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Determining a Magnetic Schrödinger Operator from Partial Cauchy Data

David Dos Santos Ferreira[1], **Carlos E. Kenig**[2], **Johannes Sjöstrand**[3], **Gunther Uhlmann**[4]

[1] Institut Galilée, Université Paris 13, 93430 Villetaneuse, France. E-mail: ddsf@math.univ-paris13.fr
[2] Department of Mathematics, University of Chicago, Chicago, IL 60637, USA
[3] C.M.L.S., Ecole Polytechnique, 91128 Palaiseau Cedex, France
[4] Department of Mathematics, University of Washington, Seattle, WA 98195-4350, USA

Abstract: In this paper we show, in dimension $n \geq 3$, that knowledge of the Cauchy data for the Schrödinger equation in the presence of a magnetic potential, measured on possibly very small subsets of the boundary, determines uniquely the magnetic field and the electric potential. We follow the general strategy of [7] using a richer set of solutions to the Dirichlet problem that has been used in previous works on this problem.

Contents

1. Introduction

Let $\Omega \subset \mathbf{R}^n$ be an open bounded set with $\mathcal{C}^\infty$ boundary; we are interested in the magnetic Schrödinger operator

$$
\begin{aligned}
\mathcal{L}_{A,q}(x, D) &= \sum_{j=1}^n (D_j + A_j(x))^2 + q(x) \\
&= D^2 + A \cdot D + D \cdot A + A^2 + q
\end{aligned}
\tag{1.1}
$$

with real magnetic potential $A = (A_j)_{1 \leq j \leq n} \in \mathcal{C}^2(\bar{\Omega}; \mathbf{R}^n)$ and bounded electric potential $q \in L^\infty(\Omega)$. As usual, $D = -i\nabla$. In this paper, we always assume the dimension to be ≥ 3. Let us introduce the

Assumption 1. 0 *is not an eigenvalue of the magnetic Schrödinger operator* $\mathcal{L}_{A,q}$:
$H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$.

Let ν be the unit exterior normal. Under Assumption 1, the Dirichlet problem

$$\begin{cases} \mathcal{L}_{A,q}u = 0 \\ u|_{\partial\Omega} = f \in H^{\frac{1}{2}}(\partial\Omega) \end{cases} \tag{1.2}$$

has a unique solution in $H^1(\Omega)$, and we can introduce the Dirichlet to Neumann map
(DN)

$$\mathcal{N}_{A,q} : H^{\frac{1}{2}}(\partial\Omega) \ni f \mapsto (\partial_\nu + iA \cdot \nu)u|_{\partial\Omega} \in H^{-\frac{1}{2}}(\partial\Omega)$$

associated to the magnetic Schrödinger operator $\mathcal{L}_{A,q}$ with magnetic potential defined
by (1.1).

The inverse problem we consider in this paper is to recover information about the
magnetic and electric potential from the DN map measured on subsets of the boundary.
As was noted in [12], the DN map is invariant under a gauge transformation of the
magnetic potential: it ensues from the identities

$$e^{-i\Psi}\mathcal{L}_{A,q}e^{i\Psi} = \mathcal{L}_{A+\nabla\Psi,q}, \quad e^{-i\Psi}\mathcal{N}_{A,q}e^{i\Psi} = \mathcal{N}_{A+\nabla\Psi,q}, \tag{1.3}$$

that $\mathcal{N}_{A,q} = \mathcal{N}_{A+\nabla\Psi,q}$ when $\Psi \in C^1(\bar{\Omega})$ is such that $\Psi|_{\partial\Omega} = 0$. Thus $\mathcal{N}_{A,q}$ carries
information about the magnetic field[1] $B = dA$. Sun showed in [12] that from this infor-
mation one can determine the magnetic field and the electric potential if the magnetic
potential is small in an appropriate class. In [8] the smallness assumption was eliminated
for smooth magnetic and electric potentials and for C^2 and compactly supported mag-
netic potential and L^∞ electrical potential. The regularity assumption on the magnetic
potential was improved in [13] to $C^{2/3+\epsilon}$, $\epsilon > 0$, and to Dini continuous in [10]. Recently
in [11] a method was given for reconstructing the magnetic field and the electric potential
under some regularity assumptions on the magnetic potential.

All of the above mentioned results rely on constructing complex geometrical optics
solutions, with a linear phase, for the magnetic Schrödinger equation. We also mention
that the inverse boundary value problem is closely related to the inverse scattering prob-
lem at a fixed energy for the magnetic Schrödinger operator. The latter was studied under
various regularity assumptions on the magnetic and electrical potentials in [9], for small
compactly supported magnetic potential and compactly supported electric potential. This
result was extended in [3] for exponentially decaying magnetic and electric potentials
with no smallness assumption.

In this paper we extend one of the main results of [7] to the case of the magnetic
Schrödinger equation. Let $x_0 \in \mathbf{R}^n \backslash \overline{\text{ch}(\Omega)}$ (where $\text{ch}(\Omega)$ denotes the convex hull of Ω),
we define the front and back sides of $\partial\Omega$ with respect to x_0 by

$$F(x_0) = \{x \in \partial\Omega : (x - x_0) \cdot \nu(x) \le 0\},$$
$$B(x_0) = \{x \in \partial\Omega : (x - x_0) \cdot \nu(x) > 0\}.$$

The following result was obtained in [7] when there are no magnetic potentials, i.e.
$A_1 = A_2 = 0$ (we then write $\mathcal{N}_{q_j} = \mathcal{N}_{0,q_j}$ for the Dirichlet to Neumann maps):

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[1] Here A is viewed as the 1-form $\sum_{j=1}^n A_j dx_j$.

Theorem (1.2 in [7]**)** (Kenig, Sjöstrand, Uhlmann). *Let Ω be an open bounded set with $\mathcal{C}^\infty$ boundary in $\mathbf{R}^n$, $n \geq 3$, let q_1, q_2 be two bounded potentials on Ω such that Assumption 1 is satisfied.*

Let $x_0 \in \mathbf{R}^n \backslash \overline{\mathrm{ch}(\Omega)}$, suppose that there exists a neighborhood $\tilde{F}$ of the front side $F(x_0)$ such that

$$\mathcal{N}_{q_1} f(x) = \mathcal{N}_{q_2} f(x) \quad \forall x \in \tilde{F}, \quad \forall f \in H^{\frac{1}{2}}(\partial\Omega),$$

then $q_1 = q_2$.

Let us now state the precise results of this article.

Theorem 1.1. *Let Ω be a simply connected open bounded set with $\mathcal{C}^\infty$ boundary in $\mathbf{R}^n$, $n \geq 3$, let A_1, A_2 be two real $\mathcal{C}^2$ vector fields on $\bar{\Omega}$ and q_1, q_2 be two bounded potentials on Ω such that Assumption 1 is satisfied.*

Let $x_0 \in \mathbf{R}^n \backslash \overline{\mathrm{ch}(\Omega)}$, suppose that the Dirichlet to Neumann maps related to the operators $\mathcal{L}_{A_1,q_1}$ and $\mathcal{L}_{A_2,q_2}$ coincide on part of the boundary near x_0 in the sense that there exists a neighborhood $\tilde{F}$ of the front side of $\partial\Omega$ with respect to x_0 such that

$$\mathcal{N}_{A_1,q_1} f(x) = \mathcal{N}_{A_2,q_2} f(x) \quad \forall x \in \tilde{F}, \quad \forall f \in H^{\frac{1}{2}}(\partial\Omega), \tag{1.4}$$

then if A_1 and A_2 are viewed as 1-forms,

$$dA_1 = dA_2 \text{ and } q_1 = q_2.$$

Remark 1.2. We only use the simple connectedness of the set Ω to obtain the equality $dA_1 = dA_2$ and to deduce that the two magnetic potentials differ from a gradient. If we already know that $A_1 - A_2 = \nabla\Psi$, we don't need the fact that Ω is simply connected, in particular, Theorem 1.1 contains Theorem 1.2 of [7].

Nevertheless Theorem 1.1 in [7] improves on this result by restricting the Dirichlet-to-Neuman maps to a space of functions on the boundary with support in a small neighborhood of the back side $B(x_0)$. We have left the corresponding result in the magnetic case open.

As in [7], we make the following definition of a strongly star shaped domain.

Definition 1.3. *An open set Ω with smooth boundary is said to be strongly star shaped with respect to $x_1 \in \partial\Omega$ if every line through x_1 which is not contained in the tangent hyperplane cuts the boundary at precisely two distinct points x_1 and x_2 with transversal intersection at x_2.*

With this definition, Theorem 1.1 implies the following corollary

Corollary 1.4. *Under the assumptions on Ω, the magnetic potentials A_1, A_2 and the electric potentials q_1, q_2 of Theorem 1.1, let $x_1 \in \partial\Omega$ be a point of the boundary such that the tangent hyperplane of $\partial\Omega$ at x_1 only intersects $\partial\Omega$ at x_1 and such that Ω is strongly star shaped with respect to x_1.*

Suppose that there exists a neighborhood $\tilde{F}$ of x_1 in $\partial\Omega$ such that (1.4) holds then

$$dA_1 = dA_2 \text{ and } q_1 = q_2.$$

We proceed as in [7] by constructing some complex geometrical optics solutions using a Carleman estimate. The construction of these solutions is fairly similar to those presented in the latter paper, except for the changes due to the presence of the magnetic potential. However, the part concerned with the recovery of the potential and the magnetic field is new.

