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Domain and Range of the Modified Wave Operator for Schrödinger Equations with a Critical Nonlinearity

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Abstract: We study the final problem for the nonlinear Schrödinger equation

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda |u|^{\frac{2}{n}} u, \quad (t,x) \in \mathbf{R} \times \mathbf{R}^n,$$

where $\lambda \in \mathbf{R}$,$n = 1, 2, 3$. If the final data $u_+ \in \mathbf{H}^{0,\alpha} = \left\{\phi \in \mathbf{L}^2 : (1+|x|)^{\alpha}\phi \in \mathbf{L}^2\right\}$ with $\frac{n}{2} < \alpha < \min\left(n, 2, 1+\frac{2}{n}\right)$ and the norm $\|\widehat{u_+}\|_{\mathbf{L}^\infty}$ is sufficiently small, then we prove the existence of the wave operator in $\mathbf{L}^2$. We also construct the modified scattering operator from $\mathbf{H}^{0,\alpha}$ to $\mathbf{H}^{0,\delta}$ with $\frac{n}{2} < \delta < \alpha$.

1. Introduction

In this paper we consider the modified wave operator for the nonlinear Schrödinger equation

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda\, |u|^{\frac{2}{n}}\, u, \quad (t,x) \in \mathbf{R} \times \mathbf{R}^n, \tag{1.1}$$

where $\lambda \in \mathbf{R}$, $n = 1, 2, 3$. Denote by $\mathcal{F}\phi$ or $\hat{\phi}$ the Fourier transform of ϕ,

$$\mathcal{F}\phi(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{-i(x\cdot\xi)}\phi(x)dx,$$

the inverse Fourier transform is denoted by $\mathcal{F}^{-1}$. Our purpose is to find the solutions of (1.1) satisfying

$$\lim_{t\to+\infty} \left(u(t) - (it)^{-\frac{n}{2}}\, e^{\frac{i|x|^2}{2t}}\, \widehat{u_+}\left(\frac{\cdot}{t}\right) \exp\left(-i\lambda \left|\widehat{u_+}\left(\frac{\cdot}{t}\right)\right|^{\frac{2}{n}} \log t\right)\right) = 0 \tag{1.2}$$

in $\mathbf{L}^2$ under the conditions that the final data

$$u_+ \in \mathbf{H}^{0,\alpha} \quad \text{with} \quad \frac{n}{2} < \alpha < \min\left\{n, 2, 1 + \frac{2}{n}\right\}$$

and the norm $\|\widehat{u_+}\|_{\mathbf{L}^\infty}$ is sufficiently small. Also we show the existence of the modified scattering operator from $\mathbf{H}^{0,\alpha}$ to $\mathbf{H}^{0,\delta}$,$\frac{n}{2} < \delta < \alpha$, under the smallness condition in $\mathbf{H}^{0,\alpha}$.

Notation and function spaces. We let $\partial_j = \partial/\partial x_j$, $\partial^l = \partial_1^{l_1} \cdots \partial_n^{l_n}$, $l \in (\mathbf{N} \cup \{0\})^n$. $\mathcal{U}(t)$ is the free Schrödinger evolution group defined by

$$\mathcal{U}(t)\phi = (2\pi i t)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{\frac{i}{2t}|x-y|^2} \phi(y)\,dy = \mathcal{F}^{-1} e^{\frac{it}{2}|\xi|^2} \mathcal{F}\phi$$
$$= \mathcal{M}(t)\,\mathcal{D}(t)\,\mathcal{F}\mathcal{M}(t),$$

where $\mathcal{M} = \mathcal{M}(t) = \exp\left(\frac{i|x|^2}{2t}\right)$ and $\mathcal{D}(t)$ is the dilation operator

$$(\mathcal{D}(t)\phi)(x) = (it)^{-\frac{n}{2}} \phi\left(\frac{x}{t}\right).$$

We note that

$$\mathcal{U}(-t) = \mathcal{M}(-t)\, i^n \mathcal{F}^{-1} \mathcal{D}\left(\frac{1}{t}\right) \mathcal{M}(-t),$$

since $(\mathcal{D}(t))^{-1} = i^n \mathcal{D}\left(\frac{1}{t}\right)$. By using the above identities we easily see that

$$\mathcal{J}(t) = \mathcal{U}(t)\, x\, \mathcal{U}(-t) = \mathcal{M}(t)\, it\nabla \mathcal{M}(-t) = x + it\nabla$$

and

$$|\mathcal{J}|^\beta(t) = \mathcal{U}(t)\,|x|^\beta\, \mathcal{U}(-t) = t^\beta \mathcal{M}(t)\,(-\Delta)^{\frac{\beta}{2}}\, \mathcal{M}(-t)$$

for $\beta \geq 0$.

We introduce some function spaces. The Lebesgue space $\mathbf{L}^p = \left\{\phi \in \mathcal{S}';\ \|\phi\|_{\mathbf{L}^p} < \infty\right\}$, where $\|\phi\|_{\mathbf{L}^p} = \left(\int_{\mathbf{R}^n} |\phi(x)|^p\,dx\right)^{1/p}$ if $1 \leq p < \infty$ and $\|\phi\|_{\mathbf{L}^\infty} = \text{ess.sup}\,\{|\phi(x)|\,;\ x \in \mathbf{R}^n\}$ if $p = \infty$. We denote by $\mathbf{W}_p^{s,a}$ the weighted Sobolev space

$$\mathbf{W}_p^{s,a} = \left\{\phi \in \mathcal{S}' : \left\|\langle\cdot\rangle^a \langle i\partial_x\rangle^s \phi\right\|_{\mathbf{L}^p} < \infty\right\}$$

for any $s, a \in \mathbf{R}$, $1 \leq p \leq \infty$, where $\langle x\rangle = \left(1 + |x|^2\right)^{\frac{1}{2}}$. In particular, we denote $\mathbf{H}^{s,a} = \mathbf{W}_2^{s,a}(\mathbf{R}^n)$, $\mathbf{W}_p^s = \mathbf{W}_p^{s,0}$ and $\mathbf{H}^s = \mathbf{W}_2^s$. By $\dot{\mathbf{B}}_{p,q}^s$ we denote the homogeneous Besov space with the semi-norm

$$\|\phi\|_{\dot{\mathbf{B}}_{p,q}^s} = \left(\int_0^\infty x^{-1-\sigma q} \sup_{|y|\leq x} \sum_{|\theta|\leq[s]} \left\|\partial^\theta(\phi_y - \phi)\right\|_{\mathbf{L}^p}^q dx\right)^{1/q},$$

where $s = [s] + \sigma$, $0 < \sigma < 1$, $\phi_y(x) = \phi(x + y)$ and $[s]$ is the largest integer less than s. We let $\mathbf{C}(\mathbf{I}; \mathbf{E})$ be the space of continuous functions from an interval $\mathbf{I}$ to a Banach space $\mathbf{E}$. Different positive constants might be denoted by the same letter C.

We now state our results in this paper. In the next theorem we prove the existence of the modified wave operator

$$\mathcal{W}_+ : u_+ \in \mathbf{H}^{0,\alpha} \to u_0 \in \mathbf{H}^{0,\beta}.$$

Theorem 1. *We assume that* $u_+ \in \mathbf{H}^{0,\alpha}$ *and* $\|\widehat{u_+}\|_{\mathbf{L}^\infty} = \varepsilon$, *where* ε *is sufficiently small and* $\frac{n}{2} < \alpha < \min\left(n, 2, 1+\frac{2}{n}\right)$. *Then there exists a unique global solution* u *of (1.1) satisfying*

$$u \in \mathbf{C}\left([0,\infty);\mathbf{L}^2\right), \quad |\mathcal{J}|^\beta u \in \mathbf{C}\left([0,\infty);\mathbf{L}^2\right),$$

where $\frac{n}{2} < \beta < \alpha$. *Moreover the estimate is true*

$$\left\|\mathcal{U}(-t)\left(u(t) - (it)^{-\frac{n}{2}} e^{\frac{i|x|^2}{2t}} \widehat{u_+}\left(\frac{\cdot}{t}\right) \exp\left(-i\lambda \left|\widehat{u_+}\left(\frac{\cdot}{t}\right)\right|^{\frac{2}{n}} \log t\right)\right)\right\|_{\mathbf{H}^{0,\delta}} \le C t^{-\frac{\beta-\delta}{2}-\mu}$$

for all $t > 0$, *where* $0 \le \delta \le \beta$, $\mu > 0$.

