots, \lambda_m^{(i)}$, $i = 1, \ldots, n, \infty$ satisfying the constraint (A.12), one can uniquely determine the conjugacy class of the n-tuple of matrices $\mathbf{A} = (A_1, \ldots, A_n)$ modulo diagonal conjugations and permutations. To this end we will prove, essentially using the technique of [11] the converse statement that shows that, for a Zariski open subset in the space $\mathbf{P}^{M-1}$ of algebraic curves of the form (A.7), the curve can be represented in the determinant form. We will also describe parameters of such determinant representations of a given curve.

Let D be a divisor of the degree g on the Riemann surface Γ. We will say that the divisor is D_∞*-non-special* if

$$\dim H^0(\Gamma, \mathcal{O}(D + \infty_i - \infty_1)) = 1, \quad i = 1, \ldots, m. \tag{A.29}$$

Theorem A.8. *Any smooth affine curve* $\hat{\mathcal{R}}(z, \hat{w}) = 0$ *of the form* (A.7) *can be represented in the determinant form* (A.4) *for a matrix* $\hat{A}(z)$ *of the form* (A.2). *For a* D_∞*-generic curve such representations, considered modulo diagonal conjugations*

$$\hat{A}(z) \mapsto K^{-1}\hat{A}(z)\,K, \quad K = \mathrm{diag}(k_1, \ldots, k_m)$$

and permutations of coordinates

$$\hat{A}(z) \mapsto P^{-1}\hat{A}(z)\,P, \quad P \in S_{m-1}$$

preserving the vector $(1, 0, \ldots, 0)$ *are in one-to-one correspondence with the degree g* D_∞*-non-special divisors D on* Γ.

Proof. Let us order the infinite points of Γ and choose nonzero sections

$$\psi_k \in H^0(\Gamma, O(D + \infty_k - \infty_1)), \quad k = 2, \ldots, m. \tag{A.30}$$

They are determined uniquely up to constant factors. Introduce a vector valued meromorphic function on Γ putting

$$\psi = (1, \psi_2, \ldots, \psi_m)^T.$$

Introduce the $m \times m$ matrix $\Psi(z) = (\Psi_{kj}(z))$ of Laurent series in $1/z$ expanding the functions ψ_k near the infinite points $\infty_j \in \Gamma$. Let $W(z) = \mathrm{diag}(W_1(z), \ldots, W_m(z))$ be the diagonal matrix obtained by taking the Laurent series of the function $\hat{w}$ on Γ near $\infty_1, \ldots, \infty_m$. Define a matrix of polynomials

$$\hat{A}(z) := \left(\Psi(z)W(z)\Psi^{-1}(z)\right)_+. \tag{A.31}$$

Here $(\ )_+$ means the polynomial part in z of the expansion. By construction

$$\hat{A}(z) = -z^{m(n-1)}\mathrm{diag}(\lambda_1^{(\infty)}, \ldots, \lambda_m^{(\infty)}) + O(z^{m(n-1)-1}).$$

Let us prove that the vector function ψ on Γ satisfies

$$\hat{A}(z)\psi = \hat{w}\,\psi.$$

This can be done using the standard arguments of Krichever's scheme [35]. Indeed, by construction all the components of the difference

$$\hat{w}\psi - \hat{A}(z)\psi$$

are analytic at the infinite points $\infty_1, \ldots, \infty_m$. Therefore they have poles only at the points of the divisor D. Due to nonspeciality of D the difference is equal to zero.

We have proved that Γ coincides with the spectral curve of the polynomial matrix $\hat{A}(z)$. By construction the divisor D we started with coincides with the one defined above. It remains to observe that, choosing another basic section

$$\tilde{\psi}_k \in H^0(\Gamma, O(D + \infty_k - \infty_1)),$$
$$\tilde{\psi}_k = c_k\psi_k, \quad k = 2, \ldots, m,$$

yields the diagonal conjugation of the polynomial matrix $\hat{A}(z)$,

$$\hat{A}(z) \mapsto C\,\hat{A}(z)\,C^{-1}, \quad C = \mathrm{diag}(1, c_2, \ldots, c_m).$$

Moreover, changing the order of the infinite points preserving ∞_1 implies a permutation. The theorem is proved. □

