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Energy Splitting, Substantial Inequality, and Minimization for the Faddeev and Skyrme Models

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Abstract: In this paper, we prove that the Faddeev energy E_1 at the unit Hopf charge is attainable. The proof is based on utilizing an important inequality called the substantial inequality in our previous paper which describes how the Faddeev energy splits into its sublevels in terms of energy and topology when compactness fails. With the help of an optimal Sobolev estimate of the Faddeev energy lower bound and an upper bound of E_1, we show that E_1 is attainable. For the two-dimensional Skyrme model, we prove that the substantial inequality is also valid, which allows us to greatly improve the range of the coupling parameters for the existence of unit-charge solitons previously guaranteed in a smaller range of the coupling parameters by the validity of the concentration-compactness method.

1. Introduction

Global energy minimizers are important in field theory as they provide leading-order contributions to the transition amplitudes calculated through functional integrals or partition functionals for the quantization of fundamental particle systems [9]. Some prototype examples include kinks, vortices, monopoles, and instantons, which are static solitons characterized by various topological invariants. Except for the one-dimensional (1D) kink case which is completely integrable, in all the other cases, global energy minimizers can only be obtained in the so-called BPS limits. The main difficulty we encounter in this kind of problems is a lack of compactness because the energy functionals are all defined over the full Euclidean spaces. For the well-known Skyrme model and the Faddeev model, the situation is even less transparent because these models do not have a BPS-limit structure. Therefore, one is forced to study the direct minimization problem for these models. From an analytic point of view, the first temptation would be to try to see whether the concentration-compactness method [14] works because this method is developed to tackle similar minimization problems defined over full spaces which says that a minimizing sequence converges (hence compactness holds) if after suitable translations it concentrates in a local region (that is, if concentration takes place). For our problems, however, it is not directly possible to establish such a concentration-compactness picture. In fact, we will have to be forced to study the situation when concentration-compactness fails and an energy splitting or dichotomy takes place. It is interesting that the topological structure of these problems now become important which allows us to deduce concentration-compactness indirectly from an inequality we call "the substantial inequality" which originates essentially from assuming dichotomy or energy splitting. We have seen in [12] that this substantial inequality method enabled us to establish a series of existence theorems for the Faddeev model [6–8] and the 3D Skyrme model [18–21, 27], which were previously unavailable. In this paper, we will use this method to establish the much anticipated existence theorem that the Faddeev energy E_1 at the unit Hopf charge is attainable. Besides, we will use the same method to establish some new existence results for the 2D Skyrme model which considerably improve the existence result previously obtained in [13] using the concentration-compactness method.

The rest of this paper is organized as follows. In the next section, we recall the existence problem of the Faddeev model and prove that the Faddeev energy E_1 at the unit Hopf charge is attainable by using the substantial inequality method. This method relies on some suitable energy estimates which are consequences of a specific topological energy lower bound and an upper estimate for E_1, which will be elaborated in detail in Sect. 3 and Sect. 4. In Sect. 5, we study the 2D Skyrme model and we prove that the substantial inequality is valid. In particular, we show that the minimization problem of the 2D Skyrme model has a solution within a suitable (but unknown) topological class. In Sect. 6, we use the substantial inequality method as we do for the Faddeev model to show the existence of a least-positive-energy minimizer for the 2D Skyrme model. We also show that an energy minimizer for the 2D Skyrme model exists at the unit topological degree when the product of the coupling constants lies in an explicit interval which greatly improves the interval we obtained in [13] by using the concentration-compactness method directly. We also remark that the values of the coupling constants in the Faddeev model and Skyrme model are not important for the understanding of their minimization problems.

2. Minimization for the Faddeev Model

Let $\mathbf{n} = (n_1, n_2, n_3) : \mathbb{R}^3 \to S^2$ be a map (from the Euclidean 3-space to the unit 2-sphere) and $F_{jk}(\mathbf{n}) = \mathbf{n} \cdot (\partial_j \mathbf{n} \wedge \partial_k \mathbf{n})$ $(j, k = 1, 2, 3)$ the induced (Faddeev) magnetic field. We follow [25] to use the renormalized Faddeev energy

$$\begin{aligned} E(\mathbf{n}) &= \int_{\mathbb{R}^3} \left\{ \sum_{1 \le k \le 3} |\partial_k \mathbf{n}|^2 + \frac{1}{2} \sum_{1 \le k < \ell \le 3} F_{k\ell}^2(\mathbf{n}) \right\} \mathrm{d}x \\ &= \int_{\mathbb{R}^3} \left(|\nabla \mathbf{n}|^2 + \frac{1}{2} |\mathbf{F}|^2 \right) \mathrm{d}x. \end{aligned} \tag{2.1}$$

Here $\mathbf{F} = \mathbf{F}(\mathbf{n}) = (\frac{1}{2}\varepsilon^{jkk'} F_{kk'}(\mathbf{n})) = (F_{23}(\mathbf{n}), -F_{13}(\mathbf{n}), F_{12}(\mathbf{n}))$. The finite-energy condition implies that $\mathbf{n}$ approaches a constant vector $\mathbf{n}_\infty$ at infinity of $\mathbb{R}^3$. Hence we may compactify $\mathbb{R}^3$ into S^3 and view the fields as maps from S^3 to S^2. As a consequence, we see that each finite-energy field configuration $\mathbf{n}$ is associated with an integer, $Q(\mathbf{n})$, in $\pi_3(S^2) = \mathbb{Z}$. In fact, such an integer $Q(\mathbf{n})$ is known as the Hopf invariant which has the following integral characterization due to Whitehead [26]: Since the vector field $\mathbf{F}$ is divergence free, we can express $\mathbf{F}$ in terms of a vector potential $\mathbf{A}$, $\mathbf{F} = \nabla \wedge \mathbf{A}$. Then the Hopf invariant or charge $Q(\mathbf{n})$ of the map $\mathbf{n}$ may be evaluated by the integral

$$Q(\mathbf{n}) = \frac{1}{16\pi^2} \int_{\mathbb{R}^3} \mathbf{A} \cdot \mathbf{F} \, \mathrm{d}x, \tag{2.2}$$

which is also a Chern–Simons index (see [10] for an interesting discussion from the point of view of a physicist) and may be interpreted as a linking number.

We are interested in the following topologically constrained minimization problem

$$E_m = \inf\{E(\mathbf{n}) \mid E(\mathbf{n}) < \infty, \;\; Q(\mathbf{n}) = m\}. \tag{2.3}$$

For $m \neq 0$, the solutions of (2.3) give rise to static solitons known as the Faddeev knots [8, 2–4]. In [12], we proved the existence of an infinite subset $\mathbb{S}$ of the set of all integers $\mathbb{Z}$ such that (2.3) is solvable for any $m \in \mathbb{S}$. Moreover, we showed also that $m_0 \in \mathbb{S}$, where $m_0 \neq 0$ is such that $E_{m_0} = \inf\{E_m \mid m \in \mathbb{Z}\backslash\{0\}\}$. We were unable to further describe the set $\mathbb{S}$. In this paper, we shall establish the much anticipated result, $1 \in \mathbb{S}$, for the above Faddeev problem [7]. That is, we shall prove

Theorem 2.1. *The Faddeev minimization problem* (2.3) *has a solution for* $m = \pm 1$.

