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On the Geometry of Closed G_2-Structures

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Abstract: We give an answer to a question posed in physics by Cvetič et al. [9] and recently in mathematics by Bryant [3], namely we show that a compact 7-dimensional manifold equipped with a G_2-structure with closed fundamental form is Einstein if and only if the Riemannian holonomy of the induced metric is contained in G_2. This could be considered to be a G_2 analogue of the Goldberg conjecture in almost Kähler geometry and was indicated by Cvetič et al. in [9]. The result was generalized by Bryant to closed G_2-structures with too tightly pinched Ricci tensor. We extend it in another direction proving that a compact G_2-manifold with closed fundamental form and divergence-free Weyl tensor is a G_2-manifold with parallel fundamental form. We introduce a second symmetric Ricci-type tensor and show that Einstein conditions applied to the two Ricci tensors on a closed G_2-structure again imply that the induced metric has holonomy group contained in G_2.

1. Introduction

A 7-dimensional Riemannian manifold is called a G_2-manifold if its structure group reduces to the exceptional Lie group G_2. The existence of a G_2-structure is equivalent to the existence of a non-degenerate three-form on the manifold, sometimes called the fundamental form of the G_2-manifold. From the purely topological point of view, a 7-dimensional paracompact manifold is a G_2-manifold if and only if it is an oriented spin manifold [23].

The geometry of G_2-structures has also attracted much attention from physicists. The central issue in physics is that connections with holonomy contained in G_2 plays a rôle in string theory [14, 9, 24, 15, 18]. The G_2-connections admitting three-form torsion have been of particular interest.

In [11], Fernández and Gray divide G_2-manifolds into 16 classes according to how the covariant derivative of the fundamental three-form behaves with respect to its decomposition into G_2-irreducible components (see also [7]). If the fundamental form is parallel with respect to the Levi-Civita connection then the Riemannian holonomy group is contained in G_2, we will say that the G_2-manifold or the G_2-structure on the manifold is *parallel*. In this case the induced metric on the G_2-manifold is Ricci-flat, a fact first observed by Bonan [1]. It was shown by Gray [17] (see also [2, 25]) that a G_2-manifold is parallel precisely when the fundamental form is harmonic. The first examples of complete parallel G_2-manifolds were constructed by Bryant and Salamon [5, 16]. Compact examples of parallel G_2-manifolds were obtained first by Joyce [19–21] and recently by Kovalev [22]. Compact parallel G_2-manifolds will be referred to as *Joyce spaces*. Examples of G_2-manifolds in other Fernández-Gray classes may be found in [10, 6]. A central point in our argument is that the Riemannian scalar curvature of a G_2-manifold may be expressed in terms of the fundamental form and its derivatives and furthermore the scalar curvature carries a definite sign for certain classes of G_2-manifolds [13, 3].

In the present paper we are interested in the geometry of closed G_2-structures i.e., G_2-manifolds with closed fundamental form (sometimes these spaces are called almost G_2-manifolds or calibrated G_2-manifolds). In the sense of the Fernández-Gray classes, this is complementary to the physicists' requirement of three-form torsion [12]. Compact examples of closed G_2-manifolds were presented by Fernández [10]. Topological quantum field theory on closed G_2-manifolds were discussed in [24]. Supersymmetric string solutions on closed G_2-manifolds were investigated in [9] where the authors indicated the G_2-analogue of the Goldberg conjecture in almost Kähler geometry. Bryant shows in [3] that if the scalar curvature of a closed G_2-structure is non-negative then the G_2-manifold is parallel. The question whether there are closed G_2-structures which are Einstein but not Ricci-flat then naturally arises. We investigate this question in the compact and in the non-compact cases.

In the first version of the present article [8] we answered negatively to the G_2-version of the Goldberg conjecture, namely, we proved that there are no closed Einstein G_2-structures (other than the parallel ones) on a compact 7-manifold. In [4] Bryant generalized this non-existence result for closed G_2-structures on compact 7-manifold whose Ricci tensor is too tightly pinched.

In the present article we obtain a non-existence result involving third derivatives of the fundamental form. Namely, we prove the following

Main Theorem. *A compact G_2-manifold with closed fundamental form and harmonic Weyl tensor* (*divergence-free Weyl tensor*) *is a Joyce space.*

The second Bianchi identity leads to

Corollary 1.1. *A compact G_2-manifold with closed fundamental form and harmonic curvature* (*divergence-free curvature tensor*) *is a Joyce space.*

Corollary 1.2. *A compact Einstein G_2-manifold with closed fundamental form is a Joyce space.*

The latter may be considered to be a G_2 analogue of the Goldberg conjecture in almost Kähler geometry (see e.g. [9]).

The representation theory of G_2 gives rise to a second symmetric Ricci type tensor on G_2-manifolds. Therefore one may consider two complementary Einstein equations. We find a connection between the two Ricci tensors and show in Theorem 5.7, with no compactness assumption, that if both Einstein conditions hold simultaneously on a G_2-manifold with closed fundamental form then the fundamental form is parallel.

Our main tool is the canonical connection of a G_2-structure and its curvature. We will show that the Ricci tensor of the canonical connection is proportional to the Riemannian

Ricci tensor. This leads to the corollary that a compact G_2-manifold with closed fundamental form which is Einstein with respect to the canonical connection is a Joyce space.

Our main technical tool is an integral formula which holds on any compact G_2-manifold with closed fundamental form. We derive the Main Theorem as a consequence of a more general result, Theorem 7.1, which shows that the vanishing of the Λ^2_7-part of the divergence of the Weyl tensor implies that a closed G_2-structure is parallel on a compact 7-manifold.

2. General Properties of G_2-Structures

We recall some notions of G_2 geometry. Endow $\mathbb{R}^7$ with its standard orientation and inner product. Let $e_1, \dots, e_7$ be an oriented orthonormal basis which we identify with the dual basis via the inner product. Write $e_{i_1 i_2 \dots i_p}$ for the monomial $e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_p}$. We shall omit the $\sum$-sign understanding summation on any pair of equal indices.

Consider the three-form ω on $\mathbb{R}^7$ given by

$$\omega = e_{124} + e_{235} + e_{346} + e_{457} + e_{561} + e_{672} + e_{713}. \tag{2.1}$$

The subgroup of $GL(7)$ fixing ω is the exceptional Lie group G_2. It is a compact, connected, simply-connected, simple Lie subgroup of $SO(7)$ of dimension 14 [2].