The plan of this article is as follows. In the second section, we prove a Carleman estimate with boundary terms for the magnetic Schrödinger operator, which will be useful both for the construction of the complex geometrical optics solutions, and for estimating boundary terms in the limit when the semi-classical parameter tends to zero. In the third section, we construct complex geometrical optics solutions by solving an eikonal and a transport equation and using the Carleman estimate derived in the preceding section. The fourth and fifth sections are devoted to the analysis of the information obtained when passing to the limit when the semi-classical parameter tends to zero. This actually provides enough information on a certain Radon transform to determine the magnetic field. In the last section, once the magnetic field has been determined, we apply the same arguments to determine the electric potential.

2. Carleman Estimate

Our first step is to construct solutions of the magnetic Schrödinger equation $\mathcal{L}_{A,q}u = 0$ of the form

$$u(x, h) = e^{\frac{1}{h}(\varphi + i\psi)}(a(x) + hr(x, h)) \tag{2.1}$$

(where φ and ψ are real functions) by use of the complex geometrical optics method: of course, ψ and a will be sought as solutions of respectively an eikonal equation and a transport equation. In order to be able to go from an approximate solution to an exact solution, one wants the conjugated operator

$$e^{\frac{\varphi}{h}}h^2\mathcal{L}_{A,q}e^{-\frac{\varphi}{h}}$$

to be locally solvable in a semi-classical sense, which means its principal symbol[2]

$$p_\varphi(x, \xi) = \xi^2 - (\nabla\varphi)^2 + 2i\nabla\varphi \cdot \xi \tag{2.2}$$

to satisfy Hörmander's condition

$$\{\operatorname{Re} p_\varphi, \operatorname{Im} p_\varphi\} \leq 0 \quad \text{when } p_\varphi = 0.$$

Since we furthermore want to obtain solutions (2.1) for both the phases φ and $-\varphi$, we will consider phases satisfying the condition

$$\{\operatorname{Re} p_\varphi, \operatorname{Im} p_\varphi\} = 0 \quad \text{when } p_\varphi = 0. \tag{2.3}$$

Definition 2.1. *A real smooth function φ on an open set $\tilde{\Omega}$ is said to be a limiting Carleman weight if it has non-vanishing gradient on $\tilde{\Omega}$ and if the symbol (2.2) satisfies condition (2.3) on $T^*(\tilde{\Omega})$. This is equivalent to say that*

$$\langle\varphi''\nabla\varphi, \nabla\varphi\rangle + \langle\varphi''\xi, \xi\rangle = 0 \quad \text{when } \xi^2 = (\nabla\varphi)^2 \quad \text{and} \quad \nabla\varphi \cdot \xi = 0. \tag{2.3'}$$

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[2] Here and throughout this article, we are using the semi-classical convention.

The appropriate tool in deducing local solvability for the conjugated operator and in proving that the geometrical optics method is effective (meaning that indeed one gains one power of h in the former asymptotics) is a Carleman estimate. The goal of this section is to prove such an estimate.

In this section, Ω is as in the introduction and $\tilde{\Omega}$ will denote an open set $\tilde{\Omega} \Supset \Omega$. We will use the following notations:

$$(u|v) = \int_\Omega u(x)\bar{v}(x)\, dx, \quad (u|v)_{\partial\Omega} = \int_{\partial\Omega} u(x)\bar{v}(x)\, d\sigma(x),$$

and $\|u\| = \sqrt{(u|u)}$ denotes the L^2 norm on Ω. We say that the estimate

$$F(u, h) \lesssim G(u, h)$$

holds for all $u \in X$ (where X is a function space, such as $L^2(\Omega)$) and for h small if there exist constants $C > 0$ and $h_0 > 0$ (possibly depending on q and A) such that for all $0 \le h \le h_0$ and for all $u \in X$, the inequality $F(u, h) \le CG(u, h)$ is satisfied.

We will make extensive use of the Green formula for the magnetic Schrödinger operator $\mathcal{L}_{A,q}$, which for the sake of convenience, we state as a lemma.

Lemma 2.2. *Let A be a real C^1 vector field on $\bar{\Omega}$ and $q \in L^\infty(\Omega)$ then we have the magnetic Green formula*

$$(\mathcal{L}_{A,q}u|v)_\Omega - (u|\mathcal{L}_{A,\bar{q}}v)_\Omega = \left(u|(\partial_\nu + i\nu \cdot A)v\right)_{\partial\Omega} - \left((\partial_\nu + i\nu \cdot A)u|v\right)_{\partial\Omega} \quad (2.4)$$

for all $u, v \in H^1(\Omega)$ such that $\Delta u, \Delta v \in L^2(\Omega)$.

Proof. Integrating by parts, we have

$$(\mathcal{L}_{A,q}u|v)_\Omega = (\nabla u|\nabla v)_\Omega + (Du|Av)_\Omega + (Au|Dv)_\Omega$$
$$+ \left((q + A^2)u|v\right)_\Omega - \left((\partial_\nu + i\nu \cdot A)u|v\right)_{\partial\Omega} \quad (2.5)$$

and by permuting u and v, replacing q by $\bar{q}$, and taking the complex conjugate of the former, we get

$$(\mathcal{L}_{A,\bar{q}}u|v)_\Omega = (\nabla u|\nabla v)_\Omega + (Du, Av)_\Omega + (Au|Dv)_\Omega$$
$$+ \left((q + A^2)u|v\right)_\Omega - \left(u|(\partial_\nu + i\nu \cdot A)v\right)_{\partial\Omega}.$$

Subtracting the former to (2.5), we end up with (2.4). □

If φ is a limiting Carleman weight, we define

$$\partial\Omega_\pm = \{x \in \partial\Omega : \pm\partial_\nu\varphi \ge 0\}.$$

Proposition 2.3. *Let φ be a C^∞ limiting Carleman weight on $\tilde{\Omega}$, let A be a C^1 vector field on $\bar{\Omega}$ and $q \in L^\infty(\Omega)$, the Carleman estimate*

$$-h(\partial_\nu\varphi\, e^{\frac{\varphi}{h}}\partial_\nu u | e^{\frac{\varphi}{h}}\partial_\nu u)_{\partial\Omega_-} + \|e^{\frac{\varphi}{h}}u\|^2 + \|e^{\frac{\varphi}{h}}h\nabla u\|^2$$
$$\lesssim h^2\|e^{\frac{\varphi}{h}}\mathcal{L}_{A,q}u\|^2 + h(\partial_\nu\varphi\, e^{\frac{\varphi}{h}}\partial_\nu u | e^{\frac{\varphi}{h}}\partial_\nu u)_{\partial\Omega_+} \quad (2.6)$$

holds for all $u \in C^\infty(\bar{\Omega}) \cap H_0^1(\Omega)$ and h small.

In particular, when $u \in \mathcal{C}_0^\infty(\Omega)$, we have the Carleman estimate

$$\|e^{\frac{\varphi}{h}}u\| + h\|e^{\frac{\varphi}{h}}\nabla u\| \lesssim h\|e^{\frac{\varphi}{h}}\mathcal{L}_{A,q}u\|. \tag{2.6'}$$

The proof of the Carleman estimate follows standard arguments. We refer the reader to Chapters 17.2 and 28 in [6] and the references therein for a general study on L^2 Carleman estimates.

Proof. Taking $v = e^{\frac{\varphi}{h}}u$, it is equivalent to prove the following *a priori* estimate

$$-h(\partial_\nu\varphi \, \partial_\nu v|\partial_\nu v)_{\partial\Omega_-} + \|v\|^2 + \|h\nabla v\|^2$$
$$\lesssim \frac{1}{h^2}\|(e^{\frac{\varphi}{h}}h^2\mathcal{L}_{A,q}e^{-\frac{\varphi}{h}})v\|^2 + h(\partial_\nu\varphi \, \partial_\nu v|\partial_\nu v)_{\partial\Omega_+} \tag{2.7}$$

since $\|he^{\frac{\varphi}{h}}\nabla u\| \lesssim \|v\| + \|h\nabla v\|$ and $(e^{\frac{\varphi}{h}}\partial_\nu u)|_{\partial\Omega} = \partial_\nu v|_{\partial\Omega}$. Conjugating the magnetic Schrödinger operator by the exponential weight gives rise to the following operator:

$$e^{\frac{\varphi}{h}}h^2\mathcal{L}_{A,q}e^{-\frac{\varphi}{h}} = P + iQ + R + h^2(q + A^2), \tag{2.8}$$

where P and Q are the self-adjoint operators

$$P = h^2D^2 - (\nabla\varphi)^2,$$
$$Q = \nabla\varphi \cdot hD + hD \cdot \nabla\varphi,$$
$$\text{and} \quad R = h(A \cdot hD + hD \cdot A) + 2ihA \cdot \nabla\varphi.$$

Our first remark concerns the fact that we may neglect the term $h^2(q + A^2)$ since the right-hand side of (2.7) may be perturbed by a term bounded by $h^2\|v\|^2$, which may be absorbed by the left-hand side if h is small enough. Omitting the term $q + A^2$ gives rise to such an error. Hence we will prove the *a priori* estimate for the operator $P + iQ + R$. The same is not true of the term R because errors of order $\|v\|^2 + \|h\nabla v\|^2$ may not be absorbed into the left hand-side.