Next we show the existence of the operator $\mathcal{W}_-^{-1}$ such that

$$\mathcal{W}_-^{-1} : u_0 \in \mathbf{H}^{0,\beta} \to u_- \in \mathbf{H}^{0,\delta},$$

where $\frac{n}{2} < \delta < \beta < \min\left(2, 1+\frac{2}{n}\right)$. Therefore we have the modified scattering operator

$$\mathcal{S}_+ = \mathcal{W}_-^{-1}\mathcal{W}_+ : \mathbf{H}^{0,\alpha} \to \mathbf{H}^{0,\delta},$$

where $\frac{n}{2} < \delta < \alpha < \min\left(n, 2, 1+\frac{2}{n}\right)$, provided that the norm $\|u_+\|_{\mathbf{H}^{0,\alpha}}$ is sufficiently small.

Theorem 2. *We assume that* $u_0 \in \mathbf{H}^{0,\beta}$ *and* $\|u_0\|_{\mathbf{H}^{0,\beta}} = \varepsilon$, *where* ε *is sufficiently small and* $\frac{n}{2} < \beta < 1+\frac{2}{n}$. *Then there exist unique functions* u_-, $h_- \in \mathbf{H}^{0,\delta}$ *with* $\frac{n}{2} < \delta < \beta$ *satisfying*

$$\left\|(\mathcal{F}\mathcal{U}(-t)u)\exp\left(i\lambda\left(h_- + |t|^{-\chi}\right)^{\frac{1}{n}}\log|t|\right) - \widehat{u_-}\right\|_{\mathbf{H}^\delta} \le C\varepsilon^{1+\frac{2}{n}}|t|^{-\mu} \tag{1.3}$$

for all $t < 0$, *with some* $\mu > 2\chi > 0$, *where* $u(t)$ *is a solution of (1.1) such that*

$$u \in \mathbf{C}\left((-\infty,0];\mathbf{L}^2\right), \quad |\mathcal{J}|^\beta u \in \mathbf{C}\left((-\infty,0];\mathbf{L}^2\right).$$

Furthermore the asymptotic representation is true

$$\left\|\mathcal{U}(-t)\left(u(t)\exp\left(i\lambda\left(h_-\left(\frac{\cdot}{t}\right) + |t|^{-\chi}\right)^{\frac{1}{n}}\log|t|\right) - (it)^{-\frac{n}{2}}\widehat{u_-}\left(\frac{\cdot}{t}\right)\right)\right\|_{\mathbf{H}^{0,\eta}} \le C\varepsilon^{1+\frac{2}{n}}|t|^{-\frac{n}{2}-\mu} \tag{1.4}$$

for all $t < 0$, *where* $\frac{n}{2} < \eta < \beta$ *with some* $\mu > 0$.

Our results are improvements of papers [3, 5, 9]. In Theorem 2 of [9], it was shown that for any $u_+ \in \mathbf{H}^{0,3} \cap \mathbf{H}^{1,2}$ with smallness condition on $\|\widehat{u_+}\|_{\mathbf{L}^\infty}$, Eq. (1.1) has a unique solution $u \in \mathbf{C}\left([0,\infty);\mathbf{H}^{1,0}\right)$ such that

$$\left\|u(t) - (it)^{-\frac{1}{2}} e^{\frac{i|x|^2}{2t}} \widehat{u_+}\left(\frac{\cdot}{t}\right)\exp\left(-i\lambda\left|\widehat{u_+}\left(\frac{\cdot}{t}\right)\right|^2\log t\right)\right\|_{\mathbf{H}^{1,0}} \le C t^{-\frac{b}{2}},$$

with $1 < b < 2$ in the one dimensional case $n = 1$. In [3], the result of [9] was improved as follows: it was shown that for any $u_+ \in \mathbf{H}^{0,3} \cap \mathbf{H}^{1,2}$ with smallness condition on $\|\widehat{u_+}\|_{\mathbf{L}^\infty}$, Eq. (1.1) has a unique solution $u \in \mathbf{C}\left([0,\infty);\mathbf{H}^{1,0} \cap \mathbf{H}^{0,1}\right)$ such that

$$\left\| u(t) - (it)^{-\frac{1}{2}} e^{\frac{i|x|^2}{2t}} \widehat{u_+}\left(\frac{\cdot}{t}\right) \exp\left(-i\lambda \left|\widehat{u_+}\left(\frac{\cdot}{t}\right)\right|^2 \log t\right) \right\|_{\mathbf{H}^{1,0}} \le Ct^{-1} \log^3 t,$$

and

$$\left\| \mathcal{U}(-t) \left(u(t) - (it)^{-\frac{1}{2}} e^{\frac{i|x|^2}{2t}} \widehat{u_+}\left(\frac{\cdot}{t}\right) \exp\left(-i\lambda \left|\widehat{u_+}\left(\frac{\cdot}{t}\right)\right|^2 \log t\right)\right) \right\|_{\mathbf{H}^{0,1}} \le Ct^{-1} \log^3 t.$$

The last estimate and the result of [7] enable us to define the modified scattering operator $\mathcal{S}_+ : \mathbf{H}^{0,3} \cap \mathbf{H}^{1,2} \to \mathbf{L}^2$ (see Corollary 2 in [3]). Their results required more smoothness conditions than those of ours since their methods are based on the substitution of an approximate solution

$$(it)^{-\frac{1}{2}} e^{\frac{i|x|^2}{2t}} \widehat{u_+}\left(\frac{\cdot}{t}\right) \exp\left(-i\lambda \left|\widehat{u_+}\left(\frac{\cdot}{t}\right)\right|^2 \log t\right)$$

to the free Schrödinger equation which implies the second differentiability of $\widehat{u_+}\left(\frac{\cdot}{t}\right)$. Note that by the method of paper [9] the condition $u_+ \in \mathbf{H}^{0,2}$ only is required for constructing the modified wave operator. In order to get the result of Theorem 1 we use the factorization of $\mathcal{U}(-t)$ and take $\mathcal{U}(t)\mathcal{F}^{-1}\widehat{u_+}\exp\left(-i\lambda|\widehat{u_+}|^{\frac{2}{n}}\log t\right)$ as an approximate solution of u. By the identity

$$\mathcal{U}(t)\mathcal{F}^{-1}\widehat{u_+}\exp\left(-i\lambda|\widehat{u_+}|^{\frac{2}{n}}\log t\right) = (it)^{-\frac{n}{2}} e^{\frac{i|x|^2}{2t}} \widehat{u_+}\left(\frac{\cdot}{t}\right) \exp\left(-i\lambda \left|\widehat{u_+}\left(\frac{\cdot}{t}\right)\right|^{\frac{2}{n}} \log t\right)$$
$$+\mathcal{M}\mathcal{D}\mathcal{F}(\mathcal{M}-1)\mathcal{F}^{-1}\widehat{u_+}\exp\left(-i\lambda|\widehat{u_+}|^{\frac{2}{n}}\log t\right)$$

we can see that the difference between the two approximate solutions is

$$\mathcal{M}\mathcal{D}\mathcal{F}(\mathcal{M}-1)\mathcal{F}^{-1}\widehat{u_+}\exp\left(-i\lambda|\widehat{u_+}|^{\frac{2}{n}}\log t\right).$$

In the proof of Theorem 2 we take a modified approximate solution

$$\mathcal{U}(t)\mathcal{F}^{-1}\widehat{u_-}\exp\left(-i\lambda\left(h_- + |t|^{-\chi}\right)^{\frac{1}{n}}\log|t|\right)$$

to avoid the loss of the differentiability. The rest of the paper is organized as follows. In Sect. 2 we prove some preliminary estimates of the nonlinearity in the Sobolev space. Section 3 is devoted to the proof of Theorem 1. Then we prove Theorem 2 in Sect. 4.

2. Lemmas

First we state the Sobolev imbedding inequality (see [4]).

Lemma 3. *Let q, r be any numbers satisfying $1 \leq q, r \leq \infty$, and let j, m be any numbers satisfying $0 \leq j < m$. If $\phi \in \mathbf{W}_r^m \cap \mathbf{L}^q$, then*

$$\left\| (-\Delta)^{\frac{j}{2}} \phi \right\|_{\mathbf{L}^p} \leq C \left\| (-\Delta)^{\frac{m}{2}} \phi \right\|_{\mathbf{L}^r}^a \|\phi\|_{\mathbf{L}^q}^{1-a},$$

where $\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + \frac{1-a}{q}$ for all a in the interval $\frac{j}{m} \leq a \leq 1$, where C is a constant depending only on n, m, j, q, r, a, with the following exception: if $m - j - \frac{n}{r}$ is a nonnegative integer, then the above estimate holds for $a = \frac{j}{m}$.

