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Global Existence and Blow-Up Phenomena for the Degasperis-Procesi Equation

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Abstract: This paper is concerned with several aspects of the existence of global solutions and the formation of singularities for the Degasperis-Procesi equation on the line. Global strong solutions to the equation are determined for a class of initial profiles. On the other hand, it is shown that the first blow-up can occur only in the form of wave-breaking. A new wave-breaking mechanism for solutions is described in detail and two results of blow-up solutions with certain initial profiles are established.

1. Introduction

Recently, Degasperis and Procesi [21] studied the following family of third order dispersive PDE conservation laws,

$$u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{txx} = \left(c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx}\right)_x, \tag{1.1}$$

where α, c_0, c_1, c_2, and c_3 are real constants and subindices denote partial derivatives. They found that there are only three equations that satisfy the asymptotic integrability condition within this family: the KdV equation, the Camassa-Holm equation and the Degasperis-Procesi equation.

With $\alpha = c_2 = c_3 = 0$ in Eq. (1.1), it becomes the well-known Korteweg-de Vries equation which describes the unidirectional propagation of waves at the free surface of shallow water under the influence of gravity: $u(t, x)$ represents the wave height above a flat bottom, x is proportional to the distance in the direction of propagation and t is proportional to the elapsed time. The KdV equation is completely integrable and its solitary waves are solitons [22, 39]. The Cauchy problem of the KdV equation has been the subject of a number of studies, and a satisfactory local or global (in time) existence theory is now in hand (for example, see [31, 43]). It is shown that the KdV equation is globally well-posed for $u_0 \in L^2(\mathbb{R})$ [43]. It is observed that the KdV equation does not accommodate wave breaking (by wave breaking we understand that the wave remains bounded but its slope becomes unbounded in finite time [45]).

For $c_1 = -\frac{3}{2}c_3/\alpha^2$ and $c_2 = c_3/2$, Eq. (1.1) becomes the Camassa-Holm equation, modeling the unidirectional propagation of shallow water waves over a flat bottom, $u(t, x)$ standing for the fluid velocity at time t in the spatial x direction and c_0 being a nonnegative parameter related to the critical shallow water speed [3, 23, 29]. The Camassa-Holm equation is also a model for the propagation of axially symmetric waves in hyperelastic rods [17, 19]. It has a bi-Hamiltonian structure [33, 26] and is completely integrable [3, 9]. Its solitary waves are smooth if $c_0 > 0$ and peaked in the limiting case $c_0 = 0$ [4]. The orbital stability of the peaked solitons is proved in [16], and that of the smooth solitons in [18]. The explicit interaction of the peaked solitons is given in [1].

The Cauchy problem of the Camassa-Holm equation has been studied extensively. It has been shown that the Camassa-Holm equation is locally well-posed [10, 34, 42] with the initial data $u_0 \in H^s(\mathbb{R}),\ s > \frac{3}{2}$. More interestingly, it has global strong solutions [7, 10] and also blow-up solutions in finite time [7, 10, 11, 14, 35] with a different class of initial profiles in the Sobolev spaces $H^s(\mathbb{R}),\ s > 3/2$. On the other hand, it has global weak solutions in $H^1(\mathbb{R})$ [2, 12, 15, 46]. It is observed that if u is the solution of the Camassa-Holm equation with the initial data u_0 in $H^1(\mathbb{R})$, we have for all $t > 0$,

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \sqrt{2}\|u(t, \cdot)\|_{H^1(\mathbb{R})} \leq \sqrt{2}\|u_0(\cdot)\|_{H^1(\mathbb{R})}.$$

The advantage of the Camassa-Holm equation in comparison with the KdV equation lies in the fact that the Camassa-Holm equation has peaked solitons and models wave breaking [4].

With $c_1 = -2c_3/\alpha^2$ and $c_2 = c_3$ in Eq. (1.1), by rescaling, shifting the dependent variable and applying a Galilean boost [20], we find the Degasperis-Procesi equation of the form

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad t > 0,\ x \in \mathbb{R}. \tag{1.2}$$

Degasperis, Holm and Hone [20] proved the formal integrability of Eq. (1.2) by constructing a Lax pair. They also showed [20] that Eq. (1.2) has bi-Hamiltonian structure and an infinite sequence of conserved quantities, and admits exact peakon solutions which are analogous to the Camassa-Holm peakons.

The Degasperis-Procesi equation can be regarded as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same as for the Camassa-Holm shallow water equation. Dullin, Gottwald and Holm [24] showed that the Degasperis-Procesi equation can be obtained from the shallow water elevation equation by an appropriate Kodama transformation. Lundmark and Szmigielski [37] presented an inverse scattering approach for computing n-peakon solutions to Eq. (1.2). Vakhnenko and Parkes [44] investigated traveling wave solutions of Eq. (1.2). Holm and Staley [28] studied stability of solitons and peakons numerically to Eq. (1.2).

After the Degasperis-Procesi equation (1.2) was derived, many papers were devoted to its study, cf. [13, 27, 32, 36, 38, 47, 48] and the citations therein. For example, Yin proved local well-posedness to Eq. (1.2) with initial data $u_0 \in H^s(\mathbb{R}),\ s > \frac{3}{2}$ on the line [47] and on the circle [48] and derived the precise blow-up scenario and a blow-up result. The global existence of strong solutions and global weak solutions to Eq. (1.2) are also investigated in [49, 50]. Recently, Lenells [32] classified all weak traveling wave solutions. Matsuno [38] studied multisoliton solutions and their peakon limit.

Analogous to the case of Camassa-Holm equation [8], Henry [27] and Mustafa [41] showed that smooth solutions to Eq. (1.2) have infinite speed of propagation.

Coclite and Karlsen [13] also obtained global existence results for entropy weak solutions belonging to the class of $L^1(\mathbb{R}) \cap BV(\mathbb{R})$ and the class of $L^2(\mathbb{R}) \cap L^4(\mathbb{R})$.

Despite the similarities to the Camassa-Holm equation, we would like to point out that these two equations are truly different. One of the important features of Eq. (1.2) is it has not only peakon solitons [20], $u(t,x) = ce^{-|x-ct|}$, $c > 0$ but also shock peakons [6, 36] of the form

$$u(t,x) = -\frac{1}{t+k}\operatorname{sgn}(x)e^{-|x|}, \quad k > 0.$$

It is easy to see from [36] that the above shock-peakon solutions can be observed by substituting $(x,t) \longmapsto (\epsilon x, \epsilon t)$ to Eq. (1.2) and letting $\epsilon \to 0$ so that it yields the "derivative Burger's equation" $(u_t + uu_x)_{xx} = 0$, from which shock waves form.

On the other hand, the isospectral problem in the Lax pair for Eq. (1.2) is the third-order equation

$$\psi_x - \psi_{xxx} - \lambda y \psi = 0$$

cf. [20], while the isospectral problem for the Camassa-Holm equation is the second order equation

$$\psi_{xx} - \frac{1}{4}\psi - \lambda y \psi = 0$$

(in both cases $y = u - u_{xx}$) cf. [3]. Another indication of the fact that there is no simple transformation of Eq. (1.2) into the Camassa-Holm equation is the entirely different form of conservation laws for these two equations [3, 20]. Furthermore, the Camassa-Holm equation is a re-expression of geodesic flow on the diffeomorphism group [13] or on the Bott-Virasoro group [40], while no geometric derivation of the Degasperis-Procesi equation is available.

The following are three useful conservation laws of the Degasperis-Procesi equation:

$$E_1(u) = \int_{\mathbb{R}} y\,dx, \quad E_2(u) = \int_{\mathbb{R}} yv\,dx, \quad E_3(u) = \int_{\mathbb{R}} u^3\,dx,$$

where $y = (1-\partial_x^2)u$ and $v = (4-\partial_x^2)^{-1}u$, while the corresponding three useful conservation laws of the Camassa-Holm equation are the following:

$$F_1(u) = \int_{\mathbb{R}} y\,dx, \quad F_2(u) = \int_{\mathbb{R}} (u^2 + u_x^2)\,dx, \quad F_3(u) = \int_{\mathbb{R}} (u^3 + uu_x^2)\,dx.$$

It is found that the corresponding conservation laws of the Degasperis-Procesi equation are much weaker than those of the Camassa-Holm equation. Therefore, the issue of if and how particular initial data generate a global solution or blow-up in finite time is more subtle.