Note that if p and q denote the principal symbol respectively of P and Q, the fact that φ is a limiting Carleman weight means that

$$\{p, q\} = 0 \text{ when } p + iq = 0.$$

This condition is not enough to obtain an *a priori* estimate for $P + iQ$, one needs to have a positive Poisson bracket. Our first step is to remedy this by using a classical convexity argument. Consider the modified Carleman weight

$$\tilde{\varphi} = \varphi + h\frac{\varphi^2}{2\varepsilon},$$

where ε is a suitable small parameter to be chosen independent of h, and denote by $\tilde{p}$ and $\tilde{q}$ the corresponding symbols, and by $\tilde{P}, \tilde{Q}, \tilde{R}$ the corresponding operators, when φ has been replaced by $\tilde{\varphi}$. Then, we have

$$\nabla\tilde{\varphi} = \left(1 + \frac{h}{\varepsilon}\varphi\right)\nabla\varphi, \quad \tilde{\varphi}'' = \left(1 + \frac{h}{\varepsilon}\varphi\right)\varphi'' + \frac{h}{\varepsilon}\nabla\varphi \otimes \nabla\varphi;$$

therefore when $\xi^2 = (\nabla\tilde\varphi)^2$ and $\nabla\tilde\varphi \cdot \xi = 0$, we have

$$\{\tilde p, \tilde q\} = 4\langle\tilde\varphi''\xi, \xi\rangle + 4\langle\tilde\varphi''\nabla\tilde\varphi, \nabla\tilde\varphi\rangle$$

$$= \frac{4h}{\varepsilon}\left(1 + \frac{h}{\varepsilon}\varphi\right)^2 (\nabla\varphi)^4 > 0, \tag{2.9}$$

since φ is a limiting Carleman weight. Furthermore, if we restrict ourselves to the hyperplane V_x orthogonal to $\nabla\varphi$, we get

$$\{\tilde p, \tilde q\}(x, \cdot)|_{V_x} = \frac{4h}{\varepsilon}\left(1 + \frac{h}{\varepsilon}\varphi\right)^2 (\nabla\varphi)^4 + a(x)(\xi^2 - (\nabla\tilde\varphi)^2)$$

with $a(x) = 4h(\nabla\tilde\varphi)^2/\varepsilon - 4\langle\tilde\varphi''\nabla\tilde\varphi, \nabla\tilde\varphi\rangle/(\nabla\tilde\varphi)^2$, and since this bracket is a quadratic polynomial with no linear part, this implies that there exists a linear form $b(x, \xi)$ in ξ such that

$$\{\tilde p, \tilde q\} = \frac{4h}{\varepsilon}\left(1 + \frac{h}{\varepsilon}\varphi\right)^2 (\nabla\varphi)^4 + a(x)\tilde p + b(x, \xi)\tilde q.$$

This computation implies on the operator level that

$$i[\tilde P, \tilde Q] = \frac{4h^2}{\varepsilon}\left(1 + \frac{h}{\varepsilon}\varphi\right)^2 (\nabla\varphi)^4 + \frac{h}{2}(a\tilde P + \tilde Pa)$$

$$+ \frac{h}{2}(b^w\tilde Q + \tilde Qb^w) + h^3c(x),$$

where the first order differential operator b^w is the semi-classical Weyl quantization[3] of b. In fact, the positivity of the bracket (2.9) essentially induces the positivity of the commutator $i[\tilde P, \tilde Q]$

$$i([\tilde P, \tilde Q]v|v) = \underbrace{\frac{4h^2}{\varepsilon}\left(1 + \frac{h}{\varepsilon}\varphi\right)^2 \|(\nabla\varphi)^2 v\|^2}_{>0}$$

$$+ h\,\mathrm{Re}(a\tilde Pv|v) + h\,\mathrm{Re}(\tilde Qv|b^w v) + h^3(cv|v) \tag{2.10}$$

(recall that $v|_{\partial\Omega} = 0$ which explains why there are no boundary terms). The former fact will be enough to obtain the *a priori* estimate on $\tilde P + i\tilde Q$.

Our last observation is that

$$\|h\nabla v\|^2 = (\tilde Pv|v) + \|\sqrt{(\nabla\varphi)^2}\,v\|^2$$

leading to

$$\|h\nabla v\|^2 \lesssim \|\tilde Pv\|^2 + \|v\|^2. \tag{2.11}$$

Now, we turn to the proof of the estimate. We have

$$\|(\tilde P + i\tilde Q)v\|^2 = \|\tilde Pv\|^2 + \|\tilde Qv\|^2 + i(\tilde Qv|\tilde Pv) - i(\tilde Pv|\tilde Qv)$$

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[3] The absence of the h^2 term is due to the use of the Weyl quantization.

and the magnetic Green formula (2.4) (used in the straightforward case with no potential $\tilde{P} = h^2 \mathcal{L}_{0, -(\nabla\tilde{\varphi})^2/h^2}$), together with the fact that $v|_{\partial\Omega} = 0$, gives

$$(\tilde{Q}v | \tilde{P}v) = (\tilde{P}\tilde{Q}v | v) - h^2(\tilde{Q}v | \partial_\nu v)_{\partial\Omega}$$
$$= (\tilde{P}\tilde{Q}v | v) + 2ih^3(\partial_\nu\tilde{\varphi}\,\partial_\nu v | \partial_\nu v)_{\partial\Omega}$$

and similarly, since $\tilde{Q}$ is first order, we get

$$(\tilde{P}v, \tilde{Q}v) = (\tilde{Q}\tilde{P}v, v).$$

Therefore we have

$$\|(\tilde{P} + i\tilde{Q})v\|^2 = \|\tilde{P}v\|^2 + \|\tilde{Q}v\|^2 + i([\tilde{P}, \tilde{Q}]v | v)$$
$$- 2h^3(\partial_\nu v | \partial_\nu\tilde{\varphi}\,\partial_\nu v)_{\partial\Omega},$$

and using (2.10), we get

$$\|(\tilde{P} + i\tilde{Q})v\|^2 + 2h^3(\partial_\nu v | \partial_\nu\tilde{\varphi}\,\partial_\nu v)_{\partial\Omega} \geq \|\tilde{P}v\|^2 + \|\tilde{Q}v\|^2 + \frac{C_1 h^2}{\varepsilon}\|v\|^2$$
$$- \underbrace{(Ch\|\tilde{P}v\|\,\|v\| + Ch\|\tilde{Q}v\|\,\|h\nabla v\|)}_{\leq \frac{1}{2}\|\tilde{P}v\|^2 + \frac{1}{2}\|\tilde{Q}v\|^2 + \frac{C_2^2 h^2}{2}\left(\|v\|^2 + \|h\nabla v\|^2\right)},$$

which combined with (2.11), gives when ε is small enough

$$\|(\tilde{P} + i\tilde{Q})v\|^2 + 2h^3(\partial_\nu v | \partial_\nu\tilde{\varphi}\,\partial_\nu v)_{\partial\Omega}$$
$$\gtrsim (1 - \mathcal{O}(\varepsilon^{-1}h^2)\|\tilde{P}v\|^2 + \|\tilde{Q}v\|^2 + \frac{h^2}{\varepsilon}\left(\|v\|^2 + \|h\nabla v\|^2\right).$$

Thus, taking h and ε small enough

$$\|(\tilde{P} + i\tilde{Q})v\|^2 + 2h^3(\partial_\nu v | \partial_\nu\tilde{\varphi}\,\partial_\nu v)_{\partial\Omega} \gtrsim \frac{h^2}{\varepsilon}\left(\|v\|^2 + \|h\nabla v\|^2\right). \qquad (2.12)$$

The last part[4] of the proof is concerned with the additional term $\tilde{R}v$ due to the magnetic potential; from the former inequality we deduce

$$\|(\tilde{P} + i\tilde{Q} + \tilde{R})v\|^2 + h^3(\partial_\nu v | \partial_\nu\tilde{\varphi}\,\partial_\nu v)_{\partial\Omega}$$
$$\gtrsim \frac{h^2}{\varepsilon}\left(\|v\|^2 + \|h\nabla v\|^2 - \mathcal{O}(\varepsilon)\|h^{-1}\tilde{R}v\|^2\right)$$

and using the fact that $\|h^{-1}\tilde{R}v\|^2 \lesssim \|v\|^2 + \|h\nabla v\|^2$, we obtain

$$\|(\tilde{P} + i\tilde{Q} + \tilde{R})v\|^2 + h^3(\partial_\nu v | \partial_\nu\tilde{\varphi}\,\partial_\nu v)_{\partial\Omega} \gtrsim \frac{h^2}{\varepsilon}\left(\|v\|^2 + \|h\nabla v\|^2\right)$$

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[4] This is the main difference with respect to the proof of the Carleman estimate in [7].