The Hodge star operator supplies the 4-form $*\omega$ given by

$$*\omega = -e_{3567} - e_{4671} - e_{5712} - e_{6123} - e_{7234} - e_{1345} - e_{2456}. \tag{2.2}$$

We let the expressions

$$\omega = \frac{1}{6}\omega_{ijk} e_{ijk},$$
$$*\omega = \frac{1}{24}\omega_{ijkl} e_{ijkl}$$

define the symbols ω_{ijk} and ω_{ijkl}. We then obtain the following set of formulae:

$$\begin{aligned}
\omega_{ipq}\omega_{jpq} &= 6\delta_{ij},\\
\omega_{ipq}\omega_{jkpq} &= -4\omega_{ijk},\\
\omega_{ijp}\omega_{klp} &= -\omega_{ijkl} + \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk},\\
\omega_{ijpq}\omega_{klpq} &= -2\omega_{ijkl} + 4(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}),\\
\omega_{ijp}\omega_{klmp} &= \delta_{ik}\omega_{jlm} - \delta_{jk}\omega_{ilm} + \delta_{il}\omega_{jmk} - \delta_{jl}\omega_{imk} + \delta_{im}\omega_{jkl} - \delta_{jm}\omega_{ikl}.
\end{aligned} \tag{2.3}$$

Definition 2.1. *A* **G_2-structure** *on a 7-manifold M is a reduction of the structure group of the tangent bundle to the exceptional group G_2. Equivalently, there exists a nowhere vanishing differential three-form ω on M and local frames of the cotangent bundle with respect to which ω takes the form* (2.1). *The three-form ω is called the* **fundamental form** *of the G_2-manifold M* [1].

We will say that the pair (M, ω) is a **G_2-manifold** *with* **G_2-structure** (*determined by*) *ω.*

Remark 2.2. Alternatively, a G_2-structure can be described by the existence of a two-fold vector cross product P on the tangent spaces of M.

The fundamental form of a G_2-manifold determines a metric through $g_{ij} = \frac{1}{6}\omega_{ikl}\omega_{jkl}$. This is referred to as the metric induced by ω. We write ∇^g for the associated Levi-Civita connection, $||.||^2$ for the tensor norm with respect to g. In addition we will freely identify vectors and co-vectors via the induced metric g.

Let (M, ω) be a G_2-manifold. The action of G_2 on the tangent space induces an action of G_2 on $\Lambda^k(M)$ splitting the exterior algebra into orthogonal subspaces, where Λ^k_l corresponds to an l-dimensional G_2-irreducible subspace of Λ^k:

$$\Lambda^1(M) = \Lambda^1_7, \quad \Lambda^2(M) = \Lambda^2_7 \oplus \Lambda^2_{14}, \quad \Lambda^3(M) = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27},$$

where

$$\begin{aligned}
\Lambda^2_7 &= \{\alpha \in \Lambda^2(M) | *(\alpha \wedge \omega) = -2\alpha\},\\
\Lambda^2_{14} &= \{\alpha \in \Lambda^2(M) | *(\alpha \wedge \omega) = \alpha\},\\
\Lambda^3_1 &= \{t.\omega |\ t \in R\},\\
\Lambda^3_7 &= \{*(\beta \wedge \omega) |\ \beta \in \Lambda^1(M)\},\\
\Lambda^3_{27} &= \{\gamma \in \Lambda^3(M) |\ \gamma \wedge \omega = 0,\ \gamma \wedge *\omega = 0\}.
\end{aligned}$$

The Hodge star $*$ gives an isometry between Λ^k_l and Λ^{7-k}_l.

More generally, $V^d_{(\lambda_1,\lambda_2)}$ will denote the G_2 representation of highest weight (λ_1, λ_2) of dimension d. Note that $V^1_{(0,0)} \cong \Lambda^3_1 \cong \Lambda^4_1$ is the trivial representation, $\Lambda^1_7 \cong V^7_{(1,0)}$ is the standard representation of G_2 on $\mathbb{R}^7$, and the adjoint representation is $\mathfrak{g}_2 \cong V^{14}_{(0,1)} \cong \Lambda^2_{14}$. Also note that $V^{27}_{(2,0)} \cong \Lambda^3_{27} \cong \Lambda^4_{27}$ is isomorphic to the space of traceless symmetric tensors $S^2_0 V^7$ on $V^7_{(1,0)}$.

3. Ricci Tensors on G_2-Manifold

Let (M, ω) be a G_2-manifold with fundamental form ω. Let g be the associated Riemannian metric;

$$R_{X,Y} = [\nabla^g{}_X, \nabla^g{}_Y] - \nabla^g{}_{[X,Y]}$$

is then the curvature tensor of the Levi-Civita connection ∇^g of the metric g. The Ricci tensor ρ is defined as usual as the contraction $\rho_{ij} = R_{sijs}$, where R_{sijs} are the components

$$R_{sijk} := g(R(e_s, e_i)e_j, e_k)$$

of the curvature tensor with respect to an orthonormal basis $e_1, \dots, e_7$.

Definition 3.1. *On (M, ω) we may define a second symmetric tensor $\rho^\star$ by*

$$\rho^\star_{sm} := R_{ijkl}\omega_{ijs}\omega_{klm}. \tag{3.4}$$

We will call the $\rho^\star$ the **$\star$-Ricci tensor** *of the G_2-manifold.*

The two Ricci tensors have common trace in the following sense. Let $s = \mathrm{tr}_g\, \rho = \rho_{ii}$ be the scalar curvature and let the trace of $\rho^\star$ be denoted by $s^\star = \mathrm{tr}_g\, \rho^\star = \rho^\star_{ii}$.

Proposition 3.2. *On a G_2-manifold we have $s^\star = -2s$.*

Proof. Apply (2.3) to the definition of $s^\star$ and use skew-symmetry of $*\omega$ and the Bianchi identity to conclude that $R_{ijkl}\omega_{ijkl} = 0$. □

Definition 3.3. *We shall use the term* **⋆-Einstein** *for G_2-manifold (M,ω) when the traceless part of the ⋆-Ricci tensor vanishes, i.e., when the equation*

$$\rho^\star = \frac{s^\star}{7} g$$

holds.

We define associated Ricci three-forms by

$$\rho^\star_{ijk} := R_{ijlm}\omega_{lmk} + R_{jklm}\omega_{lmi} + R_{kilm}\omega_{lmj},$$
$$\rho_{ijk} := \rho_{is}\omega_{sjk} + \rho_{js}\omega_{ski} + \rho_{ks}\omega_{sij}.$$

In terms of the Ricci forms, the Weitzenböck formula for the fundamental form can be written as follows:

Proposition 3.4. *On any G_2-manifold the following formula holds:*

$$d\delta\omega + \delta d\omega = \nabla^{g*}\nabla^g\omega + \rho + \rho^\star. \tag{3.5}$$

□

4. Closed G_2-Structures

Let (M^7,ω) be a G_2-manifold with closed fundamental form. The two-form $\delta\omega$ then takes values in Λ^2_{14} [3]. As a consequence we get

Proposition 4.1. *The following formulas are valid on a closed G_2-structure:*

$$\delta\omega_{ij}\omega_{ijk} = 0, \qquad \delta\omega_{ip}\omega_{pjk} + \delta\omega_{jp}\omega_{pki} + \delta\omega_{kp}\omega_{pij} = 0. \tag{4.6}$$

□

It is well-known [17] that a G_2-structure is parallel if and only if it is closed and co-closed, $d\omega = \delta\omega = 0$. The two-form $\delta\omega$ thus may be interpreted as the deviation of ω from a parallel G_2-structure. We are going to find explicit formulae for the covariant derivatives of the fundamental form of a closed G_2-structure in terms of $\delta\omega$ and its derivatives.

