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On Genus Two Riemann Surfaces Formed from Sewn Tori

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Abstract: We describe the period matrix and other data on a higher genus Riemann surface in terms of data coming from lower genus surfaces via an explicit sewing procedure. We consider in detail the construction of a genus two Riemann surface by either sewing two punctured tori together or by sewing a twice-punctured torus to itself. In each case the genus two period matrix is explicitly described as a holomorphic map from a suitable domain (parameterized by genus one moduli and sewing parameters) to the Siegel upper half plane $\mathbb{H}_2$. Equivariance of these maps under certain subgroups of $Sp(4, \mathbb{Z})$ is shown. The invertibility of both maps in a particular domain of $\mathbb{H}_2$ is also shown.

1. Introduction

This paper is the second of a series intended to develop a mathematically rigorous theory of chiral partition and n-point functions on Riemann surfaces at all genera, based on the theory of vertex operator algebras. The purpose of the paper is to provide a rigorous and explicit description of the period matrix and other data on a higher genus Riemann surface in terms of data coming from lower genus surfaces via an explicit sewing procedure. In particular, we consider and compare in some detail the construction of a genus two Riemann surface in two separate ways: either by sewing two tori together or by sewing a torus to itself. Although our primary motivation is to lay the foundations for the explicit construction of the partition and n-point functions for a vertex operator algebra on higher genus Riemann surfaces [MT2], we envisage that the results herein may also be of wider interest.

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As is well-known (cf. [H] for a systematic development), the axioms for a vertex (operator) algebra V amount to an algebraicization of aspects of the theory of gluing spheres, i.e., compact Riemann surfaces of genus zero. A quite complete theory of (bosonic) n-point functions at genus one was developed by Zhu in his well-known paper [Z]. For particularly well-behaved vertex operator algebras (we have in mind *rational* vertex operator algebras in the sense of [DLM]), Zhu essentially showed that the n-point functions, defined as particular graded traces over V, are elliptic and have certain $SL(2,\mathbb{Z})$ modular-invariance properties[1] with respect to the torus modular parameter τ. A complete description of bosonic Heisenberg and lattice VOA n -point functions is given in [MT1]. Apart from its intrinsic interest, modular-invariance is an important feature of conformal field theory and its application to string theory e.g., [P, GSW]. Mathematically, it has been valuable in recent developments concerning the structure theory of vertex operator algebras [DM1, DM2], and it may well play an important role in geometric applications such as elliptic cohomology and elliptic genus. These are all good reasons to anticipate that a theory of n -point functions at higher genus will be valuable; another is the consistency of string theory at higher loops (genera). Our ultimate goal, then, is to emulate Zhu's theory by first defining and then understanding the automorphic properties of n-point functions on a higher genus Riemann surface where the modular variable τ is replaced by the period matrix $\Omega^{(g)}$, a point in the Siegel upper half-space $\mathbb{H}_g$ at genus g.

When one tries to implement the vision outlined above at genus $g \geq 2$, difficulties immediately arise which have no analog at lower genera [T]. In Zhu's theory, there is a clear relation between the variable τ and the relevant vertex operators: it is not the definition of n-point functions, but rather the elucidation of their properties, that is difficult. At higher genus, the very definition of n-point function is less straightforward and raises interesting issues. Many (but not all) of these are already present at genus 2, and it is this case that we mainly deal with in the present paper.

A very general approach to conformal field theory on higher genus Riemann surfaces has been discussed in the physics literature [MS, So, DP, VV, P]. In particular, there has been much progress in recent years in understanding genus two superstring theory [DPI, DPVI, DGP]. The basic idea we follow is that by cutting a Riemann surface along various cycles it can be reduced to (thrice) punctured spheres, and conversely one can construct Riemann surfaces by sewing punctured spheres. It is interesting to note that Zhu's $g = 1$ theory is *not* formulated by sewing punctured spheres per se, but rather by implementing a conformal map of the complex plane onto a cylinder. Tracing over V has the effect of sewing the ends of the cylinder to obtain a torus. This idea does not generalize to the case when $g = 2$, and we must hew more closely to the sewing approach of conformal field theory. Roughly speaking, what we do is sew tori together in order to obtain a compact Riemann surface S of genus 2 and which is endowed with certain genus 1 data encoded by V via the Zhu theory [T]. There are two essentially different ways to obtain S (which, for simplicity, we take to have no punctures in this work) from genus 1 data: either by sewing a pair of once-punctured complex tori, or by sewing a twice punctured torus to itself (attaching a handle). These two sewing schemes will give rise to seemingly different theories and different definitions of n-point functions. These issues are discussed in detail in the sequel [MT2].

We now give a more technical introduction to the contents of the present paper. Notwithstanding our earlier discussion of the role of vertex operators, they do not appear explicitly in the present work! We are concerned here exclusively with setting up

[1] The general conjecture that the partition function is a modular function if V is rational remains open.

foundations so that the ideas we have been discussing are rigorous and computationally effective. Section 2 records the many modular and elliptic-type functions that we will need. In the paper [Y], which is very important for us, Yamada developed a general approach to computing the period matrix of a Riemann surface S obtained by sewing Riemann surfaces S_1, S_2 (which may coincide) of smaller genus. In Sects. 3 and 4 we develop the theory in the case that S_1 and S_2 are *distinct*. We refer to this as the ϵ *-formalism*, ϵ itself being a complex number which is a part of the data according to which the sewing is performed. (See Fig. 1 below.) We begin in Sect. 3 with some of the details of Yamada's general theory, and make some explicit computations. In particular, we introduce infinite matrices A_a for $a = 1, 2$, whose entries are certain weighted moments of the normalized differential of the second kind on S_a. These matrices determine another infinite matrix X whose entries are weighted moments of the normalized differential of the second kind on S (Proposition 1), and this in turn determines the period matrix $\Omega^{(g)}$ of S (Theorem 3). In particular, the infinite matrix

$$I - A_1A_2 \tag{1}$$

plays an important role (I is the infinite unit matrix). The entries of this matrix depend on data coming from S_a, and in particular they are power series in ϵ. We show (Theorem 2) that (1) has a well-defined determinant $\det(I - A_1A_2)$ which is *holomorphic* for small enough ϵ. Section 3 ends with some additional results concerning the holomorphy of $\det(I - A_1A_2)$ and $\Omega^{(g)}$ in various domains.

In Sect. 4 we study in more detail the case in which the S_a have genus 1, so that they have a modulus $\tau_a \in \mathbb{H}_1$. The triple $(\tau_1, \tau_2, \epsilon)$ determines a genus 2 surface as long as the three parameters in question satisfy a certain elementary inequality. This defines a manifold $\mathcal{D}^\epsilon(\tau_1, \tau_2, \epsilon) \subseteq \mathbb{H}_1 \times \mathbb{H}_1 \times \mathbb{C}$ consisting of all such *admissible triples*. Associating to this data the genus two period matrix $\Omega^{(2)} = \Omega^{(2)}(\tau_1, \tau_2, \epsilon)$ of S defines a map

$$\begin{aligned} F^\epsilon : \mathcal{D}^\epsilon(\tau_1, \tau_2, \epsilon) &\to \mathbb{H}_2 \\ (\tau_1, \tau_2, \epsilon) &\mapsto \Omega^{(2)}(\tau_1, \tau_2, \epsilon) \end{aligned} \tag{2}$$

which is important for everything that follows. When we introduce partition functions in the ϵ-formalism at $g = 2$ in the sequel to the present paper [MT2], they will be functions on $\mathcal{D}^\epsilon$, not $\mathbb{H}_2$. The map F^ϵ interpolates between the two domains. We obtain (Theorem 4) an explicit expression for the genus 2 period matrix

$$\Omega^{(2)}(\tau_1, \tau_2, \epsilon) = \begin{pmatrix} \Omega^{(2)}_{11} & \Omega^{(2)}_{12} \\ \Omega^{(2)}_{12} & \Omega^{(2)}_{22} \end{pmatrix} \tag{3}$$

determined by an admissible triple. Each $\Omega^{(2)}_{ij}$ turns out to be essentially a power series in ϵ with coefficients which are[2] quasimodular forms, i.e., certain polynomials in the Eisenstein series $E_2(\tau_i)$, $E_4(\tau_i)$, $E_6(\tau_i)$ for $i = 1, 2$. Moreover F^ϵ is an analytic map, and we show (Theorem 5) that it is equivariant with respect to the action of a group $G \cong (SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})) \wr \mathbb{Z}_2$ (the wreathed product of $SL(2, \mathbb{Z})$ and $\mathbb{Z}_2$). G embeds into $Sp(4, \mathbb{Z})$ in a standard way, and this defines the action of G on $\mathbb{H}_2$. The action on $\mathcal{D}^\epsilon$ is explained in Subsect. 4.4. These calculations are facilitated by an alternate description (Proposition 4) of the entries of (3) in terms of combinatorial gadgets that

[2] Notation for modular and elliptic-type functions is covered in Sect. 2.

we call *chequered necklaces*. They are certain kinds of graphs with nodes labelled by positive integers and edges labelled by quasimodular forms, and they play a critically important role in the sequel to the present paper. We show (Proposition 5) that about any degeneration point p where $\epsilon = 0$ (i.e., the two tori S_1, S_2 touch at a point), there is a G-invariant neighborhood of p throughout which F^ϵ is invertible.

Sections 5 and 6 are devoted to development of the corresponding formalism in the case that S is obtained by self-sewing (i.e., attaching a handle to) a surface S_1 of one lower genus. We refer to this as the ρ-formalism. Although we are able to achieve results that parallel the development of the ϵ-formalism outlined in the previous paragraph, it is fair to say that of the two, the ρ-formalism is the more complicated. In Sect. 5 we first discuss the results of Yamada (loc. cit.) in a general ρ -formalism, and calculate weighted moments as before. This leads us to introduce the analog of (1), namely

$$I - R, \tag{4}$$

where R is an infinite matrix whose entries are 2×2 block matrices determined by weighted moments of the normalized differential of the second kind on S_1. As before, the entries of R are holomorphic in ρ, and we show (Theorem 7) that $\det(I-R)$ is defined and holomorphic in a certain ρ-domain. The matrix R then determines the period matrix on S (Theorem 8). We also discuss several sewing scenarios for self-sewing a sphere, including one (Proposition 9) where the Catalan series, familiar from combinatorics [St], plays an unexpected role.

In Sect. 6 we investigate in detail the self-sewing of a twice-punctured torus with modulus $\tau \in \mathbb{H}_1$ to form a genus two Riemann surface. As before sewing determines a map

$$\begin{aligned} F^\rho : \mathcal{D}^\rho(\tau, w, \rho) &\to \mathbb{H}_2 \\ (\tau, w, \rho) &\mapsto \Omega^{(2)}(\tau, w, \rho), \end{aligned} \tag{5}$$

where now $\mathcal{D}^\rho(\tau, w, \rho) \subseteq \mathbb{H}_1 \times \mathbb{C} \times \mathbb{C}$ determines the admissible sewing parameters (w describes the relative position of the punctures). Again we obtain (Theorem 9) explicit formulas for the entries of the matrix

$$\Omega^{(2)}(\tau, w, \rho) = \begin{pmatrix} \Omega^{(2)}_{11} & \Omega^{(2)}_{12} \\ \Omega^{(2)}_{12} & \Omega^{(2)}_{22} \end{pmatrix} \tag{6}$$

and show that F^ρ is holomorphic. Roughly speaking, the entries of R and the $\Omega^{(2)}_{ij}$ in this case are power series in ρ with coefficients which are quasimodular and elliptic functions in the variables τ, w. We provide a combinatorial description of (6) in terms of a notion of chequered necklace suitably modified compared to the ϵ case. A complicating factor is that $\Omega^{(2)}_{22}$ involves a logarithm of (the inverse square of) the prime form on S. Because of this, it is necessary to pass to a covering space $\hat{\mathcal{D}}^\rho$ of $\mathcal{D}^\rho$ before equivariance properties can be considered. This is carried-out in Sect. 6.3, where we construct a diagram

$$\begin{array}{ccc} \mathcal{D}^\rho & \xrightarrow{F^\rho} & \mathbb{H}_2 \\ \nwarrow & & \nearrow \hat{F}^\rho \\ & \hat{\mathcal{D}}^\rho & \end{array}$$

We show (Theorem 11) that the map $\hat{F}^{\rho}$ is equivariant with respect to a group L described as a semi-direct product of $SL(2,\mathbb{Z})$ and the (nonabelian) *Heisenberg* group $H \cong \mathbb{Z}^{1+2}$ (a 2-step nilpotent group with center $\mathbb{Z}$). Again L acts on $\mathbb{H}_2$ via an embedding into $Sp(4,\mathbb{Z})$ and on $\hat{\mathcal{D}}^{\rho}$ in a manner prescribed in Theorem 10. In Subsect. 6.4 we obtain the expected local invertibility of F^{ρ} about a point of degeneration, which is a bit more subtle than degeneration in the ϵ-formalism. One of the reasons for establishing the local invertibility results is that once obtained, we have a way of comparing the two sewing domains $\mathcal{D}^{\epsilon}$ and $\mathcal{D}^{\rho}$, at least in some regions, by looking at

$$(F^{\rho})^{-1} \circ F^{\epsilon}. \tag{7}$$

In the final Sect. 7, (7) is briefly discussed, where we observe that it is equivariant with respect to a common subgroup of G and L isomorphic to $SL(2,\mathbb{Z})$. The Appendix concludes with the explicit formulas for $\Omega^{(2)}$ to $O(\epsilon^9)$ in the ϵ-formalism and to $O(\rho^5)$ in the ρ -formalism.

2. Some Elliptic Functions

We briefly discuss a number of modular and elliptic-type functions that we will need. The notation we introduce will be in force throughout the paper. The Weierstrass elliptic function with periods[3] $\sigma, \varsigma \in \mathbb{C}^*$ is defined by

$$\wp(z,\sigma,\varsigma) = \frac{1}{z^2} + \sum_{m,n\in\mathbb{Z},(m,n)\neq(0,0)} \left[\frac{1}{(z-m\sigma-n\varsigma)^2} - \frac{1}{(m\sigma+n\varsigma)^2}\right]. \tag{8}$$

Choosing $\varsigma = 2\pi i$ and $\sigma = 2\pi i\tau$ (τ will *always* lie in the complex upper half-plane $\mathbb{H}$), we define

$$\begin{aligned} P_2(\tau,z) &= \wp(z,2\pi i\tau,2\pi i) + E_2(\tau) \\ &= \frac{1}{z^2} + \sum_{k=2}^{\infty}(k-1)E_k(\tau)z^{k-2}. \end{aligned} \tag{9}$$

Here, $E_k(\tau)$ is equal to 0 for k odd, and for k even is the Eisenstein series [Se]

$$E_k(\tau) = -\frac{B_k}{k!} + \frac{2}{(k-1)!}\sum_{n\geq 1}\sigma_{k-1}(n)q^n.$$

Here and below, we take $q = \exp(2\pi i\tau)$; $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$, and B_k is the k^{th} Bernoulli number defined by

$$\begin{aligned} \frac{t}{e^t-1} - 1 + \frac{t}{2} &= \sum_{k\geq 2} B_k\frac{t^k}{k!} \\ &= \frac{1}{12}t^2 - \frac{1}{720}t^4 + \frac{1}{30240}t^6 + O(t^8). \end{aligned}$$

[3] The period basis is more usually denoted by ω_1, ω_2.

P_2 can be alternatively expressed as

$$P_2(\tau, z) = \frac{q_z}{(q_z - 1)^2} + \sum_{n \geq 1} \frac{nq^n}{1 - q^n}(q_z^n + q_z^{-n}), \tag{10}$$

where $q_z = \exp(z)$. If $k \geq 4$ then $E_k(\tau)$ is a holomorphic modular form of weight k on $SL(2, \mathbb{Z})$. That is, it satisfies

$$E_k(\gamma\tau) = (c\tau + d)^k E_k(\tau)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$, where we use the standard notation

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}. \tag{11}$$

On the other hand, $E_2(\tau)$ has an exceptional transformation law

$$E_2(\gamma\tau) = (c\tau + d)^2 E_2(\tau) - \frac{c(c\tau + d)}{2\pi i}. \tag{12}$$

The first three Eisenstein series $E_2(\tau)$, $E_4(\tau)$, $E_6(\tau)$ are algebraically independent and generate a weighted polynomial algebra $Q = \mathbb{C}[E_2(\tau), E_4(\tau), E_6(\tau)]$ which, following [KZ], we call the algebra of *quasimodular forms*.

We define $P_1(\tau, z)$ by

$$P_1(\tau, z) = \frac{1}{z} - \sum_{k \geq 2} E_k(\tau) z^{k-1}, \tag{13}$$

where $P_2 = -\frac{d}{dz} P_1$ and $P_1 + zE_2$ is the classical Weierstrass zeta function. P_1 is quasi-periodic with

$$\begin{aligned} P_1(\tau, z + 2\pi i) &= P_1(\tau, z), \\ P_1(\tau, z + 2\pi i\tau) &= P_1(\tau, z) - 1. \end{aligned} \tag{14}$$

We also define $P_0(\tau, z)$, up to a choice of the logarithmic branch, by

$$P_0(\tau, z) = -\log(z) + \sum_{k \geq 2} \frac{1}{k} E_k(\tau) z^k, \tag{15}$$

where $P_1 = -\frac{d}{dz} P_0$. We define the elliptic prime form $K(\tau, z)$ by [Mu]

$$K(\tau, z) = \exp(-P_0(\tau, z)), \tag{16}$$

so that $P_2 = \frac{d^2}{dz^2} \log K$. ($\exp(z^2 E_2/2) K(\tau, z)$ is the classical Weierstrass sigma function.) $K(\tau, z)$ is quasi-periodic with

$$\begin{aligned} K(\tau, z + 2\pi i) &= -K(\tau, z), \\ K(\tau, z + 2\pi i\tau) &= -q_z^{-1} q^{-1/2} K(\tau, z). \end{aligned} \tag{17}$$

$K(\tau, z)$ is an odd function of z and can be expressed as

$$K(\tau, z) = -\frac{i\theta_1(\tau, z)}{\eta(\tau)^3} = z + O(z^3), \tag{18}$$

where $\theta_1(\tau, z) = \sum_{n\in\mathbb{Z}} \exp(\pi i \tau(n + 1/2)^2 + (n + 1/2)(z + i\pi))$ and

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n\geq 1}(1 - q^n) \tag{19}$$

is the Dedekind eta function.

Define elliptic functions $P_k(\tau, z)$ for $k \geq 3$ from the analytic expansion

$$P_1(\tau, z - w) = \sum_{k\geq 1} P_k(\tau, z)w^{k-1}, \tag{20}$$

where

$$P_k(\tau, z) = \frac{(-1)^{k-1}}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}P_1(\tau, z) = \frac{1}{z^k} + E_k + O(z). \tag{21}$$

Finally, it is convenient to define for $k, l = 1, 2, \ldots,$

$$C(k, l) = C(k, l, \tau) = (-1)^{k+1}\frac{(k + l - 1)!}{(k - 1)!(l - 1)!}E_{k+l}(\tau), \tag{22}$$

$$D(k, l, z) = D(k, l, \tau, z) = (-1)^{k+1}\frac{(k + l - 1)!}{(k - 1)!(l - 1)!}P_{k+l}(\tau, z). \tag{23}$$

Note that $C(k, l) = C(l, k)$ and $D(k, l, z) = (-1)^{k+l}D(l, k, z)$. These naturally arise in the analytic expansions (in appropriate domains)

$$P_2(\tau, z - w) = \frac{1}{(z - w)^2} + \sum_{k,l\geq 1} C(k, l)z^{l-1}w^{k-1}, \tag{24}$$

and for $k \geq 1$,

$$P_{k+1}(\tau, z) = \frac{1}{z^{k+1}} + \frac{1}{k}\sum_{l\geq 1} C(k, l)z^{l-1}, \tag{25}$$

$$P_{k+1}(\tau, z - w) = \frac{1}{k}\sum_{l\geq 1} D(k, l, w)z^{l-1}. \tag{26}$$

3. The ϵ Formalism for Sewing Together Two Riemann Surfaces

In this section we review a general construction due to Yamada [Y] for "sewing" together two Riemann surfaces of genus g_1 and g_2 to form a surface of genus $g_1 + g_2$. The principle aim is to describe various structures such as the genus $g_1 + g_2$ period matrix in terms of data coming from the genus g_1 and genus g_2 surfaces. The basic method described below follows that of Yamada. However, a significant number of changes have been made in order to express the final formulas more neatly. We also discuss the holomorphic properties of the period matrix and of a certain infinite dimensional determinant. In the next section, this general formalism will be applied to the construction of a genus two Riemann surface.

