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On the Structure of Eigenfunctions Corresponding to Embedded Eigenvalues of Locally Perturbed Periodic Graph Operators

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Abstract: The article is devoted to the following question. Consider a periodic self-adjoint difference (differential) operator on a graph (quantum graph) G with a co-compact free action of the integer lattice $\mathbb{Z}^n$. It is known that a local perturbation of the operator might embed an eigenvalue into the continuous spectrum (a feature uncommon for periodic elliptic operators of second order). In all known constructions of such examples, the corresponding eigenfunction is compactly supported. One wonders whether this must always be the case. The paper answers this question affirmatively. What is more surprising, one can estimate that the eigenmode must be localized not far away from the perturbation (in a neighborhood of the perturbation's support, the width of the neighborhood dependent upon the unperturbed operator only). The validity of this result requires the condition of irreducibility of the Fermi (Floquet) surface of the periodic operator, which is known in some cases and is expected to be satisfied for periodic Schrödinger operators.

1. Introduction

Difference equations on graphs and differential equations on quantum graphs, even when they resemble Laplace or Schrödinger operators in many regards, lack one important property of second order elliptic operators, namely uniqueness of continuation. Uniqueness of continuation states that any solution of an elliptic second order equation $Au = 0$ that vanishes on an open set, is identically zero. It is known to be extremely important and has many implications, in particular in spectral theory. It is also known that elliptic equations of higher orders do not necessarily possess such a property [27], which leads to some weird spectral examples as well (e.g., [18, 19]).

This absence of uniqueness of continuation for graph operators leads for instance to the following possibility: a periodic "elliptic second order" operator on a graph (quantum graph) with a co-compact action of an abelian group can have non-empty pure point spectrum (bound states) [17]; this is an absolute no-no in the continuous case, see [18, 29, 33] and references therein. It is easy to explain this effect for instance as follows. Assume that one has a compact graph with an eigenfunction of the discrete Laplacian that vanishes at a vertex. Then one can attach this graph by that vertex to any other graph and extend the function as zero, still keeping it as an eigenfunction. This attachment can also be done in a periodic manner. Such constructions yield these "strange" eigenfunctions generated by compactly supported ones. Indeed, it has been shown that all such bound states on periodic graphs are in fact generated by the compactly supported eigenfunctions [17, 24]. It is interesting to note that the Laplace operator on the Cayley graph of an infinite discrete group can even have solely pure point spectrum [7, 12].

Using the described above attachment procedure, one can also easily construct examples when a localized perturbation of a periodic difference operator does embed an eigenvalue into absolutely continuous spectrum, which is also expected to be impossible in the continuous situation.[1] The aim of this paper is to see what can be said about the eigenfunctions corresponding to such embedded eigenvalues. We show not only that such an eigenfunction must be compactly supported, but that its support must be contained in a finite width neighborhood of the support of the perturbation, the width dependent on the unperturbed operator only. Thus, effect of a localized impurity seems to be of an extremely short range, when on the absolutely continuous spectrum of the periodic background.

In the next section, we introduce the necessary notions and state and prove the main result for the case of periodic combinatorial graphs (Theorem 5). The following section contains formulation and the proof of the analogous result for the quantum graph case (Theorems 10 – 12). The paper ends with a brief section containing some final remarks.

2. Combinatorial Graph Case

Consider an infinite combinatorial graph Γ with the set of vertices V and a faithful co-compact action of the free abelian group $G = \mathbb{Z}^n$ (i.e., the space of G-orbits is a finite graph). In fact, in this section we can think of Γ just as of a discrete set V of vertices. The graph structure is not truly relevant here, albeit the main operators of interest usually come from graphs (e.g., graph Laplacian [5]). Without loss of generality, the reader may think of the graph as a discrete subset of $\mathbb{R}^n$ invariant with respect to all integer shifts. We also consider a G-periodic finite difference operator A of a finite order acting on $l_2(V)$. Here $l_2(V)$ is the space of all square summable functions on Γ (i.e., on V). The words "finite difference operator of a finite order" mean that the value of Au at any vertex v involves the values of u at finitely many other vertices (due to periodicity, the number of these vertices is uniformly bounded). This can be easily expressed in terms of the matrix representation of the operator. Indeed, if $v_j \in V$ are the vertices of Γ, then operator A can be represented by a matrix $A = (a_{ij})$. The finite order condition in the periodic case is equivalent to this matrix having finitely many entries in each row. Such periodic operators are clearly bounded in $l_2(V)$.

We will fix a (finite) fundamental domain W for the action of $G = \mathbb{Z}^n$ on V.

Consider for instance the $\mathbb{Z}^2$-periodic sub-graph of $\mathbb{R}^2$ shown in Fig. 1, with the fundamental domain W indicated. An example of a periodic difference operator here could be the version of the Laplace operator that subtracts from the value of a function at a

[1] This is completely proven in dimensions one only [30, 31] with just a single result in higher dimension available [25, 26].