if ε is chosen small enough. Finally, with $v = e^{\frac{\varphi^2}{2\varepsilon}} e^{\frac{\varphi}{h}} u$, we get

$$\|e^{\frac{\varphi^2}{2\varepsilon}} e^{\frac{\varphi}{h}} u\|^2 + \|h\nabla e^{\frac{\varphi^2}{2\varepsilon}} e^{\frac{\varphi}{h}} u\|^2 \lesssim \frac{1}{h^2} \|e^{\frac{\varphi^2}{2\varepsilon}} e^{\frac{\varphi}{h}} h^2 \mathcal{L}_{A,q} u\|^2$$
$$+ h\big(e^{\frac{\varphi^2}{\varepsilon}} \partial_\nu(e^{\frac{\varphi}{h}} u) | \partial_\nu \tilde{\varphi} \, \partial_\nu(e^{\frac{\varphi}{h}} u)\big)_{\partial\Omega},$$

this gives the Carleman estimate (2.6) since

$$1 \leq e^{\varphi^2/2\varepsilon} \leq C, \quad \frac{1}{2} \leq \frac{\partial_\nu \tilde{\varphi}}{\partial_\nu \varphi} = 1 + \frac{h}{\varepsilon}\varphi \leq \frac{3}{2}$$

on $\bar{\Omega}$ for all h small enough. $\quad\square$

We denote by $H^1_{\mathrm{scl}}(\Omega)$ the semi-classical Sobolev space of order 1 on Ω with associated norm

$$\|u\|^2_{H^1_{\mathrm{scl}}(\Omega)} = \|h\nabla u\|^2 + \|u\|^2$$

and by $H^s_{\mathrm{scl}}(\mathbf{R}^n)$ the semi-classical Sobolev space on $\mathbf{R}^n$ with associated norm

$$\|u\|^2_{H^s_{\mathrm{scl}}(\mathbf{R}^n)} = \|\langle hD\rangle^s u\|^2_{L^2(\mathbf{R}^n)} = \int (1 + h^2\xi^2)^s |\hat{u}(\xi)|^2 \, d\xi.$$

Changing φ into $-\varphi$, we may rewrite the Carleman estimate in the following convenient way:

$$\sqrt{h}\|\sqrt{\partial_\nu \varphi}\, e^{-\frac{\varphi}{h}} \partial_\nu u\|_{L^2(\partial\Omega_+)} + \|e^{-\frac{\varphi}{h}} u\|_{H^1_{\mathrm{scl}}(\Omega)}$$
$$\lesssim h\|e^{-\frac{\varphi}{h}} \mathcal{L}_{A,q} u\| + \sqrt{h}\|\sqrt{-\partial_\nu \varphi}\, e^{-\frac{\varphi}{h}} \partial_\nu u\|_{L^2(\partial\Omega_-)}. \tag{2.13}$$

By regularization, this estimate is still valid for $u \in H^2(\Omega) \cap H^1_0(\Omega)$. A similar Carleman estimate gives the following solvability result:

Proposition 2.4. *Let φ be a limiting Carleman weight on $\tilde{\Omega}$, let A be a $\mathcal{C}^1$ vector field on $\bar{\Omega}$ and $q \in L^\infty(\Omega)$. There exists h_0 such that for all $0 \leq h \leq h_0$ and for all $w \in L^2(\Omega)$, there exists $u \in H^1(\Omega)$ such that*

$$h^2 \mathcal{L}_{A,q}(e^{\frac{\varphi}{h}} u) = e^{\frac{\varphi}{h}} w \quad and \quad h\|u\|_{H^1_{\mathrm{scl}}(\Omega)} \lesssim \|w\|.$$

Proof. We need the following Carleman estimate:

$$\|v\| \lesssim h\|e^{-\frac{\varphi}{h}} \mathcal{L}_{A,q} e^{\frac{\varphi}{h}} v\|_{H^{-1}_{\mathrm{scl}}(\mathbf{R}^n)}, \quad \forall v \in \mathcal{C}^\infty_0(\Omega). \tag{2.14}$$

Let $\Omega \Subset \hat{\Omega} \subset \tilde{\Omega}$, assume that we have extended A to a $\mathcal{C}^1$ vector field on $\hat{\Omega}$ and q to a L^∞ function on $\hat{\Omega}$. Let $\chi \in \mathcal{C}^\infty_0(\hat{\Omega})$ equal 1 on Ω. With the notations used in the proof of Proposition 2.3 we have

$$\langle hD\rangle^{-1}(\tilde{P} + i\tilde{Q})\langle hD\rangle = \tilde{P} + i\tilde{Q} + hR_1,$$

where R_1 is a semi-classical pseudo-differential operator of order 1, therefore from estimate (2.12) we deduce

$$\|(\tilde{P} + i\tilde{Q})\langle hD\rangle v\|^2_{H_{\text{scl}}^{-1}(\mathbf{R}^n)} \gtrsim \frac{h^2}{\varepsilon}(\|v\|^2_{H_{\text{scl}}^1(\mathbf{R}^n)} - \mathcal{O}(\varepsilon)\|R_1 v\|^2)$$

$$\gtrsim \frac{h^2}{\varepsilon}\|v\|^2_{H_{\text{scl}}^1(\mathbf{R}^n)}$$

for any $v \in C_0^\infty(\hat{\Omega})$, if h and ε are small enough. Besides, we have

$$\|(\tilde{R} + h^2(q + A^2))v\|_{H_{\text{scl}}^{-1}(\mathbf{R}^n)} \lesssim h\|v\|,$$

therefore if ε is small enough, we have

$$\|\langle hD\rangle v\| \lesssim h\|e^{-\frac{\tilde{\varphi}}{h}}\mathcal{L}_{A,q}e^{\frac{\tilde{\varphi}}{h}}\langle hD\rangle v\|_{H_{\text{scl}}^{-1}(\mathbf{R}^n)}$$

for any $v \in C_0^\infty(\hat{\Omega})$. Hence if $u \in C_0^\infty(\Omega)$, taking $v = \chi\langle hD\rangle^{-1}u \in C^\infty(\hat{\Omega})$ in the former estimate, and using the fact that

$$\|(1 - \chi)\langle hD\rangle^{-1}u\|_{H_{\text{scl}}^s} = \mathcal{O}(h^\infty)\|u\|$$

we obtain

$$\|u\| \lesssim h\|e^{-\frac{\tilde{\varphi}}{h}}\mathcal{L}_{A,q}e^{\frac{\tilde{\varphi}}{h}}u\|_{H_{\text{scl}}^{-1}(\mathbf{R}^n)}, \quad \forall u \in C_0^\infty(\Omega).$$

This gives the Carleman estimate (2.14) since $e^{\tilde{\varphi}/h} = e^{\varphi^2/\varepsilon}e^{\varphi/h}$. Classical arguments involving the Hahn-Banach theorem give the solvability result. $\square$

3. Construction of Solutions by Complex Geometrical Optics

The goal of this section is to construct solutions of the magnetic Schrödinger equation of the form (2.1). To do so we take ψ to be a solution of the eikonal equation

$$p(x, \nabla\psi(x)) + iq(x, \nabla\psi(x)) = 0;$$

such solutions exist since $\{p, q\} = 0$ when $p = q = 0$. More precisely, the eikonal equation reads

$$(\nabla\psi)^2 = (\nabla\varphi)^2, \quad \nabla\varphi \cdot \nabla\psi = 0. \tag{3.1}$$

In fact, in the remainder of this article, we fix the limiting Carleman weight to be

$$\varphi(x) = \frac{1}{2}\log(x - x_0)^2. \tag{3.2}$$

For such a choice of φ, the second part of the eikonal equation is merely the fact that ψ is a function of the angular variable $(x - x_0)/|x - x_0|$ and we can actually give an explicit solution of the eikonal equation

$$\psi(x) = \frac{\pi}{2} - \arctan\frac{\omega \cdot (x - x_0)}{\sqrt{(x - x_0)^2 - (\omega \cdot (x - x_0))^2}}$$

$$= d_{S^{n-1}}\left(\frac{x - x_0}{|x - x_0|}, \omega\right), \tag{3.3}$$

where $\omega \in S^{n-1}$. Let us be more precise about the set where ω may vary, keeping in mind that we want this function to be smooth — in particular, we have to ensure that $\omega \neq (x - x_0)/|x - x_0|$ whenever $x \in \bar{\Omega}$.

For that purpose, let $r_0 > 0$ be large enough so that $\bar{\Omega} \subset B(x_0, r_0)$, let H denote a hyperplane separating x_0 and $\mathrm{ch}(\Omega)$, and H^+ the open half space containing $\bar{\Omega}$ (and therefore $x_0 \notin H^+$), we set

$$\Gamma = \{\theta \in S^{n-1} : x_0 + r_0\theta \in H^+\}$$

and $\check{\Gamma}$ the image of Γ under the antipodal application. Let $\omega_0 \in S^{n-1} \setminus (\Gamma \cup \check{\Gamma})$ and Γ_0 be a neighborhood of ω_0 in $S^{n-1} \setminus (\Gamma \cup \check{\Gamma})$, then the distance $\Gamma \times \Gamma_0 \ni (\theta, \omega) \rightarrow d_{S^{n-1}}(\theta, \omega)$ is a $\mathcal{C}^\infty$ function. Moreover, $\bar{\Omega} \subset \tilde{\Omega} = x_0 + \mathbf{R}_+\Gamma$, hence we have $(x - x_0)/|x - x_0| \in \Gamma$ for all x in the open neighborhood $\tilde{\Omega}$ of $\bar{\Omega}$, thus ψ depends smoothly on the variables (x, ω) on $\tilde{\Omega} \times \Gamma_0$.