3.1. The Bilinear Form $\omega^{(g)}$ and the Period Matrix $\Omega^{(g)}$. Consider a compact Riemann surface $\mathcal{S}$ of genus g with canonical homology basis $a_1, \ldots a_g, b_1, \ldots b_g$. In general there exists g holomorphic 1-forms $\nu_i^{(g)}$, $i = 1, \ldots g$ which we may normalize by [FK1, Sp]

$$\oint_{a_i} \nu_j^{(g)} = 2\pi i \delta_{ij}. \tag{27}$$

These forms can be neatly encapsulated in a unique singular bilinear two form $\omega^{(g)}$, known as the *normalized differential of the second kind*. It is defined by the following properties [Sp, Mu, Y]:

$$\omega^{(g)}(x, y) = \left(\frac{1}{(x-y)^2} + \text{regular terms}\right) dxdy \tag{28}$$

for any local coordinates x, y, with normalization

$$\int_{a_i} \omega^{(g)}(x, \cdot) = 0 \tag{29}$$

for $i = 1, \ldots, g$. Using the Riemann bilinear relations, one finds that

$$\nu_i^{(g)}(x) = \oint_{b_i} \omega^{(g)}(x, \cdot), \tag{30}$$

with $\nu_i^{(g)}$normalized as in (27). The genus g period matrix $\Omega^{(g)}$ is then defined by

$$\Omega_{ij}^{(g)} = \frac{1}{2\pi i} \oint_{b_i} \nu_j^{(g)} \tag{31}$$

for $i, j = 1, \ldots, g$. It is useful to also introduce the *normalized differential of the third kind* [Mu, Y]

$$\omega_{p_2 - p_1}^{(g)}(x) = \int_{p_1}^{p_2} \omega^{(g)}(x, \cdot), \tag{32}$$

for which $\oint_{a_i} \omega_{p_2-p_1}^{(g)} = 0$. For a local coordinate x near p_a for $a = 1, 2$ we have

$$\omega_{p_2-p_1}^{(g)}(x) = \left(\frac{(-1)^a}{x - p_a} + \text{regular terms}\right) dx.$$

Both $\omega^{(g)}(x, y)$ and $\omega_{p_2-p_1}^{(g)}(x)$ can be expressed in terms of the *prime form* $K^{(g)}(x, y)$ $(dx)^{-1/2}(dy)^{-1/2}$, a holomorphic form of weight $(-\frac{1}{2}, -\frac{1}{2})$ with [Mu]

$$\omega^{(g)}(x, y) = \partial_x \partial_y \log K^{(g)}(x, y) dxdy, \tag{33}$$

$$\omega_{p_2-p_1}^{(g)}(x) = \partial_x \log \frac{K^{(g)}(x, p_2)}{K^{(g)}(x, p_1)} dx. \tag{34}$$

We also note that $K^{(g)}(x, y) = -K^{(g)}(y, x)$ and that $K^{(g)}(x, y) = x - y + O((x-y)^3)$.

Example 1. For the genus one Riemann torus with periods $2\pi i$ and $2\pi i\tau$ along the a and b cycles, the holomorphic 1-form satisfying (27) in the usual parameterization is $\nu_1^{(1)} = dz$. The normalized differential of the second kind is determined by $P_2(\tau, z)$ via

$$\omega^{(1)}(x, y) = P_2(\tau, x - y)dxdy. \tag{35}$$

In this case, (29) and (30) follow from (14), and $\Omega_{11}^{(1)} = \tau$. The normalized differential of the third kind is $\omega_{p_2-p_1}^{(1)}(x) = (P_1(\tau, x - p_2) - P_1(\tau, x - p_1))dx$ and the prime form is $K^{(1)}(x, y) = K(\tau, x - y)$ of (16).

It is well-known that $\Omega^{(g)}$ is a complex symmetric matrix with positive-definite imaginary part, i.e., $\Omega^{(g)} \in \mathbb{H}_g$, the genus g Siegel complex upper half-space. The intersection form Ξ is a natural non-degenerate symplectic bilinear form on the first homology group $H_1(\mathcal{S}, \mathbb{Z}) \cong \mathbb{Z}^{2g}$, satisfying

$$\Xi(a_i, a_j) = \Xi(b_i, b_j) = 0, \quad \Xi(a_i, b_j) = \delta_{ij}, \quad i, j = 1, \ldots, g.$$

The genus g symplectic group[4] is

$$Sp(2g, \mathbb{Z}) = \{\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2g, \mathbb{Z})| \\ AB^T = BA^T, CD^T = D^T C, AD^T - BC^T = I_g\}.$$

It acts on $\mathbb{H}_g$ via

$$\gamma.\Omega^{(g)} = (A\Omega^{(g)} + B)(C\Omega^{(g)} + D)^{-1}, \tag{36}$$

and naturally on $H_1(\mathcal{S}, \mathbb{Z})$, where it preserves Ξ.

3.2. The ϵ Formalism for Sewing Two Riemann Surfaces. We now discuss a general method described by Yamada [Y] for calculating the bilinear form (28) and hence the period matrix on the surface formed by sewing together two Riemann surfaces. Consider two Riemann surfaces $\mathcal{S}_a$ of genus g_a for $a = 1, 2$. Choose a local coordinate z_a on $\mathcal{S}_a$ in the neighborhood of a point p_a, and consider the closed disk $|z_a| \leq r_a$ for $r_a > 0$ sufficiently small. (Note that the choice $r_a = 1$ is made in ref. [Y]). Introduce a complex sewing parameter ϵ where $|\epsilon| \leq r_1 r_2$, and excise the disk

$$\{z_a, |z_a| \leq |\epsilon|/r_{\bar{a}}\} \subset \mathcal{S}_a$$

to form a punctured surface

$$\hat{\mathcal{S}}_a = \mathcal{S}_a \backslash \{z_a, |z_a| \leq |\epsilon|/r_{\bar{a}}\}.$$

Here and below, we use the convention

$$\overline{1} = 2, \quad \overline{2} = 1. \tag{37}$$

[4] Here and elsewhere, the transpose of a matrix or vector is denoted by T.

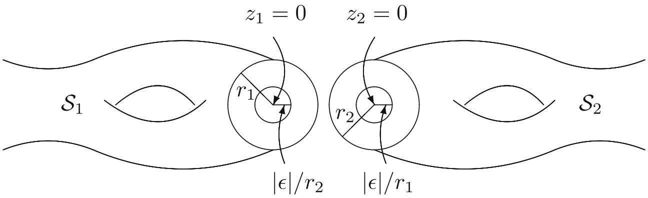

Fig. 1. Sewing Two Riemann Surfaces

Define the annulus

$$\mathcal{A}_a = \{z_a, |\epsilon|/r_{\bar{a}} \leq |z_a| \leq r_a\} \subset \hat{\mathcal{S}}_a,$$

and identify $\mathcal{A}_1$ and $\mathcal{A}_2$ as a single region $\mathcal{A} = \mathcal{A}_1 \simeq \mathcal{A}_2$ via the sewing relation

$$z_1 z_2 = \epsilon. \tag{38}$$

In this way we obtain a compact Riemann surface $\{\hat{\mathcal{S}}_1 \backslash \mathcal{A}_1\} \cup \{\hat{\mathcal{S}}_2 \backslash \mathcal{A}_2\} \cup \mathcal{A}$ of genus $g_1 + g_2$. The sewing relation (38) can be considered to be a parameterization of a cylinder connecting the two punctured Riemann surfaces. Noting the notational differences with ref. [Y], the genus $g_1 + g_2$ normalized differential of the second kind $\omega^{(g_1+g_2)}$ of (28) enjoys the following properties:

Theorem 1 (Ref. [Y], Theorem 1, Theorem 4).

(a) $\omega^{(g_1+g_2)}$ *is holomorphic in* ϵ *for* $|\epsilon| < r_1 r_2$*;*
(b) $\lim_{\epsilon \to 0} \omega^{(g_1+g_2)}(x, y) = \omega^{(g_a)}(x, y)\delta_{ab}$ *for* $x \in \hat{\mathcal{S}}_a$, $y \in \hat{\mathcal{S}}_b$, $a, b = 1, 2$.

Regarded as a power series in ϵ, the coefficients of $\omega^{(g_1+g_2)}$ can be calculated from $\omega^{(g_1)}$ and $\omega^{(g_2)}$ as follows. Let $\mathcal{C}_a(z_a) \subset \mathcal{A}_a$ denote a simple, closed, anti-clockwise oriented contour parameterized by z_a, surrounding the puncture at $z_a = 0$. Note that $\mathcal{C}_1(z_1)$ may be deformed to $-\mathcal{C}_2(z_2)$ via (38). Then one finds [Y]:

Lemma 1 (op.cit., Lemma 4).

$$\omega^{(g_1+g_2)}(x, y) = \omega^{(g_a)}(x, y)\delta_{ab} + \frac{1}{2\pi i} \oint_{\mathcal{C}_a(z_a)} (\omega^{(g_1+g_2)}(y, z_a) \int^{z_a} \omega^{(g_a)}(x, \cdot)) \tag{39}$$

for $x \in \hat{\mathcal{S}}_a$, $y \in \hat{\mathcal{S}}_b$ *and* $a, b = 1, 2$.

Define weighted moments for $\omega^{(g_1+g_2)}$ for $k, l = 1, 2, \ldots$ by

$$\begin{aligned} X_{ab}(k, l) &= X_{ab}(k, l, \epsilon) \\ &= \frac{\epsilon^{(k+l)/2}}{\sqrt{kl}} \frac{1}{(2\pi i)^2} \oint_{\mathcal{C}_a(u)} \oint_{\mathcal{C}_b(v)} u^{-k} v^{-l} \omega^{(g_1+g_2)}(u, v). \end{aligned} \tag{40}$$

The $\epsilon^{(k+l)/2}/\sqrt{kl}$ factor is introduced for later convenience. Note that

$$X_{ab}(k, l) = X_{ba}(l, k) \tag{41}$$

and that $\epsilon^{-(k+l)/2} X_{ab}(k, l, \epsilon)$ is holomorphic in ϵ for $|\epsilon| < r_1 r_2$ from Theorem 1. We define $X_{ab} = (X_{ab}(k, l))$ to be the infinite matrix indexed by k, l.

Next define a set of holomorphic 1-forms on $\hat{\mathcal{S}}_a$ by

$$a_a(k,x) = a_a(k,x,\epsilon) = \frac{\epsilon^{k/2}}{2\pi i\sqrt{k}} \oint_{\mathcal{C}_a(z_a)} z_a^{-k}\omega^{(g_a)}(x,z_a), \tag{42}$$

and define $a_a(x) = (a_a(k,x))$ to be the infinite row vector indexed by k. Note from (28) that for $x, y \in \hat{\mathcal{S}}_a$ with $x \neq 0$ we have

$$\begin{aligned}\omega^{(g_a)}(x,y) &= \sum_{k\geq 1}\left[\frac{1}{2\pi i}\oint_{\mathcal{C}_a(z_a)} z_a^{-k}\omega^{(g_a)}(x,z_a)\right] y^{k-1}dy,\\ &= \sum_{k\geq 1}\sqrt{k}\epsilon^{-k/2}a_a(k,x)y^{k-1}dy.\end{aligned} \tag{43}$$

Using Lemma 1 we have:

Lemma 2. *$\omega^{(g_1+g_2)}(x,y)$ is given by*

$$\omega^{(g_1+g_2)}(x,y) = \begin{cases}\omega^{(g_a)}(x,y) + a_a(x)X_{\bar{a}\bar{a}}a_a^T(y) & x, y \in \hat{\mathcal{S}}_a,\\ a_a(x)(-I + X_{\bar{a}a})a_{\bar{a}}^T(y) & x \in \hat{\mathcal{S}}_a, \quad y \in \hat{\mathcal{S}}_{\bar{a}}.\end{cases} \tag{44}$$

Proof. From (43) it follows that

$$\int_0^{z_a}\omega^{(g_a)}(x,\cdot) = \sum_{k\geq 1}\frac{\epsilon^{-k/2}}{\sqrt{k}}a_a(k,x)z_a^k. \tag{45}$$

Let $x, y \in \hat{\mathcal{S}}_1$. Using (38), (39) and (45) we find that $\omega^{(g_1+g_2)}(x,y) - \omega^{(g_1)}(x,y)$ is given by

$$\begin{aligned}&\sum_{k\geq 1}\frac{\epsilon^{-k/2}}{\sqrt{k}}a_1(k,x)\left(-\frac{\epsilon^k}{2\pi i}\oint_{\mathcal{C}_2(z_2)} z_2^{-k}\omega^{(g_1+g_2)}(y,z_2)\right)\\ &= \sum_{k,l\geq 1}\frac{\epsilon^{(k+l)/2}}{\sqrt{kl}}a_1(k,x)a_1(l,y)\frac{1}{(2\pi i)^2}\oint_{\mathcal{C}_2(w_2)}\oint_{\mathcal{C}_2(z_2)} z_2^{-k}w_2^{-l}\omega^{(g_1+g_2)}(z_2,w_2),\end{aligned} \tag{46}$$

giving (44) for $x, y \in \hat{\mathcal{S}}_1$.

For $x \in \hat{\mathcal{S}}_1$, $y \in \hat{\mathcal{S}}_2$ we find that $\omega^{(g_1+g_2)}(x,y)$ is given by

$$\begin{aligned}&\sum_{k\geq 1}\frac{\epsilon^{-k/2}}{\sqrt{k}}a_1(k,x)\left(-\frac{\epsilon^k}{2\pi i}\oint_{\mathcal{C}_2(z_2)} z_2^{-k}\omega^{(g_1+g_2)}(y,z_2)\right)\\ &= -\sum_{k\geq 1}a_1(k,x)a_2(k,y)\\ &\quad + \sum_{k,l\geq 1}\frac{\epsilon^{(k+l)/2}}{\sqrt{kl}}a_1(k,x)a_2(l,y)\frac{1}{(2\pi i)^2}\oint_{\mathcal{C}_1(z_1)}\oint_{\mathcal{C}_2(z_2)} z_1^{-l}z_2^{-k}\omega^{(g_1+g_2)}(z_1,z_2),\end{aligned} \tag{47}$$

giving (44). A similar analysis follows for $x, y \in \hat{\mathcal{S}}_1$ and $x \in \hat{\mathcal{S}}_2$, $y \in \hat{\mathcal{S}}_1$. □

We next compute the explicit form of the moment matrix X_{ab} in terms of the moments of $\omega^{(g_a)}$, which we denote by

$$A_a(k,l) = A_a(k,l,\epsilon) = \frac{\epsilon^{(k+l)/2}}{(2\pi i)^2\sqrt{kl}} \oint_{\mathcal{C}_a(x)} \oint_{\mathcal{C}_a(y)} x^{-k} y^{-l} \omega^{(g_a)}(x,y)$$
$$= \frac{\epsilon^{k/2}}{2\pi i\sqrt{k}} \oint_{\mathcal{C}_a(x)} x^{-k} a_a(l,x). \tag{48}$$

Note from (28) that for $x, y \in \hat{\mathcal{S}}_a$ we have

$$\omega^{(g_a)}(x,y) - \frac{dxdy}{(x-y)^2}$$
$$= \sum_{k,l\geq 1} \left\{ \frac{1}{(2\pi i)^2} \oint_{\mathcal{C}_a(u)} \oint_{\mathcal{C}_a(v)} u^{-k} v^{-l} \omega^{(g_a)}(u,v) \right\} x^{k-1} y^{l-1} dxdy,$$
$$= \sum_{k,l\geq 1} \sqrt{kl}\,\epsilon^{-(k+l)/2} A_a(k,l,\epsilon) x^{k-1} y^{l-1} dxdy. \tag{49}$$

Proposition 1. *The matrices X_{ab} are given in terms of A_a by*

$$X_{aa} = A_a(I - A_{\bar{a}} A_a)^{-1}, \tag{50}$$
$$X_{a\bar{a}} = I - (I - A_a A_{\bar{a}})^{-1}. \tag{51}$$

Here,

$$(I - A_a A_{\bar{a}})^{-1} = \sum_{n\geq 0} (A_a A_{\bar{a}})^n \tag{52}$$

and is convergent as a power series in ϵ for $|\epsilon| < r_1 r_2$.

Proof. Compute X_{11} from (46) to find

$$X_{11}(k,l) = \frac{\epsilon^{(k+l)/2}}{\sqrt{kl}} \frac{1}{(2\pi i)^2} \oint_{\mathcal{C}_1(x)} \oint_{\mathcal{C}_1(y)} x^{-k} y^{-l} \omega^{(g_1)}(x,y)$$
$$- \sum_{m\geq 1} [\frac{\epsilon^{k/2}}{2\pi i} \frac{1}{\sqrt{k}} \oint_{\mathcal{C}_1(x)} x^{-k} a_1(m,x).$$
$$\frac{\epsilon^{(m+l)/2}}{\sqrt{ml}} \frac{1}{(2\pi i)^2} \oint_{\mathcal{C}_1(y)} \oint_{\mathcal{C}_2(z_2)} y^{-l} z_2^{-m} \omega^{(g_1+g_2)}(y,z_2)],$$

and similarly for X_{22}. Thus using (48) and recalling (41) we have

$$X_{aa} = A_a(I - X_{\bar{a}a}). \tag{53}$$

We may find X_{12} from (47) as follows:

$$X_{12}(k,l) = -\sum_{m\geq 1} \frac{\epsilon^{k/2}}{2\pi i} \frac{1}{\sqrt{k}} \oint_{\mathcal{C}_1(x)} a_1(m,x) x^{-k}$$
$$\left(\frac{\epsilon^{(m+l)/2}}{\sqrt{ml}} \frac{1}{(2\pi i)^2} \oint_{\mathcal{C}_2(y)} \oint_{\mathcal{C}_2(z_2)} y^{-l} z_2^{-m} \omega^{(2)}(y,z_2) \right),$$

and similarly for X_{21}, i.e.,

$$X_{a\bar{a}} = -A_a X_{\bar{a}\bar{a}}. \tag{54}$$

Define infinite block matrices

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}, \qquad A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad Q = \begin{bmatrix} 0 & -A_1 \\ -A_2 & 0 \end{bmatrix} \tag{55}$$

so that (53, 54) can be combined as

$$X = A + QX, \tag{56}$$

so that

$$X = (I-Q)^{-1} A. \tag{57}$$

Here $(I-Q)^{-1} = \sum_{n\geq 0} Q^n$ which we now show converges for $|\epsilon| < r_1 r_2$.

Consider $X = A + AX + Q^2 X$. Then since $Q^2 = \mathrm{diag}(A_1A_2, A_2A_1)$ we obtain iterative relations

$$X_{aa} = A_a(I + A_{\bar{a}} X_{aa}), \tag{58}$$
$$I - X_{a\bar{a}} = I + A_a A_{\bar{a}}(I - X_{a\bar{a}}). \tag{59}$$

Now $\epsilon^{-(k+l)/2} X_{ab}(k,l)$ is holomorphic in ϵ for $|\epsilon| < r_1 r_2$ by Theorem 1. Therefore, $X_{ab}(k,l)$ has a series expansion in $\epsilon^{1/2}$ convergent for $|\epsilon| < r_1 r_2$. Then (58) implies $X_{aa}(k,l) = (\sum_{n=0}^{N} (A_a A_{\bar{a}})^n A_a)(k,l) + O(\epsilon^{(k+l)/2+2N+1})$ where the coefficient of each power of ϵ consists of a finite sum of finite products of A_1 and A_2. Hence

$$X_{aa} = \sum_{n=0}^{\infty} A_a (A_{\bar{a}} A_a)^n = A_a (I - A_{\bar{a}} A_a)^{-1},$$

converges for $|\epsilon| < r_1 r_2$. A similar argument holds for $X_{a\bar{a}}$ where one finds

$$X_{a\bar{a}} = \sum_{n=1}^{\infty} (A_a A_{\bar{a}})^n = I - (I - A_a A_{\bar{a}})^{-1}$$

converges for $|\epsilon| < r_1 r_2$. Finally

$$(I-Q)^{-1} = \sum_{n\geq 0} Q^n = \sum_{n\geq 0} Q^{2n}(I+Q)$$

is therefore also convergent for $|\epsilon| < r_1 r_2$. $\square$

The invertibility of the infinite matrix $I - A_1A_2$ for $|\epsilon| < r_1r_2$ is crucial in the ϵ sewing formalism. We now define an infinite determinant of $I - A_1A_2$ which we show is a holomorphic function of ϵ for $|\epsilon| < r_1r_2$. This determinant plays a dominant role in the sequel to this work [MT2]. Firstly, since $A_1(k,m)A_2(m,l) = O(\epsilon^{m+(k+l)/2})$ we may define a $(2N-3)\times(2N-3)$ matrix

$$T_N(k,l) = \sum_{1\le m\le N-(k+l)/2} A_1(k,m)A_2(m,l), \tag{60}$$

for $1 \le k, l \le 2N-3$. T_N is a truncated approximation for A_1A_2 to $O(\epsilon^N)$,

$$A_1A_2 = \begin{pmatrix} T_N & 0 \cdots \\ 0 & 0 \cdots \\ \vdots & \vdots \ddots \end{pmatrix} + O(\epsilon^{N+1}).$$

We may then define as formal power series in ϵ to $O(\epsilon^N)$ the expressions[5]

$$\det(I - A_1A_2) = \det(I_N - T_N) + O(\epsilon^{N+1}), \tag{61}$$

$$Tr\log(I - A_1A_2) = Tr\log(I_N - T_N) + O(\epsilon^{N+1}), \tag{62}$$

where $Tr\log(I_N - T_N) = -\sum_{n=1}^{N/2}\frac{1}{n}Tr(T_N^n) + O(\epsilon^{N+1})$. Comparing the finite matrix contributions of (61) and (62) we have

Lemma 3. *As formal power series in* ϵ,

$$\log\det(I - A_1A_2) = Tr\log(I - A_1A_2). \tag{63}$$

We now show that $Tr\log(I - A_1A_2)$ is holomorphic in ϵ for $|\epsilon| < r_1r_2$ so that:

Theorem 2. $\det(I - A_1A_2)$ *is non-vanishing and holomorphic in* ϵ *for* $|\epsilon| < r_1r_2$.