We denote the fractional partial derivative $\partial_{x_j}^{\beta}$ for $\beta > 0,\ j = 1, 2, \ldots, n$, as follows

$$\partial_{x_j}^{\beta} \phi(x) = \frac{2\pi}{\Gamma(1-\varrho)} \int_0^{\infty} \partial_{x_j}^k \left(\phi_{y_j} - \phi\right) y_j^{-1-\varrho} dy_j,$$

where $k = [\beta]$, $\varrho = \beta - k \in (0, 1)$, $\phi_{y_j} = \phi\left(x_1, \ldots, x_j + y_j, \ldots, x_n\right)$, Γ is the Euler gamma function (see [1, 11]).

Lemma 4. *Let $\frac{n}{2} < \beta < \min\left(n, 2, 1 + \frac{2}{n}\right)$. Then the estimates are true*

$$\left\| \phi |\phi|^{\frac{2}{n}} \right\|_{\dot{\mathbf{H}}^{\beta}} \leq C \|\phi\|_{\mathbf{L}^{\infty}}^{\frac{2}{n}} \|\phi\|_{\dot{\mathbf{H}}^{\beta}},$$

$$\left\| \phi \exp\left(i |\phi|^{\frac{2}{n}} \log \tau\right) \right\|_{\dot{\mathbf{H}}^{\beta}} \leq C \left(1 + \sum_{j=1}^{2} \|\phi\|_{\mathbf{L}^{\infty}}^{\frac{2}{n} j} (\log \tau)^j \right) \|\phi\|_{\dot{\mathbf{H}}^{\beta}},$$

and

$$\begin{aligned} \left\| |\phi|^2 \phi - |\psi|^2 \psi \right\|_{\dot{\mathbf{H}}^{\beta}} &\leq C \left(\|\phi\|_{\mathbf{L}^{\infty}}^2 + \|\psi\|_{\mathbf{L}^{\infty}}^2 \right) \|\phi - \psi\|_{\dot{\mathbf{H}}^{\beta}} \\ &\quad + C \left(\|\phi\|_{\mathbf{L}^{\infty}} + \|\psi\|_{\mathbf{L}^{\infty}} \right) \|\phi - \psi\|_{\mathbf{L}^{\infty}} \|\psi\|_{\dot{\mathbf{H}}^{\beta}} \end{aligned}$$

if $n = 1$. Also

$$\begin{aligned} &\left\| |\phi|^{\frac{2}{n}} \phi - |\psi|^{\frac{2}{n}} \psi \right\|_{\dot{\mathbf{H}}^{\beta}} \\ &\quad \leq C \|\phi\|_{\mathbf{L}^{\infty}}^{\frac{2}{n}} \left(\|\phi - \psi\|_{\dot{\mathbf{H}}^{\beta}} + s^{1-\beta} \|\phi - \psi\|_{\dot{\mathbf{H}}^1} \right) \\ &\quad + C \|\phi - \psi\|_{\mathbf{L}^{\infty}}^{\frac{2}{n}} \left(\|\psi\|_{\dot{\mathbf{H}}^{\beta}} + s^{1-\beta} \|\psi\|_{\dot{\mathbf{H}}^1} \right) \\ &\quad + C s^{\frac{2}{n}\gamma} \left(\|\phi\|_{\dot{\mathbf{H}}^{\frac{n}{2}+\gamma}}^{\frac{2}{n}} \|\phi\|_{\dot{\mathbf{H}}^{\beta}} + \|\phi\|_{\dot{\mathbf{H}}^{\frac{n}{2}+\gamma}}^{\frac{2}{n}} \|\psi\|_{\dot{\mathbf{H}}^{\beta}} + \|\psi\|_{\dot{\mathbf{H}}^{\frac{n}{2}+\gamma}}^{\frac{2}{n}} \|\psi\|_{\dot{\mathbf{H}}^{\beta}} \right) \end{aligned}$$

for all $s > 0$ if $n = 2, 3$, where $0 < \gamma \leq \min\left(\beta - \frac{n}{2}, \frac{n}{2}\left(1 + \frac{2}{n} - \beta\right)\right)$, provided that the right-hand sides are finite.

Proof. By the Taylor expansion

$$\exp\left(i\,|\phi|^{\frac{2}{n}}\log\tau\right)-1=i\,|\phi|^{\frac{2}{n}}\log\tau\int_0^1\exp\left(i\theta\,|\phi|^{\frac{2}{n}}\log\tau\right)d\theta.$$

Let us estimate

$$\left\|\phi\,|\phi|^{\frac{2}{n}}\exp\left(i\theta\,|\phi|^{\frac{2}{n}}\log\tau\right)\right\|_{\dot{\mathbf{H}}^{\beta}}$$

for $\beta > 1$ and $n = 2, 3$. By a direct computation

$$\nabla\left(\phi\,|\phi|^{\frac{2}{n}}\exp\left(i\theta\,|\phi|^{\frac{2}{n}}\log\tau\right)\right)=f(\phi)\exp\left(i\,|\phi|^{\frac{2}{n}}\log\tau\right),$$

where

$$f(\phi)=\left(\frac{2}{n}+1\right)|\phi|^{\frac{2}{n}}\nabla\phi+\frac{2}{n}\phi^{2}\,|\phi|^{\frac{2}{n}-2}\,\overline{\nabla\phi}+\frac{2i}{n}\,|\phi|^{\frac{4}{n}-2}\,\phi\left(\overline{\phi}\nabla\phi+\phi\overline{\nabla\phi}\right)\log\tau.$$

By the Hölder inequality we find

$$\begin{aligned}&\left\|f\left(\phi_{y_j}\right)\exp\left(i\theta\left|\phi_{y_j}\right|^{\frac{2}{n}}\log\tau\right)-f(\phi)\exp\left(i\theta\,|\phi|^{\frac{2}{n}}\log\tau\right)\right\|_{\mathbf{L}^2}\\&\le C\left\|f\left(\phi_{y_j}\right)-f(\phi)\right\|_{\mathbf{L}^2}+C\log\tau\left\|\phi_{y_j}-\phi\right\|^{\frac{2}{n}}_{\mathbf{L}^{\frac{2}{n}p}}\|f(\phi)\|_{\mathbf{L}^{\overline{p}}}\end{aligned}$$

with $\frac{1}{p}+\frac{1}{\overline{p}}=1$. Therefore

$$\begin{aligned}&\left\|f(\phi)\exp\left(i\theta\,|\phi|^{\frac{2}{n}}\log\tau\right)\right\|_{\dot{\mathbf{B}}^{\sigma}_{2,2}}\\&\le C\left(\int_0^{\infty}x^{-1-2\sigma}\sup_{|k|\le x}\left\|f\left(\phi_{y_j}\right)-f(\phi)\right\|^2_{\mathbf{L}^2}dx\right)^{\frac{1}{2}}\\&\quad+C\log\tau\left(\int_0^{\infty}x^{-1-2\sigma}\sup_{|k|\le x}\left\|\phi_{y_j}-\phi\right\|^{\frac{4}{n}}_{\mathbf{L}^{\frac{2}{n}p}}\|f(\phi)\|^2_{\mathbf{L}^{\overline{p}}}dx\right)^{\frac{1}{2}}\end{aligned}$$

for $0 < \sigma < 1$. Since the norm of the homogeneous Sobolev space $\dot{\mathbf{H}}^{\sigma}$ is equivalent to that of the homogeneous Besov space $\dot{\mathbf{B}}^{\sigma}_{2,2}$ (see [2]), then the first two estimates of the lemma follow by the method of proof of Lemma 3.4 in paper [6].

We now prove the last two estimates of the lemma. Since the norm of the homogeneous Sobolev space $\dot{\mathbf{H}}^{\beta}$ is equivalent to that of the homogeneous Besov space $\dot{\mathbf{B}}^{\beta}_{2,2}$ (see [2]), we have

$$\|\phi\|_{\dot{\mathbf{H}}^{\beta}}\le C\,\|\phi\|_{\dot{\mathbf{B}}^{\beta}_{2,2}}=\left(\int_0^{\infty}x^{-1-2\beta}\sup_{|y|\le x}\left\|\phi_y-\phi\right\|^2_{\mathbf{L}^2}dx\right)^{\frac{1}{2}},$$

where $0 < \beta < 1$, $\psi_y(x) = \psi(x + y)$. For $n = 1$ we represent

$$\left| \left|\phi_y\right|^2 \phi_y - \left|\psi_y\right|^2 \psi_y - |\phi|^2 \phi + |\psi|^2 \psi \right| \\ \leq C \left(\left|\phi_y\right| + \left|\psi_y\right| + |\phi| + |\psi|\right)^2 \left(\left|\left(\phi_y - \psi_y\right) - (\phi - \psi)\right| + \left|\psi_y - \psi\right|\right),$$

then we get

$$\left\| |\phi|^2 \phi - |\psi|^2 \psi \right\|_{\dot{\mathbf{H}}^\beta} \\ \leq C \left(\int_0^\infty x^{-1-2\beta} \sup_{|y| \leq x} \left\| \left|\phi_y\right|^2 \phi_y - \left|\psi_y\right|^2 \psi_y - |\phi|^2 \phi + |\psi|^2 \psi \right\|_{\mathbf{L}^2}^2 dx \right)^{\frac{1}{2}} \\ \leq C \left(\|\phi\|_{\mathbf{L}^\infty}^2 + \|\psi\|_{\mathbf{L}^\infty}^2 \right) \|\phi - \psi\|_{\dot{\mathbf{H}}^\beta} + C \left(\|\phi\|_{\mathbf{L}^\infty} + \|\psi\|_{\mathbf{L}^\infty} \right) \|\phi - \psi\|_{\mathbf{L}^\infty} \|\psi\|_{\dot{\mathbf{H}}^\beta}.$$

Thus the third estimate of the lemma is true.