As far as we know, the case of the Camassa-Holm equation is well understood by now [7, 10, 11, 14, 35] and the citations therein, while the Degasperis-Procesi equation case is the subject of this paper. The goal of this paper is to establish several new global existence and blow-up results for Eq. (1.2), and blow-up set as well so that important physical phenomena of Eq. (1.2) (such as wave breaking and shock waves) could be understood deeply. A forthcoming paper by the authors [25] deals with global weak solutions in $H^1(\mathbb{R})$ and blow-up structure for the Degasperis-Procesi equation.

As mentioned earlier, the first blow-up must occur as wave breaking and shock waves possibly appear afterwards. On the other hand, to obtain global existence from local results is a matter of a priori estimates. One approach to prove global existence or wave breaking for shallow water wave (1.2) is to try to follow an idea of Constantin [7], that is, we show that for a large class of initial profiles the corresponding solutions to Eq. (1.2) either exist globally in time or blow up in finite time by using a continuous family of diffeomorphisms of the line associated to Eq. (1.2). However, those ideas in [7] heavily depend on the conservation law $F_2(u)$ which is a H^1-norm. Although the bi-Hamiltonian structure of Eq. (1.2) provides an infinite number of conservation laws in our case, the conservation laws $E_i(u)$, can not guarantee the boundedness of the slope of wave, and there is no way to find conservation laws controlling the H^1-norm. To deal with this difficulty and make wave breaking possible, for example in Theorem 4.2, we develop a new wave-breaking mechanism for solutions in detail. We first obtain a priori estimate $L^\infty-$norm of the solution, then by the structure of the equation, we find the slope of the solution approaches infinity in finite or infinite time much faster than the solution itself even it is unbounded. As a result, this leads to wave breaking phenomenon too.

The remainder of the paper is organized as follows. In Sect. 2, we recall the local well-posedness of the Cauchy problem of Eq. (1.2) with initial data $u_0 \in H^s(\mathbb{R}),\ s > \frac{3}{2}$, the precise blow-up scenario of strong solutions, and two useful results which are crucial in the proof of global existence and blow-up phenomena for Eq. (1.2) from [47, 50]. In Sect. 3, by using a new conservation law and a very useful a priori estimate for the $L^\infty-$norm of the strong solutions to Eq. (1.2), we will present two new global existence results for strong solutions to Eq. (1.2) with certain initial profiles. The last section, Sect. 4, is devoted to establish two new blow-up results and show the existence of a breaking point where the slope of the solution becomes infinity exactly at breaking time.

Notation. As above and henceforth, we denote by $*$ the convolution. We write $\hat{f}$ as the Fourier transform of f. We also use $(\ ,\ )$ to represent the standard inner product in $L^2(\mathbb{R})$. For $1 \le p \le \infty$, the norm in the Lebesgue space L^p will be written $\|\cdot\|_{L^p}$, while $\|\cdot\|_s,\ s \ge 0$ will stand for the norm in the classical Sobolev spaces $H^s(\mathbb{R})$.

2. Preliminaries

Since we shall also use a priori estimates and further properties of solutions in $H^s(\mathbb{R}),\ s > \frac{3}{2}$, we briefly collect the needed results from [47, 50] in order to pursue our goal.

With $y := u - u_{xx}$, Eq. (1.2) takes the form of a quasi-linear evolution equation of hyperbolic type:

$$\begin{cases} y_t + uy_x + 3u_x y = 0, & t > 0,\ x \in \mathbb{R}, \\ y(0,x) = u_0(x) - u_{0,xx}(x), & x \in \mathbb{R}. \end{cases} \tag{2.1}$$

Note that if $p(x) := \frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}$, then $(1-\partial_x^2)^{-1} f = p * f$ for all $f \in L^2(\mathbb{R})$ and $p * (u - u_{xx}) = u$. Using this identity, we can rewrite Eq. (2.1) as follows:

$$\begin{cases} u_t + uu_x + \partial_x p * (\frac{3}{2}u^2) = 0, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{2.2}$$

The local well-posedness of the Cauchy problem of Eq. (2.2) with initial data $u_0 \in H^s(\mathbb{R}),\ s > \frac{3}{2}$ can be obtained by applying the Kato's theorem [30, 47]. As a result, we have the following well-posedness result.

Lemma 2.1 [47]. *Given* $u_0 \in H^s(\mathbb{R}),\ s > \frac{3}{2}$*, there exist a maximal* $T = T(u_0) > 0$ *and a unique solution* u *to Eq.* (1.2) (*or Eq.*(2.2))*, such that*

$$u = u(., u_0) \in C([0, T); H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R})).$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping $u_0 \to u(., u_0) : H^s(\mathbb{R}) \to C([0, T); H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R}))$ *is continuous and the maximal time of existence* $T > 0$ *can be chosen to be independent of* s.

By using the local well-posedness in Lemma 2.1 and the energy method, one can get the following precise blow-up scenario of strong solutions to Eq. (2.2).

Lemma 2.2 [47]. *Given* $u_0 \in H^s(\mathbb{R}),\ s > \frac{3}{2}$*, blow up of the solution* u=$u(.,u_0)$ *in finite time* $T < +\infty$ *occurs if and only if*

$$\liminf_{t \uparrow T} \{\inf_{x \in \mathbb{R}} [u_x(t, x)]\} = -\infty.$$

Remark 2.1. Lemma 2.2 shows that both the Degasperis-Procesi equation and the Camassa-Holm equation have the same blow-up scenario [10]. Since the H^1-norm of solution to the Camassa-Holm equation is conserved, we see that the slope of the solution to the Camassa-Holm equation becomes unbounded whereas its amplitude remains bounded. However, the H^1-norm of solution to the Degasperis-Procesi equation is not conserved generally, we can not infer this blow-up phenomenon for the Degasperis-Procesi equation directly from Lemma 2.2.

Consider the following differential equation:

$$\begin{cases} q_t = u(t, q), & t \in [0, T), \\ q(0, x) = x, & x \in \mathbb{R}. \end{cases} \tag{2.3}$$

Applying classical results in the theory of ordinary differential equations, one can obtain the following two results on q which are crucial in the proof of global existence and blow-up solutions.

Lemma 2.3 [50]. *Let* $u_0 \in H^s(\mathbb{R}),\ s \geq 3$*, and let* $T > 0$ *be the maximal existence time of the corresponding solution* u *to Eq.* (2.2). *Then Eq.* (2.3) *has a unique solution* $q \in C^1([0, T) \times \mathbb{R}, \mathbb{R})$. *Moreover, the map* $q(t, .)$ *is an increasing diffeomorphism of* $\mathbb{R}$ *with*

$$q_x(t, x) = \exp\left(\int_0^t u_x(s, q(s, x))ds\right) > 0, \quad \forall (t, x) \in [0, T) \times \mathbb{R}.$$

Lemma 2.4 [50]. *Let* $u_0 \in H^s(\mathbb{R}),\ s \geq 3$*, and let* $T > 0$ *be the maximal existence time of the corresponding solution* u *to Eq.* (2.2). *Setting* $y := u - u_{xx}$*, we have*

$$y(t, q(t, x))q_x^3(t, x) = y_0(x), \quad \forall (t, x) \in [0, T) \times \mathbb{R}.$$

3. Global Existence

In this section, we will begin by deriving a conservation law for strong solutions to Eq. (2.2). Using this conservation law, we then obtain a priori estimate for the L^∞−norm of the strong solutions. This enables us to establish several global existence theorems.

