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Uniform Decay of Local Energy and the Semi-Linear Wave Equation on Schwarzschild Space

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This paper is dedicated to the memory of Hope Machedon

Abstract: We provide a uniform decay estimate for the local energy of general solutions to the inhomogeneous wave equation on a Schwarzschild background. Our estimate implies that such solutions have asymptotic behavior $|\phi| = O\left(r^{-1}\big|t - |r^*|\big|^{-\frac{1}{2}}\right)$ as long as the source term is bounded in the norm $(1-\frac{2M}{r})^{-1}\cdot(1+t+|r^*|)^{-1}L^1\left(H^3_\Omega(r^2dr^*d\omega)\right)$. In particular this gives scattering at small amplitudes for non-linear scalar fields of the form $\Box_g\phi = \lambda|\phi|^p\phi$ for all $2 < p$.

1. Introduction

In this paper our goal is to give a somewhat elementary discussion of the global decay properties of general solutions to the scalar wave equation on the exterior of a Schwarzschild black hole. That is, we consider the manifold with boundary:

$$\mathcal{M} = \mathbb{R}\times\ [2M,\infty)\times\mathbb{S}^2, \tag{1}$$

with Lorentzian metric:

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2d\omega^2, \tag{2}$$

and we look at smooth functions ϕ which do not touch the boundary of (1) for each *fixed* value of the parameter t and which satisfy the inhomogeneous wave equation:

$$\Box_g\phi = \nabla^\alpha\partial_\alpha\phi = G.$$

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The main question we would like to answer here is: How quickly does the local energy of the wave ϕ dissipate over compact sets in the r coordinate, and how precisely does the dissipation depend on the source G?

In the special case of Minkowski space, $M = 0$, a quite satisfactory answer to this question is known. Here one has the classical uniform local energy decay estimate of C. Morawetz:

$$\int_{\mathbb{R}^3\times\{t\}} \left((1+\underline{u}^2)|L\phi|^2 + (1+u^2)|\underline{L}\phi|^2 + (1+u^2+\underline{u}^2)|\nabla\!\!\!/\phi|^2 + \frac{1+\underline{u}^2+u^2}{r^2}|\phi|^2\right) dx$$
$$\lesssim \left(\int_0^t \| (1+|\underline{u}|+|u|)G(s)\|_{L^2(dx)}\, ds\right)^2$$
$$+ \int_{\mathbb{R}^3\times\{0\}} (1+r^2)|\nabla_{t,x}\phi|^2\, dx\ . \qquad (3)$$

Here one sets $u = (t-r)$, $\underline{u} = (t+r)$, $L = 2\partial_{\underline{u}}$, and $\underline{L} = 2\partial_u$. For the original proof see the paper of Morawetz [10], and for an alternative proof as well as many generalizations, see the recent work [7].

The beauty of the estimate (3) is that it gives one a huge amount of information about the global dispersive properties of the function ϕ. For one, it produces a *pointwise* in time decay of the L^2_{loc} norm as well as the local energy. What's more, this local decay is given in such a way that it is clear what is happening on the whole of each time slice $t = const.$, even very far away from the origin $r = 0$. In fact, using Sobolev embeddings and *only* rotations, the Morawetz estimate is good enough to provide uniform decay at the rate of $(1+t)^{-1}$. However, perhaps the most important property of the estimate (3) is that it turns out to be incredibly useful when dealing with non-linear problems. This is because it places very simple conditions on the source term G, the kind which are relatively straightforward to recover in bootstrapping arguments given the form of the left-hand side of (3). For instance, (3) makes dealing with the global existence problem for small amplitude non-linear scalar fields of the form:

$$\Box\phi = \lambda\, |\phi|^p\phi, \qquad (4)$$

essentially trivial in the case where $2 < p$. All one has to do is to combine (3) with appropriately localized Sobolev embeddings to yield the decay estimate:

$$|\phi| \lesssim (1+r)^{-1}\cdot \min\left\{\frac{r^{\frac{1}{2}}}{1+|t-r|},\ \frac{1}{1+|t-r|^{\frac{1}{2}}}\right\},$$

which is enough to integrate the nonlinearity on the right-hand side of (4) when it appears on the right-hand side of (3). In fact, if one takes into account characteristic estimates of the form (3), see again [7], then it is possible to push the exponent p to certain values $p \leqslant 2$. We will not discuss such refinements here.

What we will show here is that for the more general case of $M \neq 0$, an estimate which is essentially of the form (3) holds in the case of general Schwartz (on each fixed time slice) functions ϕ. The proof we give is a relatively straightforward integration by parts, similar in spirit to how one proves (3). In the final section of the paper, we indicate how our estimates can be used to give a short proof of global existence and decay for non-linear scalar fields of the form (4) when $2 < p$.

To state our main theorem, we will use the following notation. We first reparametrize the radial variable in the usual way:

$$r^* = r + 2M \ln(r - 2M), \tag{5}$$

and then introduce the optical functions and null-generators for the coordinates (t, r^*, ω):

$$\underline{u} = (t + r^*), \qquad u = (t - r^*), \tag{6a}$$
$$L = \partial_t + \partial_{r^*}, \qquad \underline{L} = \partial_t - \partial_{r^*}. \tag{6b}$$

We will prove that:

Theorem 1.1 (Uniform Local Energy Decay for the Scalar Wave Equation on Schwarzschild Space)**.** *Let* (t, r^*, ω) *be the coordinates* (*as defined above*) *on the manifold* $\mathcal{M}$ (1) *with metric* (2)*. Let* ϕ *be a smooth function compactly supported on each hypersurface* $t = const.$ *and set:*

$$\Box_g \phi = G.$$

Then one has the following global estimate:

$$\begin{aligned}
&\int_{\mathbb{R}\times\mathbb{S}^2\times\{t\}} \Big((1+\underline{u}^2)(L(r\phi))^2 + (1+u^2)(\underline{L}(r\phi))^2 \\
&\quad +(1+\underline{u}^2+u^2)\Big(1-\frac{2M}{r}\Big)\cdot\Big[\frac{1}{r^2}|\nabla\!\!\!/_\omega(r\phi)|^2 + \frac{M}{r^3}(r\phi)^2\Big]\Big) dr^* d\omega \\
&\lesssim \Big(\int_0^t \|(1+|\underline{u}|+|u|)\Big(1-\frac{2M}{r}\Big) r\cdot\Big(\sqrt{1-\Delta_{sph}}G\Big)(s)\|_{L^2(dr^*d\omega)}\, ds\Big)^2 \\
&\quad + \int_{\mathbb{R}\times\mathbb{S}^2\times\{0\}} (1+(r^*)^2)\Big[\big|\nabla_{t,r^*}\sqrt{1-\Delta_{sph}}(r\phi)\big|^2 \\
&\quad + \Big(1-\frac{2M}{r}\Big)\Big(\frac{1}{r^2}\big|\nabla\!\!\!/_\omega(\sqrt{1-\Delta_{sph}}\,(r\phi))\big|^2 + \frac{M}{r^3}(\sqrt{1-\Delta_{sph}}(r\phi))^2\Big)\Big] dr^* d\omega,
\end{aligned} \tag{7}$$

where the implicit constant depends only on the mass M*. Here* Δ_{sph} *is the Laplacian in the angular variable* ω*, and* $\nabla\!\!\!/_\omega$ *is the associated gradient.*[1]

Remark 1.2. Let us first give a heuristic summary of the content of the estimate (7) and how it contrasts to the situation of flat space $M = 0$. Roughly speaking, the Schwarzschild space can be split into three pieces where one sees qualitatively different behavior in solutions to the wave equation.

The first region is very close to the boundary $r = 2M$. For the static space–time we are considering, this is quite easy to understand. Here wave propagation looks essentially trivial in that one has $\phi \sim F_1(t + r^*, \omega)$ for some smooth decaying function F_1 on the space $\mathbb{R} \times \mathbb{S}^2$. That is, wave propagation near the boundary $r = 2M$ is essentially just transport in the variable $\underline{u} = t + r^*$. The caveat is that this variable is the only one in

[1] This should not be confused with $\nabla\!\!\!/$ from line (3) which is the covariant derivative on spheres of radius r. Of course these two only differ by the factor of r^{-1}.

which it is possible to get decay for this region because $F_1(0, \omega)$ does not need to be small.

We would like to call the reader's attention to the fact that the precise decay of the function F_1 in the first variable seems to be quite a delicate issue, and we will only obtain $|F_1(\underline{u}, \omega)| \lesssim (1 + |\underline{u}|)^{-\frac{1}{2}}$. Of course, this is all one should expect given that the right-hand side of (7) is consistent with initial data decaying at this rate. Since this estimate does not ask for a lot of information, which is actually its strength in treating non-linear problems, it does not give a lot of information in return. For a much more precise asymptotic in the case of spherical symmetry, and for the more difficult case of dynamic space-times (for the spherically symmetric coupled scalar field), we refer the reader to the very deep recent work of Dafermos-Rodnianski (see [5, 6]).

The second region is in the "far exterior" $t \lesssim r^*$ where things look essentially flat. This is also fairly easy to understand. Here one expects that things look very similar to the well known case of Minkowski space.

The third region is "the boarder" close to $r = 3M$, which in Regge–Wheeler coordinates (5) we extend to the region $|r^*| \leqslant \frac{1}{2}t$. This is by far the most difficult region to understand, and where one loses regularity in the estimate (7). This loss of regularity is in sharp contrast to the estimate (3) in the case of Minkowski space, and is also something one sees only by looking at the non-spherically symmetric (functions) situation. What is happening here is that the geometry is pulling the radiation apart into the two regions just mentioned, and there is a danger that this "splitting" could allow some fairly large residual portion of the radiation to linger for a long amount of time in the transition region $r^* \sim 0$.

Now it turns out that this effect can only happen (and it does happen!) if the wave ϕ has a very high angular momentum. In this case it can concentrate on a very small set for each fixed time in the ω variable, and it will essentially rotate around the sphere $r = 3M$ for a long while before dispersing. This behavior can be understood by observing that null geodesics tangent to the surface $r = 3M$ will remain tangent to this surface [9], and that high angular momentum solutions to the wave equation will closely follow these geodesics for a long period of time before dispersing.