Remark 3.1. Suppose that $x_0 = 0$ and $\omega = (1, 0, \ldots, 0)$, which we can always assume by doing a translation and a rotation. Notice that by considering the complex variable $z = x_1 + i|x'| \in \mathbf{C}$ (with $x = (x_1, x') \in \mathbf{R} \times \mathbf{R}^{n-1}$), we have

$$\varphi = \log|z| = \mathrm{Re}\log z, \quad \psi = \frac{\pi}{2} - \arctan\frac{\mathrm{Re}\,z}{\mathrm{Im}\,z} = \mathrm{Im}\log z$$

when $\mathrm{Im}\,z > 0$ (note that $\psi = \arctan(\mathrm{Im}\,z/\mathrm{Re}\,z)$ on the first quadrant $\mathrm{Re}\,z > 0$, $\mathrm{Im}\,z > 0$) hence $\varphi + i\psi = \log z$.

With such φ and ψ, we have

$$h^2 \mathcal{L}_{A,q} e^{\frac{1}{h}(\varphi + i\psi)} = e^{\frac{1}{h}(\varphi + i\psi)}\Big(h(D + A) \cdot (\nabla\psi - i\nabla\varphi)$$
$$+ h(\nabla\psi - i\nabla\varphi) \cdot (D + A) + h^2 \mathcal{L}_{A,q}\Big),$$

thus we will have

$$h^2 \mathcal{L}_{A,q}\Big(e^{\frac{1}{h}(\varphi + i\psi)} a\Big) = \mathcal{O}(h^2) e^{\frac{1}{h}(\varphi + i\psi)}$$

if a is a $\mathcal{C}^2$ solution of the first transport equation, given by

$$\Big((D + A) \cdot (\nabla\psi - i\nabla\varphi) + (\nabla\psi - i\nabla\varphi) \cdot (D + A)\Big)a = 0.$$

We write the latter as a vector field equation

$$(\nabla\psi - i\nabla\varphi) \cdot Da + (\nabla\psi - i\nabla\varphi) \cdot Aa + \frac{1}{2i}(\Delta\psi - i\Delta\varphi)a = 0. \qquad (3.4)$$

We seek a under exponential form $a = e^\Phi$, which means finding Φ solution of

$$(\nabla\varphi + i\nabla\psi) \cdot \nabla\Phi + i(\nabla\varphi + i\nabla\psi) \cdot A + \frac{1}{2}\Delta(\varphi + i\psi) = 0 \qquad (3.5)$$

on Ω. The function Φ has $\mathcal{C}^2$ regularity since the magnetic potential A is $\mathcal{C}^2$.

Remark 3.2. Considering the complex variable $z = x_1 + i|x'|$ as in Remark 3.1 with $\varphi + i\psi = \log z$, we may seek Φ as a solution of the following Cauchy-Riemann equation in the z variable:

$$\frac{\partial \Phi}{\partial \bar{z}} - \frac{(n-2)}{2(z-\bar{z})} + \frac{1}{2} A \cdot (e_1 + i e_r) = 0,$$

where $e_r = (0, \theta)$ is the unit vector pointing in the direction of the r-axis. Indeed, if we denote by $(x_1, r, \theta) \in \mathbf{R} \times \mathbf{R}_+ \times S^{n-2}$ a choice of cylindrical coordinates on $\mathbf{R}^n$ and $z = x_1 + ir$, we have

$$\nabla(\varphi + i\psi) \cdot \nabla = \frac{\partial \log z}{\partial x_1} \frac{\partial}{\partial x_1} + \frac{\partial \log z}{\partial r} \frac{\partial}{\partial r} = \frac{2}{z} \frac{\partial}{\partial \bar{z}}$$

$$\text{and} \quad \Delta(\varphi + i\psi) = \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial r^2} + \frac{(n-2)}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{S^{n-2}} \Big)(\varphi + i\psi)$$

$$= \frac{n-2}{r} \frac{\partial}{\partial r} \log z = \frac{(n-2)i}{rz}.$$

Remark 3.3. Note that the set of solutions of (3.4) is invariant under the multiplication by a function g satisfying

$$(\nabla \varphi + i \nabla \psi) \cdot \nabla g = 0.$$

In the setting of Remark 3.1, this condition reads

$$\frac{\partial g}{\partial \bar{z}} = 0$$

on Ω, i.e. g is a holomorphic function of $z = x_1 + i|x'|$.

Having chosen the phase $\varphi + i\psi$ and the amplitude e^Φ, we obtain an approximate solution of the magnetic Schrödinger equation

$$h^2 \mathcal{L}_{A,q}(e^{\frac{1}{h}(\varphi+i\psi)} e^\Phi) = e^{\frac{1}{h}(\varphi+i\psi)} h^2 \mathcal{L}_{A,q} e^\Phi = \mathcal{O}(h^2) e^{\frac{\varphi}{h}}$$

(recall that Φ is C^2) which we can transform into an exact solution thanks to Proposition 2.4; there exists $r(x, h) \in H^1(\Omega)$ such that

$$h^2 e^{\frac{1}{h}(\varphi+i\psi)} \mathcal{L}_{A,q} r(x, h) = -e^{\frac{1}{h}(\varphi+i\psi)} h \mathcal{L}_{A,q} e^\Phi$$

and $\|r\|_{H^1_{\mathrm{scl}}(\Omega)} \lesssim \|\mathcal{L}_{A,q} e^\Phi\|$.

We sum up the result of this section in the following lemma.

Lemma 3.4. *Let $x_0 \in \mathbf{R}^n \backslash \overline{\mathrm{ch}\, \Omega}$, there exists $h_0 > 0$ and r such that $\|r\|_{H^1_{\mathrm{scl}}(\Omega)} = \mathcal{O}(1)$ and*

$$u(x, h) = e^{\frac{1}{h}(\varphi+i\psi)}(e^{\Phi(x)} + h r(x, h))$$

is a solution of the equation $\mathcal{L}_{A,q} u = 0$, when $h \leq h_0$, and φ is the limiting Carleman weight (3.2), ψ is given by (3.3) and Φ is a solution of the Cauchy-Riemann equation (3.5).

Note that with φ as in (3.2) the parts of the boundary $\partial \Omega_\pm$ delimited by the sign of $\partial_\nu \varphi$ correspond to the front and back sides of the boundary

$$\partial \Omega_- = F(x_0), \quad \partial \Omega_+ = \overline{B(x_0)}.$$

4. Towards Recovering the Magnetic Field

Let $x_0 \in \mathbf{R}^n \backslash \Omega$, suppose that the assumptions of Theorem 1.1 are fulfilled and consider

$$F_\varepsilon = \{x \in \partial\Omega : (x - x_0) \cdot \nu(x) < \varepsilon |x - x_0|^2\} \supset F(x_0)$$

with $\varepsilon > 0$ small enough so that $F_\varepsilon \subset \tilde{F}$, therefore satisfying

$$\mathcal{N}_{A_1,q_1} f(x) = \mathcal{N}_{A_2,q_2} f(x), \quad \forall x \in F_\varepsilon, \quad \forall f \in H^{\frac{1}{2}}(\partial\Omega). \tag{4.1}$$

We may assume without loss of generality that the normal components of A_1 and A_2 are equal on the boundary

$$A_1 \cdot \nu = A_2 \cdot \nu \text{ on } \partial\Omega \tag{4.2}$$

since we can do a gauge transformation in the magnetic potential

$$\mathcal{N}_{A,q} = \mathcal{N}_{A+\nabla\Psi,q}$$

(see (1.3) in the introduction) with $\Psi \in C^3(\bar{\Omega})$ such that $\Psi|_{\partial\Omega} = 0$ and $\partial_\nu \Psi$ is a pre-scribed C^2 function on the boundary [5]. We extend A_1 and A_2 as C^2 compactly supported[6] functions in $\mathbf{R}^n$.

We consider two geometrical optics solutions

$$u_j(x, h) = e^{\frac{1}{h}(\varphi_j + i\psi_j)}(e^{\Phi_j} + h r_j(x, h))$$

of the equations $\mathcal{L}_{A_1,\bar{q}_1} u_1 = 0$ and $\mathcal{L}_{A_2,q_2} u_2 = 0$ constructed in the former section with phases

$$\varphi_2(x) = -\varphi_1(x) = \varphi(x) = \log|x - x_0|,$$
$$\psi_2(x) = \psi_1(x) = \psi(x) = d_{S^{n-1}}\left(\frac{x - x_0}{|x - x_0|}, \omega\right), \tag{4.3}$$

defined on a neighborhood $\tilde{\Omega}$ of Ω (and ω varies in Γ_0), and where Φ_1 and Φ_2 are solutions of the equations

$$(\nabla\varphi - i\nabla\psi) \cdot \nabla\Phi_1 + i(\nabla\varphi - i\nabla\psi) \cdot A_1 + \frac{1}{2}\Delta(\varphi - i\psi) = 0,$$
$$(\nabla\varphi + i\nabla\psi) \cdot \nabla\Phi_2 + i(\nabla\varphi + i\nabla\psi) \cdot A_2 + \frac{1}{2}\Delta(\varphi + i\psi) = 0. \tag{4.4}$$

The remainders r_j are bounded independently of h in H^1_{scl}. Note that it implies the following estimate on u_j:

$$\|e^{-\frac{\varphi_j}{h}} u_j\|_{H^1_{\text{scl}}} = \mathcal{O}(1). \tag{4.5}$$

By w we denote the solution to the equation

$$\mathcal{L}_{A_1,q_1} w = 0, \quad w|_{\partial\Omega} = u_2|_{\partial\Omega}$$

---

[5] After the use of a partition of unity and a transfer to $\{x_1 \geq 0\}$, this is Theorem 1.3.3. in [6].
[6] Note that A_1 and A_2 do not necessarily agree on $\partial\Omega$.