The proof of this theorem follows from an inequality we derived in [12], which we called the “substantial inequality” and some suitable refined energy estimates.

Substantial Inequality [12]. For any $m \in \mathbb{Z}\backslash\{0\}$, there is a decomposition

$$m = m_1 + \cdots + m_\ell, \quad m_s \in \mathbb{S}\backslash\{0\}, \quad s = 1, \ldots, \ell, \tag{2.4}$$

so that the following sub-additivity relation

$$E_m \geq E_{m_1} + \cdots + E_{m_\ell} \tag{2.5}$$

holds.

Note that the two ingredients of (2.4) and (2.5) are that the former expresses a “charge-conservation” law and the latter says that the total mass of a multiple-particle system is at least equal to the sum of the masses of the particles that the system is made of plus a possible amount of binding energy. More precisely, such an energy splitting process may be compared with the familiar nuclear fission process. Indeed, when a nucleus undergoes fission spontaneously, it splits into several smaller fragments (or substances). The sum of the masses of these fragments is less than the original mass of the nucleus and the “missing” mass has been converted into energy according to Einstein’s equation. With this interpretation, the substantial inequality may also be called the “mass inequality” or the “fission inequality.” We thank Michael Kiessling and Zhengchao Han for their valuable comments on this interpretation.

Lower and Upper Energy Estimates. The following energy lower bound holds:

$$E(\mathbf{n}) \geq 3^{3/8} 8\sqrt{2}\pi^2 |Q(\mathbf{n})|^{3/4}. \tag{2.6}$$

Besides, the energy E_1 satisfies the upper estimate

$$E_1 \leq 32\sqrt{2}\pi^2. \tag{2.7}$$

Remarks. The lower bound (2.6) was first derived by Vakulenko and Kapitanski [24] in the form $E(\mathbf{n}) \geq C|Q(\mathbf{n})|^{3/4}$ with $C > 0$ an unspecified universal constant. Since (2.6) is an important inequality for the Faddeev model, we shall go over the details of its derivation in the next section. When we do this, we give close attention at all the steps to keeping the optimality of various constants encountered. Other related discussions can be found in [11, 17]. The upper bound (2.7) was obtained by Ward [25]. In Sect. 4, we shall follow the steps sketched in [25] to arrive at (2.7).

Proof of Theorem 1.1. Suppose that E_1 is not attainable. Then in the minimization process for E_1 concentration does not occur and there holds the nontrivial energy splitting in view of the substantial inequality (2.6) by [12]:

$$E_1 \geq E_{m_1} + \cdots + E_{m_\ell}, \tag{2.8}$$

$$1 = m_1 + \cdots + m_\ell, \quad m_s \in \mathbb{Z} \setminus \{0\}, \quad s = 1, \ldots, \ell, \tag{2.9}$$

with $\ell \geq 2$. Since each $E_{m_s} > 0$ in view of (2.6), we see from (2.8) and the fact $E_1 = E_{-1}$ that $m_s \neq \pm 1$ for $s = 1, \ldots, \ell$. In view of (2.9), one of the integers, $m_1, \ldots, m_\ell$, must be an odd number. Assume that m_1 is odd. Then $|m_1| \geq 3$. Of course, $|m_2| \geq 2$. Therefore (2.7) and (2.6) lead us to

$$32\sqrt{2}\pi^2 \geq E_1 \geq E_{m_1} + E_{m_2} \geq 3^{3/8} 8\sqrt{2}\pi^2 (3^{3/4} + 2^{3/4}), \tag{2.10}$$

which is a contradiction and the proof of the theorem follows.

3. Vakulenko–Kapitanski Inequality

First recall the sharp Sobolev inequality [1, 23] for a scalar function $f \in W^{1,p}(\mathbb{R}^n)$: if $1 < p < n$ and $1/q = 1/p - 1/n$, then

$$C_0 \|f\|_q \leq \left(\int_{\mathbb{R}^n} |\nabla f|^p \, \mathrm{d}x \right)^{1/p}, \tag{3.1}$$

where the best constant C_0 is defined by

$$C_0 = n^{1/p} \left(\frac{n-p}{p-1} \right)^{1-1/p} \left(\omega_n \frac{\Gamma(\frac{n}{p})\Gamma(n+1-\frac{n}{p})}{\Gamma(n)} \right)^{1/n}, \tag{3.2}$$

with ω_n the n-dimensional volume enclosed by the unit sphere S^{n-1} in $\mathbb{R}^n$, and $\|f\|_q$ denotes the standard $L^q(\mathbb{R}^n)$-norm. Since we need $n = 3$ and $p = 2$, we must have $q = 6$ and (3.1) and (3.2) give us the sharp Sobolev inequality in 3D (see also [16]):

$$\|f\|_6 \leq \left(\frac{4}{3\sqrt{3}\pi^2} \right)^{1/3} \left(\int_{\mathbb{R}^3} |\nabla f|^2 \, \mathrm{d}x \right)^{1/2}. \tag{3.3}$$

We now consider the vector fields $\mathbf{A}$ and $\mathbf{F}$ defined in (2.1) and (2.2). Following [24], we have

$$\left| \int_{\mathbb{R}^3} \mathbf{A} \cdot \mathbf{F} \, \mathrm{d}x \right| \leq \|\mathbf{A}\|_6 \|\mathbf{F}\|_{6/5} \leq \|\mathbf{A}\|_6 \|\mathbf{F}\|_1^{2/3} \|\mathbf{F}\|_2^{1/3}. \tag{3.4}$$

Note that we always use $\|\mathbf{A}\|_p$ to denote the $L^p(\mathbb{R}^3)$-norm for magnitude (scalar) function $|\mathbf{A}|$ of the vector field $\mathbf{A}$.

