Commun. Math. Phys. 270, 575–585 (2007)
Digital Object Identifier (DOI) 10.1007/s00220-006-0157-3

Communications in
Mathematical Physics

Nonzero Kronecker Coefficients and What They Tell us about Spectra

Matthias Christandl[1], Aram W. Harrow[2], Graeme Mitchison[1]

[1] Centre for Quantum Computation, DAMTP, University of Cambridge, Wilberforce Road, Cambridge, CB3 0WA, UK. E-mail: matthias.christandl@qubit.org; g.j.mitchison@damtp.cam.ac.uk
[2] Department of Computer Science, University of Bristol, Bristol, BS8 1UB, UK. E-mail: a.harrow@bris.ac.uk

Received: 20 February 2006 / Accepted: 27 June 2006
Published online: 9 January 2007 –

Abstract: A triple of spectra (r^A, r^B, r^{AB}) is said to be *admissible* if there is a density operator ρ^{AB} with

$$(\text{Spec } \rho^A, \text{Spec } \rho^B, \text{Spec } \rho^{AB}) = (r^A, r^B, r^{AB}).$$

How can we characterise such triples? It turns out that the admissible spectral triples correspond to Young diagrams (μ, ν, λ) with nonzero Kronecker coefficient $g_{\mu\nu\lambda}$ [5, 14]. This means that the irreducible representation of the symmetric group V_λ is contained in the tensor product of V_μ and V_ν. Here, we show that such triples form a finitely generated semigroup, thereby resolving a conjecture of Klyachko [14]. As a consequence we are able to obtain stronger results than in [5] and give a complete information-theoretic proof of the correspondence between triples of spectra and representations. Finally, we show that spectral triples form a convex polytope.

1. Introduction

A curious connection between representation theory and the spectra of operators was discovered recently. Suppose we are given a bipartite density operator ρ^{AB}, and suppose this has spectrum $r_{AB} = \text{Spec } (\rho^{AB})$. Let r^A be the spectrum of the marginal operator $\rho^A = \text{Tr }_B \rho^{AB}$, and r^B that of the other marginal operator ρ^B. Then clearly there are restrictions on the possible spectral triples (r^A, r^B, r^{AB}) as ρ^{AB} ranges over all density operators. For instance, if ρ^{AB} is pure, so $r^{AB} = (1, 0, \dots)$, then $r^A = r^B$. How does one characterise the set of possible spectral triples? One way to do this is via representation theory [5, 14]: there is a correspondence between triples of spectra and irreducible representations of the symmetric group V_μ, V_ν and V_λ, where

$$V_\lambda \subset V_\mu \otimes V_\nu. \tag{1}$$

Two rather different methods were used to prove this. In [14] a body of powerful techniques from invariant theory [11, 15, 2] were harnessed (see also [16, 7]). In [5], the approach came from the direction of quantum information theory, and a key ingredient was a theorem relating spectra and Young diagrams due to Alicki, Rudnicki and Sadowski [1] and Keyl and Werner [12]. This theorem can be given a short and elegant proof [10] (see also [5]) that has interesting parallels with classical information theory. To those with an information theory background, therefore, the approach taken in [5] has some advantages of accessibility. It is shown there that for every density operator ρ^{AB} there is a sequence of triples $(\mu^{(j)}, \nu^{(j)}, \lambda^{(j)})$ satisfying relation (1) that converges to the spectra:

$$\lim_{j\to\infty} (\bar{\mu}^{(j)}, \bar{\nu}^{(j)}, \bar{\lambda}^{(j)}) = (r^A, r^B, r^{AB}),$$

where the bar denotes normalisation. Klyachko [14] proves this as well as a converse that says that to every (μ, ν, λ) with $V_\lambda \subset V_\mu \otimes V_\nu$ there is a density operator with spectra $(\bar{\mu}, \bar{\nu}, \bar{\lambda})$.

One aim of this paper is to show that informational methods can be used to prove Klyachko's converse. On our way to this result we prove his conjecture [14, Conjecture 7.1.4] that triples (μ, ν, λ) with $V_\lambda \subset V_\mu \otimes V_\nu$ form a semigroup. We also prove that the semigroup is finitely generated. Together with our previous results on the correspondence with spectral triples this will imply that the set of admissible spectral triples is a convex polytope.

2. Background

Let us consider in more detail the relation between irreducible representations and spectra. The irreducible representations of both unitary and symmetric groups are labelled by Young diagrams. If λ denotes a Young diagram, its row lengths are $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d$ and its size is $|\lambda| := \sum_{i=1}^{d} \lambda_i$. We denote the corresponding irreducible representations of $\mathrm{U}(d)$ (or $\mathrm{GL}(d)$) with highest weight λ by U_λ^d and those of the symmetric group S_k by V_λ. Schur-Weyl duality states that $(\mathbb{C}^d)^{\otimes k}$ decomposes as a direct sum of irreducible representations:

$$(\mathbb{C}^d)^{\otimes k} \cong \bigoplus_{\lambda \in \mathrm{Par}(k,d)} U_\lambda^d \otimes V_\lambda, \tag{2}$$

where $\mathrm{Par}(k, d)$ indicates the set of partitions of k into $\leq d$ parts; i.e. the Young diagrams with no more than d rows and a total of k boxes.

