Commun. Math. Phys. 267, 515–542 (2006)
Digital Object Identifier (DOI) 10.1007/s00220-006-0055-8

Communications in
**Mathematical
Physics**

Renormalization of Non-Commutative Φ^4_4 Field Theory in x Space

Razvan Gurau[1], Jacques Magnen[2], Vincent Rivasseau[1], Fabien Vignes-Tourneret[1]

[1] Laboratoire de Physique Théorique, Bât. 210, CNRS UMR 8627 Université Paris XI, 91405 Orsay Cedex, France. E-mail: {razvan.gurau, vincent.rivasseau, fabien.vignes}@th.u-psud.fr
[2] Centre de Physique Théorique, CNRS UMR 7644 Ecole Polytechnique 91128 Palaiseau Cedex, France. E-mail: magnen@cpht.polytechnique.fr

Received: 22 December 2005 / Accepted: 27 February 2006
Published online: 15 June 2006 –

Abstract: In this paper we provide a new proof that the Grosse–Wulkenhaar non-commutative scalar Φ^4_4 theory is renormalizable to all orders in perturbation theory, and extend it to more general models with covariant derivatives. Our proof relies solely on a multiscale analysis in x space. We think this proof is simpler. It also allows direct interpretation in terms of the physical positions of the particles and should be more adapted to the future study of these theories (in particular at the non-perturbative or constructive level).

1. Introduction

In this paper we recover the proof of perturbative renormalizability of non-commutative Φ^4_4 field theory [1–3] by a method solely based on x space. In this way we avoid completely the sometimes tedious use of the matrix basis and of the associated special functions of [1–3] and we recover the more physical direct space representation of fields and particles. Moreover our proof works for the optimal range $]0, 1]$ of the parameter Ω which was restricted to a much smaller interval in a previous proof. We also extend the corresponding BPHZ theorem to the more general complex Langmann-Szabo-Zarembo $\bar{\varphi}\star\varphi\star\bar{\varphi}\star\varphi$ model with covariant derivatives, hereafter called the LSZ model. This model has a slightly more complicated propagator, and is exactly solvable in a certain limit [4].

Our method builds upon previous work of Filk and Chepelev-Roiban [5, 6]. These works however remained inconclusive [7], since these authors used the right interaction but not the right propagator, hence the problem of ultraviolet/infrared mixing prevented them from obtaining a finite renormalized perturbation series. The Grosse Wulkenhaar breakthrough was to realize that the right propagator in non-commutative field theory is not the ordinary commutative propagator, but has to be modified to obey Langmann-Szabo duality [8, 2].

Non-commutative field theories (for a general review see [9]) deserve a thorough and systematic investigation. Indeed they may be relevant for physics beyond the standard model. They are certainly effective models for certain limits of string theory [10, 11]. Also they form almost certainly the correct framework for a microscopic *ab initio* understanding of the quantum Hall effect which is currently lacking. We think that x space-methods are probably more powerful for the future systematic study of the noncommutative Langmann-Szabo covariant field theories.

Fermionic theories such as the two dimensional Gross-Neveu model can be shown to be renormalizable to all orders in their Langmann-Szabo covariant versions, using either the matrix basis or the direct space version developed here [12]. However the x-space version seems the most promising for a complete non perturbative construction, using Pauli's principle to control the apparent (fake) divergences of perturbation theory. In the case of ϕ^4_4, recall that although the commutative version has been until now fatally flawed due to the famous Landau ghost, there is some hope that the non-commutative field theory treated at the perturbative level in this paper may also exist at the constructive level [13, 14]. Again the x-space formalism is probably better than the matrix basis for a rigorous investigation of this question.

In the first section of this paper we establish the x-space power counting of the theory using the Mehler kernel form of the propagator in direct space given in [15]. In the second section we prove that the divergent subgraphs can be renormalized by counterterms of the form of the initial Lagrangian. The LSZ models are treated in the Appendix. Note that we do not prove here the exact topological power counting for irrelevant graphs. This should be doable with our methods but is not necessary for our theorem.

2. Power Counting in x-Space

2.1. Model, notations. Beware that throughout this paper we will use many different notations for position variables. To avoid any confusion for the reader we summarize these notations at the end of the paper.

The simplest noncommutative φ^4_4 theory is defined on $\mathbb{R}^4$ equipped with the associative and noncommutative Moyal product

$$(a \star b)(x) = \int \frac{d^4k}{(2\pi)^4} \int d^4y \; a(x+\tfrac{1}{2}\theta\cdot k)\, b(x+y)\, \mathrm{e}^{\imath k\cdot y} \; . \tag{2.1}$$

The renormalizable action functional introduced in [2] is

$$S[\varphi]=\int d^4x \left(\frac{1}{2}\partial_\mu\varphi \star \partial^\mu\varphi + \frac{\Omega^2}{2}(\tilde{x}_\mu\varphi)\star(\tilde{x}^\mu\varphi) + \frac{1}{2}\mu_0^2\,\varphi\star\varphi + \frac{\lambda}{4!}\varphi\star\varphi\star\varphi\star\varphi\right)(x), \tag{2.2}$$

where $\tilde{x}_\mu = 2(\theta^{-1})_{\mu\nu}x^\nu$ and the Euclidean metric is used.

In four dimensional x-space the propagator is [15]

$$C(x,x') = \frac{\tilde{\Omega}^2}{[2\pi \sinh \tilde{\Omega}t]^2} e^{-\frac{\tilde{\Omega}\coth\tilde{\Omega}t}{2}(x^2+x'^2)+\frac{\tilde{\Omega}}{\sinh\tilde{\Omega}t}x\cdot x'-\mu_0^2 t}, \tag{2.3}$$

where $\tilde{\Omega} = 2\theta^{-1}\Omega$ and the (cyclically invariant) vertex is [5]

$$V(x_1,x_2,x_3,x_4) = \delta(x_1-x_2+x_3-x_4)e^{\imath\sum_{1\leqslant i<j\leqslant 4}(-1)^{i+j+1}x_i\theta^{-1}x_j}, \tag{2.4}$$

where we note[1] $x\theta^{-1}y \equiv \frac{2}{\theta}(x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3)$.

The main result of this paper is a new proof in configuration space of

[1] Of course two different θ parameters could be used for the two symplectic pairs of variables of $\mathbb{R}^4$.

Theorem 2.1 (BPHZ Theorem for Noncommutative Φ_4^4 [2, 3]). *The theory defined by the action* (2.2) *is renormalizable to all orders of perturbation theory.*

Let G be an arbitrary connected graph. The amplitude associated with this graph is (with selfexplaining notations):

$$A_G = \int \prod_{v,i=1,\ldots 4} dx_{v,i} \prod_l dt_l \\ \times \prod_v \Big[\delta(x_{v,1} - x_{v,2} + x_{v,3} - x_{v,4}) e^{\imath \sum_{i<j} (-1)^{i+j+1} x_{v,i}\theta^{-1}x_{v,j}} \Big] \\ \times \prod_l \frac{\tilde{\Omega}^2}{[2\pi \sinh(\tilde{\Omega}t_l)]^2} e^{-\frac{\tilde{\Omega}}{2}\coth(\tilde{\Omega}t_l)(x_{v,i(l)}^2 + x_{v',i'(l)}^2) + \frac{\tilde{\Omega}}{\sinh(\tilde{\Omega}t_l)} x_{v,i(l)}\cdot x_{v',i'(l)} - \mu_0^2 t_l}. \tag{2.5}$$

For each line l of the graph joining positions $x_{v,i(l)}$ and $x_{v',i'(l)}$, we choose an orientation and we define the "short" variable $u_l = x_{v,i(l)} - x_{v',i'(l)}$ and the "long" variable $v_l = x_{v,i(l)} + x_{v',i'(l)}$. With these notations, defining $\tilde{\Omega}t_l = \alpha_l$, the propagators in our graph can be written as:

$$\int \prod_l \frac{\tilde{\Omega} d\alpha_l}{[2\pi \sinh(\alpha_l)]^2} e^{-\frac{\tilde{\Omega}}{4}\coth(\frac{\alpha_l}{2})u_l^2 - \frac{\tilde{\Omega}}{4}\tanh(\frac{\alpha_l}{2})v_l^2 - \frac{\mu_0^2}{\tilde{\Omega}}\alpha_l}. \tag{2.6}$$

2.2. Orientation and position routing. A rule to solve the δ functions at every vertex is a "position routing" exactly analog to a momentum routing in the ordinary commutative case, except for the additional difficulty of the cyclic signs which impose to orient the lines. It is well known that there is no canonical such routing but there is a routing associated to any choice of a spanning tree in G. Such a tree choice is also useful to orient the lines of the graph, hence to fix the exact sign definition of the "short" line variables u_l, and to optimize the multiscale power counting bounds below.

Let n be the number of vertices of G, N the number of its external fields, and L the number of internal lines of G. We have $L = 2n - N/2$. Let T be a rooted tree in the graph (when the graph is not a vacuum graph it is convenient to choose for the root a vertex with external fields but this is not essential). We orient first all the lines of the tree and all the remaining half-loop lines or "loop fields", following the cyclicity of the vertices. This means that starting from an arbitrary orientation of a first field at the root and inductively climbing into the tree, at each vertex we follow the cyclic order to alternate entering and exiting lines as in Fig. 1.

Every line of the tree by definition of this orientation has one end exiting a vertex and another entering another one. This may not be true for the loop lines, which join two "loop fields". Among these, some exit one vertex and enter another; they are called well-oriented. But others may enter or exit at both ends. These loop lines are subsequently referred to as "clashing lines". If there are no clashing lines, the graph is called orientable. If not, it is called non-orientable.

We will see below that non-orientable graphs are irrelevant in the renormalization group sense. In fact they do not occur at all in some particular models such as the LSZ model treated in the Appendix, or in the most natural noncommutative Gross-Neveu models [12].

For all the well-oriented lines (hence all tree propagators plus some of the loop propagators) we define in the natural way $u_l = x_{v,i(l)} - x_{v',i'(l)}$ if the line enters at $x_{v,i(l)}$

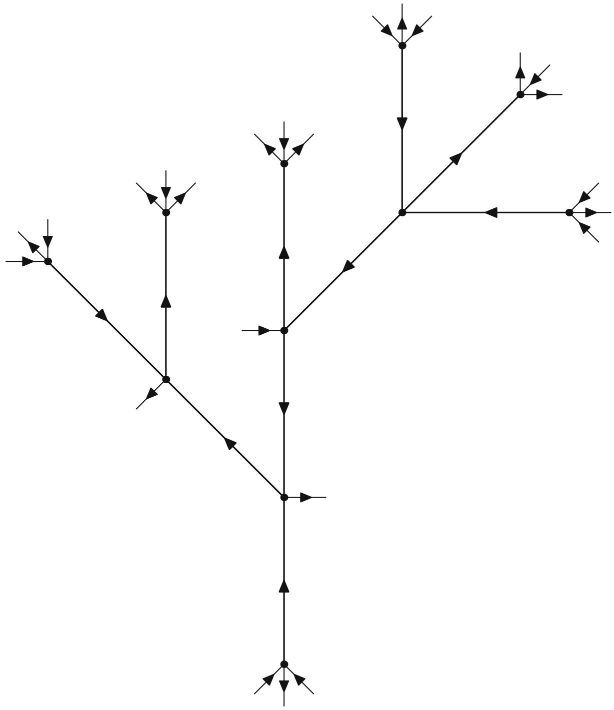

Fig. 1. Orientation of a tree

and exits from $x_{v',i'(l)}$. Finally we fix an additional (completely arbitrary) auxiliary orientation for all the clashing loop lines, and fix in the same way $u_l = x_v - x_{v'}$ with respect to this auxiliary orientation.

