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Dissipative Quasi-Geostrophic Equation for Large Initial Data in the Critical Sobolev Space

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Abstract: The critical and super-critical dissipative quasi-geostrophic equations are investigated in $\mathbb{R}^2$. We prove local existence of a unique regular solution for arbitrary initial data in $H^{2-2\alpha}$ which corresponds to the scaling invariant space of the equation. We also consider the behavior of the solution near $t = 0$ in the Sobolev space.

1. Introduction

Let us consider the two dimensional dissipative quasi-geostrophic equation:

$$\begin{cases} \dfrac{\partial \theta}{\partial t} + (-\Delta)^{\alpha}\theta + u \cdot \nabla\theta = 0 \quad \text{in} \quad \mathbb{R}^2 \times (0, \infty), \\ u = (-R_2\theta, R_1\theta) \quad \text{in} \quad \mathbb{R}^2 \times (0, \infty), \\ \theta|_{t=0} = \theta_0 \quad \text{in} \quad \mathbb{R}^2, \end{cases} \tag{DQG$_\alpha$}$$

where the scalar function θ and the vector field u denote the potential temperature and the fluid velocity, respectively, and α is a non-negative constant. $R_i = \frac{\partial}{\partial x_i}(-\Delta)^{-1/2}$ $(i = 1, 2)$ represents the Riesz transform. We are concerned with the initial value problem for this equation. It is known that (DQG$_\alpha$) is an important model in geophysical fluid dynamics. Indeed, it is derived from general quasi-geostrophic equations in the special case of constant potential vorticity and buoyancy frequency. Since there are a number of applications to the theory of oceanography and meteorology, a lot of mathematical researches have been devoted to the equation.

The case $\alpha = 1/2$ is called critical since its structure is quite similar to that of the 3-dimensional Navier-Stokes equations. The case $\alpha > 1/2$ is called sub-critical and $\alpha < 1/2$ is called super-critical, respectively. In the sub-critical cases, Constantin and Wu [5], Wu [15] proved global existence of a unique regular solution. However, in the critical and super-critical cases, global well-posedness for large initial data is still open. In the critical case, Constantin, Cordoba and Wu [4] constructed a global regular solution for the initial data in H^1 with small L^∞ norm. In both critical and super-critical cases, Chae and Lee [2] and Ju [9] proved global existence of a unique regular solution for the initial data in the Besov space $B^{2-2\alpha}_{2,1}$ and $H^{2-2\alpha}$ under the smallness assumption of each homogeneous norm, respectively. For large initial data, Cordoba-Cordoba [6] proved local existence of a regular solution for the initial data in H^s with $s > 2 - \alpha$. Ju [9, 10] improved the admissible exponent up to $s > 2 - 2\alpha$.

In this paper we show local existence of a unique regular solution with initial data in $H^{2-2\alpha}$ for both critical and super-critical cases. In Ju [10], he conjectured the local H^1 solution in the critical case without smallness assumption on the initial data. Our theorem gives a positive answer to his question. Moreover, our theorem improves the class of initial data to construct the local regular solution. Indeed, $H^{2-2\alpha}$ is larger than $H^s (s > 2 - 2\alpha)$. See Remark 1 below. Here the exponent $2 - 2\alpha$ is important, because this is the borderline case with respect to the scaling. We observe that if $\theta(x, t)$ is the solution of (DQG$_\alpha$), then $\theta_\lambda(x, t) \equiv \lambda^{2\alpha-1}\theta(\lambda x, \lambda^{2\alpha} t)$ is also a solution of (DQG$_\alpha$). Then the homogeneous space $\dot{H}^{2-2\alpha}$ is called scaling invariant, since $\|\theta_\lambda(\cdot, 0)\|_{\dot{H}^{2-2\alpha}} = \|\theta(\cdot, 0)\|_{\dot{H}^{2-2\alpha}}$ holds for all $\lambda > 0$. The scaling invariant spaces play an important role for the theory of nonlinear partial differential equations. If the equation has a class of scaling invariance, then it coincides with the most suitable space to construct the solution which is expected unique and regular. (See, e.g. Danchin [7], Koch-Tataru [11].)

We now sketch the idea of our proof. In contrast with other equations, it seems to be difficult to prove the local existence of regular solutions by the classical approach such as Fujita-Kato's argument [8]. As is pointed out in [2], we have difficulty to find an appropriate space $\mathcal{E}$ which yields the following continuous bilinear estimate of the Duhamel term:

$$\left\| \int_0^{\cdot} e^{-(\cdot-s)(-\Delta)^\alpha}(u \cdot \nabla\theta)(s)ds \right\|_{\mathcal{E}} \le C\|\theta\|^2_{\mathcal{E}}.$$

For $\alpha \le 1/2$, we see the linear part $(-\Delta)^\alpha\theta$ in (DQG$_\alpha$) is too weak to control the nonlinear term $u \cdot \nabla\theta$. In fact, the smoothing property of the semigroup $e^{-t(-\Delta)^\alpha}$ is not enough to overcome the *loss of derivatives* in the nonlinear term. To avoid this difficulty, in [2, 9] they applied the cancellation property of the equation to construct the small global solution. However, it seems to be difficult to adopt their method to deal with the large initial data. So, in this paper we introduce a modified version of Fujita-Kato's argument. To be precise, we derive a family of integral inequalities on the Littlewood-Paley decomposition of the solution, which makes it possible to utilize the cancellation property of the equation. In the usual Fujita-Kato argument, such cancellation property seems to be unavailable. In order to apply the cancellation property, we establish a new commutator estimate associated with the Littlewood-Paley operator in the Sobolev space. Such inequality plays an crucial role to estimate the nonlinear term. Combining with the cancellation property and the commutator estimate we obtain a priori estimates in the scaling invariant spaces. Thus we construct the local solution for large initial data in $H^{2-2\alpha}$. As a byproduct of our approach, we can obtain weighted (in time) estimates of the solution near $t = 0$ in higher order Sobolev spaces.

The paper is organized as follows. In Sect. 2, we define some function spaces and the precise statement of our theorem. Section 3 is devoted to establish some useful estimates such as the commutator estimate. Finally in Sect. 4 we prove the theorem.

2. Definitions and the Statement of the Theorem

In this section we define some function spaces and then state the main theorem. Throughout this paper we deal with the two-dimensional space $\mathbb{R}^2$. Let us first recall the definition of the Sobolev space. We define $\mathcal{Z}'$ as the topological dual space of $\mathcal{Z}$ defined by

$$\mathcal{Z} \equiv \{f \in \mathcal{S};\ \textstyle\int x^\alpha f(x)\,dx = 0 \quad \text{for all } \alpha \in \mathbb{N}^2\},$$

where $\mathcal{S}$ denotes the space of Schwartz functions.

Let $\{\phi_j\}_{j=-\infty}^{\infty}$ be the Littlewood-Paley decomposition of unity, i.e. $\hat{\phi} \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$, $\operatorname{supp}\hat{\phi} \subset \{\xi \in \mathbb{R}^2;\ 1/2 \le |\xi| \le 2\}$ and $\sum_{j=-\infty}^{\infty} \hat{\phi}(2^{-j}\xi) \equiv 1$ except $\xi = 0$. We define the Littlewood-Paley operator Δ_j as $\Delta_j = \phi_j *$, where $\mathcal{F}(\phi_j)(\xi) = \hat{\phi}(2^{-j}\xi)$.