This slow dispersion can also be understood by conformally changing the metric (2) by the factor $(1 - \frac{2M}{r})$. Since the coefficient of dt^2 is constant on this new manifold, the corresponding wave equation describes the time evolution of a wave on a three dimensional Riemannian manifold with metric given by the spatial portion of the conformal Lorentzian metric. A simple calculation shows that this Riemannian manifold has a totally geodesic sphere (and hence closed geodesics) at the value $r = 3M$. Now, the original wave equation is equivalent to the wave equation with respect to the conformal metric modulo a smooth potential. For very high frequencies this potential cannot compete with the principle part of the conformal wave operator, so it is not difficult to construct coherent state solutions which concentrate near the closed geodesics sitting at $r = 3M$ for a long amount of time. Therefore, from this point of view, one should look at (7) as a sort of "cheap" dispersive estimate, and it is well known that such estimates lose regularity when the underlying geodesic flow is not well behaved in the sense of spreading of the classical trajectories.

We further remark here that the nature of the surface and geodesics at $r = 3M$ can be a little confusing to discuss in the relativistic terminology. The null geodesics at $r = 3M$ which orbit the black hole form a helix in four dimensional space-time with an axis in the time direction. Although their projections onto the three dimensional coordinates (r, ω) is closed, because the t coordinate is constantly increasing they are not closed null geodesics. Also, while it is true that the hypersurface $r = 3M$ is foliated by null geodesics it is not itself a null hypersurface, which is one with a null normal direction (see [8]). Note that a normal to $r = 3M$ is ∂_{r^*} which is space like.

As far as our analysis is concerned, the presence of null geodesics at $r = 3M$ manifests itself through trapping terms which are positive for $r \sim 3M$. For a wave equation with a potential Q, we refer to $\vec{x} \cdot \nabla Q + 2Q$ as the trapping term. This expression appears as a contribution governing the growth of the conformal (Morawetz) energy. It can be seen as the main "error" which is generated by the divergence of the conformal energy density, and is given by the first two terms on the right-hand side of Eq. (50) below. This identity was first derived using a somewhat different formalism in the dissertation of the first author (see [1]).

Remark 1.3. Our proof of the bound (7) will be very general in the sense that we derive it from a fairly generic family of estimates that holds for 1-D wave equations with "strongly repulsive" potentials. It is to be hoped that this procedure can be used to accommodate other situations, such as higher spin equations on Schwarzschild space and possibly other space-times where spherical harmonic decompositions make sense. We will leave these discussions to further work.

The approach we take here is based on the previous works [2, 3] which proved space-time Morawetz type estimates on Schwarzschild-like manifolds, and the thesis [1] which proved a version of the conformal (Morawetz) energy estimate (7) with growing right-hand side. In the estimate contained in [1], the trapping term (described previously) appears integrated in space-time against the quantity $t(\phi)^2$, where ϕ is the scalar field. If the factor of t were not present and if the field ϕ were restricted to a single spherical harmonic, then the Morawetz estimate from [2] would be sufficient to control this space-time integral. However, due to the fact that the reduction to individual spherical harmonics leads to trapping terms which grow quadratically according to the angular frequency, both an additional angular derivative *and* the factor of t must be controlled.

In this paper, we present a simple argument which allows one to absorb the trapping term with the factor of t, and hence prove (7). In the dissertation [1] and the forthcoming work [4], a more involved phase space analysis is used to reduce the loss of angular regularity in the space-time Morawetz estimate and in the analogue of (7) to only ϵ powers of the operator $1 - \Delta_{sph}$. We leave the combination of these two techniques to future work.

2. Preliminary setup

In this section we will set up some preliminary notation and ideas from one dimensional wave equations on Minkowski space. This material is for the most part entirely standard, and we make no claim of originality for the basic concepts. Now, it turns out that Theorem 1.1 is actually a special case of a family of estimates which holds in the following general situation. We consider 1-D wave equations of the form:

$$\Box\psi - Q(x)\psi = H, \tag{8}$$

where $\Box = -\partial_t^2 + \partial_x^2$ and $Q(x)$ is some smooth real valued function which we assume is general for the time being. When the source term H vanishes the field (8) comes from a Lagrangian with energy momentum tensor:

$$T_{\alpha\beta}[\psi] = \partial_\alpha\psi\partial_\beta\psi - \frac{1}{2}g_{\alpha\beta}\left(\partial^\gamma\psi\partial_\gamma\psi + Q(x)\cdot(\psi)^2\right). \tag{9}$$

A quick calculation using Eq. (8) shows that one has the divergence identity:

$$\partial^\alpha T_{\alpha\beta}[\psi] = -\frac{1}{2}\partial_\beta(Q)\cdot(\psi)^2 + H\cdot\partial_\beta\psi. \tag{10}$$

Also, the trace identity:

$$g^{\alpha\beta}T_{\alpha\beta}[\psi] = -Q(x)\cdot(\psi)^2$$

follows immediately, where $g = diag(-1, 1)$ is the 1-D Minkowski metric.

The utility of the tensor (9) is that it keeps track of how the field (8) reacts to the flow of various vector-fields $X = X^\alpha\partial_\alpha$ on $\mathbb{R}\times\mathbb{R}$. In general, we form the momentum density associated to X:

$$^{(X)}P_\alpha = T_{\alpha\beta}X^\beta, \tag{11}$$

and we compute from (10) the divergence:

$$\partial^{\alpha}\ {}^{(X)}P_\alpha = -\frac{1}{2}X(Q)\cdot(\psi)^2 + \frac{1}{2}T_{\alpha\beta}{}^{(X)}\pi^{\alpha\beta} + H\cdot X(\psi), \tag{12}$$

where $^{(X)}\pi$ is the deformation tensor:

$$^{(X)}\pi_{\alpha\beta} = \partial_\alpha X_\beta + \partial_\beta X_\alpha.$$

In the next section we will use this setup to prove the following general 1-D uniform local energy decay estimate:

Theorem 2.1 (1-D Morawetz Estimate for Positive Strongly Repulsive Potentials)**.** *Let ψ be a function on $(1+1)$ Minkowski space which is compactly supported for each fixed value of the time variable t. Suppose ψ satisfies Eq.* (8) *for some smooth real valued function $Q(x)$ which satisfies all of the following conditions*:

$$\begin{aligned}
0 &\leqslant Q, && && (\textit{Positivity}) && (13)\\
0 &\leqslant -x\partial_x(Q), && && (\textit{Repulsive}1) && (14)\\
x\partial_x(Q)(x)+2Q(x) &\leqslant -Csgn(x)\,\partial_x(Q)(x), && x\notin\mathcal{B}_1, && (\textit{Repulsive}2) && (15)\\
x\partial_x(Q)(x)+2Q(x) &\leqslant C|x|^{-1}Q(x), && x\notin\mathcal{B}_2, && (\textit{Homogeneity}) && (16)\\
(1+\lambda^2)x^2 &\leqslant -Cx\partial_x(Q)(x), && x\in 2\mathcal{B}_1, && (\textit{Critical Point}) && (17)\\
C^{-1} \leqslant Q(x) &\leqslant C(1+\lambda^2), && x\in 2\mathcal{B}_1, && (\textit{Local Bounds}) && (18)
\end{aligned}$$

where C and λ are fixed non-negative constants[2] *with C strictly positive, and the $\mathcal{B}_i$ are compact intervals containing the origin. Then the following uniform local energy decay estimate of Morawetz type holds:*

$$\begin{aligned}
&\int_{\mathbb{R}\times\{t\}}\Big((1+\underline{u}^2)(L\psi)^2 + (1+u^2)(\underline{L}\psi)^2 + (1+\underline{u}^2+u^2)Q(x)\cdot(\psi)^2\Big)\,dx\\
&\lesssim (1+\lambda^2)\underline{E}(\psi(0))\\
&\quad+\int_0^t \|\,(1+|\underline{u}|+|u|)(1+\lambda)H(s)\,\|_{L^2(dx)}\cdot\|\,(1+\lambda)(|\nabla_{t,x}\psi|+Q^{\frac{1}{2}}\cdot|\psi|)(s)\,\|_{L^2(dx)}\,ds,\\
&\quad+\int_0^t \|\,(1+|\underline{u}|+|u|)\,H(s)\,\|_{L^2(dx)}\cdot\underline{E}^{\frac{1}{2}}(\psi(s))\,ds.
\end{aligned} \tag{19}$$

[2] It is important for us to point out here that while λ is a constant in this theorem, it will later be used as a parameter. Thus, one of the main points is to have bounds which are uniform in the size of (large) λ.

Here we have set:

$$\underline{u} = t + x, \qquad u = t - x,$$
$$L = \partial_t + \partial_x\ , \quad \underline{L} = \partial_t - \partial_x,$$

and

$$\underline{E}(\psi(s)) = \int_{\mathbb{R}\times\{s\}} \Big((1+\underline{u}^2)(L\psi)^2 + (1+u^2)(\underline{L}\psi)^2 + (1+\underline{u}^2+u^2)Q(x)\cdot(\psi)^2\Big)\, dx. \tag{20}$$

The implicit constant in estimate (19) *depends only on the constant C and the size of the two intervals* $\mathcal{B}_i$*, and* **not** *on the value of t or* λ *or any other property of the potential* $Q(x)$ *than those listed above.*

2.1. The case of Schwarzschild Space. Before moving on to prove the estimate (19), let us first briefly indicate how this can be used to prove the bound (7). In the (t, r^*, ω) coordinates one writes the wave operator $|g|^{-\frac{1}{2}}\partial_\alpha g^{\alpha\beta}|g|^{\frac{1}{2}}\partial_\beta$ as:

$$\left(1-\frac{2M}{r}\right)^{-1}\left(-\partial_t^2\phi + r^{-2}\partial_{r^*}(r^2\partial_{r^*}\phi)\right) + \frac{1}{r^2}\Delta_{sph}\phi = G.$$

Here Δ_{sph} is the Laplacian in the ω variable. Introducing now the quantities $\overline{\psi} = r\phi$ and $\overline{H} = (1-\frac{2M}{r})rG$ this last line becomes:

$$-\partial_t^2\overline{\psi} + \partial_{r^*}^2\overline{\psi} - r^{-1}\partial_{r^*}^2(r)\overline{\psi} + \frac{(1-\frac{2M}{r})}{r^2}\Delta_{sph}\overline{\psi} = \overline{H}\ . \tag{21}$$