Definition 4.2. *The* **canonical connection** $\tilde\nabla$ *of a closed G_2-structure may be defined by the equation*

$$g(\tilde\nabla_X Y, Z) = g(\nabla^g{}_X Y, Z) - \frac{1}{6}\delta\omega(X, e_i)\omega(e_i, Y, Z) \tag{4.7}$$

for vector fields X, Y, Z.

Using (4.6) it is easy to see that $\tilde\nabla$ is a metric G_2-connection, i.e., it satisfies

$$\tilde\nabla\omega = 0, \qquad \tilde\nabla g = 0.$$

The torsion T of $\tilde\nabla$ is determined by

$$g(T(X,Y), Z) = \frac{1}{6}\delta\omega(Z, e_i)\omega(e_i, X, Y).$$

On a compact G_2-manifold the canonical connection may be characterized as the unique G_2-connection of minimal torsion with respect to the L^2-norm on M. It may also be described by the fact that the difference $\nabla^g - \tilde\nabla$ takes values in Λ^2_7, the orthogonal complement of $\mathfrak{g}_2 \subset \Lambda^2$ with respect to the metric induced by g.

From the properties of the canonical connection and $\delta\omega$ one derives

Proposition 4.3. *For a closed G_2-structure the following relations hold:*

$$\nabla^g{}_i\omega_{jkl} = \frac{1}{2}\delta\omega_{ip}\omega_{pjkl}, \tag{4.8}$$

$$\nabla^g{}_i\omega_{jklm} = -\frac{1}{2}\left(\delta\omega_{ij}\omega_{klm} - \delta\omega_{ik}\omega_{lmj} + \delta\omega_{il}\omega_{mjk} - \delta\omega_{im}\omega_{jkl}\right) \tag{4.9}$$

and

$$\nabla^{g*}\nabla^g\omega_{jkl} = \frac{1}{4}\|\delta\omega\|^2\omega_{jkl} - \frac{1}{4}\left(\delta\omega_{ip}\delta\omega_{ij}\omega_{pkl} + \delta\omega_{ip}\delta\omega_{ik}\omega_{plj} + \delta\omega_{ip}\delta\omega_{il}\omega_{pjk}\right). \tag{4.10}$$

□

Applying (4.7) and (4.8) we get that the curvature $\tilde{R}$ of the canonical connection $\tilde{\nabla}$ is related to the curvature of the Levi-Civita connection by:

$$R_{ijkl} = \tilde{R}_{ijkl} + \frac{1}{6}\left[\nabla^g{}_i\delta\omega_{jp} - \nabla^g{}_j\delta\omega_{ip}\right]\omega_{pkl} + \frac{1}{9}\delta\omega_{is}\delta\omega_{jp}\omega_{spkl} - \frac{1}{36}\left[\delta\omega_{ik}\delta\omega_{jl} - \delta\omega_{il}\delta\omega_{jk}\right]. \tag{4.11}$$

5. Curvature of Closed G_2-Structures

From here on (M^7, ω) will be a G_2-manifold with closed G_2-structure. We have

Proposition 5.1. *The Ricci tensors of a closed G_2-structure (M, ω) are given by*

$$\rho_{lm} = -\frac{1}{4}d\delta\omega_{sjm}\omega_{sjl} + \frac{1}{2}\delta\omega_{lj}\delta\omega_{mj}; \tag{5.12}$$

$$\rho^\star_{lm} = d\delta\omega_{sjm}\omega_{sjl} + \delta\omega_{lj}\delta\omega_{mj} - \frac{1}{2}\|\delta\omega\|^2\delta_{ml}. \tag{5.13}$$

Proof. The Ricci identities for ω, $*\omega$ together with (4.8) and (4.9) lead to the following useful

Lemma 5.2. *If ω is a closed G_2-structure on M^7 then*

$$\rho_{sr}\omega_{rkl} + \frac{1}{2}R_{skir}\omega_{lir} - \frac{1}{2}R_{slir}\omega_{kir} = -\frac{1}{4}\left(d\delta\omega_{sjp} + \nabla^g{}_s\delta\omega_{jp}\right)\omega_{pjkl} + \frac{1}{2}\delta\omega_{pj}\delta\omega_{sj}\omega_{klp}, \tag{5.14}$$

$$\begin{aligned}
&-R_{sijr}\omega_{rklm} - R_{sikr}\omega_{jrlm} - R_{silr}\omega_{jkrm} - R_{simr}\omega_{jklr}\\
&= \frac{1}{2}\left[\left(\nabla^g{}_i\delta\omega_{sj} - \nabla^g{}_s\delta\omega_{ij}\right)\omega_{klm} - \left(\nabla^g{}_i\delta\omega_{sk} - \nabla^g{}_s\delta\omega_{ik}\right)\omega_{lmj}\right]\\
&+ \frac{1}{2}\left[\left(\nabla^g{}_i\delta\omega_{sl} - \nabla^g{}_s\delta\omega_{il}\right)\omega_{mjk} - \left(\nabla^g{}_i\delta\omega_{sm} - \nabla^g{}_s\delta\omega_{im}\right)\omega_{jkl}\right]\\
&- \frac{1}{4}\left[\left(\delta\omega_{ij}\delta\omega_{sp} - \delta\omega_{sj}\delta\omega_{ip}\right)\omega_{pklm} - \left(\delta\omega_{ik}\delta\omega_{sp} - \delta\omega_{sk}\delta\omega_{ip}\right)\omega_{plmj}\right]\\
&- \frac{1}{4}\left[\left(\delta\omega_{il}\delta\omega_{sp} - \delta\omega_{sl}\delta\omega_{ip}\right)\omega_{pmjk} - \left(\delta\omega_{im}\delta\omega_{sp} - \delta\omega_{sm}\delta\omega_{ip}\right)\omega_{pjkl}\right].
\end{aligned} \tag{5.15}$$

□

Using (4.7) we get

$$\nabla^g{}_k \delta\omega_{is}\omega_{ism} = \tilde{\nabla}_k \delta\omega_{is}\omega_{ism} + \frac{1}{6}\delta\omega_{kr}\delta\omega_{rq}\omega_{siq}\omega_{ism} = \delta\omega_{kr}\delta\omega_{mr}, \tag{5.16}$$

since $\tilde{\nabla}\delta\omega \in \Lambda^2_{14}$. If we multiply (5.14) by ω_{mkl} and use the Bianchi identity as well as (5.16) we obtain (5.12).