For $1 < p < \infty$, we define the homogeneous and inhomogeneous Sobolev spaces $\dot{H}^{s,p}$ and $H^{s,p}$ by

$$\dot{H}^{s,p} \equiv \left\{ f \in \mathcal{Z}';\ \|f\|_{\dot{H}^{s,p}} \equiv \left\| \left(\sum_{j\in\mathbb{Z}} (2^{sj}|\Delta_j f|)^2 \right)^{1/2} \right\|_p < \infty \right\} \quad \text{for} \quad s \in \mathbb{R},$$

and

$$H^{s,p} \equiv \left\{ f \in \mathcal{S}';\ \|f\|_{H^{s,p}} \equiv \|f\|_{L^p} + \|f\|_{\dot{H}^{s,p}} < \infty \right\} \quad \text{for} \quad s > 0,$$

respectively. We abbreviate $\dot{H}^{s,2} = \dot{H}^s$ and $H^{s,2} = H^s$.

Remark. Let $\mathcal{P}$ be the set of all polynomials. Then $\mathcal{Z}' \simeq \mathcal{S}'/\mathcal{P}$ holds. Since we cannot distinguish zero from other polynomials in $\mathcal{S}'/\mathcal{P}$, $\dot{H}^{s,p}$ seems not to be appropriate as function spaces to treat equations. Fortunately, if the exponents s and p satisfy the condition $s < 2/p$, then $\dot{H}^{s,p}$ can be regarded as a subspace of $\mathcal{S}'$. Indeed, for $s < 2/p$, we have

$$\dot{H}^{s,p} \simeq \left\{ f \in \mathcal{S}';\ \|f\|_{\dot{H}^{s,p}} < \infty \text{ and } f = \sum_{j\in\mathbb{Z}} \Delta_j f \text{ in } \mathcal{S}' \right\}.$$

For the details, see, e.g. Kozono-Yamazaki [12].

Now we state the main theorem of this paper.

Theorem 1. *Let* $0 < \alpha \le 1/2$. *Suppose that the initial data* $\theta_0 \in H^{2-2\alpha}$. *Then there exist a positive constant* T *and a unique solution* θ *of* (DQG_α) *in* $L^\infty(0,T;H^{2-2\alpha}) \cap L^2(0,T;\dot{H}^{2-\alpha})$. *Moreover such a solution* θ *belongs to* $C([0,T);H^{2-2\alpha})$ *and it satisfies the following estimate:*

$$\sup_{0<t<T} t^{\frac{\beta}{2\alpha}} \|\theta(t)\|_{\dot{H}^{2-2\alpha+\beta}} < \infty \quad for \quad 0 \le \beta < 2\alpha. \tag{2.1}$$

In particular, we have

$$\lim_{t\to 0} t^{\frac{\beta}{2\alpha}} \|\theta(t)\|_{\dot{H}^{2-2\alpha+\beta}} = 0 \quad for \quad 0 < \beta < 2\alpha. \tag{2.2}$$

Remark 1. i) Ju [9, 10] proved the local existence of a unique solution for the initial data in H^s with $s > 2 - 2\alpha$. Theorem 1 improves his result on the space of initial data. Indeed, $H^{2-2\alpha}$ is larger than H^s for $s > 2 - 2\alpha$.

ii) In contrast with Chae-Lee [2] and Ju [9], we make use of the Fujita-Kato type argument to construct the solution. This approach provides us the weighted estimate (2.1) of the solution in higher order Sobolev space.

iii) Ju [9] proved global existence of a solution for the initial data in $H^{2-2\alpha}$ with small homogeneous norm. Theorem 1 can be regarded as the local version of his result. In fact, by the argument of our proof, one can also prove the similar global existence theorem:

Corollary 1. *There exists a positive constant ε such that if the initial data $\theta_0 \in H^{2-2\alpha}$ satisfies $\|\theta_0\|_{\dot{H}^{2-2\alpha}} < \varepsilon$, then one can take $T = \infty$ in Theorem 1.*

3. Littlewood-Paley Operator and the Commutator Estimate

In this section we recall several estimates related to the Littlewood-Paley operator. Throughout this paper we denote a positive constant by C (or C', etc.) the value of which may differ from one occasion to another. On the other hand, we denote $C_i\,(i = 1, 2, \cdots)$ as the certain constants. We recall Bernstein's inequality.

Lemma 1. (i) *Let $s \in \mathbb{R}$, $1 \le p \le \infty$. Then there exist positive constants $C = C(s, p)$ and $C' = C'(s, p)$ such that*

$$C2^{js}\|\Delta_j f\|_{L^p} \le \|(-\Delta)^{s/2}\Delta_j f\|_{L^p} \le C'2^{js}\|\Delta_j f\|_{L^p}$$

holds for all $j \in \mathbb{Z}$.

(ii) *Let $1 \le p \le q \le \infty$. Then there exists a positive constant $C = C(p, q)$ such that*

$$\|\Delta_j f\|_{L^q} \le C2^{(2/p-2/q)j}\|\Delta_j f\|_{L^p}$$

holds for all $j \in \mathbb{Z}$.

We prepare various product estimates in the Sobolev space. For this purpose we recall paraproduct formula introduced by Bony [1]. Paraproduct operators are defined by

$$T_f g \equiv \sum_{j\in\mathbb{Z}} S_j f \Delta_j g,$$

$$R(f, g) \equiv \sum_{|i-j|\le 2} \Delta_i f \Delta_j g,$$

where $S_j f \equiv \sum_{k\le j-3} \Delta_k f$. Then we have the formal expression for the product:

$$fg = T_f g + T_g f + R(f, g).$$

The following estimates are fundamental properties for the paraproduct operators. For the proof see, e.g. Runst-Sickel [13].

Lemma 2. (i) *Let $s < 1, t \in \mathbb{R}$. Then there exists a positive constant $C = C(s, t)$ such that*

$$\|T_f g\|_{\dot{H}^{s+t-1}} \le C\|f\|_{\dot{H}^s}\|g\|_{\dot{H}^t}$$

holds for $f \in \dot{H}^s$ and $g \in \dot{H}^t$.

(ii) *Let* $s + t > 0$. *Then there exists a positive constant* $C = C(s, t)$ *such that*

$$\|R(f, g)\|_{\dot{H}^{s+t-1}} \le C\|f\|_{\dot{H}^s}\|g\|_{\dot{H}^t}$$

holds for $f \in \dot{H}^s$ *and* $g \in \dot{H}^t$

A direct consequence is the following product estimate in the Sobolev space:

Proposition 1. *Let* $s, t < 1$ *and* $s + t > 0$. *Then there exists a positive constant* $C = C(s, t)$ *such that*

$$\|fg\|_{\dot{H}^{s+t-1}} \le C\|f\|_{\dot{H}^s}\|g\|_{\dot{H}^t}$$

holds for $f \in \dot{H}^s$ *and* $g \in \dot{H}^t$.

Finally, we state the commutator estimate associated with the operator Δ_j, which plays an important role for the estimate of the nonlinear term.

Proposition 2. *Let* $1 \le s < 2, t < 1$ *with* $s + t > 1$. *Then there exist positive constants* $C = C(s, t)$ *such that*

$$\|[f, \Delta_j]g\|_{L^2} \le C2^{-(s+t-1)j}c_j\|f\|_{\dot{H}^s}\|g\|_{\dot{H}^t}$$

holds for $j \in \mathbb{Z}$, $f \in \dot{H}^s$ *and* $g \in \dot{H}^t$ *with* $\sum_{j\in\mathbb{Z}} c_j^2 = 1$. *Here we denote*

$$[f, \Delta_j]g = f\Delta_j g - \Delta_j(fg).$$

Proof. Let us decompose the commutator $[f, \Delta_j]g$ by paraproduct formula as follows:

$$[f, \Delta_j]g = [T_f, \Delta_j]g + R(f, \Delta_j g) - \Delta_j R(f, g) + T_{\Delta_j g}f - \Delta_j T_g f.$$

We estimate five terms on the right-hand side respectively.