Consider a density operator ρ on $\mathbb{C}^d$. We can take k copies of it and measure the label λ on $(\mathbb{C}^d)^{\otimes k}$. The estimation theorem [1, 12] states that, as k increases, the spectrum r of ρ is increasingly well approximated by the normalised row lengths of λ, i.e. by the distribution $\bar{\lambda} = \lambda/|\lambda|$. Formally:

Theorem 2.1 (Estimation Theorem). *Let P_λ be the projection onto $U_\lambda^d \otimes V_\lambda$. Then*

$$Tr\, P_\lambda \rho^{\otimes k} \leq (k+1)^{d(d-1)/2} \exp\left(-kD(\bar{\lambda}||r)\right), \tag{3}$$

where $D(p||q) = \sum_i p_i \log(p_i/q_i)$ is the Kullback-Leibler distance.

Let us now return to the case of bipartite states, and consider the Clebsch-Gordan series for the symmetric group:

$$V_\mu \otimes V_\nu \cong \bigoplus_{\lambda \in \mathrm{Par}(k,k)} g_{\mu\nu\lambda} V_\lambda,$$

where the multiplicities $g_{\mu\nu\lambda}$ are known as the *Kronecker coefficients*. Since $V_\lambda \cong V_\lambda^\star$, the Kronecker coefficients can also be defined in terms of the S_k-invariant subspace of $V_\mu \otimes V_\nu \otimes V_\lambda$, i.e.

$$g_{\mu\nu\lambda} = \dim(V_\mu \otimes V_\nu \otimes V_\lambda)^{S_k}. \tag{4}$$

There is also a way of viewing the Kronecker coefficients in terms of the irreducible representations of the general linear group. It is arrived at by equating the Schur-Weyl decompositions of $(\mathbb{C}^m \otimes \mathbb{C}^n)^{\otimes k}$ and of $(\mathbb{C}^{mn})^{\otimes k}$ (see [5]) and reads

$$U_\lambda^{mn}\!\downarrow_{\mathrm{GL}(m)\times\mathrm{GL}(n)} \cong \bigoplus_{\substack{\mu \in \mathrm{Par}(k,m) \\ \nu \in \mathrm{Par}(k,n)}} g_{\mu\nu\lambda} U_\mu^m \otimes U_\nu^n.$$

This interpretation of the Kronecker coefficients can equivalently be stated in terms of invariants as

$$g_{\mu\nu\lambda} = \dim(U_\mu^m \otimes U_\nu^n \otimes U_\lambda^{mn\star})^{\mathrm{GL}(m)\times\mathrm{GL}(n)}, \tag{5}$$

where $\mathrm{GL}(m) \times \mathrm{GL}(n)$ acts on $U_\mu^m \otimes U_\nu^n$ and simultaneously on $U_\lambda^{mn\star}$, the representation dual to U_λ^{mn}, according to the inclusion $\mathrm{GL}(m) \times \mathrm{GL}(n) \to \mathrm{GL}(m) \otimes \mathrm{GL}(n) \subset \mathrm{GL}(mn)$.

In [5] Theorem 2.1 was applied to give the following:

Theorem 2.2. *For every density operator ρ^{AB}, there is a sequence $(\mu^{(j)}, \nu^{(j)}, \lambda^{(j)})$ of partitions, labeled by natural numbers j, such that*

$$g_{\mu^{(j)},\nu^{(j)},\lambda^{(j)}} \neq 0 \quad \textit{for all} \quad j$$

and

$$\lim_{j\to\infty} \bar{\mu}^{(j)} = \mathit{Spec}\ \rho^A, \tag{6}$$

$$\lim_{j\to\infty} \bar{\nu}^{(j)} = \mathit{Spec}\ \rho^B, \tag{7}$$

$$\lim_{j\to\infty} \bar{\lambda}^{(j)} = \mathit{Spec}\ \rho^{AB}. \tag{8}$$

Klyachko [14] derived a very similar theorem:

Theorem 2.3. *For a density operator ρ^{AB} with the rational spectral triple (r^A, r^B, r^{AB}) there is an integer $m > 0$ such that $g_{mr^A,mr^B,mr^{AB}} \neq 0$.*

He also proved a converse, that is given in the following section as Theorem 3.2.

We now give a resumé of our new results.

3. Summary of the Results

Let KRON denote the set of triples (μ, ν, λ) with nonzero Kronecker coefficients. Our first result is

Theorem 3.1. KRON *is a finitely generated semigroup with respect to row-wise addition, i.e.* $g_{\mu\nu\lambda} \neq 0$ *and* $g_{\mu'\nu'\lambda'} \neq 0$ *implies* $g_{\mu+\mu',\nu+\nu',\lambda+\lambda'} \neq 0$.

This was conjectured in Klyachko's paper [14, Conjecture 7.1.4 and below]. It implies stability of the Kronecker coefficients: i.e. if $g_{\mu\nu\lambda} \neq 0$ then $g_{N\mu N\nu N\lambda} \neq 0$, for integers $N > 0$. This was announced by Kirillov [13, Theorem 2.11] but without proof. A simple corollary of stability is that non-vanishing Kronecker coefficients obey entropic relations (as explained in [5]). More importantly, it plays a key role in our information-theoretic proof of the following theorem.