Using (3.3), we have

$$\|\mathbf{A}\|_6 = \|\,|\mathbf{A}|\,\|_6 \leq \left(\frac{4}{3\sqrt{3}\pi^2}\right)^{1/3} \left(\int_{\mathbb{R}^3} |\,\nabla|\mathbf{A}|\,|^2\,\mathrm{d}x\right)^{1/2}$$
$$\leq \left(\frac{4}{3\sqrt{3}\pi^2}\right)^{1/3} \left(\int_{\mathbb{R}^3} |\nabla\mathbf{A}|^2\,\mathrm{d}x\right)^{1/2}, \qquad (3.5)$$

where $|\nabla\mathbf{A}|^2 = \sum|\nabla A_j|^2$. On the other hand, neglecting boundary terms at infinity when integrating, we have the identity $\int|\nabla\mathbf{A}|^2 = \int(\nabla\cdot\mathbf{A})^2 + \int|\nabla\wedge\mathbf{A}|^2$ (in [24], there is an additional erroneous factor $1/2$ on the right-hand side of this relation). Hence restricting to divergence-free vector field $\mathbf{A}$ as in [24] and using the relation $\mathbf{F} = \nabla\wedge\mathbf{A}$, we see that (3.5) becomes

$$\|\mathbf{A}\|_6 \leq \left(\frac{4}{3\sqrt{3}\pi^2}\right)^{1/3} \|\mathbf{F}\|_2. \qquad (3.6)$$

Inserting (3.6) into (3.4), we obtain

$$\left|\int_{\mathbb{R}^3} \mathbf{A}\cdot\mathbf{F}\,\mathrm{d}x\right| \leq \left(\frac{4}{3\sqrt{3}\pi^2}\right)^{1/3} \|\mathbf{F}\|_1^{2/3}\|\mathbf{F}\|_2^{4/3}. \qquad (3.7)$$

We now estimate $\|\mathbf{F}\|_1$ and $\|\mathbf{F}\|_2$ in terms of the Faddeev energy $E(\mathbf{n})$ given in (2.1). For convenience, we make the decomposition $E = E_{\mathrm{D}} + E_{\mathrm{S}}$ where

$$E_{\mathrm{D}}(\mathbf{n}) = \int_{\mathbb{R}^3} |\nabla\mathbf{n}|^2\,\mathrm{d}x \quad \text{and} \quad E_{\mathrm{S}}(\mathbf{n}) = \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{F}|^2\,\mathrm{d}x \qquad (3.8)$$

stand for the Dirichlet-type energy and Skyrme-type energy, respectively.

Specializing the argument of Ward [25] based on a paper of Manton [15] using symmetric polynomials, we have $|\mathbf{F}| \leq |\nabla\mathbf{n}|^2/2$. In fact, let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of the symmetric matrix $(\nabla n_j \cdot \nabla n_k)$. Then there holds the identity

$$\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 = \sum_{1\leq j<k\leq 3}^{3} \det\begin{pmatrix} \nabla n_j\cdot\nabla n_j & \nabla n_j\cdot\nabla n_k \\ \nabla n_k\cdot\nabla n_j & \nabla n_k\cdot\nabla n_k \end{pmatrix}$$
$$= \sum_{1\leq j<k\leq 3}^{3} |\partial_j\mathbf{n}\wedge\partial_k\mathbf{n}|^2 = |\mathbf{F}|^2. \qquad (3.9)$$

It can be directly checked that $\mathbf{n}$ lies in the nullspace of the matrix $(\nabla n_j \cdot \nabla n_k)$. Therefore this matrix has a zero eigenvalue. Assume $\lambda_3 = 0$. We get from (3.9) that $|\mathbf{F}| = \sqrt{\lambda_1\lambda_2} \leq \frac{1}{2}(\lambda_1+\lambda_2) = \frac{1}{2}$ the trace of $(\nabla n_j\cdot\nabla n_k) = \frac{1}{2}|\nabla\mathbf{n}|^2$ as stated.

Hence $\|\mathbf{F}\|_1 \leq \frac{1}{2}\int |\nabla \mathbf{n}|^2 = \frac{1}{2}E_{\mathrm{D}}(\mathbf{n})$. Besides, it is obvious that $\|\mathbf{F}\|_2^2 = 2E_{\mathrm{S}}(\mathbf{n})$. As a consequence, we can update (3.7) into the form

$$\begin{aligned} \left|\int_{\mathbb{R}^3} \mathbf{A}\cdot\mathbf{F}\,\mathrm{d}x\right| &\leq \left(\frac{4}{3\sqrt{3}\pi^2}\right)^{1/3}\left(\frac{1}{2}E_{\mathrm{D}}(\mathbf{n})\right)^{2/3}(2E_{\mathrm{S}}(\mathbf{n}))^{2/3} \\ &\leq \left(\frac{4}{3\sqrt{3}\pi^2}\right)^{1/3}\left(\frac{1}{2}E(\mathbf{n})\right)^{4/3} \\ &= (12\sqrt{3}\pi^2)^{-1/3}(E(\mathbf{n}))^{4/3}, \end{aligned} \tag{3.10}$$

which establishes (2.6).

4. An Upper Estimate for E_1

In this section, we follow the steps sketched in Ward [25] to derive (2.7). Note that an intermediate result (see (4.18) below) we obtain is different from that stated in [25] due to our choice of the stereographic projection for the 3-sphere. However, this result does not affect the final estimate (2.7).

Energy of the Hopf Map from S_R^3 into S^2. Consider the spheres in $\mathbb{R}^4$ and $\mathbb{R}^3$ given in terms of their respective coordinate variables by

$$\begin{aligned} S_R^3 &= \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2\right\}, \\ S^2 &= \left\{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_1^2 + y_2^2 + y_3^2 = 1\right\}. \end{aligned}$$

The Hopf map $\Phi : S_R^3 \to S^2$, $\Phi(x_1, x_2, x_3, x_4) = (y_1, y_2, y_3)$, may be defined by

$$y_1 = \frac{2}{R^2}(x_1x_3 + x_2x_4), \quad y_2 = \frac{2}{R^2}(x_2x_3 - x_1x_4), \quad y_3 = \frac{1}{R^2}\left(x_4^2 + x_3^2 - x_2^2 - x_1^2\right). \tag{4.1}$$

This map has the Hopf index one. Using the "Hopf" coordinates (θ, s, t) for which

$$\begin{aligned} &x_1 = R\sin\left(\frac{\theta}{2}\right)\sin s,\ x_2 = R\sin\left(\frac{\theta}{2}\right)\cos s,\ x_3 = \cos\left(\frac{\theta}{2}\right)\sin t, \\ &x_4 = \cos\left(\frac{\theta}{2}\right)\cos t,\ 0 \leq \theta \leq \pi, \quad -\pi \leq s \leq \pi, \quad -\pi \leq t \leq \pi, \end{aligned} \tag{4.2}$$

the Hopf map can be represented in view of (4.1) and (4.2) simply as

$$\Phi(\theta, s, t) = (\sin\theta\cos(s-t), -\sin\theta\sin(s-t), \cos\theta). \tag{4.3}$$