so that $\mathcal{N}_{A_1,q_1}(u_2|_{\partial\Omega}) = (\partial_\nu w)|_{\partial\Omega} + i A_1 \cdot \nu\, u_2|_{\partial\Omega}$. The assumption (4.1) means that

$$\partial_\nu(w - u_2)(x) = 0, \quad \forall x \in F_\varepsilon$$

(here we use the fact (4.2) that the normal components of the magnetic potentials coincide on the boundary). Besides, we have

$$
\begin{aligned}
\mathcal{L}_{A_1,q_1}(w - u_2) &= -\mathcal{L}_{A_1,q_1} u_2 = (\mathcal{L}_{A_2,q_2} - \mathcal{L}_{A_1,q_1})u_2 \\
&= (A_2 - A_1) \cdot D u_2 + D \cdot (A_2 - A_1)u_2 \\
&\quad + (A_2^2 - A_1^2 + q_2 - q_1)u_2,
\end{aligned}
\tag{4.6}
$$

hence we deduce

$$
\begin{aligned}
&\big(\mathcal{L}_{A_1,q_1}(w - u_2)\,|\,u_1\big) \\
&= \big((A_2 - A_1) \cdot D u_2\,|\,u_1\big) + \big(u_2\,|\,(A_2 - A_1) \cdot D u_1\big) \\
&\quad + \frac{1}{i}\underbrace{\big((A_2 - A_1) \cdot \nu\, u_2|u_1\big)_{\partial\Omega}}_{=0} + \big((A_2^2 - A_1^2 + q_2 - q_1)u_2|u_1\big).
\end{aligned}
\tag{4.7}
$$

The magnetic Green's formula gives

$$
\begin{aligned}
&(\mathcal{L}_{A_1,q_1}(w - u_2)|u_1) \\
&\quad = \underbrace{(w - u_2|\mathcal{L}_{A_1,\bar{q}_1}u_1)}_{=0} - \big((\partial_\nu + i A_1 \cdot \nu)(w - u_2)|u_1\big)_{\partial\Omega} \\
&\quad = -(\partial_\nu(w - u_2)|u_1)_{\partial\Omega\setminus F_\varepsilon},
\end{aligned}
\tag{4.8}
$$

and combining (4.7) and (4.8), we finally obtain

$$
\begin{aligned}
\int_{\partial\Omega\setminus F_\varepsilon} \partial_\nu(u_2 - w)\,\bar{u}_1\,d\sigma(x) &= \int_\Omega (A_2^2 - A_1^2 + q_2 - q_1)u_2\,\bar{u}_1\,dx \\
&\quad + \int_\Omega (A_2 - A_1) \cdot (D u_2\,\bar{u}_1 + u_2\,\overline{D u_1})\,dx.
\end{aligned}
\tag{4.9}
$$

With our choice of $\varphi_2 = \varphi = \log|x - x_0|$, we have

$$F_\varepsilon \supset F(x_0) = \partial\Omega_- \quad \text{thus} \quad \partial\Omega\setminus F_\varepsilon \subset \partial\Omega_+,$$

and moreover $\partial_\nu\varphi > \varepsilon$ on $\partial\Omega\setminus F_\varepsilon$, therefore the modulus of the left-hand side in (4.9) is bounded by

$$\frac{1}{\sqrt{\varepsilon}}\|\sqrt{\partial_\nu\varphi}\,e^{-\frac{\varphi}{h}}\partial_\nu(u_2 - w)\|_{\partial\Omega_+} \times \underbrace{\|e^{\Phi_1} + h r_1\|_{\partial\Omega_+}}_{\leq \|e^{\Phi_1}\|_{\partial\Omega} + \|r_1\|_{H^1_{\text{scl}}}}$$

which, by virtue of the Carleman estimate (2.13), is bounded by a constant times

$$\frac{1}{\sqrt{\varepsilon}}\Big(\sqrt{h}\|e^{-\frac{\varphi}{h}}\mathcal{L}_{A_1,q_1}(u_2 - w)\| + \underbrace{\|\sqrt{-\partial_\nu\varphi}\,e^{-\frac{\varphi}{h}}\partial_\nu(u_2 - w)\|_{\partial\Omega_-}}_{=0 \text{ because of (4.1)}}\Big).$$

In view of (4.6) and of (4.5) the former expression is $\mathcal{O}(h^{-\frac{1}{2}})$. Therefore we can conclude that the right-hand side of (4.9) is $\mathcal{O}(h^{-\frac{1}{2}})$. This constitutes an important difference with [7], where[7] the corresponding term was $\mathcal{O}(h)$.

More directly, the first and second right-hand side terms of (4.9) are respectively $\mathcal{O}(1)$ and $\mathcal{O}(h^{-1})$ as may be seen from (4.5). It turns out that the information obtained when disregarding the bounded term is enough to recover the magnetic field. We multiply (4.9) by h and let h tend to 0:

$$\lim_{h \to 0} \int_\Omega \left((A_2 - A_1) \cdot h D u_2 \, \bar{u}_1 + u_2 \, (A_2 - A_1) \cdot h \overline{D u_1} \right) dx = 0.$$

Using the explicit form of the solutions u_1 and u_2, this further means

$$\int_\Omega (A_2 - A_1) \cdot (\nabla \varphi + i \nabla \psi) e^{\bar{\Phi}_1 + \Phi_2} \, dx = 0. \tag{4.10}$$

Adding the complex conjugate of the first line of (4.4) to the second line, we see that

$$(\nabla \varphi + i \nabla \psi) \cdot \left(\nabla \bar{\Phi}_1 + \nabla \Phi_2 + i(A_2 - A_1) \right) + \Delta(\varphi + i \psi) = 0;$$

this implies that

$$(D + A_2 - A_1) \cdot (\nabla \varphi + i \nabla \psi)(e^{\bar{\Phi}_1 + \Phi_2}) = 0. \tag{4.11}$$

As observed in Remark 3.3, in the expression for u_2, we may replace e^{Φ_2} by $e^{\Phi_2} g$ if g is a solution of $(\nabla \varphi + i \nabla \psi) \cdot \nabla g = 0$. Then (4.10) can be replaced by

$$\int_\Omega (A_2 - A_1) \cdot (\nabla \varphi + i \nabla \psi) e^{\bar{\Phi}_1 + \Phi_2} g \, dx = 0.$$

From equation (4.11), we see that we can replace $A_2 - A_1$ by $i \nabla$ in the former equality

$$\int_\Omega g(x) \, \nabla \cdot \left(e^{\bar{\Phi}_1 + \Phi_2} (\nabla \varphi + i \nabla \psi) \right) dx = 0 \tag{4.12}$$

for all functions g such that $(\nabla \varphi + i \nabla \psi) \cdot \nabla g = 0$ on Ω.

Remark 4.1. An integration by parts gives

$$\int_{\partial \Omega} (\partial_\nu \varphi + i \partial_\nu \psi) e^{\bar{\Phi}_1 + \Phi_2} g \, d\sigma - \int_\Omega e^{\bar{\Phi}_1 + \Phi_2} \underbrace{(\nabla \varphi + i \nabla \psi) \cdot \nabla g}_{=0} \, dx = 0$$

hence we have $\int_{\partial \Omega} (\partial_\nu \varphi + i \partial_\nu \psi) e^{\bar{\Phi}_1 + \Phi_2} g \, d\sigma = 0.$

---

[7] See (5.16) and the subsequent lines in [7].

5. Moving to the Complex Plane

In this section, we follow Remark 3.1 and choose to work in the cylindrical coordinates. Let us be more precise: suppose that $x_0 = 0 \notin \text{ch}(\Omega)$ and that we have picked $\omega \in S^{n-1} \setminus (\Gamma \cup \check{\Gamma})$ with the notations of Sect. 3. After a rotation, we assume that $\omega = (1, 0, \ldots, 0)$, therefore we have

$$\Omega \Subset \tilde{\Omega} \subset \{x \in \mathbf{R}^n : x' \neq 0\}.$$

We choose the following cylindrical coordinates:

$$t = x_1, \quad r = |x'| > 0, \quad \theta = \Theta(x) = \frac{x'}{|x'|} \in S^{n-2}.$$

By Sard's theorem, the set of critical values of $\Theta : \Omega \mapsto S^{n-2}$ is of measure 0, therefore the set $\Omega_{\theta_0} = \Theta^{-1}(\theta_0) = \{x \in \Omega : x' = r\theta_0, r > 0\}$ is an open set with smooth boundary for almost every θ_0 in $\Theta(\Omega)$. The result obtained in the former section reads

$$\int_{S^{n-2}} \iint_{\Omega_\theta} g(x) \, \nabla_x \cdot \left(e^{\bar{\Phi}_1 + \Phi_2} (\nabla_x \varphi + i \nabla_x \psi) \right) r^{n-2} dr dt \, d\theta = 0,$$

and taking $g = g_1(t, r) \otimes g_2(\theta)$ and varying g_2 leads to

$$\iint_{\Omega_\theta} g(t, r) \, \nabla_x \cdot \left(e^{\bar{\Phi}_1 + \Phi_2} (\nabla_x \varphi + i \nabla_x \psi) \right) r^{n-2} dr dt = 0 \tag{5.1}$$

for any function g such that $(\nabla_x \varphi + i \nabla_x \psi) \cdot \nabla_x g = 0$ on Ω_θ, and this for almost every θ.