Multiplying (5.15) by ω_{mlj}, and again using the Bianchi identity (alternatively: multiply (4.11) by ω_{klm}), we get

$$R_{silr}\omega_{klr} = \left(\nabla^g{}_s \delta\omega_{ik} - \nabla^g{}_i \delta\omega_{sk}\right) + \frac{1}{4}\left(\delta\omega_{ij}\delta\omega_{sp} - \delta\omega_{sj}\delta\omega_{ip}\right)\omega_{jpk}. \tag{5.17}$$

From (5.17) we get that

$$\begin{aligned} \rho^\star_{km} &= R_{silr}\omega_{klr}\omega_{sim} \\ &= d\delta\omega_{sik}\omega_{sim} - \nabla^g{}_k \delta\omega_{is}\omega_{sim} + \frac{1}{2}\delta\omega_{ij}\delta\omega_{sp}\omega_{jpk}\omega_{sim}. \end{aligned} \tag{5.18}$$

The second term is calculated in (5.16). The last term is manipulated using (4.6) and (2.3):

$$\begin{aligned} \delta\omega_{ij}\delta\omega_{sp}\omega_{jpk}\omega_{sim} &= \left(-\delta\omega_{pj}\omega_{jki} - \delta\omega_{kj}\omega_{jip}\right)\delta\omega_{sp}\omega_{sim} \\ &= \delta\omega_{sp}\left(\delta\omega_{jp}\omega_{kij}\omega_{sim} + \delta\omega_{jk}\omega_{jip}\omega_{sim}\right) \\ &= \delta\omega_{sp}\delta\omega_{jp}\left(-\omega_{kjsm} + \delta_{ks}\delta_{jm} - \delta_{km}\delta_{js}\right) \\ &\quad + \delta\omega_{sp}\delta\omega_{jk}\left(-\omega_{jpsm} + \delta_{js}\delta_{pm} - \delta_{jm}\delta_{ps}\right) \\ &= -\left\|\delta\omega\right\|^2\delta_{km} + 4\delta\omega_{jm}\delta\omega_{jk}, \end{aligned} \tag{5.19}$$

again, since $\delta\omega \in \Lambda^2_{14}$. Substituting (5.19) and (5.16) into (5.18) we obtain (5.13). □

The equality (5.16) leads to

$$d\delta\omega_{sjm}\omega_{sjm} = 3\left\|\delta\omega\right\|^2. \tag{5.20}$$

Taking the trace in (5.12) and using (5.20), we get the formula for the scalar curvature of a closed G_2-structure discovered recently by Bryant in [3].

Corollary 5.3. *The scalar curvature of a closed G_2-structure is non-positive while the $\star$-scalar curvature is non-negative. These functions are given by*

$$s = -\frac{1}{4}\left\|\delta\omega\right\|^2, \qquad s^\star = \frac{1}{2}\left\|\delta\omega\right\|^2. \tag{5.21}$$

In view of (5.21), the trace-free part of the Ricci tensor ρ^0 has the expression

$$\rho^0 = \rho + \frac{1}{28}\left\|\delta\omega\right\|^2 g. \tag{5.22}$$

Definition 5.4. *The canonical connection gives us a third Ricci tensor which we denote by $\tilde{\rho}$:*

$$\tilde{\rho}_{ij} = \tilde{R}_{sijs}.$$

Corollary 5.5. *On a 7-manifold with closed G_2-structure the Ricci tensor of the canonical connection is related to the Riemannian Ricci tensor through the following formula:*

$$\tilde{\rho} = \frac{2}{3}\rho.$$

Proof. Taking the trace of (4.11) we get

$$\rho_{il} = \tilde{\rho}_{il} - \frac{1}{12} d\delta\omega_{pji}\omega_{pjl} + \frac{1}{6}\delta\omega_{is}\delta\omega_{ls}.$$

This equality and (5.12) completes the proof. □

Furthermore, we have

Proposition 5.6. *The Ricci three-forms of a closed G_2-structure are given by*

$$\begin{aligned}\rho_{jlk} = &-\frac{1}{4}\|\delta\omega\|^2\omega_{jlk} - d\delta\omega_{jlk} - \frac{1}{2}\big(\delta\omega_{li}\delta\omega_{pi}\omega_{pjk} + \delta\omega_{ki}\delta\omega_{pi}\omega_{plj} \\ &+ \delta\omega_{ji}\delta\omega_{pi}\omega_{pkl}\big),\end{aligned} \tag{5.23}$$

$$\rho^{\star}_{sik} = 2d\delta\omega_{sik} + \frac{1}{4}\big(\delta\omega_{ij}\delta\omega_{pj}\omega_{psk} + \delta\omega_{kj}\delta\omega_{pj}\omega_{pis} + \delta\omega_{sj}\delta\omega_{pj}\omega_{pki}\big). \tag{5.24}$$

Proof. Substitute (4.10) and (5.24) into the Weitzenböck formula (3.5) to get (5.23). The cyclic sum in the equality (5.17) gives (5.24). □

Theorem 5.7. *Let (M^7, ω) be a G_2-manifold with closed fundamental form. If (M, ω) is Einstein and $\star$-Einstein then M is parallel.*

Proof. Let (M, ω) be a G_2-manifold with closed G_2-structure ω. Suppose that both the Einstein and $\star$-Einstein equations are satisfied. Proposition 5.1 in this case yields

$$\delta\omega_{ij}\delta\omega_{pj} = \frac{1}{7}\|\delta\omega\|^2\delta_{ip}, \qquad d\delta\omega_{ijk}\omega_{ijm} = \frac{3}{7}\|\delta\omega\|^2\delta_{km}. \tag{5.25}$$

Taking into account (5.25) and the equalities (4.10), (5.23), we obtain

$$\begin{aligned}\rho^{\star}_{ijk} &= 2d\delta\omega_{ijk} - \frac{3}{28}\|\delta\omega\|^2\omega_{ijk}, \\ \nabla^{g*}\nabla^{g}\omega_{ijk} &= \frac{1}{7}\|\delta\omega\|^2\omega_{ijk}, \\ \rho_{ijk} &= -\frac{3}{28}\|\delta\omega\|^2\omega_{ijk}.\end{aligned} \tag{5.26}$$

In view of (5.26), the Weitzenböck formula (3.5) gives

$$d\delta\omega = \frac{1}{14}\|\delta\omega\|^2\omega. \tag{5.27}$$

Bryant shows in [3] that on an Einstein manifold with closed G_2-structure the Λ^3_{27}-part of $d\delta\omega$ is given by the Λ^3_{27}-part of $*(\delta\omega\wedge\delta\omega)$. Comparing with (5.27), we see that the Λ^3_{27}-part of $*(\delta\omega\wedge\delta\omega)$ vanishes. We need the following algebraic

Lemma 5.8. *Let α be a two-form in Λ^2_{14}. Then the Λ^3_{27}-part of $*(\alpha\wedge\alpha)$ vanishes if and only if $\alpha = 0$.*

Proof. First note that if α is a two-form in Λ^2_{14} then $\alpha \otimes \alpha \in S^2\Lambda^2_{14} \subset S^2(\Lambda^2)$. The space of symmetric tensors on Λ^2_{14} decomposes as follows:

$$S^2\Lambda^2_{14} = V^{77}_{(0,2)} + V^{27}_{(0,2)} + V^{1}_{(0,0)}.$$

Recall that the map $S^2\Lambda^2 \to \Lambda^4$ given by $\beta \vee \gamma \mapsto \beta \wedge \gamma$ is surjective and equivariant. By Schur's Lemma we may conclude that $\alpha \wedge \alpha \in \Lambda^4_{27} \oplus \Lambda^4_1$ if $\alpha \in \Lambda^2_{14}$.