Theorem 3.2. *Let* μ, ν *and* λ *be diagrams with k boxes and at most m, n and mn rows, respectively. If* $g_{\mu\nu\lambda} \neq 0$, *then there exists a density operator* ρ^{AB} *on* $\mathcal{H}_A \otimes \mathcal{H}_B = \mathbb{C}^m \otimes \mathbb{C}^n$ *with spectra*

$$\text{Spec } \rho^A = \bar{\mu}, \tag{9}$$

$$\text{Spec } \rho^B = \bar{\nu}, \tag{10}$$

$$\text{Spec } \rho^{AB} = \bar{\lambda}. \tag{11}$$

We also give a compact version of the proof of Theorem 2.2, which was presented in [5]. In this way we obtain a simple proof of the full correspondence between Kronecker coefficients and admissible spectral triples.

Furthermore, it will allow us to draw the following corollary.

Corollary 3.3. *Theorem* 2.2 *and Theorem* 2.3 *are equivalent.*

Using the correspondences to spectral triples, the fact that KRON is a finitely generated semigroup can be given the following geometrical interpretation.

Theorem 3.4. Spect, *the set of admissible spectral triples, is a convex polytope.*

4. The Set of Nonzero Kronecker Coefficients is a Finitely Generated Semigroup

In order to prove Theorem 3.1, we introduce a representation of $\mathrm{GL}(n)$ which we call the *Cartan product ring*

$$Q^m := \bigoplus_{k\geq 0} \bigoplus_{\mu\in \mathrm{Par}(k,m)} U^m_\mu.$$

Define Q^n similarly and also

$$Q^{mn\star} := \bigoplus_{k\geq 0} \bigoplus_{\lambda\in \mathrm{Par}(k,mn)} U^{mn\star}_\lambda,$$

where $U^{mn\star}_\lambda$ is the representation dual to U^{mn}_λ. We assume here that μ_m, ν_n and λ_{mn} are non-negative because we are ultimately interested in combining irreducible representations of S_k, which are only defined for nonnegative λ. However, all of our results can be easily generalized for dominant weights μ, ν, λ without the non-negativity restrictions. To establish Q^n as a graded ring, we introduce the *Cartan product* [8] that maps $U_\mu \otimes U_\nu$ to $U_{\mu+\nu}$ by projecting onto the unique $U_{\mu+\nu}$-isotypic subspace of $U_\mu \otimes U_\nu$. We denote the Cartan product by $\circ$, so that for $|u_\mu\rangle \in U_\mu$, $|u_\nu\rangle \in U_\nu$, $|u_\mu\rangle \circ |u_\nu\rangle$ is defined to be the projection of $|u_\mu\rangle \otimes |u_\nu\rangle$ onto the $U_{\mu+\nu}$-isotypic subspace of $U_\mu \otimes U_\nu$. Clearly Q^n is graded under the action of $\circ$, $\mathrm{GL}(n)$ preserves this grading and $\mathrm{GL}(n)$ acts properly on products, i.e. $g(|\alpha\rangle \circ |\beta\rangle) = (g|\alpha\rangle) \circ (g|\beta\rangle)$.

The proof of Theorem 3.1 now rests on the following lemma:

Lemma 4.1. (a) *Q^n has no zero divisors. That is, if $|\alpha\rangle, |\beta\rangle \in Q^n$ are nonzero, then $|\alpha\rangle \circ |\beta\rangle \neq 0$.*

(b) *$Q^m \otimes Q^n \otimes (Q^{mn})^\star$ has no zero divisors.*

Here we have defined $(Q^n)^\star = \bigoplus_\lambda (U^n_\lambda)^\star$ with the corresponding Cartan product $(U^n_\mu)^\star \circ (U^n_\nu)^\star \to (U^n_{\mu+\nu})^\star$ and we have extended the Cartan product to tensor products in the natural way.

Proof. Although only statement (b) of the lemma is used in the proof of the theorem, for ease of exposition we will prove part (a) and then sketch how similar arguments can establish (b). Our proof follows the treatment of the Borel-Weil theorem in [3, p. 115], with notational changes (e.g. we write $\langle\alpha|g|v\rangle$ for what would be called $\alpha(gv)$ there).

Let $|v_\lambda\rangle$ be a highest weight vector for U_λ. For any $|\alpha\rangle \in U_\lambda$, note that $\langle\alpha|g|v_\lambda\rangle$ is

(i) a polynomial in the matrix elements of g,
(ii) identically zero only if $|\alpha\rangle = 0$ (due to the irreducibility of U_λ).

Now define the set $X_\alpha := \{g \in \mathrm{GL}(n) | \langle\alpha|g|v_\lambda\rangle = 0\}$. The above two claims mean that X_α is a proper closed subset of $\mathrm{GL}(n)$ in the Zariski topology whenever $|\alpha\rangle \neq 0$.

Similarly, if $|\beta\rangle \in U_{\lambda'}$ and $|v_{\lambda'}\rangle$ is a highest weight vector for $U_{\lambda'}$ then $X_\beta := \{g \in \mathrm{GL}(n) | \langle\beta|g|v_{\lambda'}\rangle = 0\}$ is a proper Zariski-closed subset of $\mathrm{GL}(n)$ if and only if $|\beta\rangle \neq 0$.