By the definition of paraproduct and localization in frequency, we have

$$[T_f, \Delta_j]g = \sum_{|k-j|\le 3} [S_k f, \Delta_j]\Delta_k g.$$

Applying the mean value theorem, we see that the right-hand side is equal to

$$\begin{aligned}&\sum_{|k-j|\le 3} \int\int_0^1 \phi_j(y)(y\cdot(S_k\nabla f)(x-\tau y))\Delta_k g(x-y)d\tau dy\\ &= 2^{-j}\sum_{|k-j|\le 3}\int\int_0^1 \phi(y)(y\cdot(S_k\nabla f)(x-2^{-j}\tau y))\Delta_k g(x-2^{-j}y)d\tau dy.\end{aligned}$$

Since $\int |y||\phi(y)|dy < \infty$, we have

$$\|[T_f, \Delta_j]g\|_{L^2} \le C2^{-j}\sum_{|k-j|\le 3}\|S_k\nabla f\|_{L^p}\|\Delta_k g\|_{L^{p^*}},$$

where we have taken $p < \infty$ as $s' \equiv s + 2/p < 2$ and $1/p + 1/p^* = 1/2$. We can choose such p by the assumption of s. Then Hölder's inequality yields

$$
\begin{aligned}
S_k \nabla f &= \sum_{l \le k-3} 2^{(2-s')l} 2^{(s'-2)l} \Delta_l \nabla f \\
&\le C 2^{(2-s')k} \left(\sum_{l \in \mathbb{Z}} (2^{(s'-2)l} \Delta_l \nabla f)^2 \right)^{1/2}.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\| S_k \nabla f \|_{L^p} &\le C 2^{-(s'-2)k} \left\| \left(\sum_{l \in \mathbb{Z}} (2^{(s'-2)l} \Delta_l \nabla f)^2 \right)^{1/2} \right\|_{L^p} \\
&\le C 2^{-(s'-2)k} \| f \|_{\dot{H}^{s'-1,p}} \\
&\le C 2^{-(s'-2)k} \| f \|_{\dot{H}^s}.
\end{aligned}
$$

By finiteness of the number of the sum on k, we can estimate as follows:

$$
\begin{aligned}
\| [T_f, \Delta_j] g \|_{L^2} &\le C 2^{-(s'-1)j} \| f \|_{\dot{H}^s} \sum_{|k-j| \le 3} \| \Delta_k g \|_{L^{p^*}} \\
&\le C 2^{-(s'-1)j} \| f \|_{\dot{H}^s} \| \Delta_j g \|_{L^{p^*}} \\
&\le C 2^{-(s+t-1)j} \| f \|_{\dot{H}^s} 2^{j(s-s'+t)} \| \Delta_j g \|_{L^{p^*}} \\
&\le C 2^{-(s+t-1)j} \| f \|_{\dot{H}^s} 2^{jt} \| \Delta_j g \|_{L^2} \\
&\le C 2^{-(s+t-1)j} c_j \| f \|_{\dot{H}^s} \| g \|_{\dot{H}^t},
\end{aligned}
$$

where we define $c_j = (2^{jt} \| \Delta_j g \|_{L^2}) / \| g \|_{\dot{H}^t}$. Thus, we obtain the estimate for the first term.

Let $\tilde{\Delta}_k = \sum_{|k-j| \le 2} \Delta_j$. Then we observe that

$$
R(f, \Delta_j g) = \sum_{|k-j| \le 2} \tilde{\Delta}_k f \Delta_k \Delta_j g,
$$

which yields the estimate of the second term:

$$
\begin{aligned}
\| R(f, \Delta_j g) \|_{L^2} &\le \sum_{|k-j| \le 2} \| \tilde{\Delta}_k f \Delta_k \Delta_j g \|_{L^2} \\
&\le \sum_{|k-j| \le 2} \| \tilde{\Delta}_k f \|_{L^p} \| \Delta_k \Delta_j g \|_{L^{p^*}} \\
&\le 2^{-(s+t-1)j} \sum_{|k-j| \le 2} 2^{k(s'-1)} \| \tilde{\Delta}_k f \|_{L^p} 2^{(s-s'+t)k} \| \Delta_k \Delta_j g \|_{L^{p^*}} \\
&\le C 2^{-(s+t-1)j} c_j \| f \|_{\dot{H}^s} \| g \|_{\dot{H}^t},
\end{aligned}
$$

where p, s' and c_j are chosen as above.

Since $s + t > 0$, we can apply Lemma 2 to the third term:

$$
\| \Delta_j R(f, g) \|_{L^2} \le C c'_j 2^{-(s+t-1)j} \| f \|_{\dot{H}^s} \| g \|_{\dot{H}^t}
$$

with $c'_j = (2^{(s+t-1)j}\|\Delta_j R(f,g)\|_{L^2})/\|fg\|_{\dot{H}^{s+t-1}}$.

For the fourth term, we observe that

$$T_{\Delta_j g} f = \sum_{k \geq j-2} S_k \Delta_j g \Delta_k f,$$

which yields

$$\begin{aligned}\|T_{\Delta_j g} f\|_{L^2} &\leq C\|\sum_{k\geq j-2} M(\Delta_j g)|\Delta_k f|\|_{L^2}\\ &\leq \|\Delta_j g\|_{L^{p^*}}\|\sum_{k\geq j-2}|\Delta_k f|\|_{L^p}\\ &\leq C2^{-(s-s'+t)j} c_j \|g\|_{\dot{H}^t}\|\sum_{k\geq j-2}|\Delta_k f|\|_{L^p}.\end{aligned}$$

In the above inequalities, we have used the L^{p^*}-boundedness of the Hardy-Littlewood maximal operator M, where $Mf(x) \equiv \sup_{r>0} 1/|B(x,r)|\int_{B(x,r)}|f(y)|dy$.

Since $s' = s + 2/p > 1$, we have

$$\begin{aligned}\sum_{k\geq j-2}|\Delta_k f| &= \sum_{k\geq j-2} 2^{-(s'-1)k}2^{(s'-1)k}|\Delta_k f|\\ &\leq C2^{-(s'-1)j}\left(\sum_{k\geq j-2} 2^{2(s'-1)k}|\Delta_k f|^2\right)^{1/2}.\end{aligned}$$

Thus we can estimate the fourth term.

Finally, since $t < 1$, Lemma 2 shows that

$$\|\Delta_j T_g f\|_{L^2} \leq Cc''_j 2^{-(s+t-1)j}\|f\|_{\dot{H}^s}\|g\|_{\dot{H}^t}$$

with $c''_j = (2^{(s+t-1)j}\|\Delta_j T_g f\|_{L^2})/\|fg\|_{\dot{H}^{s+t-1}}$. □

4. Proof of Theorem

4.1. Linear estimates. In this subsection, we consider the linear dissipative equation. The following lemma is closely related to Chemin [3, Prop. 2.1], which characterizes the evolution of the solution to the linear equation.

Lemma 3. *Let $e^{-t(-\Delta)^\alpha}a \equiv \mathcal{F}^{-1}(e^{-t|\cdot|^{2\alpha}}\hat{a})$, where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. Then there exist positive constants λ and $\lambda'(\lambda < \lambda')$ depending only on $\alpha > 0$ such that*

$$e^{-2^{2\alpha j}\lambda' t}\|\Delta_j a\|_{L^2} \leq \|e^{-t(-\Delta)^\alpha}\Delta_j a\|_{L^2} \leq e^{-2^{2\alpha j}\lambda t}\|\Delta_j a\|_{L^2}$$

for all $t > 0$.