Lemma 3.1. *If $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, then as long as the solution $u(t,x)$ given by Lemma* 2.1 *exists, we have*

$$\int_{\mathbb{R}} y(t,x)v(t,x)dx = \int_{\mathbb{R}} y_0(x)v_0(x)dx,$$

where $y(t,x) = u(t,x) - u_{xx}(t,x)$ and $v(t,x) = (4-\partial_x^2)^{-1}u$. Moreover, we have

$$\|u(t)\|_{L^2}^2 \leq 4\|u_0\|_{L^2}^2.$$

Proof. Applying Lemma 2.1 and a simple density argument, we only need to show that the above theorem with some $s > \frac{3}{2}$. Here we assume $s = 3$ to prove the above theorem. Let $T > 0$ be the maximal time of existence of the solution u to Eq. (2.2) with initial data $u_0 \in H^3(\mathbb{R})$ such that $u \in C([0,T); H^3(\mathbb{R})) \cap C^1([0,T); H^2(\mathbb{R}))$, which is guaranteed by Lemma 2.1. By Eq. (2.2), we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} yv\,dx &= \frac{1}{2}\int_{\mathbb{R}} y_t v\,dx + \frac{1}{2}\int_{\mathbb{R}} yv_t dx = \int_{\mathbb{R}} y_t\, v\,dx \\
&= -\int_{\mathbb{R}} vy_x u\,dx - 3\int_{\mathbb{R}} vyu_x dx \\
&= -\int_{\mathbb{R}} v(yu)_x\,dx - 2\int_{\mathbb{R}} vyu_x dx.
\end{aligned}$$

Using the relations $y = u - u_{xx}$ and $4v - v_{xx} = u$, it yields that

$$\begin{aligned}
\int_{\mathbb{R}} v(yu)_x\,dx &= -\int_{\mathbb{R}} v_x yu\,dx = -\int_{\mathbb{R}} v_x u^2\,dx + \int_{\mathbb{R}} v_x uu_{xx} dx \\
&= -\int_{\mathbb{R}} v_x u^2\,dx - \int_{\mathbb{R}} (v_x u)_x\, u_x\,dx \\
&= -\int_{\mathbb{R}} v_x u^2\,dx - \int_{\mathbb{R}} v_{xx} uu_x\,dx - \int_{\mathbb{R}} v_x u_x^2 dx \\
&= -\int_{\mathbb{R}} v_x u^2\,dx + \frac{1}{2}\int_{\mathbb{R}} v_{xxx} u^2\,dx - \int_{\mathbb{R}} v_x u_x^2 dx \\
&= -\int_{\mathbb{R}} v_x u^2\,dx + \frac{1}{2}\int_{\mathbb{R}} (4v_x - u_x)u^2\,dx - \int_{\mathbb{R}} v_x u_x^2 dx \\
&= -\int_{\mathbb{R}} v_x u^2\,dx + 2\int_{\mathbb{R}} v_x u^2\,dx - \int_{\mathbb{R}} v_x u_x^2 dx.
\end{aligned}$$

On the other hand,

$$2\int_{\mathbb{R}} v\,y\,u_x\,dx = 2\int_{\mathbb{R}} vuu_x\,dx - 2\int_{\mathbb{R}} vu_{xx}u_x dx = -\int_{\mathbb{R}} v_x\,u^2\,dx + \int_{\mathbb{R}} v_x u_x^2\,dx.$$

Combining the above three relations, we deduce that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} yv\,dx = -\int_{\mathbb{R}} v(yu)_x\,dx - 2\int_{\mathbb{R}} vyu_x dx = 0.$$

Consequently, this implies the desired conserved quantity. In view of the above conservation law, it then follows that

$$\begin{aligned}\|u(t)\|_{L^2}^2 = \|\hat{u}(t)\|_{L^2}^2 &\le 4\int_{\mathbb{R}} \frac{1+\xi^2}{4+\xi^2}|\hat{u}(t,\xi)|^2\,d\xi = 4(\hat{y}(t),\ \hat{v}(t))\\ &= 4(y(t),\ v(t)) = 4(y_0,\ v_0) = 4(\hat{y}_0,\ \hat{v}_0)\\ &\le 4\int_{\mathbb{R}} \frac{1+\xi^2}{4+\xi^2}|\hat{u}_0(\xi)|^2\,d\xi \le 4\|\hat{u}_0\|_{L^2}^2 = 4\|u_0\|_{L^2}^2.\end{aligned}$$

This completes the proof of Lemma 3.1.

The following important estimate can be obtained by Lemma 3.1.

Lemma 3.2. *Assume $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. Let T be the maximal existence time of the solution u to the Eq.* (2.2) *guaranteed by Lemma* 2.1. *Then we have*

$$\|u(t,x)\|_{L^\infty} \le 3\|u_0(x)\|_{L^2}^2 t + \|u_0(x)\|_{L^\infty}, \quad \forall t \in [0,T].$$

Proof. Applying Lemma 2.1 and a simple density argument, it suffices to consider $s = 3$ to prove the above theorem. Let $T > 0$ be the maximal time of existence of the solution u to Eq. (2.2) with the initial data $u_0 \in H^3(\mathbb{R})$ such that $u \in C([0,T); H^3(\mathbb{R})) \cap C^1([0,T); H^2(\mathbb{R}))$, which is guaranteed by Lemma 2.1. By (2.2), we have

$$u_t + uu_x = -\partial_x p * \left(\frac{3}{2}u^2\right) = -3p * (uu_x). \tag{3.1}$$

Note that

$$\begin{aligned}-3p * (uu_x) &= -\frac{3}{2}\int_{-\infty}^{+\infty} e^{-|x-\eta|}uu_\eta d\eta\\ &= -\frac{3}{2}\int_{-\infty}^{x} e^{-x+\eta}uu_\eta d\eta - \frac{3}{2}\int_{x}^{+\infty} e^{x-\eta}uu_\eta d\eta\\ &= \frac{3}{4}\int_{-\infty}^{x} e^{-|x-\eta|}u^2 d\eta - \frac{3}{4}\int_{x}^{+\infty} e^{-|x-\eta|}u^2 d\eta.\end{aligned}$$

By (2.3), we have

$$\frac{du(t,q(t,x))}{dt} = u_t(t,q(t,x)) + u_x(t,q(t,x))\frac{dq(t,x)}{dt} = (u_t + uu_x)(t,q(t,x)).$$

It then follows from (3.1) that

$$-\frac{3}{4}\int_{q(t,x)}^{+\infty} e^{-|q(t,x)-\eta|}u^2 d\eta \le \frac{du(t,q(t,x))}{dt} \le \frac{3}{4}\int_{-\infty}^{q(t,x)} e^{-|q(t,x)-\eta|}u^2 d\eta.$$

It thus transpires that

$$\left|\frac{du(t,q(t,x))}{dt}\right| \le \frac{3}{4}\int_{-\infty}^{+\infty} e^{-|q(t,x)-\eta|}u^2 d\eta \le \frac{3}{4}\int_{-\infty}^{+\infty} u^2(t,\eta)d\eta.$$

In view of Lemma 3.1, we have

$$-3\|u_0\|_{L^2}^2 \le \frac{du(t,q(t,x))}{dt} \le 3\|u_0\|_{L^2}^2.$$

Integrating the above inequality with respect to $t < T$ on $[0,t]$ yields

$$-3\|u_0\|_{L^2}^2 t + u_0(x) \le u(t,q(t,x)) \le 3\|u_0\|_{L^2}^2 t + u_0(x).$$

Thus,

$$|u(t,q(t,x))| \le \|u(t,q(t,x))\|_{L^\infty} \le 3\|u_0\|_{L^2}^2 t + \|u_0\|_{L^\infty}. \tag{3.2}$$

Using the Sobolev embedding to ensure the uniform boundedness of $u_x(s,\eta)$ for $(s,\eta) \in [0,t]\times\mathbb{R}$ with $t \in [0,T)$, in view of Lemma 2.3, we get for every $t \in [0,T)$ a constant $C(t) > 0$ such that

$$e^{-C(t)} \le q_x(t,x) \le e^{C(t)}, \qquad x \in \mathbb{R}.$$

We deduce from the above equation that the function $q(t,\cdot)$ is strictly increasing on $\mathbb{R}$ with $\lim_{x\to\pm\infty} q(t,x) = \pm\infty$ as long as $t \in [0,T)$. Thus, by (3.2) we can obtain

$$\|u(t,x)\|_{L^\infty} = \|u(t,q(t,x))\|_{L^\infty} \le 3\|u_0\|_{L^2}^2 t + \|u_0\|_{L^\infty}. \tag{3.3}$$

This completes the proof of Lemma 3.2.