It is also convenient to define the set of "branches" associated to the rooted tree T. There are $n-1$ such branches $b(l)$, one for each of the $n-1$ lines l of the tree, plus the full tree itself, called the root branch, and noted b_0. Each such branch is made of the subgraph G_b containing all the vertices "above l" in T, plus the tree lines and loop lines joining these vertices. It has also "external fields" which are the true external fields hooked to G_b, plus the loop fields in G_b for the loops with one end (or "field") inside and one end outside G_b, plus the upper end of the tree line l itself to which b is associated. In the particular case of the root branch, $G_{b_0} = G$ and the external fields for that branch are simply all true external fields. We call X_b the set of all external fields f of b.

We can now describe the position routing associated to T. There are n δ functions in (2.5), hence n linear equations for the $4n$ positions, one for each vertex. The momentum routing associated to the tree T solves this system by passing to another equivalent system of n linear equations, one for each branch of the tree. This equivalent system is obtained by summing the arguments of the δ functions of the vertices in each branch. Obviously the Jacobian of this transformation is 1, so we simply get another equivalent set of n δ functions, one for each branch.

Let us describe more precisely the positions summed in these branch equations, using the orientation. Fix a particular branch G_b, with its subtree T_b. In the branch sum we find a sum over all the u_l short parameters of the lines l in T_b and no v_l long parameters since l both enters and exits the branch. This is also true for the set L_b of well-oriented loops lines with both fields in the branch. For the set $L_{b,+}$ of clashing loops lines with both fields entering the branch, the short variable disappears and the long variable remains; the same is true but with a minus sign for the set $L_{b,-}$ of clashing loops lines with both fields exiting the branch. Finally we find the sum of positions of all external fields for the branch (with the signs according to entrance or exit). For instance in the particular case of Fig. 2, the delta function is

$$\delta\big(u_{l_1} + u_{l_2} + u_{l_3} + u_{L_1} + u_{L_3} - v_{L_2} + X_1 - X_2 + X_3 + X_4\big). \tag{2.7}$$

The position routing is summarized by:

Lemma 2.1 (Position Routing). *We have, calling I_G the remaining integrand in* (2.5)*:*

$$A_G = \int \Big[\prod_v \big[\delta(x_{v,1} - x_{v,2} + x_{v,3} - x_{v,4})\big]\Big] I_G(\{x_{v,i}\})$$
$$= \int \prod_b \delta\left(\sum_{l\in T_b\cup L_b} u_l + \sum_{l\in L_{b,+}} v_l - \sum_{l\in L_{b,-}} v_l + \sum_{f\in X_b} \epsilon(f) x_f\right) I_G(\{x_{v,i}\}), \tag{2.8}$$

where $\epsilon(f)$ is ± 1 depending on whether the field f enters or exits the branch.

Using the above equations one can at least solve all the long tree variables v_l in terms of external variables, short variables and long loop variables, using the $n-1$ non-root branches. To this end, recall that the unique X_i which is at the upper end of each tree line should be written in (2.7) as $1/2(v_l \pm u_l)$. There remains then the root branch δ function. If G_b is orientable, this δ function of branch b_0 contains only short and external variables, since $L_{b,+}$ and $L_{b,-}$ are empty. If G_b is non-orientable one can solve for an

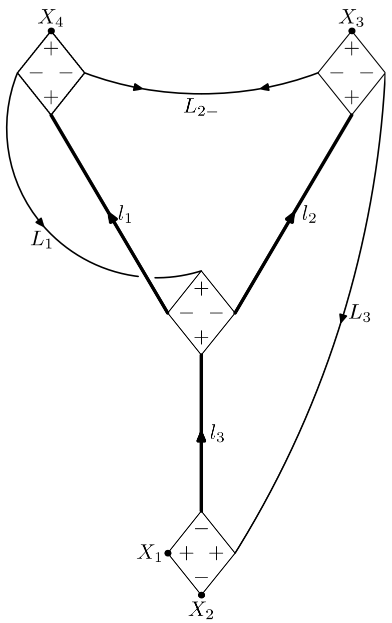

Fig. 2. A branch

additional "clashing" long loop variable. We can summarize these observations in the following lemma:

Lemma 2.2. *The position routing solves any long tree variable v_l as a function of:*

- *the short tree variable u_l of the line l itself,*
- *the short tree and loop variables with both ends in $G_{b(l)}$,*
- *the long loop variables of the clashing loops with both ends in $G_{b(l)}$ (if any),*
- *the short and long variables of the loop lines with one end inside $G_{b(l)}$ and the other outside,*
- *the true external variables x hooked to $G_{b(l)}$.*

The last equation corresponding to the root branch is particular. In the orientable case it does not contain any long variable, but gives a linear relation among the short variables and the external positions. In the non-orientable case it gives a linear relation between the long variables w of all the clashing loops in the graph, some short variables u, and all the external positions.

From now on, each time we use this lemma to solve the long tree variables v_l in terms of the other variables, we shall call w_l rather than v_l the remaining $n+1-N/2$ independent long loop variables. Hence looking at the long variables' names the reader can check whether Lemma 2.2 has been used or not.

2.3. Multiscale analysis and crude power counting. In this section we follow the standard procedure of multiscale analysis [16]. First the parametric integral for the propagator is sliced in the usual way:

$$C(u, v) = C^0(u, v) + \sum_{i=1}^{\infty} C^i(u, v), \tag{2.9}$$

with

$$C^i(u, v) = \int_{M^{-2i}}^{M^{-2(i-1)}} \frac{\tilde{\Omega} d\alpha}{[2\pi \sinh \alpha]^2} e^{-\frac{\tilde{\Omega}}{4} \coth \frac{\alpha}{2} u^2 - \frac{\tilde{\Omega}}{4} \tanh \frac{\alpha}{2} v^2 - \frac{\mu_0^2}{\tilde{\Omega}} \alpha_l}. \tag{2.10}$$

Lemma 2.3. *For some constants K (large) and c (small) :*

$$C^i(u, v) \leqslant K M^{2i} e^{-c[M^i \|u\| + M^{-i} \|v\|]} \tag{2.11}$$

(*which a posteriori justifies the terminology of "long" and "'short" variables*).

The proof is elementary. For $i \geqslant 1$, it relies only on second order approximation of the hyperbolic functions near the origin. This bound is also true for the first slice $i = 0$ with K depending on μ_0.

Taking absolute values, hence neglecting all oscillations, leads to the following crude bound:

$$|A_G| \leqslant \sum_{\mu} \int du_l dv_l \prod_l C^{i_l}(u_l, v_l) \prod_v \delta_v, \tag{2.12}$$

where μ is the standard assignment of an integer index i_l to each propagator of each internal line l of the graph G, which represents its "scale". We will consider only amputated graphs. Therefore we have no external propagators, but only external vertices of the graph; in the renormalization group spirit, the convenient convention is to assign all external indices of these external fields to a fictitious -1 "background" scale.

To any assignment μ and scale i are associated the standard connected components G^i_k, $k = 1, \ldots, k(i)$ of the subgraph G^i made of all lines with scales $j \geqslant i$. These tree components are partially ordered according to their inclusion relations and the (abstract) tree describing these inclusion relations is called the Gallavotti-Nicolò tree [17]; its nodes are the G^i_k's and its root is the complete graph G (see Fig. 3).

More precisely for an arbitrary subgraph g one defines:

$$i_g(\mu) = \inf_{l \in g} i_l(\mu), \quad e_g(\mu) = \sup_{l \text{ external line of } g} i_l(\mu). \tag{2.13}$$

The subgraph g is a G^i_k for a given μ if and only if $i_g(\mu) \geqslant i > e_g(\mu)$. As is well known in the commutative field theory case, the key to optimize the bound over spatial integrations is to choose the real tree T compatible with the abstract Gallavotti-Nicolò tree, which means that the restriction T^i_k of T to any G^i_k must still span G^i_k. This is always possible (by a simple induction from leaves to root). In Fig. 3a, an example of such a compatible tree is given with bold lines. We pick such a compatible tree T and use it both to orient the graph as in the previous section and to solve the associated branch system of δ functions according to Lemma 2.2. We obtain:

$$\begin{aligned} |A_{G,\mu}| &\leqslant K^n \prod_l M^{2i_l} \int du_l dv_l \prod_l e^{-c[M^{i_l}\|u_l\| + M^{-i_l}\|v_l\|]} \prod_b \delta_b. \\ &\leqslant K^n \prod_l M^{2i_l} \int du_l dw_l \prod_l e^{-c[M^{i_l}\|u_l\| + M^{-i_l}\|v_l(u,w,x)\|]} \delta_{b_0}. \end{aligned} \tag{2.14}$$

The key observation is to remark that any long variable integrated at scale i costs KM^{4i} whereas any short variable integrated at scale i brings KM^{-4i}, and the variables "solved" by the δ functions bring or cost nothing. For an orientable graph the optimal solution is easy: we should solve the $n-1$ long variables v_l's of the tree propagators in terms of the other variables, because this is the maximal number of long variables that we can solve, and they have highest possible indices because T has been chosen compatible with the Gallavotti-Nicolò tree structure. Finally we still have the last δ_{b_0} function (equivalent to the overall momentum conservation in the commutative case). It is optimal to use it to solve one external variable (if any) in terms of all the short variables and the external ones. Since external variables are typically smeared against unit scale test functions, this leaves power counting invariant.[2]

The non-orientable case is slightly more subtle. We remarked that in this case the system of branch equations allows to solve n long variables as a function of all the others.

[2] In the case of a vacuum graph, there are no external variables and we must therefore use the last δ_{b_0} function to solve the lowest possible short variable in terms of all others. In this way, we lose the M^{-4i} factor for this short integration. This is why the power counting of a vacuum graph at scale i is not given by the usual formula $M^{(4-N)i} = M^{4i}$ below at $N = 0$, but is in M^{8i}, hence worse by M^{4i}. This is of course still much better than the commutative case, because in that case and in the analog conditions, that is without a fixed internal point, vacuum graphs would be worse than the others by an ... infinite factor, due to translation invariance! In any case vacuum graphs are absorbed in the normalization of the theory, hence play no role in the renormalization.