Proof. Let $u(t) \equiv e^{-t(-\Delta)^\alpha}\Delta_j a$. Then u satisfies

$$\begin{cases} \dfrac{\partial u}{\partial t} + (-\Delta)^\alpha u = 0 \quad \text{in} \quad \mathbb{R}^2 \times (0,\infty), \\ u|_{t=0} = \Delta_j a \quad \text{in} \quad \mathbb{R}^2. \end{cases}$$

Taking the inner product in L^2 with the first equation and u, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|(-\Delta)^{\alpha/2}u\|_{L^2}^2 = 0.$$

By Lemma 1, there exist positive constants λ and $\lambda'(\lambda < \lambda')$ such that

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \lambda 2^{2\alpha j}\|u\|_{L^2}^2 \le 0,$$

and

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \lambda' 2^{2\alpha j}\|u\|_{L^2}^2 \ge 0.$$

Dividing the above inequalities by $\|u\|_{L^2}$ and then integrating on the interval $(0,t)$, we have

$$e^{-2^{2\alpha j}\lambda' t}\|u(0)\|_{L^2} \le \|u(t)\|_{L^2} \le e^{-2^{2\alpha j}\lambda t}\|u(0)\|_{L^2}.$$

By definition of u, we obtain the desired result. □

Now we state the smoothing estimates.

Proposition 3. *For $\alpha > 0$ and $s \ge 0$, there exists a positive constant $C = C(s,\alpha)$ such that*

$$\sup_{t>0} t^{\frac{s}{2\alpha}}\|e^{-t(-\Delta)^\alpha}a\|_{\dot{H}^s} \le C\|a\|_{L^2} \tag{4.1}$$

for all $a \in L^2$. In particular, we have

$$\lim_{t\to 0} t^{\frac{s}{2\alpha}}\|e^{-t(-\Delta)^\alpha}a\|_{\dot{H}^s} = 0, \tag{4.2}$$

for all $a \in L^2$. Moreover, if $0 \le s \le \alpha$, then we have

$$\|e^{-t(-\Delta)^\alpha}a\|_{L^{2\alpha/s}(0,\infty;\dot{H}^s)} \le C\|a\|_{L^2} \tag{4.3}$$

for all $a \in L^2$.

Proof. We have

$$\|e^{-t(-\Delta)^\alpha}a\|_{\dot{H}^s} = \left(\sum_{j\in\mathbb{Z}} 2^{2js}\left\|e^{-t(-\Delta)^\alpha}\Delta_j a\right\|_{L^2}^2\right)^{1/2}.$$

On the other hand, it follows from the previous lemma that

$$\|e^{-t(-\Delta)^\alpha}\Delta_j a\|_{L^2}^2 \le e^{-2^{2\alpha j+1}\lambda t}\|\Delta_j a\|_{L^2}^2.$$

Here, we observe that

$$\sup_{j\in\mathbb{Z}} 2^{2js}e^{-\lambda t 2^{2\alpha j+1}} \le Ct^{-\frac{s}{\alpha}},$$

which yields

$$\sup_{0<t<T} t^{\frac{s}{2\alpha}}\|e^{-t(-\Delta)^\alpha}a\|_{\dot{H}^s} \le C\left(\sum_{j\in\mathbb{Z}}\|\Delta_j a\|_{L^2}^2\right)^{1/2} \le \|a\|_{L^2}.$$

To prove (4.2), for any $\varepsilon > 0$ we choose the function $a_\varepsilon \in C_0^\infty$ satisfying

$$\|a - a_\varepsilon\|_{L^2} < \varepsilon/2.$$

Then it follows from (4.1) that

$$\begin{aligned} t^{\frac{s}{2\alpha}}\|e^{-t(-\Delta)^\alpha}a\|_{\dot{H}^s} &\le t^{\frac{s}{2\alpha}}\left(\|e^{-t(-\Delta)^\alpha}(a-a_\varepsilon)\|_{\dot{H}^s} + \|e^{-t(-\Delta)^\alpha}a_\varepsilon\|_{\dot{H}^s}\right) \\ &< \|a-a_\varepsilon\|_{L^2} + t^{\frac{s}{2\alpha}}\|a_\varepsilon\|_{\dot{H}^s}. \end{aligned}$$

Let $T' = \varepsilon/(2\|a_\varepsilon\|_{\dot{H}^s})$. Then the left-hand side of the above inequality is bounded by ε if $t < T'$. This proves (4.2).

Next we prove (4.3). Let $v_j(t) \equiv e^{-2^{2\alpha j}\lambda t}\|\Delta_j a\|_{L^2}$, and v_j satisfies

$$\partial_t v_j + \lambda 2^{2\alpha j}v_j = 0 \quad \text{for} \quad t > 0 \quad \text{and} \quad j \in \mathbb{Z}.$$

Multiplying this inequality by $v_j^{2\alpha/s-1}$ and then integrating the above identity in time, we have

$$\int_0^\infty \lambda 2^{2\alpha j}v_j(t)^{2\alpha/s}dt = Cv_j(0)^{2\alpha/s},$$

that is,

$$\|2^{sj}v_j\|_{L^{2\alpha/s}} = Cv_j(0).$$

Taking l^2-norm on both sides of this estimate, we obtain

$$\left(\sum_{j\in\mathbb{Z}}\|2^{sj}v_j\|_{L^{2\alpha/s}}\right)^{1/2} \le C\left(\sum_{j\in\mathbb{Z}}v_j^2(0)\right)^{1/2}.$$

Since $\alpha/s \geq 1$, the left-hand side is estimated from below as follows:

$$\begin{aligned}\left(\sum_{j\in\mathbb{Z}} \|2^{sj} v_j\|_{L^{2\alpha/s}}^2\right)^{1/2} &= \left(\sum_{j\in\mathbb{Z}} \|2^{2sj} v_j^2\|_{L^{\alpha/s}}\right)^{1/2} \\ &\geq \left\|\sum_{j\in\mathbb{Z}} 2^{2sj} v_j^2\right\|_{L^{\alpha/s}}^{1/2} \\ &= \left\|\left(\sum_{j\in\mathbb{Z}} 2^{2sj} v_j^2\right)^{1/2}\right\|_{L^{2\alpha/s}}.\end{aligned}$$

So we have

$$\left\|\left(\sum_{j\in\mathbb{Z}} 2^{2sj} v_j^2\right)^{1/2}\right\|_{L^{2\alpha/s}} \leq C\left(\sum_{j\in\mathbb{Z}} v_j^2(0)\right)^{1/2}.$$

From Lemma 3, we obtain (4.3). □

4.2 Proof of Theorem 1

Step 1. A priori estimates. We first show an a priori estimate in $L^3(0, T; \dot{H}^{2-4\alpha/3})$. More precisely, we will prove that there exist a positive constant C_1 and a bounded function $I_1 = I_1(T)$ with

$$I_1(T) \leq C\|\theta_0\|_{\dot{H}^{2-2\alpha}} \quad \text{and} \quad \lim_{T\to 0} I_1(T) = 0 \tag{4.4}$$

such that

$$\|\theta\|_{L^3_T \dot{H}^{2-4\alpha/3}} \leq I_1(T) + C_1\|\theta\|^2_{L^3_T \dot{H}^{2-4\alpha/3}} \tag{4.5}$$

holds for all solutions θ of (DQG$_\alpha$). Here we write the space $L^p(0, T; \dot{H}^s)$ as $L^p_T \dot{H}^s$.