So $|\Phi_\theta|^2 = 1$, $|\Phi_s|^2 = \sin^2\theta$, and $|\Phi_t|^2 = \sin^2\theta$. Besides, with the notation $x = (x_1, x_2, x_3, x_4)$ and the coordinate representation (4.2), we can calculate the induced metric components for S_R^3 directly: $g_{\theta\theta} = R^2/4$, $g_{ss} = R^2\sin^2(\theta/2)$, $g_{tt} = R^2\cos^2(\theta/2)$, and $g_{\theta s} = g_{\theta t} = g_{st} = 0$. Consequently, $g^{\theta\theta} = 4/R^2$, $g^{ss} = 1/R^2\sin^2(\theta/2)$,

$g^{tt} = 1/R^2\cos^2(\theta/2)$, $g^{\theta s} = g^{\theta t} = g^{st} = 0$, and the Dirichlet energy density of the Hopf map over S^3_R, $\mathcal{E}_{\rm D}(\Phi; S^3_R)$ takes the constant value,

$$\mathcal{E}_{\rm D}\left(\Phi; S^3_R\right) = g^{jk}\partial_j\Phi\cdot\partial_k\Phi \quad (j,k=\theta,s,t) = \frac{8}{R^2}, \tag{4.4}$$

as stated in [25]. Similarly, we can evaluate the Skyrme energy density, $\mathcal{E}_{\rm S}(\Phi; S^3_R)$. We easily see in view of (4.3) that the respective components of the Faddeev magnetic field $F_{jk}(\Phi) = \Phi\cdot(\partial_j\Phi\wedge\partial_k\Phi)$ are $F_{\theta s}(\Phi) = -\sin\theta$, $F_{\theta t}(\Phi) = \sin\theta$, and $F_{st}(\Phi) = 0$. Therefore

$$\mathcal{E}_{\rm S}\left(\Phi; S^3_R\right) = \frac{1}{4}g^{j\ell}g^{km}F_{jk}(\Phi)F_{\ell m}(\Phi) = \frac{8}{R^4}, \tag{4.5}$$

also as stated in [25]. Integrating (4.4) and (4.5) over S^3_R and using the fact that the total volume of S^3_R is $2\pi^2R^3$, we arrive at the following Ward's number [25] for the intrinsic Faddeev energy of the Hopf map $\Phi : S^3_R \to S^2$:

$$E\left(\Phi; S^3_R\right) \equiv \int_{S^3_R}\left(\mathcal{E}_{\rm D}(\Phi; S^3_R) + \mathcal{E}_{\rm S}\left(\Phi; S^3_R\right)\right)\,{\rm d}V_{S^3_R} = 16\pi^2\left(R + \frac{1}{R}\right), \tag{4.6}$$

where we use ${\rm d}V_{S^3_R}$ to denote the canonical volume element of S^3_R.

Stereographic Coordinates. We need the stereographic projection from S^3_R to $\mathbb{R}^3$ so that the inverse of this projection can be viewed as a specific coordinate chart for S^3_R:

$$x_1 = \frac{2R^2}{r^2+R^2}\xi, \quad x_2 = \frac{2R^2}{r^2+R^2}\zeta, \quad x_3 = \frac{2R^2}{r^2+R^2}\eta, \quad x_4 = \left(\frac{r^2-R^2}{r^2+R^2}\right)R,$$
$$(\xi,\zeta,\eta)\in\mathbb{R}^3, \quad r^2 = \xi^2+\zeta^2+\eta^2. \tag{4.7}$$

In terms of this coordinate system, we see that the respective components of the canonical metric tensor of S^3_R become $g_{\xi\xi} = g_{\zeta\zeta} = g_{\eta\eta} = 4R^4/(r^2+R^2)^2$ and $g_{\xi\zeta} = g_{\xi\eta} = g_{\zeta\eta} = 0$. Consequently, ${\rm d}V_{S^3_R} = (8R^6/(r^2+R^2)^3){\rm d}\xi{\rm d}\zeta{\rm d}\eta$, $g^{\xi\xi} = g^{\zeta\zeta} = g^{\eta\eta} = (r^2+R^2)^2/4R^4$, and $g^{\xi\zeta} = g^{\xi\eta} = g^{\zeta\eta} = 0$. Now let $\mathbf{n} : \mathbb{R}^3 \to S^2$ be a map of finite Faddeev energy, which may be viewed as a map from S^3_R into S^2 represented through the above stereographic coordinates. We have

$$\begin{aligned} E_{\rm D}(\mathbf{n}; S^3_R) &\equiv \int_{S^3_R} g^{jk}\partial_j\mathbf{n}\cdot\partial_k\mathbf{n}\,{\rm d}V_{S^3_R} \quad (j,k=\xi,\zeta,\eta) \\ &= \int_{\mathbb{R}^3}\frac{2R^2}{r^2+R^2}|\nabla\mathbf{n}|^2\,{\rm d}\xi{\rm d}\zeta{\rm d}\eta \\ &\to 2\int_{\mathbb{R}^3}|\nabla\mathbf{n}|^2\,{\rm d}\xi{\rm d}\zeta{\rm d}\eta \quad \text{as } R\to\infty, \end{aligned} \tag{4.8}$$

which is twice the standard Dirichlet energy over $\mathbb{R}^3$. Similarly, for the Skyrme energy part, we have

$$
\begin{aligned}
E_{\mathrm{S}}(\mathbf{n}; S^3_R) &\equiv \int_{S^3_R} \frac{1}{4} g^{j\ell} g^{km} F_{jk}(\mathbf{n}) F_{\ell m}(\mathbf{n})\, \mathrm{d}V_{S^3_R} \quad (j,k=\xi,\zeta,\eta)\\
&= \frac{1}{2}\int_{\mathbb{R}^3} \frac{(r^2+R^2)}{R^2}\left(\frac{1}{4}\delta^{j\ell}\delta^{km} F_{jk}(\mathbf{n}) F_{\ell m}(\mathbf{n})\right) \mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}\eta\\
&\to \frac{1}{4}\int_{\mathbb{R}^3} |\mathbf{F}(\mathbf{n})|^2\, \mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}\eta \quad \text{as } R\to\infty,
\end{aligned}
\tag{4.9}
$$

which is half of the standard Skyrme energy over $\mathbb{R}^3$. Hence we arrive at the weighted limit

$$
E(\mathbf{n}) = \lim_{R\to\infty}\left\{\frac{1}{2}E_{\mathrm{D}}\left(\mathbf{n}; S^3_R\right) + 2E_{\mathrm{S}}\left(\mathbf{n}; S^3_R\right)\right\}. \tag{4.10}
$$

Note that the above weighted limit is a result of our choice of the stereographic projection (4.7) which maps S^3_R onto the extended plane through its equator. However, when we use the stereographic projection which maps S^3_R onto the extended plane tangential to its north pole, we shall not need to place weights and we get the same intermediate result as that stated in Ward [25], instead of (4.18) below.