Now suppose $(\alpha \wedge \alpha)_{\Lambda^4_{27}} = 0$. We may then conclude that $\alpha \wedge \alpha = c{*}\omega$ for some constant c. However, as α is a two-form on an odd-dimensional space it is degenerate. Let $X \in \mathbb{R}^7$ be a non-zero vector such that $X \lrcorner \alpha = 0$. Then

$$cX \lrcorner *\omega = X \lrcorner (\alpha \wedge \alpha) = 2(X \lrcorner \alpha) \wedge \alpha = 0.$$

But the left-hand side vanishes only if $c = 0$.

Now, Lemma 5.8 implies $\delta\omega = 0$, whence $\nabla^g\omega = 0$. □

Remark 5.9. Bryant observe [4] that the following identity holds.

$$\|\alpha \wedge \alpha\|^2 = 6\,\|\alpha\|^4, \quad \alpha \in \Lambda^2_{14}. \tag{5.28}$$

Clearly (5.28) implies the lemma. Note that the constants have been changed to fit our conventions.

6. An Integral Formula on Closed G_2-Manifold

Our main technical tool to handle the closed G_2-structure on a compact manifold is the next

Proposition 6.1. *Let (M, ω, g) be a compact G_2-manifold with closed fundamental form. Then the following integral formula holds:*

$$\int_M \Big(\frac{1}{24}\,\|\delta\omega\|^4 + \frac{28}{9}||\rho^0||^2 - \frac{7}{18}\rho^0_{pl}\delta\omega_{bl}\delta\omega_{bp} - \frac{7}{9}(\delta R)_{bkl}\omega_{jkl}\delta\omega_{bj}\Big)\, dV = 0. \tag{6.29}$$

Proof. The first Pontrjagin form $p_1(\nabla)$ of a connection ∇ may be defined by

$$p_1(\nabla) := \frac{1}{16\pi^2} R_{ijab}R_{klab}e_i \wedge e_j \wedge e_k \wedge e_l.$$

The first Pontrjagin class of TM which is the de Rham cohomology class whose representative element is the first Pontrjagin form of a some affine connection on M is independent of the connection on M. This implies that $\big(p_1(\nabla^g) - p_1(\tilde{\nabla})\big)$ is an exact 4-form. Since the fundamental form ω is closed, the wedge product $\big(p_1(\nabla^g) - p_1(\tilde{\nabla})\big)\wedge\omega$ is exact. From Stokes' theorem we obtain

$$\int_M \big(p_1(\nabla^g) - p_1(\tilde{\nabla})\big) \wedge \omega = 0. \tag{6.30}$$

However, we may also express the integrand in terms of the curvatures of ∇^g and $\tilde{\nabla}$ using that

$$16\pi^2 p_1(\nabla^g) \wedge \omega = R_{ijab}R_{klab}\omega_{ijkl} = R_{abij}R_{klab}\omega_{ijkl},$$

and

$$16\pi^2 p_1(\tilde{\nabla}) \wedge \omega = \tilde{R}_{ijab}\tilde{R}_{klab}\omega_{ijkl}.$$

From this point on the proof is essentially a brute force calculation reducing the difference of these two expressions to the form (6.29).

Since $\tilde{\nabla}$ is a G_2-connection we get

$$\tilde{R}_{abij}\omega_{ijkl} = 2\tilde{R}_{abkl}. \tag{6.31}$$

Using (4.11), (2.3) and (6.31) we calculate

$$16\pi^2 p_1(\nabla^g) \wedge \omega = 2R_{klab}\tilde{R}_{abkl} - \frac{2}{3}\big(d\delta\omega_{abp} - \nabla^g{}_p\delta\omega_{ab}\big)\omega_{pkl}R_{klab} - \frac{5}{18}\delta\omega_{ai}\delta\omega_{bj}\omega_{ijkl}R_{klab} + \frac{8}{9}\delta\omega_{ak}\delta\omega_{bl}R_{klab}. \tag{6.32}$$

Applying (4.11) to (6.32) and using (2.3) we obtain after some calculations that

$$\begin{aligned}16\pi^2 p_1(\nabla^g) \wedge \omega &= 2\tilde{R}_{klab}\tilde{R}_{abkl} + \big(d\delta\omega_{abp} - \nabla^g{}_p\delta\omega_{ab}\big)\omega_{pkl}\Big(-\frac{2}{3}R_{klab} + \frac{1}{3}\tilde{R}_{klab}\Big)\\ &+\delta\omega_{ai}\delta\omega_{bj}\omega_{ijkl}\Big(-\frac{5}{18}R_{klab} + \frac{2}{9}\tilde{R}_{klab}\Big) + \delta\omega_{ak}\delta\omega_{bl}\Big(\frac{8}{9}R_{klab} - \frac{1}{9}\tilde{R}_{klab}\Big).\end{aligned} \tag{6.33}$$

To calculate the $p_1(\tilde{\nabla})$ term of the integrand we first observe that (4.11) implies

$$\begin{aligned}\tilde{R}_{ijkl} &= \tilde{R}_{klij} + \frac{1}{6}\big(d\delta\omega_{klp} - \nabla^g{}_p\delta\omega_{kl}\big)\omega_{pij}\\ &-\frac{1}{6}\big(d\delta\omega_{ijp} - \nabla^g{}_p\delta\omega_{ij}\big)\omega_{pkl} + \frac{1}{9}\delta\omega_{ks}\delta\omega_{lp}\omega_{spij} - \frac{1}{9}\delta\omega_{is}\delta\omega_{jp}\omega_{spkl}.\end{aligned} \tag{6.34}$$

Taking into account (6.31), (6.34) and (2.3), we calculate

$$16\pi^2 p_1(\tilde{\nabla}) \wedge \omega = 2\tilde{R}_{klab}\tilde{R}_{abkl} - \frac{2}{3}\big(d\delta\omega_{abp} - \nabla^g{}_p\delta\omega_{ab}\big)\omega_{pkl}\tilde{R}_{klab} - \frac{4}{9}\delta\omega_{ai}\delta\omega_{bj}\omega_{ijkl}\tilde{R}_{klab} + \frac{8}{9}\delta\omega_{ak}\delta\omega_{bl}\tilde{R}_{klab}. \tag{6.35}$$

Subtracting (6.35) from (6.33) we get

$$16\pi^2\big(p_1(\nabla^g) - p_1(\tilde{\nabla})\big) \wedge \omega = A + B + C, \tag{6.36}$$

where A, B, C are given by

$$A = \big(d\delta\omega_{abp} - \nabla^g{}_p\delta\omega_{ab}\big)\omega_{pkl}\Big(\frac{1}{3}R_{klab} - \big(R_{klab} - \tilde{R}_{klab}\big)\Big), \tag{6.37}$$

$$B = \delta\omega_{ai}\delta\omega_{bj}\omega_{ijkl}\Big(\frac{7}{18}R_{klab} - \frac{2}{3}\big(R_{klab} - \tilde{R}_{klab}\big)\Big), \tag{6.38}$$

$$C = \delta\omega_{ak}\delta\omega_{bl}\Big(-\frac{1}{9}R_{klab} + \big(R_{klab} - \tilde{R}_{klab}\big)\Big). \tag{6.39}$$

We shall calculate each term in (6.36).