The fact that $|v_\lambda\rangle$ and $|v_{\lambda'}\rangle$ are highest weight vectors means that $|v_\lambda\rangle \otimes |v_{\lambda'}\rangle = |v_{\lambda+\lambda'}\rangle \in U_{\lambda+\lambda'}$ and thus

$$\begin{aligned} \langle\alpha|g|v_\lambda\rangle\langle\beta|g|v_{\lambda'}\rangle &= (\langle\alpha| \otimes \langle\beta|) g (|v_\lambda\rangle \otimes |v_{\lambda'}\rangle) \\ &= (\langle\alpha| \circ \langle\beta|) g |v_{\lambda+\lambda'}\rangle. \end{aligned} \tag{12}$$

We are free to replace $\langle\alpha| \otimes \langle\beta|$ with $\langle\alpha| \circ \langle\beta|$ in the last step because we are taking the inner product with a vector that lies entirely in $U_{\lambda+\lambda'}$. Now suppose $|\alpha\rangle \circ |\beta\rangle = 0$. Then for all g at least one of the terms on the LHS of Eq. (12) vanishes, and thus $\mathrm{GL}(n) = X_\alpha \cup X_\beta$. Since $\mathrm{GL}(n)$ is irreducible, it cannot be the union of two proper closed subsets, and we conclude $|\alpha\rangle$ and $|\beta\rangle$ cannot both be nonzero.

The proof of (b) is almost identical, but consider instead $|\alpha\rangle \in U_\mu \otimes U_\nu \otimes U^\star_\lambda$, $|\beta\rangle \in U_{\mu'} \otimes U_{\nu'} \otimes U^\star_{\lambda'}$ and the group $\mathrm{GL}(m) \times \mathrm{GL}(n) \times \mathrm{GL}(mn)$ (which is still irreducible). $\square$

Note that we could relax the restriction that $\lambda_n \geq 0$ by multiplying the inner products of the form $\langle\alpha|g|v_\lambda\rangle$ by a high enough power of $\det g$ (guaranteed to be nonzero for $g \in \mathrm{GL}(n)$) to obtain a polynomial in the matrix elements of g.

Proof of Theorem 3.1. Given any ring R with an action of G on it, let R^G denote the ring of G-invariants in R. Now recall Eq. (5):

$$g_{\mu\nu\lambda} = \dim(U_\mu \otimes U_\nu \otimes U^\star_\lambda)^{\mathrm{GL}(m)\times\mathrm{GL}(n)}.$$

If $g_{\mu\nu\lambda} \neq 0$ and $g_{\mu'\nu'\lambda'} \neq 0$ then according to this equation there exist nonzero vectors $|u_{\mu\nu\lambda}\rangle \in (U_\mu \otimes U_\nu \otimes U^\star_\lambda)^{\mathrm{GL}(m)\times\mathrm{GL}(n)}$ and $|u_{\mu'\nu'\lambda'}\rangle \in (U_{\mu'} \otimes U_{\nu'} \otimes U^\star_{\lambda'})^{\mathrm{GL}(m)\times\mathrm{GL}(n)}$. Define $|u_{\mu+\mu',\nu+\nu',\lambda+\lambda'}\rangle = |u_{\mu,\nu,\lambda}\rangle \circ |u_{\mu',\nu',\lambda'}\rangle$. Then $|u_{\mu+\mu',\nu+\nu',\lambda+\lambda'}\rangle \neq 0$ by part (b) of Lemma 4.1 and $|u_{\mu+\mu',\nu+\nu',\lambda+\lambda'}\rangle \in U_{\mu+\mu'} \otimes U_{\nu+\nu'} \otimes U^\star_{\lambda+\lambda'}$ is $\mathrm{GL}(m) \times \mathrm{GL}(n)$-invariant, since this property is preserved by the Cartan product. Thus $(U_{\mu+\mu'} \otimes U_{\nu+\nu'} \otimes U^\star_{\lambda+\lambda'})^{\mathrm{GL}(m)\times\mathrm{GL}(n)} \neq 0$ and we conclude that $g_{\mu+\mu',\nu+\nu',\lambda+\lambda'} \neq 0$.

We continue to show that KRON is finitely generated. Note that Q^m, Q^n and $Q^{mn\star}$ are each generated by a basis for the respective fundamental representations[1] and therefore are *finitely generated*. This implies that also $R = Q^m \otimes Q^n \otimes Q^{mn\star}$ is finitely generated. We now consider the ring of invariants R^G of R under the simultaneous action of the group $G = \mathrm{GL}(m) \times \mathrm{GL}(n)$ on $Q^m \otimes Q^n$ and as a subgroup of $\mathrm{GL}(mn)$ on $Q^{mn\star}$. This action is algebraic, i.e. every element of R is contained in a finite-dimensional representation of G. Thus we can apply a generalisation of Hilbert's theorem [18, Theorem 3.6] to conclude that R^G is finitely generated. Since

$$R^G \cong \bigoplus_{\mu,\nu,\lambda} (U^m_\mu \otimes U^n_\nu \otimes U^{mn\star}_\lambda)^{\mathrm{GL}(m)\times\mathrm{GL}(n)}$$

and $g_{\mu\nu\lambda} = \dim(U^m_\mu \otimes U^n_\nu \otimes U^{n\star}_\lambda)^{\mathrm{GL}(m)\times\mathrm{GL}(n)}$ we see that KRON is finitely generated, too. □

A similar proof was given in [6] which proved that under the diagonal action of a connected reductive group G, the triples (μ, ν, λ) with $\dim(U_\mu \otimes U_\nu \otimes U_\lambda)^G \neq 0$ form a finitely generated semigroup.