Proof. Let $\omega^{(g_1+g_2)} = f(z_1, z_2, \epsilon)dz_1dz_2$ for $|z_a| \le r_a$. Then $f(z_1, z_2, \epsilon)$ is holomorphic in ϵ for $|\epsilon| \le r$ for $r < r_1r_2$ from Theorem 1. Apply Cauchy's inequality to the coefficient functions for $f(z_1, z_2, \epsilon) = \sum_{n\ge0} f_n(z_1, z_2)\epsilon^n$ to find

$$|f_n(z_1, z_2)| \le \frac{M}{r^n}, \tag{64}$$

for $M = \sup_{|z_a|\le r_a, |\epsilon|\le r} |f(z_1, z_2, \epsilon)|$. Consider

$$\mathcal{I} = \frac{1}{(2\pi i)^2}\oint_{\mathcal{C}_{r_1}(z_1)}\oint_{\mathcal{C}_{r_2}(z_2)} \omega^{(g_1+g_2)}(z_1, z_2)\log(1 - \frac{\epsilon}{z_1z_2}), \tag{65}$$

for $\mathcal{C}_{r_a}(z_a)$ the contour with $|z_a| = r_a$. Then using (64) we find

$$\begin{aligned} |\mathcal{I}| &\le \sum_{n\ge0}\frac{|\epsilon|^n}{(2\pi)^2}\oint_{\mathcal{C}_{r_1}(z_1)}\oint_{\mathcal{C}_{r_2}(z_2)} |f_n(z_1, z_2)\log(1 - \frac{\epsilon}{z_1z_2})dz_1dz_2| \\ &\le \sum_{n\ge0} M.\frac{|\epsilon|^n}{r^n}.|\log\left(1 - \frac{|\epsilon|}{r_1r_2}\right)|.r_1r_2, \end{aligned}$$

[5] For the sake of notational simplicity we denote *both* the usual finite dimensional and the defined infinite dimensional determinants by det.

i.e., $\mathcal{I}$ is absolutely convergent and thus holomorphic in ϵ for $|\epsilon| \leq r < r_1 r_2$. Since $|z_1 z_2| = r_1 r_2$ we may alternatively expand in $\epsilon/z_1 z_2$ to obtain

$$\begin{aligned}\mathcal{I} &= -\sum_{k\geq 1} \frac{\epsilon^k}{k} \frac{1}{(2\pi i)^2} \oint_{\mathcal{C}_{r_1}(z_1)} \oint_{\mathcal{C}_{r_2}(z_2)} \omega^{(g_1+g_2)}(z_1, z_2) z_1^{-k} z_2^{-k} \\ &= -Tr X_{12},\end{aligned}$$

where $Tr X_{12} = \sum_{k\geq 1} X_{12}(k,k)$ for X_{12} of (51). But (59) implies

$$Tr X_{12} = -\sum_{n\geq 1} Tr((A_1 A_2)^n),$$

is absolutely convergent for $|\epsilon| < r_1 r_2$. Hence we find

$$Tr \log(I - A_1 A_2) = -\sum_{n\geq 1} \frac{1}{n} Tr((A_1 A_2)^n),$$

is also absolutely convergent for $|\epsilon| < r_1 r_2$. Thus from Lemma 3, $\det(I - A_1 A_2)$ is non-vanishing and holomorphic for $|\epsilon| < r_1 r_2$. □

These determinant properties can also be expressed in terms of the block matrix Q using[6]

Lemma 4. $\det(I \pm Q) = \det(I - A_1 A_2)$.

Proof. Let Q_N be the truncated approximation for Q to $O(\epsilon^N)$. Then one finds $\det(I + Q_N) = \det(I - Q_N)$. But $\det(I + Q_N)\det(I - Q_N) = \det(I - Q_N^2) = \det(I - T_N)^2$ for T_N of (60) and the result follows. □

The sewn genus $g_1 + g_2$ Riemann surface naturally inherits a basis of cycles labelled $\{a_{s_1}, b_{s_1} | s_1 = 1, \ldots, g_1\}$ and $\{a_{s_2}, b_{s_2} | s_2 = g_1 + 1, \ldots, g_1 + g_2\}$ from the genus g_1 and genus g_2 surfaces respectively. Integrating $\omega^{(g_1+g_2)}$ along the b cycles then gives us the holomorphic 1-forms and period matrix. For $a = 1, 2$ we define

$$\alpha_{s_a}^{(g_a)}(k) = \oint_{b_{s_a}} a_a(k), \tag{66}$$

and the infinite vector $\alpha_{s_a}^{(g_a)} = (\alpha_{s_a}^{(g_a)}(k))$. Then we find using (30) and (31) together with Lemma 2 and Proposition 1 that [Y]:

Theorem 3 (op. cit. Theorem 4.). *$\Omega^{(g_1+g_2)}$ is holomorphic in ϵ for $|\epsilon| < r_1 r_2$, and is given by*

$$2\pi i \Omega_{s_1 t_1}^{(g_1+g_2)} = 2\pi i \Omega_{s_1 t_1}^{(g_1)} + \alpha_{s_1}^{(g_1)}(A_2(I - A_1 A_2)^{-1})\alpha_{t_1}^{(g_1)T}, \tag{67}$$

$$2\pi i \Omega_{s_2 t_2}^{(g_1+g_2)} = 2\pi i \Omega_{s_2 t_2}^{(g_2)} + \alpha_{s_2}^{(g_2)}(A_1(I - A_1 A_2)^{-1})\alpha_{t_2}^{(g_2)T}, \tag{68}$$

$$\begin{aligned}2\pi i \Omega_{s_1 s_2}^{(g_1+g_2)} &= -\alpha_{s_1}^{(g_1)}(I - A_2 A_1)^{-1}\alpha_{s_2}^{(g_2)T} \\ &= -\alpha_{s_2}^{(g_2)}(I - A_1 A_2)^{-1}\alpha_{s_1}^{(g_1)T},\end{aligned} \tag{69}$$

with $s_1, t_1 \in \{1, \ldots, g_1\}$ and $s_2, t_2 \in \{g_1 + 1, \ldots, g_1 + g_2\}$.

[6] See the previous footnote.

Example 2. Let $\mathcal{S}_1$ be a genus g surface and $\mathcal{S}_2$ the Riemann sphere $\mathbb{C} \cup \{\infty\}$ with bilinear form

$$\omega^{(0)}(x, y) = \frac{dxdy}{(x - y)^2}, \quad x, y \in \mathcal{S}_2. \tag{70}$$

Choose $p_2 = 0$ with $z_2 \in \mathbb{C}$ as the local coordinate on $\mathcal{S}_2$. Then $a_2(k, x) = \sqrt{k}\epsilon^{k/2}x^{-k-1}$ dx from (42) and hence $A_2 = 0$ from (48). Thus $X_{22} = A_1$ and $X_{11} = X_{12} = X_{21} = 0$. Then one can check that the RHS of (44) reproduces $\omega^{(g)}$ directly for $a = b = 1$ whereas, using (38), it follows for $a = b = 2$ from (49) and for $a \neq b$ from (43).

Let us now consider the holomorphic properties of $\Omega^{(g_1+g_2)}$. From Theorem 1, $\omega^{(g_1+g_2)}(x, y)$ is holomorphic in ϵ for $|\epsilon| < r_1 r_2$ and therefore $\Omega^{(g_1+g_2)}$ is also. We now show that if $\omega^{(g_a)}$ is holomorphic with respect to a complex parameter (such as one of the modular parameters of the Riemann surface $\mathcal{S}_a$) then $\omega^{(g_1+g_2)}$ and therefore $\Omega^{(g_1+g_2)}$ is also holomorphic with respect to that parameter. To this end we firstly prove the following elementary lemma:

Lemma 5. *Let $f(x, y)$ be a complex function holomorphic in x for $|x| < R$ with expansion $f(x, y) = \sum_{m \geq 0} c_m(y)x^m$. Suppose that each $c_m(y)$ is holomorphic and that $f(x, y)$ is continuous in y for $|y| < S$. Then $f(x, y)$ is also holomorphic in y for $|y| < S$.*

Proof. Define the compact region $\mathcal{R} = \{(x, y) : |x| \leq R_-, |y| \leq S_-\}$ for $R_- = R - \delta_1$ and $S_- = S - \delta_2$ for $\delta_1, \delta_2 > 0$. f is continuous in the compact region $\mathcal{R}$ and hence $|f(x, y)| \leq M \equiv \sup_{\mathcal{R}} |f|$. Apply Cauchy's inequality to f as a holomorphic function of x for $|x| \leq R_-$ to find

$$|c_m(y)| \leq \frac{\sup_{|x| \leq R_-} |f(x, y)|}{R_-^m} \leq \frac{M}{R_-^m}.$$

But $c_m(y)$ is holomorphic for $|y| \leq S_-$ with expansion $c_m(y) = \sum_{n \geq 0} c_{mn} y^n$ so that applying Cauchy's inequality again gives

$$|c_{mn}| \leq \frac{\sup_{|y| \leq S_-} |c_m(y)|}{S_-^n} \leq \frac{M}{R_-^m S_-^n}.$$

Thus $f(x, y) = \sum_{m \geq 0} \sum_{n \geq 0} c_{mn} x^m y^n$ is absolutely convergent for $|x| < R_-, |y| < S_-$. Hence $c_n(x) = \sum_{m \geq 0} c_{mn} x^m$ converges for $|x| < R_-$ and f is holomorphic in y with convergent expansion $f(x, y) = \sum_{n \geq 0} c_n(x) y^n$ for $|y| < S_-$. We may then choose δ_1, δ_2 sufficiently small to show that the result follows for all $|x| < R$, $|y| < S$. □

Proposition 2. *Suppose that $\omega^{(g_a)}$ is a holomorphic function of a complex parameter μ for $|\mu| < S$. Then for $|\epsilon| < r_1 r_2$, $\omega^{(g_1+g_2)}$ is also holomorphic in μ for $|\mu| < S$.*

Proof. Suppose that $\omega^{(g_1)}$ is holomorphic in μ wlog. Then $a_1(k)$ and $A_1(k, l)$ are holomorphic (and continuous) in μ for $|\mu| < S$. We now show that X_{ab} is holomorphic in μ for $|\mu| < S$ using Lemma 5. Using continuity of A_1 and (58) of Proposition 1 we find that for $|\mu + \delta| < S$,

$$(I - A_2 A_1) \lim_{\delta \to 0} (X_{11}(\mu + \delta) - X_{11}(\mu)) = 0.$$

But $(I - A_2A_1)$ is invertible for $|\epsilon| < r_1r_2$ from Proposition 1 and so $X_{11}(\mu)$ is continuous for $|\mu| < S$. A similar result holds for X_{12}, X_{21} and X_{22}. From Theorem 1, $\epsilon^{-(k+l)/2}X_{ab}(k,l)$ is holomorphic in ϵ for $|\epsilon| < r_1r_2$. Furthermore, as explained in Proposition 1, the ϵ expansion coefficients consist of a finite sum of finite products of A_1 and A_2 terms and thus they are holomorphic in μ for $|\mu| < S$. We may therefore apply Lemma 5 to $\epsilon^{-(k+l)/2}X_{ab}(k,l)$ which is continuous in μ for $|\mu| < S$ and holomorphic in ϵ for $|\epsilon| < R = r_1r_2$ with ϵ expansion coefficients holomorphic in μ for $|\mu| < S$. Thus $\epsilon^{-(k+l)/2}X_{ab}(k,l)$ and therefore $X_{ab}(k,l)$ is holomorphic in μ for $|\mu| < S$.

Finally consider $\omega^{(g_1+g_2)}$ as given in (44) of Lemma 2. Using arguments similar to those above we see that $\omega^{(g_1+g_2)}$ is continuous in μ for $|\mu| < S$ and holomorphic in ϵ for $|\epsilon| < r_1r_2$ with ϵ expansion coefficients holomorphic in μ for $|\mu| < S$. Thus $\omega^{(g_1+g_2)}$ is holomorphic in μ for $|\mu| < S$. □

Corollary 1. *Given the previous conditions then $\Omega^{(g_1+g_2)}$ is holomorphic in μ for $|\mu| < S$ where $|\epsilon| < r_1r_2$.*

In conclusion, we similarly find by applying Proposition 2 to (65) that

Proposition 3. *Suppose that $\omega^{(g_a)}$ is holomorphic in μ for $|\mu| < S$. Then* $\det(I - A_1A_2)$ *is non-vanishing and holomorphic in μ for $|\mu| < S$ and $|\epsilon| < r_1r_2$.*

4. Sewing Two Tori to Form a Genus Two Riemann Surface

We now specialize to the case of two tori sewn together to form a genus two Riemann surface. We first consider an elementary description of a disk on a torus compatible with $SL(2,\mathbb{Z})$ modular-invariance. We then apply the ϵ formalism in order to sew two punctured tori with modular parameters τ_1 and τ_2 together to form a genus two Riemann surface with period matrix $\Omega^{(2)}(\tau_1,\tau_2,\epsilon) \in \mathbb{H}_2$, where $\Omega^{(2)}$ is holomorphic in $(\tau_1,\tau_2,\epsilon) \in \mathcal{D}^\epsilon$ for a suitably defined domain $\mathcal{D}^\epsilon$. We provide an alternative description of $\Omega^{(2)}$ in terms of the sum of weights of particular "necklace" graphs. We then describe the equivariance properties of this holomorphic mapping from $\mathcal{D}^\epsilon$ to $\mathbb{H}_2$ with respect to a certain subgroup $G \subseteq Sp(4,\mathbb{Z})$, and prove that it is invertible in a certain G-invariant domain.

4.1. A Closed Disk on a Torus. A complex torus $\mathcal{S}$ (that is, a compact Riemann surface of genus 1), can be represented as a quotient $\mathbb{C}/\Lambda$, where Λ is a lattice in $\mathbb{C}$. Moreover, two such tori $\mathcal{S}_a, a = 1,2$ are isomorphic if, and only if, the lattices are *homothetic* , that is, there is a $\xi \in \mathbb{C}^*$ such that $\Lambda_2 = \xi\Lambda_1$.

A *framing* of $\mathcal{S} = \mathbb{C}/\Lambda$ is a choice of basis (σ,ς) such that the modulus $\tau = \sigma/\varsigma$ satisfies $\tau \in \mathbb{H}_1$. We say that the basis (σ,ς) is *positively oriented* in this case. A pair of framed tori $(\mathbb{C}/\Lambda_a, \sigma_a, \varsigma_a), a = 1,2$ are isomorphic if, and only if, there is a ξ as above such that $(\sigma_2,\varsigma_2) = \xi(\sigma_1,\varsigma_1)$. The modulus τ depends only on the isomorphism class of the framed torus, and there is a *bijection*

$$\begin{aligned}\{\text{isomorphism classes of framed tori}\} &\to \mathbb{H}_1, \\ (\mathbb{C}/\Lambda,\sigma,\varsigma) &\mapsto \sigma/\varsigma.\end{aligned} \tag{71}$$

$SL(2,\mathbb{Z})$ is the group of automorphisms of Λ which preserves oriented bases. It acts on isomorphism classes of framed tori via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (\mathbb{C}/\Lambda,\sigma,\varsigma) \mapsto (\mathbb{C}/\Lambda, a\sigma + b\varsigma, c\sigma + d\varsigma),$$

and on $\mathbb{H}_1$ via fractional linear transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tau \mapsto \frac{a\tau + b}{c\tau + d}.$$

With respect to these two actions, the bijection (71) is $SL(2, \mathbb{Z})$-equivariant.

In the following it is convenient to identify $\mathcal{S}$ with the standard fundamental region for Λ determined by the basis, and with appropriate identifications of boundary. To describe a well-defined disk in $\mathcal{S}$, define the minimal length of Λ as

$$D(\Lambda) = \min_{0 \neq \lambda \in \Lambda} |\lambda|. \tag{72}$$

It obviously satisfies

$$D(\xi\Lambda) = |\xi| D(\Lambda). \quad (\xi \neq 0). \tag{73}$$

We may now describe a closed disk on $\mathcal{S}$. The proof follows from the triangle inequality.

Lemma 6. *For $p \in \mathcal{S}$, the points $z \in \mathcal{S}$ satisfying $|z - p| \leq kD(\Lambda)$ define a closed disk centered at p provided $k < \frac{1}{2}$.*

Let $\mathcal{S}$ be a complex torus of modulus τ. Among all homothetic lattices Λ for which we define $\mathcal{S} \cong \mathbb{C}/\Lambda$, it will be convenient to work with the lattice Λ_τ which has basis $(2\pi i\tau, 2\pi i)$. Note that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ we have

$$D(\Lambda_{\gamma\tau}) = \frac{1}{|c\tau + d|} D(\Lambda_\tau). \tag{74}$$

4.2. The Genus Two Period Matrix in the ϵ Formalism. We now apply the ϵ-formalism to a pair of tori $\mathcal{S}_a = \mathbb{C}/\Lambda_a$ with local co-ordinates z_a, where Λ_a has oriented basis (σ_a, ς_a) and $\tau_a = \sigma_a/\varsigma_a \in \mathbb{H}_1$ for $a = 1, 2$. We shall sometimes refer to $\mathcal{S}_1$ and $\mathcal{S}_2$ as the left and right torus respectively. After Lemma 6 we may consider the annuli $\mathcal{A}_a$ centred at the origin of $\mathcal{S}_a$ described above, with outer radius $r_a < \frac{1}{2} D(\Lambda_a)$. Following the prescription of Subsect. 3.2, we sew the two tori by identifying the annuli $\mathcal{A}_1$ and $\mathcal{A}_2$ via the relation $z_1 z_2 = \epsilon$ as in (38), where $|\epsilon| \leq r_1 r_2 < \frac{1}{4} D(\Lambda_1) D(\Lambda_2)$.

As discussed in Subsect. 4.1 we take $(\sigma_a, \varsigma_a) = (2\pi i\tau_a, 2\pi i)$ and $q_a = \exp(2\pi i\tau_a)$ for $a = 1, 2$. Define the domain[7]

$$\mathcal{D}^\epsilon = \{(\tau_1, \tau_2, \epsilon) \in \mathbb{H}_1 \times \mathbb{H}_1 \times \mathbb{C} \mid |\epsilon| < \frac{1}{4} D(\Lambda_{\tau_1}) D(\Lambda_{\tau_2})\}. \tag{75}$$

We now explicitly determine the period matrix[8].

[7] The superscript ϵ merely denotes that we are working in the ϵ-formalism, and should not be interpreted as a variable of any kind.

[8] The genus two period matrix is also described in [T] without proof and in a different notation.

Theorem 4. *Sewing determines a holomorphic map*

$$F^{\epsilon} : \mathcal{D}^{\epsilon} \to \mathbb{H}_2,$$
$$(\tau_1, \tau_2, \epsilon) \mapsto \Omega^{(2)}(\tau_1, \tau_2, \epsilon). \tag{76}$$

Moreover $\Omega^{(2)} = \Omega^{(2)}(\tau_1, \tau_2, \epsilon)$ *is given by*

$$2\pi i \Omega^{(2)}_{11} = 2\pi i \tau_1 + \epsilon(A_2(I - A_1 A_2)^{-1})(1, 1), \tag{77}$$
$$2\pi i \Omega^{(2)}_{22} = 2\pi i \tau_2 + \epsilon(A_1(I - A_2 A_1)^{-1})(1, 1), \tag{78}$$
$$2\pi i \Omega^{(2)}_{12} = -\epsilon(I - A_1 A_2)^{-1}(1, 1). \tag{79}$$

Notation here is as follows: $A_a(\tau_a, \epsilon)$ *is the infinite matrix with* (k, l)*-entry,*

$$A_a(k, l, \tau_a, \epsilon) = \frac{\epsilon^{(k+l)/2}}{\sqrt{kl}} C(k, l, \tau_a); \tag{80}$$

$(1, 1)$ *refers to the* $(1, 1)$*-entry of a matrix with* $C(k, l, \tau)$ *of (22).*

Proof. The bilinear two form $\omega^{(1)}$ is given by (35). Using (20), the basis of 1-forms (42) with periods $(2\pi i \tau_a, 2\pi i)$ is then given by

$$a_a(k, x) = \frac{\epsilon^{k/2} dx}{2\pi i \sqrt{k}} \oint_{\mathcal{C}_a(z)} z^{-k} P_2(\tau_a, x - z) dz,$$
$$= \sqrt{k} \epsilon^{k/2} P_{k+1}(\tau_a, x) dx. \tag{81}$$

Now (80) follows from (48), (25) and (22). Note that (14) implies that $\alpha_a(k)$ of (66) is

$$\alpha_a(k) = \epsilon^{1/2} \delta_{k,1}. \tag{82}$$

We therefore find $\Omega^{(2)}$ to be given by (77)–(79) for $|\epsilon| < r_1 r_2 < \frac{1}{4} D(\Lambda_{\tau_1}) D(\Lambda_{\tau_2})$.