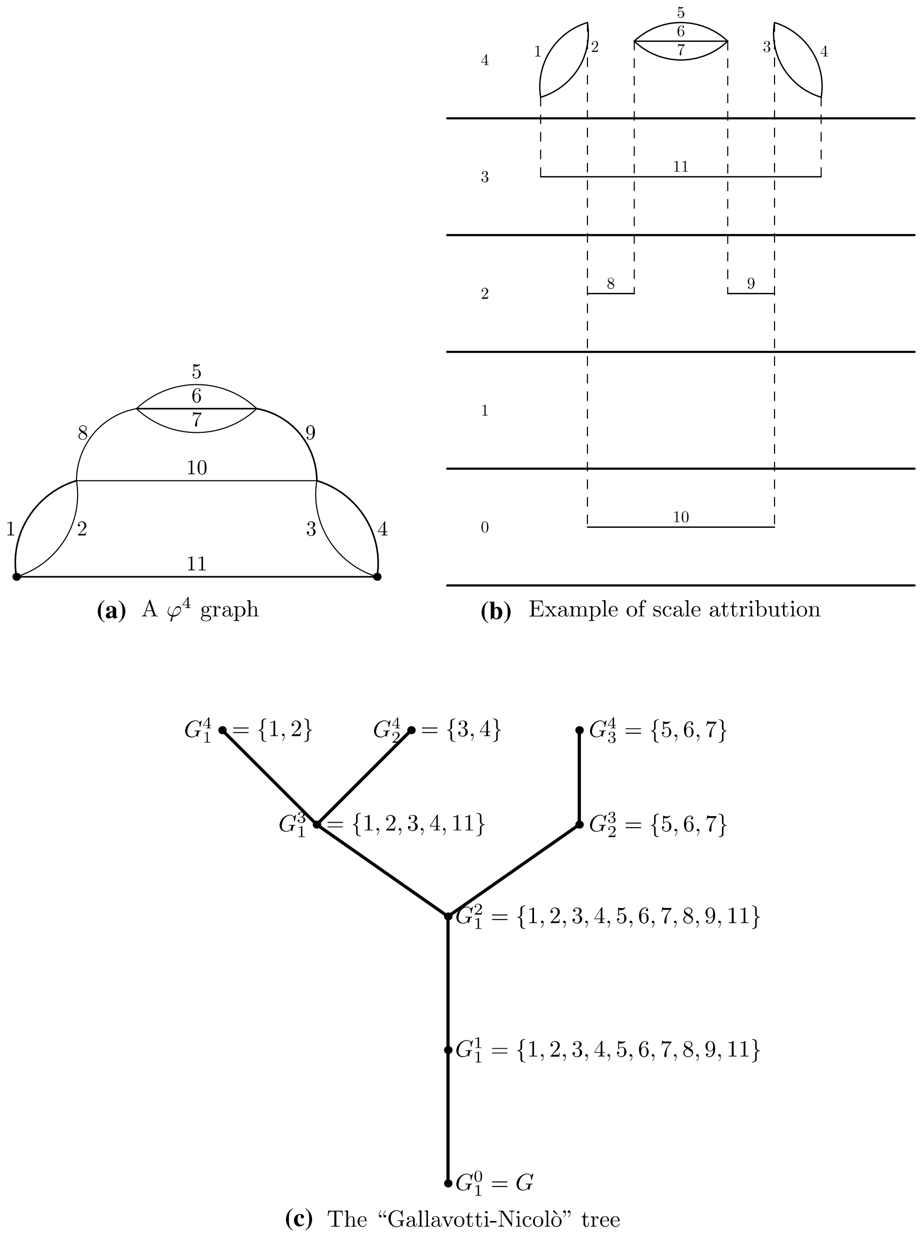

Fig. 3. (a) A φ^4 graph (b) Example of scale attribution (c) The "Gallavotti-Nicolò" tree

Should we always choose these n long variables as the $n-1$ long tree variables plus one long loop variable? This is *not always* the optimal choice. Indeed when several disjoint G_k^i subgraphs are non-orientable it is better to solve more long clashing loop variables, essentially one per disjoint non-orientable G_k^i, because they spare higher costs than if tree lines were chosen instead. We now describe the optimal procedure, using words rather than equations to facilitate the reader's understanding.

Let $\mathcal{C}$ be the set of all the clashing loop lines. Each clashing loop line has a certain scale i, therefore belongs to one and only one G^i_k and consequently to all $G^j_{k'} \supset G^i_k$. We now define the set S of n long variables to be solved via the δ functions. First we put in S all the $n-1$ long tree variables v_l. Then we scan all the connected components G^i_k starting from the leaves towards the root, and we add a clashing line to S each time some new non-orientable component G^i_k appears. We also remove $p-1$ tree lines from S each time $p \geqslant 2$ non-orientable components merge into a single one. In the end we obtain a new set S of exactly n long variables.

More precisely suppose some G^i_k at scale i is a "non-orientable leaf", which means that it contains some clashing lines at scale i but none at scales $j > i$. We then choose one (arbitrary) such clashing line and put it in the set S. Once a clashing line is added to S in this way it is never removed and no other clashing line is chosen in any of the G^j_k at lower scales $j < i$ to which the chosen line belongs. (The reader should be aware that this process allows nevertheless several clashing lines of S to belong to a single G^i_k, provided they were added to different connected components at upper scales.) When $p \geqslant 2$ non-orientable components merge at scale i into a single non-orientable G^i_k, we can find $p-1$ lines in the part of the tree T^i_k joining them together, (e.g. taking them among the first lines on the unique paths in T from these p components towards the root) and remove them from S.

We see that if we have added in all q clashing lines to the set S, we have eliminated $q-1$ tree lines. The final set S thus obtained in the end has exactly n elements. The non-trivial statement is that thanks to inductive use of Lemma 2.2 in each G^i_k, we can solve all the long variables in the set S with the branch system of δ functions associated to T.

We perform now all remaining integrations. This spares the corresponding M^{4i} integration cost for each long variable in S. For any line not in S we see that the net power counting is 1, since the cost of the long variable integration exactly compensates the gain of the short variable integration. But for any line in S we earn the M^{-4i} power counting of the corresponding short variable u without paying the M^{4i} cost of the long variable.

Gathering all the corresponding factors together with the propagators' prefactors M^{2i} leads to the following bound:

$$|A_{G,\mu}| \leqslant K^n \prod_l M^{2i_l} \prod_{l \in S} M^{-4i_l}. \tag{2.15}$$

Remark that if the graph is well-oriented this formula remains true but the set S consists of only the $n-1$ tree lines.

In the usual way of [16] we write

$$\prod_l M^{2i_l} = \prod_l \prod_{i=1}^{i_l} M^2 = \prod_{i,k} \prod_{l \in G^i_k} M^2 = \prod_{i,k} M^{2l(G^i_k)} \tag{2.16}$$

and

$$\prod_{l \in S} \prod_{i=1}^{i_l} M^{-4i_l} = \prod_{i,k} \prod_{l \in G^i_k \cap S} M^{-4}, \tag{2.17}$$

and we must now only count the number of elements in $G^i_k \cap S$.

If G^i_k is orientable, it contains no clashing lines, hence $G^i_k \cap S = T^i_k$, and the cardinal of T^i_k is $n(G^i_k) - 1$.

If G^i_k contains one or more clashing lines and p clashing lines $l_1, \ldots, l_p$ in G^i_k have been chosen to belong to S, then $p-1$ tree variables in T^i_k have also been removed from S and $G^i_k \cap S = T^i_k \cup \{l_1, \ldots, l_p\} - \{\text{p} - 1 \text{ tree variables}\}$, hence the cardinal of $G^i_k \cap S$ is $n(G^i_k)$.

Using the fact that $2l(G^i_k) - 4n(G^i_k) = -N(G^i_k)$ we can summarize these results in the following lemma:

Lemma 2.4. *The following bound holds for a connected graph (with external arguments integrated against fixed smooth test functions):*

$$|A_{G,\mu}| \leqslant K^n \prod_{i,k} M^{-\omega(G^i_k)} \tag{2.18}$$

for some (large) constant K, with $\omega(G^i_k) = N(G^i_k) - 4$ if G^i_k is orientable and $\omega(G^i_k) = N(G^i_k)$ if G^i_k is non-orientable.

This lemma is optimal *if vertices' oscillations are not taken into account*, and proves that non-orientable subgraphs are irrelevant. But it is not yet sufficient for a renormalization theorem to all orders of perturbation.

2.4. Improved power counting. Recall that for any non-commutative Feynman graph G we can define the genus of the graph, called g and the number of faces "broken by external legs", called B [2, 3]. We have $g \geqslant 0$ and $B \geqslant 1$. The power counting established with the matrix basis in [2, 3], rewritten in the language of this paper [3] is:

$$\omega(G) = N - 4 + 8g + 4(B-1), \tag{2.19}$$

hence we must (and can) renormalize only 2 and 4 point subgraphs with $g = 0$ and $B = 1$, which we call *planar regular*. They are the only non-vacuum graphs with $\omega \leqslant 0$.

In the previous section we established that

$$\omega(G) \geqslant N - 4, \text{ if } G \text{ orientable}, \quad \omega(G) \geqslant N, \text{ if } G \text{ non-orientable}. \tag{2.20}$$

It is easy to check that planar regular subgraphs are orientable, but the converse is not true. Hence to prove that *orientable non-planar* subgraphs or *orientable planar* subgraphs with $B \geqslant 2$ are irrelevant requires to use a bit of the vertices oscillations to improve Lemma 2.4 and get:

Lemma 2.5. *For orientable subgraphs with $g \geqslant 1$ we have*

$$\omega(G) \geqslant N + 4. \tag{2.21}$$

For orientable subgraphs with $g = 0$ and $B \geqslant 2$ we have

$$\omega(G) \geqslant N. \tag{2.22}$$

[3] Beware that the factor i in [3] is now $2i$, and that the ω used here is the convergence rather than divergence degree. Hence there is both a sign change and a factor 2 of difference between the ω's of this paper and the ones of [3].

This lemma although still not giving (2.19) is sufficient for the purpose of this paper. For instance it implies directly that graphs which contain only irrelevant subgraphs in the sense of (2.19) have finite amplitudes uniformly bounded by K^n, using the standard method of [16] to bound the assignment sum over μ in (2.12).

The rest of this subsection is essentially devoted to the proof of this Lemma 2.5.

We return before solving δ functions, hence to the v variables. We will need only to compute in a precise way the oscillations which are quadratic in the long variables v to prove (2.21) and the linear oscillations in $v\theta^{-1}x$ to prove (2.22). Fortunately an analog problem was solved in momentum space by Filk and Chepelev-Roiban [5, 6], and we need only a slight adaptation of their work to position space. In fact in this subsection short variables are quite inessential but it is convenient to treat on the same footing the long $1/2\, v$ and the external x variables, so we introduce a new global notation y for all these variables. The vertices are rewritten as

$$\prod_v \delta\left(y_1 - y_2 + y_3 - y_4 + \frac{1}{2}\epsilon^i u_i\right) e^{\imath\left(\sum_{i<j}(-1)^{i+j+1} y_i\theta^{-1} y_j + yQu + uRu\right)} \tag{2.23}$$

for some inessential signs ϵ^i and some symplectic matrices Q and R.

Since we are not interested in the precise oscillations in the short u variables we will denote in the sequel quite sloppily by E_u any linear combination of the u variables. Let's consider the first Filk reduction [5], which contracts tree lines of the graph. It creates progressively generalized vertices with even number of fields. At a given induction step and for a tree line joining two such generalized vertices with respectively p and $q-p+1$ fields (p is even and q is odd), we assume by induction that the two vertices are

$$\begin{aligned} &\delta(y_1 - y_2 + y_3 \ldots - y_p + E_u)\delta(y_p - y_{p+1} + \ldots - y_q + E_u) \\ &e^{\imath\left(\sum_{1\leqslant i<j\leqslant p}(-1)^{i+j+1} y_i\theta^{-1} y_j + \sum_{p\leqslant i<j\leqslant q}(-1)^{i+j+1} y_i\theta^{-1} y_j + yQu + uRu\right)}. \end{aligned} \tag{2.24}$$

Using the second δ function we see that:

$$y_p = y_{p+1} - y_{p+2} + \ldots + y_q - E_u. \tag{2.25}$$

Substituting this expression in the first δ function we get:

$$\begin{aligned} &\delta(y_1 - y_2 + \ldots - y_{p+1} + .. - y_q + E_u)\delta(y_p - y_{p+1} + \ldots - y_q + E_u) \\ &e^{\imath\left(\sum_{1\leqslant i<j\leqslant p}(-1)^{i+j+1} y_i\theta^{-1} y_j + \sum_{p\leqslant i<j\leqslant q}(-1)^{i+j+1} y_i\theta^{-1} y_j + yQu + uRu\right)}. \end{aligned} \tag{2.26}$$

The quadratic terms which include y_p in the exponential are (taking into account that p is an even number):

$$\sum_{i=1}^{p-1}(-1)^{i+1} y_i\theta^{-1} y_p + \sum_{j=p+1}^{q}(-1)^{j+1} y_p\theta^{-1} y_j. \tag{2.27}$$

Using the expression (2.25) for y_p we see that the second term gives only terms in yLu. The first term yields:

$$\sum_{i=1}^{p-1}\sum_{j=p+1}^{q}(-1)^{i+1+j+1} y_i\theta^{-1} y_j = \sum_{i=1}^{p-1}\sum_{k=p}^{q-1}(-1)^{i+k+1} y_i\theta^{-1} y_k, \tag{2.28}$$

which reconstitutes the crossed terms, and we have recovered the inductive form of the larger generalized vertex.