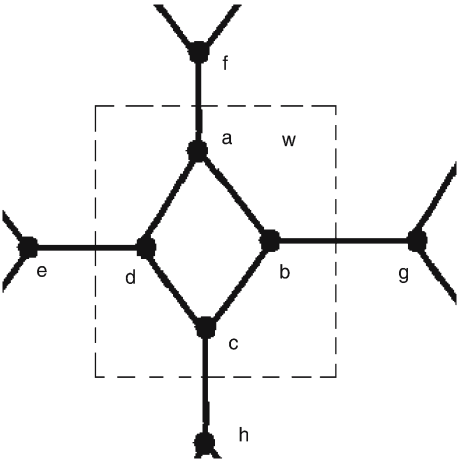

Fig. 1. A periodic graph

vertex its average value at all vertices adjacent to this one: $\Delta f(v) = f(v) - \frac{1}{d_v} \sum\limits_{u \sim v} f(u)$, where d_v is the degree of the vertex v. This operator is clearly a finite difference operator, periodic with respect to the group action.

We will need to measure the sizes of finite subsets $S \subset \Gamma$ by the number and locations of the shifts of the fundamental domain W that are needed to cover S. Given a finite subset S of Γ, we will call its *radius* the number

$$r(S) = \min \left\{ N \in \mathbb{Z}^+ \mid S \subset \bigcup_{\gamma \in [-N,N]^n \subset \mathbb{Z}^n} \gamma W \right\}. \tag{1}$$

We will also need to define two notions of support of a finite difference periodic operator A. First, let v be a vertex of Γ. Then we introduce the notion of the v-support of A as follows

$$supp_v(A) = \{u \in V \mid (A\delta_u)(v) \neq 0\}. \tag{2}$$

Here δ_u is the delta function on V supported at the vertex u, i.e. $\delta_u(v) = \delta_{u,v}$ for $u, v \in V$. To put it differently, the v-support of A consists of all points u, values at which of a function ψ influence the values of $A\psi$ at v. In the terms of the matrix $A = (a_{ij})$, one has $supp_{v_i}(A) = \{v_j \in V \mid a_{ij} \neq 0\}$.

We also define the W-support of A as

$$\begin{aligned} supp_W(A) &= \bigcup_{v \in W} supp_v(A) \\ &= \{u \in V \mid \ A\delta_u|_W \neq 0\}. \end{aligned} \tag{3}$$

In other words, the W-support of A consists of all points u values at which of a function ψ influence the values of $A\psi$ on W.

As always, dealing with a periodic problem, it is advantageous to use the basic transformations of *Floquet theory* (e.g., [18, 29]). Namely, for any compactly supported (or sufficiently fast decaying) function $f(v)$ on V, we define its *Floquet transform* as follows:

$$f(v) \mapsto \hat{f}(v, z) = \sum_{g \in \mathbb{Z}^n} f(gv) z^{-g}, \tag{4}$$

where gv denotes the action of $g \in \mathbb{Z}^n$ on the point $v \in v$, $z = (z_1, \dots, z_n) \in (\mathbb{C}^*)^n = (\mathbb{C}\backslash 0)^n$, and $z^g = z_1^{g_1} \times \cdots \times z_n^{g_n}$. This is clearly just the Fourier transform on the group G of periods.

One can notice the easily verifiable cyclic (or Floquet) identity

$$\hat{f}(gv, z) = z^g \hat{f}(v, z) \tag{5}$$

satisfied for any $v \in V$ and $g \in G$. The vector z is sometimes called *Floquet multiplier*, and if it is represented as $z = e^{ik} = (e^{ik_1}, e^{ik_2}, \dots, e^{ik_n})$, the vector k is said to be the *quasi-momentum* (e.g., [2, 18, 29]).

Relation (5) implies that in order to know all the values of the function $\hat{f}(v, z)$, it is sufficient to know them for only one representative v from each G-orbit, i.e. for v from a fundamental domain of the G-action[2]. Thus, we fix such a fundamental domain W (which is a finite set (graph)) and consider only $v \in W$ in $\hat{f}(v, z)$. We will also denote $\hat{f}(v, z)$ by $\hat{f}(z)$, where the latter expression is a function on W depending on the parameter z.

The following statement is immediate:

Lemma 1. *The images under the Floquet transform of the compactly supported functions on* Γ *are exactly all finite Laurent series*[3] *in* z *with coefficients in* $\mathbb{C}^{|W|}$. *Moreover, for a compactly supported function* f, *the Laurent series of* $\widehat{f}$ *includes only powers* z^g *that satisfy*

$$\|g\|_\infty := \max(|g_j|) \le r(supp(f)), \tag{6}$$

where $r(S)$ *is the radius of a set* S *introduced in* (1).

We will also need the unit torus

$$\mathbb{T}^n = \{z \in \mathbb{C}^n \mid |z_j| = 1, j = 1, \dots, n\} \subset \mathbb{C}^n.$$

It is well known and easy to prove [8, 18, 29] that the Floquet transform (4) extends to an isometry (up to a possible constant factor) between $l_2(V)$ and $L_2(\mathbb{T}^n, \mathbb{C}^{|W|})$.