By Theorem 3, $\Omega^{(2)}$ is holomorphic in ϵ for $|\epsilon| < r_1 r_2 < \frac{1}{4} D(\Lambda_{\tau_1}) D(\Lambda_{\tau_2})$. The left torus bilinear form $\omega^{(1)}(x, y, \tau_1)$ is holomorphic in some neighborhood $|\tau_1 - \tau_1^0| < S$ of any point $\tau_1^0 \in \mathbb{H}_1$. Hence we may apply Corollary 1 with $\mu = \tau_1 - \tau_1^0$ for $|\mu| < S$ to conclude that $\Omega^{(2)}$ is holomorphic in τ_1 . Similarly $\Omega^{(2)}$ is holomorphic in τ_2, and by Hartog's theorem (e.g., [Gu]) $\Omega^{(2)}$ is holomorphic on $\mathcal{D}^{\epsilon}$. □

The infinite matrices $A_a(\tau_a, \epsilon)$ will play a crucial rôle in the further analysis of the ϵ formalism. Dropping the subscript, they are symmetric and have the form:

$$A(\tau, \epsilon) = \begin{pmatrix} \epsilon E_2(\tau) & 0 & \sqrt{3}\epsilon^2 E_4(\tau) & 0 & \cdots \\ 0 & -3\epsilon^2 E_4(\tau) & 0 & -5\sqrt{2}\epsilon^3 E_6(\tau) & \cdots \\ \sqrt{3}\epsilon^2 E_4(\tau) & 0 & 10\epsilon^3 E_6(\tau) & 0 & \cdots \\ 0 & -5\sqrt{2}\epsilon^3 E_6(\tau) & 0 & -35\epsilon^4 E_8(\tau) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

4.3. Chequered Necklace Expansion for $\Omega^{(2)}$. It is useful to introduce an interpretation for the expressions for $\Omega^{(2)}$ found above in terms of the sum of weights of certain graphs. Let us introduce the set of *chequered necklaces* $\mathcal{N}$. By definition, these are connected graphs with $n \geq 2$ nodes, $(n-2)$ of which have valency 2 and two of which have valency 1 (these latter are the *end nodes*), together with an orientation, say from left to right, on the edges. Moreover vertices are labelled by positive integers and edges are labelled alternatively by 1 or 2 as one moves along the graph, e.g.,

$$\overset{k_1}{\bullet} \overset{1}{\longrightarrow} \overset{k_2}{\bullet} \overset{2}{\longrightarrow} \overset{k_3}{\bullet} \overset{1}{\longrightarrow} \overset{k_4}{\bullet} \overset{2}{\longrightarrow} \overset{k_5}{\bullet} \overset{1}{\longrightarrow} \overset{k_6}{\bullet}$$

We also define the *degenerate necklace* N_0 to be a single node with no edges. Define a weight function

$$\omega : \mathcal{N} \longrightarrow \mathbb{C}[E_2(\tau_a), E_4(\tau_a), E_6(\tau_a), \epsilon \mid a = 1, 2],$$

as follows: if a chequered necklace N has edges E labelled as $\overset{k}{\bullet} \overset{a}{\longrightarrow} \overset{l}{\bullet}$ then we define

$$\begin{aligned} \omega(E) &= A_a(k, l, \tau_a, \epsilon), \\ \omega(N) &= \prod \omega(E), \end{aligned} \tag{83}$$

where $A_a(k, l, \tau_a, \epsilon)$ is given by (80) and the product is taken over all edges E of N. We further define $\omega(N_0) = 1$.

Among all chequered necklaces there is a distinguished set for which both end nodes are labelled by 1. There are four types of such chequered necklaces, which may be further distinguished by the labels of the two edges at the extreme left and right. We use the notation (37) for $a = 1, 2$, and say that the chequered necklace

$$\overset{1}{\bullet} \overset{a}{\longrightarrow} \overset{i}{\bullet} \ldots \overset{j}{\bullet} \overset{b}{\longrightarrow} \overset{1}{\bullet}$$

is of *type* $\overline{a}\overline{b}$. We then set

$$\begin{aligned} \mathcal{N}_{ab} &= \{\text{isomorphism classes of chequered necklaces of type } ab\}, \\ \omega_{ab} &= \sum_{N \in \mathcal{N}_{ab}} \omega(N), \end{aligned}$$

where ω_{ab} is considered as an element in $\mathbb{C}[E_2(\tau_a), E_4(\tau_a), E_6(\tau_a), \epsilon \mid a = 1, 2]$. It is clear that we may use this formalism to represent matrix expressions like those appearing earlier. Then we have

Lemma 7. *For* $a = 1, 2$ *we have*

$$\begin{aligned} \omega_{a\bar{a}} &= \omega_{\bar{a}a} = (I - A_a A_{\bar{a}})^{-1}(1, 1), \\ \omega_{aa} &= (A_{\bar{a}}(I - A_a A_{\bar{a}})^{-1})(1, 1). \end{aligned}$$

Thus we may conclude from Theorem 4 that

Proposition 4. *For* $a = 1, 2$ *we have*

$$\begin{aligned} \Omega^{(2)}_{aa} &= \tau_a + \frac{\epsilon}{2\pi i}\omega_{aa}, \\ \Omega^{(2)}_{a\bar{a}} &= -\frac{\epsilon}{2\pi i}\omega_{a\bar{a}}. \end{aligned}$$

4.4. Equivariance of F^{ϵ}. In Theorem 4 we established the existence of the analytic map F^{ϵ}. Here we establish the equivariance of this map with respect to a certain subgroup G of $Sp(4, \mathbb{Z})$. We will employ the graphical representation for $\Omega = \Omega^{(2)}(\tau_1, \tau_2, \epsilon)$ in terms of chequered necklaces discussed in the last subsection.

As an abstract group, G is isomorphic to $(SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})) \rtimes \mathbb{Z}_2$, i.e., the direct product of two copies of $SL(2, \mathbb{Z})$ which are interchanged upon conjugation by an involution. There is a natural injection $G \to Sp(4, \mathbb{Z})$ in which the two $SL(2, \mathbb{Z})$ subgroups are mapped to

$$\Gamma_1 = \left\{ \begin{bmatrix} a_1 & 0 & b_1 & 0 \\ 0 & 1 & 0 & 0 \\ c_1 & 0 & d_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\}, \quad \Gamma_2 = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & b_2 \\ 0 & 0 & 1 & 0 \\ 0 & c_2 & 0 & d_2 \end{bmatrix} \right\}, \tag{84}$$

and the involution is mapped to

$$\beta = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{85}$$

In this way we obtain a natural action of G on $\mathbb{H}_2$. The action on $\mathcal{D}^{\epsilon}$ is described in the next lemma.

Lemma 8. *G has a left action on $\mathcal{D}^{\epsilon}$ as follows:*

$$\gamma_1.(\tau_1, \tau_2, \epsilon) = (\gamma_1\tau_1, \tau_2, \frac{\epsilon}{c_1\tau_1 + d_1}), \tag{86}$$

$$\gamma_2.(\tau_1, \tau_2, \epsilon) = (\tau_1, \gamma_2\tau_2, \frac{\epsilon}{c_2\tau_2 + d_2}), \tag{87}$$

$$\beta.(\tau_1, \tau_2, \epsilon) = (\tau_2, \tau_1, \epsilon), \tag{88}$$

for $(\gamma_1, \gamma_2) \in SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$.

Proof. It is straightforward (and quite standard) to see that (86) - (88) formally define a (left) action of G on $\mathbb{H}_1 \times \mathbb{H}_1 \times \mathbb{C}$. What we must show is that this action preserves the domain $\mathcal{D}^{\epsilon}$. For β this is obvious, and for elements (γ_1, γ_2) it follows from (74). □

We now establish the following result:

Theorem 5. *F^{ϵ} is equivariant with respect to the action of G, i.e., there is a commutative diagram for $\gamma \in G$,*

$$\begin{array}{ccc} \mathcal{D}^{\epsilon} & \stackrel{F^{\epsilon}}{\to} & \mathbb{H}_2 \\ \gamma \downarrow & & \downarrow \gamma \\ \mathcal{D}^{\epsilon} & \stackrel{F^{\epsilon}}{\to} & \mathbb{H}_2 \end{array}$$

Proof. Fix $(\tau_1, \tau_2, \epsilon) \in \mathcal{D}^{\epsilon}$, with $\Omega = F^{\epsilon}(\tau_1, \tau_2, \epsilon) = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12} & \Omega_{22} \end{pmatrix}$. Of course, each Ω_{ij} is a function of $(\tau_1, \tau_2, \epsilon)$. The action of G on $\mathbb{H}_2$ is given in (36), and in particular $\beta : \Omega \mapsto \begin{pmatrix} \Omega_{22} & \Omega_{12} \\ \Omega_{12} & \Omega_{11} \end{pmatrix}$. Therefore from (88) we have

$$F^{\epsilon}(\beta(\tau_1, \tau_2, \epsilon)) = F^{\epsilon}(\tau_2, \tau_1, \epsilon) = \begin{pmatrix} \Omega_{22} & \Omega_{12} \\ \Omega_{12} & \Omega_{11} \end{pmatrix} = \beta(F^{\epsilon}(\tau_1, \tau_2, \epsilon)).$$

So the theorem is true in case $\gamma = \beta$. To complete the proof of the theorem, it suffices to consider the case when $\gamma = \gamma_1$ lies in the 'left' modular group acting on τ_1. From (36),

$$\gamma_1 : \Omega \mapsto \begin{pmatrix} \frac{a_1\Omega_{11}+b_1}{c_1\Omega_{11}+d_1} & \frac{\Omega_{12}}{c_1\Omega_{11}+d_1} \\ \frac{\Omega_{12}}{c_1\Omega_{11}+d_1} & \Omega_{22} - \frac{c_1\Omega_{12}^2}{c_1\Omega_{11}+d_1} \end{pmatrix}, \tag{89}$$

and we are obliged to show that the matrix in display (89) coincides with $F^{\epsilon}(\gamma_1(\tau_1, \tau_2, \epsilon)) = F^{\epsilon}(\gamma_1\tau_1, \tau_2, \frac{\epsilon}{c_1\tau_1+d_1})$. In other words, we must establish the following identities:

$$\frac{a_1\Omega_{11}+b_1}{c_1\Omega_{11}+d_1} = \Omega_{11}(\gamma_1\tau_1, \tau_2, \frac{\epsilon}{c_1\tau_1+d_1}), \tag{90}$$

$$\frac{\Omega_{12}}{c_1\Omega_{11}+d_1} = \Omega_{12}(\gamma_1\tau_1, \tau_2, \frac{\epsilon}{c_1\tau_1+d_1}), \tag{91}$$

$$\Omega_{22} - \frac{c_1\Omega_{11}^2}{c_1\Omega_{11}+d_1} = \Omega_{22}(\gamma_1\tau_1, \tau_2, \frac{\epsilon}{c_1\tau_1+d_1}). \tag{92}$$

$A_a(k, l, \tau_a, \epsilon)$ of (80) is a modular form of weight $k + l$ for $k + l > 2$, whereas $A_a(1, 1, \tau_a, \epsilon) = \epsilon E_2(\tau_a)$ enjoys an exceptional transformation law thanks to (12).

Using Lemma 8 we then find that

$$A_1(k, l, \gamma_1\tau_1, \frac{\epsilon}{c_1\tau_1+d_1})) = (c_1\tau_1+d_1)^{(k+l)/2}(A_1(\tau_1, \epsilon) + \kappa\delta_{k1}\delta_{l1}), \tag{93}$$

$$A_2(k, l, \tau_2, \frac{\epsilon}{c_1\tau_1+d_1})) = (c_1\tau_1+d_1)^{-(k+l)/2}A_2(\tau_2, \epsilon), \tag{94}$$

where

$$\kappa = -\frac{\epsilon}{2\pi i}\frac{c_1}{c_1\tau_1+d_1}. \tag{95}$$

It follows from Proposition 4 both that

$$1 - \kappa\omega_{11} = \frac{c_1\Omega_{11}+d_1}{c_1\tau_1+d_1}, \tag{96}$$

and

$$\Omega_{11}\left(\gamma_1\tau_1, \tau_2, \frac{\epsilon}{c_1\tau_1+d_1}\right) = \frac{1}{c_1\tau_1+d_1}\left(a_1\tau_1 + b_1 + \frac{\epsilon}{2\pi i}\omega_{11}\left(\gamma_1\tau_1, \tau_2, \frac{\epsilon}{c_1\tau_1+d_1}\right)\right). \tag{97}$$

Consider a necklace $N \in \mathcal{N}_{11}$ of weight $\omega(N)$ and let $\mathcal{S}_{11}(N)$ denote the set of all "broken" graphs formed from N by deleting any n edges of type $\overset{1}{\bullet} \overset{1}{\longrightarrow} \overset{1}{\bullet}$ for all $n \geq 0$. Every such graph consists of $n + 1$ connected graphs $N_1, \ldots N_{n+1}$ of type 11. From (93) and (94) it therefore follows that

$$\omega(N)(\gamma_1\tau_1, \tau_2, \frac{\epsilon}{c_1\tau_1+d_1}) = \frac{1}{c_1\tau_1+d_1}\sum_{n\geq 0}\kappa^n \sum_{N_1,\ldots N_{n+1}} \omega(N_1)\ldots\omega(N_{n+1}).$$

Summing over all N we then find

$$\begin{aligned}\omega_{11}(\gamma_1\tau_1, \tau_2, \frac{\epsilon}{c_1\tau_1 + d_1}) &= \frac{1}{(c_1\tau_1 + d_1)} \sum_{n\geq 0} \kappa^n \omega_{11}^{n+1} \\ &= \frac{1}{(c_1\tau_1 + d_1)} \frac{\omega_{11}}{1 - \kappa\omega_{11}} \\ &= \frac{\omega_{11}}{c_1\Omega_{11} + d_1},\end{aligned}$$

where for the last equality we used (96). Now (97) yields

$$\begin{aligned}\Omega_{11}\left(\gamma_1\tau_1, \tau_2, \frac{\epsilon}{c_1\tau_1 + d_1}\right) &= \frac{1}{c_1\tau + d_1}\left(a_1\tau_1 + b_1 + \frac{\Omega_{11} - \tau_1}{c_1\Omega_{11} + d_1}\right) \\ &= \frac{a_1\Omega_{11} + b_1}{c_1\Omega_{11} + d_1},\end{aligned}$$

which is the desired (90). Similarly from Proposition 4 we have

$$\Omega_{12}\left(\gamma_1\tau_1, \tau_2, \frac{\epsilon}{c_1\tau_1 + d_1}\right) = -\frac{1}{(c_1\tau_1 + d_1)} \frac{\epsilon}{2\pi i} \omega_{12}\left(\gamma_1\tau_1, \tau_2, \frac{\epsilon}{c_1\tau_1 + d_1}\right).$$

Breaking necklaces of type 12 results in products over necklaces of type 11 together with one necklace of type 12. Hence by a similar argument to that above we find

$$\begin{aligned}\omega_{12}\left(\gamma_1\tau_1, \tau_2, \frac{\epsilon}{c_1\tau_1 + d_1}\right) &= \frac{\omega_{12}}{1 - \kappa\omega_{11}} \\ &= \frac{(c_1\tau_1 + d_1)\omega_{12}}{c_1\Omega_{11} + d_1},\end{aligned}$$

so that $\Omega_{12}(\gamma_1\tau_1, \tau_2, \frac{\epsilon}{c\tau_1 + d_1})$ is as in (91). Finally,

$$\Omega_{22}(\gamma_1\tau_1, \tau_2, \frac{\epsilon}{c\tau_1 + d_1}) = \tau_2 + \frac{1}{c_1\tau_1 + d_1} \frac{\epsilon}{2\pi i} \omega_{22}\left(\gamma_1\tau_1, \tau_2, \frac{\epsilon}{c\tau_1 + d_1}\right).$$

Breaking necklaces of type 22 results in products over necklaces of type 11 together with one necklace of type 12 and another of type 21. Hence by a similar argument to that above we find

$$\begin{aligned}&\frac{1}{(c_1\tau_1 + d_1)} \frac{\epsilon}{2\pi i} \omega_{22}(\gamma_1\tau_1, \tau_2, \frac{\epsilon}{c\tau_1 + d_1}) = \frac{\epsilon}{2\pi i}\left(\omega_{22} + \frac{\kappa\omega_{12}^2}{1 - \kappa\omega_{11}}\right) \\ &= \Omega_{22} - \tau_2 - \frac{c_1\Omega_{12}^2}{c_1\Omega_{11} + d_1},\end{aligned}$$

leading to (92). This completes the proof of the theorem. □

4.5. Local Invertibility of F^ϵ about the Two Tori Degeneration Point $\epsilon = 0$. Let $\mathcal{D}_0^\epsilon$ be the subset of $\mathcal{D}^\epsilon$ for which $\epsilon = 0$. From Theorem 4 it is clear that the restriction of F^ϵ to $\mathcal{D}_0^\epsilon$ induces the natural identification

$$F^\epsilon : \mathcal{D}_0^\epsilon \overset{\sim}{\to} \mathbb{H}_1 \times \mathbb{H}_1 \subseteq \mathbb{H}_2,$$
$$(\tau_1, \tau_2, 0) \mapsto \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}. \tag{98}$$

$\mathcal{D}_0^\epsilon$ corresponds to the set of points where the genus 2 Riemann surface degenerates into a pair of genus 1 surfaces with $\Omega^{(2)} = \mathrm{diag}(\Omega_{11}^{(2)}, \Omega_{22}^{(2)})$. We will consider the invertibility of the map F^ϵ in a neighborhood of a point in $\mathcal{D}_0^\epsilon$. First we prepare a lemma.

Recall (e.g., [FK2]) that a group H of homeomorphisms of a space X is said to act *discontinuously* on X if each point $x \in X$ has a *precisely invariant open neighborhood* under the action of H in the following sense (loc. cit.): the stabilizer $\mathrm{Stab}(x)$ of x in H is finite, and there is an open neighborhood N of x such that $hN \cap N = \phi$ if $h \notin \mathrm{Stab}(x)$ and $hN = N$ if $h \in \mathrm{Stab}(x)$.

Lemma 9. *Suppose that H acts discontinuously on a pair of spaces X, Y, and that $F : X \to Y$ is a continuous H-equivariant map. Then the following hold:*
(a) *If $x \in X$ and $F(x) = y$ then there are precisely invariant open neighborhoods (under the action of H) $U \subseteq X$ and $V \subseteq Y$ of x and y respectively with $F(U) \subseteq V$;*
(b) *Suppose further that $Stab(x) = Stab(y)$ and that the restriction of F to U is $1-1$. Then F is $1-1$ on the H-invariant domain $\bigcup_{h\in H} hU$.*

Proof. For part (a), let V be a precisely invariant open neighborhood of y in Y, U' a precisely invariant neighborhood of x in X, and set $U = F^{-1}(V) \cap U'$. Because F is H-equivariant then $\mathrm{Stab}(x) \subseteq \mathrm{Stab}(y)$, and from this it follows that U is also precisely invariant under the action of H. Now (a) follows.

As for (b), suppose the contrary so that there exist $u_1, u_2 \in U$ and $h_1, h_2 \in H$ such that $h_1u_1 \neq h_2u_2$ and $F(h_1u_1) = F(h_2u_2)$. Thanks to the equivariance of F it is no loss to assume that $h_2 = 1$, so that $h_1u_1 \neq u_2$ and $F(h_1u_1) = F(u_2)$. From the last equality we see that $h_1V \cap V \neq \phi$, so that $h_1V = V$ and $h_1 \in \mathrm{Stab}(y)$.

Therefore, $h_1 \in \mathrm{Stab}(x)$ by hypothesis, and therefore $h_1U = U$. But then h_1u_1 and $u_2 \in U$ are distinct points of U on which F takes the same value. This contradicts the assumption that F is $1-1$ on U, and completes the proof of the lemma. □

We now have

Proposition 5. *Let $x \in \mathcal{D}_0^\epsilon$. Then there exists a $G-$invariant neighborhood $\mathcal{N}_x^\epsilon \subseteq \mathcal{D}^\epsilon$ of x throughout which F^ϵ is invertible.*

Proof. Let $x = (\tau_1, \tau_2, 0)$. From Theorem 4, the Jacobian of F^ϵ at x satisfies

$$\left|\frac{\partial(\Omega_{11}, \Omega_{22}, \Omega_{12})}{\partial(\tau_1, \tau_2, \epsilon)}\right|_x = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = -1.$$

By the inverse function theorem, there exists an open neighborhood of x in $\mathcal{D}^\epsilon$ throughout which F^ϵ is invertible. Set $F^\epsilon(x) = y$. It follows immediately from (98) that the stabilizers (in G) of x and y are equal.