5. The Correspondence of Nonzero Kronecker Coefficients to Spectra

Proof of Theorem 2.2. Rather than working with the mixed state ρ^{AB} we will consider a purification $|\psi\rangle^{ABC}$ of ρ^{AB}, which has $\mathrm{Spec}\ \rho^C = \mathrm{Spec}\ \rho^{AB}$. Let $r^A := \mathrm{Spec}\ \rho^A$, $r^B := \mathrm{Spec}\ \rho^B$ and $r^C := \mathrm{Spec}\ \rho^C$. P^A_μ denotes the projector onto the Young subspace $U^m_\mu \otimes V_\mu$ in system A, and P^B_ν, P^C_λ are the corresponding projectors onto Young subspaces in B and C, respectively. As a consequence of Theorem 2.1 (see [5, Corollary 2]), for given $\epsilon > 0$ one can find a k_0 such that the following inequalities hold simultaneously for all $k \geq k_0$:

$$\mathrm{Tr}\, P_X(\rho^A)^{\otimes k} \geq 1-\epsilon, \quad P_X := \sum_{\mu:\|\bar{\mu}-r^A\|_1\leq\epsilon} P^A_\mu, \tag{13}$$

$$\mathrm{Tr}\, P_Y(\rho^B)^{\otimes k} \geq 1-\epsilon, \quad P_Y := \sum_{\nu:\|\bar{\nu}-r^B\|_1\leq\epsilon} P^B_\nu, \tag{14}$$

$$\mathrm{Tr}\, P_Z(\rho^C)^{\otimes k} \geq 1-\epsilon, \quad P_Z := \sum_{\lambda:\|\bar{\lambda}-r^C\|_1\leq\epsilon} P^C_\lambda. \tag{15}$$

For $0 < \epsilon < \frac{1}{3}$, the estimates (13)-(15) can be combined to yield

$$\mathrm{Tr}\,[(P_X \otimes P_Y \otimes P_Z)\,(|\psi\rangle\langle\psi|^{ABC})^{\otimes k}] \geq 1-3\epsilon > 0. \tag{16}$$

[1] The fundamental representations of $\mathrm{GL}(d)$ are those with Young diagrams of the form $(1^\ell, 0^{d-\ell})$ for $\ell \in \{1,\ldots,d\}$.

Since $(|\psi\rangle^{ABC})^{\otimes k}$ is evidently invariant under permutation of its k subsystems, it takes the form

$$(|\psi\rangle^{ABC})^{\otimes k} = \sum_{\mu\nu\lambda} |\alpha_{\mu\nu\lambda}\rangle,$$

where $|\alpha_{\mu\nu\lambda}\rangle \in U_\mu^m \otimes U_\nu^n \otimes U_\lambda^{mn} \otimes (V_\mu \otimes V_\nu \otimes V_\lambda)^{S_k}$. Equation (16) then implies that there must be at least one triple (μ, ν, λ) with $\|\bar{\mu} - r^A\|_1 \leq \epsilon$, $\|\bar{\nu} - r^B\|_1 \leq \epsilon$, $\|\bar{\lambda} - r^C\|_1 \leq \epsilon$ and $|\alpha_{\mu\nu\lambda}\rangle \neq 0$. Thus $(V_\mu \otimes V_\nu \otimes V_\lambda)^{S_k} \neq 0$ implies that $g_{\mu\nu\lambda} \neq 0$. It remains to pick a sequence of decreasing ϵ_j with corresponding triples $(\mu^{(j)}, \nu^{(j)}, \lambda^{(j)})$. □

It has been observed in different contexts that the speed of convergence in Theorem 2.1 and consequently in Theorem 2.2 is proportional to $1/\sqrt{k}$ [1, 9].

We will now prove Corollary 3.3, the equivalence of Theorem 2.2 and 2.3.

Proof of Corollary 3.3. We start by showing how Theorem 2.3 follows from Theorem 2.2.

Let $(r^A, r^B, r^{AB}) \in$ Spect, the set of admissible spectral triples. According to Theorem 2.2, there is a sequence of elements in KRON, whose normalised values converge to (r^A, r^B, r^{AB}). By Theorem 3.1, the set KRON is a finitely generated semigroup. With a finite set of generators $(\mu^{(i)}, \nu^{(i)}, \lambda^{(i)})$ of KRON we can therefore express (r^A, r^B, r^{AB}) in the form

$$(r^A, r^B, r^{AB}) = \sum_i x_i (\bar{\mu}^{(i)}, \bar{\nu}^{(i)}, \bar{\lambda}^{(i)}), \tag{17}$$

for a set of nonnegative numbers x_i which sum to one. Since the union of the $t+1$-vertex simplices equals the whole polytope, every point in it can be taken to be the sum of just $t+1$ normalised generators $(\bar{\mu}^{(i)}, \bar{\nu}^{(i)}, \bar{\lambda}^{(i)})$ (cf. Carathéodory's theorem). From the set of $m+n+mn$ equations in the variables x_i in Eq. (17), choose a set of t linearly independent ones, add the $(t+1)^{\text{th}}$ constraint $\sum_i x_i = 1$ and write the set of equations as $M\vec{x} = \vec{r}$, i.e. $\vec{r} = (r_1, \ldots, r_t, 1)$ for $r_j \in \{r_1^A, \ldots, r_m^A, r_1^B, \ldots r_n^B, r_1^{AB}, \ldots, r_{mn}^{AB}\}$ and $\vec{x} = (x_1, \ldots, x_{t+1})$.