Remark 3.1. Although the H^1-norm of solution to the Degasperis-Procesi equation is not conserved generally, Lemmas 2.2 and Lemma 3.2 ensure that the slope of the solution to the Degasperis-Procesi equation becomes unbounded in finite time whereas its amplitude remains bounded in finite time.

We now present the first global existence result.

Theorem 3.1. *Assume $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. If $y_0 = u_0 - u_{0,xx}$ does not change sign on $\mathbb{R}$, then Eq.* (2.2) *has a global strong solution*

$$u = u(.,u_0) \in C([0,\infty); H^s(\mathbb{R})) \cap C^1([0,\infty); H^{s-1}(\mathbb{R})).$$

Moreover, $E_2(u) = \int_{\mathbb{R}} y\ v\ dx$ is a conservation law, where $y = u - u_{xx}$ and $v = (4-\partial_x^2)^{-1}u$, and we have for all $t \in \mathbb{R}_+$,

(i) $|u_x(t,\cdot)| \le |u(t,\cdot)|$ *on* $\mathbb{R}$,
(ii) $\|u\|_1^2 \le 6\|u_0\|_{L^2}^4 t^2 + 4\|u_0\|_{L^2}^2\|u_0\|_{L^\infty} t + \|u_0\|_1^2$.

Proof. We only assume $s = 3$ to prove the above theorem. Let T be the maximal time of existence of the solution u to Eq. (2.2) with initial data $u_0 \in H^3(\mathbb{R})$. We first consider the case where $y_0 \ge 0$ on $\mathbb{R}$. If $y_0 \ge 0$, then Lemma 2.3 and Lemma 2.4 ensure that $y \ge 0$ for all $t \in [0,T)$. Using $u = p * y$ and the positivity of p, we infer that $u(t,\cdot) \ge 0$ for all $t \ge 0$. Note that

$$u(t,x) = \frac{e^{-x}}{2}\int_{-\infty}^{x} e^{\eta} y(t,\eta)d\eta + \frac{e^x}{2}\int_x^{\infty} e^{-\eta} y(t,\eta)d\eta \tag{3.4}$$

and

$$u_x(t,x) = -\frac{e^{-x}}{2}\int_{-\infty}^{x} e^{\eta} y(\eta)d\eta + \frac{e^x}{2}\int_x^{\infty} e^{-\eta} y(\eta)d\eta. \tag{3.5}$$

From the above two equations, we deduce that

$$\begin{aligned} u(t,x)+u_x(t,x) &= e^x \int_x^{\infty} e^{-\eta} y(t,\eta)d\eta, \\ u(t,x)-u_x(t,x) &= e^{-x} \int_{-\infty}^{x} e^{\eta} y(t,\eta)d\eta. \end{aligned} \tag{3.6}$$

By (3.6) and $y \geq 0$ for all $t \in [0,T)$, we obtain for $t \in [0,T)$,

$$|u_x(t,x)| \leq u(t,x) \qquad \forall (t,x) \in [0,T) \times \mathbb{R}.$$

This proves (i). By Lemma 3.2, we have

$$|u_x(t,x)| \leq u(t,x) \leq \left(3\|u_0\|_{L^2}^2 t + \|u_0\|_{L^\infty}\right), \quad \forall (t,x) \in [0,T) \times \mathbb{R}.$$

The above inequality and Lemma 2.2 imply $T = \infty$. This proves that the solution u exists globally in time.

Multiplying Eq. (1.2) by u and integrating by parts, in view of (i) and Lemma 3.1 and Lemma 3.2, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \left(u^2+u_x^2\right) dx &= -4\int_{\mathbb{R}} u^2 u_x dx + 3\int_{\mathbb{R}} u u_x u_{xx} dx + \int_{\mathbb{R}} u^2 u_{xxx} dx \\ &= -\frac{1}{2}\int_{\mathbb{R}} u_x^3 dx \leq \frac{1}{2}\int_{\mathbb{R}} u^3 dx \leq \frac{1}{2}\|u\|_{L^\infty}\int_{\mathbb{R}} u^2 dx \\ &= \frac{1}{2}\left(12\|u_0\|_{L^2}^2 t + 4\|u_0\|_{L^\infty}\right)\int_{\mathbb{R}} u_0^2 dx. \end{aligned}$$

Integrating the above inequality with respect to t, we have

$$\|u\|_1^2 \leq 6\|u_0\|_{L^2}^4 t^2 + 4\|u_0\|_{L^2}^2\|u_0\|_{L^\infty} t + \|u_0\|_1^2.$$

This proves (ii) and completes the proof of the theorem with the assumption $y_0 \geq 0$ on $\mathbb{R}$. In the case when $y_0(x) \leq 0$ on $\mathbb{R}$, one can repeat the above proof to get the desired result.

Remark 3.2. Theorem 3.1 improves the previous global existence result in Theorem 3.4 (see [47]), where the additional assumptions $u_0 \in L^3(\mathbb{R})$ and $y_0 = (u_0 - u_{0,xx}) \in L^1(\mathbb{R})$ are needed. Since we have used a new conservation law and a priori estimate for the $L^\infty-$norm of strong solution to Eq. (2.2) in Lemma 3.1 and Lemma 3.2, it enables us to eliminate the additional assumptions $u_0 \in L^3(\mathbb{R})$ and $y_0 = (u_0 - u_{0,xx}) \in L^1(\mathbb{R})$.

We now present the second global existence result.

Theorem 3.2. *Assume $u_0 \in H^s(\mathbb{R}),\ s > \frac{3}{2}$ and there exists $x_0 \in \mathbb{R}$ such that*

$$\begin{cases} y_0(x) \leq 0 & \text{if} \quad x \leq x_0, \\ y_0(x) \geq 0 & \text{if} \quad x \geq x_0. \end{cases}$$

Then Eq. (2.2) *has a unique global strong solution*

$$u = u(., u_0) \in C([0,\infty); H^s(\mathbb{R})) \cap C^1([0,\infty); H^{s-1}(\mathbb{R})).$$

Moreover, $E_2(u) = \int_{\mathbb{R}} yv\, dx$ is a conservation law, where $y = (1-\partial_x^2)u$ and $v = (4-\partial_x^2)^{-1}u$, and for all $t \in \mathbb{R}_+$ we have

(i) $u_x(t,\cdot) \geq -|u(t,\cdot)|$ *on* $\mathbb{R}$,
(ii) $\|u\|_1^2 \leq 6\|u_0\|_{L^2}^4 t^2 + 4\|u_0\|_{L^2}^2\|u_0\|_{L^\infty} t + \|u_0\|_1^2$.

Proof. We only assume $s = 3$ to prove the above theorem. Note that the function $q(t, x)$ is an increasing diffeomorphism of $\mathbb{R}$ with $q_x(t, x) > 0$ with respect to time t. We infer from the assumptions of the theorem and Lemma 2.3 and Lemma 2.4 that for $t \in [0, T)$ we have

$$\begin{cases} y(t,x) \leq 0 & \text{if} \quad x \leq q(t, x_0), \\ y(t,x) \geq 0 & \text{if} \quad x \geq q(t, x_0), \end{cases} \tag{3.7}$$

and $y(t, q(t, x_0)) = 0,\ t \in [0, T)$.