Next, it is well-known that the action (36) of $Sp(2g,\mathbb{Z})$ on $\mathbb{H}_g$ is discontinuous. In particular, the action of G on $\mathbb{H}_2$ is discontinuous: furthermore from the case $g=1$ it is easy to see that the action of G on $\mathbb{H}_1\times\mathbb{H}_1\times\mathbb{C}$ (and hence also on D^{ϵ}) is also discontinuous. Choose precisely invariant neighborhoods (under the action of G) U, V of x, respectively y such that the conditions of part (a) of Lemma 9 hold. It is clear from the non-vanishing of the Jacobian that we may also assume that F^{ϵ} is $1-1$ on U. Thus, we have achieved the hypotheses of part (b) of Lemma 9. That result tells us that the open neighborhood

$$\mathcal{N}_x^{\epsilon}=\bigcup_{\gamma\in G}\gamma U$$

has the desired properties. □

We conclude this section with the explicit form of $\Omega=\Omega^{(2)}(\tau_1,\tau_2,\epsilon)$ to order ϵ^3. We have from (77) to (79) that

$$2\pi i\Omega_{11}=2\pi i\tau_1+\epsilon^2E_2(\tau_2)+O(\epsilon^4), \tag{99}$$

$$2\pi i\Omega_{22}=2\pi i\tau_2+\epsilon^2E_2(\tau_1)+O(\epsilon^4), \tag{100}$$

$$2\pi i\Omega_{12}=-\epsilon(1+\epsilon^2E_2(\tau_1)E_2(\tau_2)+O(\epsilon^4)). \tag{101}$$

It is straightforward to check the equivariance properties described in Theorem 5 to the given order. In Appendix A more detailed expansions are provided. We may invert this relationship using Proposition 5 to find to order Ω_{12}^3 that

$$\tau_1=\Omega_{11}-2\pi i\Omega_{12}^2E_2(\Omega_{22})+O(\Omega_{12}^4), \tag{102}$$

$$\tau_2=\Omega_{22}-2\pi i\Omega_{12}^2E_2(\Omega_{11})+O(\Omega_{12}^4), \tag{103}$$

$$\epsilon=-2\pi i\Omega_{12}(1-(2\pi i)^2\Omega_{12}^2E_2(\Omega_{11})E_2(\Omega_{22})+O(\Omega_{12}^4)). \tag{104}$$

5. The ρ Formalism for Self-Sewing a Riemann Surface

5.1. The General ρ Formalism. In this section we review the general Yamada construction [Y] for sewing a Riemann surface of genus g to itself to form a surface of genus $g+1$. We consider examples of sewing a Riemann sphere to itself in some detail where the Catalan series arise in a surprising way. In the next section, this general formalism will be applied to the construction of a genus two surface where the Catalan series again plays an important role.

Consider a Riemann surface $\mathcal{S}$ of genus g and let z_1, z_2 be local coordinates on $\mathcal{S}$ in the neighborhood of two separated points p_1 and p_2. Consider two disks $|z_a|\leq r_a$ for $r_a>0$ sufficiently small and $a=1,2$. Note that r_1, r_2 must be sufficiently small to also ensure that the disks do not intersect. Introduce a complex parameter ρ where $|\rho|\leq r_1r_2$ and excise the disks

$$\{z_a, |z_a|<|\rho|r_{\bar{a}}^{-1}\}\subset\mathcal{S}$$

for $a=1,2$ to form a twice-punctured surface

$$\hat{\mathcal{S}}=\mathcal{S}\backslash\bigcup_{a=1,2}\{z_a, |z_a|<|\rho|r_{\bar{a}}^{-1}\}.$$

Here we again use the convention (37). We define annular regions $\mathcal{A}_a \subset \hat{\mathcal{S}}$ with $\mathcal{A}_a = \{z_a, |\rho| r_{\bar{a}}^{-1} \leq |z_a| \leq r_a\}$ and identify them as a single region $\mathcal{A} = \mathcal{A}_1 \simeq \mathcal{A}_2$ via the sewing relation

$$z_1 z_2 = \rho, \tag{105}$$

to form a compact Riemann surface $\hat{\mathcal{S}} \backslash \{\mathcal{A}_1 \cup \mathcal{A}_2\} \cup \mathcal{A}$ of genus $g+1$. The sewing relation (105) can be considered to be a parameterization of a cylinder connecting the punctured Riemann surface to itself. Using the Yamada formalism [Y], and noting the notational differences, the genus $g+1$ normalized differential of the second kind $\omega^{(g+1)}$ of (28) obeys

Theorem 6 (Ref. [Y], Theorem 1, Theorem 4).

(a) $\omega^{(g+1)}$ *is holomorphic in* ρ *for* $|\rho| < r_1 r_2$.
(b) $\lim_{\rho \to 0} \omega^{(g+1)}(x, y) = \omega^{(g)}(x, y)$ *for* $x, y \in \hat{\mathcal{S}}$.

Regarded as a power series in ρ, the coefficients of the analytic expansion of $\omega^{(g+1)}$ in ρ can be calculated from $\omega^{(g)}$. Let $\mathcal{C}_a(z_a) \subset \mathcal{A}_a$ denote a closed anti-clockwise oriented contour parameterized by z_a surrounding the puncture at $z_a = 0$ on $\hat{\mathcal{S}}$. Note that $\mathcal{C}_1(z_1)$ may be deformed to $-\mathcal{C}_2(z_2)$. Then similarly to Lemma 1 we find [Y]

Lemma 10.

$$\omega^{(g+1)}(x, y) = \omega^{(g)}(x, y) + \frac{1}{2\pi i} \sum_{a=1,2} \oint_{\mathcal{C}_a(z)} (\omega^{(g+1)}(y, z) \int^{z} \omega^{(g)}(x, \cdot)), \tag{106}$$

for $x, y \in \hat{\mathcal{S}}$.

For $a, b = 1, 2$ and $k, l = 1, 2, \ldots$ we define weighted moments

$$Y_{\bar{a}b}(k, l) = \frac{\rho^{(k+l)/2}}{\sqrt{kl}} \frac{1}{(2\pi i)^2} \oint_{\mathcal{C}_a(u)} \oint_{\mathcal{C}_b(v)} u^{-k} v^{-l} \omega^{(g+1)}(u, v). \tag{107}$$

Note that $Y_{ab}(k, l) = Y_{\bar{b}\bar{a}}(l, k)$. We also define $Y = (Y_{ab}(k, l))$ to be the infinite matrix indexed by the pairs a, k and b, l. We define a set of holomorphic 1-forms on $\hat{\mathcal{S}}$,

$$a_a(k, x) = \frac{\rho^{k/2}}{2\pi i \sqrt{k}} \oint_{\mathcal{C}_a(z_a)} z_a^{-k} \omega^{(g)}(x, z_a), \tag{108}$$

and define $a(x) = (a_a(k, x))$ and $\bar{a}(x) = (a_{\bar{a}}(k, x))$ to be the infinite row vectors indexed by a, k . In a similar way to Lemma 2 we then have

Lemma 11. $\omega^{(g+1)}(x, y)$ *for* $x, y \in \hat{\mathcal{S}}$ *is given by*

$$\omega^{(g+1)}(x, y) = \omega^{(g)}(x, y) - a(x)(I - Y)\bar{a}(y)^T. \tag{109}$$

We next compute the explicit form of Y in terms of the following weighted moments of $\omega^{(g)}$:

$$R_{\bar{a}b}(k,l) = -\frac{\rho^{(k+l)/2}}{\sqrt{kl}} \frac{1}{(2\pi i)^2} \oint_{\mathcal{C}_a(x)} \oint_{\mathcal{C}_b(y)} x^{-k} y^{-l} \omega^{(g)}(x,y)$$
$$= -\frac{\rho^{k/2}}{\sqrt{k}} \frac{1}{2\pi i} \oint_{\mathcal{C}_a(x)} x^{-k} a_b(l,x), \tag{110}$$

where $R_{ab}(k,l) = R_{\bar{b}\bar{a}}(l,k)$ and the extra minus sign is introduced for later convenience. We may consider R as an infinite block matrix (similar to Q of (55))

$$R = (R_{ab}(k,l)) = -\begin{bmatrix} B & A \\ A & B^T \end{bmatrix}, \tag{111}$$

with

$$A(k,l) = A(k,l,\rho) = \frac{\rho^{(k+l)/2}}{(2\pi i)^2 \sqrt{kl}} \oint_{\mathcal{C}_1(x)} \oint_{\mathcal{C}_1(y)} x^{-k} y^{-l} \omega^{(g)}(x,y),$$
$$B(k,l) = B(k,l,\rho) = \frac{\rho^{(k+l)/2}}{(2\pi i)^2 \sqrt{kl}} \oint_{\mathcal{C}_2(x)} \oint_{\mathcal{C}_1(y)} x^{-k} y^{-l} \omega^{(g)}(x,y). \tag{112}$$

Similarly to Proposition 1 we find:

Proposition 6. *$Y_{ab}(k,l)$ is given in terms of R by*

$$I - Y = (I - R)^{-1}. \tag{113}$$

Here

$$(I - R)^{-1} = \sum_{n \geq 0} R^n$$

and is convergent in ρ for $|\rho| < r_1 r_2$.

Likewise, similarly to Theorem 2 we may define $\det(I - R)$ and find:

Theorem 7. $\det(I - R)$ *is non-vanishing and holomorphic in ρ for $|\rho| < r_1 r_2$.*

We can define a standard basis of cycles $\{a_1, b_1, \ldots a_{g+1}, b_{g+1}\}$ on the sewn genus $g+1$ surface as follows, where the set $\{a_1, b_1, \ldots a_g, b_g\}$ is the original basis. Then a_{g+1} is defined as the contour $\mathcal{C}_2$ on $\hat{\mathcal{S}}$ whereas b_{g+1} is defined to be a path chosen in $\hat{\mathcal{S}}$ from $z_1 = z_0$ to $z_2 = \rho/z_0$ which points are identified on the sewn surface. Integrating (109) along a b_r cycle on $\mathcal{S}$ for $r = 1, \ldots g$ gives g holomorphic 1-forms for $x \in \hat{\mathcal{S}}$,

$$\nu_r^{(g+1)}(x) = \nu_r^{(g)}(x) - a(x)(I - R)^{-1} \bar{\alpha}_r^T, \tag{114}$$

where $\bar{\alpha}_r = (\alpha_{r,\bar{a}}(k))$ with

$$\alpha_{r,a}(k) = \oint_{b_r} a_a(x,k). \tag{115}$$

We then find from (30), (31), (114) and (115) that for $r, s = 1, \ldots g$,

$$2\pi i \Omega_{rs}^{(g+1)} = 2\pi i \Omega_{rs}^{(g)} - \alpha_r (I - R)^{-1} \bar{\alpha}_s^T .$$

The remaining normalized holomorphic one form $\nu_{g+1}^{(g+1)}$ can be expressed in terms of the normalized differential of the third kind $\omega_{p_2-p_1}^{(g)}$ of (32) with weighted moments

$$\beta_a(k) = \frac{\rho^{k/2}}{\sqrt{k}} \frac{1}{2\pi i} \int_{\mathcal{C}_a(z_a)} (\omega_{p_2-p_1}^{(g)} + (-1)^{1+a} \frac{dz_a}{z_a}) z_a^{-k}. \tag{116}$$

Then by Cauchy's theorem we find that [Y]

Lemma 12 (op. cit, Corollary 5). *The normalized holomorphic one form* $\nu_{g+1}^{(g+1)}$ *is given by*

$$\nu_{g+1}^{(g+1)}(x) = \omega_{p_2-p_1}^{(g)}(x) + \frac{1}{2\pi i} \sum_{a=1,2} \oint_{\mathcal{C}_a(z)} \omega^{(g+1)}(x, z) \int^{z} (\omega_{p_2-p_1}^{(g)} + (-1)^{1+a} \frac{dz_a}{z_a}). \tag{117}$$

Hence integrating (117) over a b_r cycle and using (109) we find for $r = 1, \ldots, g$ that

$$2\pi i \Omega_{rg+1}^{(g+1)} = \int_{p_1}^{p_2} \nu_r^{(g)} - \alpha_r (I - R)^{-1} \bar{\beta}^T .$$

Finally $\Omega_{g+1g+1}^{(g+1)}$ is described in [Y]:

Lemma 13 (op. cite. Lemma 5). $\Omega_{g+1g+1}^{(g+1)}$ *is given by*

$$2\pi i \Omega_{g+1g+1}^{(g+1)} = \log(\frac{\rho}{z_0^2}) + \int_{z_1^{-1}(z_0)}^{z_2^{-1}(z_0)} \omega_{p_2-p_1}^{(g)}$$
$$+ \sum_{a=1,2} \frac{1}{2\pi i} \oint_{\mathcal{C}_a} \nu_{g+1}^{(g+1)}(z) \int_{z_a^{-1}(z_0)}^{z} (\omega_{p_2-p_1}^{(g)} + (-1)^{1+a} \frac{dz_a}{z_a}),$$

where the logarithmic branch is determined by the choice of the cycle b_{g+1} *as a path in* $\hat{\mathcal{S}}$ *from* $z_1 = z_0$ *to* $z_2 = \rho/z_0$.

Substituting $\nu_{g+1}^{(g+1)}$ from (117) one eventually obtains [Y]

$$2\pi i \Omega_{g+1g+1}^{(g+1)} = \log \rho + C_0 - \beta (I - R)^{-1} \bar{\beta}^T ,$$

where

$$C_0 = \lim_{u \to 0} \left[\int_{z_1^{-1}(u)}^{z_2^{-1}(u)} \omega_{p_2-p_1}^{(g)} - 2 \log u \right] .$$

However from (34) we may express C_0 in terms of the prime form

$$C_0 = \lim_{u\to 0} \log \frac{K^{(g)}(z_2^{-1}(u), p_2)K^{(g)}(z_1^{-1}(u), p_1)}{u^2 K^{(g)}(z_2^{-1}(u), p_1)K^{(g)}(z_1^{-1}(u), p_2)}$$

$$= -\log(-z_1'(p_1)z_2'(p_2)K^{(g)}(p_2, p_1)^2),$$

where $\frac{d}{du}z_a^{-1}(u)|_{u=0} = 1/z_a'(p_a)$ and using $K^{(g)}(p_2, p_1) = -K^{(g)}(p_1, p_2)$. We therefore find altogether that

Theorem 8. *The genus $g+1$ period matrix for $|\rho| < r_1 r_2$ is given by*

$$2\pi i \Omega_{rs}^{(g+1)} = 2\pi i \Omega_{rs}^{(g)} - \alpha_r (I - R)^{-1} \bar{\alpha}_s^T, \quad r, s = 1, \ldots, g, \tag{118}$$

$$2\pi i \Omega_{r g+1}^{(g+1)} = \int_{p_1}^{p_2} \nu_r^{(g)} - \beta(I - R)^{-1} \bar{\alpha}_r^T, \quad r = 1, \ldots, g, \tag{119}$$

$$2\pi i \Omega_{g+1 g+1}^{(g+1)} = \log\left(\frac{-\rho}{z_1'(p_1)z_2'(p_2)K^{(g)}(p_2, p_1)^2}\right) - \beta(I - R)^{-1}\bar{\beta}^T, \tag{120}$$

where $\Omega^{(g+1)}$ is holomorphic in ρ for $0 < |\rho| < r_1 r_2$ and the logarithmic branch is determined by the choice of the cycle b_{g+1}.

We finally obtain the following holomorphic properties for $\omega^{(g+1)}$, $\Omega^{(g+1)}$ and $\det(I - R)$. The proof follows a similar argument to that for Propositions 2 and 3.

Proposition 7. *Suppose that $\omega^{(g)}$ is a holomorphic function of a complex parameter μ for $|\mu| < S$. Then $\det(I - R)$ is non-vanishing and both $\omega^{(g+1)}$ and $\det(I - R)$ are holomorphic in μ for $|\mu| < S$ with $|\rho| < r_1 r_2$ whereas $\Omega^{(g+1)}$ is holomorphic in μ for $|\mu| < S$ with $0 < |\rho| < r_1 r_2$.*

5.2. Self-Sewing a Sphere to form a Torus. It is instructive to consider two separate examples of sewing a Riemann sphere to itself to form a torus. The first is mainly illustrative whereas the second is related to some later genus two considerations wherein the Catalan numbers arise in an interesting and surprising way. In both cases $\omega^{(1)}$ is given by (35) with an appropriately identified modular parameter τ.

5.2.1. Simplest Case. Let $\mathcal{S}_0 = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere with bilinear form (70). Choose local coordinates $z_1 = z$ in the neighborhood of the origin and $z_2 = 1/z'$ for z' in the neighborhood of the point at infinity. Identify the annular regions $|q| r_{\bar{a}}^{-1} \le |z_a| \le r_a$ for a complex parameter q obeying $|q| \le r_1 r_2$ via the sewing relation

$$z = qz'. \tag{121}$$

Note that the annular regions do not intersect on the sphere provided $r_1 r_2 < 1$ so that $|q| < 1$. We then find [Y]

Proposition 8. *$q = \exp(2\pi i\tau)$, where τ is the torus modular parameter.*

Proof. The 1-forms (108) are

$$a_1(k,x) = \sqrt{k}q^{k/2}x^{-k-1}dx,$$
$$a_2(k,x) = -\sqrt{k}q^{k/2}x^{k-1}dx,$$

so that $A(k,l) = 0$ and $B(k,l) = q^k\delta_{k,l}$ in (111) giving

$$I - R = \mathrm{diag}(1-q, 1-q, \ldots, 1-q^k, 1-q^k, \ldots).$$

Hence Lemma 11 gives for $x, y \in \hat{\mathcal{S}}_0$,

$$\omega^{(1)}(x,y) = \left\{ \frac{xy}{(x-y)^2} + \sum_{k\geq 1} \frac{kq^k}{1-q^k}\left[\left(\frac{x}{y}\right)^k + \left(\frac{y}{x}\right)^k\right]\right\} \frac{dxdy}{xy}.$$

Under the conformal map $z \to \log z$ we then verify $\omega^{(1)}(u,v) = P_2(\tau, u-v)dudv$ with $u = \log x$ and $v = \log y$ using (10), where $q = \exp(2\pi i\tau)$. The sewing relation (121) is then just the standard torus periodicity relation $\log z = \log z' + 2\pi i\tau$.

Alternatively, we may apply (120) of Theorem 8 using $z_2 = 1/z - 1/p_2$ and then consider $p_2 \to \infty$. Then $K^{(0)}(p_2, 0) = p_2$ with $\omega^{(1)}_{p_2-0}(x) = (\frac{1}{x-p_2} - \frac{1}{x})dx$ so that $\beta_a(k) = 0$ and $z_1'(0)z_2'(p_2)K^{(0)}(p_2,0)^2 = -1$ independent of p_2. This implies that $2\pi i\tau = 2\pi i\Omega^{(1)}_{11} = \log q$ again. □

Remark 1. The modular transformation $\tau \to \tau + 1$ is generated by a continuous variation in the sewing parameter $\exp(i\theta)q$ for $0 \leq \theta \leq 2\pi$. This corresponds to a Dehn twist $b_1 \to a_1 + b_1$ in the b_1 cycle chosen in Lemma 13 and Theorem 8 so that $2\pi i\Omega^{(1)}_{11} = \log q$ is evaluated on the next logarithmic branch.

Remark 2. $\omega^{(1)}(x,y)$ and $\det(1-R) = \prod_{k\geq 1}(1-q^k)^2$ are clearly holomorphic for $|q| < 1$ as expected from Theorems 6 and 7.

5.2.2. General Self-Sewing of a Sphere and the Catalan Series. For $z \in \mathcal{S}_0$ choose local coordinates $z_1 = z$ in the neighborhood of the origin and $z_2 = z' - w$ for z' in the neighborhood of $w \in \mathcal{S}_0$. Identify the annuli $|\rho|r_2^{-1} \leq |z| \leq r_1$ and $|\rho|r_1^{-1} \leq \left|z' - w\right| \leq r_2$ for $|\rho| \leq r_1 r_2$ via the sewing relation

$$z(z' - w) = \rho. \tag{122}$$

The two annular regions do not intersect provided $|w| > r_1 + r_2 \geq r_1 + |\rho|r_1^{-1} \geq 2|\rho|^{1/2}$. The lower bound occurs for $r_1 = r_2 = |\rho|^{1/2}$ and is realized when the two annuli become degenerate (infinitesimally thin) and touch at the point $z_1 = -z_2 = w/2$ with $w^2 = -4\rho$. Thus defining

$$\chi = -\frac{\rho}{w^2},$$

then $\chi = \frac{1}{4}$ is the degenerate point. We define the Catalan series[9] to be the series $f(\chi)$ convergent for $|\chi| < \frac{1}{4}$ satisfying

$$\chi = \frac{f}{(1+f)^2}. \tag{123}$$

Thus

$$\begin{aligned} f(\chi) &= \frac{1-\sqrt{1-4\chi}}{2\chi} - 1 = \sum_{n\geq 1} \frac{1}{n}\binom{2n}{n+1}\chi^n \\ &= \chi + 2\chi^2 + 5\chi^3 + 14\chi^4 + O\left(\chi^5\right). \end{aligned} \tag{124}$$

The coefficients $\frac{1}{n}\binom{2n}{n+1}$ are the Catalan numbers which occur in a remarkably wide range of combinatorial settings e.g., [St].

Proposition 9. *For the sewing described by (122), the torus modular parameter is* $q = f(\chi)$*, the Catalan series.*

Proof. Define the Möbius transformation

$$z \mapsto \gamma.z = \frac{w}{1+f}(\frac{z-f}{z-1}), \tag{125}$$

where $f = f(\chi)$. Then with $z = \gamma.Z$ and $z' = \gamma.Z'$ the sewing relation (122) becomes, on using (123),

$$Z = fZ'.$$

Thus we recover the earlier sewing relation of (121) with modular parameter $q = f(\chi)$.