First observe that (5.28) implies the useful identity

$$4\delta\omega_{ai}\delta\omega_{bi}\delta\omega_{aj}\delta\omega_{bj} = ||\delta\omega||^4. \tag{6.40}$$

Taking into account (4.11), (5.17), (5.12), (5.16), (5.19), (5.20) and (6.40) we obtain after some calculation that

$$\begin{aligned} A = &-\frac{19}{24.7}\|\delta\omega\|^4 + \frac{1}{3}\left\|d\delta\omega_{abp} - \nabla^g{}_p\delta\omega_{ab}\right\|^2 - \frac{8}{3}||\rho^0||^2 \\ &+\frac{4}{3}\rho^0_{sp}\delta\omega_{st}\delta\omega_{pt} - \frac{1}{9}\left(d\delta\omega_{abp} - \nabla^g{}_p\delta\omega_{ab}\right)\omega_{ijp}\delta\omega_{ai}\delta\omega_{bj} \\ &-\frac{1}{9}\left(d\delta\omega_{abp} - \nabla^g{}_p\delta\omega_{ab}\right)\omega_{pkl}\omega_{smab}\delta\omega_{ks}\delta\omega_{lm}. \end{aligned} \tag{6.41}$$

Let $X_s = R_{akbl}\omega_{sak}\delta\omega_{bl}, \quad Y_s = \delta\omega_{bl}\delta\omega_{li}\delta\omega_{bp}\omega_{ips}$. Using the first and second Bianchi identity as well as (5.17), we get

$$\begin{aligned} R_{klab}\delta\omega_{ak}\delta\omega_{bl} &= \frac{1}{2}R_{akbl}\delta\omega_{ak}\delta\omega_{bl} \\ &= -\frac{1}{2}R_{akbl}\nabla^g{}_s\omega_{sak}\delta\omega_{bl} \\ &= \frac{1}{2}\delta X + \frac{1}{2}R_{akbl}\omega_{sak}\nabla^g{}_s\delta\omega_{bl} \\ &= \frac{1}{2}\delta X + \frac{1}{6}\|d\delta\omega\|^2 - \frac{1}{2}\left\|\nabla^g\delta\omega\right\|^2 + \frac{1}{4}\nabla^g{}_s\delta\omega_{bl}\delta\omega_{li}\delta\omega_{bp}\omega_{ips} \\ &= \frac{1}{2}\delta X - \frac{1}{4}\delta Y + \frac{1}{6}\|d\delta\omega\|^2 - \frac{1}{2}\left\|\nabla^g\delta\omega\right\|^2 \\ &\quad -\frac{1}{8}||\delta\omega||^4 + \frac{1}{4}d\delta\omega_{sil}\delta\omega_{bl}\delta\omega_{bp}\omega_{ips}. \end{aligned} \tag{6.42}$$

Applying (5.12) to (6.42), we obtain

$$\begin{aligned} R_{klab}\delta\omega_{ak}\delta\omega_{bl} = &\frac{1}{2}\delta X - \frac{1}{4}\delta Y + \frac{1}{6}\|d\delta\omega\|^2 \\ &-\frac{1}{2}\left\|\nabla^g\delta\omega\right\|^2 + \frac{1}{28}\|\delta\omega\|^4 - \rho^0_{sp}\delta\omega_{st}\delta\omega_{pt}. \end{aligned} \tag{6.43}$$

Remark 6.2. Notice that (6.43) is the Weitzenböck formula

$$\delta d\delta\omega = \nabla^{g*}\nabla^g\delta\omega - \frac{1}{14}\|\delta\omega\|^2\delta\omega + \rho^0(\delta\omega, .) + \rho^0(., \delta\omega) + R(\delta\omega)$$

for the 2-form $\delta\omega$ on a closed G_2-structure.

Using (4.11), (5.28) and (6.43) we get that

$$\begin{aligned} C = &-\frac{1}{36}\delta(2X - Y) - \frac{25}{63.16}\|\delta\omega\|^4 - \frac{1}{54}\|d\delta\omega\|^2 + \frac{1}{18}\left\|\nabla^g\delta\omega\right\|^2 \\ &+\frac{1}{9}\rho^0_{sp}\delta\omega_{st}\delta\omega_{pt} + \frac{1}{6}\left(d\delta\omega_{kls} - \nabla^g{}_s\delta\omega_{kl}\right)\omega_{sab}\delta\omega_{ak}\delta\omega_{bl}. \end{aligned} \tag{6.44}$$

Now we calculate B. Denote $Z_a = R_{klab}\omega_{jkl}\delta\omega_{bj}$. Using (4.8) and (5.17) we have the following sequence of equalities:

$$\begin{aligned}\frac{1}{2}R_{klab}\delta\omega_{ai}\omega_{ijkl}\delta\omega_{bj} &= R_{klab}\nabla^g{}_a\omega_{jkl}\delta\omega_{bj}\\ &= -\delta Z + (\delta R)_{bkl}\omega_{jkl}\delta\omega_{bj} - R_{klab}\omega_{jkl}\nabla^g{}_a\delta\omega_{bj}\\ &= -\delta Z + (\delta R)_{bkl}\omega_{jkl}\delta\omega_{bj} - \frac{1}{6}\|d\delta\omega\|^2 - \frac{1}{2}\left\|\nabla^g\delta\omega\right\|^2\\ &\quad + \frac{1}{4}\left(d\delta\omega_{abj} - \nabla^g{}_j\delta\omega_{ab}\right)\omega_{slj}\delta\omega_{as}\delta\omega_{bl}. \qquad (6.45)\end{aligned}$$

Applying (4.11), we get

$$\begin{aligned}&-\frac{2}{3}\left(R_{klab} - \tilde{R}_{klab}\right)\delta\omega_{ai}\delta\omega_{bj}\omega_{ijkl}\\ &\quad= -\frac{1}{9}\left(d\delta\omega_{kls} - \nabla^g{}_s\delta\omega_{kl}\right)\omega_{sab}\omega_{ijkl}\delta\omega_{ai}\delta\omega_{bj}\\ &\quad -\frac{2}{27}\delta\omega_{ks}\delta\omega_{lm}\delta\omega_{ai}\delta\omega_{bj}\omega_{smab}\omega_{ijkl}. \qquad (6.46)\end{aligned}$$