Now we consider the complex variable $z = t + ir \in \mathbf{C}_+ = \{w \in \mathbf{C} : \text{Im } w > 0\}$ and write our results in this setting. Let us recall the results of the computations made in Remarks 3.1 and 3.2:

$$\nabla_x (\varphi + i\psi) \cdot \nabla_x = \frac{2}{z} \frac{\partial}{\partial \bar{z}} \quad \text{and} \quad \Delta_x (\varphi + i\psi) = \frac{(n-2)i}{rz} \tag{5.2}$$

$$\text{and thus} \quad \nabla_x \cdot \nabla_x \circ (\varphi + i\psi) = \frac{2}{z} \frac{\partial}{\partial \bar{z}} + \frac{(n-2)i}{rz}.$$

Similarly, the functions Φ_j satisfy

$$\frac{\partial}{\partial \bar{z}} (\bar{\Phi}_1 + \Phi_2) = \frac{(n-2)}{(z - \bar{z})} + \frac{1}{2} (A_1 - A_2) \cdot (e_1 + i e_r). \tag{5.3}$$

Finally, (5.1) reads

$$\iint_{\Omega_\theta} g(z) \frac{1}{z} \left(\frac{\partial}{\partial \bar{z}} - \frac{(n-2)}{(z-\bar{z})} \right) (e^{\bar{\Phi}_1 + \Phi_2})(z - \bar{z})^{n-2} \, d\bar{z} \wedge dz = 0, \tag{5.4}$$

for any $g \in \mathcal{H}(\Omega_\theta)$. Replacing the holomorphic function g/z on Ω_θ by g, we can drop the factor $1/z$.

If g is a holomorphic function, we have[8]

$$d\left(e^{\bar{\Phi}_1+\Phi_2}g(z)(z-\bar{z})^{n-2}dz\right)$$

$$= \frac{\partial}{\partial\bar{z}}\left((z-\bar{z})^{n-2}e^{\bar{\Phi}_1+\Phi_2}\right)g(z)d\bar{z}\wedge dz$$

$$= \left(\frac{\partial}{\partial\bar{z}} - \frac{n-2}{z-\bar{z}}\right)\left(e^{\bar{\Phi}_1+\Phi_2}\right)g(z)(z-\bar{z})^{n-2}d\bar{z}\wedge dz,$$

therefore the Stokes' formula implies

$$\iint_{\Omega_\theta} g(z)\left(\frac{\partial}{\partial\bar{z}} - \frac{(n-2)}{(z-\bar{z})}\right)(e^{\bar{\Phi}_1+\Phi_2})(z-\bar{z})^{n-2}\,d\bar{z}\wedge dz$$

$$= \int_{\partial\Omega_\theta} g(z)e^{\bar{\Phi}_1+\Phi_2}(z-\bar{z})^{n-2}\,dz.$$

Together with (5.4) this gives

$$\int_{\partial\Omega_\theta} g(z)e^{\bar{\Phi}_1+\Phi_2}(z-\bar{z})^{n-2}\,dz = 0 \qquad (5.5)$$

for any $g \in \mathcal{H}(\Omega_\theta)$.

Lemma 5.1. *There exists a non-vanishing holomorphic function F on Ω_θ, continuous on $\bar{\Omega}_\theta$, whose restriction to $\partial\Omega_\theta$ is equal to $(z-\bar{z})^{n-2}e^{\bar{\Phi}_1+\Phi_2}$.*

Proof. We denote $f(z) = (z-\bar{z})^{n-2}e^{\bar{\Phi}_1+\Phi_2}$ and consider the Cauchy integral operator

$$C(f)(z) = \frac{1}{2\pi i}\int_{\partial\Omega_\theta} \frac{f(\zeta)}{\zeta-z}\,d\zeta, \quad \forall z \in \mathbf{C}\backslash\partial\Omega_\theta.$$

The function $C(f)$ is holomorphic inside and outside Ω_θ and the Plemelj-Sokhotski-Privalov formula reads on the boundary

$$\lim_{\substack{z\to z_0 \\ z\in\Omega_\theta}} C(f)(z) - \lim_{\substack{z\to z_0 \\ z\notin\Omega_\theta}} C(f)(z) = f(z_0), \quad \forall z_0 \in \partial\Omega_\theta. \qquad (5.6)$$

The function $\zeta \to (\zeta-z)^{-1}$ is holomorphic on Ω_θ when $z \notin \Omega_\theta$ hence (5.5) implies that $C(f)(z) = 0$ when $z \notin \Omega_\theta$. The second limit in (5.6) is 0, thus $F = C(f)$ is a holomorphic function on Ω_θ whose restriction to the boundary agrees with f.

It remains to prove that F does not vanish on Ω_θ. This is clear by the argument principle since

$$\operatorname*{vararg}_{\partial\Omega_\theta} F = \operatorname*{vararg}_{\partial\Omega_\theta} f = 0$$

and F is holomorphic. □

---

[8] The result of this computation is the transcription of the fact that the formal adjoint of $(\nabla\varphi + i\nabla\psi)\cdot\nabla$ is $\nabla\cdot(\nabla\varphi + i\nabla\psi)$ in the complex setting, where the measure is $r^{n-2}d\bar{z}\wedge dz$.

In particular, with the former function, we have on the boundary

$$(z - \bar{z})^{n-2} e^{\bar{\Phi}_1 + \Phi_2} = F(z, \theta), \quad \forall z \in \partial\Omega_\theta.$$

We want to prove that F admits a holomorphic logarithm[9]; to do so, we consider the one form on Ω

$$\alpha = \frac{dF}{F}$$

(recall that F is non-vanishing on Ω) which is closed. Since Ω is simply connected, α has a primitive $\alpha = da$, which defines a logarithm of F. The equation $F = e^a$ implies that a is holomorphic with respect to z, hence one can define a holomorphic logarithm of F. This implies

$$\bar{\Phi}_1 + \Phi_2 + \log(z - \bar{z})^{n-2} = \log F(z), \quad \forall z \in \partial\Omega_\theta$$

with $\log F$ a holomorphic function on Ω_θ, and therefore

$$\int_{\partial\Omega_\theta} g(z)\left(\bar{\Phi}_1 + \Phi_2 + \log(z - \bar{z})^{n-2}\right) dz = 0.$$

An application of Stokes' formula gives

$$\iint_{\Omega_\theta} g(z)\left(\frac{\partial}{\partial\bar{z}}(\bar{\Phi}_1 + \Phi_2) - \frac{n-2}{z - \bar{z}}\right) d\bar{z} \wedge dz = 0,$$

hence using equation (5.3), this implies

$$\iint_{\Omega_\theta} g(z)(A_1 - A_2) \cdot (e_1 + ie_r) \, d\bar{z} \wedge dz = 0. \tag{5.7}$$

With $g = 1$, the former equality reads

$$\iint_{\Omega_\theta} (t + ir)(\nabla_x \varphi + i\nabla_x \psi) \cdot (A_1 - A_2) \, dt \, dr = 0.$$

Denote by $P_\theta = \mathrm{span}(\omega, e_r)$ the plane along the axis directed by $\omega = (1, 0, \ldots, 0)$, and by P_θ^+ the half plane where $x \cdot e_r > 0$, then $\Omega_\theta = \Omega \cap \{x = (x_1, x') \in \mathbf{R}^n : x' = r\theta, r > 0\} = \Omega \cap P_\theta^+$. Let π_θ be the projection on P_θ and $d\lambda_\theta$ the measure on the plane, then (5.7) (with $g = 1$) implies

$$\int_{P_\theta \cap \Omega} \pi_\theta (A_1 - A_2) \, d\lambda_\theta = 0,$$

for almost every $\theta \in S^{n-2}$, hence for all $\theta \in S^{n-2}$ by continuity. The former may be rephrased under the form

$$\int_{x_0 + P} \xi \cdot \left(1_\Omega (A_1 - A_2)\right) d\lambda_P = 0, \quad \forall \xi \in P$$

for all linear planes P containing $\omega = (1, 0, \ldots, 0)$. We can also let x_0 vary in a small neighborhood of $0 \notin \mathrm{ch}(\Omega)$ and ω vary in a neighborhood Γ_0 of $(1, 0, \ldots, 0)$ on the sphere S^{n-1}.

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[9] This is obvious when Ω_θ is simply connected, which is the case if Ω is for instance convex.

Lemma 5.2. *Let A be a C^1 vector field on $\bar{\Omega}$. If*

$$\int_{P \cap \Omega} \xi \cdot A \, d\lambda_P = 0, \quad \forall \xi \in T_x(P) \tag{5.8}$$

for all planes P such that $d((0, e_1), T(P)) < \delta$ then $dA = 0$ on Ω.

The proof of this lemma is based on the following microlocal version of Helgason's support theorem.

Theorem 5.3. *Let $f \in C^0(\mathbf{R}^n)$, suppose that the Radon transform of f satisfies*

$$\mathcal{R}f(H) = \int_H f \, d\lambda_H = 0$$

for all hyperplanes H in some neighbourhood of a hyperplane H_0, then

$$N^*(H_0) \cap \mathrm{WF}_a(f) = \varnothing,$$

where $N^*(H_0)$ denotes the conormal bundle of H_0.

The proof of this result may be found in [2] (see Proposition 1) or in [5] (see Sect. 6). We will also need the microlocal version of Holmgren's theorem (see [5], Sect. 1 or [6] Sect. 8.5).