This result can be verified from (120) of Theorem 8 as follows. With $\omega^{(0)}$ of (70) the basis of 1-forms (108) is given by

$$\begin{aligned} a_1^{(0)}(k,x) &= \sqrt{k}\rho^{k/2}x^{-k-1}dx, \\ a_2^{(0)}(k,x) &= \sqrt{k}\rho^{k/2}(x-w)^{-k-1}dx, \end{aligned} \tag{126}$$

with

$$\begin{aligned} R^{(0)} &= -\begin{bmatrix} B^{(0)} & A^{(0)} \\ A^{(0)} & B^{(0)T} \end{bmatrix}, \\ A^{(0)}(k,l) &= 0, \quad B^{(0)}(k,l) = \frac{(-\chi)^{(k+l)/2}}{\sqrt{kl}} \frac{(-1)^{k+1}(k+l-1)!}{(k-1)!(l-1)!}, \\ \omega_{w-0}^{(0)}(x) &= (\frac{1}{x-w} - \frac{1}{x})dx, \\ K^{(0)}(w,0) &= w, \\ \beta^{(0)}(k) &= \frac{(-\chi)^{k/2}}{\sqrt{k}}[-1,(-1)^k], \end{aligned} \tag{127}$$

[9] The Catalan series is more usually defined to be $1 + f(\chi)$.

where the 0 superscript indicates the genus of the sphere. After some calculation, we find that τ is given by

$$2\pi i\tau = 2\pi i\Omega_{11}^{(1)} = \log\chi + 2\sum_{k\geq 1}\frac{1}{k}\chi^k\sum_{n\geq 1}S_{n,k}(\chi),$$

where $S_{1,k}(\chi) = 1$ and

$$S_{n,k}(\chi) = \sum_{k_{n-1},\ldots k_1\geq 1}\chi^{k_{n-1}+\ldots+k_1}\binom{k+k_{n-1}-1}{k_{n-1}}\binom{k_{n-1}+k_{n-2}-1}{k_{n-2}} \ldots\binom{k_2+k_1-1}{k_1}, \tag{128}$$

for $n > 1$. We will show below that

$$\sum_{n\geq 1}S_{n,k}(\chi) = (1+f(\chi))^k, \tag{129}$$

which implies $\sum_{k\geq 1}\frac{1}{k}\chi^k\sum_{n\geq 1}S_{n,k}(\chi) = -\log(1-\chi(1+f)) = \log(1+f)$ from (124). Therefore $2\pi i\tau = \log\chi + 2\log(1+f) = \log f$ so that $q = f$ as claimed.

It remains to prove (129). Since $\sum_{k_1\geq 1}\chi^{k_1}\binom{k_2+k_1-1}{k_1} = (1-\chi)^{-k_2} - 1$ we find for $n > 1$ that

$$S_{n,k}(\chi) = \sum_{k_{n-1}\geq 1}\chi^{k_{n-1}}\binom{k+k_{n-1}-1}{k_{n-1}}\ldots \sum_{k_2\geq 1}\left(\frac{\chi}{1-\chi}\right)^{k_2}\binom{k_3+k_2-1}{k_2} - S_{n-1,k}(\chi).$$

Repeating this process leads to

$$\sum_{n=1}^{N}S_{n,k}(\chi) = \left(\left[\frac{1}{1-\frac{\chi}{1-\chi/\cdots}}\right]_N\right)^k,$$

where $[\frac{1}{1-\frac{\chi}{1-\chi/\cdots}}]_N$ denotes the N^{th} term in the continued fraction expansion of $F = 1/(1-\chi F)$ whose solution from (124) is $F = 1+f$. □

Remark 3. The modular transformation $\tau\to\tau+1$ is generated by a continuous variation in the sewing parameter $\exp(i\theta)\rho$ for $0\leq\theta\leq 2\pi$.

Using Lemma 11 and comparing to $\omega^{(1)}$ of (35) results in novel expressions for Eisenstein series $E_n(q)$ for $q = f(\chi)$. Thus, for example, one finds

Proposition 10.

$$E_2(q = f(\chi)) = -\frac{1}{12} + \frac{2\chi}{1-4\chi}(1+B^{(0)})^{-1}(1,1), \tag{130}$$

where $(1,1)$ *refers to the* $(k,l) = (1,1)$ *element of the infinite matrix* $(1+B^{(0)})^{-1}$.

Proof. From (109) and (126) we have

$$\omega^{(1)}(x, y) = \frac{dxdy}{(x-y)^2} - a^{(0)}(x)(I - R^{(0)})^{-1}(\bar{a}^{(0)})^T(y).$$

But $\omega^{(1)}(x, y) = \omega^{(1)}(u, v) = P_2(u - v, \tau)dudv$ with $\tau = \frac{1}{2\pi i}\log f(\chi)$ from Proposition 9 with $x = \gamma.e^u$ and $y = \gamma.e^v$ using (125). Then, on substituting for u, v into $\omega^{(1)}(x, y)$ one eventually finds using (127) that

$$\omega^{(1)}(x, y) = \frac{e^{u-v}dudv}{(e^{u-v}-1)^2} - \sum_{k,l\geq 1}(1 + B^{(0)})^{-1}(k, l)\sqrt{kl}\left(-\frac{\chi}{1-4\chi}\right)^{(k+l)/2}$$
$$.(e^u - 1)^{k-1}(e^v - 1)^{l-1}\left[\left(\frac{1-f}{e^u - f}\right)^{k+1}\left(\frac{1-f}{1-fe^v}\right)^{l+1}\right.$$
$$\left.+(-1)^{k+l}\left(\frac{1-f}{1-fe^u}\right)^{k+1}\left(\frac{1-f}{e^v - f}\right)^{l+1}\right]e^{u+v}dudv,$$

using $1 - f = (1 + f)\sqrt{1 - 4\chi}$. Expanding in u, v we then find that

$$\omega^{(1)}(x, y) = [\frac{1}{(u-v)^2} - \frac{1}{12} + \frac{2\chi}{1-4\chi}(1 + B^{(0)})^{-1}(1, 1) + O(u, v)]dudv, \quad (131)$$

from which the result follows on comparison with (9). □

6. Self-Sewing a Torus to Form a Genus Two Riemann Surface

6.1. The Genus Two Period Matrix in the ρ Formalism. We now apply the ρ-formalism to sew a twice punctured torus with modulus τ and punctures separated by w to form a genus two Riemann surface with period matrix $\Omega^{(2)}(\tau, w, \rho)$. We will see that $\Omega^{(2)}$ is holomorphic for (τ, w, ρ) in an appropriate domain $\mathcal{D}^\rho$. We again provide a description of $\Omega^{(2)}$ in terms of a sum of weights of *necklaces*. There is a holomorphic mapping $F^\rho : \mathcal{D}^\rho \to \mathbb{H}_2$, and we describe its equivariance properties with respect to a certain group. The logarithmic contribution $\log(-\rho/K^2)$ to $\Omega^{(2)}_{22}$ in (120) gives rise to a subtle analytic structure which we discuss in some detail. Finally, we prove that F^ρ is invertible in a certain domain.

Consider a framed torus (cf. Subsect. 4.1) $\mathcal{S} = \mathbb{C}/\Lambda$, where $\Lambda \subseteq \mathbb{C}$ is a lattice with positively oriented basis (σ, ς) and modulus $\tau = \sigma/\varsigma \in \mathbb{H}_1$. Define annuli $\mathcal{A}_a$, $a = 1, 2$, centered at $z = 0$ and $z = w$ of $\mathcal{S}$ with local coordinates $z_1 = z$ and $z_2 = z - w$ respectively. Take the outer radius of $\mathcal{A}_a$ to be $r_a < \frac{1}{2}D(\Lambda)$ and the inner radius to be $|\rho|/r_{\bar{a}}$, with $|\rho| \leq r_1 r_2 < \frac{1}{4}D(\Lambda)^2$ (cf. Lemma 6). Identifying the annuli according to the sewing relation (105) $z_1 z_2 = \rho$ gives rise to a compact Riemann surface of genus 2.

As in the remarks following Lemma 6, we now take $\Lambda = \Lambda_\tau$ with basis $(2\pi i\tau, 2\pi i)$ and with w in the fundamental parallelogram for Λ_τ with sides $(2\pi i\tau, 2\pi i)$. As with the sphere example above, the two annuli must not intersect. This requires the inequalities $|w - \lambda| > r_1 + r_2 \geq 2|\rho|^{1/2}$ to hold for $\lambda \in \Lambda_\tau$. Thus we find

$$2|\rho|^{1/2} < |w| < D(\Lambda_\tau) - 2|\rho|^{1/2}.$$

Notice that this implies $|\rho| < \frac{1}{16}D(\Lambda_\tau)^2$, which refines the inequality satisfied by ρ discussed above. As a result of this discussion, we see that the relevant domain in the ρ -formalism is the following:[10]

$$\mathcal{D}^\rho = \{(\tau, w, \rho) \in \mathbb{H}_1 \times \mathbb{C} \times \mathbb{C} \mid |w - \lambda| > 2|\rho|^{1/2} > 0,\ \lambda \in \Lambda_\tau\}. \tag{132}$$

We may apply Theorem 8 to determine $\Omega^{(2)}(\tau, w, \rho)$. We find:

Theorem 9. *Sewing determines a holomorphic map*

$$\begin{aligned} F^\rho &: \mathcal{D}^\rho \to \mathbb{H}_2, \\ (\tau, w, \rho) &\mapsto \Omega^{(2)}(\tau, w, \rho). \end{aligned} \tag{133}$$

Proposition 11. $\Omega^{(2)} = \Omega^{(2)}(\tau, w, \rho)$ *is given by*

$$2\pi i \Omega^{(2)}_{11} = 2\pi i \tau - \rho\sigma((I - R)^{-1}(1, 1)), \tag{134}$$

$$2\pi i \Omega^{(2)}_{12} = w - \rho^{1/2}\sigma((\beta(I - R)^{-1}(1)), \tag{135}$$

$$2\pi i \Omega^{(2)}_{22} = \log(-\frac{\rho}{K(\tau, w)^2}) - \beta(I - R)^{-1}\bar{\beta}^T, \tag{136}$$

where the branch of the log *function in* (136) *is determined by the choice of the cycle* b_2. *Here,* $R = R(\tau, w, \rho) = (R_{ab}(k, l))$ *is an infinite matrix with indices* $k, l = 1, 2, 3, \ldots$ *and* $a, b = 1, 2$; $\beta = \beta(\tau, w, \rho) = (\beta_a(k))$ *is an infinite row vector;* $(1, 1)$ *and* (1) *are the* $(1, 1)$- *and* (1)- *(block) entries of a matrix;* $\sigma(M)$ *denotes sum over the entries of a finite matrix; and*

$$R(k, l) = -\frac{\rho^{(k+l)/2}}{\sqrt{kl}} \begin{bmatrix} D(k, l, \tau, w) & C(k, l, \tau) \\ C(k, l, \tau) & D(l, k, \tau, w) \end{bmatrix}, \tag{137}$$

$$\beta(k) = \frac{\rho^{k/2}}{\sqrt{k}}(P_k(\tau, w) - E_k(\tau))[-1, (-1)^k], \tag{138}$$

with notation as in Sect. 2.

Proof. Since $\omega^{(1)}(x, y) = P_2(x - y)dxdy$ from (20) we find that the set of 1-forms (108) with periods $(2\pi i\tau, 2\pi i)$ is given by

$$\begin{aligned} a_1(k, x) &= a_1(k, x, \tau, \rho) = \sqrt{k}\rho^{k/2} P_{k+1}(\tau, x)dx, \\ a_2(k, x) &= a_2(k, x, \tau, \rho) = a_1(k, x - w). \end{aligned}$$

The matrices $A(k, l)$, $B(k, l)$ in (112) are given directly from the expansions (25) and (26) which are convergent on $\mathcal{D}^\rho$ resulting in (137). $\alpha_{1,a}(k)$ of (115) is independent of $a = 1, 2$ with

$$\alpha_{1,a}(k) = \oint_{b_1} a_a(k, \cdot) = \rho^{1/2}\delta_{k,1}.$$

Hence $2\pi i\Omega^{(2)}_{11}$ is as stated from (118) of Theorem 8 for $(\tau, w, \rho) \in \mathcal{D}^\rho$. From Example 1 we know that $\omega^{(1)}_{w-0}(x) = (P_1(\tau, x - w) - P_1(\tau, x))dx$ and the prime form is

[10] The footnote relating to (75) concerning notation applies here too.

$K^{(1)}(x, y) = K(\tau, x - y)$. We obtain the given moments (138) of $\omega^{(1)}_{w-0}(x)$ from (20). Hence since $\nu^{(1)}(x) = dx$, we find $2\pi i \Omega^{(2)}_{12}$ is as given from (119) of Theorem 8 for $(\tau, w, \rho) \in \mathcal{D}^\rho$. Finally applying (120) with $K^{(1)}(w, 0) = K(\tau, w)$ we obtain (136) for $(\tau, w, \rho) \in \mathcal{D}^\rho$.

$\Omega^{(2)}_{ij}(\tau, w, \rho)$ is holomorphic in ρ for $0 < |\rho| < r_1 r_2$ from Theorem 8. Proposition 7 then states that $\Omega^{(2)}_{ij}(\tau, w, \rho)$ is also holomorphic in $\tau \in \mathbb{H}_1$. We also need to show that $\Omega^{(2)}_{ij}(\tau, w, \rho)$ is holomorphic in w. Since $Y = (I - R)^{-1}$ converges for $|\rho| < r_1 r_2$ then, following an argument similar to that in Proposition 2, we find that $\Omega^{(2)}_{ij}(\tau, w, \rho)$ is continuous in w for $(\tau, w, \rho) \in \mathcal{D}^\rho$. The Weierstrass functions $P_k(\tau, w)$ for $k \geq 1$ are holomorphic in w. Hence the ρ expansion coefficients $\Omega^{(2)}_{ij}(\tau, w, \rho)$ are holomorphic functions in w since they consist of finite sums and products of these Weierstrass functions. Hence, by Lemma 5, $\Omega^{(2)}_{ij}(\tau, w, \rho)$ is holomorphic in w. Finally, by Hartog's Theorem, $\Omega^{(2)}_{ij}$ is holomorphic on $\mathcal{D}^\rho$. □

6.2. Necklace Expansion for $\Omega^{(2)}$. We introduce a graphical interpretation for the ρ period matrix formulas analogous to that described earlier for the ϵ-expansion. Consider the set of *necklaces* $\mathcal{N} = \{N\}$: they are connected graphs with $n \geq 2$ nodes, $n - 2$ of which have valency 2 and two of which have valency 1, together with an orientation, say from left to right. Furthermore, each vertex carries two labels k, a with k a positive integer and $a = 1$ or 2. A typical necklace in the ρ-formalism looks as follows:

$$\overset{k_1,a_1}{\bullet} \longrightarrow \overset{k_2,a_2}{\bullet} \longrightarrow \overset{k_3,a_3}{\bullet} \longrightarrow \overset{k_4,a_4}{\bullet}$$

We define the degenerate necklace N_0 to be a single node with no edges. Next we define a weight function

$$\omega : \mathcal{N} \longrightarrow \mathbb{C}[P_2(\tau, w), P_3(\tau, w), E_2(\tau), E_4(\tau), E_6(\tau), \rho^{1/2}].$$

If $N \in \mathcal{N}$ has edges E labelled as $\overset{k,a}{\bullet} \longrightarrow \overset{l,b}{\bullet}$ then we define

$$\omega(E) = R_{ab}(k, l, \tau, w, \rho),$$
$$\omega(N) = \prod \omega(E),$$

with $R_{ab}(k, l)$ as in (137) and where the product is taken over all edges of N. We further define $\omega(N_0) = 1$.

The necklaces with prescribed end nodes labelled $(k, a; l, b)$ look as follows:

$$\overset{k,a}{\bullet} \longrightarrow \overset{k_1,a_1}{\bullet} \ldots \overset{k_2,a_2}{\bullet} \longrightarrow \overset{l,b}{\bullet} \qquad \text{(type } (k, a; l, b)\text{)}.$$

We set

$$\mathcal{N}_{k,a;l,b} = \{\text{isomorphism classes of necklaces of type } (k, a; l, b)\}.$$

As in Lemma 7 we obtain

Lemma 14. *We have for* $k, l \geq 1$,

$$(I - R)^{-1}_{ab}(k, l) = \sum_{N \in \mathcal{N}_{k,a;l,b}} \omega(N).$$

Finally it is convenient to define

$$\begin{aligned}
\omega_{11} &= \sum_{a,b=1,2} \sum_{N \in \mathcal{N}_{1,a;1,b}} \omega(N),\\
\omega_{\beta 1} &= \sum_{a,b=1,2} \sum_{k\geq 1} \beta_a(k) \sum_{N \in \mathcal{N}_{k,a;1,b}} \omega(N),\\
\omega_{1\bar{\beta}} &= \sum_{a,b=1,2} \sum_{k\geq 1} \bar{\beta}_b(k) \sum_{N \in \mathcal{N}_{1,a;k,b}} \omega(N),\\
\omega_{\beta\bar{\beta}} &= \sum_{a,b=1,2} \sum_{k,l\geq 1} \beta_a(k)\bar{\beta}_b(l) \sum_{N \in \mathcal{N}_{k,a;l,b}} \omega(N).
\end{aligned} \tag{139}$$

Note that $R_{ab}(k, l) = R_{\bar{b}\bar{a}}(l, k)$, so that $\omega_{\beta 1} = \omega_{1\bar{\beta}}$. Then from Theorem 9 we have

Proposition 12.

$$\begin{aligned}
2\pi i \Omega^{(2)}_{11} &= 2\pi i\tau - \rho\omega_{11},\\
2\pi i \Omega^{(2)}_{12} &= w - \rho^{1/2}\omega_{\beta 1},\\
2\pi i \Omega^{(2)}_{22} &= \log(-\frac{\rho}{K(\tau, w)^2}) - \omega_{\beta\bar{\beta}}. \quad \square
\end{aligned}$$

6.3. Equivariance of F^ρ. In Subsect. 4.4 we defined a subgroup $G \subset Sp(4, \mathbb{Z})$ which preserves the domain $\mathcal{D}^\epsilon$, and proved the equivariance of F^ϵ under the action of G. In this section we wish to establish analogous equivariance properties in the ρ-formalism. With this in mind, one might expect that the map F^ρ occurring in Theorem 9 is the correct analog of F^ϵ. However, because of the logarithmic branch structure of $\Omega^{(2)}_{22}$, it is necessary to lift F^ρ to a single-valued function $\hat{F}^\rho$ on a certain covering space $\hat{\mathcal{D}}^\rho$ for $\mathcal{D}^\rho$ before the correct analogs can be established.

6.3.1. Some Heisenberg and Jacobi-type groups. In this subsection we consider some groups relevant to our enterprise, and start with certain subgroups of $Sp(4, \mathbb{Z})$. For $(a, b, c) \in \mathbb{Z}^3$ set

$$\mu(a, b, c) = \begin{pmatrix} 1 & 0 & 0 & b \\ a & 1 & b & c \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{140}$$

with

$$A = \mu(1, 0, 0),\ B = \mu(0, 1, 0),\ C = \mu(0, 0, 1).$$

The matrices (140) form a subgroup $\hat{H} \subseteq Sp(4,\mathbb{Z})$ which is a 2-step nilpotent group with center isomorphic to $\mathbb{Z}$ and generated by C, and central quotient isomorphic to $\mathbb{Z}^2$. Note that we have the presentation

$$\hat{H} = \langle A, B, C \mid [A,B]C^{-2} = [A,B,C] = 1\rangle. \tag{141}$$

The 'left' modular group Γ_1 (84) is also a subgroup of $Sp(4,\mathbb{Z})$, and indeed it normalizes $\hat{H}$ according to the conjugation formula

$$\gamma^{-1}\mu(u,v,w)\gamma = \mu((u,v)\gamma, w),\ \gamma \in \Gamma_1. \tag{142}$$

Here it was convenient to abuse notation, taking

$$\gamma = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma_1 \quad \text{and} \quad (u,v)\gamma = (u,v)\begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{143}$$

In this way we get a subgroup $L = \hat{H}\Gamma_1 \subseteq Sp(4,\mathbb{Z})$ which is a split extension of $SL(2,\mathbb{Z})$ by $\hat{H}$. Note that $Z(L) = \langle C\rangle$, and that the central quotient $J = L/Z(L) \cong \mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$ is the Jacobi group which figures in the transformation laws of Jacobi forms ([EZ]).

Let $H = \langle A, B\rangle$ be the subgroup of $\hat{H}$ generated by A and B. It follows from (141) that H has the presentation

$$H = \langle A, B \mid [A,B] = C', [A,C'] = [B,C'] = 1\rangle,$$

where we have set $C' = C^2$. We call H the *Heisenberg* group, though H and $\hat{H}$ (which are *not* isomorphic) are often confused in this regard. We see from (142) that $|\hat{H} : H| = 2$ and that Γ_1 normalizes H. Thus $L_0 = H\Gamma_1$ is a subgroup of L of index 2.

Lemma 15. *L acts on $\mathcal{D}^\rho$ as follows*:

$$\mu(a,b,c).(\tau,w,\rho) = (\tau, w + 2\pi i a\tau + 2\pi i b, \rho), \tag{144}$$

$$\gamma.(\tau,w,\rho) = \left(\frac{a\tau+b}{c\tau+d}, \frac{w}{c\tau+d}, \frac{\rho}{(c\tau+d)^2}\right). \tag{145}$$

The kernel of the action is $Z(L)$, so that the effective action is that of $J = L/Z(L)$.