By (3.6) and (3.7), we obtain for $t \in [0, T)$,

$$\begin{cases} u_x(t,x) \geq u(t,x) & \text{if} \quad x \leq q(t, x_0), \\ u_x(t,x) \geq -u(t,x) & \text{if} \quad x \geq q(t, x_0). \end{cases} \tag{3.8}$$

Therefore, $u_x(t,\cdot) \geq -|u(t,\cdot)|$ on $\mathbb{R}$ for all $t \in [0, T)$. This proves (i).

By Lemma 3.2 and (i), we have

$$u_x(t,\cdot) \geq -|u(t,\cdot)| \geq -\left(3\|u_0\|_{L^2}^2 t + \|u_0\|_{L^\infty}\right), \quad \forall t \in [0, T].$$

The above inequality and Lemma 2.2 imply $T = \infty$. This proves that the solution u exists globally in time.

Multiplying Eq. (1.2) by u and integrating by parts, in view of (i) and Lemma 3.2, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \left(u^2 + u_x^2\right) dx &= -4\int_{\mathbb{R}} u^2 u_x dx + 3\int_{\mathbb{R}} u u_x u_{xx} dx + \int_{\mathbb{R}} u^2 u_{xxx} dx \\ &= -\frac{1}{2}\int_{\mathbb{R}} u_x^3 dx \leq \frac{1}{2}\int_{\mathbb{R}} |u|^3 dx \leq \frac{1}{2}\|u\|_{L^\infty}\int_{\mathbb{R}} u^2 dx \\ &= \frac{1}{2}\left(12\|u_0\|_{L^2}^2 t + 4\|u_0\|_{L^\infty}\right)\int_{\mathbb{R}} u_0^2\, dx. \end{aligned}$$

Integrating the above inequality with respect to $t < T$ on $[0, t]$, we have

$$\|u\|_1^2 \leq 6\|u_0\|_{L^2}^4 t^2 + 4\|u_0\|_{L^2}^2\|u_0\|_{L^\infty} t + \|u_0\|_1^2.$$

This proves (ii) and completes the proof of the theorem.

Remark 3.3. Note that the previous blow-up result in Theorem 3.2 (see [47]) showed that if $u_0 \in H^s(\mathbb{R}),\ s > \frac{3}{2}$ is odd and $u_0'(0) < 0$, then the corresponding solution of Eq. (2.2) does not exist globally in time, while Theorem 3.2 implies that if $u_0 \in H^s(\mathbb{R}),\ s > \frac{3}{2}$ such that $y_0 = (u_0 - u_{0,xx}) \not\equiv 0$ is odd, $y_0 \leq 0$ on $\mathbb{R}_-$ and $y_0 \geq 0$ on $\mathbb{R}_+$, then the corresponding solutions of Eq. (2.2) exist globally in time. Since $u_0 = p * y_0$ with $p = \frac{1}{2}e^{-|x|},\ x \in \mathbb{R}$, one can verify that u_0 is also odd. However, from (3.5) we have $u_0'(0) > 0$.

4. Blow-up Phenomena

Our purpose here is to establish two new blow-up results for Eq. (2.2) with certain initial profiles and show that there is at least one point where the slope of the solution becomes infinity exactly at breaking time.

We are now in the position to present the first blow-up result.

Theorem 4.1. *Let $\varepsilon > 0$ and $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. Assume there is $x_0 \in \mathbb{R}$ such that*

$$u_0'(x_0) < -\frac{(1+\varepsilon)\sqrt{6}}{4}\left(\|u_0\|_{L^\infty} + (2\sqrt{6}\|u_0\|_{L^2}^2 \ln\left(1+\frac{2}{\varepsilon}\right) + \|u_0\|_{L^\infty}^2)^{\frac{1}{2}}\right).$$

Then the corresponding solution to Eq. (2.2) *blows up in finite time. Moreover, the maximal time of existence is estimated above by*

$$\frac{\left(2\sqrt{6}\|u_0\|_{L^2}^2 \ln\left(1+\frac{2}{\varepsilon}\right) + \|u_0\|_{L^\infty}^2\right)^{\frac{1}{2}} - \|u_0\|_{L^\infty}}{6\|u_0\|_{L^2}^2}.$$

Proof. As mentioned earlier, here we only need to show that the above theorem holds for $s = 3$. Let $T > 0$ be the maximal time of existence of the solution u to Eq. (2.2) with the initial data $u_0 \in H^3(\mathbb{R})$. Differentiating Eq. (2.2) with respect to x, in view of $\partial_x^2 p * f = p * f - f$, we have

$$u_{tx} = -u_x^2 - uu_{xx} + \frac{3}{2}u^2 - p * \left(\frac{3}{2}u^2\right). \tag{4.1}$$

Note that

$$\begin{aligned}\frac{du_x(t,q(t,x))}{dt} &= u_{xt}(t,q(t,x)) + u_{xx}(t,q(t,x))\frac{dq(t,x)}{dt}\\ &= u_{tx}(t,q(t,x)) + u(t,q(t,x))u_{xx}(t,q(t,x)).\end{aligned} \tag{4.2}$$

By (4.1) and (4.2), in view of $p * \left(\frac{3}{2}u^2\right)(t,q(t,x)) \geq 0$, we deduce that

$$\begin{aligned}\frac{du_x(t,q(t,x))}{dt} &= -u_x^2(t,q(t,x)) + \frac{3}{2}u^2(t,q(t,x)) - p * \left(\frac{3}{2}u^2(t,q(t,x)\right)\\ &\leq -u_x^2(t,q(t,x)) + \frac{3}{2}u^2(t,q(t,x))\\ &\leq -u_x^2(t,q(t,x)) + \frac{3}{2}\left(3\|u_0\|_{L^2}^2 t + \|u_0\|_{L^\infty}\right)^2.\end{aligned} \tag{4.3}$$

Set $m(t) = u_x(t,q(t,x_0))$ and fix $\varepsilon > 0$. Taking

$$T_1 = \frac{\left(2\sqrt{6}\|u_0\|_{L^2}^2 \ln\left(1+\frac{2}{\varepsilon}\right) + \|u_0\|_{L^\infty}^2\right)^{\frac{1}{2}} - \|u_0\|_{L^\infty}}{6\|u_0\|_{L^2}^2}$$

and

$$K(T_1) = \frac{\sqrt{6}}{2}\left(3\|u_0\|_{L^2}^2 T_1 + \|u_0\|_{L^\infty}\right),$$

it is found that

$$2K(T_1)T_1 - \ln\left(1 + \frac{2}{\varepsilon}\right) \geq 0. \tag{4.4}$$

By the assumption of the theorem, we have

$$m(0) < -(1+\varepsilon)K(T_1).$$

This implies that

$$0 < \frac{m(0) - K(T_1)}{m(0) + K(T_1)} = 1 - \frac{2K(T_1)}{m(0) + K(T_1)} \leq 1 + \frac{2}{\varepsilon}.$$

It then follows from the above inequality and (4.4) that

$$\frac{1}{2K(T_1)} \ln \frac{m(0) - K(T_1)}{m(0) + K(T_1)} \leq T_1.$$

In view of (4.3), we have

$$\frac{dm(t)}{dt} \leq -m^2(t) + K^2(T_1), \quad \forall t \in [0, T_1] \cap [0, T). \tag{4.5}$$

Note that $m(0) < -(1+\varepsilon)K(T_1) < -K(T_1)$ and $\frac{1}{2K(T_1)} \ln \frac{m(0)-K(T_1)}{m(0)+K(T_1)} \leq T_1$. Thus the standard argument of continuity shows $m(t) \leq -K(T_1)$, for all $t \in [0, T_1] \cap [0, T)$. By solving the inequality (4.5), we can obtain

$$\frac{m(0) + K(T_1)}{m(0) - K(T_1)} e^{2K(T_1)t} - 1 \leq \frac{2K(T_1)}{m(t) - K(T_1)} \leq 0.$$

Since $0 < \frac{m(0)+K(T_1)}{m(0)-K(T_1)} < 1$, there exists

$$0 < T < \frac{1}{2K(T_1)} \ln\left(\frac{m(0) - K(T_1)}{m(0) + K(T_1)}\right) \leq T_1,$$

such that $\lim_{t \uparrow T} m(t) = -\infty$. This completes the proof of the theorem.