Upper Bound by Rescaling/Dilation. From (4.6), we see that we cannot take the $R\to\infty$ limit directly for the Hopf map. On the other hand, however, the limit (4.10) suggests that for a suitably chosen map, the limit $R\to\infty$ taken over S^3_R may allow us to recover the Faddeev energy over the Euclidean space $\mathbb{R}^3$. In the following, we use a rescaling/dilation argument of Ward [25] to get a suitable Hopf map over S^3_R which allows us to take the $R\to\infty$ limit. In this way, we arrive at the upper bound (2.7) stated for E_1 in [25].

We again use (ξ,ζ,η) to denote the stereographic coordinates defined in (4.7) and Φ the Hopf map from S^3_R to S^2 defined in (4.1). We introduce the deformed (dilated) map $\Phi_\lambda : S^3_R \to S^2$ given by

$$
\Phi_\lambda(\xi,\zeta,\eta) = \Phi(\lambda\xi,\lambda\zeta,\lambda\eta) = (y_1,y_2,y_3)\in S^2, \tag{4.11}
$$

where, in view of (4.1) and (4.7), the image coordinates y_1, y_2, y_3 are given by

$$
\begin{aligned}
y_1 &= \frac{4\lambda R}{(\lambda^2 r^2+R^2)^2}(2\lambda R\xi\eta + [\lambda^2 r^2 - R^2]\zeta),\\
y_2 &= \frac{4\lambda R}{(\lambda^2 r^2+R^2)^2}(2\lambda R\zeta\eta - [\lambda^2 r^2 - R^2]\xi),\\
y_3 &= 1 - \frac{8\lambda^2 R^2}{(\lambda^2 r^2+R^2)^2}(\eta^2 - r^2).
\end{aligned}
\tag{4.12}
$$

Let $R = \lambda a$. Then the above representation simplifies to

$$\begin{aligned} y_1 &= \frac{4a}{(r^2+a^2)^2}(2a\,\xi\eta + [r^2 - a^2]\zeta), \\ y_2 &= \frac{4a}{(r^2+a^2)^2}(2a\,\zeta\eta - [r^2 - a^2]\xi), \\ y_3 &= 1 - \frac{8a^2}{(r^2+a^2)^2}(\eta^2 - r^2), \end{aligned} \tag{4.13}$$

which is the Hopf map from S_a^3 to S^2. This property allows us to evaluate the energy densities on S_R^3 easily. Indeed, we have in view of the conformality of the stereographic coordinates and (4.13) the relations

$$\begin{aligned} \mathcal{E}_{\mathrm{D}}\left(\Phi_\lambda; S_R^3\right) &= \frac{(r^2+R^2)^2}{4R^4} \cdot \frac{4a^4}{(r^2+a^2)^2}\mathcal{E}_{\mathrm{D}}\left(\Phi; S_a^3\right) \\ &= \left(\frac{r^2+R^2}{r^2+a^2}\right)^2 \cdot \frac{a^4}{R^4} \cdot \frac{8}{a^2}, \end{aligned} \tag{4.14}$$

$$\begin{aligned} \mathcal{E}_{\mathrm{S}}\left(\Phi_\lambda; S_R^3\right) &= \left(\frac{(r^2+R^2)^2}{4R^4}\right)^2 \cdot \left(\frac{4a^4}{(r^2+a^2)^2}\right)^2 \mathcal{E}_{\mathrm{S}}\left(\Phi; S_a^3\right) \\ &= \left(\frac{r^2+R^2}{r^2+a^2}\right)^4 \cdot \frac{a^8}{R^8} \cdot \frac{8}{a^4}. \end{aligned} \tag{4.15}$$

Hence, integrating (4.14) and (4.15) against the volume element $\mathrm{d}V_{S_R^3} = (8R^6/(r^2 + R^2)^3)\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}\eta$ over the (ξ, ζ, η)-space $\mathbb{R}^3$, we obtain

$$E_{\mathrm{D}}\left(\Phi_\lambda; S_R^3\right) = \frac{64\pi^2\lambda R}{(1+\lambda)^2}, \quad E_{\mathrm{S}}\left(\Phi_\lambda; S_R^3\right) = \frac{8\pi^2(1+\lambda^2)}{\lambda R}. \tag{4.16}$$

It can be checked that, as a function of $\lambda > 0$, the global minimum of

$$\tilde{E}\left(\Phi_\lambda; S_R^3\right) = \frac{1}{2}E_{\mathrm{D}}\left(\Phi_\lambda; S_R^3\right) + 2E_{\mathrm{S}}\left(\Phi_\lambda; S_R^3\right) \tag{4.17}$$

is

$$\min\left\{\tilde{E}\left(\Phi_\lambda; S_R^3\right) \mid \lambda > 0\right\} = 32\sqrt{2}\pi^2 - \frac{32\pi^2}{R}, \tag{4.18}$$

which is achieved at

$$\lambda = \lambda_R \equiv \frac{R}{\sqrt{2}} - 1 + \sqrt{\frac{1}{2}R^2 - \sqrt{2}R}. \tag{4.19}$$

With this choice of the dilation parameter $\lambda = \lambda_R$, the Hopf map (4.12) under the (ξ, ζ, η)-coordinates can be rewritten as

$$\begin{aligned} y_1^R &= \frac{4(R/\lambda_R)}{(r^2+(R/\lambda_R)^2)^2}(2(R/\lambda_R)\xi\eta + [r^2 - (R/\lambda_R)^2]\zeta), \\ y_2^R &= \frac{4(R/\lambda_R)}{(r^2+(R/\lambda_R)^2)^2}(2(R/\lambda_R)\zeta\eta - [r^2 - (R/\lambda_R)^2]\xi), \\ y_3^R &= 1 - \frac{8(R/\lambda_R)^2}{(r^2+(R/\lambda_R)^2)^2}(\eta^2 - r^2). \end{aligned} \tag{4.20}$$

Using (4.19) in (4.20), we see that, as $R \to \infty$, the map $y^R = (y_1^R, y_2^R, y_3^R) : \mathbb{R}^3 \to S^2$ converges rapidly to $\Psi \circ P^{-1} \equiv \mathbf{N} : \mathbb{R}^3 \to S^2$, where Ψ is the Hopf map from $S^3_{1/\sqrt{2}}$ to S^2 defined in (4.1) and $P : S^3_{1/\sqrt{2}} \to \mathbb{R}^3$ is the stereographic projection defined in (4.7), respectively, with $R = 1/\sqrt{2}$. Hence, setting $\lambda = \lambda_R$ in (4.17) and letting $R \to \infty$, we obtain by virtue of (4.18) that

$$\begin{aligned} E(\mathbf{N}) &= \int_{\mathbb{R}^3} \left\{ |\nabla \mathbf{N}|^2 + \frac{1}{4}\delta^{j\ell}\delta^{km} F_{jk}(\mathbf{N}) F_{\ell m}(\mathbf{N}) \right\} \mathrm{d}\xi \, \mathrm{d}\zeta \, \mathrm{d}\eta \\ &= \lim_{R\to\infty} \int_{\mathbb{R}^3} \left\{ \frac{R^2}{r^2+R^2} |\nabla y^R|^2 + \frac{(r^2+R^2)}{R^2} \cdot \frac{1}{4}\delta^{j\ell}\delta^{km} F_{jk}(y^R) F_{\ell m}(y^R) \right\} \mathrm{d}\xi \, \mathrm{d}\zeta \, \mathrm{d}\eta \\ &= \lim_{R\to\infty} \left\{ \frac{1}{2} E_{\mathrm{D}} \left(\Phi_{\lambda_R}; S^3_R \right) + 2 E_{\mathrm{S}} \left(\Phi_{\lambda_R}; S^3_R \right) \right\} = 32\sqrt{2}\pi^2. \end{aligned} \tag{4.21}$$

Of course, $Q(\mathbf{N}) = 1$. Therefore the upper bound (2.7) follows.