Corollary A.9. *The map*

$$[A(z)] \mapsto \mathcal{S}pec \tag{A.32}$$

is a birational isomorphism of the space of classes of equivalence of rational matrix-valued functions of the form

$$A(z) = \sum_{i=1}^{n} \frac{A_i}{z - u_i}, \quad A_\infty = -(A_1 + \cdots + A_n)$$

with diagonal A_∞ considered modulo diagonal conjugations and the space of spectral data with the coordinates

$$\left(u_1, \ldots, u_n, \mathrm{Spec}\, A_1, \ldots, \mathrm{Spec}\, A_n, \mathrm{Spec}\, A_\infty, \gamma_1, \mu_1, \ldots, \gamma_g, \mu_g\right) \in \mathcal{S}pec. \tag{A.33}$$

In particular, $(\gamma_1, \mu_1, \ldots, \gamma_g, \mu_g)$ are coordinates on the reduced symplectic leaves of the Poisson bracket (3.2).

We will now prove that $\gamma_i = \gamma_i(\mathbf{A})$, $\mu_i = \mu_i(\mathbf{A})$, $i = 1, \ldots, g$ are *canonical coordinates* on the reduced symplectic leaves of the Poisson bracket. It will be convenient to represent the Poisson bracket (3.2) in the following well known r-matrix form (see [15] regarding the definitions and notations).

Lemma A.10. *The Poisson bracket* (3.2) *can be represented in the form*

$$\left\{A(z_1) \mathop{\otimes}_{,} A(z_2)\right\} = \left[A(z_1) \otimes \mathbb{1} + \mathbb{1} \otimes A(z_2), r(z_1 - z_2)\right], \tag{A.34}$$

where $r(z)$ is a classical r-matrix, i.e. a solution of the linearized Yang – Baxter equation, given by

$$r_{ik}^{jl}(z) = \frac{\delta_i^l \delta_k^j}{z}.$$

Equivalently, (A.35) *reads*

$$\left\{A^i_j(z_1),\, A^k_l(z_2)\right\} = \frac{1}{z_1 - z_2}\left[\delta^i_l\left(A^k_j(z_1) - A^k_j(z_2)\right) - \left(A^i_l(z_1) - A^i_l(z_2)\right)\delta^k_j\right]. \tag{A.35}$$

Proof. By the definition

$$\left\{A(z_1) \underset{,}{\otimes} A(z_2)\right\}^{ik}_{jl} : \sum_p \frac{A_p{}^k_j\delta^i_l - A_p{}^i_l\delta^k_j}{(z_1 - u_p)(z_2 - u_p)}.$$

So

$$\sum_p \frac{A_p{}^k_j\delta^i_l - A_p{}^i_l\delta^k_j}{(z_1 - u_p)(z_2 - u_p)} = \frac{1}{(z_2 - z_1)}\sum_p\left(\frac{1}{z_1 - u_p} - \frac{1}{z_2 - u_p}\right)\left(A_p{}^k_j\delta^i_l - A_p{}^i_l\delta^k_j\right)$$
$$= \frac{1}{(z_2 - z_1)}\left(A^k_j(z_1)\delta^i_l - A^i_l(z_1)\delta^k_j + A^i_l(z_2)\delta^k_j - A^k_j(z_2)\delta^i_l\right)$$
$$= [A(z_1)\otimes \mathbb{1} + \mathbb{1}\otimes A(z_2), r(z_1 - z_2)]^{ik}_{jl}\,.$$

This concludes the proof. □

Theorem A.11. *The functions* $\lambda^{(k)}_i = \lambda^{(k)}_i(\mathbf{A})$*,* $i = 1, \dots, m$*,* $k = 1, \dots, n, \infty$*,* $\gamma_i = \gamma_i(\mathbf{A})$*,* $\mu_i = \mu_i(\mathbf{A})$*,* $g = 1, \dots, g$ *on the space of* $m \times m$ *matrices* $(A_1, \dots, A_n) =: \mathbf{A}$ *have the following canonical Poisson brackets w.r.t. the structure* (A.35)*:*

$$\{\gamma_i, \mu_j\} = \delta_{ij},$$

all other Poisson brackets vanish.