After the Floquet transform, A becomes the operator of multiplication in $L_2(\mathbb{T}^n, \mathbb{C}^{|W|})$ by a rational $|W| \times |W|$ matrix function $A(z)$. To make this clearer, let us consider the Laplace operator Δ for the graph shown in Fig. 1. We compute the value of Δu on a function u that satisfies the cyclic condition (5). We notice that for such a function, one has $u(f) = z_2 u(c)$, $u(g) = z_1 u(d)$, $u(h) = z_2^{-1} u(a)$, $u(e) = z_1^{-1} u(b)$. Thus, writing the values of u as a vector $(u(a), u(b), u(c), u(d))^t$, the action of Δ on $u|_W$ becomes multiplication by the matrix $A(z)$,

$$\begin{pmatrix} 1 & -1/3 & -1/3z_2 & -1/3 \\ -1/3 & 1 & -1/3 & -1/3z_1 \\ -1/3z_2^{-1} & -1/3 & 1 & -1/3 \\ -1/3 & -1/3z_1^{-1} & -1/3 & 1 \end{pmatrix}. \tag{7}$$

[2] In some cases one has to take a more sophisticated approach and treat $\hat{f}(v, z)$ as a section of a naturally defined (depending on z) line bundle over Γ/G.

[3] By *Laurent series* we mean here expansions into powers z^g with $g \in G = \mathbb{Z}^n$.

In other words, $A(z)$ is the restriction of A to the space of all (not square summable) functions f satisfying the cyclic condition (5).

To formulate our result, we need to introduce another notion.

Definition 2. *Let* $\lambda \in \mathbb{C}$. *We call* **the Floquet surface** $\Phi_{A,\lambda} \subset (\mathbb{C}^*)^n$ **of the operator** A **at the energy** λ *the set of all* $z \in (\mathbb{C}^*)^n$*, such that the matrix* $A(z) - \lambda$ *is not invertible* (*i.e.,* $\det(A(z) - \lambda) = 0$).

The term *Floquet surface* is non-standard. If one considers quasi-momenta k instead of the Floquet multipliers z, one arrives at the standard in solid state physics and theory of periodic equations notion of *Fermi surface* $F_{A,\lambda}$ [2, 18]. So, the Floquet surface is the Fermi one with the natural periodicity with respect to quasimomenta k being factored out.

It is clear from the definition that the Floquet surface for a periodic difference operator is an algebraic subset[4] of dimension $n-1$ in $\mathbb{C}^n$. We also look at its intersection with the torus $\mathbb{T}^n$,

$$\Phi^R_{A,\lambda} = \Phi_{A,\lambda} \cap \mathbb{T}^n,$$

which we will call the *real Floquet surface*. The name comes from the fact that it corresponds to real quasimomenta from the Fermi surface.

The following standard fact [8, 18, 29] is easy to prove:

Lemma 3. *The point* λ *belongs to the spectrum of the operator* A *if and only if* $\Phi^R_{A,\lambda} \neq \emptyset$.

We will also need to introduce some additional notions originating from the solid state physics [2]. Consider for any $z \in \mathbb{T}^n$ the defined above finite dimensional self-adjoint operator $A(z)$. It has a finite spectrum $\{\lambda_j(z)\}$, which can be considered as the graph of a multiple-valued function $\sigma(A(z))$. This function is said to be the *dispersion relation* and its graph the *dispersion curve*. The preceding lemma says that the range of this function coincides with the spectrum of A in $l_2(\Gamma)$. Arranging the eigenvalues in non-decreasing order splits this curve into continuous (in fact, piecewise-analytic [18, 29, 35]) branches $\lambda_j(z)$. Their ranges are finite closed segments of the spectral axis called *spectral bands*, union of which comprises the whole spectrum $\sigma(A)$. This is the so-called band-gap structure of the spectrum [8, 18, 29].

The (complex) Floquet surface $\Phi_{A,\lambda}$ is never empty. When λ changes, it moves around. The lemma says that, whenever $\Phi_{A,\lambda}$ hits the torus $\mathbb{T}^n$, λ belongs to the spectrum. It is natural to expect that when λ is a generic point in the interior of the spectrum, then the real Floquet surface will be a variety of the maximal possible real dimension $n-1$ in the torus. This is confirmed by the following statement.

Lemma 4. *If* λ *belongs to the interior of a spectral band of the operator* A*, then the real Floquet surface* $\Phi^R_{A,\lambda}$ *has a part that is a smooth* $n-1$*-dimensional hyper-surface in* $\mathbb{T}^n$.

Proof of the Lemma. Let λ belong to the interior of the band formed by the branch $\lambda_j(z)$. Then function $\lambda_j(z) - \lambda$ changes sign on $\mathbb{T}^n$. Thus, the real Floquet surface separates $\mathbb{T}^n$. Since the Floquet surface is an analytic set[5], this implies the conclusion of the lemma (see more details of this simple part of the argument in [25]). □

[4] Its analog for quantum graphs will be only analytic, not algebraic.

[5] In fact, in the case of a discrete graph that we currently consider, it is even algebraic. This, however, will change to analyticity only in the case of quantum graphs.

In what follows, we will need to assume that the Floquet surface $\Phi_{A,\lambda}$ is irreducible as an analytic variety[6]. This condition does not necessarily hold in general, but it has been conjectured that it is always true if A is the discrete Schrödinger operator on $\mathbb{Z}^2$ with a potential periodic with respect to a sublattice [11]. This is probably also true in any dimension. It was shown in [11] that in $2D$ irreducibility holds for all but finitely many values of the spectral parameter λ. Examples of some separable cases when irreducibility has been proven can be found in [3, 11, 25, 26].