Theorem 5.4. *Let $f \in \mathcal{E}'(\mathbf{R}^n)$ then we have*

$$N(\mathrm{supp}\, f) \subset \mathrm{WF}_a(f),$$

where $N(\mathrm{supp}\, f)$ is the normal set of the support of f.

These two results may be combined to provide a proof of Helgason's support theorem (see [2] and [5]). We also refer to the book [4] for a review on Radon transforms.

Proof of Lemma 5.2. Let $\chi \in C_0^\infty(|x| < \frac{1}{2})$ and $\chi_\varepsilon = \varepsilon^{-n} \chi(\cdot/\varepsilon)$ be a standard regularization, one has

$$\int_P \xi \cdot (\chi_\varepsilon * 1_\Omega A) \, d\lambda_P = \varepsilon^{-n} \int \chi\left(\frac{y}{\varepsilon}\right) \left(\int_{(-y+P) \cap \Omega} \xi \cdot A \, d\lambda_{-y+P}\right) dy = 0$$

when $d((0, e_1), T(P)) < \delta - \varepsilon$. Therefore it suffices to prove the result when $\Omega = \mathbf{R}^n$ and $A \in C_0^\infty(\mathbf{R}^n; \mathbf{R}^n)$ since $d(\chi_\varepsilon * 1_\Omega A)$ tends to dA as a distribution when ε tends to 0.

Our first step is to prove that

$$\iota_H^* dA = 0 \tag{5.9}$$

for any subspace $H \subset \mathbf{R}^n$ of dimension 3 such that $d((0, e_1), T(H)) < \delta$. Here ι_H denotes the injection of H in $\mathbf{R}^n$. For any plane $P \subset H$ such that $d((0, e_1), T(P)) < \delta$, we have

$$\int_P \langle dA, \xi \wedge \eta \rangle \, d\lambda_P = \frac{d}{dt}\left(\int_{t\xi+P} \eta \cdot A \, d\lambda_P - \int_{t\eta+P} \xi \cdot A \, d\lambda_P\right)_{t=0}$$

when $\xi, \eta \in T_x(H)$. The space H is of dimension 3 so we can assume that either η or ξ belongs to $T_x(P)$, thus the former expression is zero because of (5.8) and of the fact that whenever $\eta \in T_x(P)$

$$\int_{t\eta+P} \xi \cdot A \, d\lambda_P \quad \text{is constant.}$$

Therefore, if $\mathcal{R}_H$ denotes the Radon transform in H, we obtain

$$\mathcal{R}_H\big(\langle \iota_H^* dA, \xi \wedge \eta \rangle\big)(P) = 0$$

for any plane $P \subset H$ such that $d((0, e_1), T(P)) < \delta$ and for any $\xi, \eta \in T_x(H)$. Combining Theorems 5.3 and 5.4 we obtain

$$N^*(P) \cap N(\mathrm{supp}\langle \iota_H^* dA, \xi \wedge \eta \rangle) = \varnothing$$

for any plane $P \subset H$ such that $d((0, e_1), T(P)) < \delta$. This gives (5.9) since such a family of planes sweeps other $\Omega \cap H$ and the support of A is on one side of at least one such plane.

The result (5.9) implies in particular that

$$\langle dA(x), \xi \wedge \eta \rangle = 0, \quad \forall x \in \mathbf{R}^n, \forall (\xi, \eta) \in S^n \times \mathbf{R}^n, \ |\xi - e_1| < \delta,$$

and therefore $dA = 0$ by linearity. $\quad\square$

6. Recovering the Potential

End of the proof of Theorem 1.1. Applying this lemma, we finally obtain

$$dA_1 = dA_2 \quad \text{on } \Omega,$$

therefore the difference of the two potentials is a gradient $A_1 - A_2 = \nabla\Psi$ (recall that Ω is simply connected). The identity (5.7) now reads

$$\iint_{\Omega_\theta} g(z)\partial_{\bar z}\Psi(z, \theta) \, d\bar z \wedge dz = 0$$

for any holomorphic function g on Ω_θ and by Stokes' theorem we get

$$\int_{\partial\Omega_\theta} g(z)\Psi(z, \theta) \, dz = 0.$$

Reasoning as in the beginning of Lemma 5.1, there exists a holomorphic function $\tilde\Psi \in \mathcal{H}(\Omega_\theta)$ such that $\tilde\Psi|_{\partial\Omega_\theta} = \Psi|_{\partial\Omega_\theta}$. Now Ψ is real-valued, and since $\tilde\Psi$ is real-valued on $\partial\Omega_\theta$ and harmonic, it is real-valued everywhere. The only real-valued holomorphic functions are the constant ones, so $\tilde\Psi$ and hence Ψ is constant on $\partial\Omega_\theta$. Varying x_0 and ω, we get that Ψ is constant on the boundary $\partial\Omega$.

Indeed, the fact that Ψ is constant on $\partial\Omega_\theta$ means that $X \cdot \Psi = 0$ when X is a tangent vector field to $\partial\Omega_\theta$, hence it suffices to show that when varying x_0 and ω, one generates enough directions to span the tangent space to $\partial\Omega$ at a given point $a \in \partial\Omega$. Note that the plane P_θ belonging to the family of rotating planes passing through $a \in \partial\Omega$ is spanned by the vectors ω and $e_r = \pi_\omega(a - x_0)/|\pi_\omega(a - x_0)|$, where π_ω is the orthogonal projection onto the space $\omega^\perp$. A variation of x_0 in a small neighborhood of 0 induces a variation

of e_r in a neighborhood of $\pi_\omega(a)/|\pi_\omega(a)|$. If one picks ω to be outside the tangent space $T_a(\partial\Omega)$, then the intersection of $T_a(\partial\Omega)$ with the union of planes generated by ω and e_r when e_r varies in a neighborhood of $\pi_\omega(a)/|\pi_\omega(a)|$, contains enough independent vectors to span $T_a(\partial\Omega)$. Therefore Ψ is constant on $\partial\Omega$, and we may as well assume that $\Psi = 0$ on $\partial\Omega$.

By a gauge transformation, we may assume that $\Psi = 0$, thus $A_1 = A_2$. We could almost directly apply the result in [7] to recover the identity of the two potentials $q_1 = q_2$, if it were not for the presence of the two magnetic potentials in the equations. Instead we go back to the limit induced by (4.9). The second right-hand side term is now zero. The left-hand side is now $\mathcal{O}(\sqrt{h})$ since the $\mathcal{O}(h^{-1})$ term in (4.6) is zero and we can reproduce the arguments given after (4.9). Therefore we obtain

$$\lim_{h\to 0} \int_\Omega (q_2 - q_1)u_2\bar{u}_1 \, dx = 0, \tag{6.1}$$

thus

$$\int_\Omega (q_2 - q_1)e^{\bar{\Phi}_1 + \Phi_2} \, dx = 0.$$

As observed in Remark 3.3, we may replace e^{Φ_2} by $e^{\Phi_2}g$ if g is a solution of $(\nabla\varphi + i\nabla\psi) \cdot \nabla g = 0$. Then the former can be replaced by

$$\int_\Omega (q_2 - q_1)e^{\bar{\Phi}_1 + \Phi_2}g(x) \, dx = 0.$$

Moving to the complex plane, as in Sect. 5, this reads

$$\int_\Omega (q_2 - q_1)g(z)e^{\bar{\Phi}_1 + \Phi_2}(z - \bar{z})^{n-2} \, d\bar{z} \wedge dz \wedge d\theta = 0$$

for any holomorphic function g on Ω_θ. But the transport equation (5.3) now reads

$$\frac{\partial}{\partial\bar{z}}\left((z - \bar{z})^{n-2}e^{\bar{\Phi}_1 + \Phi_2}\right) = 0,$$

therefore if we take $g = (z - \bar{z})^{-n+2}e^{-\bar{\Phi}_1 - \Phi_2}$, and write $q = 1_\Omega(q_1 - q_2)$, we obtain

$$\int_{\mathbf{R}^2 \times S^{n-1}} q(t, r, \theta)g(\theta) \, dt \, dr \, d\theta = 0 \tag{6.2}$$

for any (say smooth) function $g(\theta)$. Varying x_0 slightly, this remains true for the translated functions $q(\cdot - y)$, when y is small, hence for regularisations of q,

$$\chi_\varepsilon * q = \varepsilon^{-n} \int \chi\left(\frac{y}{\varepsilon}\right)q(x - y) \, dy,$$

therefore it suffices to assume that q is smooth. When q is smooth, varying g, (6.2) implies

$$\int_{\mathbf{R}^2} q(t, r, \theta) \, dt \, dr = 0 \tag{6.3}$$

for all $\theta \in S^{n-2}$.

As in Sect. 5, varying x_0 and ω, (6.3) may be interpreted as

$$\int_P q \, d\lambda_P = 0$$

for any plane P such that $d((0, e_1), T(P)) < \delta$. This implies that for any subspace H of dimension 3 such that $d((0, e_1), T(H)) < \delta$ we have

$$\mathcal{R}_H q(P) = \int_P q \, d\lambda_P = 0$$

for any plane such that $d((0, e_1), T(P)) < \delta$. Applying Theorems 5.3 and 5.4 as in the proof of Lemma 5.2, we get

$$N^*(P) \cap N(\operatorname{supp} q|_H) = \varnothing,$$

and therefore $q = 0$ on H leading to $q = 0$. This ends the proof of Theorem 1.1. $\quad\square$

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