One should be aware that y_p has disappeared from the final result, but that all the subsequent $y_{s>p}$ have changed sign. This complication arises because of the cyclicity of the vertex. As p was chosen to be even (which implies q odd) we see that $q-1$ is even as it should be. Consequently by this procedure we will always treat only even vertices. We finally rewrite the product of the two vertices as:

$$\delta(y_1 - y_2 + \ldots + y_{p-1} - y_{p+1} + .. - y_q + E_u)\delta(y_p - y_{p-1} + \ldots - y_q + E_u)$$
$$\times e^{\imath\left(\sum_{1\leqslant i<j\leqslant q}(-1)^{i+j+1} y_i\theta^{-1}y_j + yQu+uRu\right)}, \qquad (2.29)$$

where the exponential is written in terms of the *reindexed* vertex variables. In this way we can contract all lines of a spanning tree T and reduce G to a single vertex with "tadpole loops" called a "rosette graph" [6]. In this rosette to keep track of cyclicity is essential so rather than the "point-like" vertex of [6] we prefer to draw the rosette as a cycle (which is the border of the former tree) bearing loops lines on it (see Fig. 4). Remark that the rosette can also be considered as a big vertex, with $r = 2n+2$ fields, on which N are external fields with external variables x and $2n+2-N$ are loop fields for the corresponding $n+1-N/2$ loops. When the graph is orientable (which is the case to consider in Lemma 2.5, the fields alternatively enter and exit, and correspond to the fields on the border of the tree T, which we meet turning around counterclockwise in Fig. 1. In the rosette the long variables y_l for l in T have disappeared. Let us call z the set of remaining long loop and external variables. Then the rosette vertex factor is

$$\delta(z_1 - z_2 + \ldots - z_r + E_u)e^{\imath\left(\sum_{1\leqslant i<j\leqslant r}(-1)^{i+j+1} z_i\theta^{-1}z_j + zQu+uRu\right)}. \qquad (2.30)$$

The initial product of δ functions has not disappeared so we can still write it as a product over branches like in the previous section and use it to solve the y_l variables in terms of the z variables and the short u variables. *The net effect of the Filk first reduction was simply to rewrite the root branch δ function and the combination of all vertices oscillations* (*using the other δ functions*) *as the new big vertex or rosette factor* (2.30).

The second Filk reduction [5] further simplifies the rosette factor by erasing the loops of the rosette which do not cross any other loops or arch over external fields. Here again

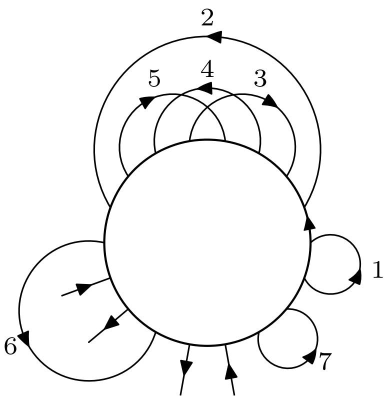

Fig. 4. A typical rosette

the same operation is possible. Consider indeed such a rosette loop l (for instance loop 2 in Fig. 4). This means that on the rosette cycle there is an even number of vertices in betwen the two ends of that loop and moreover that the sum of z's in betwen these two ends must be zero, since they are loop variables which both enter and exit between these ends. Putting together all the terms in the exponential which contain z_l we conclude exactly as in [5] that these long z variables completely disappear from the rosette oscillation factor, which simplifies as in [6] to

$$\delta(z_1 - z_2 + \ldots - z_r + E_u) e^{\imath \left(z\mathcal{I}z + zQu + uRu \right)}, \tag{2.31}$$

where $\mathcal{I}_{ij}$ is the antisymmetric "intersection matrix" of [6] (up to a different sign convention). Here $\mathcal{I}_{ij} = +1$ if oriented loop line i crosses oriented loop line j coming from its right, $\mathcal{I}_{ij} = -1$ if i crosses j coming from its left, and $\mathcal{I}_{ij} = 0$ if i and j do not cross. These formulas are also true for i external line and j loop line or the converse, provided one extends the external lines from the rosette circle radially to infinity to see their crossing with the loops. Finally when i and j are external lines one should define $\mathcal{I}_{ij} = (-1)^{p+q+1}$ if p and q are the numbering of the lines on the rosette cycle (starting from an arbitrary origin).

If a node G^i_k of the Gallavotti–Nicolò tree is orientable but non-planar ($g \geqslant 1$), there must therefore exist two intersecting loop lines in the rosette corresponding to this G^i_k, with long variables w_1 and w_2. Moreover since G^i_k is orientable, none of the long loop variables associated with these two lines belongs to the set S of long variables eliminated by the δ constraints. Therefore, after integrating the variables in S the basic mechanism to improve the power counting of a single non planar subgraph is the following:

$$\begin{aligned} &\int dw_1 dw_2 e^{-cM^{-2i_1} w_1^2 - cM^{-2i_2} w_2^2 - i w_1 \theta^{-1} w_2 + w_1 . E_1(x,u) + w_2 E_2(x,u)} \\ &= \int dw_1' dw_2' e^{-cM^{-2i_1}(w_1')^2 - cM^{-2i_2}(w_2')^2 + i w_1' \theta^{-1} w_2' + (u,x) Q(u,x)} \\ &= K M^{4i_1} \int dw_2' e^{-(M^{2i_1} + M^{-2i_2})(w_2')^2} = K. \end{aligned} \tag{2.32}$$

In these equations we used for simplicity M^{-2i} instead of the correct but more complicated factor $(\tilde{\Omega}/4)\tanh(\alpha/2)$ (see 2.6) (of course this does not change the argument) and we performed a unitary linear change of variables $w_1' = w_1 + \ell_1(x,u)$, $w_2' = w_2 + \ell_2(x,u)$ to compute the oscillating w_1' integral. The gain in (2.32) is $M^{-4(i_1+i_2)}$, which is the difference between K and the normal factor $M^{4(i_1+i_2)}$ that the w_1 and w_2 integrals would have cost if we had done them with the regular $e^{-cM^{-2i_1}w_1^2 - cM^{-2i_2}w_2^2}$ factor for long variables. Beware that in (2.32) our constant c depends on θ and that our bounds are singular in the limit $\theta \to 0$.

This basic argument must then be generalized to each non-planar leaf in the Gallavotti-Nicolò tree. This is done exactly in the same way as the inductive definition of the set A of clashing lines in the non-orientable case. In any orientable non-planar 'primitive" G^i_k node (i.e. not containing sub-non-planar nodes) we can choose an arbitrary pair of crossing loop lines which will be integrated as in (2.32) using this oscillation. The corresponding improvements are independent.

This leads to an improved amplitude bound:

$$|A_{G,\mu}| \leqslant K^n \prod_{i,k} M^{-\omega(G^i_k)}, \tag{2.33}$$

where now $\omega(G^i_k) = N(G^i_k) + 4$ if G^i_k is orientable and non planar (i.e. $g \geqslant 1$). This bound proves (2.21).

Finally it remains to consider the case of nodes G^i_k which are planar orientable but with $B \geqslant 2$. In that case there are no crossing loops in the rosette but there must be at least one loop line arching over a non trivial subset of external legs in the G^i_k rosette (see line 6 in Fig. 4). We have then a non trivial integration over at least one external variable, called x, of at least one long loop variable called w. This "external" x variable without the oscillation improvement would be integrated with a test function of scale 1 (if it is a true external line of scale 1) or better (if it is a higher long loop variable).[4] But we get now

$$\int dxdwe^{-M^{-2i}w^2 - iw\theta^{-1}x + w.E_1(x',u)} = KM^{4i}\int dxe^{-M^{+2i}x^2} = K', \tag{2.34}$$

so that a factor M^{4i} in the former bound becomes $O(1)$, hence is improved by M^{-4i}. This proves (2.22), hence completes the proof of Lemma 2.5. □

This method could be generalized to get the true power counting (2.19). One simply needs a better description of the rosette oscillating factors when g or B increase. We conjecture that it is in fact possible to "disentangle" the rosette by some kind of "third Filk move". Indeed the rank of the long variables' quadratic oscillations is exactly the genus [7], and the rank of the linear term coupling these long variables to the external ones is exactly $B - 1$. So one can through a unitary change of variables on the long variables inductively disentangle adjacent crossing pairs of loops in the rosette. This means that it is possible to diagonalize the rosette symplectic form through explicit moves of the loops along the rosette. Once oscillations are factorized in this way, the single improvements shown in this section generalize to one improvement of M^{-8i} per genus and one improvement of M^{-4i} per broken face. In this way the exact power counting (2.19) should be recovered by pure x-space techniques which never require the use of the matrix basis. This study is more technical and not really necessary for the BPHZ theorem proved in this paper.

3. Renormalization

In this section we need to consider only divergent subgraphs, namely the planar two and four point subgraphs with a single external face ($g = 0$, $B = 1$, $N = 2$ or 4). We shall prove that they can be renormalized by appropriate counterterms of the form of the initial Lagrangian. We compute first the oscillating factors Q and R of the short variables in (2.31) for these graphs. This is not truly necessary for what follows, but is a good exercise.

3.1. The oscillating rosette factor. In this subsection we define another more precise representation for the rosette factor obtained after applying the first Filk moves to a graph of order n. We rewrite in terms of u_l and v_l the coordinates of the ends of the tree lines l, $l = 1, \ldots, n-1$ (those contracted in the first Filk moves), but keep as variables called $s_1, \ldots, s_{2n+2}$ the positions of all external fields and all ends of loop lines (those not contracted in the first Filk moves).

[4] Since the loop line arches over a non trivial (i.e. neither full nor empty) subset of external legs of the rosette, the variable x cannot be the full combination of external variables in the "root" δ function.

We start from the root and turn around the tree in the trigonometrical sense. We number separately all the fields as $1, \ldots, 2n+2$ and all the tree lines as $1, \ldots, n-1$ in the order they are met, but we also define a global ordering $\prec$ on the set of all the fields and tree lines according to the order in which they are met (see Fig. 5). In this way we know whether the field number p is met before or after tree line number q. For example, in Fig. 5, field number $8 \prec$ tree line number 6.

Lemma 3.1. *The rosette contribution after a complete first Filk reduction is exactly*:

$$\delta\left(s_1 - s_2 + \cdots - s_{2n+2} + \sum_{l\in T} u_l\right) e^{\imath \sum_{0\leqslant i<j\leqslant 2n+2}(-1)^{i+j+1} s_i\theta^{-1}s_j}$$
$$\times e^{-\imath\sum_{l\prec l'} u_l\theta^{-1}u_{l'}}\, e^{-\imath\sum_l \epsilon(l)\frac{u_l\theta^{-1}v_l}{2}}\, e^{\imath\sum_{l,i\prec l}(-1)^i s_i\theta^{-1}u_l+\imath\sum_{l,i\succ l} u_l\theta^{-1}(-1)^i s_i}, \quad (3.1)$$

where $\epsilon(l)$ is -1 if the tree line l is oriented towards the root and $+1$ if it is not.