We now follow the usual procedure of projecting this equation onto individual spherical harmonics. Since all of our estimates are both L^2 and rotation invariant, there is absolutely no harm in doing this. We write:

$$\overline{\psi} = \sum_{\lambda,i} \psi_{\lambda,i} Y_\lambda^i,$$

where the Y_λ^i form an orthonormal basis for the space $\Delta_{sph}Y_\lambda = -\lambda^2 Y_\lambda$. On each harmonic Eq. (21) becomes:

$$-\partial_t^2\psi_{\lambda,i} + \partial_{r^*}^2\psi_{\lambda,i} - r^{-1}\partial_{r^*}^2(r)\psi_{\lambda,i} - \frac{\lambda^2(1-\frac{2M}{r})}{r^2}\psi_{\lambda,i} = H_{\lambda,i}\ . \tag{22}$$

Dropping the (λ, i) indices, labeling $r^* = x$, and using the notation:

$$Q_\lambda = r^{-1}\partial_{r^*}^2(r) + \frac{\lambda^2(1-\frac{2M}{r})}{r^2}, \tag{23}$$

Eq. (22) becomes:

$$-\partial_t^2\psi + \partial_x^2\psi - Q_\lambda(x)\psi \ = \ H.$$

We now wish to apply the estimate (19) to each of these equations, after we apply a spatial translation by a quantity $x_0(\lambda)$ which will be determined in a moment. The resulting family of estimates can then be safely added to obtain the full estimate (7) as long as one can produce a *single* point $x_0(\infty)$ such that $x_0(\lambda) \to x_0(\infty)$, and a *single* set of objects $(C, \mathcal{B}_1, \mathcal{B}_2)$ such that the conditions (13)–(18) hold for $(C, \mathcal{B}_1 + x_0(\lambda), \mathcal{B}_2 + x_0(\lambda))$. Luckily, for the family of potentials (23) where $0 \leqslant \lambda$ is any real number, this is easy to show. The reader should keep in mind that the reason this is possible is that the conditions (13)–(18) are not really *size* conditions on the potential Q_λ, but are actually conditions on the *sign* of Q_λ and its first derivative. This type of condition is very stable under multiplication by large positive constants, so it is not hard to get uniform behavior for large λ. We will state this result as follows:

Lemma 2.2. *Let Q_λ be the family of potentials defined on line* (23) *above, and set $x_0(\infty) = 3M$. Then there exists a constant C, a pair of sets $\mathcal{B}_1, \mathcal{B}_2$, and a family of points $x_0(\lambda) \to x_0(\infty)$ such that the potentials $Q_\lambda\,(x + x_0(\lambda))$ satisfy the conditions* (13)–(18) *for the triple* $(C, \mathcal{B}_1 + x_0(\lambda), \mathcal{B}_2 + x_0(\lambda))$. *All of these objects are completely determined by the value of M.*

Proof. First, notice that condition (13) is immediate. Next, recall that in the current notation we have $x = r(x) + 2M\ln(r(x) - 2M)$. We now write:

$$Q_\lambda(x) = \left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r^3} + \frac{\lambda^2}{r^2}\right). \tag{24}$$

The proof centers around showing that Q_λ has an isolated critical point. We compute the first derivative:

$$\begin{aligned} Q'_\lambda &= \frac{2M}{r^2}(1 - \frac{2M}{r})\left(\frac{2M}{r^3} + \frac{\lambda^2}{r^2}\right) - (1 - \frac{2M}{r})^2\left(\frac{6M}{r^4} + \frac{2\lambda^2}{r^3}\right), \\ &= -\frac{2}{r^5}(1 - \frac{2M}{r}) \cdot \left[\lambda^2 r^2 - 3M(\lambda^2 - 1)r - 8M^2\right]. \end{aligned} \tag{25}$$

The polynomial on the right-hand side of this last expression has exactly one root for positive values of r. This is given by the quadratic formula:

$$r(\lambda) = \frac{3M(\lambda^2 - 1) + M\sqrt{9(\lambda^2 - 1)^2 + 32\lambda^2}}{2\lambda^2}. \tag{26}$$

We now show that this positive root is trapped inside the interval $[\frac{8M}{3}, 3M]$. Since it is clear from (26) that asymptotically $r(\lambda) \to 3M$, it suffices to show that $r(\lambda)$ is an increasing function for $0 \leqslant \lambda$. This follows at once from differentiating the polynomial on line (25) with respect to the parameter λ which yields the identity:

$$\dot{r}(\lambda) = \frac{6M\lambda r - 2\lambda r^2}{2\lambda^2 r - 3M(\lambda^2 - 1)}.$$

A simple calculation shows that this quantity is indeed positive whenever $r \in [\frac{8}{3}M, 3M]$. Therefore, we shall pick our sequence of points $x_0(\lambda)$ according to the rule $r(x_0(\lambda)) = r(\lambda)$. This immediately gives the condition (14) for the family of translated potentials $Q_\lambda(x + x_0(\lambda))$. Also, note that the pointwise bound (18) is immediate for any compact interval $\mathcal{B}_1$.

We now show the critical point behavior (17). This boils down to the fact that the polynomial on line (25) has a simple root at $r(\lambda)$. In fact, taking the second derivative of the potential Q_λ with respect to x and evaluating at the point $x_0(\lambda)$ we have that:

$$Q''_\lambda(x_0(\lambda)) = -\frac{2}{r^5(\lambda)}\left(1-\frac{2M}{r(\lambda)}\right)^2 \cdot \left[2\lambda^2 r(\lambda) - 3M(\lambda^2-1)\right].$$

Notice that this quantity never vanishes, and is $O(\lambda^2)$ as $\lambda \to \infty$, so one has (17) for *any* compact set $\mathcal{B}_1$ given a suitable constant C, independent of the value of λ.

It remains for us to show the "strongly repulsive" conditions (15)–(16) hold for a uniform constant C and pair of sets $\mathcal{B}_i$. This follows from direct inspection of the formulas (24) and (25). We consider the cases of $x \to \pm\infty$ separately. In the case of $x \to \infty$ we also have that $r \to \infty$, and we have the two asymptotic formulas (with uniform constants in λ depending only on the mass M):

$$\begin{aligned}(x - x_0(\lambda)) \cdot Q'_\lambda(x) \;&=\; -\frac{2\lambda^2}{r^2} - \frac{6M}{r^3} + O\left(\frac{\lambda^2}{r^3}\right) + O\left(\frac{1}{r^4}\right) + \{\text{something negative}\},\\ 2Q_\lambda(x) \;&=\; \frac{4M}{r^3} + \frac{2\lambda^2}{r^2} + \{\text{something negative}\}.\end{aligned}$$

Notice that the {something negative} terms on the right-hand side of the first line above contain logarithmic corrections of the form $\ln(x)/x^4$ and $\lambda^2\ln(x)/x^3$, which come from the second summand on the right-hand side of (5). It is important these come with a good sign. Now, combining these last two lines we have that as $x \to \infty$:

$$(x - x_0(\lambda)) \cdot Q'_\lambda(x) + 2Q_\lambda(x) \;\leqslant\; -\frac{2M}{r^3} + O\left(\frac{\lambda^2}{r^3}\right) + O\left(\frac{1}{r^4}\right).$$

This is enough[3] to imply (15)–(16) because as $x \to \infty$ we also have the following strict lower bounds:

$$\frac{\frac{1}{2}\lambda^2}{x^3} \;\leqslant\; -Q', \qquad\qquad \frac{\frac{1}{2}\lambda^2}{x^3} \;\leqslant\; \frac{1}{x}Q_\lambda(x).$$

Finally, we deal with the bounds (15)–(16) as $x \to -\infty$. This is even easier to treat. Notice simply that both Q and $\partial_x(Q)$ are $O\left((1+\lambda^2)(1-\frac{2M}{r})\right)$, while the factor $(x - x_0(\lambda))$ goes to $-\infty$. This means that the first term on the left-hand side of both (15)–(16) is a very large negative multiple of the second. Therefore, the bounds (15)–(16) are trivial because the left-hand side is asymptotically negative. This completes our demonstration of Lemma 2.2. □

To wrap things up for this section, let us just mention briefly how one can sum the estimate (19) over the angular frequency localized pieces $\psi_{\lambda,i}$. The key thing here is that the estimate (19) has been set up in such a way that one can use the Cauchy–Schwartz inequality to deal with the terms on the right-hand side of (19). Specifically, summing

[3] The reader should note that since r in Eq. (25) is an implicit function of x, that computing a precise and optimal value for the size of the $\mathcal{B}_i$ would be rather tedious. Suffice it to say, if $M = 1$, then one should be able to take $1000 = |\mathcal{B}_i|$ in the above argument.

this bound over (λ, i) indices and using the fact that the $\{Y_\lambda^i\}$ form an orthonormal system on the sphere $\mathbb{S}^2$ we have that:

$$\sup_{0\leqslant s\leqslant t} \underline{E}(\phi(s)) = \sup_{0\leqslant s\leqslant t} \int_{\mathbb{R}\times\mathbb{S}^2\times\{s\}} \Big((1+\underline{u}^2)(L(r\phi))^2 + (1+u^2)(\underline{L}(r\phi))^2$$
$$+(1+\underline{u}^2+u^2)\cdot\Big(\frac{|\nabla\!\!\!/_\omega(r\phi)|^2}{r^2} + \frac{M}{r^3}(r\phi)^2\Big)\Big)\, dr^*d\omega$$
$$\lesssim \sum_{\lambda,i}\Big[\, (1+\lambda^2)\underline{\underline{E}}(0)(\psi_{\lambda,i})$$
$$+\int_0^t \|\,(1+|\underline{u}|+|u|)(1+\lambda)\overline{H}_{\lambda,i}(s)\,\|_{L^2(dr^*)}\cdot\|\,(1+\lambda)(|\nabla_{t,r^*}\psi_{\lambda,i}|$$
$$+\Big(1-\frac{2M}{r}\Big)^{\frac12}\Big(\frac{\lambda^2}{r^2}+\frac{M}{r^3}\Big)^{\frac12}|\psi_{\lambda,i}|)(s)\,\|_{L^2(dr^*)}\,ds$$
$$+\int_0^t \|(1+|\underline{u}|+|u|)\,\overline{H}_{\lambda,i}(s)\|_{L^2(dr^*)}\cdot\underline{\underline{E}}^{\frac12}(\psi_{\lambda,i}(s))\,ds\,\Big],\tag{27}$$

where we are defining:

$$\underline{\underline{E}}(\psi_{\lambda,i}(s)) = \int_{\mathbb{R}\times\{s\}}\Big((1+\underline{u}^2)(L\psi_{\lambda,i})^2 + (1+u^2)(\underline{L}\psi_{\lambda,i})^2$$
$$+(1+\underline{u}^2+u^2)\Big(1-\frac{2M}{r}\Big)\Big(\frac{\lambda^2}{r^2}+\frac{M}{r^3}\Big)\cdot(\psi_{\lambda,i})^2\Big)\,dr^*.$$