After all this preparation, let us now move to the formulation of the main problem being addressed in this paper. Consider any *local* difference operator B, i.e. such that its action on a function u involves only the function's values on a finite set $S \subset \Gamma$ and the resulting function Bu is supported on S as well. In terms of the matrix $B = (b_{ij})$ this means that it has only finitely many non-zero entries. We are interested in the perturbation of the spectrum of A that occurs when the operator is perturbed by adding B: $A + B$. If we assume at this point that A is self-adjoint, it is a general fact that an additional point spectrum might arise (e.g., [29]). In the case of second order elliptic periodic PDEs, it is also the common expectation that this impurity spectrum should not be embedded into the continuous spectrum of A. This is proven for localized perturbations of a homogeneous background (see the book [9] for a detailed survey, as well as [6]). In the case of localized perturbations of a periodic background, absence of embedded eigenvalues is proven for periodic Schrödinger operators in $1D$ [30, 31]. Albeit the same must surely be true in any dimension, the problem in dimensions higher than 1 is hard and only one limited result is known [25, 26]. In the discrete (graph) situation, embedded eigenvalues can arise, due to non-trivial graph topology. Examples of such compactly supported eigenfunctions can be easily constructed using the attachment procedure described before. One might want to ask whether compactness of support of the eigenfunctions corresponding to embedded eigenvalues is the only possibility, and if yes, whether there are any *a priori* bounds on the size of their supports. A somewhat surprising answer is given by the following result.

Theorem 5. *Let B be a local perturbation supported on a finite set $S \subset \Gamma$ (i.e., $supp(Bf) \subset S$ for any f) of a periodic operator A. Let λ belong to the interior of a spectral band of A, the corresponding Floquet surface be irreducible, and λ be an embedded eigenvalue for $A + B$. Then the corresponding eigenfunction $f \in l_2(V)$ of $A + B$ is compactly supported and moreover,*

$$r(supp\ f) < r(S) + r(supp_W(A))(2|W| + 1)).$$

Here $supp_W(A)$ is defined in (2).

So, the effect of the impurity seems to be of very short range. This theorem will be deduced from the following more general statement:

Theorem 6. *Let λ belong to the interior of a spectral band of A, the corresponding Floquet surface be irreducible, and ψ be a compactly supported function on the graph. Assume that the equation $Au - \lambda u = \psi$ has an l^2-solution u. Then $u \in l_2(V)$ is compactly supported and moreover,*

$$r(supp\ f) < r(supp\ \psi) + r(supp_W(A))(2|W| + 1)).$$

[6] We remind the reader what this means: it cannot be represented as the union of two strictly smaller analytic varieties.

Remark 7. The constant $r(supp_W(A))(2|W|+1))$ in the previous two theorems can be often improved for specific periodic difference operators A.

Proof of Theorem 6. Since function ψ is compactly supported, its Floquet transform $\widehat{\psi}(v,z) = \sum \psi_g z^g$ is a Laurent polynomial with degrees g bounded by $\|g\|_\infty := \max_i |g_i| \leq r := r(supp(\psi))$. Let us denote by $\mathbf{e}$ the vector $(1,\dots,1) \in \mathbb{Z}^n$ and introduce $R := r(supp_W(A))$. We can represent $\widehat{\psi}(z)$ as $z^{-r\mathbf{e}}\phi(z)$, where $\phi(z) = z^{r\mathbf{e}}\widehat{\psi}(z)$ is a polynomial that involves only (non-negative) degrees $g \in \mathbb{Z}^n$ with $\|g\|_\infty \leq 2r$.

Taking the Floquet transform in Eq. (10), we rewrite it as

$$(A(z)-\lambda)\widehat{f}(z) = z^{-r\mathbf{e}}\phi(z). \tag{8}$$

We can rewrite the Laurent matrix function $A(z)-\lambda$ as $z^{-R\mathbf{e}}A_1(z,\lambda)$, where the matrix function $A_1(z,\lambda)$ is a polynomial in z involving only the powers z^g with $\|g\|_\infty \leq 2R$. Then its inverse can be represented as $z^{R\mathbf{e}}\frac{B(z)}{\Delta(z)}$, where $B(z)$ is a matrix polynomial (transposed to the co-factor matrix of A_1) and $\Delta(z)$ is a scalar polynomial (determinant of A_1), which vanishes exactly on the Floquet surface. Thus,