To prove the last estimate of the lemma we represent

$$\partial_{x_j} \left(|\phi|^{\frac{2}{n}} \phi - |\psi|^{\frac{2}{n}} \psi \right) = \left(1 + \frac{2}{n} \right) \left(|\phi|^{\frac{2}{n}} \partial_{x_j} (\phi - \psi) + \left(|\phi|^{\frac{2}{n}} - |\psi|^{\frac{2}{n}} \right) \partial_{x_j} \psi \right)$$

for $n = 2, 3$. Then we get

$$\left\| \left|\partial_{x_j}\right|^{\beta-1} \left(|\phi|^{\frac{2}{n}} \partial_{x_j} (\phi - \psi) \right) \right\|_{\mathbf{L}^2} \\ = C \left\| |\phi|^{\frac{2}{n}} \int_0^\infty \partial_{x_j} \left(\left(\phi_{y_j} - \psi_{y_j}\right) - (\phi - \psi) \right) y_j^{-\beta} dy_j \right. \\ \left. + \int_0^\infty \left(\left|\phi_{y_j}\right|^{\frac{2}{n}} - |\phi|^{\frac{2}{n}} \right) \partial_{x_j} \left(\phi_{y_j} - \psi_{y_j}\right) y_j^{-\beta} dy_j \right\|_{\mathbf{L}^2} \\ \leq C \|\phi\|_{\mathbf{L}^\infty}^{\frac{2}{n}} \|\phi - \psi\|_{\dot{\mathbf{H}}^\beta} \\ + C \int_0^\infty \left\| \left(\left|\phi_{y_j}\right|^{\frac{2}{n}} - |\phi|^{\frac{2}{n}} \right) \partial_{x_j} \left(\phi_{y_j} - \psi_{y_j}\right) \right\|_{\mathbf{L}^2} y_j^{-\beta} dy_j.$$

By Lemma 3 we obtain

$$\left\| \left(\left|\phi_{y_j}\right|^{\frac{2}{n}} - |\phi|^{\frac{2}{n}} \right) \partial_{x_j} \varphi \right\|_{\mathbf{L}^2} \leq C \left\| \left|\phi_{y_j}\right|^{\frac{2}{n}} - |\phi|^{\frac{2}{n}} \right\|_{\mathbf{L}^{\frac{n}{\beta-1}}} \|\varphi\|_{\dot{\mathbf{H}}^\beta} \\ \leq C \left\| \phi_{y_j} - \phi \right\|_{\dot{\mathbf{H}}^\sigma}^{\frac{2}{n}} \|\varphi\|_{\dot{\mathbf{H}}^\beta} \leq C y_j^{\frac{2}{n}\gamma + \beta - 1} \|\phi\|_{\dot{\mathbf{H}}^{\frac{n}{2}+\gamma}}^{\frac{2}{n}} \|\varphi\|_{\dot{\mathbf{H}}^\beta}, \tag{2.1}$$

where $\sigma = \frac{n}{2} - \frac{n}{2}(\beta - 1) \geq 0,\ 0 < \gamma \leq \min\left(\beta - \frac{n}{2}, \frac{n}{2}\left(1 + \frac{2}{n} - \beta\right)\right)$, for $n = 2, 3$. Therefore we find

$$\int_0^\infty \left\| \left(\left|\phi_{y_j}\right|^{\frac{2}{n}} - |\phi|^{\frac{2}{n}} \right) \partial_{x_j} \varphi \right\|_{\mathbf{L}^2} y_j^{-\beta} dy_j$$
$$\leq C \|\phi\|_{\dot{\mathbf{H}}^{\frac{n}{2}+\gamma}}^{\frac{2}{n}} \|\varphi\|_{\dot{\mathbf{H}}^\beta} \int_0^s y_j^{\frac{2}{n}\gamma - 1} dy_j + C \|\phi\|_{\mathbf{L}^\infty}^{\frac{2}{n}} \|\varphi\|_{\dot{\mathbf{H}}^1} \int_s^\infty y_j^{-\beta} dy_j$$
$$\leq C s^{\frac{2}{n}\gamma} \|\phi\|_{\dot{\mathbf{H}}^{\frac{n}{2}+\gamma}}^{\frac{2}{n}} \|\varphi\|_{\dot{\mathbf{H}}^\beta} + C s^{1-\beta} \|\phi\|_{\mathbf{L}^\infty}^{\frac{2}{n}} \|\varphi\|_{\dot{\mathbf{H}}^1}$$

for all $s > 0$ so that

$$\left\| \left|\partial_{x_j}\right|^{\beta-1} \left(|\phi|^{\frac{2}{n}} \partial_{x_j} (\phi - \psi) \right) \right\|_{\mathbf{L}^2} \leq C \|\phi\|_{\mathbf{L}^\infty}^{\frac{2}{n}} \left(\|\phi - \psi\|_{\dot{\mathbf{H}}^\beta} + s^{1-\beta} \|\phi - \psi\|_{\dot{\mathbf{H}}^1} \right)$$
$$+ C s^{\frac{2}{n}\gamma} \|\phi\|_{\dot{\mathbf{H}}^{\frac{n}{2}+\gamma}}^{\frac{2}{n}} \|\phi - \psi\|_{\dot{\mathbf{H}}^\beta}. \tag{2.2}$$

In the same manner

$$\left\| \left|\partial_{x_j}\right|^{\beta-1} \left(\left(|\phi|^{\frac{2}{n}} - |\psi|^{\frac{2}{n}} \right) \partial_{x_j} \psi \right) \right\|_{\mathbf{L}^2}$$
$$\leq C \left\| \left(|\phi|^{\frac{2}{n}} - |\psi|^{\frac{2}{n}} \right) \int_0^\infty \partial_{x_j} \left(\psi - \psi_{y_j} \right) y_j^{-\beta} dy_j \right\|_{\mathbf{L}^2}$$
$$+ C \left\| \int_0^\infty \left(|\phi|^{\frac{2}{n}} - \left|\phi_{y_j}\right|^{\frac{2}{n}} - |\psi|^{\frac{2}{n}} + \left|\psi_{y_j}\right|^{\frac{2}{n}} \right) \partial_{x_j} \psi_{y_j} y_j^{-\beta} dy_j \right\|_{\mathbf{L}^2}.$$

For the first summand we have

$$\left\| \left(|\phi|^{\frac{2}{n}} - |\psi|^{\frac{2}{n}} \right) \int_0^\infty \partial_{x_j} \left(\psi - \psi_{y_j} \right) y_j^{-\beta} dy_j \right\|_{\mathbf{L}^2}$$
$$\leq C \left\| |\phi|^{\frac{2}{n}} - |\psi|^{\frac{2}{n}} \right\|_{\mathbf{L}^\infty} \|\psi\|_{\dot{\mathbf{H}}^\beta} \leq C \|\phi - \psi\|_{\mathbf{L}^\infty}^{\frac{2}{n}} \|\psi\|_{\dot{\mathbf{H}}^\beta}.$$

As in (2.1) we obtain

$$\left\| \int_0^\infty \left(|\phi|^{\frac{2}{n}} - \left|\phi_{y_j}\right|^{\frac{2}{n}} - |\psi|^{\frac{2}{n}} + \left|\psi_{y_j}\right|^{\frac{2}{n}} \right) \partial_{x_j} \psi_{y_j} y_j^{-\beta} dy_j \right\|_{\mathbf{L}^2}$$
$$\leq C \left(\|\phi\|_{\dot{\mathbf{H}}^{\frac{n}{2}+\gamma}}^{\frac{2}{n}} + \|\psi\|_{\dot{\mathbf{H}}^{\frac{n}{2}+\gamma}}^{\frac{2}{n}} \right) \|\psi\|_{\dot{\mathbf{H}}^\beta} \int_0^s y_j^{\frac{2}{n}\gamma - 1} dy_j$$
$$+ C \|\phi - \psi\|_{\mathbf{L}^\infty}^{\frac{2}{n}} \|\psi\|_{\dot{\mathbf{H}}^1} \int_s^\infty y_j^{-\beta} dy_j$$
$$\leq C s^{\frac{2}{n}\gamma} \left(\|\phi\|_{\dot{\mathbf{H}}^{\frac{n}{2}+\gamma}}^{\frac{2}{n}} + \|\psi\|_{\dot{\mathbf{H}}^{\frac{n}{2}+\gamma}}^{\frac{2}{n}} \right) \|\psi\|_{\dot{\mathbf{H}}^\beta} + C s^{1-\beta} \|\phi - \psi\|_{\mathbf{L}^\infty}^{\frac{2}{n}} \|\psi\|_{\dot{\mathbf{H}}^1}.$$