Applying the operator Δ_j to (DQG$_\alpha$), we obtain

$$\partial_t\theta_j + (-\Delta)^\alpha\theta_j = -\Delta_j(u\cdot\nabla\theta),$$

where we denote $\theta_j \equiv \Delta_j\theta$. Adding $u\cdot\nabla\Delta_j\theta$ on both sides, we have

$$\partial_t\theta_j + (-\Delta)^\alpha\theta_j + u\cdot\nabla\Delta_j\theta = [u, \Delta_j]\nabla\theta.$$

Taking the inner product with the above inequality and θ_j, and then applying Lemma 1, we obtain from the divergence free condition that

$$\frac{1}{2}\frac{d}{dt}\|\theta_j\|^2_{L^2} + \lambda 2^{2\alpha j}\|\theta_j\|^2_{L^2} \leq \|[u, \Delta_j]\nabla\theta\|_{L^2}\|\theta_j\|_{L^2}.$$

Dividing both sides by $\|\theta_j\|_{L^2}$, we have

$$\frac{d}{dt}\|\theta_j\|_{L^2} + \lambda 2^{2\alpha j}\|\theta_j\|_{L^2} \leq \|[u, \Delta_j]\nabla\theta\|_{L^2}.$$

Applying Proposition 2 with $s = 2 - 4\alpha/3$ and $t = 1 - 4\alpha/3$ and Calderón-Zygmund's inequality, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\theta_j\|_{L^2} + \lambda 2^{2\alpha j}\|\theta_j\|_{L^2} &\leq \|[u, \Delta_j]\nabla\theta\|_{L^2} \\ &\leq Cc_j 2^{-(2-8\alpha/3)j}\|u\|_{\dot{H}^{2-4\alpha/3}}\|\nabla\theta\|_{\dot{H}^{1-4\alpha/3}} \\ &\leq Cc_j 2^{-(2-8\alpha/3)j}\|\theta\|^2_{\dot{H}^{2-4\alpha/3}}.\end{aligned}$$

Integrating both sides in time on the interval $(0, t)$, we have

$$\|\theta_j(t)\|_{L^2} \leq e^{-2^{2\alpha j}\lambda t}\|\theta_j(0)\|_{L^2} + Cc_j 2^{-(2-8\alpha/3)j}\int_0^t e^{-2^{2\alpha j}\lambda(t-s)}\|\theta(s)\|^2_{\dot{H}^{2-\alpha}}ds.$$

Multiplying the above inequality by $2^{(2-4\alpha/3)j}$ and then taking the l^2-norm with respect to j, we can estimate the $\dot{H}^{2-4\alpha/3}$ norm of θ as:

$$\begin{aligned}\|\theta(t)\|_{\dot{H}^{2-4\alpha/3}} &\leq \left(\sum_{j\in\mathbb{Z}} 2^{2(2-4\alpha/3)j}e^{-2^{2\alpha j+1}\lambda t}\|\theta_j(0)\|^2_{L^2}\right)^{1/2} \\ &\quad +C\left(\sum_{j\in\mathbb{Z}}\left(c_j 2^{4\alpha j/3}\int_0^t e^{-2^{2\alpha j}\lambda(t-s)}\|\theta(s)\|^2_{\dot{H}^{2-4\alpha/3}}ds\right)^2\right)^{1/2} \\ &\equiv I + II.\end{aligned}$$

In order to show (4.5), we need to estimate L^3_T norm of the right-hand side.

According to Lemma 3 and (4.3), we see that the first term is estimated as

$$\left\|\left(\sum_{j\in\mathbb{Z}}\left(2^{(2-4\alpha/3)j}e^{-2^{2\alpha j}\lambda t}\|\theta_j(0)\|_{L^2}\right)^2\right)^{1/2}\right\|_{L^3_T} \leq C\|\theta_0\|_{\dot{H}^{2-2\alpha}}.$$

Let

$$I_1(T) \equiv \left\|\left(\sum_{j\in\mathbb{Z}}\left(2^{(2-4\alpha/3)j}e^{-2^{2\alpha j}\lambda t}\|\theta_j(0)\|_{L^2}\right)^2\right)^{1/2}\right\|_{L^3_T}.$$

Then absolute continuity of the integral yields (4.4).

Since

$$\sup_j 2^{4\alpha j/3}e^{-2^{2\alpha j}\lambda(t-s)} < C(t-s)^{-2/3},$$

we can estimate the second term as:

$$\begin{aligned}\|II\|_{L^3_T} &= C\left\|\left(\sum_{j\in\mathbb{Z}}\left(c_j 2^{4\alpha j/3}\int_0^t e^{-2^{2\alpha j}\lambda(t-s)}\|\theta(s)\|^2_{\dot{H}^{2-4\alpha/3}}ds\right)^2\right)^{1/2}\right\|_{L^3_T} \\ &\leq C\left\|\int_0^t (t-s)^{-2/3}\|\theta(s)\|^2_{\dot{H}^{2-4\alpha/3}}ds\right\|_{L^3_T} \\ &\leq C\|\theta\|^2_{L^3_T\dot{H}^{2-4\alpha/3}},\end{aligned}$$

where we used Hardy-Littlewood-Sobolev's inequality in the last inequality. Therefore we obtain the a priori estimate (4.5).

Similarly to the previous arguments, we can also show that there exists a bounded function $I_2 = I_2(T)$ with

$$I_2(T) \le C\|\theta_0\|_{\dot{H}^{2-2\alpha}} \quad \text{and} \quad \lim_{T\to 0} I_2(T) = 0$$

satisfying

$$\|\theta\|_{L^2_T \dot{H}^{2-\alpha}} \le I_2(T) + C\|\theta\|^2_{L^3_T \dot{H}^{2-4\alpha/3}}. \tag{4.6}$$

Moreover, we have

$$\|\theta\|_{L^\infty_T \dot{H}^{2-2\alpha}} \le \|\theta_0\|_{\dot{H}^{2-2\alpha}} + C\|\theta\|^2_{L^2_T \dot{H}^{2-\alpha}}.$$

Combining the above estimates with the maximum principle [6]

$$\|\theta(t)\|_{L^2} \le \|\theta_0\|_{L^2},$$

we obtain the following estimate

$$\|\theta\|_{L^\infty_T H^{2-2\alpha}} \le \|\theta_0\|_{H^{2-2\alpha}} + C\|\theta\|^2_{L^2_T \dot{H}^{2-\alpha}}. \tag{4.7}$$

Step 2. Convergence of approximation sequences. To construct the solution, we consider the following successive approximation:

$$\begin{cases} \partial_t\theta^0 + (-\Delta)^\alpha\theta^0 = 0 & \text{in} \quad \mathbb{R}^2 \times \mathbb{R}_+, \\ \theta^0|_{t=0} = \theta_0 & \text{in} \quad \mathbb{R}^2 \end{cases}$$

and

$$\begin{cases} \partial_t\theta^{n+1} + (-\Delta)^\alpha\theta^{n+1} + u^n\cdot\nabla\theta^{n+1} = 0 & \text{in} \quad \mathbb{R}^2 \times \mathbb{R}_+, \\ u^n = (-R_2\theta^n, R_1\theta^n) & \text{in} \quad \mathbb{R}^2 \times \mathbb{R}_+, \\ \theta^{n+1}|_{t=0} = \theta_0 & \text{in} \quad \mathbb{R}^2, \end{cases}$$

for $n = 0, 1, 2\ldots$.