$$\widehat{f}(z) = z^{(R-r)\mathbf{e}}\frac{B(z)\phi(z)}{\Delta(z)}. \tag{9}$$

Notice that B involves only powers z^g with $\|g\|_\infty \leq 2R(|W|-1)$. We know that $\widehat{f}(z)$ is an L^2-function on $\mathbb{T}^n$. On the other hand, the right hand side of (9) is, up to the factor $z^{(R-r)\mathbf{e}}$, the ratio of two holomorphic polynomials in $\mathbb{C}^n$. We also know that zeros of the denominator $\Delta(z)$ in $(\mathbb{C}^*)^n$ are irreducible and intersect the torus $\mathbb{T}^n$ over an $(n-1)$-dimensional variety. This means that unless the numerator $B\phi$ vanishes on $\mathbb{T}^n$ at these zeros to their degrees, the ratio has a singularity that is not square integrable on the torus. Thus, the numerator vanishes to that degree, and due to the irreducibility of zeros, the same is true for all zeros in $(\mathbb{C}^*)^n$ (see [25] for the details of this simple argument). If there were no zeros of the denominator in $\mathbb{C}^n \setminus (\mathbb{C}^*)^n$, then, as a corollary of Hilbert's Nullstellensatz, the ratio would be a holomorphic polynomial of z. We cannot, however, exclude existence of a factor like z^q, $q \in (\mathbb{Z}^+)^n$ in $\Delta(z)$. If it does exist, we have $\|q\|_\infty \leq 2R|W|$ (since each term in Δ is like that). Factoring this power out, we represent Δ as $z^q\Delta_1(z)$, where zeros of Δ and Δ_1 in $(\mathbb{C}^*)^n$ are the same (including their orders), and thus our ratio is a holomorphic polynomial times z^{-q}. Notice that division does not increase the degree of a polynomial with respect to any variable. The degree of $\phi(z)$ has been estimated as $\|g\|_\infty \leq 2r$. The additional degree acquired during multiplication by B and division by Δ_1 does not exceed $2R(|W|-1)$. Thus, the ratio $B\phi/\Delta_1$ is a polynomial involving the degrees z^g with $\|g\|_\infty \leq 2r + 2R(|W|-1)$ only. One calculates now that the effect of the outside factor of $z^{(R-r)\mathbf{e}}$ and of z^{-q} coming from the denominator is to reduce the expression to a Laurent polynomial with degrees z^g, $g \in \mathbb{Z}^n$ such that $\|g\|_\infty \leq r + R(2|W|+1)$. We see that $\widehat{f}(z)$ is a Laurent polynomial which contains powers of z estimated by $r(\psi) + r(supp_W(A))(2|W|+1)$. Reversing the Floquet transform, we get the statement of the theorem. □

Let us now address the **proof of Theorem 5**, which is rather simple. Indeed, the conditions of the theorem imply the equality $Af + Bf = \lambda f$, or in a form more convenient for us,

$$(A-\lambda)f = -Bf := \psi. \tag{10}$$

The function $\psi(v)$ is supported on S. Thus Theorem 6 applies and proves the statement. □

3. Quantum Graph Case

We now switch to the case of a perturbed periodic quantum graph. We will remind of the reader of the main definitions concerning quantum graphs[7]. A quantum graph Γ has each its edge e equipped with a coordinate x_e (when no confusion is possible, we use just x instead). This coordinate identifies e with a segment $[0, l_e]$ of the real line. We will also assume that a Schrödinger operator $H = -\frac{d^2}{dx^2} + V(x)$, $V \in L^2_{loc}(\Gamma)$ with appropriate vertex conditions (all such self-adjoint conditions are described in [14, 16, 22]) is defined on Γ. The results of this section hold for any such conditions, however just for simplicity of presentation we will assume that the conditions at each vertex are the "standard" Neumann-Kirchhoff ones:

$$f \text{ is continuous and } \sum \frac{df}{dx_e} = 0 \text{ at each vertex,} \tag{11}$$

where the sum is taken over all edges incident with the vertex and the derivatives are taken away from the vertex.

As in the previous section, we assume that the graph is acted upon freely and co-compactly by the group $G = \mathbb{Z}^n$ that leaves the graph structure (including edges' lengths) and the Hamiltonian H invariant. We use the same letter W as before for a fundamental domain of this action.

One can now introduce the notions of the Floquet transform and Floquet variety of H analogously to the way it was done in the previous section. For instance,

Definition 8. *Let $\lambda \in \mathbb{C}$. We call* **the Floquet surface** $\Phi_{H,\lambda} \subset (\mathbb{C}^*)^n$ **of the operator H at the energy** λ *the set of all $z \in (\mathbb{C}^*)^n$ such that the equation $(H - \lambda)u = 0$ has a non-trivial solution u that is cyclic with the Floquet multiplier z, i.e. such that $u(gx) = z^g u(x)$, where $x \in \Gamma$ and $g \in \mathbb{Z}^n$. Here, as before, $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$.*

The following statement is standard in Floquet theory and can be proven the same way as for elliptic periodic PDEs [18] (or by reduction to the discrete case, as described below).

Lemma 9. *The Floquet surface $\Phi_{H,\lambda}$ is an analytic subset of $(\mathbb{C}^*)^n$.*

Due to this lemma, the notion of irreducibility of the Floquet surface makes sense.

The main result of this section is the following quantum graph analog of Theorem 5:

Theorem 10. *Let $w(x) \in L^2(\Gamma)$ be supported on a finite set S of edges. Let λ belong to the interior of a spectral band of H, the corresponding Floquet surface be irreducible, and λ be an embedded eigenvalue for $H + w$. Then the corresponding eigenfunction $f \in L_2(\Gamma)$ of $H + w$ is compactly supported and moreover,*

$$r(supp\ f) < r(\widetilde{S}) + C.$$

Here C is a constant depending on the unperturbed operator H only and for any set of vertices S we define $\widetilde{S}$ as the set of all vertices that are either in S, or adjacent to the ones in S.