Now, bringing the sum under the integral sign in the two terms on the right-hand side of (27) above and then using the L^1–L^∞ Hölder inequality yields the bound:

$$\sup_{0\leqslant s\leqslant t}\underline{E}(\phi(s)) \lesssim \underline{E}((1-\Delta_{sph})^{\frac12}\phi(0))$$
$$+\sup_{0\leqslant s\leqslant t}\big[\underline{E}(\phi(s)) + E((1-\Delta_{sph})^{\frac12}\phi(s))\big]^{\frac12}$$
$$\cdot\int_0^t \|\,(1+|\underline{u}|+|u|)\Big(1-\frac{2M}{r}\Big)\, r\cdot\Big(\sqrt{1-\Delta_{sph}}G\Big)(s)\|_{L^2(dr^*d\omega)}ds,\tag{28}$$

where the usual energy is given by:

$$E(\phi(s)) = \int_{\mathbb{R}\times\mathbb{S}^2\times\{s\}}\Big(|\nabla_{t,r^*}(r\phi)|^2 + \Big(1-\frac{2M}{r}\Big)\Big(\frac{|\nabla\!\!\!/_\omega(r\phi)|^2}{r^2}+\frac{M}{r^3}(r\phi)^2\Big)\Big)\,dr^*d\omega.$$

The estimate (7) now follows from (28) and taking an angular (momentum) derivative of the basic energy estimate (see the next section for a proof):

$$\sup_{0\leqslant s\leqslant t} E(\phi(s)) \lesssim E(\phi(0)) + \Big(\int_0^t \|\,\Big(1-\frac{2M}{r}\Big)rG\,\|_{L^2(dr^*d\omega)}\,ds\Big)^2.$$

3. Proof of the Main Estimate

We now turn to the proof of Theorem 2.1. This will be accomplished in a series of three steps, each of which represents a tightening of the usual energy estimate. These are:

(1) Usual conservation of energy.
(2) Weak local decay of energy.
(3) Strong uniform local decay of energy.

Steps (1) and (3) involve a direct use of the energy-momentum tensor identities recorded in the previous section applied to various vector-fields X which are associated with the various types of decay as just listed. To prove item (2) above we use a Soffer–Morawetz type multiplier similar to what was done in [2, 3].

Step 1. Conservation of energy. This is well known. In the current setup it comes from setting $X = T = \partial_t$ in (11). Because $Q(x)$ is time independent and since T is Killing we end up with an essentially divergence free quantity:

$$\partial^{\alpha}{}^{(T)}P_\alpha = H \cdot \partial_t(\psi).$$

Integrating this over a time slab we arrive at the energy estimate:

$$\begin{aligned}
&\int_{\mathbb{R}\times\{t\}} \left((L\psi)^2 + (\underline{L}\psi)^2 + Q(x)\cdot(\psi)^2\right)\, dx \\
&\lesssim \int_0^t \| H(s) \|_{L^2(dx)} \cdot \| \partial_t \psi(s) \|_{L^2(dx)}\, ds \\
&\quad + \int_{\mathbb{R}\times\{0\}} \left((L\psi)^2 + (\underline{L}\psi)^2 + Q(x)\cdot(\psi)^2\right)\, dx, \qquad (29)
\end{aligned}$$

where the implicit constant is fixed and does not depend on Q (it is easy to calculate but we will not bother).

Step 2. Weak Local Decay of Energy. In this subsection, we prove that the local L^1 norm of the quantity $Q \cdot (\psi)^2$ decays sufficiently fast in an average sense. Our bound will be rather weak in that we allow the right-hand side of the estimate to grow like λt. However, this weak bound will be precisely what we need in the next subsection when we prove the strong uniform local decay of energy. What we propose to show is the following estimate for integers $1 \leqslant N$:

$$\begin{aligned}
&\int_0^t \int_{\mathcal{B}_1} (1+s) Q\cdot(\psi)^2 dx ds - \int_0^t \int_{\mathbb{R}\setminus\mathcal{B}_1} (1+s)\chi_1\left(\frac{10x}{1+s}\right) \mathrm{sgn}(x)\partial_x(Q)\cdot(\psi)^2 dx\, ds \\
&\lesssim \sup_{0\leqslant s\leqslant t} N^{-1}\underline{E}(s) + N(1+\lambda^2)E(0) + N\int_0^t \int_{\mathbb{R}} \| (1+s)(1+\lambda)H(s) \|_{L^2(dx)} \\
&\quad \cdot \| (1+\lambda)(|\nabla_{t,x}\psi| + Q^{\frac{1}{2}}\cdot|\psi|)(s) \|_{L^2(dx)}\, ds. \qquad (30)
\end{aligned}$$

Here the implicit constant depends *only* on the constants C and the lengths of the interval $\mathcal{B}_1$ from lines (13)–(18). We are defining $\underline{E}$ as the Morawetz type energy from line (20) above. Finally, χ_1 denotes a smooth bump adapted to the interval $[-1, 1]$, and E denotes the basic energy from Eq. (29) above.

In our proof of (30) it will be convenient for us to make the assumption that the local bound (17) drastically improves if we restrict to very small sets containing $x = 0$. In particular, we will assume that:

$$\frac{\epsilon^{-4}}{C} \cdot x^2 \leqslant -x\partial_x(Q)(x), \qquad |x| \leqslant c_{\mathcal{B}_1}\epsilon, \tag{31}$$

for some sufficiently small parameter ϵ which will be chosen in a moment. It is crucial for us to point out here that our choice of ϵ will only be dictated by the constant C and the size of $\mathcal{B}_1$, and will not depend on any other property of Q. Also, it is immediate that the assumption (31) in fact involves no loss of generality. This is because Eq. (8) rescales as follows:

$$\psi(t,x) \rightsquigarrow \psi(\epsilon^{-1}t, \epsilon^{-1}x), \quad Q(x) \rightsquigarrow \epsilon^{-2}Q(\epsilon^{-1}x), \quad H(x) \rightsquigarrow \epsilon^{-2}H(\epsilon^{-1}t, \epsilon^{-1}x).$$

Notice that the conditions (13)–(18) adapt to the rescaled situation in obvious ways. In particular one has (31) on the set $\widetilde{\mathcal{B}}_1 = \epsilon \cdot \mathcal{B}_1$. For the rest of this subsection we will work in the rescaled situation where we assume all of (13)–(18) as well as (31). Of course once one has (30) in this rescaled situation, one can recover the same bound for the original potential Q by scaling back. This will create constants which depend on ϵ, but we choose this parameter *only* to overcome two things. The first is the possibly large constant C on the right-hand side of (17) (which is actually only a problem when λ is small). The second is the fact that the original $\mathcal{B}_1$ may be small, so the constant $c_{\mathcal{B}_1}$ on Eq. (31) where our improved bound holds is also small. Of course both C and $c_{\mathcal{B}_1}$ are fixed no matter how much we rescale, so these can be made up for by taking ϵ sufficiently small. The main thing to keep in mind here is that our rescaling will never create constants in our estimates which depend in other ways on the shape of Q, other than the original assumptions we have made (13)–(18).

To prove (30), we use the following growth multiplier of Soffer–Morawetz type:

$$A(s,x)\psi = (1+s)\chi_1\left(\frac{10x}{1+s}\right)[\varphi\partial_x\psi + \partial_x(\varphi\psi)],$$

where φ is defined as follows:

$$\varphi(x) = \int_0^x \frac{1}{(1+|y|)^k}\, dy, \tag{32}$$

where $1 < k$ is a fixed constant. In practice the smaller the value of k the more favorable the estimates, so the reader may assume that $k = 2$. However, we will do all of our calculations in the general case so the overall structure is more apparent. The estimate (30) will follow from the usual procedure of directly calculating the integral:

$$\begin{aligned} I &= -\int_0^t \int_{\mathbb{R}} H \cdot A(s,x)\psi\, dx\, ds, \\ &= \int_0^t \int_{\mathbb{R}} \partial_t^2\psi \cdot A(s,x)\psi\, dx\, ds + \int_0^t \int_{\mathbb{R}} \left(-\partial_x^2\psi + Q\psi\right) \cdot A(s,x)\psi\, dx\, ds, \\ &= I_1 + I_2, \end{aligned} \tag{33}$$

and then using a Poincaré type lemma near the critical point of $Q(x)$ to get rid of the factor $-x\partial_x(Q)$ in favor of Q. We now compute the terms I_i separately and in order. The first term I_1 is the pure error. We first integrate by parts with respect to ∂_t which yields the identity:

$$I_1 = -\int_0^t \int_{\mathbb{R}} \partial_t\psi \cdot A(s)(\partial_t\psi)\, dx\, ds - \int_0^t \int_{\mathbb{R}} \partial_t\psi \cdot \dot{A}(s)(\psi)\, dx\, ds + \int_{\mathbb{R}\times\{t\}} \partial_t\psi \cdot A(t)(\psi)\, dx - \int_{\mathbb{R}\times\{0\}} \partial_t\psi \cdot A(0)(\psi)\, dx. \qquad (34)$$

Here the operator $\dot{A}(s)$ is given by:

$$\dot{A}(s)\psi = \left[\chi_1(\frac{10x}{1+s}) - \frac{10x}{1+s}\chi_1'(\frac{10x}{1+s})\right] \cdot \left(2\varphi\partial_x\psi + \frac{1}{(1+|x|)^k}\psi\right).$$