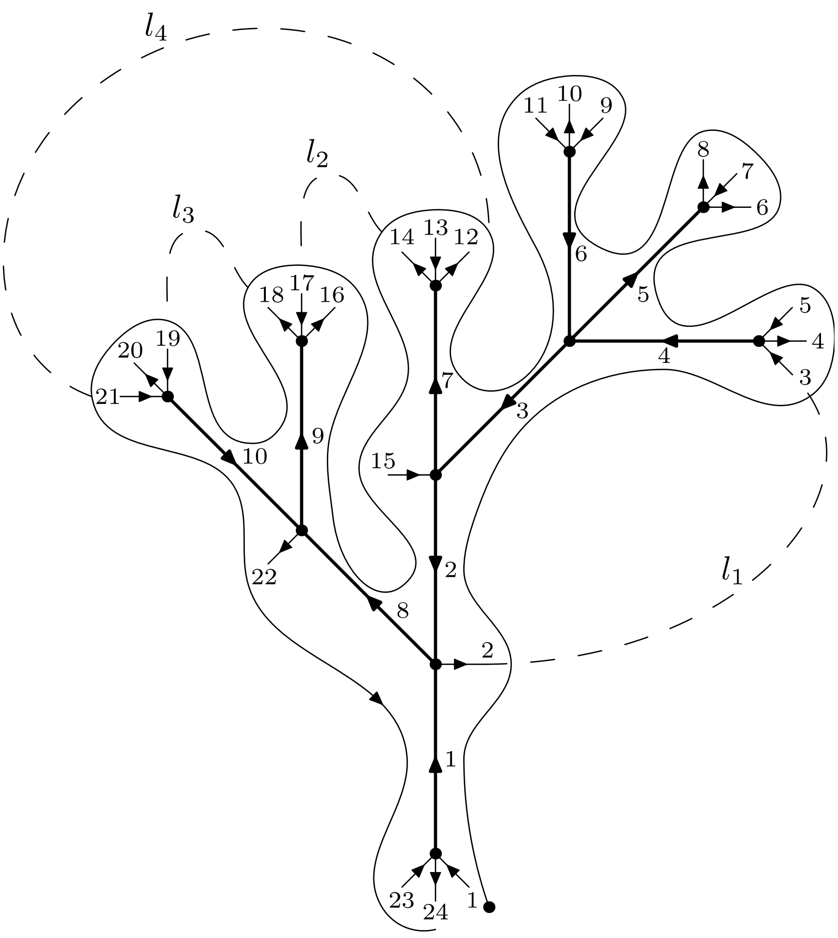

Fig. 5. Total ordering of the tree lines and fields

Proof. We proceed by induction. We contract the tree lines according to their ordering. In this way, at any step k we contract a generalized vertex with $2k+2$ external fields corresponding to the contraction of the $k-1$ first lines with a usual four-vertex with $r=4$, and obtain a new generalized vertex with $2k+4$ fields.

We suppose inductively that the generalized vertex has the above form and prove that it keeps this form after the contraction. We denote the external coordinates of this vertex as $s_1,\dots,s_{2k+2}$ and the coordinates of the four-vertex as $t_1,\dots,t_4$. We contract the propagator (s_p,t_q) with associated variables $v=s_p+t_q$ and $u=(-1)^{p+1}s_p+(-1)^{q+1}t_q$. We also note that, since the tree is orientable, $p+q$ is odd.

Adding the arguments of the two δ functions gives the global δ function. We have the two equations:

$$s_1-s_2+\cdots-s_{2k+2}+\sum u_s=0 \quad , \quad t_1-t_2+t_3-t_4=0. \tag{3.2}$$

Using the invariance of the t vertex we can always eliminate the contribution of t_q in the phase factor. We therefore have:

$$\begin{aligned}\varphi &= [s_1-s_2+\cdots+(-1)^p s_{p-1}]\theta^{-1}(-1)^p s_p\\ &\quad +(-1)^p s_p\theta^{-1}[(-1)^{p+2}s_{p+1}+\cdots-s_{2k+2}]\\ &= [s_1-s_2+\cdots+(-1)^p s_{p-1}]\theta^{-1}[-u+(-1)^{q+1}t_q]\\ &\quad +[-u+(-1)^{q+1}t_q]\theta^{-1}[(-1)^{p+2}s_{p+1}+\dots.-s_{2k+2}].\end{aligned} \tag{3.3}$$

As $(-1)^{q+1}t_q=\sum_{i=1,i\neq q}^4(-1)^i t_i$ we see that the $s\theta^{-1}t_q$ terms in the above expression reproduce exactly the crossed terms needed to complete the first exponential. We rewrite the other terms as:

$$\begin{aligned}&[s_1-s_2+\cdots+(-1)^p s_{p-1}]\theta^{-1}(-u)+(-u)\theta^{-1}[(-1)^{p+2}s_{p+1}+\cdots-s_{2k+2}]\\ &\quad =[s_1-s_2+\cdots+(-1)^p s_{p-1}]\theta^{-1}(-u)\\ &\qquad +(-u)\theta^{-1}[-s_1+s_2\cdots+(-1)^p s_p-\sum_s u_s]\\ &\quad =2[s_1-s_2+\cdots+(-1)^p s_{p-1}]\theta^{-1}(-u)+(-u)\theta^{-1}(-1)^p s_p+u\theta^{-1}\sum_s u_s\\ &\quad =2\sum_{i\prec l}(-1)^i s_i\theta^{-1}u+(-1)^{p+1}\frac{u\theta^{-1}v}{2}+\sum_s u\theta^{-1}u_s,\end{aligned} \tag{3.4}$$

where we have used $(-1)^p s_p=(-1)^p(v-u)/2$.

Note that further contractions will not involve $s_1\dots s_{p-1}$. After collecting all the contractions and using the global delta function we write:

$$2\sum_{l,i\prec l}(-1)^i s_i\theta^{-1}u_l=\sum_{l,i\prec l}(-1)^i s_i\theta^{-1}u_l+\sum_{l,i\succ l}u_l\theta^{-1}(-1)^i s_i+\sum_{l,l'}u_l\theta^{-1}u_{l'}, \tag{3.5}$$

and the last term is zero by the antisymmetry of θ^{-1}. □

We denote by $\mathcal{L}$ the set of loop lines, and analyze now further the rosette contribution for planar graphs. We call now $x_i,\ i=1,\dots,N$ the N external positions. We choose as first external field 1 an arbitrary entering external line. We define an ordering among the set of all lines, writing $l' \prec l$ if both ends of l' are before the first end of l when turning around the tree as in Fig. 5, where $l_1 \prec l_2$. Analogously we define $l \prec j$ when j is an external vertex ($l_1 \prec x_4$ in Fig. 5). We define $l' \subset l$ if both ends of l' lie in between the ends of l on the rosette ($l_2 \subset l_4$ in Fig. 5). We count a loop line as positive if it turns in the trigonometric sense like the rosette and negative if it turns clockwise. Each loop line $l \in \mathcal{L}$ has now a sign $\epsilon(l)$ associated with this convention, and we now make explicit its end variables in terms of u_l and w_l.

With these conventions we prove the following lemma:

Lemma 3.2. *The vertex contribution for a planar regular graph is exactly:*

$$\delta\Big(\sum_i(-1)^{i+1}x_i+\sum_{l\in T\cup\mathcal{L}}u_l\Big)e^{\imath\sum_{i,j}(-1)^{i+j+1}x_i\theta^{-1}x_j}$$
$$\times e^{\imath\sum_{l\in T\cup\mathcal{L},\, l\prec j}u_l\theta^{-1}(-1)^j x_j+\imath\sum_{l\in T\cup\mathcal{L},\, l\succ j}(-1)^j x_j\theta^{-1}u_l}$$
$$\times e^{-\imath\sum_{l,l'\in T\cup\mathcal{L},\, l\prec l'}u_l\theta^{-1}u_{l'}-\imath\sum_{l\in T}\frac{u_l\theta^{-1}v_l}{2}\epsilon(l)-\imath\sum_{l\in\mathcal{L}}\frac{u_l\theta^{-1}w_l}{2}\epsilon(l)}$$
$$\times e^{-\imath\sum_{l\in\mathcal{L},\, l'\in\mathcal{L}\cup T;\, l'\subset l}u_{l'}\theta^{-1}w_l\epsilon(l)}\ . \tag{3.6}$$

Proof. We see that the global root δ function has the argument:

$$\sum_i(-1)^{i+1}x_i+\sum_{l\in\mathcal{L}\cup T}u_l. \tag{3.7}$$

Since the graph has one broken face we always have an even number of vertices on the external face between two external fields. We express all the internal loop variables as functions of u's and w's. Using Lemma 3.1, we regroup the terms which still contain the external points which we relabel x in one quadratic and one linear form in the external positions. The quadratic term can be written as:

$$\sum_{i<j}(-1)^{i+j+1}x_i\theta^{-1}x_j\ . \tag{3.8}$$

The linear term in the external vertices is:

$$\begin{aligned}
&\sum_{i<j}(-1)^{i+1}s_i\theta^{-1}(-1)^j x_j+\sum_{i>j}(-1)^j x_j\theta^{-1}(-1)^{i+1}s_i\\
&\quad+\sum_{l\in T,l\succ j}(-1)^j x_j\theta^{-1}u_l+\sum_{l\in T,l\prec j}u_l\theta^{-1}(-1)^j x_j\\
&=\sum_{l'\in\mathcal{L},l'\succ j}u_{l'}\theta^{-1}(-1)^j x_j+\sum_{l'\in\mathcal{L},l'\succ j}(-1)^j x_j\theta^{-1}u_{l'}\\
&\quad+\sum_{l\in T,l\succ j}(-1)^j x_j\theta^{-1}u_l+\sum_{l\in T,l\prec j}u_l\theta^{-1}(-1)^j x_j\ .
\end{aligned} \tag{3.9}$$

Consider a loop line from s_p to s_q with $p < q$. Its contribution to the vertex amplitude decomposes in a "loop-loop" term and a "loop-tree" term. The first one is:

$$\sum_{i<p}(-1)^{i+1}s_i\theta^{-1}(-1)^p s_p + \sum_{\substack{p<i\\ i\neq q}}(-1)^p s_p\theta^{-1}(-1)^{i+1}s_i + s_p\theta^{-1}s_q$$
$$+\sum_{\substack{i<q\\ i\neq p}}(-1)^{i+1}s_i\theta^{-1}(-1)^q s_q + \sum_{q<i}(-1)^p s_q\theta^{-1}(-1)^{i+1}s_i$$
$$= \sum_{i<p}(-1)^{i+1}s_i\theta^{-1}[(-1)^p s_p + (-1)^q s_q]$$
$$+\sum_{q<i}[(-1)^p s_p + (-1)^q s_q]\theta^{-1}(-1)^{i+1}s_i$$
$$+\sum_{p<i<q}(-1)^{i+1}s^i\theta^{-1}[(-1)^{p+1}s_p + (-1)^q s_q] + s_p\theta^{-1}s_q. \qquad (3.10)$$

Taking into account that $(-1)^{i+1}s_i + (-1)^{j+1}s_j = u_{l'}$ if s_i and s_j are the two ends of the loop line l', we can rewrite the above expression as:

$$\sum_{l'\prec l}u_{l'}\theta^{-1}(-u_l) + \sum_{l'\succ l}(-u_l)\theta^{-1}u_{l'} + \sum_{l'\subset l}u_{l'}\theta^{-1}(-1)^{p+1}w_l$$
$$+(-1)^{p+1}\frac{u_l\theta^{-1}w_l}{2} + \sum_{l',l\subset l'}u_l\theta^{-1}(-1)^{i+1}w_{l'}, \qquad (3.11)$$

where l is fixed in all the above expressions. Summing the contributions of all the lines (being careful not to count the same term twice) we get the final result:

$$-\sum_{l'\prec l}u_{l'}\theta^{-1}u_l - \sum_{l,l'\subset l}u_{l'}\theta^{-1}w_l\,\epsilon(l) - \sum_l\frac{u_l\theta^{-1}w_l\,\epsilon(l)}{2}. \qquad (3.12)$$