Then we find

$$\left\| \left| \partial_{x_j} \right|^{\beta-1} \left(\left(|\phi|^{\frac{2}{n}} - |\psi|^{\frac{2}{n}} \right) \partial_{x_j} \psi \right) \right\|_{\mathbf{L}^2} \leq C \left\| \phi - \psi \right\|_{\mathbf{L}^\infty}^{\frac{2}{n}} \left(\left\| \psi \right\|_{\dot{\mathbf{H}}^\beta} + s^{1-\beta} \left\| \psi \right\|_{\dot{\mathbf{H}}^1} \right) + C s^{\frac{2}{n}\gamma} \left(\left\| \phi \right\|_{\dot{\mathbf{H}}^{\frac{n}{2}+\gamma}}^{\frac{2}{n}} + \left\| \psi \right\|_{\dot{\mathbf{H}}^{\frac{n}{2}+\gamma}}^{\frac{2}{n}} \right) \left\| \psi \right\|_{\dot{\mathbf{H}}^\beta} . \tag{2.3}$$

By (2.2) and (2.3) the last estimate of the lemma follows. Lemma 4 is proved. □

3. Modified Wave Operator

We denote the first approximation for the solutions of (1.1) by

$$u_1(t) = \mathcal{M}(t) \mathcal{D}(t) \widehat{w}(t), \ \widehat{w}(t) = \widehat{u_+} \exp\left(-i\lambda \left| \widehat{u_+} \right|^{\frac{2}{n}} \log t \right).$$

The free Schrödinger evolution group can be decomposed as

$$\mathcal{U}(t)\phi = \mathcal{M}(t)\mathcal{D}(t)\widehat{\phi} + \mathcal{R}(t)\widehat{\phi},$$

where $\mathcal{R}(t) = \mathcal{M}(t)\mathcal{D}(t)\mathcal{F}(\mathcal{M}(t) - 1)\mathcal{F}^{-1}$.

To prove Theorem 1 we define the following function space:

$$\mathbf{X} = \left\{ \phi \in \mathbf{C}\left([T, \infty); \mathbf{L}^2 \right); \left\| \phi(t) - u_1(t) \right\|_{\mathbf{X}} < \infty \right\}$$

with the norm

$$\left\| \phi \right\|_{\mathbf{X}} = \sup_{t \in [T,\infty)} \left(t^{\frac{\beta}{2}+\mu} \left\| \phi(t) \right\|_{\mathbf{L}^2} + t^{\mu} \left\| |\mathcal{J}|^{\beta} \phi(t) \right\|_{\mathbf{L}^2} \right),$$

where $\frac{n}{2} < \beta < \alpha < \min\left(n, 2, 1 + \frac{2}{n}\right)$, $\alpha - \beta > \mu > 0$ is sufficiently small.

Multiplying both sides of (1.1) by $\mathcal{F}\mathcal{U}(-t)$, we obtain

$$i\partial_t \left(\mathcal{F}\mathcal{U}(-t) u \right) = \lambda \mathcal{F}\mathcal{U}(-t) |u|^{\frac{2}{n}} u. \tag{3.1}$$

Note that $\widehat{w}(t) = \widehat{u_+} \exp\left(-i\lambda \left| \widehat{u_+} \right|^{\frac{2}{n}} \log t \right)$ satisfies the equation

$$i\partial_t \widehat{w} = \frac{\lambda}{t} |\widehat{w}|^{\frac{2}{n}} \widehat{w}. \tag{3.2}$$

By (3.1) and (3.2) we have

$$\begin{aligned} & i\partial_t \left(\mathcal{F}\mathcal{U}(-t) u - \widehat{w} \right) \\ & = \lambda \mathcal{F}\mathcal{U}(-t) \left(|u|^{\frac{2}{n}} u - \frac{1}{t} \mathcal{M}\mathcal{D} |\widehat{w}|^{\frac{2}{n}} \widehat{w} - \frac{1}{t} \mathcal{R} |\widehat{w}|^{\frac{2}{n}} \widehat{w} \right) \\ & = \lambda \mathcal{F}\mathcal{U}(-t) \left(|u|^{\frac{2}{n}} u - |u_1|^{\frac{2}{n}} u_1 \right) - \frac{\lambda}{t} \mathcal{F}\mathcal{U}(-t) \mathcal{R} |\widehat{w}|^{\frac{2}{n}} \widehat{w}. \end{aligned} \tag{3.3}$$

Since

$$\mathcal{F}\mathcal{U}(-t) u - \widehat{w} = \mathcal{F}\mathcal{U}(-t) \left(u - u_1 - \mathcal{R}\widehat{w} \right),$$

by integrating (3.3) in time and by using condition (1.2) we obtain

$$u(t) - u_1(t) = -i\lambda \int_t^\infty \mathcal{U}(t-\tau)\left(|u|^{\frac{2}{n}} u - |u_1|^{\frac{2}{n}} u_1\right) d\tau$$
$$+\mathcal{R}\widehat{w} + i\lambda \int_t^\infty \mathcal{U}(t-\tau)\,\mathcal{R}(\tau)\,|\widehat{w}|^{\frac{2}{n}}\,\widehat{w}\frac{d\tau}{\tau}. \tag{3.4}$$

Equation (3.4) is the integral equation for (1.1) with condition (1.2). Let us consider the linearized version of (3.4),

$$u(t) - u_1(t) = -i\lambda \int_t^\infty \mathcal{U}(t-\tau)\left(|v|^{\frac{2}{n}} v - |u_1|^{\frac{2}{n}} u_1\right) d\tau$$
$$+\mathcal{R}\widehat{w} + i\lambda \int_t^\infty \mathcal{U}(t-\tau)\,\mathcal{R}(\tau)\,|\widehat{w}|^{\frac{2}{n}}\,\widehat{w}\frac{d\tau}{\tau}, \tag{3.5}$$

where $v \in \mathbf{X}_\rho \equiv \left\{\phi \in \mathbf{X};\, \|\phi\|_{\mathbf{X}} \le \rho\right\}$ and $\rho \le C\,\|u_+\|_{\mathbf{H}^{0,\alpha}}$.

Since $\mathcal{R} = \mathcal{M}\mathcal{D}\mathcal{F}(\mathcal{M}-1)\mathcal{F}^{-1}$, by Lemma 4 the remainder terms are estimated as

$$\left\|\mathcal{R}(t)(-\Delta)^{\frac{\delta}{2}}\,\widehat{w}\right\|_{\mathbf{L}^2} = \left\|(\mathcal{M}-1)\,\mathcal{F}^{-1}(-\Delta)^{\frac{\delta}{2}}\,\widehat{w}\right\|_{\mathbf{L}^2}$$
$$\le Ct^{-\frac{\alpha-\delta}{2}}\,\|\widehat{w}\|_{\mathbf{H}^\alpha} \le C\rho t^{-\frac{\alpha-\delta}{2}}\log^2 t \tag{3.6}$$

and

$$\int_t^\infty \left\|\mathcal{R}(\tau)(-\Delta)^{\frac{\delta}{2}}\,|\widehat{w}|^{\frac{2}{n}}\,\widehat{w}\right\|_{\mathbf{L}^2}\frac{d\tau}{\tau} \le C\varepsilon^{\frac{2}{n}}\rho\int_t^\infty \tau^{-1-\frac{\alpha-\delta}{2}}\log^2\tau\, d\tau$$
$$\le C\varepsilon^{\frac{2}{n}}\rho t^{-\frac{\alpha-\delta}{2}}\log^2 t, \tag{3.7}$$

where $0 \le \delta \le \beta < \alpha$. Also by virtue of Lemma 3 we obtain

$$\begin{aligned}
\|v\|_{\mathbf{L}^\infty} &\le \|v-u_1\|_{\mathbf{L}^\infty} + \|u_1\|_{\mathbf{L}^\infty}\\
&\le Ct^{-\frac{n}{2}}\left\||\mathcal{J}|^\beta(v-u_1)\right\|_{\mathbf{L}^2}^{\frac{n}{2\beta}}\|v-u_1\|_{\mathbf{L}^2}^{1-\frac{n}{2\beta}} + Ct^{-\frac{n}{2}}\|\widehat{u_+}\|_{\mathbf{L}^\infty}\\
&\le Ct^{-\frac{n}{2}}\left(\rho t^{-\mu}+\varepsilon\right)
\end{aligned}$$