Also, one has the adjoint formula:

$$A^*(s)\psi = -A\psi - 20\chi_1'(\frac{10x}{1+s})\varphi \cdot \psi.$$

Therefore, a bound for the absolute value of the right-hand side of (34) above is:

$$|I_1| \lesssim \int_0^t \int_{\mathbb{R}} \widetilde{\chi}_1(\frac{10x}{1+s}) \cdot \left((\partial_t\psi)^2 + (\partial_x\psi)^2 + \frac{1}{(1+|x|)^{2k}}(\psi)^2\right) dx\, ds + \sup_{0\leqslant s\leqslant t} \int_{\mathbb{R}\times\{s\}} (1+s)\widetilde{\chi}_1(\frac{10x}{1+s}) \cdot \left((\partial_t\psi)^2 + (\partial_x\psi)^2 + \frac{1}{(1+|x|)^{2k}}(\psi)^2\right) dx, \qquad (35)$$

where $\widetilde{\chi}_1$ is another $[-1, 1]$ adapted smooth bump. To deal with the terms involving the inverse $|x|$ weight we use the Poincaré type estimate:

$$\int_{-x_0}^{x_0} \frac{1}{(1+|x|)^2}\, (\psi)^2\, dx \lesssim (\psi)^2(0) + \int_{-x_0}^{x_0} (\partial_x\psi)^2\, dx. \qquad (36)$$

This follows at once from evaluation of the integral:

$$\frac{1}{(1+|x_0|)}\left((\psi)^2(x_0) + (\psi)^2(-x_0)\right) - 2(\psi)^2(0) = \int_{-x_0}^{x_0} \mathrm{sgn}(x)\partial_x\left[\frac{1}{(1+|x|)}\, (\psi)^2\right] dx,$$

and using the Cauchy–Schwartz inequality. Using now (36) and the condition (18) it is easy to bound:

$$\int_{\mathbb{R}} \widetilde{\chi}_1(\frac{10x}{1+s}) \cdot \frac{1}{(1+|x|)^{2k}}(\psi)^2\, dx \lesssim \int_{\mathbb{R}} \widetilde{\chi}_1(\frac{5x}{1+s}) \cdot \left((\partial_x\psi)^2 + Q\cdot(\psi)^2\right) dx, \qquad (37)$$

by using a partition of unity on $\mathcal{B}_1$ and $\mathbb{R}\setminus\mathcal{B}_1$.

Our next step is to use the bound:

$$\int_{\mathbb{R}} (1+s)\widetilde{\chi}_1(\frac{5x}{1+s}) \cdot \left((\partial_t\psi)^2 + (\partial_x\psi)^2 + Q\cdot(\psi)^2\right) dx \lesssim (1+s)^{-1} \cdot \underline{E}(s),$$

where the right-hand side is the Morawetz type energy from Eq. (20) above. This and the bound (35) allows us to estimate:

$$\begin{aligned} |I_1| \lesssim & \int_{N(1+\lambda)}^{t} (1+s)^{-2} \cdot \underline{E}(s)\, ds + \sup_{N(1+\lambda)\leqslant s\leqslant t} (1+s)^{-1} \cdot \underline{E}(s) \\ & + \int_0^{N(1+\lambda)} E(s)\, ds + \sup_{0\leqslant s\leqslant N(1+\lambda)} (1+s) \cdot E(s), \\ \lesssim & \sup_{0\leqslant s\leqslant t} (N(1+\lambda))^{-1} \underline{E}(s) + \sup_{0\leqslant s\leqslant t} N(1+\lambda) E(s). \end{aligned}$$

Using now the energy inequality (29) to deal with the second term on the right-hand side of this last line we arrive at the bound:

$$|I_1| \lesssim (1+\lambda)^{-1}(\text{R.H.S.})(30). \tag{38}$$

In a moment we will need to multiply all of our estimates through by the factor $(1+\lambda)$, so (38) is of the correct form.

Before moving on to the second integral on Eq. (33) above, we mention briefly how to take care of the first integral on the right-hand side immediately above that line. Applying the Cauchy–Schwartz inequality we have the bound:

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}} |H(s)| \cdot |A(s)\psi|\, dx\, ds \\ & \quad \lesssim \int_0^t \| (1+s)H(s) \|_{L^2(dx)} \cdot \| (|\partial_x \psi| + \chi_1\, (1+|x|)^{-k} |\psi|)(s) \|_{L^2(dx)}\, ds\,. \end{aligned} \tag{39}$$

Using now a Poincaré type estimate of the form (37) to deal with the last term on the right-hand side of (39) easily yields:

$$(\text{L.H.S.})(39) \lesssim (1+\lambda)^{-2}\, (\text{R.H.S.})(30),$$

which is sufficient for our purposes.

Finally, we deal with the integral I_2 on Eq. (33). After several integration by parts (this is an essentially well known calculation) we arrive at the identity:

$$I_2 = \sum_{j=1}^{4} K_j,$$

where the integrals K_i are:

$$\begin{aligned} K_1 &= \int_0^t \int_{\mathbb{R}} 10\chi_1'(\frac{10x}{1+s}) \left[\varphi \partial_x \psi + \varphi' \psi\right] \cdot \partial_x \psi\, dx, \\ K_2 &= -\int_0^t \int_{\mathbb{R}} 5\chi_1'(\frac{10x}{1+s})\, \varphi''\, (\psi)^2\, dx, \\ K_3 &= -\int_0^t \int_{\mathbb{R}} 10\chi_1'(\frac{10x}{1+s})\, \varphi\, Q \cdot (\psi)^2\, dx, \\ K_4 &= \int_0^t \int_{\mathbb{R}} (1+s)\chi_1(\frac{10x}{1+s}) \left[2\varphi'(\partial_x\psi)^2 - \varphi\partial_x(Q)\cdot(\psi)^2 - \frac{1}{2}\varphi'''(\psi)^2\right] dx. \end{aligned}$$

Bounding the first three terms above is essentially the same as what we have just done for the term I_1 above. One simply uses Cauchy–Schwartz, the Poincaré estimate (36), and the definitions of the two energies E and $\underline{E}$ to prove that:

$$|K_1| + |K_2| + |K_3| \lesssim \sup_{0\leqslant s\leqslant t} (N(1+\lambda))^{-1}\underline{E}(s) + \sup_{0\leqslant s\leqslant t} N(1+\lambda)E(s). \tag{40}$$

Therefore, the heart of the matter now is to obtain a positive lower bound for the quantity K_4 in such a way that we can estimate the left-hand side of (30).

Before continuing with the proof, let us make a further simplification. Without loss of generality we may assume that the cutoff function χ_1 is the square of yet another smooth cutoff function, say $\widetilde{\chi}_1$. This allows us to replace ψ by $\widetilde{\chi}_1\psi$ in K_4 above modulo a term involving $[\partial_x, \widetilde{\chi}_1] = O(\frac{1}{1+s})$ which is also cutoff where $|x| \leqslant 10^{-1}t$. It is clear that this will again be of the form $(1+\lambda)^{-1}$(R.H.S.)(30), so we can just tack this error on to (40) above.

Thus, what we will need to show is that there exists a sufficiently small constant c such that the following reverse bound holds for compactly supported functions ψ:

$$\int_{\mathbb{R}} \left[2\varphi'(\partial_x\psi)^2 - \varphi\partial_x(Q)\cdot(\psi)^2 - \frac{1}{2}\varphi'''(\psi)^2\right] dx \geqslant c \int_{\mathbb{R}} \left[\varphi'(\partial_x\psi)^2 - \frac{1}{2}\varphi\partial_x(Q)\cdot(\psi)^2\right] dx. \tag{41}$$

Once this is established, the bound (30) will follow from combining the bounds (38) and (40) with (41) and the following estimate which also holds for smooth compactly supported functions ψ:

$$\int_{\mathcal{B}_1} Q\cdot(\psi)^2\, dx - \int_{\mathbb{R}\setminus\mathcal{B}_1} \operatorname{sgn}(x)\partial_x(Q)\cdot(\psi)^2\, dx \lesssim (1+\lambda) \int_{\mathbb{R}} \left[\varphi'(\partial_x\psi)^2 - \frac{1}{2}\varphi\partial_x(Q)\cdot(\psi)^2\right] dx. \tag{42}$$

We first prove (41). The overall strategy for this is very simple. The main thing we will establish is that the form of the weight function (32) reduces everything to having a "good" bound for the function ψ at $x = 0$ in terms of the left-hand side of (41). This latter task is relatively easy to accomplish because assumption (31) essentially means that $-x\partial_x(Q) \sim \epsilon^{-1}\delta_0$, where δ_0 is the unit mass at the origin. This means that the potential term on the right-hand side of (41) will give us a bound on $\psi(0)$ with an $O(\epsilon^{\frac{1}{2}})$ constant. The details of this procedure are as follows. We first compute:

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \partial_x\left[\varphi''\,(\psi)^2\right] dx, \\ &= \int_{\mathbb{R}} \varphi'''\,(\psi)^2\, dx + 2\int_{\mathbb{R}} \varphi''\,\psi\partial_x\psi\, dx. \end{aligned} \tag{43}$$

It will now be useful to have the identities:

$$\varphi''(x) = \frac{-k\cdot\operatorname{sgn}(x)}{(1+|x|)^{k+1}}, \qquad \varphi'''(x) = -2k\delta_0 + \frac{k(k+1)}{(1+|x|)^{k+2}}.$$

Therefore, the right-hand side of (43) and a Cauchy–Schwartz gives us the bound:

$$\int_{\mathbb{R}} \frac{k(k+1)}{(1+|x|)^{k+2}} (\psi)^2\, dx,$$
$$\leqslant 2\left(\frac{k}{k+1}\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \frac{1}{(1+|x|)^k} (\partial_x \psi)^2\, dx\right)^{\frac{1}{2}} \cdot \left(\int_{\mathbb{R}} \frac{k(k+1)}{(1+|x|)^{k+2}} (\psi)^2\, dx\right)^{\frac{1}{2}} + 2k(\psi)^2(0),$$
$$= 2\left(\frac{k}{k+1}\right)^{\frac{1}{2}} A^{\frac{1}{2}} \cdot B^{\frac{1}{2}} + C.$$

We may now assume without loss of generality that in this last bound we have $C \leqslant B$, otherwise there is nothing to prove on Eq. (41). Therefore, dividing through by $B^{\frac{1}{2}}$ and squaring this last line we arrive at the bound:

$$\left|\int_{\mathbb{R}} \varphi''' (\psi)^2\, dx\right| = B - C \ \leqslant\ 4\frac{k}{k+1} A + 4\left(\frac{k}{k+1}\right)^{\frac{1}{2}} A^{\frac{1}{2}} \cdot C^{\frac{1}{2}}. \tag{44}$$