It is possible to give some explicit estimates for the constant C, similarly to how it was done for the discrete case. However, the situation depends on whether or not λ is the Dirichlet eigenvalue of H on an edge of the graph. Here by a Dirichlet eigenvalue of H on an edge e we mean an eigenvalue of the operator $-\frac{d^2}{dx^2} + V(x)$ on $[0, l_e]$ with zero Dirichlet conditions at the ends.

[7] One can find more details in [16, 20–24, 28].

Theorem 11. *Let $w(x) \in L^2(\Gamma)$ be supported on a finite set S of edges. Assume that λ belongs to the interior of a spectral band of H and is not a Dirichlet eigenvalue on any of the edges, the corresponding Floquet surface is irreducible, and λ is an embedded eigenvalue for $H + w$. Then the corresponding eigenfunction $f \in L_2(\Gamma)$ of $H + w$ is compactly supported and moreover,*

$$r(supp\ f) < r(\widetilde{S}) + r(\widetilde{W})(2|\widetilde{W}| + 1).$$

Here we consider W as a set of vertices.

In the case when λ does belong to the Dirichlet spectrum of at least one of the edges, the situation is different, and one needs to modify the graph somewhat. We would like to guarantee that λ does not belong to the Dirichlet spectra of H and of $H + w$ on any of the edges of the graph. This is easy to achieve by introducing "fake" vertices. Indeed, if all the edges are sufficiently short, this condition is satisfied. Now, modulo the periodicity, there are only finitely many edges in the graph. Hence, one can introduce a finite set of periodic families of interior points on the edges, such that including these points as new vertices of degree two, one makes the lengths of all edges sufficiently small, so λ is below the Dirichlet spectra of both operators on any edge. If one imposes Neumann-Kirchhoff conditions (11) at these new vertices, which in the case of a vertex of degree two just means enforcing continuity of the function and its derivative, these additional vertices do not influence the spectra of H and of $H + w$ at all. This reduces the situation to the case of Theorem 11, however with an increased number of vertices in the fundamental domain. Let us call this new set of vertices in the fundamental domain W_1. Then Theorem 11, if proven, implies the next theorem, and thus also Theorem 10:

Theorem 12. *Let $w(x) \in L^2(\Gamma)$ be supported on a finite set S of edges. Assume that λ belongs to the interior of a spectral band of H and is not a Dirichlet eigenvalue on any of the edges, the corresponding Floquet surface is irreducible, and λ is an embedded eigenvalue for $H + w$. Then the corresponding eigenfunction $f \in L_2(\Gamma)$ of $H + w$ is compactly supported and moreover,*

$$r(supp\ f) < r(\widetilde{S}) + r(\widetilde{W_1})(2|\widetilde{W_1}| + 1).$$

Proof of Theorem 11. (and therefore also of Theorems 12 and 10) is based upon its reduction to its discrete version given in Theorem 5.

Assume that one solves the following problem on the graph:

$$\begin{cases} -\frac{d^2 f}{dx^2} + V(x) f = \lambda f \text{ on each edge} \\ \text{f is continuous and } \sum \frac{df}{dx_e} = 0 \text{ at each vertex.} \end{cases} \tag{12}$$

Since we are guaranteed that a neighborhood of λ is free of Dirichlet spectra of individual edges, one can use the standard procedure of reducing the spectral problems for H and for $H + w$ for the quantum graph to the one for a combinatorial one (e.g., [1, 4, 10, 22, 24]). This is how this is done. Consider an edge e and identify it with the segment $[0, l_e]$. Since λ is not in the Dirichlet spectrum on the edge, one can solve uniquely the first equation of (12) on this edge, assuming that the values $f(0)$, $f(l_e)$ of the function f at the ends of the edge are known. The resulting function along the edge can be represented as

$$f(x) = f(0)g_0(\lambda, x) + f(l_e)g_1(\lambda, x), \tag{13}$$

where g_0 (g_1) takes value 1 at 0 and 0 at l_e (correspondingly 0 at 0 and 1 at l_e). The functions $g_j(\lambda, x)$ are meromorphic with respect to λ with singularities at the Dirichlet spectrum of the edge only. In particular, they are analytic in the region of our interest. Now one obtains the derivatives of the function f at the vertices incident to e as follows:

$$\begin{aligned} f'(0) &= f(0)g_0'(\lambda, 0) + f(l_e)g_1'(\lambda, 0), \\ f'(l_e) &= f(0)g_0'(\lambda, l_e) + f(l_e)g_1'(\lambda, l_e). \end{aligned} \tag{14}$$

One can do this on each edge. If now f is defined on each edge according to the formulas (13), then the edge equations in (12) are satisfied. The only condition in (12) to be satisfied is the one requiring that the outgoing derivatives at each vertex add up to zero. Substituting into this condition the expressions of the derivatives from (14), one obtains an equation on the vertex values $f(v)$ of the form

$$\sum_{v\sim u} a_{u,v}(\lambda) f(v) = 0, \tag{15}$$

where functions $a_{u,v}(\lambda)$ are meromorphic with poles at the edges' Dirichlet eigenvalues and are non-zero for adjacent pairs of vertices (u, v) only. One sees that this can be written as a second order difference equation $A(\lambda)f = 0$ on the combinatorial graph.