If (r^A, r^B, r^{AB}) is rational, the x_i will be rational as well, since M is rational. This shows that $(r^A, r^B, r^{AB}) = \sum_i \frac{n_i}{n}(\bar{\mu}^{(i)}, \bar{\nu}^{(i)}, \bar{\lambda}^{(i)})$, where we set $x_i = \frac{n_i}{n}$ for $n_i, n \in \mathbb{N}$. Multiplication by $|\mu|n$ results in

$$|\mu|n(r^A, r^B, r^{AB}) = \sum_i n_i(\mu^{(i)}, \nu^{(i)}, \lambda^{(i)}).$$

Since the right-hand side of this equation is certainly an element of KRON this shows that for rational (r^A, r^B, r^{AB}) there is a number $m := |\mu|n$ such that $g_{mr^A, mr^B, mr^{AB}} \neq 0$.

It remains to show that Theorem 2.2 follows from Theorem 2.3. Suppose (r^A, r^B, r^{AB}) is a spectral triple corresponding to some ρ^{AB}. Then we can construct a series of rational triples $(r^{A(j)}, r^{B(j)}, r^{AB(j)})$ that approaches (r^A, r^B, r^{AB}) and by Theorem 2.3, there exists a series $(\mu^{(j)}, \nu^{(j)}, \lambda^{(j)})$ such that $(\bar{\mu}^{(j)}, \bar{\nu}^{(j)}, \bar{\lambda}^{(j)})$ approaches (r^A, r^B, r^{AB}) and $g_{\mu^{(j)}, \nu^{(j)}, \lambda^{(j)}} \neq 0$ for all j. □

Note that there are two ways in which Klyachko's Theorem 2.3 does not quite give the full strength of Theorem 2.2. First, it does not guarantee the speed of convergence. Second, it does not imply that, in the case of rational triples (r^A, r^B, r^{AB}), there is an increasing sequence of values of k for which $g_{kr^A,kr^B,kr^{AB}} \neq 0$; this follows from Theorem 2.2 and can be thought of as a sort of stability obtained without appeal to Theorem 3.1.

We now turn our attention to Theorem 3.2, beginning with an asymptotic result.

Lemma 5.1. *Let μ, ν and λ be diagrams with k boxes and at most m, n and mn rows, respectively. If $g_{\mu\nu\lambda} \neq 0$, then there exists a density operator ρ^{AB} on $\mathbb{C}^m \otimes \mathbb{C}^n$ with spectra satisfying*

$$\|Spec\ \rho^A - \bar{\mu}\|_1 \leq \delta, \tag{18}$$

$$\|Spec\ \rho^B - \bar{\nu}\|_1 \leq \delta, \tag{19}$$

$$\|Spec\ \rho^{AB} - \bar{\lambda}\|_1 \leq \delta, \tag{20}$$

for $\delta = O(mn\sqrt{(\log k)/k})$.

Here if p, q are probability distributions then $\|p - q\|_1 := \sum_x |p(x) - q(x)|$.

Proof. It will suffice to construct a pure state $|\varphi\rangle^{ABC} \in \mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}^{mn}$ with $\|\text{Spec}\ \varphi^A - \bar{\mu}\|_1 \leq \delta$, $\|\text{Spec}\ \varphi^B - \bar{\nu}\|_1 \leq \delta$ and $\|\text{Spec}\ \varphi^C - \bar{\lambda}\|_1 \leq \delta$, since $\text{Spec}\ \varphi^{AB} = \text{Spec}\ \varphi^C$.

Since $g_{\mu\nu\lambda} \neq 0$, there exists a unit vector $|\psi\rangle \in (V_\mu \otimes V_\nu \otimes V_\lambda)^{S_k}$. Writing $\ell = m^2n^2$,

$$|\psi\rangle \in ((\mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}^{mn})^{\otimes k})^{S_k} = ((\mathbb{C}^\ell)^{\otimes k})^{S_k} \cong U^\ell_\sigma,$$

where U^ℓ_σ denotes the symmetric representation of $\text{GL}(\ell)$, and the superscript ℓ emphasizes which $\text{GL}(\cdot)$ we are using. Denote the projector onto $U^\ell_\sigma \subset (\mathbb{C}^\ell)^{\otimes k}$ by P^ℓ_σ. Note that $\text{Tr}\ P^\ell_\sigma = \dim U^\ell_\sigma = \binom{k+\ell-1}{\ell-1} \leq k^\ell$. Fix a vector $|\phi_0\rangle \in \mathbb{C}^\ell$ and let dU denote a Haar measure for the unitary group $U(\ell)$ with normalisation $\int dU = 1$. Then by Schur's lemma

$$P^\ell_\sigma = \dim U^\ell_\sigma \int_{g \in \mathcal{U}(\ell)} dg\ (g|\phi_0\rangle\langle\phi_0|g^\dagger)^{\otimes k}.$$