The dangerous term is now the second one on the right-hand side above. This needs to be controlled in terms of a sufficiently small constant. In fact, we will show that it is $O(\epsilon^{\frac{1}{2}})$ times the (R.H.S.)(41), which implies that it may be safely absorbed into half of the remaining portion of A and a small amount of the potential term on (R.H.S.)(41). The bound which allows us to do this is the following:

$$(\psi)^2(0) \ \lesssim\ \epsilon \left(\int_{\mathbb{R}} \frac{1}{(1+|x|)^k} (\partial_x \psi)^2\, dx + \int_{\mathbb{R}} -\varphi \partial_x(Q) \cdot (\psi)^2\, dx\right).$$

From assumption (31), this last estimate follows from:

$$(\psi)^2(0) \ \lesssim\ \epsilon \left(\int_{\mathbb{R}} \frac{1}{(1+|x|)^k} (\partial_x \psi)^2\, dx + \epsilon^{-4} \int_{\mathbb{R}} x^2 \chi(\epsilon^{-1} x) \cdot (\psi)^2\, dx\right), \tag{45}$$

where χ is some smooth $O(1)$ bump function whose support depends on the size of the set $\mathcal{B}_1$ from Eq. (17). Estimate (45) is essentially scale invariant, so it suffices to show that:

$$(\psi)^2(0) \ \lesssim\ \int_{\mathbb{R}} \frac{1}{(1+\epsilon|x|)^k} (\partial_x \psi)^2\, dx + \int_{\mathbb{R}} x^2 \chi(x) \cdot (\psi)^2\, dx. \tag{46}$$

This, in turn, follows from cutting things off and using the usual Sobolev embedding once we have the bound:

$$\| \widetilde{\chi}^{\frac{1}{2}} \psi \|_{L^2}^2 \ \lesssim\ \int_{\mathbb{R}} \frac{1}{(1+\epsilon|x|)^k} (\partial_x \psi)^2\, dx + \int_{\mathbb{R}} x^2 \chi(x) \cdot (\psi)^2\, dx, \tag{47}$$

for some slightly smaller cutoff function $\widetilde{\chi}$. This last bound can be proved in two steps. We first show the estimate:

$$\| |x|^{\frac{1}{2}} \widetilde{\widetilde{\chi}}^{\frac{1}{2}} \psi \|_{L^2}^2 \ \lesssim\ \int_{\mathbb{R}} \frac{1}{(1+\epsilon|x|)^k} (\partial_x \psi)^2\, dx + \int_{\mathbb{R}} x^2 \chi(x) \cdot (\psi)^2\, dx, \tag{48}$$

for some intermediate cutoff $\widetilde{\widetilde{\chi}}$. This bound follows at once from evaluating the integral:

$$0 = \int_{\mathbb{R}} \mathrm{sgn}(x)\, \partial_x \left[x^2\, \widetilde{\widetilde{\chi}} \cdot (\psi)^2\right] dx,$$

and using the Cauchy–Schwartz inequality to bound the error terms by (R.H.S.)(48). Having now established (48) we can prove (47) by applying the same procedure to the integral:

$$0 = \int_{\mathbb{R}} \partial_x \left[x \ \widetilde{\chi} \cdot (\psi)^2 \right] \ dx.$$

This completes our proof of (45), and hence our demonstration of the main commutator estimate (41).

Having now dealt with the bound (41), the only thing left for us to do in our proof of (30) is to show the bound (42). Notice that the bound for the second term on the left-hand side of that estimate follows at once from the fact that $\mathrm{sgn}(x) \lesssim \varphi(x)$ whenever $x \in \mathbb{R} \setminus \mathcal{B}_1$. Therefore, it remains to bound the first term on the left-hand side of (42). This is where we pick up the extra factor of $(1+\lambda)$. The proof is essentially identical to what was done to establish (47) above. Using the two conditions (17)–(18), it suffices to multiply through the following estimate by the quantity $(1+\lambda)^2$:

$$\| \chi_{\mathcal{B}_1}^{\frac{1}{2}} \psi \|_{L^2}^2 \ \lesssim \ \| \, |x| \chi_{\mathcal{B}_1}^{\frac{1}{2}} \psi \|_{L^2} \cdot \| \chi_{\mathcal{B}_1}^{\frac{1}{2}} \partial_x \psi \|_{L^2} + \| \, |x| \widetilde{\chi}_{\mathcal{B}_1}^{\frac{1}{2}} \psi \|_{L^2}^2. \tag{49}$$

Here the functions $\chi_{\mathcal{B}_1}$ and $\widetilde{\chi}_{\mathcal{B}_1}$ are cutoffs which are $\equiv 1$ on the set $\mathcal{B}_1$ and which vanish outside of $2\mathcal{B}_1$. The bound (49) follows from evaluation of the integral:

$$0 = \int_{\mathbb{R}} \partial_x \left[x \ \chi_{\mathcal{B}_1} \cdot (\psi)^2 \right] \ dx,$$

and using Cauchy–Schwartz as well as the bound $|x \chi'_{\mathcal{B}_1}| \lesssim x^2 \widetilde{\chi_{\mathcal{B}_1}}$ for a suitable cutoff $\widetilde{\chi}_{\mathcal{B}_1}$. We have now finished our proof of the weak local energy decay estimate (30).

Remark 3.1. We note here that it is possible to prove (30) without rescaling the potential Q into the condition (31). This can be accomplished by using the weight function:

$$\varphi_\epsilon(x) = \int_0^x \frac{1}{(1+\epsilon|y|)^k} \ dy,$$

in place of (32) above. This yields a small factor in front of $|\psi|(0)$ when it appears in the $C^{\frac{1}{2}}$ term on the right-hand side of line (44) above, so one can proceed directly to the estimate (46) to control things. We leave the details to the interested reader.

Step 3. Strong Uniform Decay of Local Energy. We are now ready to prove the main Morawetz estimate (19). With the assumptions (13)–(18) in hand, as well as the weak local energy decay estimate (30), this becomes an essentially standard calculation. We will contract the energy-momentum tensor (9) with the conformal Killing vector-field:

$$K_0 = (t^2 + x^2)\partial_t + 2tx\partial_x = \frac{1}{2}\underline{u}^2 L + \frac{1}{2}u^2 \underline{L}.$$

The deformation tensor of this is computed to be:

$$^{(K_0)}\pi = 4tg.$$

Therefore, we may form the momentum density ${}^{(K_0)}P_\alpha = T_{\alpha\beta}K_0^\beta$ and from Eq. (12) we compute the divergence:

$$\partial^{\alpha}\ {}^{(K_0)}P_\alpha = -tx\partial_x(Q)\cdot(\psi)^2 - 2tQ\cdot(\psi)^2 + K_0(\psi)\cdot H. \tag{50}$$

By simply integrating this last line over various time slabs of the form $0 \leqslant s \leqslant t$ and using the Cauchy–Schwartz inequality we arrive at the bound:

$$\begin{aligned}\sup_{0\leqslant s\leqslant t}\ {}^{(K_0)}P_0(s) \ &\leqslant \int_0^t\int_{\mathbb{R}} |H(s)|\cdot|K_0(\psi)(s)|\,dxds + {}^{(K_0)}P_0(0)\\ &\quad+\int_0^t\int_{\mathbb{R}}\left[sx\partial_x(Q)\cdot(\psi)^2 + 2sQ\cdot(\psi)^2\right]\,dx\,ds.\end{aligned} \tag{51}$$

Using now the identity:

$${}^{(K_0)}P_0 = \frac{1}{4}\underline{u}^2(L\psi)^2 + \frac{1}{4}u^2(\underline{L}\psi)^2 + \frac{1}{4}(\underline{u}^2+u^2)Q\cdot(\psi)^2,$$

we see that (51) in conjunction with the energy estimate (29) implies the bound:

$$\begin{aligned}\sup_{0\leqslant s\leqslant t}\ \underline{E}(s) \ &\lesssim \int_0^t\int_{\mathbb{R}} \|\,(1+|\underline{u}|+|u|)H(s)\,\|_{L^2(dx)}\cdot\underline{E}^{\frac{1}{2}}(s)\,ds + \underline{E}(0)\\ &\quad+\int_0^t\int_{\mathbb{R}}\left[sx\partial_x(Q)\cdot(\psi)^2 + 2sQ\cdot(\psi)^2\right]\,dx\,ds.\end{aligned} \tag{52}$$

The last thing we need to do here is to bound the last term on the right-hand side of the previous expression. We will show the bound:

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}}\left[sx\partial_x(Q)\cdot(\psi)^2 + 2sQ\cdot(\psi)^2\right]\,dx\,ds\\ &\lesssim \sup_{0\leqslant s\leqslant t} N^{-1}\underline{E}(s) \ + \ N(1+\lambda^2)E(0) + N\int_0^t\int_{\mathbb{R}} \|\,(1+s)(1+\lambda)H(s)\,\|_{L^2(dx)}\\ &\quad\cdot\|\,(1+\lambda)(|\nabla_{t,x}\psi| + Q^{\frac{1}{2}}\cdot|\psi|)\,(s)\,\|_{L^2(dx)}\,ds,\end{aligned} \tag{53}$$

where the implicit constant is independent of the large parameter N. Notice that this bound substituted into (52) immediately implies (19) for sufficiently large N.