5. Two-Dimensional Skyrme Model

With the notation in [13], the two-dimensional Skyrme energy functional governing a configuration map $u : \mathbb{R}^3 \to S^2$ is given by

$$E(u) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2} |\nabla u|^2 + \frac{\lambda}{4} |\partial_1 u \wedge \partial_2 u|^2 + \frac{\mu}{16} |\mathbf{n} - u|^4 \right\} \mathrm{d}x, \tag{5.1}$$

where $\mathbf{n} = (0, 0, 1)$ is the north pole of S^2 in $\mathbb{R}^3$, and λ, μ are positive coupling constants. Finite-energy condition implies that u tends to $\mathbf{n}$ as $|x| \to \infty$. Therefore u may be viewed as a map from S^2 to itself which defines a homotopy class in $\pi_2(S^2) = \mathbb{Z}$, whose integer representative is the Brouwer degree of u with the integral representation

$$\deg(u) = \frac{1}{4\pi} \int_{\mathbb{R}^2} u \cdot (\partial_1 u \wedge \partial_2 u) \, \mathrm{d}x. \tag{5.2}$$

Like before, we are interested in the minimization problem

$$E_k = \inf\{E(u) \mid E(u) < \infty, \deg(u) = k\}, \tag{5.3}$$

where $k \in \mathbb{Z}$. Of course, $E_k = E_{|k|}$ for all $k \in \mathbb{Z}$. The main existence result of [13] states that if the coupling constants λ and μ satisfy

$$\lambda\mu \leq 48, \tag{5.4}$$

then the minimization problem (5.3) has a solution for $k = \pm 1$.

A direct consequence of the form of the energy (5.1) and the topological integral (5.2) is the following standard topological energy lower bound:

$$\frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \, \mathrm{d}x \geq 4\pi |\deg(u)|. \tag{5.5}$$

Hence, it follows from (5.5) that if $\deg(u) \neq 0$, then

$$E(u) > 4\pi|\deg(u)|. \tag{5.6}$$

Besides, using stereographic projection of S^2 as a trial field configuration, it can be shown [13] that there holds the following upper estimate for E_1:

$$E_1 \leq 4\pi\left(1 + \frac{1}{2}\sqrt{\frac{\lambda\mu}{3}}\right). \tag{5.7}$$

Minimization and Concentration-Compactness. Let $\{u_n\}$ be a minimizing sequence of the problem (5.3). Then, passing to a subsequence if necessary, we may assume that $u_n \to$ some u weakly in a well-understood sense and (cf. [5, 22])

$$E(u) \leq \liminf_{n\to\infty} E(u_n) = E_k. \tag{5.8}$$

Hence, in order to show that (5.3) is solved by the map u, it remains to show that u carries the same topology, $\deg(u) = k$, which is the main difficulty one encounters in this type of problems.

For the minimizing sequence $\{u_n\}$, set

$$f_n(x) = \left(\frac{1}{2}|\nabla u_n|^2 + \frac{\lambda}{4}|\partial_1 u_n \wedge \partial_2 u_n|^2 + \frac{\mu}{16}|\mathbf{n} - u_n|^4\right)(x), \quad n = 1, 2, \ldots. \tag{5.9}$$

Then $f_n \in L^1(\mathbb{R}^2)$, $\|f_n\|_{L^1} \geq 4\pi|k|$, and we can assume $\|f_n\|_{L^1} \leq E_k + 1$ (say), $n = 1, 2, \ldots$.

Use $D(y, R)$ to denote the disk in $\mathbb{R}^2$ centered at y and of radius $R > 0$: $D(y, R) = \{x \in \mathbb{R}^2 \mid |x - y| < R\}$ (we also use the simplified notation $D_R = \{x \in \mathbb{R}^2 \mid |x| < R\}$). Then, according to Lions [14], one of the following three situations must occur (principle of concentration-compactness):

(a) Compactness: There is a sequence $\{y_n\}$ in $\mathbb{R}^2$ such that, for any $\varepsilon > 0$, there exists an $R > 0$, such that

$$\sup_n \int_{\mathbb{R}^2\setminus D(y_n,R)} f_n(x)\,\mathrm{d}x \leq \varepsilon. \tag{5.10}$$

(b) Vanishing: For any $R > 0$,

$$\lim_{n\to\infty}\left(\sup_{y\in\mathbb{R}^2}\int_{D(y,R)} f_n(x)\,\mathrm{d}x\right) = 0. \tag{5.11}$$

(c) Dichotomy: There is a sequence $\{y_n\} \subset \mathbb{R}^2$ and a positive number $t \in (0, 1)$ such that for any $\varepsilon > 0$, there is an $R > 0$ and a sequence of positive numbers $\{R_n\}$ satisfying $\lim_{n\to\infty} R_n = \infty$ so that

$$\left|\int_{D(y_n,R)} f_n(x)\,\mathrm{d}x - t\|f_n\|_{L^1}\right| \leq \varepsilon,$$
$$\left|\int_{\mathbb{R}^2\setminus D(y_n,R_n)} f_n(x)\,\mathrm{d}x - (1-t)\|f_n\|_{L^1}\right| \leq \varepsilon. \tag{5.12}$$

It is not hard to show that if (a) is the case, then (5.3) has a solution. Besides, it can also be shown that (b) does not happen for $k \neq 0$. See [13]. Therefore, we are left with the remaining case (c) to consider.