Thus

$$\begin{aligned} 1 &= \text{Tr}\ P^\ell_\sigma |\psi\rangle\langle\psi| \\ &= \dim U^\ell_\sigma \int_{g \in \mathcal{U}(\ell)} dg\ \text{Tr}\,|\psi\rangle\langle\psi|(g|\phi_0\rangle\langle\phi_0|g^\dagger)^{\otimes k} \\ &\leq \dim U^\ell_\sigma \max_{g \in \mathcal{U}(\ell)} \text{Tr}\,|\psi\rangle\langle\psi|(g|\phi_0\rangle\langle\phi_0|g^\dagger)^{\otimes k}. \end{aligned}$$

Let $g \in U(\ell)$ be the unitary operator achieving the above maximisation, and define $|\varphi\rangle := g|\phi_0\rangle$. Then

$$|\langle\psi|\left(|\varphi\rangle^{\otimes k}\right)|^2 \geq \frac{1}{\dim U^\ell_\sigma} \geq k^{-\ell}.$$

Let P_μ^m, P_ν^n and P_λ^{mn} denote the projectors onto $U_\mu^m \otimes V_\mu \subset (\mathbb{C}^m)^{\otimes k}$, $U_\nu^n \otimes V_\nu \subset (\mathbb{C}^n)^{\otimes k}$ and $U_\lambda^{mn} \otimes V_\lambda \subset (\mathbb{C}^{mn})^{\otimes k}$, respectively. Then by construction $\left(P_\mu^m \otimes P_\nu^n \otimes P_\lambda^{mn}\right)|\psi\rangle = |\psi\rangle$, so $|\psi\rangle\langle\psi| \leq P_\mu^m \otimes P_\nu^n \otimes P_\lambda^{mn}$ and

$$\begin{aligned}\mathrm{Tr}\left(P_\mu^m \otimes P_\nu^n \otimes P_\lambda^{mn}\right)|\varphi\rangle\langle\varphi|^{\otimes k} &\geq \mathrm{Tr}\,(|\psi\rangle\langle\psi|)|\varphi\rangle\langle\varphi|^{\otimes k} \\ &= |\langle\psi|\left(|\varphi\rangle^{\otimes k}\right)|^2 \\ &\geq \frac{1}{\dim U_\sigma^\ell} \geq k^{-\ell}.\end{aligned}$$

Focussing for now on the A subsystem, we have

$$\mathrm{Tr}\,P_\mu^m(\varphi^A)^{\otimes k} \geq k^{-\ell}. \tag{21}$$

On the other hand, Theorem 2.1 (Spectrum Estimation) states that

$$\mathrm{Tr}\,P_\mu^m(\varphi^A)^{\otimes k} \leq (k+1)^{m(m-1)/2}\exp(-kD(\bar{\mu}\|\mathrm{Spec}\,\varphi^A)). \tag{22}$$

Combining Eqs. (21) and (22), we find that

$$D(\bar{\mu}\|\mathrm{Spec}\,\varphi^A) \leq \frac{\frac{1}{2}m(m-1)\log(k+1) + \ell\log k}{k}$$

and for $k > 1$, we can apply Pinsker's inequality [17] (which states that $\|p-q\|_1^2/2 \leq D(p\|q)$ for any probability distributions p, q) to bound

$$\|\bar{\mu} - \mathrm{Spec}\,\varphi^A\|_1 \leq 3mn\sqrt{(\log k)/k}.$$

This proves Eq. (18). Equations (19) and (20) follow by repeating this argument (starting with Eq. (21)) for P_ν^n and P_λ^{mn}. □

Theorem 3.2 now follows by observing that, if $g_{\mu,\nu,\lambda} \neq 0$, then $g_{j\mu,j\nu,j\lambda} \neq 0$ for any integer $j \geq 1$, because of the semigroup property (Theorem 3.1). The above lemma then gives us a sequence of density operators $\rho_{(j)}^{AB}$ whose spectra converge to $(\bar{\mu}, \bar{\nu}, \bar{\lambda})$, and, by compactness of the set of density matrices on $\mathbb{C}^m \otimes \mathbb{C}^n$, the sequence $\rho_{(j)}^{AB}$ has a limit ρ^{AB} satisfying Eqs. (9), (10) and (11).

6. Convexity

Let us now gather together some implications of the theorems. Let Kron denote the normalised triples $(\bar{\mu}, \bar{\nu}, \bar{\lambda})$, where $(\mu, \nu, \lambda) \in$ KRON. From Theorem 2.2 we know that any admissible spectral triple, i.e. any point in Spect, can be approximated by a sequence in Kron and therefore lies in $\overline{\mathrm{Kron}}$, the closure of Kron; thus Spect $\subseteq \overline{\mathrm{Kron}}$. From Theorem 3.2 we know that Kron $\subseteq$ Spect, and hence, since Spect is closed, $\overline{\mathrm{Kron}} \subseteq$ Spect. Thus we have

$$\mathrm{Spect} = \overline{\mathrm{Kron}}.$$

Note that Kron consists of rational points (normalised row lengths of diagrams) and there are certainly operators with irrational spectra. So Kron, unlike its closure, is a proper subset of Spect.