To make this clearer, let us consider an example of a graph whose edges are all of the same length l and assume zero potential $V(x)$ in (12). Then (13) and (14) take the following forms correspondingly:

$$f(x) = \frac{1}{\sin\lambda l}\left(f(0)\sin\lambda(l - x) + f(l)\sin\lambda x\right), \tag{16}$$

$$f'(0) = \frac{\lambda}{\sin\lambda l}\left(-f(0)\cos\lambda l + f(l)\right). \tag{17}$$

Thus, the whole problem (12) boils down to the equation

$$\frac{\lambda}{\sin\lambda l}\left(d_v\cos\lambda l f(v) - \sum_{u\sim v} f(u)\right) = 0 \tag{18}$$

satisfied at each vertex v, where $\sim$ denotes adjacency of vertices. Thus, in this case the only non-zero matrix elements of $A(\lambda)$ are: $a_{vv} = d_v\lambda\cot\lambda l$ for any vertex v and $a_{vu} = -\frac{\lambda}{\sin\lambda l}$ for adjacent vertices u and v.

Notice that the matrix $A(\lambda)$ is not algebraic, but analytic with respect to λ. This construction also shows that $supp_W(A(\lambda)) \subset \widetilde{W}$.

Analogously, the perturbed equation can be rewritten as $A_1(\lambda)f = 0$. This leads to the two combinatorial counterparts of our periodic and perturbed spectral problems:

$$A(\lambda)f = 0,\ A_1(\lambda)f = 0. \tag{19}$$

In order to prove the theorem, we will need some simple auxiliary statements collected in the following:

Lemma 13. 1. *If a function f on the quantum graph satisfies $Hf = \lambda f$ (resp. $(H + w)f = \lambda f$), then its vertex values satisfy the difference equations $A(\lambda)f = 0$ (resp. $A_1(\lambda)f = 0$). Conversely, if a vector f of vertex values satisfies $A(\lambda)f = 0$ (resp. $A_1(\lambda)f = 0$), it can be uniquely extended to a solution of $Hf = \lambda f$ (resp. $(H + w)f = \lambda f$).*

2. *If the values of such a solution f at both vertices of an edge are equal to zero, then f is zero on this edge. In particular, f is compactly supported if and only if its vertex values are compactly supported, and both supports are of equivalent sizes* (*i.e., their radii are the same*).
3. *The operator $A(\lambda)$ is periodic.*
4. *The difference operator $B = A_1(\lambda) - A(\lambda)$ is supported only on the vertices that are incident to the edges where w has a non-empty support. In particular, $A_1(\lambda)$ is a compactly supported perturbation of $A(\lambda)$ with the size of the support of the perturbation controlled by the size of the support of w.*
5. *The Floquet surfaces satisfy the following relation:*

$$\Phi_{H,\lambda} = \Phi_{A(\lambda),0}. \tag{20}$$

The proof of the lemma is rather straightforward. Indeed, the way the operators A and A_1 are defined, implies the direct part of the first claim of the lemma. The converse part is also simple. Indeed, if a vector f of vertex values satisfies $A(\lambda)f = 0$, let us solve the equation $Hu = \lambda u$ on each edge taking f as Dirichlet boundary values (this is possible due to our avoidance of Dirichlet spectra). The resulting function satisfies the equations on the edges and continuity condition by construction. The Neumann condition at the vertices is now equivalent to $A(\lambda)f = 0$.

The second claim of the lemma follows from the same avoidance of the Dirichlet spectra.

The third statement follows from periodicity of H.

The fourth statement is straightforward from the definitions of $A(\lambda)$ and $A_1(\lambda)$.

Let us prove the important (albeit still simple) last statement. If $z \in \Phi_{H,\lambda}$, this means, by the definition of $\Phi_{H,\lambda}$, that there exists a non-zero function f satisfying the equation $Hf = \lambda f$ and such that $f(gx) = z^g f(x)$ for any $g \in \mathbb{Z}^n$. Thus, as explained before the lemma, the vector u of vertex values of f satisfies the equation $A(\lambda)u = 0$. The cyclic relation $f(gx) = z^g f(x)$ in particular holds at the vertices, which implies that $z \in \Phi_{A(\lambda),0}$. Conversely, if $z \in \Phi_{A(\lambda),0}$, then there is a cyclic vertex function u with the Floquet multiplier z such that $A(\lambda)u = 0$. Let us use it as Dirichlet data on each edge to solve $-\frac{d^2 f}{dx^2} + V(x)f = \lambda f$ on each edge. The first claim of the lemma guarantees that we get a solution f of $Hf = \lambda f$. We claim that f is cyclic with the Floquet multiplier z. Indeed, for any $g \in \mathbb{Z}^n$ the functions $f(gx)$ and $z^g f(x)$ satisfy the same equation $-\frac{d^2 f}{dx^2} + V(x)f = \lambda f$ on each edge and have the same Dirichlet data. Since we avoided Dirichlet spectrum, we conclude that $f(gx) = z^g f(x)$. This proves the lemma. □