The Substantial Inequality Implied by a Technical Lemma. Let Ω be a subdomain in $\mathbb{R}^2$ and define

$$E(u;\Omega) = \int_\Omega \left\{ \frac{1}{2}|\nabla u|^2 + \frac{\lambda}{4}|\partial_1 u \wedge \partial_2 u|^2 + \frac{\mu}{16}|\mathbf{n} - u|^4 \right\} \mathrm{d}x. \tag{5.13}$$

In [13], we proved the following technical lemma for the functional $E(u;\Omega)$:

Lemma 5.1. *For any* $\varepsilon \in (0,1)$ *and* $R \geq 1$, *let* $u : \overline{D_{2R} \setminus D_R} \to S^2$ *satisfy*

$$E(u; D_{2R} \setminus D_R) < \varepsilon. \tag{5.14}$$

Then there is a map $\tilde{u} : \overline{D_{2R} \setminus D_R} \to S^2$ *such that*

(i) $\tilde{u}|_{\partial D_R} = u$,
(ii) $\tilde{u}|_{\partial D_{2R}} = \mathbf{n}$,
(iii) $E(\tilde{u}; D_{2R} \setminus D_R) < C\varepsilon^{1/2}$ *where* $C > 0$ *is an absolute constant independent of* R, ε, *and* u.

Likewise, one can also obtain a modified map $\tilde{u} : \overline{D_{2R} \setminus D_R} \to S^2$ *such that* $\tilde{u}|_{\partial D_R} = \mathbf{n}$, $\tilde{u}|_{\partial D_{2R}} = u$, *and* (iii) *holds as well.*

The above lemma seems to be naturally true for higher dimensional problems such as the Faddeev problem and the (classical 3D) Skyrme problem. However, we are only able to prove it in our 2D situation here. It will be seen that this lemma is crucial in our proof of the substantial inequality for the 2D Skyrme model as stated in the next theorem.

Theorem 5.2. *Let k be a nonzero integer and* $\{u_n\}$ *a minimizing sequence of the problem* (5.3). *Then either* (a) *holds* (*hence a subsequence of* $\{u_n\}$ *converges weakly to a solution of* (5.3)) *or there are nonzero integers* k_1 *and* k_2 *such that*

$$k = k_1 + k_2 \quad \textit{and} \quad E_k \geq E_{k_1} + E_{k_2}. \tag{5.15}$$

As a consequence, if $\mathbb{S}$ *denotes the subset of* $\mathbb{Z} \setminus \{0\}$ *for which every member* $k \in \mathbb{S}$ *makes* (5.3) *solvable, then* $\mathbb{S} \neq \emptyset$. *In particular, for any* $k \in \mathbb{Z} \setminus \{0\}$, *there are integers* $k_1, \ldots, k_\ell \in \mathbb{S}$ *such that*

$$k = k_1 + \cdots + k_\ell \quad \textit{and} \quad E_k \geq E_{k_1} + \cdots + E_{k_\ell}. \tag{5.16}$$

Proof. Suppose (c) (dichotomy) occurs. Then, after making translations, we may assume that there is a number $t \in (0,1)$ such that for any $\varepsilon > 0$ there is an $R > 0$ and a sequence of positive numbers $\{R_n\}$ satisfying $\lim_{n\to\infty} R_n = \infty$ so that

$$\left| \int_{D_R} f_n(x)\,\mathrm{d}x - tE(u_n) \right| < \varepsilon, \tag{5.17}$$

$$\left| \int_{\mathbb{R}^2 \setminus D_{R_n}} f_n(x)\,\mathrm{d}x - (1-t)E(u_n) \right| < \varepsilon. \tag{5.18}$$

Without loss of generality, we may assume that $R_n > 2R$ for all n.

From (5.17), (5.18), and the decomposition

$$E(u_n) = \int_{D_R} f_n(x)\,\mathrm{d}x + \int_{\mathbb{R}^2 \setminus D_{R_n}} f_n(x)\,\mathrm{d}x + E(u_n; D_{R_n} \setminus D_R), \tag{5.19}$$

we have

$$E(u_n; D_{2R} \setminus D_R) \le E(u_n; D_{R_n} \setminus D_R) < 2\varepsilon. \tag{5.20}$$

Using (5.20) and Lemma 5.1, we can find maps v_n and w_n from $\mathbb{R}^2$ into S^2 such that

$$\begin{aligned} v_n &= u_n \quad \text{in } D_R, \quad v_n = \mathbf{n} \quad \text{in } \mathbb{R}^2 \setminus D_{2R}, \quad E(v_n; D_{2R} \setminus D_R) < C\varepsilon^{1/2}, \\ w_n &= u_n \quad \text{in } \mathbb{R}^2 \setminus D_{R_n}, \quad w_n = \mathbf{n} \quad \text{in } D_{R_n/2}, \quad E(w_n; D_{R_n} \setminus D_{R_n/2}) < C\varepsilon^{1/2}, \end{aligned}$$

where $C > 0$ is a constant independent of R, u_n, and ε. Therefore, with the notation $F(u) = u \cdot (\partial_1 u \wedge \partial_2 u)$, we have

$$\begin{aligned} &4\pi\,|\deg(u_n) - (\deg(v_n) + \deg(w_n))| \\ &\le \int_{D_{R_n} \setminus D_R} |F(u_n)|\,\mathrm{d}x + \int_{D_{2R} \setminus D_R} |F(v_n)|\,\mathrm{d}x + \int_{D_{R_n} \setminus D_{R_n/2}} |F(w_n)|\,\mathrm{d}x \\ &\le E(u_n; D_{R_n} \setminus D_R) + E(v_n; D_{2R} \setminus D_R) + E(w_n; D_{R_n} \setminus D_{R_n/2}) \\ &\le 2\varepsilon + 2C\varepsilon^{1/2}. \end{aligned}$$

Since ε can be made arbitrarily small and $\deg(u_n)$, $\deg(v_n)$, and $\deg(w_n)$ are integers, we may assume

$$k = \deg(u_n) = \deg(v_n) + \deg(w_n), \quad \forall n. \tag{5.21}$$

On the other hand, since

$$\begin{aligned} 4\pi\,|\deg(v_n)| \le E(v_n) &= E(u_n; D_R) + E(v_n; D_{2R} \setminus D_R) \\ &\le E(u_n) + C\varepsilon^{1/2} \le (k+1) + C\varepsilon^{1/2}, \end{aligned}$$

we see that $\{\deg(v_n)\}$ is bounded.

We claim that $\deg(v_n) \neq 0$ for n sufficiently large. Indeed, if $\deg(v_n) = 0$ for infinitely many n's, then by going to a subsequence when necessary, we may assume that $\deg(v_n) = 0$ for all n. Thus in view of (5.21) we see that $\deg(w_n) = k$ for all n and

$$E(w_n) \le E(u_n; \mathbb{R}^2 \setminus D_{R_n}) + C\varepsilon^{1/2} = \int_{\mathbb{R}^2 \setminus D_{R_n}} f_n(x)\,\mathrm{d}x + C\varepsilon^{1/2}. \tag{5.22}$$

Using (5.18) and (5.22), we arrive at

$$E_k \le \limsup_{n\to\infty} E(w_n) \le (1-t)\lim_{n\to\infty} E(u_n) + \varepsilon + C\varepsilon^{1/2} \le (1-t)E_k + \varepsilon + C\varepsilon^{1/2}. \tag{5.23}$$

Since $0 < t < 1$ and ε can be made arbitrarily small, (5.23) implies $E_k = 0$ which contradicts the topological lower bound $E_k \ge 4\pi|k|$ as stated in (5.6).