Proof. We already know that the eigenvalues $\lambda^{(k)}_i$ of the matrices A_k are Casimirs of the Poisson bracket. It remains to compute the Poisson brackets of the functions $\mu_i(\mathbf{A})$ and $\gamma_j(\mathbf{A})$.

Let us introduce the following notations. Let z not be a ramification point for the Riemann surface Γ. Let us fix some ordering of the sheets of the Riemann surface. Denote

$$|a>, \quad a = 1, \dots, m$$

the basis of eigenvectors of the matrix $A = A(z)$,

$$A|a> = w_a|a> \tag{A.36}$$

normalized by the condition

$$< b_0|a> = 1, \quad b_0 = (1, 0, \dots, 0). \tag{A.37}$$

Here

$$w_a = w_a(z), \quad a = 1, \dots, m$$

are the roots of the characteristic equation $\det(w - A(z)) = 0$. Denote $< a|$ the dual basis of row-vectors

$$< a|b >= \delta_{ab}. \tag{A.38}$$

Due to (A.37) one has

$$b_0 = \sum_{a=1}^{m} < a|. \tag{A.39}$$

□

Lemma A.12. *The following formulae for the Poisson brackets hold true:*

$$\{\log \Delta_0(z), \log \Delta_0(z')\} = 0, \tag{A.40}$$

$$\{w_a(z), w_b(z')\} = 0, \tag{A.41}$$

$$\{\log \Delta_0(z), w_c(z')\} = \frac{1}{(z - z')} \sum_{a \neq b} < a|c' >< c'|b > . \tag{A.42}$$

In this formulae the primes mean that the corresponding function is computed at the point z', i.e.

$$A(z')|c' >= w'_c|c' > \quad \text{where } w'_c = w_c(z').$$

Proof. The following well known variational formulae will be useful in the computations of the Poisson brackets

$$\delta w_a =< a|\delta A|a >, \tag{A.43}$$

$$< b|\delta a >= \frac{< b|\delta A|a >}{w_a - w_b}, \quad b \neq a, \tag{A.44}$$

$$< a|\delta a >= \sum_{b \neq a} \frac{< b|\delta A|a >}{w_b - w_a}. \tag{A.45}$$

Here $|\delta a >$ is the variation of the eigenvector $|a >$. In the derivation of the last formula we have used the normalization (A.37).

Denote $\Psi(z)$ the matrix with the columns $|1 >, \ldots, |m >$. The rows of the inverse matrix Ψ^{-1} coincide with the bra-vectors $< 1|, \ldots, < m|$. From (A.39) it easily follows that the matrix $B(z)$ is equal to the product of the Vandermonde matrix of the pairwise distinct numbers $w_1, \ldots, w_m$ by $\Psi^{-1}(z)$. So

$$\det B(z) = \prod_{i<j} (w_i - w_j) \det{}^{-1} \Psi(z).$$

Using the Liouville formula

$$\delta \log(\det \Psi) = \operatorname{tr} \Psi^{-1} \delta \Psi = \sum_{a=1}^{m} < a|\delta a >$$

yields

$$\delta \log \Delta_0(z) = \frac{1}{2} \sum_{a \neq b} \frac{< a|\delta A|a > - < b|\delta A|b > - < a|\delta A|b >}{w_a - w_b}. \tag{A.46}$$

It is understood that the values of the variables u_i are fixed during the variation. From the above formulae for $\delta\Delta_0(z)$, $\delta w_c(z')$ we derive, by a straightforward calculation, the brackets (A.40) – (A.42). The lemma is proved. $\square$

Proof of the theorem. From (A.40) and from the representation

$$\delta \log \Delta_0(z) = -\sum_{i=1}^{m} \frac{\delta\gamma_i}{z-\gamma_i} + \text{regular terms}$$

it easily follows that

$$\{\gamma_i, \gamma_j\} = 0.$$

The commutation rule

$$\{\mu_i, \mu_j\} = 0$$

follows from (A.41). Let us compute the brackets $\{\gamma_i, \mu_j\}$. Due to Theorem A.8 we may assume that the projections $z = \gamma_i$ of the points of the divisor D onto the z-plane are all pairwise distinct, they are distinct from u_j and from the ramification points of the Riemann surface (cf. Assumption 2 above). Consider first the case $j \neq i$. Assume that the numeration of the sheets of the Riemann surface at the neighborhoods of the points $z = \gamma_i$ and $z' = \gamma_j$ is done in such a way that the pole of the eigenvector ψ of the matrix $A(z)$ belongs to the sheet labeled by c. That means that the ket-vector $|c>$ has a simple pole at $z \to \gamma_i$,

$$|c> = \frac{|\tilde{c}>}{z-\gamma_i} + O(1), \quad z \to \gamma_i. \tag{A.47}$$