We still have to add the "loop-tree" contribution. It reads:

$$\sum_{l'\in T,l'\prec p}u_{l'}\theta^{-1}(-1)^p s_p + \sum_{l'\in T,l'\succ p}(-1)^p s_p\theta^{-1}u_{l'}$$
$$+\sum_{l'\in T,l'\prec q}u_{l'}\theta^{-1}(-1)^q s_q + \sum_{l'\in T,l'\succ q}(-1)^q s_q\theta^{-1}u_{l'}$$
$$= \sum_{l'\in T;l'\prec p,q}u_{l'}\theta^{-1}[(-1)^p s_p + (-1)^q s_q] + \sum_{l'\in T;l'\succ p,q}[(-1)^p s_p + (-1)^q s_q]\theta^{-1}u_{l'}$$
$$+\sum_{l'\in T;\,p\prec l'\prec q}u_{l'}\theta^{-1}[(-1)^{p+1}s_p + (-1)^q s_q]$$
$$= \sum_{l'\in T;l'\prec l}u_{l'}\theta^{-1}(-u_l) + \sum_{l'\in T;l'\succ l}(-u_l)\theta^{-1}u_{l'} + \sum_{l'\in T;l'\subset l}u_{l'}\theta^{-1}(-1)^{p+1}w_l. \qquad (3.13)$$

Collecting all the factors proves the lemma □

3.2. Renormalization of the four-point function. Consider a 4 point subgraph which needs to be renormalized, hence is a node of the Gallavotti-Nicolò tree. This means that there is (i,k) such that $N(G^i_k)=4$. The four external positions of the amputated graph are labeled x_1, x_2, x_3 and x_4. We also define Q, R and S as three skew-symmetric matrices of respective sizes $4\times l(G^i_k)$, $l(G^i_k)\times l(G^i_k)$ and $[n(G^i_k)-1]\times l(G^i_k)$, where we recall that $n(G)-1$ is the number of loops of a 4 point graph with n vertices. The amplitude associated to the connected component G^i_k is then

$$A(G^i_k)(x_1,x_2,x_3,x_4)=\int\prod_{\ell\in T^i_k}du_\ell C_\ell(x,u,w)\prod_{l\in G^i_k,\, l\notin T}du_l dw_l C_l(u_l,w_l)$$
$$\times\delta\left(x_1-x_2+x_3-x_4+\sum_{l\in G^i_k}u_l\right)e^{\imath\left(\sum_{p<q}(-1)^{p+q+1}x_p\theta^{-1}x_q+XQU+URU+USW\right)}\qquad(3.14)$$

The exact form of the factor $\sum_{p<q}(-1)^{p+q+1}x_p\theta^{-1}x_q$ follows from Lemma 3.2. From this lemma, and (3.15) below would also follow exact expressions for Q, R and S, but we won't need them. The important fact is that there are no quadratic oscillations in X times W (because $B=1$) nor in W times W (because $g=0$). C_l is the propagator of the line l. For loop lines C_l is expressed in terms of u_l and w_l by formula (2.6), (with v replaced by our notation w for long variables of loop lines). But for tree lines $\ell\in T^i_k$ recall that the solution of the system of branch δ functions for T has reexpressed the corresponding long variables v_ℓ in terms of the short variables u, and the external and long loop variables of the branch graph G_ℓ which lies "over" ℓ in the rooted tree T. This is the essential content of Subsect. 2.2. More precisely consider a line $\ell\in T^i_k$ with scale $i(\ell)\geqslant i$; we can write

$$v_\ell=X_\ell+W_\ell+U_\ell,\qquad(3.15)$$

where

$$X_\ell=\sum_{e\in E(\ell)}\epsilon_{\ell,e}x_e\qquad(3.16)$$

is a linear combination on the set of external variables of the branch graph G_ℓ with the correct alternating signs $\epsilon_{\ell,e}$,

$$W_\ell=\sum_{l\in\mathcal{L}(\ell)}\epsilon_{\ell,l}w_l\qquad(3.17)$$

is a linear combination over the set $\mathcal{L}(\ell)$ of long loop variables for the external lines of G_ℓ (and $\epsilon_{\ell,l}$ are other signs), and

$$U_\ell=\sum_{l'\in S(\ell)}\epsilon_{\ell,l'}u_{l'}\qquad(3.18)$$

is a linear combination over a set S_ℓ of short variables that we do not need to know explicitly. The tree propagator for line ℓ then is

$$C_\ell(u_\ell,X_\ell,U_\ell,W_\ell)=\int_{M^{-2i(\ell)}}^{M^{-2(i(\ell)-1)}}\frac{\tilde\Omega d\alpha_\ell e^{-\frac{\tilde\Omega}{4}\{\coth(\frac{\alpha_\ell}{2})u_l^2+\tanh(\frac{\alpha_\ell}{2})[X_\ell+W_\ell+U_\ell]^2\}}}{[2\pi\sinh(\alpha_\ell)]^2}.\qquad(3.19)$$

To renormalize, let us call $e = \max e_p$, $p = 1, \ldots, 4$ the highest external index of the subgraph G^i_k. We have $e < i$ since G^i_k is a node of the Gallavotti-Nicolò tree. We evaluate $A(G^i_k)$ on external fields[5] $\varphi^{\leqslant e}(x_p)$ as:

$$
\begin{aligned}
A(G^i_k) &= \int \prod_{p=1}^{4} dx_p \varphi^{\leqslant e}(x_p)\, A(G^i_k)(x_1, x_2, x_3, x_4) \\
&= \int \prod_{p=1}^{4} dx_p \varphi^{\leqslant e}(x_p)\, e^{\imath \mathrm{Ext}} \prod_{\ell \in T^i_k} du_\ell C_\ell(u_\ell, tX_\ell, U_\ell, W_\ell) \\
&\quad \times \prod_{l \in G^i_k \;\; l \notin T} du_l dw_l C_l(u_l, w_l)\, \delta\Big(\Delta + t \sum_{l \in G^i_k} u_l\Big) e^{\imath t XQU + \imath URU + \imath USW}\Big|_{t=1}
\end{aligned}
\tag{3.20}
$$

with $\Delta = x_1 - x_2 + x_3 - x_4$ and $\mathrm{Ext} = \sum_{p<q=1}^{4} (-1)^{p+q+1} x_p \theta^{-1} x_q$.

This formula is designed so that at $t = 0$ all dependence on the external variables x factorizes out of the u, w integral in the desired vertex form for renormalization of the $\varphi \star \varphi \star \varphi \star \varphi$ interaction in the action (2.2). We now perform a Taylor expansion to first order with respect to the t variable and prove that the remainder term is irrelevant. Let $\mathfrak{U} = \sum_{l \in G^i_k} u_l$, and

$$
\begin{aligned}
\mathfrak{R}(t) &= -\sum_{\ell \in T^i_k} \frac{\tilde{\Omega}}{4} \tanh\Big(\frac{\alpha_\ell}{2}\Big) \Big\{ t^2 X_\ell^2 + 2t X_\ell \big[W_\ell + U_\ell\big] \Big\} \\
&\equiv -t^2 \mathcal{A} X.X - 2t \mathcal{A} X.(W + U),
\end{aligned}
\tag{3.21}
$$

where $\mathcal{A}_\ell = \frac{\tilde{\Omega}}{4} \tanh(\frac{\alpha_\ell}{2})$, and $X.Y$ means $\sum_{\ell \in T^i_k} X_\ell Y_\ell$. We have

$$
\begin{aligned}
A(G^i_k) &= \int \prod_{p=1}^{4} dx_p \varphi^{\leqslant e}(x_p)\, e^{\imath \mathrm{Ext}} \prod_{\ell \in T^i_k} du_\ell\, C_\ell(u_\ell, U_\ell, W_\ell) \\
&\quad \times \Bigg[\prod_{l \in G^i_k \;\; l \notin T} du_l dw_l C_l(u_l, w_l) \Bigg] e^{\imath URU + \imath USW} \\
&\quad \times \Bigg\{ \delta(\Delta) + \int_0^1 dt \Big[\mathfrak{U}.\nabla \delta(\Delta + t\mathfrak{U}) + \delta(\Delta + t\mathfrak{U}) [\imath XQU + \mathfrak{R}'(t)] \Big] \\
&\qquad e^{\imath t XQU + \mathfrak{R}(t)} \Bigg\},
\end{aligned}
\tag{3.22}
$$

where $C_\ell(u_\ell, U_\ell, W_\ell)$ is given by (3.19) but taken at $X_\ell = 0$.

The first term, denoted by τA, is of the desired form (2.4) times a number independent of the external variables x. It is asymptotically constant in the slice index i, hence

[5] For the external index to be exactly e the external smearing factor should be in fact $\prod_p \varphi^{\leqslant e}(x_p) - \prod_p \varphi^{\leqslant e-1}(x_p)$ but this subtlety is inessential.

the sum over i at fixed e is logarithmically divergent: this is the divergence expected for the four-point function. It remains only to check that $(1-\tau)A$ converges as $i-e \to \infty$. But we have three types of terms in $(1-\tau)A$, each providing a specific improvement over the regular, log-divergent power counting of A:

- The term $\mathfrak{U}.\nabla\delta(\Delta + t\mathfrak{U})$. For this term, integrating by parts over external variables, the ∇ acts on external fields $\varphi^{\leqslant e}$, hence brings at most M^e to the bound, whether the $\mathfrak{U}$ term brings at least M^{-i}.
- The term XQU. Here X brings at most M^e and U brings at least M^{-i}.
- The term $\mathfrak{R}'(t)$. It decomposes into terms in $\mathcal{A}X.X$, $\mathcal{A}X.U$ and $\mathcal{A}X.W$. Here the $\mathcal{A}_\ell$ brings at least $M^{-2i(\ell)}$, X brings at worst M^e, U brings at least M^{-i} and $X_\ell W_\ell$ brings at worst $M^{e+i(\ell)}$. This last point is the only subtle one: if $\ell \in T^i_k$, remark that because T^i_k is a sub-tree within each Gallavotti-Nicolò subnode of G^i_k, in particular all parameters $w_{l'}$ for $l' \in \mathcal{L}(\ell)$ which appear in W_ℓ must have indices lower or equal to $i(\ell)$ (otherwise they would have been chosen instead of ℓ in T^i_k).

In conclusion, since $i(\ell) \geqslant i$, the Taylor remainder term $(1-\tau)A$ improves the power-counting of the connected component G^i_k by a factor at least $M^{-(i-e)}$. This additional $M^{-(i-e)}$ factor makes $(1-\tau)A(G^i_k)$ convergent and irrelevant as desired.