We will establish uniform estimates on θ^n. Similarly to the arguments in Step 1, we can show that there exists a bounded function I_1 with $\lim_{T\to 0} I_1(T) = 0$ such that

$$\begin{aligned} \|\theta^0\|_{L^3_T \dot{H}^{2-4\alpha/3}} &\le I_1(T), \\ \|\theta^{n+1}\|_{L^3_T \dot{H}^{2-4\alpha/3}} &\le I_1(T) + C_1\|\theta^n\|_{L^3_T \dot{H}^{2-4\alpha/3}}\|\theta^{n+1}\|_{L^3_T \dot{H}^{2-4\alpha/3}} \end{aligned}$$

for $n = 0, 1, 2\ldots$. Taking $T_0 > 0$ so small that $I_1(T_0) \le 1/(4C_1)$, we have

$$\|\theta^n\|_{L^3_T \dot{H}^{2-4\alpha/3}} \le 2I_1(T) \quad \text{for} \quad T < T_0.$$

By (4.6), we can also show that there exists a bounded function I_2 with $\lim_{T\to 0} I_2(T) = 0$ such that

$$\|\theta^n\|_{L^2_T \dot{H}^{2-\alpha}} \le I_2(T) + C(I_1(T))^2 \quad \text{for} \quad T < T_0. \tag{4.8}$$

Moreover, (4.7) yields

$$\|\theta^n\|_{L^\infty_T H^{2-2\alpha}} \le \|\theta_0\|_{H^{2-2\alpha}} + C(I_3(T))^2 \quad \text{for} \quad T < T_0, \tag{4.9}$$

where we write $I_3(T) \equiv I_2(T) + C(I_1(T))^2$. Using (4.8), we will prove the convergence of the sequence θ^n in $L^4_T \dot{H}^{3/4}$.

Let $\delta\theta^{n+1} = \theta^{n+1} - \theta^n$, $\delta u^{n+1} = u^{n+1} - u^n$, $\delta\theta^0 = \theta^0$ and $\delta u^0 = u^0$, and we have following equations of the differences:

$$\begin{cases} \partial_t \delta\theta^{n+1} + (-\Delta)^\alpha \delta\theta^{n+1} + u^n \cdot \nabla\delta\theta^{n+1} + \delta u^n \cdot \nabla\theta^n = 0 \quad \text{in } \mathbb{R}^2 \times \mathbb{R}_+, \\ \delta u^n = (-R_2\delta\theta^n, R_1\delta\theta^n) \quad \text{in } \mathbb{R}^2 \times \mathbb{R}_+, \\ \delta\theta^{n+1}|_{t=0} = 0 \quad \text{in } \mathbb{R}^2, \end{cases}$$

for $n = 0, 1, 2 \dots$.

Similarly to the arguments in Step 1, we have

$$\frac{1}{2}\frac{d}{dt}\left\|\delta\theta_j^{n+1}\right\|_{L^2}^2 + \lambda 2^{2\alpha j}\left\|\delta\theta_j^{n+1}\right\|_{L^2}^2 \le -\left\langle \Delta_j(u^n \cdot \nabla\delta\theta^{n+1}) + \Delta_j(\delta u^n \cdot \nabla\theta^n), \delta\theta_j^{n+1}\right\rangle,$$

where $\delta\theta_j^{n+1} \equiv \Delta_j\theta^{n+1} - \Delta_j\theta^n$. Since $\operatorname{div} u^n = 0$, we have

$$\left\langle u^n \cdot \nabla\delta\theta_j^{n+1}, \delta\theta_j^{n+1}\right\rangle = 0.$$

By Hölder's inequality, we have

$$\frac{d}{dt}\left\|\delta\theta_j^{n+1}\right\|_{L^2} + \lambda 2^{2\alpha j}\left\|\delta\theta_j^{n+1}\right\|_{L^2} \le \|[u^n, \Delta_j]\nabla\delta\theta^{n+1}\|_{L^2} + \|\Delta_j(\delta u^n \cdot \nabla\theta^n)\|_{L^2},$$

which yields

$$\left\|\delta\theta_j^{n+1}(t)\right\|_{L^2} \le C\int_0^t e^{-2^{2\alpha j}\lambda(t-s)}\left(\left\|\left[u^n, \Delta_j\right]\nabla\delta\theta^{n+1}\right\|_{L^2} + \|\Delta_j(\delta u^n \cdot \nabla\theta^n)\|_{L^2}\right) ds. \tag{4.10}$$

By $s = 2 - \alpha$ and $t = -1/4$ in Proposition 2, we have

$$\begin{aligned}\left\|\left[u^n, \Delta_j\right]\nabla\delta\theta^{n+1}\right\|_{L^2} &\le C2^{-(3/4-\alpha)j} c_j \|u^n\|_{\dot{H}^{2-\alpha}} \|\nabla\delta\theta^{n+1}\|_{\dot{H}^{-1/4}} \\ &\le C2^{-(3/4-\alpha)j} c_j \|\theta^n\|_{\dot{H}^{2-\alpha}} \|\delta\theta^{n+1}\|_{\dot{H}^{3/4}}.\end{aligned}$$

On the other hand, by Proposition 1, we have

$$\begin{aligned}\left\|\Delta_j(\delta u^n \cdot \nabla\theta^n)\right\|_{L^2} &\le C2^{-(3/4-\alpha)j} c'_j \|\delta u^n \cdot \nabla\theta^n\|_{\dot{H}^{3/4-\alpha}} \\ &\le C2^{-(3/4-\alpha)j} c'_j \|\delta u^n\|_{\dot{H}^{3/4}} \|\theta^n\|_{\dot{H}^{2-\alpha}},\end{aligned}$$

where $\sum_j {c'_j}^2 = 1$. Multiplying (4.10) by $2^{3/4j}$, and then taking the l^2-norm with respect to j, we have

$$\begin{aligned}&\|\delta\theta^{n+1}(t)\|_{\dot{H}^{3/4}} \\ &\le C\left(\sum_{j\in\mathbb{Z}}\left(2^{\alpha j}\int_0^t e^{-2^{2\alpha j}\lambda(t-s)}\|\theta^n\|_{\dot{H}^{2-\alpha}}(c_j\|\delta\theta^{n+1}\|_{\dot{H}^{3/4}} + c'_j\|\delta u^n\|_{\dot{H}^{3/4}}) ds\right)^2\right)^{1/2},\end{aligned}$$

which yields

$$\begin{aligned}\|\delta\theta^{n+1}\|_{L^4_T\dot H^{3/4}} &\le C\left(\|\theta^n\|_{L^2_T\dot H^{2-\alpha}}\|\delta\theta^{n+1}\|_{L^4_T\dot H^{3/4}} + \|\delta\theta^n\|_{L^4_T\dot H^{3/4}}\|\theta^n\|_{L^2_T\dot H^{2-\alpha}}\right)\\ &\le C_2\|\theta^n\|_{L^2_T\dot H^{2-\alpha}}\left(\|\delta\theta^{n+1}\|_{L^4_T\dot H^{3/4}} + \|\delta\theta^n\|_{L^4_T\dot H^{3/4}}\right).\end{aligned}$$

By (4.8), there exists $T_1 > 0$ such that $\|\theta^n\|_{L^2_T\dot H^{2-\alpha}} < 1/(3C_2)$ for all $n = 0, 1, 2 \ldots$. Hence we have

$$\begin{aligned}\|\delta\theta^{n+1}\|_{L^4_{T_1}\dot H^{3/4}} &\le \frac{1}{2}\|\delta\theta^n\|_{L^4_{T_1}\dot H^{3/4}}\\ &\le \frac{1}{2^{n+1}}\|\theta^0\|_{L^4_{T_1}\dot H^{3/4}}\\ &\le \frac{C}{2^{n+1}}\|\theta_0\|_{\dot H^{3/4-\alpha/2}}\\ &\le \frac{C}{2^{n+1}}\|\theta_0\|_{H^{2-2\alpha}}.\end{aligned}$$

This shows the existence of the function $\theta \in L^4_{T_1}\dot H^{3/4}$ satisfying $\lim_{n\to\infty}\theta^n = \theta$ in $L^4_{T_1}\dot H^{3/4}$. Furthermore, the uniform estimates (4.8) and (4.9) show that θ also belongs to $L^\infty_{T_1}H^{2-2\alpha} \cap L^2_{T_1}\dot H^{2-\alpha}$. We can also prove the uniqueness by similar arguments as above. Here we can easily check that θ satisfies (DQG$_\alpha$).