since $v \in \mathbf{X}_\rho$. Then by (3.5) the $\mathbf{L}^2$-norm can be estimated as

$$\begin{aligned}\|u(t) - u_1(t)\|_{\mathbf{L}^2} &\le C \int_t^\infty \left(\|v\|_{\mathbf{L}^\infty}^{\frac{2}{n}} + \|u_1\|_{\mathbf{L}^\infty}^{\frac{2}{n}} \right) \|v - u_1\|_{\mathbf{L}^2}\, d\tau \\ &\quad + \|\mathcal{R}\widehat{w}\|_{\mathbf{L}^2} + C \int_t^\infty \left\| \mathcal{R}(\tau) |\widehat{w}|^{\frac{2}{n}} \widehat{w} \right\|_{\mathbf{L}^2} \frac{d\tau}{\tau} \\ &\le C\varepsilon^{\frac{2}{n}} \int_t^\infty \|v - u_1\|_{\mathbf{L}^2} \frac{d\tau}{\tau} + C\rho^{\frac{2}{n}} \int_t^\infty \|v - u_1\|_{\mathbf{L}^2} \frac{d\tau}{\tau^{1+\mu}} \\ &\quad + C\rho t^{-\frac{\alpha}{2}} \log^2 t \le C\rho t^{-\frac{\beta}{2}-\mu} \end{aligned} \tag{3.8}$$

for all $t \ge T$ if $T > 0$ is sufficiently large.

Note that $|\mathcal{J}|^\beta \mathcal{R}(\tau) = \mathcal{R}(\tau)(-\Delta)^{\frac{\beta}{2}}$. Then multiplying (3.5) by $|\mathcal{J}|^\beta = t^\beta \mathcal{M}(t)$ $(-\Delta)^{\frac{\beta}{2}} \mathcal{M}(-t)$, we obtain

$$\begin{aligned}|\mathcal{J}|^\beta (u(t) - u_1(t)) &= -i\lambda \int_t^\infty \mathcal{U}(t-\tau) \left(|\mathcal{J}(\tau)|^\beta \left(|v|^{\frac{2}{n}} v - |u_1|^{\frac{2}{n}} u_1 \right) \right) d\tau \\ &\quad + \mathcal{R}(t)(-\Delta)^{\frac{\beta}{2}} \widehat{w} + i\lambda \int_t^\infty \mathcal{U}(t-\tau) \mathcal{R}(\tau) (-\Delta)^{\frac{\beta}{2}} |\widehat{w}|^{\frac{2}{n}} \widehat{w} \frac{d\tau}{\tau}.\end{aligned}$$

Then by (3.6) and (3.7) we find

$$\left\| |\mathcal{J}|^\beta (u(t) - u_1(t)) \right\|_{\mathbf{L}^2} \le \int_t^\infty \left\| |\mathcal{J}|^\beta \left(|v|^{\frac{2}{n}} v - |u_1|^{\frac{2}{n}} u_1 \right) \right\|_{\mathbf{L}^2} d\tau + C\rho t^{-\frac{\alpha-\beta}{2}} \log^2 t. \tag{3.9}$$

Applying Lemma 4 we have

$$\begin{aligned}&\left\| |\mathcal{J}|^\beta \left(|v|^2 v - |u_1|^2 u_1 \right) \right\|_{\mathbf{L}^2} \\ &\quad = C\tau^\beta \left\| \overline{\mathcal{M}} v \left| \overline{\mathcal{M}} v \right|^2 - \overline{\mathcal{M}} u_1 \left| \overline{\mathcal{M}} u_1 \right|^2 \right\|_{\dot{\mathbf{H}}^\beta} \\ &\quad \le C\rho\tau^{-1} \left(\varepsilon + \rho\tau^{-\mu} \right) \left(\left\| |\mathcal{J}|^\beta (v - u_1) \right\|_{\mathbf{L}^2} + \|v - u_1\|_{\mathbf{L}^\infty} \right) \\ &\quad \le C\rho\tau^{-1-\mu} \left(\varepsilon + \rho\tau^{-\mu} \right) \end{aligned} \tag{3.10}$$

in the case $n = 1$ and also

$$
\begin{aligned}
&\left\| |\mathcal{J}|^{\beta} \left(|v|^{\frac{2}{n}} v - |u_1|^{\frac{2}{n}} u_1 \right) \right\|_{\mathbf{L}^2} \\
&= C\tau^{\beta} \left\| \overline{\mathcal{M}} v \left| \overline{\mathcal{M}} v \right|^{\frac{2}{n}} - \overline{\mathcal{M}} u_1 \left| \overline{\mathcal{M}} u_1 \right|^{\frac{2}{n}} \right\|_{\dot{\mathbf{H}}^{\beta}} \\
&\le C \|v\|_{\mathbf{L}^\infty}^{\frac{2}{n}} \left\| |\mathcal{J}|^{\beta} (v - u_1) \right\|_{\mathbf{L}^2} + C \|v - u_1\|_{\mathbf{L}^\infty}^{\frac{2}{n}} \left\| |\mathcal{J}|^{\beta} u_1 \right\|_{\mathbf{L}^2} \\
&\quad + C s^{1-\beta} \tau^{\beta - 1} \left(\|v\|_{\mathbf{L}^\infty}^{\frac{2}{n}} \|\mathcal{J}(v - u_1)\|_{\mathbf{L}^2} + \|v - u_1\|_{\mathbf{L}^\infty}^{\frac{2}{n}} \|\mathcal{J} u_1\|_{\mathbf{L}^2} \right) \\
&\quad + C s^{\frac{2}{n}\gamma} \tau^{-1-\frac{2}{n}\gamma} \left(\left\| |\mathcal{J}|^{\frac{n}{2}+\gamma} v \right\|_{\mathbf{L}^2}^{\frac{2}{n}} \left\| |\mathcal{J}|^{\beta} v \right\|_{\mathbf{L}^2} + \left\| |\mathcal{J}|^{\frac{n}{2}+\gamma} v \right\|_{\mathbf{L}^2}^{\frac{2}{n}} \left\| |\mathcal{J}|^{\beta} u_1 \right\|_{\mathbf{L}^2} \right. \\
&\qquad \left. + \left\| |\mathcal{J}|^{\frac{n}{2}+\gamma} u_1 \right\|_{\mathbf{L}^2}^{\frac{2}{n}} \left\| |\mathcal{J}|^{\beta} u_1 \right\|_{\mathbf{L}^2} \right) \qquad (3.11)
\end{aligned}
$$

for all $s > 0$ if $n = 2, 3$, where $0 < \gamma \le \min\left(\beta - \frac{n}{2}, \frac{n}{2}\left(1 + \frac{2}{n} - \beta\right)\right)$. Since $v \in \mathbf{X}_{\rho}$, by using the estimates

$$
\begin{aligned}
\|v - u_1\|_{\mathbf{L}^\infty} &\le C\tau^{-\frac{n}{2}} \left\| |\mathcal{J}|^{\beta} (v - u_1) \right\|_{\mathbf{L}^2}^{\frac{n}{2\beta}} \|v - u_1\|_{\mathbf{L}^2}^{1 - \frac{n}{2\beta}} \\
&\le C\rho\tau^{-\frac{n}{2} - \mu - \frac{1}{2}\left(\beta - \frac{n}{2}\right)}
\end{aligned}
$$

and $\|\mathcal{J}(v - u_1)\|_{\mathbf{L}^2} \le C\rho\tau^{-\mu - \frac{\beta - 1}{2}}$, we get from (3.11)

$$
\begin{aligned}
&\left\| |\mathcal{J}|^{\beta} \left(|v|^{\frac{2}{n}} v - |u_1|^{\frac{2}{n}} u_1 \right) \right\|_{\mathbf{L}^2} \\
&\le C\rho\tau^{-1-\mu} \left(\varepsilon^{\frac{2}{n}} + \rho^{\frac{2}{n}} \tau^{-\mu} \right) + C\rho^{1+\frac{2}{n}} \tau^{-1-\frac{2}{n}\mu - \frac{2}{n}\left(\beta - \frac{n}{2}\right)} \\
&\quad + C\rho^{1+\frac{2}{n}} s^{1-\beta} \tau^{\beta-1} \left(\tau^{-1-\mu-\frac{\beta-1}{2}} + \tau^{-1-\frac{2}{n}\mu - \frac{2}{n}\left(\beta - \frac{n}{2}\right)} \right) \\
&\quad + C\rho^{1+\frac{2}{n}} s^{\frac{2}{n}\gamma} \tau^{-1-\frac{2}{n}\gamma} \le C\rho\tau^{-1-\mu} \left(\varepsilon^{\frac{2}{n}} + \rho^{\frac{2}{n}} \tau^{-\mu} \right) \qquad (3.12)
\end{aligned}
$$

if we take $s = \tau^{1-\nu}$, $\gamma\nu \ge n\mu$ and $(\beta - 1)\nu + 2\mu \le \frac{1}{n}\left(\beta - \frac{n}{2}\right)$ in the cases $n = 2, 3$. (For example, we can choose $\nu = \frac{\gamma}{4}$ and $\mu = \nu^2$.)