3.3. Renormalization of the two-point function. We consider now the nodes such that $N(G^i_k) = 2$. We use the same notations as in the previous subsection. The two external points are labeled x and y. Using the global δ function, which is now $\delta(x - y + \mathfrak{U})$, we remark that the external oscillation $e^{\imath x\theta^{-1}y}$ can be absorbed in a redefinition of the term $e^{\imath tXQU}$, which we do from now on. Also we want to use expressions symmetrized over x and y. The full amplitude is

$$A(G^i_k) = \int dxdy\varphi^{\leqslant e}(x)\varphi^{\leqslant e}(y)\delta\Big(x - y + \mathfrak{U}\Big) \times \prod_{l\in G^i_k,\, l\notin T} du_l dw_l C_l(u_l, w_l) \times \prod_{\ell\in T^i_k} du_\ell C_\ell(u_\ell, X_\ell, U_\ell, W_\ell)\, e^{\imath XQU+\imath URU+\imath USW}. \tag{3.23}$$

First we write the identity

$$\varphi^{\leqslant e}(x)\varphi^{\leqslant e}(y) = \frac{1}{2}\Big[[\varphi^{\leqslant e}(x)]^2 + [\varphi^{\leqslant e}(y)]^2 - [\varphi^{\leqslant e}(y) - \varphi^{\leqslant e}(x)]^2\Big], \tag{3.24}$$

we develop it as

$$\varphi^{\leqslant e}(x)\varphi^{\leqslant e}(y) = \frac{1}{2}\Bigg\{[\varphi^{\leqslant e}(x)]^2 + [\varphi^{\leqslant e}(y)]^2 - \Bigg[(y-x)^\mu.\nabla_\mu\varphi^{\leqslant e}(x) + \int_0^1 ds(1-s)(y-x)^\mu(y-x)^\nu\nabla_\mu\nabla_\nu\varphi^{\leqslant e}(x + s(y-x))\Bigg]^2\Bigg\}, \tag{3.25}$$

and substitute into (3.23). The first term A_0 is a symmetric combination with external fields at the same argument. Consider the case with the two external legs at x, namely the term in $[\varphi^{\leqslant e}(x)]^2$. For this term we integrate over y. This uses the δ function. We perform then a Taylor expansion in t at order 3 of the remaining function

$$f(t) = e^{\imath t XQU + \Re(t)}, \tag{3.26}$$

where we recall that $\Re(t) = -[t^2 \mathcal{A}X.X + 2t\mathcal{A}X.(W+U)]$. We get

$$\begin{aligned} A_0 = \frac{1}{2}\int dx[\varphi^{\leqslant e}(x)]^2\, e^{\imath(URU+USW)} \\ \times \prod_{l\in G_k^j,\, l\notin T} du_l dw_l C_l(u_l, w_l) \prod_{\ell\in T_k^i} du_\ell C_\ell(u_\ell, U_\ell, W_\ell) \\ \times \left(f(0) + f'(0) + \frac{1}{2}f''(0) + \frac{1}{2}\int_0^1 dt\,(1-t)^2 f^{(3)}(t)\right). \end{aligned} \tag{3.27}$$

In order to evaluate that expression, let $A_{0,0}$, $A_{0,1}$, $A_{0,2}$ be the zeroth, first and second order terms in this Taylor expansion, and $A_{0,R}$ be the remainder term. First,

$$\begin{aligned} A_{0,0} = \int dx\,[\varphi^{\leqslant e}(x)]^2\, e^{\imath(URU+USW)} \prod_{l\in G_k^i\,, l\notin T} du_l dw_l C_l(u_l, w_l) \\ \times \prod_{\ell\in T_k^i} du_\ell C_\ell(u_\ell, U_\ell, W_\ell) \end{aligned} \tag{3.28}$$

is quadratically divergent and exactly of the expected form for the mass counterterm. Then

$$\begin{aligned} A_{0,1} = \frac{1}{2}\int dx[\varphi^{\leqslant e}(x)]^2\, e^{\imath(URU+USW)} \prod_{l\in G_k^i,\, l\notin T} du_l dw_l C_l(u_l, w_l) \\ \times \prod_{\ell\in T_k^i} du_\ell C_\ell(u_\ell, U_\ell, W_\ell)\Big(\imath XQU + \Re'(0)\Big) \end{aligned} \tag{3.29}$$

vanishes identically. Indeed all the terms are odd integrals over the u, w-variables. $A_{0,2}$ is more complicated:

$$\begin{aligned} A_{0,2} = \frac{1}{2}\int dx[\varphi^{\leqslant e}(x)]^2\, e^{\imath(URU+USW)} \prod_{l\in G_k^i,\, l\notin T} du_l dw_l C_l(u_l, w_l) \\ \times \prod_{\ell\in T_k^i} du_\ell C_\ell(u_\ell, U_\ell, W_\ell)\Big(-(XQU)^2 \\ -4\imath XQU\mathcal{A}X.(W+U) - 2\mathcal{A}X.X + 4[\mathcal{A}X.(W+U)]^2\Big). \end{aligned} \tag{3.30}$$

The four terms in $(XQU)^2$, $XQU\mathcal{A}X.W$, $\mathcal{A}X.X$ and $[\mathcal{A}X.W]^2$ are logarithmically divergent and contribute to the renormalization of the harmonic frequency term $\tilde{\Omega}$ in

(2.2). The terms in $x^\mu x^\nu$ with $\mu \neq \nu$ do not survive by parity and the terms in $(x^\mu)^2$ have obviously the same coefficient. The other terms in $XQU\mathcal{A}X.U$, $(\mathcal{A}X.U)(\mathcal{A}X.W)$ and $[\mathcal{A}X.U]^2$ are irrelevant. Similarly the terms in $A_{0,R}$ are all irrelevant.

For the term in $A_0(y)$ in which we have $\int dx[\varphi^{\leqslant e}(y)]^2$ we have to perform a similar computation, but beware that it is now x which is integrated with the δ function so that Q, S, R and $\mathfrak{R}$ change, but not the conclusion.

Next we have to consider the term in $\left[(y-x)^\mu.\nabla_\mu\varphi^{\leqslant e}(x)\right]^2$ in (3.25), for which we need to develop the f function only to first order. Integrating over y replaces each $y-x$ by a $\mathfrak{U}$ factor so that we get a term

$$A_1 = \frac{1}{2}\int dx\left[\mathfrak{U}^\mu.\nabla_\mu\varphi^{\leqslant e}(x)\right]^2 e^{\imath(URU+USW)} \prod_{l\in G^i_k,\, l\notin T} du_l dw_l C_l(u_l,w_l)$$
$$\times \prod_{\ell\in T^i_k} du_\ell C_\ell(u_\ell,U_\ell,W_\ell)\left(f(0)+\int_0^1 dt f'(t)dt\right). \tag{3.31}$$

The first term is

$$A_{1,0} = \frac{1}{2}\int dx\left[\mathfrak{U}^\mu.\nabla_\mu\varphi^{\leqslant e}(x)\right]^2 e^{\imath(URU+USW)} \prod_{l\in G^i_k,\, l\notin T} du_l dw_l C_l(u_l,w_l)$$
$$\times \prod_{\ell\in T^i_k} du_\ell C_\ell(u_\ell,U_\ell,W_\ell). \tag{3.32}$$

The terms with $\mu \neq \nu$ do not survive by parity. The other ones reconstruct a counterterm proportional to the Laplacian. The power-counting of this factor $A_{1,0}$ is improved, with respect to A, by a factor $M^{-2(i-e)}$ which makes it only logarithmically divergent, as it should be for a wave-function counterterm.

The remainder term in $A^x_{1,R}$ has an additional factor at worst $M^{-(i-e)}$ coming from the $\int_0^1 dt f'(t)dt$ term, hence is irrelevant and convergent.

Finally the remainder terms A_R with three or four gradients in (3.25) are also irrelevant and convergent. Indeed we have terms of various types:

- There are terms in U^3 with ∇^3. The ∇ act on the variables x, hence on external fields, hence bring at most M^{3e} to the bound, whether the $\mathfrak{U}^3$ brings at least M^{-3i}.
- Finally there are terms with 4 gradients which are still smaller.

Therefore for the renormalized amplitude A_R the power-counting is improved, with respect to A_0, by a factor $M^{-3(i-e)}$, and becomes convergent.

Putting together the results of the two previous sections, we have proved that the usual effective series which expresses any connected function of the theory in terms of an infinite set of effective couplings, related one to each other by a discretized flow [16], have finite coefficients to all orders. Reexpressing these effective series in terms of the renormalized couplings would reintroduce in the usual way the Zimmermann's forests of "useless" counterterms and build the standard "old-fashioned" renormalized series. The most explicit way to check finiteness of these renormalized series in order to complete the "BPHZ theorem" is to use the standard "classification of forests" which distributes

Zimmermann's forests into packets such that the sum over assignments in each packet is finite [16].[6] This part is completely standard and identical to the commutative case. Hence the proof of Theorem 2.1 is completed.

A. The LSZ Model

In this section we prove the perturbative renormalizability of a generalized Langmann-Szabo-Zarembo model [18]. It consists in a bosonic complex scalar field theory in a fixed magnetic background plus an harmonic oscillator. The quartic interaction is of the Moyal type. The action functional is given by

$$S = \int \frac{1}{2}\bar{\varphi}\big(- D^{\mu} D_{\mu} + \tilde{\Omega}^2 x^2 + \mu_0^2\big)\varphi + \lambda\, \bar{\varphi} \star \varphi \star \bar{\varphi} \star \varphi, \tag{A.1}$$

where $D_\mu = \partial_\mu - \imath B_{\mu\nu} x^\nu$ is the covariant derivative. The $1/2$ factor is somewhat unusual in a complex theory but it allows us to recover exactly the results given in [15] with $\tilde{\Omega}^2 \to \omega^2 = \tilde{\Omega}^2 + B^2$. By expanding the quadratic part of the action, we get a Φ^4-like kinetic part plus an angular momentum term:

$$\bar{\varphi} D^{\mu} D_{\mu}\varphi + \tilde{\Omega}^2 x^2 \bar{\varphi}\varphi = \bar{\varphi}\big(\Delta - \omega^2 x^2 - 2BL_5\big)\varphi \tag{A.2}$$

with $L_5 = x^1 p_2 - x^2 p_1 + x^3 p_4 - x^4 p_3 = x \wedge \nabla$. Here the skew-symmetric matrix B has been put in its canonical form

$$B = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & (0) \\ & (0) & 0 & -1 \\ & & 1 & 0 \end{pmatrix}. \tag{A.3}$$

In x space, the interaction term is exactly the same as (2.4). The complex conjugation of the fields only selects the orientable graphs.

At $\tilde{\Omega} = 0$, the model is similar to the Gross-Neveu theory. Its renormalization is therefore harder [12] and is not treated in this paper. If we additionally set $B = \theta^{-1}$ we recover the integrable LSZ model [18].

A.1. Power counting. The propagator corresponding to the action (A.1) has been calculated in [15] in the two-dimensional case. The generalization to higher dimensions, e.g. four, is straightforward:

$$C(x, y) = \int_0^\infty dt\, \frac{\omega^2}{(2\pi \sinh \omega t)^2}\, \exp -\frac{\omega}{2}\bigg(\frac{\cosh Bt}{\sinh \omega t}(x-y)^2 + \frac{\cosh \omega t - \cosh Bt}{\sinh \omega t}(x^2 + y^2) + \imath\, \frac{\sinh Bt}{\sinh \omega t} x\theta^{-1} y\bigg). \tag{A.4}$$

[6] One could also use the popular inductive scheme of Polchinski, which however does not extend yet to non-perturbative "constructive" renormalization

Note that the sliced version of (A.4) obeys the same bound (2.11) as the φ^4 propagator. Moreover the additional oscillating phases $\exp\imath x\theta^{-1}y$ are of the form $\exp\imath\, u_l\theta^{-1}v_l$. Such terms played no role in the power counting of the Φ^4 theory. They were bounded by one. This allows to conclude that Lemmas 2.4 and 2.5 hold for the generalized LSZ model. Note also that in this case, the theory contains only orientable graphs due to the use of complex fields.

A.2. Renormalization. As for the noncommutative Φ^4 theory, we only need to renormalize the planar ($g = 0$) two and four-point functions with only one external face.

Recall that the oscillating factors of the propagators are

$$\exp\imath\frac{\sinh Bt}{2\sinh\omega t}u_l\theta^{-1}v_l. \tag{A.5}$$

After resolving the $v_\ell,\ \ell\in T$ variables in terms of X_ℓ, W_ℓ and U_ℓ, they can be included in the vertices oscillations by a redefinition of the Q, S and R matrices (see (3.14)). For the four-point function, we can then perform the same Taylor subtraction as in the Φ^4 case.