We next prove continuity of the solution with values in $H^{2-2\alpha}$. By the standard bootstrap argument, it suffices to show the right continuity at $t = 0$. For the purpose, firstly we prove continuity of the solution with values in H^r for $0 \le r < 1-\alpha$. Indeed, since u and θ satisfy

$$u,\ \theta \in L^2(0, T_1; H^{2-\alpha})$$

and

$$\partial_t\theta = -(-\Delta)^\alpha\theta - u\cdot\nabla\theta,$$

we easily see that the right-hand side of the above identity belongs to $L^1_{T_1}H^r$ for $0 \le r < 1-\alpha$, which yields that $\theta \in C([0, T_1); H^r)$. From the fact that θ belongs to $L^\infty_{T_1}H^{2-2\alpha}$, Lemma 1.4 in [14, Chap. 3] shows that $\theta \in C_w([0, T); H^{2-2\alpha})$. By (4.7), we have

$$\begin{aligned}\|\theta(t) - \theta_0\|^2_{H^{2-2\alpha}} &\le \|\theta(t)\|^2_{H^{2-2\alpha}} + \|\theta_0\|^2_{H^{2-2\alpha}} - 2\langle\theta(t), \theta_0\rangle_{H^{2-2\alpha}}\\ &\le 2\|\theta_0\|^2_{H^{2-2\alpha}} - 2\langle\theta(t), \theta_0\rangle_{H^{2-2\alpha}} + C\|\theta\|^2_{L^2_tH^{2-2\alpha}},\end{aligned}$$

where $\langle\cdot,\cdot\rangle_{H^{2-2\alpha}}$ is the inner product of $H^{2-2\alpha}$. Since θ is weakly continuous, the second term converges to $2\|\theta_0\|^2_{H^{2-2\alpha}}$ as t tends to 0. On the other hand, the third term converges to 0 as t tends to 0 because of absolute continuity of the L^2_t-norm on $t > 0$. This shows continuity of the solution at $t = 0$ with values in $H^{2-2\alpha}$.

Step 3. Weighted estimates. For the proof of (2.1) and (2.2), it suffices to show

$$\lim_{t\to 0}\sup_{n\ge 0} t^{\frac{\beta}{2\alpha}}\|\theta^n(t)\|_{\dot H^{2-2\alpha+\beta}} = 0 \tag{4.11}$$

for $0 < \beta < 2\alpha$. We divide the proof into two cases $0 < \beta < \alpha$ and $\alpha \le \beta < 2\alpha$.

Case 1. We prove (4.11) for $0 < \beta < \alpha$. For $n = 0$ (4.1) shows that

$$\sup_{0<t<T} t^{\frac{\beta}{2\alpha}} \|\theta^0(t)\|_{\dot{H}^{2-2\alpha+\beta}} \leq C\|\theta_0\|_{\dot{H}^{2-2\alpha}}. \tag{4.12}$$

In particular, it follows from (4.2) that $J_1(T) \equiv \sup_{0<t<T} t^{\frac{\beta}{2\alpha}} \|\theta^0(t)\|_{\dot{H}^{2-2\alpha+\beta}}$ satisfies

$$J_1(T) \leq C\|\theta_0\|_{\dot{H}^{2-2\alpha}} \quad \text{and} \quad \lim_{T\to 0} J_1(T) = 0.$$

For $n \geq 0$, θ_j^{n+1} satisfies

$$\frac{d}{dt}\left\|\theta_j^{n+1}\right\|_{L^2} + \lambda 2^{2\alpha j}\left\|\theta_j^{n+1}\right\|_{L^2} \leq \left\|[u^n, \Delta_j]\nabla\theta^{n+1}\right\|_{L^2}. \tag{4.13}$$

Applying Proposition 2 with $s = 2 - 2\alpha + \beta$ and $t = 1 - 2\alpha + \beta$, we have

$$\left\|\left[u^n, \Delta_j\right]\nabla\theta^{n+1}\right\|_{L^2} \leq Cc_j 2^{-(2-4\alpha+2\beta)j}\|\theta^n\|_{\dot{H}^{2-2\alpha+\beta}}\left\|\theta^{n+1}\right\|_{\dot{H}^{2-2\alpha+\beta}}.$$

Hence from (4.13) we obtain

$$\begin{aligned}
&\|\theta_j^{n+1}(t)\|_{L^2} \leq e^{-2^{2\alpha j}\lambda t}\|\theta_j(0)\|_{L^2} \\
&+Cc_j 2^{-(2-4\alpha+2\beta)j}\int_0^t e^{-2^{2\alpha j}\lambda(t-s)}\|\theta^n(s)\|_{\dot{H}^{2-2\alpha+\beta}}\|\theta^{n+1}(s)\|_{\dot{H}^{2-2\alpha+\beta}}ds.
\end{aligned}$$

Similarly to the arguments in Step 1, we have

$$\begin{aligned}
t^{\frac{\beta}{2\alpha}}\|\theta^{n+1}(t)\|_{\dot{H}^{2-2\alpha+\beta}} &\leq t^{\frac{\beta}{2\alpha}}\left(\sum_{j\in\mathbb{Z}}\left(2^{(2-2\alpha+\beta)j}e^{-2^{2\alpha j}\lambda t}\|\theta_j(0)\|_{L^2}\right)^2\right)^{1/2} \\
&+Ct^{\frac{\beta}{2\alpha}}\left(\sum_{j\in\mathbb{Z}}\left(c_j 2^{(2\alpha-\beta)j}\int_0^t e^{-2^{2\alpha j}\lambda(t-s)}\|\theta^n(s)\|_{\dot{H}^{2-2\alpha+\beta}}\|\theta^{n+1}(s)\|_{\dot{H}^{2-2\alpha+\beta}}ds\right)^2\right)^{1/2} \\
&\equiv I + II.
\end{aligned}$$

The first term is estimated as in (4.12). Indeed, Lemma 3 and Proposition 3 yield

$$I \leq \sup_{0<t<T} t^{\frac{\beta}{2\alpha}}\left(\sum_{j\in\mathbb{Z}}\left(2^{(2-2\alpha+\beta)j}e^{-2^{2\alpha j}\lambda t}\|\theta_j(0)\|_{L^2}\right)^2\right)^{1/2} \leq CJ_1(T).$$

Since

$$\sup_{j\in\mathbb{Z}} 2^{(2\alpha-\beta)j}e^{-2^{2\alpha j}\lambda(t-s)} < C(t-s)^{-1+\frac{\beta}{2\alpha}},$$

we can estimate the second term as follows:

$$
\begin{aligned}
II &\le Ct^{\frac{\beta}{2\alpha}} \int_0^t (t-s)^{-1+\frac{\beta}{2\alpha}} \|\theta^n(s)\|_{\dot H^{2-2\alpha+\beta}} \|\theta^{n+1}(s)\|_{\dot H^{2-2\alpha+\beta}} ds \\
&\le C\left(\sup_{0<t<T} t^{\frac{\beta}{2\alpha}} \|\theta^n(t)\|_{\dot H^{2-2\alpha+\beta}}\right)\left(\sup_{0<t<T} t^{\frac{\beta}{2\alpha}} \|\theta^{n+1}(t)\|_{\dot H^{2-2\alpha+\beta}}\right) \\
&\quad \times t^{\frac{\beta}{2\alpha}} \int_0^t (t-s)^{-1+\frac{\beta}{2\alpha}} s^{-\frac{\beta}{\alpha}} ds \\
&\le C\left(\sup_{0<t<T} t^{\frac{\beta}{2\alpha}} \|\theta^n(t)\|_{\dot H^{2-2\alpha+\beta}}\right)\left(\sup_{0<t<T} t^{\frac{\beta}{2\alpha}} \|\theta^{n+1}(t)\|_{\dot H^{2-2\alpha+\beta}}\right)
\end{aligned}
$$

for $0 < t < T$, where we have used the assumption $0 < \beta < \alpha$ in the last line. Thus we have

$$
\begin{aligned}
&\sup_{0<t<T} t^{\frac{\beta}{2\alpha}} \|\theta^{n+1}(t)\|_{\dot H^{2-2\alpha+\beta}} \\
&\le C_3 J_1(T) + C_4 \left(\sup_{0<t<T} t^{\frac{\beta}{2\alpha}} \|\theta^n(t)\|_{\dot H^{2-2\alpha+\beta}}\right)\left(\sup_{0<t<T} t^{\frac{\beta}{2\alpha}} \|\theta^{n+1}(t)\|_{\dot H^{2-2\alpha+\beta}}\right).
\end{aligned}
$$