Remark 4.1. Note that if $\varepsilon > 0$ goes to positive infinity and the assumption of Theorem 4.1 still holds, then the maximal time of existence of the solution will tend to zero. This means that the steeper the slope of solution at some point is, the quicker the solution blows up.

We now present the second blow-up result.

Theorem 4.2. *Let $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. Assume there exists $x_0 \in \mathbb{R}$ such that*

$$\begin{cases} y_0(x) = u_0(x) - u_{0,xx}(x) \geq 0 & \text{if} \quad x \leq x_0, \\ y_0(x) = u_0(x) - u_{0,xx}(x) \leq 0 & \text{if} \quad x \geq x_0, \end{cases}$$

and y_0 changes sign. Then, the corresponding solution to Eq.(2.2) *blows up in finite time.*

Proof. Again, we only need to show that the above theorem holds for $s = 3$. Let $T > 0$ be the maximal time of existence of the solution u to Eq. (2.2) with the initial data $u_0 \in H^3(\mathbb{R})$.

In view of (4.1), we have

$$\begin{aligned} u_{tx} + uu_{xx} &= -u_x^2 + \frac{3}{2}u^2 - p * \left(\frac{3}{2}u^2\right) \\ &= -u_x^2 + \frac{3}{2}u^2 - p * \left(\frac{1}{2}u_x^2 + u^2\right) + \frac{1}{2}p * \left(u_x^2 - u^2\right). \end{aligned}$$

Due to $p * \left(\frac{1}{2}u_x^2 + u^2\right)(t,x) \geq \frac{1}{2}u^2(t,x)$, $\forall (t,x) \in [0,T) \times \mathbb{R}$ (see p. 347, line 11 in [7]), it follows from the above relation that

$$u_{tx} + uu_{xx} \leq -u_x^2 + u^2 + \frac{1}{2}p * \left(u_x^2 - u^2\right). \tag{4.6}$$

Note that the function $q(t,x)$ is an increasing diffeomorphism of $\mathbb{R}$ with $q_x(t,x) > 0$ with respect to time t. We infer from the assumption of the theorem and Lemma 2.4 that for $t \in [0,T)$ we have

$$\begin{cases} y(t,x) \geq 0 & \text{if} \quad x \leq q(t,x_0), \\ y(t,x) \leq 0 & \text{if} \quad x \geq q(t,x_0), \end{cases} \tag{4.7}$$

and $y(t, q(t,x_0)) = 0,\ t \in [0,T)$. Define

$$M(t,x) := e^{-x} \int_{-\infty}^{x} e^{\eta} y(t,\eta)d\eta, \quad t \in [0,T), \tag{4.8}$$

and

$$N(t,x) := e^{x} \int_{x}^{\infty} e^{-\eta} y(t,\eta)d\eta, \quad t \in [0,T). \tag{4.9}$$

By (4.8) and (4.9), in view of (4.7), we have

$$\begin{aligned} &M(t,q(t,x_0))N(t,q(t,x_0)) \\ &\quad = \int_{-\infty}^{q(t,x_0)} e^{\eta} y(t,\eta)d\eta \int_{q(t,x_0)}^{\infty} e^{-\eta} y(t,\eta)d\eta < 0, \quad t \in [0,T). \end{aligned} \tag{4.10}$$

On the other hand, from (3.6) we have

$$M(t,q(t,x_0)) = u(t,q(t,x_0)) - u_x(t,q(t,x_0)), \quad t \in [0,T), \tag{4.11}$$

and

$$N(t,q(t,x_0)) = u(t,q(t,x_0)) + u_x(t,q(t,x_0)) \quad t \in [0,T). \tag{4.12}$$

It is observed from (4.10)-(4.12) that

$$0 > M(t,q(t,x_0))N(t,q(t,x_0)) = u^2(t,q(t,x_0)) - u_x^2(t,q(t,x_0)). \tag{4.13}$$

Since $y(t,q(t,x_0)) = 0$ on $[0,T)$, one can obtain by taking derivative with respect to t on $[0,T)$,

$$\frac{dM(t,q(t,x_0))}{dt} = -q_t(t,x_0)M(t,q(t,x_0)) + e^{-q(t,x_0)} \int_{-\infty}^{q(t,x_0)} e^{\eta} y_t(t,\eta)d\eta. \tag{4.14}$$

Using (2.1) and integrating by parts, in view of $y = u - u_{xx}$, we obtain

$$\begin{aligned}
\int_{-\infty}^{q(t,x_0)} e^{\eta} y_t(t,\eta)d\eta &= -\int_{-\infty}^{q(t,x_0)} e^{\eta}\big(y(t,\eta)u(t,\eta)\big)_{\eta} d\eta \\
&\quad -2\int_{-\infty}^{q(t,x_0)} e^{\eta} y(t,\eta)u_{\eta}(t,\eta)d\eta \\
&= \int_{-\infty}^{q(t,x_0)} e^{\eta} y(t,\eta)u(t,\eta)d\eta - 2\int_{-\infty}^{q(t,x_0)} e^{\eta} u(t,\eta)u_{\eta}(t,\eta)d\eta \\
&\quad +2\int_{-\infty}^{q(t,x_0)} e^{\eta} u_{\eta}(t,\eta)u_{\eta\eta}(t,\eta)d\eta \\
&= \int_{-\infty}^{q(t,x_0)} e^{\eta} u^2(t,\eta)d\eta - \int_{-\infty}^{q(t,x_0)} e^{\eta} u(t,\eta)u_{\eta\eta}(t,\eta)d\eta \\
&\quad -2\int_{-\infty}^{q(t,x_0)} e^{\eta} u(t,\eta)u_{\eta}(t,\eta)d\eta \\
&\quad +2\int_{-\infty}^{q(t,x_0)} e^{\eta} u_{\eta}(t,\eta)u_{\eta\eta}(t,\eta)d\eta \\
&= \int_{-\infty}^{q(t,x_0)} e^{\eta} u^2(t,\eta)d\eta - \int_{-\infty}^{q(t,x_0)} e^{\eta} u(t,\eta)u_{\eta}(t,\eta)d\eta \\
&\quad -e^{q(t,x_0)}u(t,q(t,x_0))u_x(t,q(t,x_0)) + e^{q(t,x_0)}u_x^2(t,q(t,x_0)) \\
&= \frac{3}{2}\int_{-\infty}^{q(t,x_0)} e^{\eta} u^2(t,\eta)d\eta - e^{q(t,x_0)}\frac{1}{2}u^2(t,q(t,x_0)) \\
&\quad -e^{q(t,x_0)}u(t,q(t,x_0))u_x(t,q(t,x_0)) + e^{q(t,x_0)}u_x^2(t,q(t,x_0)).
\end{aligned} \tag{4.15}$$