Denote $V_k = \omega_{mab}\delta\omega_{lm}\delta\omega_{bj}\delta\omega_{ai}\omega_{ijkl}$. Using (4.8), we obtain

$$\begin{aligned}\frac{1}{2}\delta\omega_{ks}\omega_{smab}\delta\omega_{lm}\delta\omega_{ai}\delta\omega_{bj}\omega_{ijkl} &= \nabla^g{}_k\omega_{mab}\delta\omega_{lm}\delta\omega_{ai}\delta\omega_{bj}\omega_{ijkl}\\ &= -\delta V - \frac{3}{2}\left(d\delta\omega_{klm} - \nabla^g{}_m\delta\omega_{kl}\right)\omega_{mab}\omega_{ijkl}\delta\omega_{ai}\delta\omega_{bj}.\end{aligned} \qquad (6.47)$$

Getting together (6.45), (6.46) and (6.47) we obtain

$$\begin{aligned}B &= -\frac{1}{27}\delta(21Z - 4V) + \frac{7}{9}(\delta R)_{bkl}\omega_{jkl}\delta\omega_{bj} - \frac{7}{54}\|d\delta\omega\|^2 - \frac{7}{18}\left\|\nabla^g\delta\omega\right\|^2\\ &\quad + \frac{1}{9}\left(d\delta\omega_{klm} - \nabla^g{}_m\delta\omega_{kl}\right)\omega_{mab}\omega_{ijkl}\delta\omega_{ai}\delta\omega_{bj}\\ &\quad + \frac{7}{36}\left(d\delta\omega_{abj} - \nabla^g{}_j\delta\omega_{ab}\right)\omega_{slj}\delta\omega_{as}\delta\omega_{bl}. \qquad (6.48)\end{aligned}$$

Collecting terms from (6.41), (6.44) and (6.48) we obtain the following expression:

$$\begin{aligned}16\pi^2\left(p_1(\nabla^g) - p_1(\tilde{\nabla})\right)\wedge\omega &= \delta\left(-\frac{1}{18}X + \frac{1}{36}Y - \frac{7}{9}Z + \frac{4}{27}V\right)\\ &\quad - \frac{139}{18.56}\|\delta\omega\|^4 - \frac{1}{27}\|d\delta\omega\|^2 - \frac{8}{3}||\rho^0||^2\\ &\quad + \frac{7}{9}(\delta R)_{bkl}\omega_{jkl}\delta\omega_{bj} + \frac{1}{4}\big(d\delta\omega_{kls}\\ &\quad - \nabla^g{}_s\delta\omega_{kl}\big)\omega_{sab}\delta\omega_{ak}\delta\omega_{bl} + \frac{13}{9}\rho^0_{pl}\delta\omega_{bl}\delta\omega_{bp}. \qquad (6.49)\end{aligned}$$

We express the last two terms in a more tractable form. Applying (4.6) we have

$$\begin{aligned}d\delta\omega_{abp}\delta\omega_{ai}\delta\omega_{bj}\omega_{pij} &= -d\delta\omega_{abp}\delta\omega_{ai}\left(\delta\omega_{pj}\omega_{ibj} + \delta\omega_{ij}\omega_{bpj}\right)\\ &= -d\delta\omega_{apb}\delta\omega_{ai}\delta\omega_{bj}\omega_{ipj} - d\delta\omega_{abp}\delta\omega_{ai}\delta\omega_{ij}\omega_{bpj},\end{aligned}$$

whence

$$d\delta\omega_{abp}\delta\omega_{ai}\delta\omega_{bj}\omega_{pij} = \frac{1}{2}d\delta\omega_{abp}\omega_{bpj}\delta\omega_{ai}\delta\omega_{ji} = -2\rho_{aj}\delta\omega_{ai}\delta\omega_{ji} + \frac{1}{4}||\delta\omega||^4, \tag{6.50}$$

where we used (6.40) and (5.12).

The next equalities are a consequence of (5.12), (5.28) and (5.16),

$$\begin{aligned}\nabla^g{}_s\delta\omega_{kl}\omega_{sab}\delta\omega_{ak}\delta\omega_{bl} &= \delta Y + \frac{1}{4}||\delta\omega||^4 - \left(d\delta\omega_{sak} - \nabla^g{}_a\delta\omega_{ks}\right)\omega_{sab}\delta\omega_{kl}\delta\omega_{bl}\\ &= \delta Y + 4\rho_{kb}\delta\omega_{kl}\delta\omega_{bl}.\end{aligned} \tag{6.51}$$

Substitute (6.50), (6.51) into (6.49), use (5.12) and integrate over the compact M to get

$$\begin{aligned}\int_M \Big(-\frac{11}{18.28}||\delta\omega||^4 - \frac{1}{27}||d\delta\omega||^2 - \frac{8}{3}||\rho^0||^2 - \frac{1}{18}\rho^0_{pl}\delta\omega_{bl}\delta\omega_{bp}\\ + \frac{7}{9}(\delta R)_{bkl}\omega_{jkl}\delta\omega_{bj}\Big)\, dV = 0.\end{aligned} \tag{6.52}$$

We calculate from (5.23) that

$$||d\delta\omega||^2 = \frac{15}{28}||\delta\omega||^4 + 12||\rho^0||^2 - 12\rho^0_{pl}\delta\omega_{bl}\delta\omega_{bp}. \tag{6.53}$$

Substitute (6.53) into (6) to obtain (6.29). □

Corollary 6.3. (**Integral Weitzenböck formula.**) *On a compact G_2-manifold with closed fundamental form we have*

$$\int_M R_{akbl}\delta\omega_{ak}\delta\omega_{bl}\, dV = \int_M \left(\frac{1}{3}\,\|d\delta\omega\|^2 - \left\|\nabla^g\delta\omega\right\|^2 - 2\rho_{sp}\delta\omega_{st}\delta\omega_{pt}\right) dV.$$

Proof. The first Bianchi identity implies $R_{akbl}\delta\omega_{ak}\delta\omega_{bl} = 2R_{klab}\delta\omega_{ak}\delta\omega_{bl}$. Apply (5.22) to (6.43), multiply by two and integrate the obtained equality over the compact manifold to get the result. □

7. Proof of the Main Theorem

We consider the co-differential of the Weyl tensor, δW as an element of $T^*M \otimes \Lambda^2(T^*M)$. According to the splitting of the space of two forms, δW splits as follows:

$$\delta W = \delta W^2_{14} \oplus \delta W^2_7,$$

where δW^2_{14} is a section of $T^*M \otimes \Lambda^2_{14}(T^*M) \cong \mathbb{R}^7 \otimes \mathfrak{g}_2$ while δW^2_7 is a section of $T^*M \otimes \Lambda^2_7(T^*M)$.