Taking $T_2 > 0$ sufficiently small, we obtain $J_1(T) < 1/(4C_3C_4)$ for $T < T_2$. Hence we conclude that

$$
\sup_{0<t<T} t^{\frac{\beta}{2\alpha}} \|\theta^n(t)\|_{\dot H^{2-2\alpha+\beta}} \le 2J_1(T) \quad \text{for} \quad T < T_2 \quad \text{and} \quad n = 0, 1, 2 \ldots,
$$

which yields (4.11).

Case 2. We next prove (4.11) for $\alpha \le \beta < 2\alpha$. For $n = 0$, again by Proposition 3, there exists a bounded function $J_2 = J_2(T)$ with

$$
J_2(T) \le C\|\theta_0\|_{\dot H^{2-2\alpha}} \quad \text{and} \quad \lim_{T\to 0} J_2(T) = 0
$$

such that

$$
\sup_{0<t<T} t^{\frac{\beta}{2\alpha}} \|\theta^0(t)\|_{\dot H^{2-2\alpha+\beta}} \le J_2(T). \tag{4.14}
$$

For $n \ge 0$, we apply Proposition 2 with $s = 2 - 3\alpha/2 + \beta/4$ and $s = 1 - 3\alpha/2 + \beta/4$ to the right-hand side of (4.13), and it holds

$$
\frac{d}{dt}\left\|\theta_j^{n+1}\right\|_{L^2} + \lambda 2^{2\alpha j}\left\|\theta_j^{n+1}\right\|_{L^2} \le Cc_j 2^{-(2-3\alpha+\beta/2)j} \|\theta^n\|_{\dot H^{2-3\alpha/2+\beta/4}} \|\theta^{n+1}\|_{\dot H^{2-3\alpha/2+\beta/4}}.
$$

Similarly to the previous arguments, we have

$$
\begin{aligned}
t^{\frac{\beta}{2\alpha}} \|\theta^{n+1}(t)\|_{\dot H^{2-2\alpha+\beta}} &\le t^{\frac{\beta}{2\alpha}} \left(\sum_{j\in\mathbb{Z}} \left(2^{(2-2\alpha+\beta)j} e^{-2^{2\alpha j}\lambda t} \|\theta_j(0)\|_{L^2}\right)^2\right)^{1/2} \\
&+ Ct^{\frac{\beta}{2\alpha}} \left(\sum_{j\in\mathbb{Z}} \left(c_j 2^{(\alpha-\beta/2)j} \int_0^t e^{-2^{2\alpha j}\lambda(t-s)} \|\theta^n(s)\|_{\dot H^{2-3\alpha/2+\beta/4}} \|\theta^{n+1}(s)\|_{\dot H^{2-3\alpha/2+\beta/4}} ds\right)^2\right)^{1/2} \\
&\equiv I + II.
\end{aligned}
$$

The first term I is estimated as in (4.14). So we need to treat only the second term II. Since

$$\sup_{j\in\mathbb{Z}} 2^{(\alpha-\beta/2)j} e^{-2^{2\alpha j}\lambda(t-s)} < C(t-s)^{-\frac{1}{2}-\frac{\beta}{4\alpha}},$$

we have

$$\begin{aligned} II &\le Ct^{\frac{\beta}{2\alpha}} \int_0^t (t-s)^{-\frac{1}{2}-\frac{\beta}{4\alpha}} \|\theta^n(s)\|_{\dot{H}^{2-3\alpha/2+\beta/4}} \|\theta^{n+1}(s)\|_{\dot{H}^{2-3\alpha/2+\beta/4}} ds \\ &\le C \left(\sup_{0<t<T} t^{\frac{1}{4}+\frac{\beta}{8\alpha}} \|\theta^n(t)\|_{\dot{H}^{2-3\alpha/2+\beta/4}} \right) \left(\sup_{0<t<T} t^{\frac{1}{4}+\frac{\beta}{8\alpha}} \|\theta^{n+1}(t)\|_{\dot{H}^{2-3\alpha/2+\beta/4}} \right) \end{aligned}$$

for $0 < t < T$. Since $0 < 1/4 + \beta/(8\alpha) < \alpha$, it follows from the previous case that

$$\sup_{0<t<T} t^{\frac{1}{4}+\frac{\beta}{8\alpha}} \|\theta^n(t)\|_{\dot{H}^{2-3\alpha/2+\beta/4}} \le 2J_1(T) \quad \text{for} \quad T < T_2.$$

Hence the second term is bounded by $4C(J_1(T))^2$ for $T < T_2$.

From the above estimates, we see estimate (4.11) holds for $\alpha \le \beta < 2\alpha$. □

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References

1. Bony, J.-M.: Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. Ann. Sci. École Norm. Sup. **14**, 209–246 (1981)
2. Chae, D., Lee, J.: Global well-posedness in the super-critical dissipative quasi-geostrophic equations. Commun. Math. Phys. **233**, 297–311 (2003)
3. Chemin, Y.: Théorèmes d'unicité pour le système de Navier-Stokes tridimensionnel. J. Anal. Math. **77**, 27–50 (1999)
4. Constantin, P., Cordoba, D., Wu, J.: On the critical dissipative quasi-geostrophic equation. Indiana Univ. Math. J. **50**, 97–107 (2001)
5. Constantin, P., Wu, J.: Behavior of solutions of 2D quasi-geostrophic equations. SIAM J. Math. Anal. **30**, 937–948 (1999)
6. Cordoba, A., Cordoba, D.: A maximum principle applied to quasi-geostrophic equations. Commun. Math. Phys. **249**, 511–528 (2004)
7. Danchin, R.: Local theory in critical spaces for compressible viscous and heat-conductive gases. Comm. Partial Differ. Eq. **26**, 1183–1233 (2001)
8. Fujita, H., Kato, T.: On the Navier-Stokes initial value problem I. Arch. Rat. Mech. Anal. **16**, 269–315 (1964)
9. Ju, N.: Existence and uniqueness of the solution to the dissipative 2D quasi-geostrophic equations in the Sobolev space. Commun. Math. Phys. **251**, 365–376 (2004)
10. Ju, N.: On the two dimensional quasi-geostrophic equations. Indiana Univ. Math. J. **54**, 897–926 (2005)
11. Koch, H., Tataru, D.: Well-posedness for the Navier-Stokes equations. Adv. Math. **157**, 22–35 (2001)
12. Kozono, H., Yamazaki, M.: Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data. Comm. Partial Differ. Eq. **19**, 959–1014 (1994)
13. Runst, T., Sickel, W.: Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations. de Gruyter Series in Nonlinear Analysis and Applications **3**, Berlin: Walter de Gruyter & Co., 1996
14. Temam, R: Navier-Stokes equations. Theory and numerical analysis. Providence, RI: AMS Chelsea Publishing, 2001
15. Wu, J.: Dissipative quasi-geostrophic equations with L^p data. Electron. J. Differ. Eq. **56**, 1–13 (2001)

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