To prove (53) we will chop the left-hand side up into three pieces. The first is the "bad" set $\mathcal{B}_1$. This is where most of the positivity of (L.H.S.) (53) can be found. The second set is where $x \notin \mathcal{B}_1$ and $|x| \leqslant \frac{1}{20}t$. Here we use the strongly repulsive condition (15). Finally, in the exterior of the influence of the potential when $t \lesssim |x|$ we can simply integrate things using the homogeneity bound (16). The details of this procedure are as follows:

On the set $\mathcal{B}_1$ we use the repulsive condition (14) and the first term on the left-hand side of (30) to bound:

$$\begin{aligned}&\int_0^t\int_{\mathcal{B}_1}\left[sx\partial_x(Q)\cdot(\psi)^2 + 2sQ\cdot(\psi)^2\right]\,dx\,ds\ ,\\ &\lesssim \int_0^t\int_{\mathcal{B}_1}(1+s)Q\cdot(\psi)^2\,dx\,ds,\\ &\lesssim \text{(R.H.S.)(53)}.\end{aligned}$$

Next, we work in the set $\mathbb{R} \setminus \mathcal{B}_1$ but cutoff according to how large t is. Here we make use of the condition (15):

$$\int_0^t \int_{\mathbb{R}\setminus\mathcal{B}_1} \chi_1\left(\frac{10x}{1+s}\right)\left[sx\partial_x(Q)\cdot(\psi)^2 + 2sQ\cdot(\psi)^2\right] dx\, ds,$$
$$\lesssim -\int_0^t \int_{\mathbb{R}\setminus\mathcal{B}_1} (1+s)\chi_1\left(\frac{10x}{1+s}\right)\mathrm{sgn}(x)\partial_x(Q)\cdot(\psi)^2\, dx\, ds,$$
$$\lesssim (\text{R.H.S.})(53).$$

Finally, in the exterior where $t \lesssim |x|$ we use the condition (16) and the following bound which holds for parameters N such that $|\mathcal{B}_2| \leqslant N$ (where the implicit constant of course depends on $|\mathcal{B}_2|$):

$$\int_0^t \int_{\mathbb{R}} \left(1-\chi_1\left(\frac{10x}{1+s}\right)\right)\left[sx\partial_x(Q)\cdot(\psi)^2 + 2sQ\cdot(\psi)^2\right] dx\, ds,$$
$$\lesssim \int_0^{20|\mathcal{B}_2|}\int_{\mathbb{R}} 2sQ\cdot(\psi)^2\, dx\, ds + \int_0^N \int_{\mathbb{R}} Q\cdot(\psi)^2\, dx\, ds + \int_N^t \int_{\mathbb{R}} Q\cdot(\psi)^2\, dx\, ds$$
$$\lesssim N \sup_{0\leqslant s\leqslant t} E(s) \;+\; N^{-1} \sup_{0\leqslant s\leqslant t} \underline{E}(s),$$
$$\lesssim (\text{R.H.S.})(53).$$

This completes our demonstration of (53), and hence our proof the main estimate (19).

4. Scattering for Small Amplitude Non-Linear Scalar Fields

We will be brief here and leave many of the details to the reader. The main result of this section is the following:

Theorem 4.1 (Scattering for Scalar Fields)**.** *Consider the Cauchy problem:*

$$\Box_g\phi \;=\; \lambda|\phi|^p\phi, \qquad \phi(0) \;=\; f, \qquad \partial_t\phi(0) \;=\; g, \tag{54}$$

for compactly supported functions (f, g). *Define the regularity space:*

$$\|\,\phi\,\|^2_{\mathcal{H}^k_\Omega} = \underline{E}\left((1-\Delta_{sph})^{\frac{k}{2}}\phi\right),$$

where $\underline{E}$ *is the Morawetz type energy from Eq.* (7)*. Then if* $2 < p$*, there exists a universal set of positive constants* $\mathcal{E}$ *and* C *depending only on* p *such that if:*

$$\|\,\phi(0)\,\|_{\mathcal{H}^3_\Omega} \;\leqslant\; \mathcal{E}\,,$$

then a unique solution to the problem (54) *exists for all values of the variable* t *and it obeys the bound:*

$$\|\,\phi(t)\,\|_{\mathcal{H}^2_\Omega} \;\leqslant\; C\mathcal{E}.$$

In particular, one has the following uniform point-wise bounds:

$$|\phi| \;\lesssim\; \mathcal{E}\, r^{-1}\cdot\min\{\,1,\; \big|t-|r^*|\big|^{-\frac{1}{2}}\,\}. \tag{55}$$

A previous result of this type was recently obtained by Dafermos and Rodnianski in the case of spherical symmetry and powers $3 < p$ (see [6]).

To prove Theorem (4.1) we need four ingredients. The first is the Morawetz estimate (7). The second is a Poincaré type estimate which will allow us to control the L^2 norm of our function in terms of the energy $\underline{E}$. The third is a paraproduct bound which allows us to concentrate all of our angular derivatives on a single term of the non-linearity $\lambda|\phi|^p\phi$. And the final is a global Sobolev inequality which will give us the bound (55) in terms of our energy space $\mathcal{H}^2_\Omega$. We now state the last three of these in order:

Lemma 4.2 (Poincaré type estimate for the weights $\underline{u}$ and u)**.** *Let ψ be a function of the variables (t, r^*), and define the weights $\underline{u}$ and u as on Eqs.* (6)*. Then the following estimate holds:*

$$\int_{\mathbb{R}} (\psi)^2\, dr^* \lesssim \int_{\mathbb{R}} \Big(\underline{u}^2 (L\psi)^2 + u^2 (\underline{L}\psi)^2 + \frac{(1+\underline{u}^2+u^2)(1-\frac{2M}{r})}{r^3}(\psi)^2 \Big)\, dr^*. \tag{56}$$

Lemma 4.3 (Paraproduct bounds)**.** *On the sphere $\mathbb{S}^2$ the following estimates holds:*

$$\| (1-\Delta_{sph})^{\frac{k}{2}} (|F|^p F) \|_{L^2(\mathbb{S}^2)} \lesssim \| F \|^p_{L^\infty(\mathbb{S}^2)} \cdot \| (1-\Delta_{sph})^{\frac{k}{2}} F \|_{L^2(\mathbb{S}^2)}, \tag{57}$$

for all test functions F and integers $0 \leqslant k \leqslant p+1$.

Lemma 4.4 (A global Sobolev inequality)**.** *Let ϕ be a function of the variables (r^*, ω). Then one has the following global bounds:*

$$|r\phi| \lesssim \min\{\, 1,\ r^{\frac{1}{2}}(1-\frac{2M}{r})^{-\frac{1}{4}} \big| t - |r^*| \big|^{-1},\ \big| t-|r^*|\big|^{-\frac{1}{2}} \,\} \cdot \| \phi \|_{\mathcal{H}^2_\Omega}. \tag{58}$$

We now give short proofs of these three lemmas:

Proof of Estimate (56). The proof will follow from cutting the function ψ into three pieces. We write:

$$\psi = \chi_{r^*<-1}\psi + \chi_{-1<r^*<1}\psi + \chi_{r^*>1}\psi,$$

where the χ form a smooth partition of unity such that the $\partial_{r^*}\chi_{r^*<\pm 1}$ are supported on the interval $[-2, 2]$. For the left-hand portion we compute that:

$$0 = \int_{\mathbb{R}} \partial_{r^*}\Big[(t+r^*)\chi_{r^*<-1}(\psi)^2 \Big]\, dr^* = \int_{\mathbb{R}} \chi_{r^*<-1}(\psi)^2\, dr^* + \int_{\mathbb{R}} (t+r^*)\chi'_{r^*<-1}(\psi)^2\, dr^* + 2\int_{\mathbb{R}} (t+r^*)\chi_{r^*<-1}\psi\,\partial_{r^*}\psi\, dr^*.$$

Collecting terms and applying the Cauchy–Schwartz inequality we arrive at the bound:

$$\int_{\mathbb{R}} \chi_{r^*<-1}(\psi)^2\, dr^* \lesssim \int_{\mathbb{R}} \frac{(1+\underline{u}^2+u^2)(1-\frac{2M}{r})}{r^3}(\psi)^2\, dr^* + \Big(\int_{\mathbb{R}} \chi_{r^*<-1}(\psi)^2\, dr^* \Big)^{\frac{1}{2}} \cdot \Big(\int_{\mathbb{R}} \underline{u}^2\, \chi_{r^*<-1}(\partial_{r^*}\psi)^2\, dr^* \Big)^{\frac{1}{2}}.$$

This easily proves (56) for the left-hand portion of things because here one has the bound:

$$\underline{u}^2\ \chi_{r^*<-1}(\partial_{r^*}\psi)^2\ \lesssim\ \underline{u}^2(L\psi)^2+u^2(\underline{L}\psi)^2. \tag{59}$$

The proof of (56) for the right-hand term $\chi_{r^*>1}\psi$ follows from an identical argument. We leave this to the reader. □

Proof of Estimate (57). Because k is chosen to be an integer, it is easy to reduce this estimate to the case of Euclidean space. First of all, note that one has the following bound:

$$\|\,(1-\Delta_{sph})^{\frac{k}{2}}(|F|^pF)\,\|_{L^2(\mathbb{S}^2)}\ \lesssim\ \sum_{|\alpha|\leqslant k}\|\,\Omega_{ij}^{\alpha}(|F|^pF)\,\|_{L^2(\mathbb{S}^2)},$$

where α is a multiindexing of the rotation generators $\{\Omega_{ij}\}$. Via a partition of unity, the desired bound easily reduces to proving that for $k\in\mathbb{N}$ one has:

$$\|\,|F|^pF\,\|_{H^k}\ \lesssim\ \|\,F\,\|_{L^\infty}^p\cdot\|\,F\,\|_{H^k},\qquad\qquad 0\leqslant k\leqslant p+1.$$

Estimates of this type are well known and easy to prove (see e.g. [11]). □

Proof of Estimate (58). We first make a preliminary reduction. Because we are including two angular (momentum) derivatives in the norm $\mathcal{H}^2_\Omega$, via the Sobolev embedding on the sphere $\mathbb{S}^2$ it suffices to prove the following global Sobolev estimate for functions ψ of the variable r^*:

$$|\psi|\ \lesssim\ \min\{\,1,\ r^{\frac{1}{2}}(1-\frac{2M}{r})^{-\frac{1}{4}}\big|t-|r^*|\big|^{-1},\ \big|t-|r^*|\big|^{-\frac{1}{2}}\,\}\cdot\|\,\phi\,\|_{\mathcal{H}}, \tag{60}$$

where $\mathcal{H}$ is the Hilbert Space:

$$\|\,\psi\,\|^2_{\mathcal{H}}=\int_{\mathbb{R}}\left[(1+\underline{u}^2)(L\psi)^2+(1+u^2)(\underline{L}\psi)^2+(1+\underline{u}^2+u^2)\frac{(1-\frac{2M}{r})}{r^3}(\psi)^2\right]dr^*.$$

Using now the Poincaré estimate (56) and bounds of the form (59) we see that we have:

$$\int_{\mathbb{R}}\left[(1+(t-|r^*|)^2)(\partial_r^*\psi)^2\ +\ (\psi)^2\ +\ (1+\underline{u}^2+u^2)\frac{(1-\frac{2M}{r})}{r^3}(\psi)^2\right]dr^*\lesssim\|\,\psi\,\|^2_{\mathcal{H}}. \tag{61}$$

The first two terms on the left-hand side of this last expression are enough to get the basic weight in the estimate (60). Specifically, one has the bound:

$$(1+(t-|r^*|)^2)^{\frac{1}{2}}(\psi)^2\ \lesssim\ \int_{\mathbb{R}}\left[(1+(t-|r^*|)^2)(\partial_r^*\psi)^2\ +\ (\psi)^2\right]dr^*.$$

The proof of this kind of estimate is completely standard and reduces to the usual Sobolev bound after decomposing things via cutoffs on intervals of dyadic sizes according to the value of the weight $(1+(t-|r^*|)^2)^{\frac{1}{2}}$. See [7] for details on this procedure.