Theorem 3.1 allows us to say more about $\overline{\text{Kron}}$, and hence Spect. The semigroup property of KRON (Theorem 3.1) implies that if $(\bar{\mu}, \bar{\nu}, \bar{\lambda}), (\bar{\mu}', \bar{\nu}', \bar{\lambda}') \in$ Kron, then

$$(p\bar{\mu} + (1-p)\bar{\mu}', p\bar{\nu} + (1-p)\bar{\nu}', p\bar{\lambda} + (1-p)\bar{\lambda}') \in \text{Kron},$$

for every p with $0 \leq p \leq 1$. Thus $\overline{\text{Kron}}$ is convex. Furthermore, Theorem 3.1 implies that there is a finite set of generators $(\mu^{(i)}, \nu^{(i)}, \lambda^{(i)})$ of KRON, so any $(\bar{\mu}, \bar{\nu}, \bar{\lambda}) \in \overline{\text{Kron}}$ can be written

$$(\bar{\mu}, \bar{\nu}, \bar{\lambda}) = \sum_i x_i (\bar{\mu}^{(i)}, \bar{\nu}^{(i)}, \bar{\lambda}^{(i)}).$$

Thus $\overline{\text{Kron}}$ is a convex polytope. We enshrine this in Theorem 3.4.

An alternative proof for Theorem 3.4 that makes use of Kirwan's convexity theorem for moment maps can be found in [4, Chap. 2.3.6].

Acknowledgements. The inspiration for this paper came from a discussion by one of us (MC) with Allen Knutson, who essentially sketched the arguments of Theorems 3.1. He gave us further helpful advice at several points, always worded in a lively and indeed unforgettable way. We also had valuable advice from Graeme Segal. Finally, we thank Koenraad Audenaert for useful pointers to the literature, Alexander Klyachko for many stimulating discussions, and an anonymous referee for suggestions that simplified the proof of Theorem 3.1.

This project was supported by the EU under projects PROSECCO (IST-2001-39227), RESQ (IST-2001-37559) and the Integrated Project FET/QIPC SCALA. MC acknowledges the support of a DAAD Doktorandenstipendium, the U.K. Engineering and Physical Sciences Research Council and a Nevile Research Fellowship, which he holds at Magdalene College Cambridge. AWH thanks the Centre for Quantum Computation for hospitality while completing this work and acknowledges partial support from the ARO and ARDA under ARO contract DAAD19-01-1-06.

References

1. Alicki, R., Rudnicki, S., Sadowski, S.: Symmetry properties of product states for the system of N n-level atoms. J. Math. Phys. **29**(5), 1158–1162 (1988)
2. Berenstein, A., Sjamaar, R.: Coadjoint orbits, moment polytopes and the Hilbert-Mumford criterion. J. Amer. Math. Soc. **13**(2), 433–466 (2000)
3. Carter, R., Segal, G., MacDonald, I.: Lectures on Lie Groups and Lie Algebras. Volume **32** of *London Mathematical Society Student Texts*. 1st edition, Cambridge Univ. Press, (1995)
4. Christandl, M.: The Structure of Bipartite Quantum States: Insights from Group Theory and Cryptography. *PhD thesis*, University of Cambridge, February 2006. Available at http://arxiv.org/abs/quant-ph/0604183, 2001
5. Christandl, M., Mitchison, G.: The spectra of density operators and the Kronecker coefficients of the symmetric group. Commun. Math. Phys. **261**(3), 789–797 (2005)
6. Elashvili, A.G.: Invariant algebras. Advances in Soviet Math. **8**, 57–64 (1992)
7. Franz, M.: Moment polytopes of projective G-varieties and tensor products of symmetric group representations. J. Lie Theory **12**, 539–549 (2002)
8. Fulton, W., Harris, J.: *Representation Theory: A First Course*. New York: Springer, (1991)
9. Harrow, A.: Applications of coherent classical communication and the Schur transform to quantum information theory. *Thesis, Doctor of philosophy in physics*, Massachusetts Institute of Technology, September 2005, available at http://arxiv.org/abs/quant-ph/0512255, 2005
10. Hayashi, M., Matsumoto, K.: Quantum universal variable-length source coding. Phys. Rev. A **66**(2), 022311 (2002)
11. Heckman, G.J.: Projections of orbits and asymptotic behaviour of multiplicities for compact connected Lie groups. Invent. Math. **67**, 333–356 (1982)
12. Keyl, M., Werner, R.F.: Estimating the spectrum of a density operator. Phys. Rev. A **64**(5), 052311 (2001)
13. Kirillov, A.N.: An invitation to the generalized saturation conjecture. http://arxiv.org/abs/math.CO/0404353, 2004
14. Klyachko, A.: Quantum marginal problem and representations of the symmetric group http://arxiv.org/list/quant-ph/0409113, 2004

15. Mumford, D., Fogarty, J., Kirwan, F.: Geometric invariant theory, Volume **34** of *Ergebnisse der Mathematic und ihrer Grenzgebiete (2)*. Third edition, Berlin: Springer-Verlag, (1994)
16. Ness, L.: A stratification of the null cone via the moment map. Amer. J. Math. **106**, 1281–1329 (1984)
17. Pinsker, M.S.: Information and Information Stability of Random Variables and Processes. San Francisco: Holden-Day, 1964
18. Popov, V.L., Vinberg, E.B.: Algebraic Geometry IV. Volume **55** of *Encyclopaedia of Mathematical Sciences, Chapter II*. Invariant Theory, Berlin-Heidelberg-New York: Springer-Verlag 1994, pp. 123–278

Communicated by M.B. Ruskai