Similarly, we see that the sequence $\{\deg(w_n)\}$ is also bounded and $\deg(w_n) \neq 0$ for n sufficiently large.

Hence, by going to subsequences if necessary, we may assume that there are integers $k_1 \neq 0$ and $k_2 \neq 0$ such that

$$\deg(v_n) = k_1 \quad \text{and} \quad \deg(w_n) = k_2 \quad \forall n. \tag{5.24}$$

Now we have

$$\begin{aligned} &E(v_n) + E(w_n) \\ &\quad = E(u_n; D_R) + E(u_n; \mathbb{R}^2 \setminus D_{R_n}) + E(v_n; D_{2R} \setminus D_R) + E(w_n; D_{R_n} \setminus D_{R_n/2}) \\ &\quad \leq E(u_n) + 2C\varepsilon^{1/2}. \end{aligned} \tag{5.25}$$

Therefore, it follows from (5.25) directly that

$$E_{k_1} + E_{k_2} \leq \lim_{n\to\infty} E(u_n) + 2C\varepsilon^{1/2} = E_k + 2C\varepsilon^{1/2}. \tag{5.26}$$

Combining (5.21), (5.24), and (5.26), we see that (5.15) is established.

If (a) (compactness) does not occur at $k = k_1$ or $k = k_2$ for the minimization problem (5.3), we can continue our splitting in the above fashion. This splitting procedure will have to stop after finitely many steps because E_k is a finite number and the splitting cannot go on forever. In other words, we will have to stop at an inequality of the type $E_k \geq E_{k_1} + \cdots + E_{k_\ell}$ with $k = k_1 + \cdots + k_\ell$ ($k_s \neq 0, s = 1, \ldots, \ell$) and no splitting of the energies $E_{k_1}, \ldots, E_{k_\ell}$ will be possible. Therefore, (a) (compactness) must occur for the minimization problem (5.3) for $k = k_1, \ldots, k = k_\ell$. In other words, (5.3) is solvable for $k = k_1, \ldots, k = k_\ell$ and (5.16) is established as well.

6. Least-Positive-Energy and Unit-Charge Solitons

Using the inequality $E_k \geq 4\pi|k|$ (cf. (5.6)), we see that $\{E_k\}_{k\in\mathbb{Z}\setminus\{0\}} \subset [4\pi, \infty)$ and that there is an integer $k_0 \geq 1$ such that

$$E_{k_0} = \min\{E_k \mid k \in \mathbb{Z} \setminus \{0\}\}. \tag{6.1}$$

That is, E_{k_0} is the least possible positive energy of the 2D Skyrme model (5.1). For this energy value, we have

Theorem 6.1. *The least positive energy E_{k_0} of the 2D Skyrme model is attainable. In other words, for $k = k_0$, the minimization problem* (5.3) *has a solution.*

Proof. We use Theorem 5.2. If (a) (compactness) does not occur when taking $k = k_0$ in the minimization problem (5.3), then in view of Theorem 5.2 we can find two nonzero integers k_1 and k_2 such that $E_{k_0} \geq E_{k_1} + E_{k_2}$, which is false because $E_{k_1} \geq E_{k_0}$, $E_{k_2} \geq E_{k_0}$, and $E_{k_0} > 0$.

Next, we use Theorem 5.2 to study the attainability of E_1 for the 2D Skyrme model following the substantial inequality method used in Sect. 2 for the Faddeev model.

We can state

Theorem 6.2. *For the 2D static Skyrme model* (5.1), *the energy* E_1 *is attainable provided that the coupling constants* λ *and* μ *satisfy the bound*

$$\lambda\mu \leq 192. \tag{6.2}$$

In other words, the minimization problem (5.3) *is solvable for* $k = \pm 1$ *under the condition* (6.2).

Proof. First recall that we established [13] a stronger version of the topological lower bound (5.6) which states that there is a positive constant $C(\lambda, \mu, k)$ (i.e., the constant only depends on the coupling parameters λ and μ and the nonzero integer k) such that $E(u) \geq 4\pi|\deg(u)| + C(\lambda, \mu, \deg(u))$ ($\deg(u) \neq 0$). In particular, we have

$$E_k > 4\pi|k|, \quad k \neq 0. \tag{6.3}$$

Now assume (6.2). If for $k = 1$ the compactness (the alternative (a)) for a minimizing sequence of (5.3) does not occur, then by Theorem 5.2 there are nonzero integers k_1 and k_2 so that $1 = k_1 + k_2$ and

$$E_1 \geq E_{k_1} + E_{k_2}. \tag{6.4}$$

Since $E_{k_1} > 0$ and $E_{k_2} > 0$, we see from (6.4) that $k_1 \neq \pm 1$ and $k_2 \neq \pm 1$. However, one of the k_1 and k_2 must be odd. Assume k_1 is odd. Then $|k_1| \geq 3$. Since k_2 must be even, so $|k_2| \geq 2$. Using these facts, (5.7), (6.4), and (6.3), we get

$$4\pi\left(1 + \frac{1}{2}\sqrt{\frac{\lambda\mu}{3}}\right) \geq E_1 > 4\pi(3+2), \tag{6.5}$$

which contradicts the condition (6.2).

Note that (6.2) enlarges the range of the product of the coupling parameters λ and μ stated in (5.4) (obtained earlier in [13]) by three times. Thus we see again that the method of substantial inequality is rather powerful.

It may be interesting to know whether the minimization problem for the Faddeev model or the Skyrme model in 3D may be modified by introducing coupling parameters in the energy functional as in the 2D Skyrme model. To answer this question, we use the notation (3.8), modify the Faddeev energy as

$$E_{\lambda\mu}(\mathbf{n}) = \lambda E_{\mathrm{D}}(\mathbf{n}) + \mu E_{\mathrm{S}}(\mathbf{n}), \tag{6.6}$$

where $\lambda, \mu > 0$ are constants, and consider the minimization problem

$$(E_{\lambda\mu})_m = \inf\{E_{\lambda\mu}(\mathbf{n}) \mid E_{\lambda\mu}(\mathbf{n}) < \infty,\ Q(\mathbf{n}) = m\}. \tag{6.7}$$

Then, using the conformal properties of $E_{\mathrm{D}}(\mathbf{n})$ and $E_{\mathrm{S}}(\mathbf{n})$, we can establish the factorization relation $(E_{\lambda\mu})_m = \sqrt{\lambda\mu}E_m$, where E_m denotes the energy infimum stated in (2.3). In other words, the effect of the coupling constants can always be factored away for the minimization problem.

Note that the same relation is also valid for the classical 3D Skyrme model.

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