It remains for us to prove the estimate (60) for the more refined weight. It is clear from the form of the energy (61) that this bound holds when $r^*\in[-1,1]$. Therefore we only need to establish it for the cases $r^* < -1$ and $r* > 1$. We will do this separately by directly evaluating some weighted integrals similar to what we have done many times now. For values $t < r_0^* < -1$ we compute that (note that the weight r is essentially a constant here so we can safely disregard it):

$$(1 - \frac{2M}{r})^{\frac{1}{2}}(t + r_0^*)^2(\psi)^2(r_0^*)$$
$$= - \int_{r_0^*}^{0} \partial_{r^*}\left[(1 - \frac{2M}{r})^{\frac{1}{2}}(t + r^*)^2(\psi)^2\right] dr^* \ + \ (1 - \frac{2M}{r})^{\frac{1}{2}} \, t^2 \, (\psi)^2(0).$$

By collecting terms and using the Cauchy–Schwartz inequality this gives us the bound:

$$(1 - \frac{2M}{r})^{\frac{1}{2}}(t + r_0^*)^2(\psi)^2(r_0^*) \ \lesssim \ (1 - \frac{2M}{r})^{\frac{1}{2}} \, t^2 \, (\psi)^2(0)$$
$$+ \parallel (1 - \frac{2M}{r})^{\frac{1}{2}}(t + r^*)\psi \parallel_{L^2([0,r_0^*])} \cdot \parallel (t + r^*)\partial_{r^*}\psi \parallel_{L^2([0,r_0^*])} .$$

The crucial thing to notice here is that when the derivative falls on the weight $(1 - \frac{2M}{r})(t + r^*)^2$ it creates a *positive* term which can be collected with the left-hand side. This last line gives us the desired bound because all the L^2 type norms are covered by the energy $\mathcal{H}$, and so is the bound for ψ at the origin.

To wrap things up here we need to prove the refined bound in (60) for the region where $1 \leqslant r^*$. Notice that it suffices to do this for $r^* \leqslant t$ because otherwise the simpler weight on the right-hand side of (60) is favorable. Also, in this case we may safely ignore the weight $(1 - \frac{2M}{r})$ because it is essentially a non-zero constant. The desired bound will drop out from computing the following integral for fixed points $0 \leqslant r_0^* \leqslant t$:

$$r^{-1}(t - r_0^*)^2(\psi)^2(r_0^*) = \int_0^{r_0^*} \partial_{r^*}\left[r^{-1}(t - r^*)^2(\psi)^2\right] dr^* \ + \ r^{-1} \, t^2 \, (\psi)^2(0).$$

Computing the integral on the right-hand side of this last expression, throwing away *only* the term which results when the derivative falls on the weight $(t - r^*)^2$, collecting the terms of like sign to the left-hand side, and applying the Cauchy–Schwartz inequality we arrive at the bound:

$$r^{-1}(t - r_0^*)^2(\psi)^2(r_0^*) \ + \ \parallel r^{-1}(t - r^*)\psi \parallel^2_{L^2([0,r_0^*])} \lesssim \ r^{-1} \, t^2 \, (\psi)^2(0)$$
$$+ \parallel r^{-1}(t - r^*)\psi \parallel_{L^2([0,r_0^*])} \cdot \parallel (t - r^*)\partial_{r^*}\psi \parallel_{L^2([0,r_0^*])}.$$

From this one easily derives the bound:

$$r^{-1}(t - r_0^*)^2(\psi)^2(r_0^*) \ \lesssim \ r^{-1} \, t^2 \, (\psi)^2(0) \ + \ \parallel (t - r^*)\partial_{r^*}\psi \parallel_{L^2([0,r_0^*])},$$

which is itself bounded by the energy $\mathcal{H}$. This completes our proof of the estimate (60), and hence our demonstration of the global Sobolev bound (58). □

Proof of Theorem 4.1. We now use the previous three lemmas and the main decay estimate (7) to prove the global regularity result. This will follow by bootstrapping the usual local existence theorem. We will not state or prove this local result here because even in this context it is an elementary application of Picard iteration and energy estimates (in fact, one can use the estimates developed here to set up a global Picard iteration because the precise structure of the non-linearity does not need to be preserved). Now, our theorem will follow if we can show that for each fixed time t up to which we have existence the weak bound $\sup_{0\leqslant s\leqslant t} \| \phi(s) \|_{\mathcal{H}^2_\Omega} \leqslant 2C\mathcal{E}$ implies the stronger bound $\sup_{0\leqslant s\leqslant t} \| \phi(s) \|_{\mathcal{H}^2_\Omega} \leqslant C\mathcal{E}$. From the estimate (7), we see that we can provide this as long as we can show the bound:

$$\int_0^t \| (1+|\underline{u}|+|u|)(1-\frac{2M}{r})\, r \cdot (\sqrt{1-\Delta_{sph}})^{\frac{3}{2}}(|\phi|^p\phi)\ (s) \|_{L^2(dr^*d\omega)}\, ds \lesssim \sup_{0\leqslant s\leqslant t} \| \phi(s) \|^{p+1}_{\mathcal{H}^2_\Omega}. \tag{62}$$

To compute the integral on the left-hand side, we use the paraproduct estimate (57) which gives us the bound:

$$\| (\sqrt{1-\Delta_{sph}})^{\frac{3}{2}}(|\phi|^p\phi)\ (s) \|_{L^2(d\omega)} \lesssim \| \phi(s) \|^p_{L^\infty_\omega} \cdot \| (\sqrt{1-\Delta_{sph}})^{\frac{3}{2}}\phi\ (s) \|_{L^2(d\omega)}.$$

Using now the definition of the energy $\mathcal{H}^2_\Omega$ and the Poincaré estimate (56) to control the L^2 norm of the zero harmonic of ϕ, we see that we can bound:

$$\text{(L.H.S.)(62)} \lesssim \sup_{0\leqslant s\leqslant t} \| \phi(s) \|_{\mathcal{H}^2_\Omega} \cdot \int_0^t \| (1+|\underline{u}|+|u|)(1-\frac{2M}{r})^{\frac{1}{2}}|\phi|^p\ (s) \|_{L^\infty_{(r^*,\omega)}} ds.$$

The claim (62) will now follow once we can show the fixed time bound:

$$\| (1+|\underline{u}|+|u|)(1-\frac{2M}{r})^{\frac{1}{2}}|\phi|^p\ (s) \|_{L^\infty_{(r^*,\omega)}} \lesssim (1+s)^{-1-\frac{1}{2}(p-2)} \| \phi(s) \|^p_{\mathcal{H}^2_\Omega}. \tag{63}$$

This last bound follows easily from the global Sobolev estimate (58). To see this, it is convenient to split things into the three regions:

$$\mathcal{R}_1 = \{r^* < -\frac{1}{2}s\}, \qquad \mathcal{R}_2 = \{|r^*| \leqslant \frac{1}{2}s\} \qquad \mathcal{R}_3 = \{\frac{1}{2}s \leqslant r^*\}.$$

In the region $\mathcal{R}_1$ the bound on (63) is completely trivial. Here the non-linear interaction breaks down entirely. All one has to do is to use the bound:

$$(1-\frac{2M}{r}) \lesssim e^{\frac{1}{2M}r^*}, \qquad r^* \leqslant 0,$$

and the fact that the ϕ are uniformly bounded (which is the best we can do!).

In the transition region $\mathcal{R}_2$, we use the fine bound contained on the right-hand side of (58). This gives us that:

$$(1-\frac{2M}{r})^{\frac{1}{2}}|\phi|^2 \lesssim (1+s)^{-2}.$$

Taking on the weight $(1+|\underline{u}|+|u|)$ which is $\sim (1+s)$ in this region, and using the fact that $|\phi|^{p-2} \lesssim (1+s)^{-\frac{1}{2}(p-2)}$ in $\mathcal{R}_2$ we again have (63).

Finally, in the Minkowski like region $\mathcal{R}_3$ we have from (58) the uniform decay estimate $|\phi| \lesssim (1+t+r^*)^{-1}$. This easily implies (63). This completes our proof of the bootstrapping estimate (62) and hence our demonstration of Theorem 4.1. □

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References

1. Blue, P.: Decay estimates and phase space analysis for wave equations on some black hole metrics. *Ph.D. thesis*, Rutgers, The State University of New Jersey, New Brunswick, NJ, October 2004
2. Blue, P, Soffer, A.: Semilinear wave equations on the Schwarzschild manifold. I. Local decay estimates. Adv. Differ. Eq. **8**(5), 595–614 (2003)
3. Blue, P., Soffer, A.: The wave equation on the Schwarzschild metric. II. Local decay for the spin-2 Regge-Wheeler equation. J. Math. Phys. **46**(1), 012502 (2005)
4. Blue, P., Soffer, A.: Phase space analysis on some black hole manifolds. submitted
5. Dafermos, M., Rodnianski, I.: A proof of Price's law for the collapse of a self-gravitating scalar field. Invent. Math. **162**(2), 381–457 (2005)
6. Dafermos, M., Rodnianski, I.: Small-amplitude nonlinear waves on a black hole background. J. Math. Pures Appl. (9) **84**(9), 1147–1172 (2005)
7. Lindblad, H., Sterbenz, J.: Global Stability for Charged Scalar Fields on Minkowski Space. Submitted
8. Hawking, S.W., Ellis, G.F.R.: The Large Scale Structure of Space-time. *Cambridge Monographs on Mathematical Physics*, No. 1. London-New York: Cambridge University Press, 1973
9. Misner, C.W., Thorne, K.S., Wheeler, J.A.: *Gravitation*. San Francisco, CA: W. H. Freeman and Co., 1973
10. Morawetz, C.: The limiting amplitude principle. Comm. Pure Appl. Math. **15**, 349–361 (1962)
11. Taylor, M.E.: Tools for PDE. Pseudodifferential operators, paradifferential operators, and layer potentials. *Mathematical Surveys and Monographs* **81**. Providence, RI: American Mathematical Society, 2000

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