Note that $p * \left(\frac{1}{2}u_x^2 + u^2\right)(t,x) \geq \frac{1}{2}u^2(t,x),\ \forall (t,x) \in [0,T) \times \mathbb{R}$. Substituting (4.15) into the last term in the expression (4.14) yields

$$\begin{aligned}
\frac{dM(t,q(t,x_0))}{dt} &= -u(t,x_0)M(t,q(t,x_0)) - \frac{1}{2}u^2(t,q(t,x_0)) \\
&\quad -u(t,q(t,x_0))u_x(t,q(t,x_0)) + u_x^2(t,q(t,x_0)) \\
&\quad + e^{-q(t,x_0)}\int_{-\infty}^{q(t,x_0)} \frac{3}{2}e^{\eta}u^2(t,\eta)d\eta \\
&= u_x^2(t,q(t,x_0)) - \frac{3}{2}u^2(t,q(t,x_0)) \\
&\quad + e^{-q(t,x_0)}\int_{-\infty}^{q(t,x_0)} \frac{3}{2}e^{\eta}u^2(t,\eta)d\eta \\
&= u_x^2(t,q(t,x_0)) + \frac{1}{2}e^{-q(t,x_0)}\int_{-\infty}^{q(t,x_0)} e^{\eta}\left(u^2(t,\eta) - u_{\eta}^2(t,\eta)\right)d\eta \\
&\quad - \frac{3}{2}u^2(t,q(t,x_0)) + e^{-q(t,x_0)}\int_{-\infty}^{q(t,x_0)} e^{\eta}\left(\frac{1}{2}u_{\eta}^2(t,\eta) + u^2(t,\eta)\right)d\eta
\end{aligned}$$

$$
\begin{aligned}
&\geq u_x^2(t, q(t, x_0)) - u^2(t, q(t, x_0)) \\
&\quad + \frac{1}{2} e^{-q(t,x_0)} \int_{-\infty}^{q(t,x_0)} e^{\eta} \left(u^2(t,\eta) - u_\eta^2(t,\eta) \right) d\eta \\
&= -M(t, q(t, x_0)) N(t, q(t, x_0)) \\
&\quad + \frac{1}{2} e^{-q(t,x_0)} \int_{-\infty}^{q(t,x_0)} e^{\eta} \left(u^2(t,\eta) - u_\eta^2(t,\eta) \right) d\eta. \qquad (4.16)
\end{aligned}
$$

We claim that

$$
e^{-q(t,x_0)} \int_{-\infty}^{q(t,x_0)} e^{\eta} \left(u^2(t,\eta) - u_\eta^2(t,\eta) \right) d\eta \geq M(t, q(t, x_0)) N(t, q(t, x_0)). \quad (4.17)
$$

In fact, in view of (4.7), (4.8), (4.9) and (3.6), we deduce

$$
\begin{aligned}
& e^{-q(t,x_0)} \int_{-\infty}^{q(t,x_0)} e^{\eta} \left(u^2(t,\eta) - u_\eta^2(t,\eta) \right) d\eta \\
&= e^{-q(t,x_0)} \int_{-\infty}^{q(t,x_0)} e^{\eta} \big(M(t,\eta) N(t,\eta) \big) d\eta \\
&= e^{-q(t,x_0)} \int_{-\infty}^{q(t,x_0)} e^{\eta} \left(\int_{\eta}^{\infty} e^{-\xi} y(t,\xi) d\xi \right) \left(\int_{-\infty}^{\eta} e^{\xi} y(t,\xi) d\xi \right) d\eta \\
&= e^{-q(t,x_0)} \int_{-\infty}^{q(t,x_0)} e^{\eta} \left(\int_{q(t,x_0)}^{\infty} e^{-\xi} y(t,\xi) d\xi \right) \left(\int_{-\infty}^{\eta} e^{\xi} y(t,\xi) d\xi \right) d\eta \\
&\quad + e^{-q(t,x_0)} \int_{-\infty}^{q(t,x_0)} e^{\eta} \left(\int_{\eta}^{q(t,x_0)} e^{-\xi} y(t,\xi) d\xi \right) \left(\int_{-\infty}^{\eta} e^{\xi} y(t,\xi) d\xi \right) d\eta \\
&\geq e^{-q(t,x_0)} \int_{-\infty}^{q(t,x_0)} e^{\eta} \left(\int_{q(t,x_0)}^{\infty} e^{-\xi} y(t,\xi) d\xi \right) \left(\int_{-\infty}^{\eta} e^{\xi} y(t,\xi) d\xi \right) d\eta \\
&\geq \int_{q(t,x_0)}^{\infty} e^{-\xi} y(t,\xi) d\xi \int_{-\infty}^{q(t,x_0)} e^{\xi} y(t,\xi) d\xi \\
&= M(t, q(t, x_0)) N(t, q(t, x_0)). \qquad (4.18)
\end{aligned}
$$

Combining (4.16) with (4.18), we get

$$
\frac{dM(t, q(t, x_0))}{dt} \geq -\frac{1}{2} M(t, q(t, x_0)) N(t, q(t, x_0)) > 0. \qquad (4.19)
$$

In an analogous way one has

$$
\begin{aligned}
\frac{dN(t, q(t, x_0))}{dt} &= q_t(t, x_0) N(t, q(t, x_0)) - e^{-q(t,x_0)} \int_{-\infty}^{q(t,x_0)} e^{\eta} y_t(t,\eta) d\eta \\
&= \frac{3}{2} u^2(t, q(t, x_0)) - u_x^2(t, q(t, x_0)) - e^{q(t,x_0)} \int_{q(t,x_0)}^{\infty} \frac{3}{2} e^{-\eta} u^2(t,\eta) d\eta \\
&\leq u^2(t, q(t, x_0)) - u_x^2(t, q(t, x_0)) \\
&\quad - \frac{1}{2} e^{q(t,x_0)} \int_{q(t,x_0)}^{\infty} e^{-\eta} \left(u^2(t,\eta) - u_\eta^2(t,\eta) \right) d\eta
\end{aligned}
$$

$$
\begin{aligned}
&= M(t, q(t, x_0))N(t, q(t, x_0)) \\
&\quad -\frac{1}{2}e^{q(t,x_0)} \int_{q(t,x_0)}^{\infty} e^{-\eta} \left(u^2(t, \eta) - u_\eta^2(t, \eta)\right) d\eta.
\end{aligned} \tag{4.20}
$$

Following the similar argument of (4.18), it is found that

$$
e^{q(t,x_0)} \int_{q(t,x_0)}^{\infty} e^{-\eta} \left(u^2(t, \eta) - u_\eta^2(t, \eta)\right) d\eta \geq M(t, q(t, x_0))N(t, q(t, x_0)). \tag{4.21}
$$

Combining (4.20) with (4.21), we get

$$
\frac{dN(t, q(t, x_0))}{dt} \leq \frac{1}{2}M(t, q(t, x_0))N(t, q(t, x_0)) < 0. \tag{4.22}
$$

The differential inequalities (4.19) and (4.22) show therefore that $M(t, q(t, x_0))$ is strictly increasing while $N(t, q(t, x_0))$ is strictly decreasing on $[0, T)$. The assumptions of the theorem ensure $M(0, x_0) > 0$ and $N(0, x_0) < 0$ so that

$$
M(t, q(t, x_0))N(t, q(t, x_0)) < M(0, x_0)N(0, x_0) < 0, \quad t \in [0, T). \tag{4.23}
$$

By (4.6) and (4.13), we have

$$
\begin{aligned}
\frac{du_x(t, q(t, x_0))}{dt} &\leq u^2(t, q(t, x_0)) - u_x^2(t, q(t, x_0)) - \frac{1}{2}p * \left(u^2 - u_x^2\right)(t, q(t, x_0)) \\
&= M(t, q(t, x_0))N(t, q(t, x_0)) \\
&\quad -\frac{1}{2}p * \left(u^2 - u_x^2\right)(t, q(t, x_0)).
\end{aligned} \tag{4.24}
$$

By definition of $p(x)$, in view of (4.17) and (4.21), we have

$$
\begin{aligned}
p * (u^2 - u_x^2)(t, q(t, x_0)) &= \frac{1}{2} \int_{-\infty}^{\infty} e^{-|q(t,x_0)-\eta|} \left(u^2(t, \eta) - u_\eta^2(t, \eta)\right) d\eta \\
&= \frac{1}{2}e^{-q(t,x_0)} \int_{-\infty}^{q(t,x_0)} e^{\eta} \left(u^2(t, \eta) - u_\eta^2(t, \eta)\right) d\eta \\
&\quad +\frac{1}{2}e^{q(t,x_0)} \int_{q(t,x_0)}^{\infty} e^{-\eta} \left(u^2(t, \eta) - u_\eta^2(t, \eta)\right) d\eta \\
&\geq M(t, q(t, x_0))N(t, q(t, x_0)).
\end{aligned} \tag{4.25}
$$

Combining (4.24) with (4.25), in view of (4.23), we obtain

$$
\begin{aligned}
\frac{df(t)}{dt} &\leq \frac{1}{2} \left(u^2(t, q(t, x_0)) - u_x^2(t, q(t, x_0))\right) \\
&= \frac{1}{2}M(t, q(t, x_0))N(t, q(t, x_0)) < \frac{1}{2}M(0, x_0)N(0, x_0) < 0,
\end{aligned} \tag{4.26}
$$

where the function $f(t)$ is defined by $f(t) = u_x(t, q(t, x_0))$.