Proof. Let us first work with the larger domain whereby we allow the triple (τ,w,ρ) to lie in $\mathbb{H}_1 \times \mathbb{C} \times \mathbb{C}$. Then it is easy to see that the first equality defines an action of $\hat{H}$ with kernel $\langle C\rangle$, and that the second equality defines a faithful action of $SL(2,\mathbb{Z})$.

Next we show that these two actions jointly define an action of the group L . To this end it is useful to rewrite (145) more functorially in terms of the cocycle $j(\gamma,\tau) = c\tau + d$, which satisfies

$$j(\gamma_1\gamma_2,\tau) = j(\gamma_1,\gamma_2\tau)j(\gamma_2,\tau),\quad \gamma_1,\gamma_2 \in \Gamma_1. \tag{146}$$

Thus

$$\gamma.(\tau,w,\rho) = \left(\gamma\tau, \frac{w}{j(\gamma,\tau)}, \frac{\rho}{j(\gamma,\tau)^2}\right),$$

and we have to show that

$$\gamma^{-1}\mu(x, y, z)\gamma.(\tau, w, \rho) = \mu((x, y)\gamma, z).(\tau, w, \rho). \tag{147}$$

The right-hand-side of (147) is equal to

$$(\tau, w + 2\pi i((ax + cy)\tau + bx + dy), \rho).$$

The left-hand-side is equal to

$$\begin{aligned}
&\gamma^{-1}\mu(x, y, z).\left(\gamma\tau, \frac{w}{j(\gamma,\tau)}, \frac{\rho}{j(\gamma,\tau)^2}\right) \\
&= \gamma^{-1}.\left(\gamma\tau, \frac{w}{j(\gamma,\tau)} + 2\pi i(x\gamma\tau + y), \frac{\rho}{j(\gamma,\tau)^2}\right) \\
&= \gamma^{-1}.\left(\gamma\tau, \frac{w + 2\pi i(x(a\tau+b) + y(c\tau+d))}{j(\gamma,\tau)}, \frac{\rho}{j(\gamma,\tau)^2}\right) \\
&= \left(\tau, \frac{w + 2\pi i(x(a\tau+b) + y(c\tau+d))}{j(\gamma,\tau)j(\gamma^{-1},\gamma\tau)}, \frac{\rho}{(j(\gamma,\tau)j(\gamma^{-1},\gamma\tau))^2}\right) \\
&= (\tau, w + 2\pi i(x(a\tau+b) + y(c\tau+d)), \rho)\,,
\end{aligned}$$

where we used (146) to get the last equality. This confirms (147).

It remains to show that the action of L preserves $\mathcal{D}^\rho$, and for this it is enough to prove it for a set of generators. Bearing in mind the definition of $\mathcal{D}^\rho$ (132), the result is clear for $\mu(x, y, z)$. To prove it for $\gamma \in \Gamma_1$, we must show that if $(\tau, w, \rho) \in \mathcal{D}^\rho$ then

$$\left|\frac{w}{j(\gamma,\tau)} - \lambda\right| > 2\left|\frac{\rho}{(j(\gamma,\tau))^2}\right|^{1/2} > 0 \tag{148}$$

for all $\lambda \in \Lambda_{\gamma\tau}$. But $\lambda = \frac{1}{j(\gamma,\tau)}\lambda'$ for some $\lambda' \in \Lambda_\tau$, whence (148) reduces to $|j(\gamma,\tau)|^{-1}|w-\lambda'| > 2|j(\gamma,\tau)|^{-1}|\rho|^{1/2} > 0$. This follows from the fact that $(\tau, w, \rho) \in \mathcal{D}^\rho$, and the proof of the lemma is complete. □

6.3.2. Some covering spaces. One sees that projection onto the first coordinate

$$\begin{aligned}
pr_1 : \mathcal{D}^\rho &\to \mathbb{H}_1, \\
(\tau, w, \rho) &\mapsto \tau,
\end{aligned}$$

is *locally trivial* with contractible base $\mathbb{H}_1$. From the long exact sequence associated to a fibration we obtain an exact sequence $0 = \pi_2(\mathbb{H}_1) \to \pi_1(F) \to \pi_1(\mathcal{D}^\rho) \to \pi_1(\mathbb{H}_1) = 0$, where F is the fiber. Thus, we have $\pi_1(\mathcal{D}^\rho) \cong \pi_1(F)$. From Lemma 6.5, there is a free action of $\mathbb{Z}^2 = \hat{H}/Z(L)$ on each fiber $pr_1^{-1}(\tau)$. Furthermore, from the definition of $\mathcal{D}^\rho$ we see that

$$\begin{aligned}
\pi_1(\mathcal{D}^\rho/\mathbb{Z}^2) &\cong \pi_1(\mathbb{C}/\Lambda_\tau \setminus \{0\}) \times \pi_1(\mathbb{C} \setminus \{0\}) \\
&\cong H \times \mathbb{Z}.
\end{aligned}$$

Here, H is the Heisenberg group of the previous subsection.

We need to describe this identification carefully. Consider the usual realization of $\mathbb{C}/\Lambda_\tau$ as the fundamental parallelogram for Λ_τ with identification of sides, and let α, β be the cycles along the sides with periods $2\pi i\tau, 2\pi i$ respectively. Define δ to be a closed *clockwise* contour about an interior point of the parallelogram with local coordinate $w = 0$. Then there is an isomorphism of groups

$$\pi_1(\mathbb{C}/\Lambda_\tau \setminus \{0\}) \overset{\cong}{\rightarrow} H,$$
$$\alpha \mapsto A, \beta \mapsto B, \delta \mapsto C'.$$

Similarly, let η denote a closed *anti-clockwise* contour about $\rho = 0$ in the complex plane. Then $\pi_1(\mathbb{C} \setminus \{0\}) = \langle \eta \rangle$.

Let $\tilde{\mathcal{D}}^\rho$ be a universal covering space of $\mathcal{D}^\rho$ with covering projection

$$p_1 : \tilde{\mathcal{D}}^\rho \to \mathcal{D}^\rho.$$

There is a free action of the fundamental group $H \times \mathbb{Z}$ on $\tilde{\mathcal{D}}^\rho$, and we define

$$\hat{\mathcal{D}}^\rho = \tilde{\mathcal{D}}^\rho / \langle \eta^{-2}\delta \rangle.$$

Thus we have a sequence of covering projections

$$\tilde{\mathcal{D}}^\rho \overset{p_3}{\longrightarrow} \hat{\mathcal{D}}^\rho \overset{p_4}{\longrightarrow} \mathcal{D}^\rho \overset{p_2}{\longrightarrow} \mathcal{D}^\rho/\mathbb{Z}^2, \tag{149}$$

where $p_1 = p_4 \circ p_3$. The action of Γ_1 on $\mathcal{D}^\rho$ lifts (modulo the fundamental group) to an action on the universal cover. That is, there is a group G acting on $\tilde{\mathcal{D}}^\rho$, where G fits into a short exact sequence

$$1 \to H \times \mathbb{Z} \to G \to \Gamma_1 \to 1.$$

We have $Z(G) = Z(H) \times \mathbb{Z}$, in particular $\eta^{-2}\delta \in Z(G)$. It follows that G acts on $\hat{\mathcal{D}}^\rho$, and there is a sequence of surjective group maps

$$G \to G/\langle \eta^{-2}\delta \rangle \to L \to \Gamma_1 \tag{150}$$

in which the four groups act on the corresponding spaces in (149).

6.3.3. Lifting the logarithm $l(x)$. From (136), the logarithmic contribution to $\Omega_{22}^{(2)}$ is

$$l(x) = \log\left(-\frac{\rho}{K(\tau, w)^2}\right), \ x = (\tau, w, \rho) \in \mathcal{D}^\rho. \tag{151}$$

The remaining parts ω_{11}, $\omega_{\beta 1}$ and $\omega_{\beta\bar{\beta}}$ of $\Omega^{(2)}$ are single-valued on $\mathcal{D}^\rho$ since they are expressible in terms of the Weierstrass functions and Eisenstein series. Now $K(\tau, w)^2 = -\theta_1(\tau, w)^2/\eta(\tau)^6$ is a Jacobi form of weight -2 and index 1 [EZ], so that $\exp l(x) = \frac{-\rho}{K(\tau,w)^2}$ is single-valued. The way it transforms under the Jacobi group J can be read-off of Lemma 15. We find that

$$\exp l((a, b).x) = \exp(2\pi a^2 i\tau + 2aw) \exp l(x), \quad (a, b) \in \mathbb{Z}^2, \tag{152}$$
$$\exp l(\gamma_1.x) = \exp\left(-\frac{1}{2\pi i}\frac{c_1 w^2}{c_1\tau + d_1}\right) \exp l(x), \quad \gamma_1 \in \Gamma_1, \tag{153}$$

where (a, b) is the image of $\mu(a, b, c)$ in J. For a given choice of the branch $l(x)$, we therefore find that

$$l((a, b).x) = l(x) + 2\pi a^2 i\tau + 2aw + 2\pi i N(a, b), \quad (a, b) \in \mathbb{Z}^2,$$

for some $N(a, b) \in \mathbb{Z}$.

Let $\tilde{l}(\tilde{x})$ be a lifting of $l(x)$ to a single-valued function on $\tilde{\mathcal{D}}^{\rho}$. $K(\tau, w)^2$ is holomorphic for $(\tau, w) \in \mathbb{H}_1 \times \mathbb{C}$ with a zero of order two for each $w \in \Lambda_\tau$ (see (18)). Let $\tilde{x} \in \tilde{\mathcal{D}}^{\rho}$ and $p_1(\tilde{x}) = x = (\tau, w, \rho)$. Using (152) we find:

$$\begin{aligned}
\tilde{l}(\alpha.\tilde{x}) &= \tilde{l}(\tilde{x}) + 2\pi i \tau + 2w + 2\pi i N_\alpha,\\
\tilde{l}(\beta.\tilde{x}) &= \tilde{l}(\tilde{x}) + 2\pi i N_\beta,\\
\tilde{l}(\eta.\tilde{x}) &= \tilde{l}(\tilde{x}) + 2\pi i,\\
\tilde{l}(\delta.\tilde{x}) &= \tilde{l}(\tilde{x}) + 4\pi i,
\end{aligned}$$

for some $N_\alpha, N_\beta \in \mathbb{Z}$. In particular, note that by composing these transformations we confirm the relation $[\alpha, \beta] = \delta$. We may define new generators $\alpha' = \alpha\eta^{-N_\alpha}$, $\beta' = \beta\eta^{-N_\beta}$, which satisfy the same relations and for which $N_{\alpha'} = N_{\beta'} = 0$. Relabelling, we then obtain

Lemma 16. *With previous notation, we have*

$$\tilde{l}(\alpha^a \beta^b \gamma^c \delta^d.\tilde{x}) = \tilde{l}(\tilde{x}) + 2\pi i a^2 \tau + 2aw + 2\pi i (c + 2(ab + d)), \tag{154}$$

for $a, b, c, d \in \mathbb{Z}$. In particular,

$$\tilde{l}(\eta^{-2}\delta.\tilde{x}) = \tilde{l}(\tilde{x}),$$

so that $\tilde{l}$ pushes down to a single-valued function $\hat{l}$ on $\hat{\mathcal{D}}^{\rho}$.

From (153) we find for $\gamma \in \Gamma_1$ that

$$l(\gamma_1.x) = l(x) - \frac{1}{2\pi i}.\frac{c_1 w^2}{c_1\tau + d_1} + 2\pi i N(\gamma_1), \tag{155}$$

for some $N(\gamma_1) \in \mathbb{Z}$. It is easy to see that (155) is consistent with respect to the composition of $\gamma_1, \gamma_2 \in \Gamma_1$ with a trivial cocycle condition $N(\gamma_1\gamma_2) = N(\gamma_1) + N(\gamma_2)$. Thus the extension (150) splits, in particular G contains the subgroup $L = \hat{H}\Gamma_1$ and $G = L \times \mathbb{Z}$. Since $L \cap \langle \eta^{-2}\delta \rangle = 1$ there is an injection

$$L \longrightarrow G/\langle \eta^{-2}\delta \rangle,$$

and through this map L acts on $\hat{\mathcal{D}}^{\rho}$. We can now read-off from (154) and (155) that $\hat{l}$ transforms as follows:

Theorem 10. *The action of L on $\hat{\mathcal{D}}^{\rho}$ satisfies*

$$\hat{l}(\mu(a, b, c).\hat{x}) = \hat{l}(\hat{x}) + 2\pi i a^2 \tau + 2aw + 2\pi i (ab + c), \quad \mu(a, b, c) \in \hat{H}, \tag{156}$$

$$\hat{l}(\gamma_1.\hat{x}) = \hat{l}(\hat{x}) - \frac{1}{2\pi i}.\frac{c_1 w^2}{c_1\tau + d_1}, \quad \gamma_1 \in \Gamma_1, \tag{157}$$

where $\hat{x} \in \hat{\mathcal{D}}^{\rho}$, $p_4(\hat{x}) = (\tau, w, \rho)$.

6.3.4. Equivariance of $\hat{F}^{\rho}$ and F^{ρ}. After the results of the previous subsection we know that F^{ρ} lifts to a single-valued holomorphic function $\hat{F}^{\rho}$ on $\hat{\mathcal{D}}^{\rho}$:

$$\begin{aligned} \hat{F}^{\rho} : \hat{\mathcal{D}}^{\rho} &\to \mathbb{H}_2, \\ \hat{x} &\mapsto \hat{\Omega}^{(2)}(\hat{x}). \end{aligned} \tag{158}$$

By Proposition 12 we have

$$\begin{aligned} 2\pi i \hat{\Omega}^{(2)}_{11}(\hat{x}) &= 2\pi i \tau - \rho \omega_{11}(x), \\ 2\pi i \hat{\Omega}^{(2)}_{12}(\hat{x}) &= w - \rho^{1/2} \omega_{\beta 1}(x), \\ 2\pi i \hat{\Omega}^{(2)}_{22}(\hat{x}) &= \hat{l}(\hat{x}) - \omega_{\beta\bar{\beta}}(x), \end{aligned}$$

where $(\tau, w, \rho) = x = p_4(\hat{x})$.

Theorem 11. *$\hat{F}^{\rho}$ is equivariant with respect to the action of L. Thus, there is a commutative diagram for $\gamma \in L$,*

$$\begin{array}{ccc} \hat{\mathcal{D}}^{\rho} & \xrightarrow{\hat{F}^{\rho}} & \mathbb{H}_2 \\ \gamma \downarrow & & \downarrow \gamma \\ \hat{\mathcal{D}}^{\rho} & \xrightarrow{\hat{F}^{\rho}} & \mathbb{H}_2 \end{array}$$

Proof. It suffices to consider the separate actions of $\hat{H}$ and Γ_1, and we first consider that of $\hat{H}$. From (36), elements of $\hat{H}$ act as follows:

$$\mu(a, b, c) : \hat{\Omega} \mapsto \begin{pmatrix} \hat{\Omega}_{11}, & \hat{\Omega}_{12} + a\hat{\Omega}_{11} + b \\ \hat{\Omega}_{12} + a\hat{\Omega}_{11} + b, & \hat{\Omega}_{22} + a^2\hat{\Omega}_{11} + 2a\hat{\Omega}_{12} + ab + c \end{pmatrix}. \tag{159}$$

We must show that the matrix in (159) coincides with $\hat{F}^{\rho}(\mu(a, b, c).\hat{x})$. Set $x = (\tau, w, \rho)$. Using (144), the periodicity of $P_k(\tau, w)$ in w for $k > 1$, and the quasi-periodicity of $P_1(\tau, w)$ (14), we find that $R(k, l)$ and $\beta(k)$ satisfy

$$R(k, l)((a, b).x) = R(k, l)(x), \tag{160}$$
$$\beta(k)((a, b).x) = \beta(k)(x) + a\rho^{1/2}\delta_{k,1}. \tag{161}$$

Thus ω_{11}, $\omega_{\beta 1}$ and $\omega_{\beta\bar{\beta}}$ satisfy

$$\begin{aligned} \omega_{11}((a, b).x) &= \omega_{11}(x), \\ \omega_{\beta 1}((a, b).x) &= \omega_{\beta 1}(x) + a\rho^{1/2}\omega_{11}(x), \\ \omega_{\beta\bar{\beta}}((a, b).x) &= \omega_{\beta\bar{\beta}}(x) + a^2\rho\omega_{11}(x) + 2a\rho^{1/2}\omega_{\beta 1}(x). \end{aligned}$$

We therefore find

$$\hat{\Omega}_{11}(\mu(a, b, c).\hat{x}) = \tau - \frac{\rho}{2\pi i}\omega_{11}((a, b).x) = \hat{\Omega}_{11}(\hat{x}).$$

Similarly, we have

$$\begin{aligned} \hat{\Omega}_{12}(\mu(a, b, c).\hat{x}) &= \frac{1}{2\pi i}(w + 2\pi i a\tau + 2\pi i b - \rho^{1/2}\omega_{\beta 1}((a, b).x)) \\ &= \hat{\Omega}_{12}(\hat{x}) + a\hat{\Omega}_{11}(\hat{x}) + b. \end{aligned}$$

Now application of (156) yields

$$\begin{aligned}\hat{\Omega}_{22}(\mu(a,b,c).\hat{x}) &= \frac{1}{2\pi i}(\hat{l}(\mu(a,b,c).\hat{x}) - \omega_{\beta\bar{\beta}}((a,b).x)) \\ &= \frac{1}{2\pi i}(\hat{l}(\hat{x}) + 2\pi i a^2\tau + 2aw + 2\pi i(ab+c) \\ &\quad -\omega_{\beta\bar{\beta}}(x) - a^2\rho\omega_{11}(x) - 2a\rho^{1/2}\omega_{\beta 1}(x)) \\ &= \hat{\Omega}_{22} + a^2\hat{\Omega}_{11} + 2a\hat{\Omega}_{12} + ab + c.\end{aligned}$$

This establishes equivariance of $\hat{F}^\rho$ with respect to $\hat{H}$.

As in the ϵ-formalism, the exceptional transformation law (12) for E_2 plays a critical rôle in establishing Γ_1-equivariance of $\hat{F}^\rho$. Consider the action (89) of Γ_1 on $\mathbb{H}_2$. Since E_2 appears only in $R(1,1)$ and $\beta(1)$, (145) implies that

$$R(k,l)(\gamma_1.x) = R(k,l)(x) + \kappa\delta_{k,1}\delta_{l,1}, \tag{162}$$

$$\beta(k)(\gamma_1.x) = \beta(k)(x) - \kappa\frac{w}{\rho^{1/2}}\delta_{k,1}, \tag{163}$$

$$\kappa = \frac{c_1}{c_1\tau + d_1}\frac{\rho}{2\pi i}. \tag{164}$$

We then have

$$\hat{\Omega}_{11}(\gamma_1.\hat{x}) = \frac{1}{c_1\tau + d_1}(a_1\tau + b_1 - \frac{1}{2\pi i}\frac{\rho}{c_1\tau + d_1}\omega_{11}(\gamma_1.x)).$$

Similarly to Theorem 5 in the ϵ formalism, (162) implies that the transformation under γ_1 of the weight $\omega(N)$ for $N \in \mathcal{N}_{1,a;1,b}$ is the sum of the weights of the product over all possible necklaces in $\mathcal{N}_{1,a;1,b}$ formed from N by deleting the edges of type $\overset{1,a_1}{\bullet} \longrightarrow \overset{1,a_2}{\bullet}$ and multiplying by a κ factor for each such deletion. From Proposition 12 and (164) we obtain

$$1 - \kappa\omega_{11} = (c_1\hat{\Omega}_{11} + d_1)/(c_1\tau + d_1).$$

Then we find, much as before, that

$$\begin{aligned}\omega_{11}(\gamma_1.x) &= \sum_{n\geq 0}\kappa^n\omega_{11}^{n+1}(x) \\ &= \frac{(c_1\tau + d_1)\omega_{11}(x)}{c_1\hat{\Omega}_{11} + d_1}.\end{aligned} \tag{165}$$

Then $\hat{\Omega}_{11}(\gamma_1.\hat{x})$ is as given in (89).

We next have

$$\hat{\Omega}_{12}(\gamma_1.\hat{x}) = \frac{1}{c_1\tau + d_1}\frac{1}{2\pi i}(w - \rho^{1/2}\omega_{\beta 1}(\gamma_1.x)).$$

Equations (162) and (163) imply that $\sum_k \beta_a(k)\omega(N)$ for $N \in \mathcal{N}_{k,a;1,b}$ transforms under γ_1 as a sum over the product with κ factors of weights of necklaces in $\mathcal{N}_{1,a;1,b}$ and at most one necklace in $\mathcal{N}_{k,a;1,b}$ with a $\beta_a(k)$ factor. Then one finds

$$\rho^{1/2}\omega_{\beta 1}(\gamma_1.x) = \frac{\rho^{1/2}\omega_{\beta 1}(x) - \kappa w \omega_{11}(x)}{1-\kappa\omega_{11}(x)}$$
$$= w - 2\pi i(c_1\tau + d_1).\frac{\hat{\Omega}_{12}}{c_1\hat{\Omega}_{11} + d_1}.$$

This implies $\hat{\Omega}_{12}(\gamma_1.\hat{x})$ is as given in (89).