Then by virtue of (3.10) and (3.12) we find from (3.9),

$$
\begin{aligned}
\left\| |\mathcal{J}(t)|^{\beta} (u(t) - u_1(t)) \right\|_{\mathbf{L}^2} &\le C\rho \int_t^\infty \tau^{-1-\mu} \left(\varepsilon^{\frac{2}{n}} + \rho^{\frac{2}{n}} \tau^{-\mu} \right) d\tau + C\rho t^{-\frac{\alpha - \beta}{2}} \log^2 t \\
&\le C\rho\tau^{-\mu}. \qquad (3.13)
\end{aligned}
$$

In view of (3.8) and (3.13) we find that there exists a time T such that $u \in \mathbf{X}_{\rho}$. In the same manner we can prove the estimate

$$
\|u - \widetilde{u}\|_{\mathbf{X}} \le \frac{1}{2} \|v - \widetilde{v}\|_{\mathbf{X}},
$$

where $\widetilde{u}$ is defined by (3.5) with v replaced by $\widetilde{v}$. Therefore (3.5) defines a contraction mapping. Hence there exists a unique global solution $u \in \mathbf{C}([T, \infty) ; \mathbf{L}^2)$ of the integral equation (3.4) satisfying the estimate

$$\|u(t) - u_1(t)\|_{\mathbf{L}^2} \leq Ct^{-\frac{n}{4}-\mu}.$$

Arguing in the same way as in the proof of [12] we can extend the existence time to zero. Theorem 1 is proved.

4. Modified Scattering Operator

To prove Theorem 2 let us consider the Cauchy problem for Eq. (1.1) with initial data $u_0 \in \mathbf{H}^{0,\beta}$ with $\frac{n}{2} < \beta < 1 + \frac{2}{n}$ and with sufficiently small norm $\varepsilon = \|u_0\|_{\mathbf{H}^{0,\beta}}$. In [7] it was proved that there exists a unique global solution u of the Cauchy problem for Eq. (1.1) satisfying

$$\mathcal{U}(-t)\,u \in \mathbf{C}\left((-\infty, 0] ; \mathbf{H}^{0,\beta}\right),$$

and the following estimates:

$$\|u(t)\|_{\mathbf{L}^2} \leq \|u_0\|_{\mathbf{L}^2}, \quad \left\||\mathcal{J}(t)|^{\beta} u(t)\right\|_{\mathbf{L}^2} \leq C\varepsilon\,|t|^{\epsilon} \tag{4.1}$$

for all $t \leq 0$, where $\epsilon = C\varepsilon^{\frac{2}{n}} > 0$ is small. From estimates (4.1) and the identity

$$u(t) = \mathcal{MDFU}(-t)\,u(t) + \mathcal{MDF}(\mathcal{M} - 1)\,\mathcal{U}(-t)\,u(t)$$

by Lemma 3 it follows that

$$\begin{aligned}
\|u\|_{\mathbf{L}^\infty} &\leq Ct^{-\frac{n}{2}} \|\mathcal{FU}(-t)\,u\|_{\mathbf{L}^\infty} \\
&\quad + Ct^{-\frac{n}{2}} \left\|(-\Delta)^{\frac{\beta}{2}} \mathcal{F}(\mathcal{M} - 1)\,\mathcal{U}(-t)\,u\right\|_{\mathbf{L}^2}^{\frac{n}{2\beta}} \|\mathcal{F}(\mathcal{M} - 1)\,\mathcal{U}(-t)\,u\|_{\mathbf{L}^2}^{1-\frac{n}{2\beta}} \\
&\leq Ct^{-\frac{n}{2}} \|\mathcal{FU}(-t)\,u\|_{\mathbf{L}^\infty} + Ct^{-\frac{n}{2}-\frac{\beta}{2}\left(1-\frac{n}{2\beta}\right)} \left\||\mathcal{J}|^{\beta} u\right\|_{\mathbf{L}^2}^{\frac{n}{2\beta}} \left\||x|^{\beta}\,\mathcal{U}(-t)\,u\right\|_{\mathbf{L}^2}^{1-\frac{n}{2\beta}} \\
&\leq Ct^{-\frac{n}{2}} \|\mathcal{FU}(-t)\,u\|_{\mathbf{L}^\infty} + Ct^{-\frac{n}{4}-\frac{\beta}{2}} \left\||\mathcal{J}|^{\beta} u\right\|_{\mathbf{L}^2}.
\end{aligned} \tag{4.2}$$

By (3.1) we have for the function $\widehat{w}(t) = \mathcal{FU}(-t)\,u(t)$,

$$\widehat{w}_t = -\frac{i\lambda}{t} |\widehat{w}|^{\frac{2}{n}} \widehat{w} + R_1 + R_2, \tag{4.3}$$

where the remainder terms

$$R_1 = -\frac{i\lambda}{t}\left(|\mathcal{FM}w|^{\frac{2}{n}} \mathcal{FM}w - |\widehat{w}|^{\frac{2}{n}} \widehat{w}\right)$$

and

$$R_2 = -\frac{i\lambda}{t}\mathcal{F}\left(\mathcal{M}^{-1} - 1\right)\mathcal{F}^{-1} |\mathcal{FM}w|^{\frac{2}{n}} \mathcal{FM}w.$$

By using Lemma 4 we have

$$\begin{aligned}
\|R_1\|_{\dot{\mathbf{H}}^\delta} &= |t|^{-1} \left\| |\mathcal{F}\mathcal{M}w|^{\frac{2}{n}} \mathcal{F}\mathcal{M}w - |\widehat{w}|^{\frac{2}{n}} \widehat{w} \right\|_{\dot{\mathbf{H}}^\delta} \\
&\le C |t|^{-1} \left(\|\mathcal{F}\mathcal{M}w\|^2_{\mathbf{L}^\infty} + \|\widehat{w}\|^2_{\mathbf{L}^\infty} \right) \|\mathcal{F}(\mathcal{M}-1) w\|_{\dot{\mathbf{H}}^\delta} \\
&\quad + C |t|^{-1} (\|\mathcal{F}\mathcal{M}w\|_{\mathbf{L}^\infty} + \|\widehat{w}\|_{\mathbf{L}^\infty}) \|\mathcal{F}(\mathcal{M}-1) w\|_{\mathbf{L}^\infty} \|\widehat{w}\|_{\dot{\mathbf{H}}^\delta} \\
&\le C |t|^{-1} \|(\mathcal{M}-1) w\|_{\dot{\mathbf{H}}^{0,\delta}} \left\| |\mathcal{J}|^\beta u \right\|^{\frac{n}{\beta}}_{\mathbf{L}^2} \|u\|^{2-\frac{n}{\beta}}_{\mathbf{L}^2} \\
&\quad + C |t|^{-1} \|(\mathcal{M}-1) w\|^{\frac{n}{2\delta}}_{\dot{\mathbf{H}}^{0,\delta}} \|(\mathcal{M}-1) w\|^{1-\frac{n}{2\delta}}_{\mathbf{L}^2} \\
&\quad \times \left\| |\mathcal{J}|^\beta u \right\|^{\frac{n}{2\beta}}_{\mathbf{L}^2} \|u\|^{1-\frac{n}{2\beta}}_{\mathbf{L}^2} \left\| |\mathcal{J}|^\beta u \right\|^{\frac{\delta}{\beta}}_{\mathbf{L}^2} \|u\|^{1-\frac{\delta}{\beta}}_{\mathbf{L}^2} \\
&\le C\varepsilon |t|^{-1-\frac{\beta-\delta}{2}+\left(1+\frac{n}{\beta}\right)\epsilon}
\end{aligned}$$