We can now finish the proof of the Theorem 11. Indeed, the previous lemma guarantees that switching from the differential periodic and perturbed problems $Hf = \lambda f$ and $(H + w)f = \lambda f$ to the combinatorial problems $A(\lambda)f = 0$ and $A_1(\lambda)f = 0$, one lands in the conditions of Theorem 5. Now the same lemma implies that the conclusion of Theorem 5 about the vertex values implies the conclusion of Theorem 10 about the whole function f. □

4. Remarks

- The notion of the "radius" $r(S)$ of a finite set in Γ depends on the choice of a fundamental domain W. Indeed, choosing W further away from S increases $r(S)$. Thus, the optimal way to use the estimates of the main theorems is to choose a fundamental domain W in such a way that $r(S)$ is the smallest possible for a given support S of the perturbation. This would lead to the best localization estimate for the embedded eigenfunctions.

- As it has already been mentioned in the previous section, the Neumann-Kirchhoff conditions (11) are chosen for simplicity of exposition sake only. Since the results concerning combinatorial graphs are obtained under very general conditions on the periodic operator A, the statement of Theorem 10 and its proof carry through for the general self-adjoint vertex conditions (described for instance in [14, 16, 22]). One might have to deal with matrix difference operators A though, which causes no problem. However, the specific estimates of the constant C of Theorem 10 given in Theorems 11 and 12 will have to change depending on the vertex conditions.
- A deficiency of the results of this paper (as well as of the results of [25, 26]) is that our technique does not let us treat the case of eigenvalues embedded at spectral edges.
- It is clear from both this paper and [25, 26] that question of irreducibility of the Floquet surface (equivalently, of the Fermi surface, modulo natural periodicity) is intimately related to the problem of existence and behavior of embedded eigenvalues and corresponding eigenfunctions. This does not look like an artifact of the techniques used. It is clear that not arbitrary periodic difference operator satisfies this condition (also higher order periodic elliptic differential operators do not necessarily do [18]). As we have mentioned before, the book [11] contains both positive results and conjectures concerning irreducibility.

  Irreducibility is also known for operators (both discrete and continuous) with separable potentials [3, 11] (see also [25]). For instance, in dimension three it is sufficient that the potential separates as $V_1(x_1) + V_2(x_2, x_3)$ [3, 11, 25]. This can be deduced from the known results on irreducibility of Bloch variety in dimension two [15].

  An advantage of dealing with a difference operator is a possibility of sometimes explicitly computing the determinant $\Delta(z)$ and thus checking its irreducibility.

  In fact, examination of the proofs of this text, as well as of [25, 26] shows that we do not need complete irreducibility. What is truly required, is that every irreducible component of the Floquet variety intersects the torus $\mathbb{T}^n$ over an $n-1$ dimensional set. However, it is not clear how to control this property, and thus it is doubtful that such a weaker condition will work better in specific examples, than the full irreducibility.
- As it has been mentioned already, pathologies like pure point spectrum of periodic operators and embedded eigenvalues might and do sometimes appear in a discrete or quantum graph situation. However, they do not necessarily have to. Indeed, it is known [17] that the discrete Schrödinger operator $-\Delta + V(x)$ on the lattice $\mathbb{Z}^n$ with a potential periodic with respect to a sublattice, has absolutely continuous spectrum. This can be proven by L. Thomas' standard argument [33]. Similarly, there are some cases when one can prove that embedded eigenvalues do not arise from local perturbations of periodic discrete operators. Assume for instance that a difference operator P on the integer lattice $\mathbb{Z}^n$ (the operator could in particular be our perturbed operator $A + B$) has the following property: there exists an oriented hyperplane L in $\mathbb{R}^n$ such that for any point $y \in \Gamma$ there exists a point $x \in \Gamma$ such that $supp_x(P)$ contains the point y and lies completely on the "positive" side of the parallel shift L_y to the point y, with the only intersection with L_y at y. Then the equation $Pf = 0$ has no compactly supported solutions. Indeed, if there were such a solution f, consider a support hyperplane L_y to $supp(f)$ such that the whole $supp(f)$ is on the negative side of L_y and $y \in supp(f)$. Consider the point x that serves y as described above. Then the equality $(Pf)(x) = 0$ clearly implies that $f(y) = 0$, which is a contradiction.

This in particular proves the quoted above statement about absence of point spectrum for periodic Schrödinger operators on integer lattices.

- It would be interesting to understand how much the assumption of commutativity of the group of periods influences the validity of the results of this paper. We do not know the answer to this question, but one probably should not expect to be able to go beyond the class of groups of polynomial growth (and hence, according to M. Gromov's result [13], virtually nilpotent ones). Indeed, the results already quoted about the unusual spectral behavior of the lamplighter group (which is of an intermediate growth) [7, 12] show that one might expect surprises there.
- The approach used in this work has been previously used by the authors in different circumstances in [25, 26] (see also [32]). Its idea originates from the paper [34] of the second author.

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