Finally, using (157) we have

$$\hat{\Omega}_{22}(\gamma_1.\hat{x}) = \frac{1}{2\pi i}(\hat{l}(\hat{x}) + \kappa\frac{w^2}{\rho} - \omega_{\beta\bar{\beta}}(\gamma_1.x)).$$

Using a similar argument, (162) and (163) imply that

$$\omega_{\beta\bar{\beta}}(\gamma_1.x) = \omega_{\beta\bar{\beta}}(x) + \frac{\kappa\omega_{\beta 1}^2(x) - 2\kappa\frac{w}{\rho^{1/2}}\omega_{\beta 1}(x) + \kappa^2\frac{w^2}{\rho}\omega_{11}(x)}{1-\kappa\omega_{11}}$$
$$= \omega_{\beta\bar{\beta}}(x) - \kappa\frac{w^2}{\rho} + 2\pi i\frac{c_1\hat{\Omega}_{12}^2}{c_1\hat{\Omega}_{11} + d_1}.$$

Hence $\hat{\Omega}_{22}(\gamma_1.\hat{x})$ is as given in (89) and hence Γ_1 acts equivariantly. This completes the proof of the theorem. □

Remark 4. Much the same as in Remark 1, $\Omega_{22} \to \Omega_{22} + 1$ is generated by C corresponding to a Dehn twist in the connecting cylinder.

We may also choose a branch of $l(x)$ and consider the equivariance of F^ρ on $\mathcal{D}^\rho$ under the action of the subgroup Γ_1.

Corollary 2. *For any choice of branch for $l(x)$, F^ρ is equivariant with respect to the action of Γ_1. Thus, there is a commutative diagram for $\gamma \in \Gamma_1$,*

$$\begin{array}{ccc} \mathcal{D}^\rho & \stackrel{F^\rho}{\to} & \mathbb{H}_2 \\ \gamma\downarrow & & \downarrow\gamma \\ \mathcal{D}^\rho & \stackrel{F^\rho}{\to} & \mathbb{H}_2 \end{array}$$
□

6.4. Local Invertibility About the Two Tori Degeneration Limiting Point. We now consider the invertibility of the F^ρ map in the neighborhood of a degeneration point. In the ρ-formalism this degeneration is more subtle than that of the ϵ-formalism discussed in Subsect. 4.5. Define the Γ_1 -invariant parameter

$$\chi = -\frac{\rho}{w^2}. \tag{166}$$

We will show that $\rho, w \to 0$ for fixed χ is the 2 -torus degeneration limit. From (132) we have $|w| > 2|\rho|^{1/2}$ (for $\lambda = 0$) so that $|\chi| < \frac{1}{4}$. (Similarly to Subsect. 5.2.2, $\chi = \frac{1}{4}$

is a singular point where two degenerate annuli touch at $z_1 = -z_2 = w/2$.) To describe this limit more precisely, we introduce the domain

$$\mathcal{D}^\chi = \{(\tau, w, \chi) \in \mathbb{H}_1 \times \mathbb{C} \times \mathbb{C} \mid (\tau, w, -w^2\chi) \in \mathcal{D}^\rho, 0 < |\chi| < \frac{1}{4}\}. \quad (167)$$

Thanks to Theorem 9 and Corollary 2, there is a Γ_1-equivariant holomorphic map

$$\begin{aligned} F^\chi &: \mathcal{D}^\chi \to \mathbb{H}_2, \\ (\tau, w, \chi) &\mapsto \Omega^{(2)}(\tau, w, -w^2\chi). \end{aligned} \quad (168)$$

Let

$$\mathcal{D}_0^\chi = \{(\tau, 0, \chi) \in \mathbb{H}_1 \times \mathbb{C} \times \mathbb{C} | 0 < |\chi| < \frac{1}{4}\}$$

denote the space of limit points where $\rho, w \to 0$ for fixed $\chi \neq 0$. Then

Proposition 13. *For* $(\tau, w, \chi) \in \mathcal{D}^\chi \cup \mathcal{D}_0^\chi$ *we have*

$$\begin{aligned} 2\pi i \Omega_{11}^{(2)} &= 2\pi i \tau + w^2(1 - 4\chi)G(\chi) + O(w^4), \\ 2\pi i \Omega_{12}^{(2)} &= w\sqrt{1 - 4\chi}(1 + w^2(1 - 4\chi)E_2(\tau)G(\chi) + O(w^4)), \\ 2\pi i \Omega_{22}^{(2)} &= \log f(\chi) + w^2(1 - 4\chi)E_2(\tau) + O(w^4), \end{aligned} \quad (169)$$

where

$$G(\chi) = \frac{1}{12} + E_2(q = f(\chi)),$$

and $f(\chi)$ *is the Catalan series (124).*

Proof. Note that $P_n(\tau, w) = \frac{1}{w^n}(1 + w^2 E_2(\tau)(\delta_{n,2} - \delta_{n,1}) + O(w^4))$ from (9) and (13). Then (137) and (138) imply

$$\begin{aligned} R(k, l) &= R^{(0)}(k, l) + w^2 \chi E_2(\tau)\delta_{k,1}\delta_{l,1} + O(w^4), \\ \beta(k) &= \beta^{(0)}(k)(1 - w^2 E_2(\tau)\delta_{k,1}) + O(w^4), \\ \log\left(-\frac{\rho}{K(\tau, w)^2}\right)) &= \log \chi + w^2 E_2(\tau) + O(w^4), \end{aligned}$$

using (127). For $w = 0$ these expressions are exactly those found in Proposition 9 for a torus formed from a sphere by sewing an annulus centered at $z = 0$ to another centered at $z = w$. Expanding (134) with $\rho = -\chi w^2$ to order w^2 implies

$$2\pi i \Omega_{11}^{(2)} = 2\pi i \tau + w^2 \chi \sigma((I - R^{(0)})^{-1}(1, 1)) + O(w^4).$$

But (130) implies

$$\sigma((I - R^{(0)})^{-1}(1, 1)) = 2(I + B^{(0)})^{-1}(1, 1) = \frac{(1 - 4\chi)}{\chi} G(\chi), \quad (170)$$

leading to the stated result for $\Omega_{11}^{(2)}$. Similarly, for $\Omega_{12}^{(2)}$ we find

$$\begin{aligned} 2\pi i \Omega_{12}^{(2)} = w[1 &- (-\chi)^{1/2}\sigma(\beta^{(0)}(1 - R^{(0)})^{-1}(1))], \\ &[1 + w^2 E_2(\tau)\chi\sigma((1 - R^{(0)})^{-1}(1, 1)) + O(w^4)]. \end{aligned}$$

After some algebra and using (123), (128) and (129) one finds

$$\begin{aligned}1-(-\chi)^{1/2}\sigma(\beta^{(0)}(1-R^{(0)})^{-1}(1)) &= 1-2\chi\sum_{n\geq 1}S_{n,1}(\chi)\\ &= 1-2\chi(1+f(\chi))\\ &= \sqrt{1-4\chi}.\end{aligned}$$

The stated result for $\Omega_{12}^{(2)}$ then follows on using (170) again. Finally, for $\Omega_{22}^{(2)}$ we find as above, using Proposition 9, that

$$\begin{aligned}2\pi i\Omega_{22}^{(2)} &= \log\chi-\beta^{(0)}(1-R^{(0)})^{-1}\bar{\beta}^{(0)T}\\ &\quad +E_2(\tau)w^2[1-(-\chi)^{1/2}\sigma(\beta^{(0)}(1-R^{(0)})^{-1}(1))]^2+O(w^4)\\ &= \log f(\chi)+w^2(1-4\chi)E_2(\tau)+O(w^4). \quad \square\end{aligned}$$

The restriction of F^χ to $\mathcal{D}_0^\chi$ induces the natural identification

$$\begin{aligned}F^\chi : \mathcal{D}_0^\chi &\tilde{\to} \mathbb{H}_1\times\mathbb{H}_1\subseteq\mathbb{H}_2\\ (\tau,0,\chi) &\mapsto \begin{pmatrix}\tau & 0\\ 0 & \frac{1}{2\pi i}\log f(\chi)\end{pmatrix},\end{aligned} \tag{171}$$

i.e., $\Omega^{(2)} = \mathrm{diag}(\Omega_{11}^{(2)},\Omega_{22}^{(2)})$. The invertibility of the Γ_1-equivariant map F^χ in a neighborhood of a point in $\mathcal{D}_0^\chi$ then follows:

Proposition 14. *Let $x\in\mathcal{D}_0^\chi$. Then there exists a Γ_1−invariant neighborhood $\mathcal{N}_x^\chi\subseteq\mathcal{D}^\chi$ of x throughout which F^χ is invertible.*

Proof. The proof is very similar to Proposition 5. Let $x=(\tau,0,\chi)$. The Jacobian of the F^χ map at x is from (169)

$$\begin{aligned}\left|\frac{\partial(\Omega_{11},\Omega_{12},\Omega_{22})}{\partial(\tau,w,\chi)}\right|_x &= \begin{vmatrix}1 & 0 & 0\\ 0 & \frac{1}{2\pi i}\sqrt{1-4\chi} & 0\\ 0 & 0 & \frac{1}{2\pi i}\frac{f'(\chi)}{f(\chi)}\end{vmatrix}\\ &= \frac{1}{4\pi^2\chi}\neq 0,\end{aligned}$$

using $f'(\chi) = f(\chi)/(\chi\sqrt{1-4\chi})$. By the inverse function theorem, there exists an open neighborhood of $x\in\mathcal{D}_0^\chi$ throughout which F^χ is invertible. The result then follows by choosing precisely invariant neighborhoods (under the action of Γ_1) U, V of x, respectively $y=F^\chi(x)$ such that the conditions of part (a) of Lemma 9 hold. The open neighborhood

$$\mathcal{N}_x^\chi=\bigcup_{\gamma\in\Gamma_1}\gamma U$$

has the required properties. $\square$

7. Mapping Between the ϵ and ρ Parameterizations

We have described in the previous sections two separate parameterizations for the genus two period matrix $\Omega^{(2)}$ based on either sewing two punctured tori in the ϵ-formalism or sewing a twice-punctured torus to itself in the ρ-formalism. In this final section we show that there is a 1-1 mapping between suitable Γ_1-invariant domains in both parameterizations.

Theorem 12. *There exists a 1-1 holomorphic mapping between Γ_1-invariant open domains $\mathcal{I}^\chi \subset \mathcal{D}^\chi$ and $\mathcal{I}^\epsilon \subset \mathcal{D}^\epsilon$, where $\mathcal{I}^\chi$ and $\mathcal{I}^\epsilon$ are open neighborhoods of a 2-torus degeneration point.*

Proof. From Proposition 5 there exists a G-invariant (and thus Γ_1-invariant) open domain $\mathcal{N}^\epsilon \subset \mathcal{D}^\epsilon$ such that the holomorphic map $F^\epsilon : \mathcal{N}^\epsilon \to F^\epsilon(\mathcal{N}^\epsilon)$ is invertible with $F^\epsilon(\mathcal{N}^\epsilon)$ an open neighborhood of a given 2-torus degeneration point $\Omega^{(2)} = \mathrm{diag}(\Omega_{11}^{(2)}, \Omega_{22}^{(2)})$. Similarly, from Proposition 14 there exists a Γ_1-invariant open domain $\mathcal{N}^\chi \subset \mathcal{D}^\chi$ such that the holomorphic map $F^\chi : \mathcal{N}^\chi \to F^\chi(\mathcal{N}^\chi)$ is invertible with $F^\chi(\mathcal{N}^\chi)$ an open neighborhood of $\mathrm{diag}(\Omega_{11}^{(2)}, \Omega_{22}^{(2)})$. Define a Γ_1-invariant open neighborhood of $\mathrm{diag}(\Omega_{11}^{(2)}, \Omega_{22}^{(2)})$ by $\mathcal{I}^\Omega = F^\epsilon(\mathcal{N}^\epsilon)\cap F^\chi(\mathcal{N}^\chi)$. Hence, defining Γ_1-invariant open domains $\mathcal{I}^\chi = (F^\chi)^{-1}(\mathcal{I}^\Omega)$ and $\mathcal{I}^\epsilon = (F^\epsilon)^{-1}(\mathcal{I}^\Omega)$, we find $(F^\epsilon)^{-1}\circ F^\chi : \mathcal{I}^\chi \to \mathcal{I}^\epsilon$ is holomorphic and 1-1. □

We conclude by displaying the explicit form of the 1-1 mapping to order w^3, obtained by combining the expansions of (102)-(104) and Proposition 13:

$$\begin{aligned}\tau_1(\tau, w, \chi) &= \tau + \frac{1}{2\pi i} w^2(1-4\chi)\frac{1}{12} + O(w^4),\\ \tau_2(\tau, w, \chi) &= \frac{1}{2\pi i}\log(f(\chi)) + O(w^4),\\ \epsilon(\tau, w, \chi) &= -w\sqrt{1-4\chi}(1 + w^2 E_2(\tau)(1-4\chi) + O(w^4)).\end{aligned} \tag{172}$$

It is then straightforward to check that these relations are invariant under the action of Γ_1 to the given order using (12), (86) and (145).

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8. Appendix

In this appendix we supply the explicit form of the genus two period matrix $\Omega^{(2)}$ of Theorem 4 in the ϵ -formalism to $O(\epsilon^9)$ and of Theorem 9 in the ρ-formalism to $O(\rho^5)$.

$$\begin{aligned}2\pi i\Omega_{11}^{(2)}(\tau_1, \tau_2, \epsilon) &= 2\pi i\Omega_{22}^{(2)}(\tau_2, \tau_1, \epsilon)\\ &= 2\pi i\tau_1 + F_2\epsilon^2 + E_2{F_2}^2\epsilon^4 + ({E_2}^2{F_2}^3 + 6\, E_4F_2F_4)\epsilon^6\\ &\quad + ({E_2}^3{F_2}^4 + 12\, E_2E_4{F_2}^2F_4 + 10\, E_6F_2F_6 + 30\, E_6{F_4}^2)\epsilon^8\\ &\quad + O(\epsilon^{10}),\\ 2\pi i\Omega_{12}^{(2)}(\tau_1, \tau_2, \epsilon) &= -\epsilon[1 + E_2F_2\epsilon^2 + ({E_2}^2{F_2}^2 + 3\, E_4F_4)\epsilon^4\\ &\quad + ({E_2}^3{F_2}^3 + 9\, E_2E_4F_2F_4 + 5\, E_6F_6)\epsilon^6\\ &\quad + ({E_2}^4{F_2}^4 + 15\, {E_2}^2E_4{F_2}^2F_4 + 5\, E_2E_6F_2F_6 + 30\, E_2E_6{F_4}^2\\ &\quad + 30\, {E_4}^2F_2F_6 + 9\, {E_4}^2{F_4}^2 + 7\, E_8F_8)\epsilon^8] + O(\epsilon^{11}),\end{aligned}$$

where for brevity's sake we have defined $E_k = E_k(\tau_1)$ and $F_k = E_k(\tau_2)$.

Similarly, in the ρ-formalism we find that $\Omega^{(2)}(\tau, w, \rho)$ to $O(\rho^4)$ is as follows:

$$\begin{aligned}
2\pi i \Omega_{11}^{(2)} &= 2\pi i \tau - 2\,\rho + 2\,(\,P_2 + E_2)\,\rho^2 - 2\,\,(P_2 + E_2)^2\,\rho^3 \\
&\quad +2(\,(P_2 + E_2)^3 + 2\,P_3{}^2)\rho^4 + O(\rho^5), \\
2\pi i \Omega_{12}^{(2)} &= w + 2\,P_1 \rho - 2\,P_1\,(P_2 + E_2)\,\rho^2 \\
&\quad +2[\,P_1\,(P_2 + E_2)^2 + \,P_3\,(P_2 - E_2)]\rho^3 - 2[\,P_3\,(P_4 + E_4) \\
&\quad +P_1\,(P_2 + E_2)^3 + 2\,P_1 P_3{}^2 + \,P_3(P_2{}^2 - E_2{}^2)]\rho^4 + O(\rho^5), \\
2\pi i \Omega_{22}^{(2)} &= \log(-\frac{\rho}{K^2(w,\tau)}) - 2\,P_1{}^2\rho + [2\,P_1{}^2\,(P_2 + E_2) + (P_2 - E_2)^2]\rho^2 \\
&\quad -[2\,P_1{}^2\,(P_2 + E_2)^2 + 2/3\,P_3{}^2 + 4\,P_1 P_3\,(P_2 - E_2)]\rho^3 \\
&\quad +[1/2\,P_4{}^2 + 1/2\,E_4{}^2 + 3\,\,(P_4 - E_4)\,(P_2 - E_2)^2 + 2\,P_1{}^2\,(P_2 + E_2)^3 \\
&\quad -E_4 P_4 + 4\,P_3 P_1 (P_1 P_3 + E_4 + P_4 + P_2{}^2 - E_2{}^2)]\rho^4 + O(\rho^5),
\end{aligned}$$

where $E_k = E_k(\tau)$ and $P_k = P_k(w, \tau)$.

References

[DGP] D'Hoker, E., Gutperle, M., Phong, D.H.: Two-loop superstrings and s-duality. Nucl. Phys. B **722**, 81–118 (2005)

[DLM] Dong, C., Li, H., Mason, G.: Regularity of rational vertex operator algebras. Adv. Math. **132**(1), 148–166 (1997)

[DM1] Dong, C., Mason, G.: Rational vertex operator algebras and the effective central charge. Int. Math. Res. Not. No. **56**, 2989–3008 (2004)

[DM2] Dong, C., Mason, G.: Holomorphic vertex operator algebras of small central charge. Pac. J. Math. **213**(2), 253–266 (2004)

[DP] D'Hoker, E., Phong, D.H.: The geometry of string perturbation theory. Rev. Mod. Phys. **60**, 917–1065 (1988)

[DPI] D'Hoker, E., Phong, D.H.: Two-loop superstrings. i: main formulas. Phys. Lett. B **529**, 241–255 (2002)

[DPVI] D'Hoker, E., Phong, D.H.: Two-loop superstrings. vi: non-renormalization theorems and the 4-point function. Nucl. Phys. B **715**, 3–90 (2005)

[EZ] Eichler, M, Zagier, D: The Theory of Jacobi Forms. Birkhäuser, Boston (1985)

[FK1] Farkas, HM, Kra, I: Riemann Surfaces. Springer-Verlag, New York (1980) Farkas, H.M., Kra, I.: Riemann Surfaces. New York: Springer-Verlag 1980

[FK2] Farkas, H.M., Kra, I.: Theta Constants, Riemann Surfaces and the Modular Group. Graduate Studies in Mathematics, Providence, RI: Amer. Math. Soc. 2001

[GSW] Green, M., Schwartz, J., Witten, E.: Superstring Theory Vol. 1. Cambridge: Cambridge University Press, 1987

[Gu] Gunning, R.C.: Introduction to Holomorphic Functions of Several Variables Vol 1. Belmont, CA: Wadsworth & Brooks/Cole (1990)

[H] Huang, Y: Two-Dimensional Conformal Geometry and Vertex Operator Algebras. Birkhäuser, Boston (1997)

[KZ] Kaneko, M., Zagier, D.: A generalized Jacobi theta function and quasimodular forms. In: The Moduli Space of Curves (Texel Island, 1994), Progr. in Math. **129**, Boston: Birkhauser, 1995

[MS] Moore, G., Seiberg, N.: Classical and quantum conformal field theory. Commun. Math. Phys. **123**, 177–254 (1989)

[MT1] Mason, G., Tuite, M.P.: Torus chiral n-point functions for free boson and lattice vertex operator algebras. Commun. Math. Phys. **235**, 47–68 (2003)

[MT2] Mason, G., Tuite, M.P.: The genus two partition function for free bosonic and lattice vertex operator algebras. To appear

[Mu] Mumford, D.: Tata Lectures on Theta I and II. Boston: Birkhäuser, 1983

[P] Polchinski, J.: String Theory, Volume **I**. Cambridge: Cambridge University Press, 1998

[Se] Serre, J-P.: A Course in Arithmetic. Berlin: Springer-Verlag, 1978

[St] Stanley, R.P.: Enumerative Combinatorics, Volume **2**, Cambridge: Cambridge University Press, 1999

[So] Sonoda, H.: Sewing conformal field theories I. Nucl. Phys. **B311**, 401–416 (1988); Sewing conformal field theories II. Nucl. Phys. **B311**, 417–432 (1988)
[Sp] Springer, G.: Introduction to Riemann Surfaces. Reading, MA:, Addison-Wesley, (1957)
[T] Tuite, M.P: Genus two meromorphic conformal field theory, CRM Proceedings and Lecture Notes **30**, Providence, RI: Amet. Math. Soc., 2001, pp. 231–251
[VV] Verlinde, E.P., Verlinde, H.L.: Chiral bosonization, determinants and the string partition function. Nucl. Phys. B **288**, 357–396 (1987)
[Y] Yamada, A.: Precise variational formulas for abelian differentials. Kodai Math. J. **3**, 114–143 (1980)
[Z] Zhu, Y.: Modular invariance of characters of vertex operator algebras. J. Amer. Math. Soc. **9**, 237–302 (1996)

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