All other ket- and bra-vectors are analytic in z near this point and the corresponding bra-vector $<c|$ has a simple zero

$$<c| = (z-\gamma_i) <\tilde{c}| + O\left((z-\gamma_i)^2\right). \tag{A.48}$$

From the already proven commutation rule of the coordinates γ_i it follows that

$$\begin{aligned}\{\gamma_i, \mu_j\} &= -\lim_{z\to\gamma_i}\lim_{z'\to\gamma_j}(z-\gamma_i)\{\log\Delta_0(z), w_c(z')\} \\ &= -\frac{1}{2(\gamma_i-\gamma_j)}\lim_{z\to\gamma_i}\lim_{z'\to\gamma_j}(z-\gamma_i)\sum_{a\neq b} <a|c'><c'|b>.\end{aligned}$$

Due to (A.47), (A.48) the r.h.s. is analytic at the point $z = \gamma_j$. The singularity at $z = \gamma_i$ can come only from the terms with $b = c$. So, the singular part in the sum equals

$$\sum_{a\neq b} <a|c'><c'|b> = \sum_{a\neq c} <a|c'><c'|c> + \text{regular}.$$

Using the z-independent normalization (A.37) we rewrite the singular term in the form

$$\sum_{a\neq c} <a|c'><c'|c> = -<c|c'><c'|c>.$$

Using again (A.47), (A.48) we establish analyticity of the last expression also at $z = \gamma_i$. Therefore $\{\gamma_i, \mu_j\} = 0$ for $i \neq j$.

To compute $\{\gamma_i, \mu_i\}$ we will first calculate the limit

$$\lim_{z' \to z} \{\log \Delta_0(z), w_c(z')\}.$$

Observe that, for $z' = z$ the numerator of formula (A.42) vanishes

$$\left(\sum_{a \neq b} < a|c' >< c'|b > \right)_{z'=z} = \sum_{a \neq b} < a|c >< c|b >= \sum_{a \neq b} \delta_{ac}\delta_{cb} = 0.$$

So the needed limit is equal to the derivative

$$\lim_{z' \to z} \{\log \Delta_0(z), w_c(z')\} = - \left(\frac{d}{dz'} \sum_{a \neq b} < a|c' >< c'|b > \right)_{z'=z} .$$

Let us denote

$$|\dot{c} >:= \frac{d}{dz'} |c' > |_{z'=z}, \quad < \dot{c}| := \frac{d}{dz'} < c'| \,|_{z'=z}.$$

We obtain, using again the z'-independent normalization (A.37),

$$\begin{aligned} \frac{d}{dz'} \left(\sum_{a \neq b} < a|c' >< c'|b > \right)_{z'=z} &= \sum_{a \neq c} < a|\dot{c} > + \sum_{b \neq c} < \dot{c}|b > \\ &= - < c|\dot{c} > + \sum_{b \neq c} < \dot{c}|b > . \end{aligned}$$

Therefore

$$\{\gamma_i, \mu_i\} = \lim_{z \to \gamma_i} (z - \gamma_i) \left[- < c|\dot{c} > + \sum_{b \neq c} < \dot{c}|b > \right].$$

The last term in the brackets is analytic at the point $z = \gamma_i$. For the first one we obtain, using (A.47), (A.48),

$$< c|\dot{c} >= -\frac{< \tilde{c}|\tilde{c} >}{z - \gamma_i} + \text{regular terms}.$$

The last step is to use the normalization

$$< c|c > \equiv 1$$

to derive that

$$< \tilde{c}|\tilde{c} >= 1.$$

Hence

$$\{\gamma_i, \mu_i\} = 1.$$

The theorem is proved.

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