The two-point function case is more subtle. Let us consider the generic amplitude

$$\begin{aligned}A(G_k^i) = \int dxdy\bar\varphi^{\leqslant e}(x)\varphi^{\leqslant e}(y)\delta\big(x-y+\mathfrak{U}\big)\\ \times\prod_{l\in G_k^i,\,l\notin T} du_l dw_l C_l(u_l,w_l)\\ \times\prod_{\ell\in T_k^i} du_\ell C_\ell(u_\ell,X_\ell,U_\ell,W_\ell)\,e^{\imath XQU+\imath URU+\imath USW}.\end{aligned} \tag{A.6}$$

The symmetrization procedure (3.24) over the external fields is not possible anymore, the theory being complex. Nevertheless we can decompose $\bar\varphi(x)\varphi(y)$ in a symmetric and an anti-symmetric part:

$$\begin{aligned}\bar\varphi(x)\varphi(y) &= \tfrac12\big(\bar\varphi(x)\varphi(y)+\bar\varphi(y)\varphi(x)+\bar\varphi(x)\varphi(y)-\bar\varphi(y)\varphi(x)\big)\\ &\overset{\text{def}}{=} \big(\mathcal{S}+\mathcal{A}\big)\bar\varphi(x)\varphi(y).\end{aligned} \tag{A.7}$$

The symmetric part of A, called A_s, will lead to the same renormalization procedure as the Φ^4 case. Indeed,

$$\begin{aligned}\mathcal{S}\bar\varphi(x)\varphi(y) &= \tfrac12\big(\bar\varphi(x)\varphi(y)+\bar\varphi(y)\varphi(x)\big)\\ &= \tfrac12\big\{\bar\varphi(x)\varphi(x)+\bar\varphi(y)\varphi(y)-\big(\bar\varphi(x)-\bar\varphi(y)\big)\big(\varphi(x)-\varphi(y)\big)\big\}\end{aligned} \tag{A.8}$$

which is the complex equivalent of (3.24).

In the anti-symmetric part of A, called A_a, the linear terms $\bar{\varphi}\nabla\varphi$ do not compensate:

$$\begin{aligned}\mathcal{A}\bar{\varphi}(x)\varphi(y) &= \tfrac{1}{2}\big(\bar{\varphi}(x)\varphi(y) - \bar{\varphi}(y)\varphi(x)\big)\\ &= \tfrac{1}{2}(\bar{\varphi}(x)(y-x).\nabla\varphi(x) - (y-x).\nabla\bar{\varphi}(x)\varphi(x)\\ &\quad +\tfrac{1}{2}\bar{\varphi}(x)((y-x).\nabla)^2\varphi(x) - \tfrac{1}{2}((y-x).\nabla)^2\bar{\varphi}(x)\varphi(x)\\ &\quad +\tfrac{1}{2}\int_0^1 ds(1-s)^2\bar{\varphi}(x)((y-x).\nabla)^3\varphi(x+s(y-x))\\ &\quad -((y-x).\nabla)^3\bar{\varphi}(x+s(y-x))\varphi(x)).\end{aligned} \tag{A.9}$$

We decompose A_a into five parts following the Taylor expansion (A.9):

$$\begin{aligned}A_a^{1+} &= \int dxdy\, \bar{\varphi}(x)(y-x).\nabla\varphi(x)\delta\big(x-y+\mathfrak{U}\big)\\ &\quad\times \prod_{l\in G^i_k,\, l\notin T} du_l dw_l C_l(u_l, w_l)\\ &\quad\times \prod_{\ell\in T^i_k} du_\ell C_\ell(u_\ell, X_\ell, U_\ell, W_\ell)\, e^{\imath XQU+\imath URU+\imath USW}\\ &= \int dx\, \bar{\varphi}(x)\, \mathfrak{U}.\nabla\varphi(x) \prod_{l\in G^i_k,\, l\notin T} du_l dw_l C_l(u_l, w_l)\\ &\quad\times \prod_{\ell\in T^i_k} du_\ell C_\ell(u_\ell, X'_\ell, U'_\ell, W_\ell)\, e^{\imath XQ'U+\imath URU+\imath USW},\end{aligned} \tag{A.10}$$

where we performed the integration over y thanks to the delta function. The changes have been absorbed in a redefinition of X_ℓ, U_ℓ and Q. From now on X_ℓ (and X) contain only x (if x is hooked to the branch $b(l)$) and we forget the primes for Q and U_ℓ. We expand the function f defined in (3.26) up to order 2:

$$\begin{aligned}A_a^{1+} &= \int \bar{\varphi}(x)\, \mathfrak{U}.\nabla\varphi(x) \prod_{l\in G^i_k,\, l\notin T} du_l dw_l C_l(u_l, w_l)\\ &\quad\times \prod_{\ell\in T^i_k} du_\ell C_\ell(u_\ell, U_\ell, W_\ell)\, e^{\imath URU+\imath USW}\\ &\quad\times \left(f(0) + f'(0) + \int_0^1 dt\,(1-t) f''(t)\right).\end{aligned} \tag{A.11}$$

The zeroth order term vanishes thanks to the parity of the integrals with respect to the u and w variables. The first order term contains

$$\bar{\varphi}(x)\, \mathfrak{U}^\mu \nabla_\mu\varphi(x)\big(\imath XQU + \mathfrak{R}'(0)\big). \tag{A.12}$$

The first term leads to $(\mathfrak{U}^1\nabla_1 + \mathfrak{U}^2\nabla_2)\varphi(x^1U^2 - x^2U^1)$ with the same kind of expressions for the two other dimensions. Due to the odd integrals, only the terms of the form $(U^1)^2x^2\nabla_1 - (U^2)^2x^1\nabla_2$ survive. We are left with integrals like

$$\int (u^1_\ell)^2 \prod_{l\in G^i_k,\, l\notin T} du_l dw_l C_l(u_l, w_l) \prod_{\ell\in T^i_k} du_\ell C_\ell(u_\ell, U_\ell, W_\ell) e^{\imath URU + \imath USW}. \quad \text{(A.13)}$$

To prove that these terms give the same coefficient (in order to reconstruct a $x \wedge \nabla$ term), note that, apart from the $(u^1_\ell)^2$, the involved integrals are actually invariant under an overall rotation of the u and w variables. Then by performing rotations of $\pi/2$, we prove that the counterterm is of the form of the Lagrangian. The $\mathfrak{R}'(0)$ and the remainder term in A^{1+}_a are irrelevant.

Let us now study the other terms in A_a.

$$A^{1-}_a = -\int dx\, \mathfrak{U}.\nabla\bar\varphi(x)\,\varphi(x) \prod_{l\in G^i_k,\, l\notin T} du_l dw_l C_l(u_l, w_l) \times \prod_{\ell\in T^i_k} du_\ell C_\ell(u_\ell, X_\ell, U_\ell, W_\ell)\, e^{\imath XQU + \imath URU + \imath USW}. \quad \text{(A.14)}$$

Once more we decouple the external variables from the internal ones by Taylor expanding the function f. Up to irrelevant terms, this only doubles the $x \wedge \nabla$ term in A^{1+}_a,

$$A^{2+}_a = \frac{1}{2}\int \bar\varphi(x)\,(\mathfrak{U}.\nabla)^2\varphi(x) \prod_{l\in G^i_k,\, l\notin T} du_l dw_l C_l(u_l, w_l) \times \prod_{\ell\in T^i_k} du_\ell C_\ell(u_\ell, U_\ell, W_\ell)\, e^{\imath URU + \imath USW}\Big(f(0) + \int_0^1 dt\, f'(t)\Big). \quad \text{(A.15)}$$

The $f(0)$ term renormalizes the wave-function. The remainder term in (A.15) is irrelevant. A^{2-}_a doubles the A^{2+}_a contribution. Finally the last remainder terms (the last two lines in (A.9)) are irrelevant too. This completes the proof of the perturbative renormalizability of the LSZ models.

Remark that if we had considered a real theory with a covariant derivative which corresponds to a neutral scalar field in a magnetic background, the angular momentum term wouldn't renormalize. Only the harmonic potential term would. It seems that the renormalization "distinguishes" the true theory in which a *charged* field should couple to a magnetic field. It would be interesting to study the renormalization group flow of these kind of models along the lines of [13].

B. Notations of Positions

- The letter x is used for the four initial positions of a vertex
- the letter X is used solely for external positions of the considered graph or subgraphs
- the letters v and u are used for the sum and difference of two positions joined by an internal line
- the letter w is used solely as another name for a v variable which corresponds to a loop line (not a tree line) once a tree has been chosen
- the letter y is used for the collective of long and external variables.
- z is to y what w is to v, namely a name for the external variables or long loop variables
- s and t are names for external variables and ends of loop lines variables in rosette vertices.

Hence the same complete set of $4n$ variables for a graph with n vertices depending on context can be denoted x ; X, u and v ; y and u ; X, u, w and the v of the tree lines ; z, u and the v of the tree lines. The s and t are only used in Subsect. 3.1.

Acknowledgements. We thank V. Gayral and R. Wulkenhaar for useful discussions on this work.

References

1. Grosse, H., Wulkenhaar, R.: Power-counting theorem for non-local matrix models and renormalisation. Commun. Math. Phys. **254**(1), 91–127 (2005)
2. Grosse, H., Wulkenhaar, R.: Renormalisation of ϕ^4-theory on noncommutative $\mathbb{R}^4$ in the matrix base. Commun. Math. Phys. **256**(2), 305–374 (2005)
3. Rivasseau, V., Vignes-Tourneret, F., Wulkenhaar, R.: Renormalization of noncommutative ϕ^4-theory by multi- scale analysis. Commun. Math. Phys. **262**, 565–594 (2006)
4. Langmann, E., Szabo, R.J., Zarembo, K.: Exact solution of quantum field theory on noncommutative phase spaces. JHEP **01**, 017 (2004)
5. Filk, T.: Divergencies in a field theory on quantum space. Phys. Lett. **B376**, 53–58 (1996)
6. Chepelev, I., Roiban, R.: Convergence theorem for non-commutative Feynman graphs and renormalization. JHEP **03**, 001 (2001)
7. Chepelev, I., Roiban, R.: Renormalization of quantum field theories on noncommutative $\mathbb{R}^d$. i: Scalars. JHEP **05**, 037 (2000)
8. Langmann, E., Szabo, R.J.: Duality in scalar field theory on noncommutative phase spaces. Phys. Lett. **B533**, 168–177 (2002)
9. Douglas, M.R., Nekrasov, N.A.: Noncommutative field theory. Rev. Mod. Phys. **73**, 977–1029 (2001)
10. Connes, A., Douglas, M.R., Schwarz, A.: Noncommutative geometry and matrix theory: Compactification on tori. JHEP **02**, 003 (1998)
11. Seiberg, N., Witten, E.: String theory and noncommutative geometry. JHEP **09**, 032 (1999)
12. Vignes-Tourneret, F.: Perturbative renormalizability of the noncommutative Gross-Neveu model. Work in progress
13. Grosse, H., Wulkenhaar, R.: The beta-function in duality-covariant noncommutative ϕ^4-theory. Eur. Phys. J. **C35**, 277–282 (2004)
14. Rivasseau, V., Vignes-Tourneret, F.: Non-commutative renormalization. To appear in the Proceedings of Rigorous Quantum Field Theory: A Symposium in Honor of Jacques Bros, Paris, France, 19-21 Jul 2004 (2004), Available at http://www.arXiv.org/list/hep-th/0409312, 2004
15. Gurau, R., Rivasseau, V., Vignes-Tourneret, F.: Propagators for noncommutative field theories. http://www.arXiv.org/list/hep-th/0512071, 2005. To appear in accepteded by Ann. H. Poincaré
16. Rivasseau, V.: From Perturbative to Constructive Renormalization. Princeton Series in Physics. Princeton, NJ: Princeton University Press, 1991, 336 pp.
17. Gallavotti, G., Nicolò, F.: Renormalization theory in four-dimensional scalar fields. i. Commun. Math. Phys. **100**, 545–590 (1985)
18. Langmann, E.: Interacting fermions on noncommutative spaces: Exactly solvable quantum field theories in 2n+1 dimensions. Nucl. Phys. **B654**, 404–426 (2003)

Communicated by A. Connes