The Main Theorem is a consequence of the following general

Theorem 7.1. *Let (M, ω, g) be a compact G_2-manifold with closed fundamental form ω. Suppose that the Λ^2_7-part of the co-differential of the Weyl tensor vanishes, $\delta W = \delta W^2_{14}$. Then (M, ω, g) is a Joyce space.*

Proof. On a 7-dimensional Riemannian manifold the Weyl tensor is expressed in terms of the normalized Ricci tensor $h = -\frac{1}{5}\left(\rho - \frac{s}{12}g\right)$ as follows:

$$W_{ijkl} = R^g_{ijkl} - h_{ik}g_{jl} + h_{jk}g_{il} - h_{jl}g_{ik} + h_{il}g_{jk}.$$

The second Bianchi identity implies

$$\nabla^g{}_i\rho_{jk} - \nabla^g{}_j\rho_{ik} = (\delta R)_{kij}, \quad ds_i = 2\nabla^g{}_j\rho_{ji}, \quad (\delta W)_{kij} = -4\Big(\nabla^g{}_i h_{jk} - \nabla^g{}_j h_{ik}\Big). \tag{7.54}$$

The condition of the theorem reads

$$0 = (\delta W)_{kij}\omega_{ijl} = \frac{4}{5}\Big(\nabla^g{}_i\rho_{jk} - \nabla^g{}_j\rho_{ik}\Big)\omega_{ijl} - \frac{2}{15}ds_i\omega_{ikl}.$$

Consequently,

$$\frac{4}{5}(\delta R)_{kij}\omega_{ijl}\delta\omega_{kl} = \Big(\nabla^g{}_i\rho_{jk} - \nabla^g{}_j\rho_{ik}\Big)\omega_{ijl}\delta\omega_{kl} = 0, \tag{7.55}$$

since $\delta\omega \in \Lambda^2_{14}$.

To apply effectively our integral formula (6.29) we have to evaluate one more term. Denote $K_i = \omega_{ijl}\rho_{jk}\delta\omega_{kl}$. Equation (7.55) together with (5.12) leads to the equality

$$\delta K = -\frac{1}{2}\rho_{jk}\delta\omega_{jl}\delta\omega_{kl} - 2||\rho||^2$$

which implies, by an integration over the compact M, that

$$\int_M \rho_{jk}\delta\omega_{jl}\delta\omega_{kl}\, dV = -4\int_M ||\rho||^2\, dV. \tag{7.56}$$

The equalities (7.55), (7.56), (5.22) and (6.29) yield

$$\int_M \Big(\frac{1}{24}||\delta\omega||^4 + \frac{14}{3}||\rho^0||^2\Big)\, dV = 0.$$

Hence, Theorem 7.1 follows. □

Clearly, our Main Theorem follows from Theorem 7.1.

Corollary 5.5 and the main theorem lead to

Theorem 7.2. *Any compact 7-manifold with closed G_2-structure which is Einstein with respect to the canonical connection is a Joyce space.*

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References

1. Bonan, E.: Sur le variétés riemanniennes a groupe d'holonomie G_2 ou $Spin(7)$. C. R. Acad. Sci. Paris **262**, 127–129 (1966)
2. Bryant, R.: Metrics with exceptional holonomy. Ann. Math. **126**, 525–576 (1987)
3. Bryant, R.: Some remarks on G_2-structures. http://arxiv.org/list/math.DG/0305124, 2003
4. Bryant, R.: Some remarks on G_2-structures. To appear in the 2005 Gkova Geometry/Topology Conference Proceedings, Cambridge, MA: International Press
5. Bryant, R., Salamon, S.: On the construction of some complete metrics with exceptional holonomy. Duke Math. J. **58**, 829–850 (1989)
6. Cabrera, F., Monar, M., Swann, A.: Classification of G_2-structures. J. London Math. Soc. **53**, 407–416 (1996)
7. Chiossi, S., Salamon, S.: The intrinsic torsion of SU(3) and G_2-structures. In: *Differential Geometry*, Valencia 2001, River Edge, NJ: World Sci. Publishing, 2002, pp. 115–133
8. Cleyton, R., Ivanov, S.: On the geometry of closed G_2- structures. http://arxiv.org/list/math.DG/0306362, 2003
9. Cvetič, M., Gibbons, G.W., Lu, H., Pope, C.N.: Almost Special Holonomy in Type IIA&M Theory. Nucl. Phys. **B638**, 186–206 (2002)
10. Fernández, M.: An Example of compact calibrated manifold associated with the exceptional Lie group G_2. J. Diff. Geom. **26**, 367–370 (1987)
11. Fernández, M., Gray, A.: Riemannian manifolds with structure group G_2. Ann. Mat. Pura Appl. (4)**32**, 19–45 (1982)
12. Friedrich, Th., Ivanov S.: Parallel spinors and connections with skew-symmetric torsion in string theory. Asian J. Math. **6**, 303–336 (2002)
13. Friedrich, Th., Ivanov S.: Killing spinor equations in dimension 7 and geometry of integrable G_2 manifolds. J. Geom. Phys. **48**, 1–11 (2003)
14. Gauntlett, J., Kim, N., Martelli, D., Waldram, D.: Fivebranes wrapped on SLAG three-cycles and related geometry. JHEP **0111** 018 (2001)
15. Gauntlett, J., Martelli, D., Waldram, D.: Superstrings with Intrinsic torsion. Phys. Rev. **D69** 086002 (2004)
16. Gibbons, G.W., Page, D.N., Pope, C.N.: Einstein metrics on S^3, $\mathbb{R}^3$, and $\mathbb{R}^4$ bundles. Commun. Math. Phys. **127**, 529–553 (1990)
17. Gray, A.: Vector cross product on manifolds. Trans. Am. Math. Soc. **141**, 463–504 (1969); Correction **148**, 625 (1970)
18. Ivanov, P., Ivanov, S.: $SU(3)$-instantons and G_2, $Spin(7)$-heterotic string solitons. Commun. Math. Phys. **259**, 79–102 (2005)
19. Joyce, D.: Compact Riemannian 7-manifolds with holonomy G_2. I. J. Diff. Geom. **43**, 291–328 (1996)
20. Joyce, D.: Compact Riemannian 7-manifolds with holonomy G_2. II. J. Diff. Geom. **43**, 329–375 (1996)
21. Joyce, D.: *Compact Riemannian manifolds with special holonomy*. Oxford: Oxford University Press, 2000
22. Kovalev, A.: Twisted connected sums and special Riemannian holonomy. J. Reine Angew. Math. **565**, 125–160 (2003)
23. Lawson, B., Michelsohn, M.-L.: *Spin Geometry*. Princeton, NJ: Princeton University Press, 1989
24. Leung, N.C.: TQFT for Calabi-Yau three folds and G_2 manifolds. Adv. Theor. Math. Phys. **6**, 575–591 (2002)
25. Salamon, S.: *Riemannian geometry and holonomy groups*. London: Pitman Res. Notes Math. Ser., 201 1989

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