Assume that the solution $u(t)$ of Eq. (2.2) exists globally in time $t \in [0, \infty)$, that is, $T = \infty$. We show this leads to a contradiction. We first claim that there exists $t_1 > 0$ such that

$$
f^2(t) \geq 2u^2(t, q(t, x_0)), \quad t \geq t_1. \tag{4.27}
$$

Note that $M(0, x_0) > 0$ and $N(0, x_0) < 0$. By means of Gronwall's inequality, it follows from (4.19) and (4.22) that

$$M(t, q(t, x_0)) \geq M(0, x_0)e^{-\frac{1}{2}N(0,x_0)t} > 0,$$
$$-N(t, q(t, x_0)) \geq -N(0, x_0)e^{\frac{1}{2}M(0,x_0)t} > 0.$$

From the above two inequalities, in view of (4.13), we get

$$\begin{aligned} u_x^2(t, q(t, x_0)) - u^2(t, q(t, x_0)) &= -M(t, q(t, x_0))N(t, q(t, x_0)) \\ &\geq -M(0, x_0)N(0, x_0)e^{\frac{1}{2}(M(0,x_0)-N(0,x_0))t}. \end{aligned} \tag{4.28}$$

On the other hand, it follows from (3.2) that

$$u^2(t, q(t, x_0)) \leq \left(3\|u_0(x)\|_{L^2}^2 t + \|u_0(x)\|_{L^\infty}\right)^2. \tag{4.29}$$

Comparing (4.28) with (4.29), it is found that there exists $t_1 > 0$ such that

$$u_x^2(t, q(t, x_0)) - u^2(t, q(t, x_0)) \geq u^2(t, q(t, x_0)), \quad t \geq t_1.$$

This proves (4.27).

Combining (4.26) with (4.27), we obtain

$$\frac{d}{dt}f(t) \leq \frac{1}{2}u^2(t, q(t, x_0)) - \frac{1}{2}f^2(t) \leq -\frac{1}{4}f^2(t), \quad t \in [t_1, \infty). \tag{4.30}$$

By the assumptions of the theorem, we have

$$f(0) = u_x(0, x_0) = -\frac{1}{2}e^{-x_0}\int_{-\infty}^{x_0} e^{\eta}y_0(\eta)d\eta + \frac{1}{2}e^{x_0}\int_{x_0}^{\infty} e^{-\eta}y_0(\eta)d\eta < 0.$$

It then follows from (4.26) that

$$f(t) < f(0) + M(0, x_0)N(0, x_0)t < 0, \quad \text{for } t \geq 0.$$

Thus, solving the differential inequality (4.30) yields

$$\frac{1}{f(t_1)} - \frac{1}{f(t)} + \frac{1}{4}(t - t_1) \leq 0, \quad t \geq t_0.$$

Note that $-\frac{1}{f(t)} > 0$. Then we have

$$\frac{1}{f(t_1)} + \frac{1}{4}(t - t_1) < \frac{1}{f(t_1)} - \frac{1}{f(t)} + \frac{1}{4}(t - t_1) \leq 0, \quad t \geq t_0,$$

which leads to a contradiction as $t \to \infty$. This proves that $T < \infty$ and completes the proof of the theorem.

Remark 4.2. We note that Zhou claimed in Theorem 2.3 (see [51]) the same conclusion as Theorem 4.2. Unfortunately, the proof of Theorem 2.3 was incorrect. In particular, the key estimate in (2.13) (see [51]) was simply wrong.

Remark 4.3. By Theorem 3.1, Theorem 3.2, Theorem 4.1 and Theorem 4.2, it is shown that the lifespan of strong solutions of the Degasperis-Procesi equation is not affected by the smoothness and the size of the initial data, but affected by the shape of the initial data.

Attention is now turned to the blow-up set of a breaking solution for the Degasperis-Procesi equation. We will show that there is at least one point where the slope of the solution becomes infinity exactly at breaking time.

Theorem 4.3. *Assume $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$ and there exists $x_0 \in \mathbb{R}$ such that*

$$\begin{cases} y_0(x) = u_0(x) - u_{0,xx}(x) \geq 0 & \text{if} \quad x \leq x_0, \\ y_0(x) = u_0(x) - u_{0,xx}(x) \leq 0 & \text{if} \quad x \geq x_0, \end{cases}$$

and y_0 changes sign. Let $T < \infty$ be the finite blow-up time of the corresponding solution of Eq. (2.2). *Then we have*

$$\lim_{t \to T} u_x(t, q(t, x_0)) = -\infty.$$

Proof. Fix $t \in [0, T)$. In view of (4.7), it follows from the relations (4.9) and (4.12) that for any $x \leq q(t, x_0)$,

$$\begin{aligned} u_x(t,x) &= -u(t,x) + e^x \int_x^\infty e^{-\eta} y(t,\eta) d\eta \\ &= -u(t,x) + e^x \int_x^{q(t,x_0)} e^{-\eta} y(t,\eta) d\eta + e^x \int_{q(t,x_0)}^\infty e^{-\eta} y(t,\eta) d\eta \\ &\geq -u(t,x) + e^x \int_{q(t,x_0)}^\infty e^{-\eta} y(t,\eta) d\eta \\ &\geq -u(t,x) + e^{q(t,x_0)} \int_{q(t,x_0)}^\infty e^{-\eta} y(t,\eta) d\eta \\ &= -u(t,x) + u_x(t, q(t,x_0)) + u(t, q(t,x_0)). \end{aligned}$$

If $x \geq q(t, x_0)$ we have by the relations (4.8) and (4.11) that

$$\begin{aligned} u_x(t,x) &= u(t,x) - e^{-x} \int_{-\infty}^x e^{\eta} y(t,\eta) d\eta \\ &= u(t,x) - e^{-x} \int_{q(t,x_0)}^x e^{\eta} y(t,\eta) d\eta - e^{-x} \int_{-\infty}^{q(t,x_0)} e^{-\eta} y(t,\eta) d\eta \\ &\geq u(t,x) - e^{-x} \int_{\infty}^{q(t,x_0)} e^{-\eta} y(t,\eta) d\eta \\ &\geq u(t,x) - e^{-q(t,x_0)} \int_{\infty}^{q(t,x_0)} e^{-\eta} y(t,\eta) d\eta \\ &= u(t,x) + u_x(t, q(t,x_0)) - u(t, q(t,x_0)). \end{aligned}$$

From the above two inequalities we deduce that for $(t, x) \in [0, T) \times \mathbb{R}$,

$$\begin{aligned} u_x(t,x) &\geq u_x(t, q(t,x_0)) - 2\|u(t,\cdot)\|_{L^\infty} \\ &\geq u_x(t, q(t,x_0)) - 2\left(3\|u_0(x)\|_{L^2}^2 t + \|u_0(x)\|_{L^\infty}\right). \end{aligned} \tag{4.31}$$

Since $T < \infty$, it follows from Lemma 2.2 that

$$\liminf_{t\to T}(\inf_{x\in\mathbb{R}} u_x(t,x)) = -\infty.$$

Thus, from (4.31) it is easy to conclude $\lim_{t\to T} u_x(t, q(t, x_0)) = -\infty$. This completes the proof of the theorem.

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