if $n = 1$. Also

$$\begin{aligned}
\|R_1\|_{\dot{\mathbf{H}}^\delta} &= C |t|^{-1} \left\| |\mathcal{F}\mathcal{M}w|^{\frac{2}{n}} \mathcal{F}\mathcal{M}w - |\widehat{w}|^{\frac{2}{n}} \widehat{w} \right\|_{\dot{\mathbf{H}}^\delta} \\
&\le C |t|^{-1} \|\mathcal{F}\mathcal{M}w\|^{\frac{2}{n}}_{\mathbf{L}^\infty} \left(\|(\mathcal{M}-1) w\|_{\dot{\mathbf{H}}^{0,\delta}} + s^{1-\beta} \|(\mathcal{M}-1) w\|_{\dot{\mathbf{H}}^{0,1}} \right) \\
&\quad + C |t|^{-1} \|\mathcal{F}(\mathcal{M}-1) w\|^{\frac{2}{n}}_{\mathbf{L}^\infty} \left(\|w\|_{\dot{\mathbf{H}}^{0,\delta}} + s^{1-\beta} \|w\|_{\dot{\mathbf{H}}^{0,1}} \right) \\
&\quad + C |t|^{-1} s^{\frac{2}{n}\gamma} \|w\|^{\frac{2}{n}}_{\dot{\mathbf{H}}^{0,\frac{n}{2}+\gamma}} \|w\|_{\dot{\mathbf{H}}^{0,\delta}}
\end{aligned}$$

for all $s > 0$ if $n = 2, 3$, where $0 < \gamma \le \min\left(\beta - \frac{n}{2}, \frac{n}{2}\left(1 + \frac{2}{n} - \beta\right)\right)$. Then using the estimates

$$\|\mathcal{F}\mathcal{M}w\|_{\mathbf{L}^\infty} \le C\varepsilon |t|^{\frac{n}{2\beta}\epsilon}, \ \|\mathcal{F}(\mathcal{M}-1) w\|_{\mathbf{L}^\infty} \le C\varepsilon |t|^{-\frac{\beta}{2}\left(1-\frac{n}{2\beta}\right)+\epsilon},$$
$$\|w\|_{\dot{\mathbf{H}}^{0,\alpha}} \le C\varepsilon |t|^{\frac{\alpha}{\beta}\epsilon}, \ \|(\mathcal{M}-1) w\|_{\dot{\mathbf{H}}^{0,\alpha}} \le C\varepsilon |t|^{-\frac{\beta-\alpha}{2}+\epsilon}$$

for $0 \le \alpha \le \beta$ we get

$$\begin{aligned}
\|R_1\|_{\dot{\mathbf{H}}^\delta} &\le C |t|^{-1-\frac{\beta-\delta}{n}+2\epsilon} + C |t|^{-1+2\epsilon} \left(s^{1-\beta} |t|^{-\frac{1}{n}\left(\beta-\frac{n}{2}\right)} + s^{\frac{2}{n}\gamma} \right) \\
&\le C |t|^{-1-\frac{\beta-\delta}{n}+2\epsilon} + C |t|^{-1-\frac{2\gamma}{n}\nu+2\epsilon}
\end{aligned}$$

if we take $s = t^{-\nu}$, with $\nu = \frac{\frac{1}{n}\left(\beta-\frac{n}{2}\right)}{\beta-1+\frac{2\gamma}{n}}$.

Also by using Lemma 4 we have

$$\begin{aligned}
\|R_2\|_{\dot{\mathbf{H}}^\delta} &\le C |t|^{-1} \left\| \mathcal{F}\left(\mathcal{M}^{-1} - 1\right) \mathcal{F}^{-1} |\mathcal{F}\mathcal{M}w|^{\frac{2}{n}} \mathcal{F}\mathcal{M}w \right\|_{\dot{\mathbf{H}}^\delta} \\
&\le C |t|^{-1-\frac{\beta-\delta}{2}} \left\| |\mathcal{F}\mathcal{M}w|^{\frac{2}{n}} \mathcal{F}\mathcal{M}w \right\|_{\dot{\mathbf{H}}^\beta} \le C |t|^{-1-\frac{\beta-\delta}{2}} \|\mathcal{F}\mathcal{M}w\|^{\frac{2}{n}}_{\mathbf{L}^\infty} \|w\|_{\dot{\mathbf{H}}^{0,\beta}} \\
&\le C |t|^{-1-\frac{\beta-\delta}{2}} \|w\|^{1+\frac{n}{\beta}}_{\dot{\mathbf{H}}^{0,\beta}} \|w\|^{\frac{2}{n}-\frac{n}{\beta}}_{\mathbf{L}^2} \le C |t|^{-1-\frac{\beta-\delta}{2}} \left\| |\mathcal{J}|^\beta u \right\|^{1+\frac{n}{\beta}}_{\mathbf{L}^2} \|u\|^{\frac{2}{n}-\frac{n}{\beta}}_{\mathbf{L}^2} \\
&\le C\varepsilon |t|^{-1-\frac{\beta-\delta}{2}+\left(1+\frac{n}{\beta}\right)\epsilon}.
\end{aligned}$$

Thus we have

$$\|R_1\|_{\dot{\mathbf{H}}^\delta} + \|R_2\|_{\dot{\mathbf{H}}^\delta} \le C\varepsilon\, |t|^{-1-\mu} \tag{4.4}$$

for $\frac{n}{2} < \delta < \beta$ and some $0 < \mu < \frac{\beta-\delta}{n}$.

Multiplying both sides of Eq. (4.3) by $\overline{\widehat{w}}$ and taking the real part of the result we get for $h(t) = |\widehat{w}(t)|^2$,

$$h_t = 2\mathrm{Re}\left(\overline{\widehat{w}}\,(R_1 + R_2)\right),$$

hence integrating with respect to time we find

$$h(t) - h(s) = 2\mathrm{Re}\int_s^t \left(\overline{\widehat{w}}\,(R_1 + R_2)\right) d\tau.$$

By estimate (4.4) we have

$$\left\|h(t) - h\left(t'\right)\right\|_{\mathbf{H}^\delta} \le C\varepsilon^2 \int_{t'}^t |\tau|^{-1-\mu+\epsilon}\, d\tau \le C\varepsilon^2\, |t|^{\epsilon-\mu}$$

for all $t' < t < -1$. Then we see that there exists a unique limit $h_- \in \mathbf{H}^\delta$ such that

$$\|h(t) - h_-\|_{\mathbf{H}^\delta} \le C\varepsilon^2\, |t|^{\epsilon-\mu} \tag{4.5}$$

for all $t < -1$.

Multiplying both sides of Eq. (4.3) by $E(t) \equiv \exp\left(i\lambda\left(h_- + |t|^{-\chi}\right)^{\frac{1}{n}} \log|t|\right)$, with a small $\chi > 0$ we get

$$\partial_t\left(\widehat{w}E\right) = F, \tag{4.6}$$

where

$$F(t) = -i\lambda\left(t^{-1}\,|h(t)|^{\frac{1}{n}} - \partial_t\left(\left(h_- + |t|^{-\chi}\right)^{\frac{1}{n}} \log|t|\right)\right)\widehat{w}E + (R_1 + R_2)\,E.$$

Note that by Lemma 4 we find

$$\begin{aligned}\|F(t)\|_{\mathbf{H}^\delta} &\le C\left\|\left(t^{-1}\,|h(t)|^{\frac{1}{n}} - \partial_t\left(\left(h_- + |t|^{-\chi}\right)^{\frac{1}{n}} \log|t|\right)\right)\widehat{w}E\right\|_{\mathbf{H}^\delta}\\ &\quad + \|(R_1 + R_2)\,E\|_{\mathbf{H}^\delta} \le C\varepsilon^{1+\frac{2}{n}}\, |t|^{-1-\mu+2\chi+\epsilon}.\end{aligned}$$

Therefore integrating (4.6) with respect to time, we obtain

$$\begin{aligned}\left\|\widehat{w}\left(t'\right)E\left(t'\right) - \widehat{w}(t)\,E(t)\right\|_{\mathbf{H}^\delta} &= \left\|\int_{t'}^t F(\tau)\,d\tau\right\|_{\mathbf{H}^\delta}\\ &\le C\varepsilon^{1+\frac{2}{n}} \int_{t'}^t |\tau|^{-1-\mu+2\chi+\epsilon}\, d\tau\\ &\le C\varepsilon^{1+\frac{2}{n}}\, |t|^{2\chi+\epsilon-\mu}\end{aligned} \tag{4.7}$$

for all $t' < t < -1$. Since the norm $\|u_0\|_{\mathbf{H}^{0,\beta}}$ is sufficiently small, then we see that there exists a unique function $\widehat{u_-} \in \mathbf{H}^{\delta}$ such that

$$\|\widehat{w}(t)\, E(t) - \widehat{u_-}\|_{\mathbf{H}^{\delta}} \leq C\varepsilon^{1+\frac{2}{n}}\, |t|^{2\chi+\epsilon-\mu}$$

for all $t < -1$. This implies (1.3). Furthermore the asymptotic representation (1.4) is true. Theorem 2 is proved.

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