A Generalization of Hawking's Black Hole Topology Theorem to Higher Dimensions

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Abstract: Hawking's theorem on the topology of black holes asserts that cross sections of the event horizon in 4-dimensional asymptotically flat stationary black hole spacetimes obeying the dominant energy condition are topologically 2-spheres. This conclusion extends to outer apparent horizons in spacetimes that are not necessarily stationary. In this paper we obtain a natural generalization of Hawking's results to higher dimensions by showing that cross sections of the event horizon (in the stationary case) and outer apparent horizons (in the general case) are of positive Yamabe type, i.e., admit metrics of positive scalar curvature. This implies many well-known restrictions on the topology, and is consistent with recent examples of five dimensional stationary black hole spacetimes with horizon topology $S^2 \times S^1$. The proof is inspired by previous work of Schoen and Yau on the existence of solutions to the Jang equation (but does not make direct use of that equation).

1. Introduction

A basic result in the theory of black holes is Hawking's theorem [11, 13] on the topology of black holes, which asserts that cross sections of the event horizon in 4-dimensional asymptotically flat stationary black hole spacetimes obeying the dominant energy condition are spherical (i.e., topologically S^2). The proof is a beautiful variational argument, showing that if a cross section has genus ≥ 1 then it can be deformed along a null hypersurface to an outer trapped surface outside of the event horizon, which is forbidden by standard results on black holes [13].[1] In [12], Hawking showed that his black hole topology result extends, by a similar argument, to outer apparent horizons in black hole spacetimes that are not necessarily stationary. (A related result had been shown by Gibbons [8] in the time-symmetric case.) Since Hawking's arguments rely on the Gauss-Bonnet theorem, these results do not directly extend to higher dimensions.

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[1] Actually the torus T^2 arises as a borderline case in Hawking's argument, but can occur only under special circumstances.

Given the current interest in higher dimensional black holes, it is of interest to determine which properties of black holes in four spacetime dimensions extend to higher dimensions. In this note we obtain a natural generalization of Hawking's theorem on the topology of black holes to higher dimensions. The conclusion in higher dimensions is not that the horizon topology is spherical; that would be too strong, as evidenced by the striking example of Emparan and Reall [7] of a stationary vacuum black hole spacetime in five dimensions with horizon topology $S^2 \times S^1$. The natural conclusion in higher dimensions is that cross sections of the event horizon (in the stationary case), and outer apparent horizons (in the general case) are of positive Yamabe type, i.e. admit metrics of positive scalar curvature. As noted in [6], in the time symmetric case this follows from the minimal surface methodology of Schoen and Yau [18] in their treatment of manifolds of positive scalar curvature. The main point of the present paper is to show that this conclusion remains valid without any condition on the extrinsic curvature of space. That such a result might be expected to hold is suggested by work in [19, Sect. 4], which implies that the apparent horizons corresponding to the blow-up of solutions of the Jang equation, as described in [19], are of positive Yamabe type. We emphasize, however, that we do not need to make use of the Jang equation here.[2]

Much is now known about the topological obstructions to the existence of metrics of positive scalar curvature in higher dimensions. While the first major result along these lines is the famous theorem of Lichnerowicz [16] concerning the vanishing of the $\hat{A}$ genus, a key advance in our understanding was made in the late 70's and early 80's by Schoen and Yau [17, 18], and Gromov and Lawson [9, 10]. A brief review of these results, relevant to the topology of black holes, was considered in [6]. We shall recall the situation in five spacetime dimensions in the next section, after the statement of our main result.

2. The Main Result

Let V^n be an n-dimensional, $n \geq 3$, spacelike hypersurface in a spacetime (M^{n+1}, g). Let Σ^{n-1} be a closed hypersurface in V^n, and assume that Σ^{n-1} separates V^n into an "inside" and an "outside". Let N be the outward unit normal to Σ^{n-1} in V^n, and let U be the future directed unit normal to V^n in M^{n+1}. Then $K = U + N$ is an outward null normal field to Σ^{n-1}, unique up to scaling.

The null second fundamental form of Σ with respect to K is, for each $p \in \Sigma$, the bilinear form defined by,

$$\chi : T_p \Sigma \times T_p \Sigma \to \mathbb{R}, \qquad \chi(X, Y) = \langle \nabla_X K, Y \rangle, \tag{2.1}$$

where $\langle \,, \rangle = g$ and ∇ is the Levi-Civita connection, of M^{n+1}. Then the null expansion of Σ is defined as $\theta = \mathrm{tr}\, \chi = h^{AB} \chi_{AB} = \mathrm{div}\,_\Sigma K$, where h is the induced metric on Σ.

We shall say Σ^{n-1} is an *outer apparent horizon* in V^n provided, (i) Σ is marginally outer trapped, i.e., $\theta = 0$, and (ii) there are no outer trapped surfaces outside of Σ. The latter means that there is no $(n - 1)$-surface Σ' contained in the region of V^n outside of Σ which is homologous to Σ and which has negative expansion $\theta < 0$ with respect to its outer null normal (relative to Σ). Heuristically, Σ is the "outer limit" of outer trapped surfaces in V.

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[2] In any case, the parametric estimates of [19] which are used to construct solutions of the Jang equation asymptotic to vertical cylinders over apparent horizons are generally true only in low dimensions.

Finally, a spacetime (M^{n+1}, g) satisfying the Einstein equations

$$R_{ab} - \frac{1}{2}Rg_{ab} = T_{ab} \tag{2.2}$$

is said to obey the dominant energy condition provided the energy-momentum tensor $\mathcal{T}$ satisfies $\mathcal{T}(X, Y) = T_{ab}X^aY^b \geq 0$ for all future pointing causal vectors X, Y.

We are now ready to state the main theorem.

Theorem 2.1. *Let (M^{n+1}, g), $n \geq 3$, be a spacetime satisfying the dominant energy condition. If Σ^{n-1} is an outer apparent horizon in V^n then Σ^{n-1} is of positive Yamabe type, unless Σ^{n-1} is Ricci flat (flat if $n = 3, 4$) in the induced metric, and both χ and $\mathcal{T}(U, K) = T_{ab}U^aK^b$ vanish on Σ.*

Thus, except under special circumstances, Σ^{n-1} is of positive Yamabe type. As noted in the introduction, this implies various restrictions on the topology of Σ. Let us focus on the case dim $M = 5$, and hence dim $\Sigma = 3$, and assume, by taking a double cover if necessary, that Σ is orientable. Then by well-known results of Schoen-Yau [18] and Gromov-Lawson [10], topologically, Σ must be a finite connected sum of spherical spaces (homotopy 3-spheres, perhaps with identifications) and $S^2 \times S^1$'s. Indeed, by the prime decomposition theorem, Σ can be expressed as a connected sum of spherical spaces, $S^2 \times S^1$'s, and $K(\pi, 1)$ manifolds (manifolds whose universal covers are contractible). But as Σ admits a metric of positive scalar curvature, it cannot have any $K(\pi, 1)$'s in its prime decomposition. Thus, the basic horizon topologies in dim $M = 5$ are S^3 and $S^2 \times S^1$, both of which are realized by nontrivial black hole spacetimes. Under stringent geometric assumptions on the horizon, a related conclusion is arrived at in [14].

Proof of the theorem. We consider normal variations of Σ in V, i.e., variations $t \to \Sigma_t$ of $\Sigma = \Sigma_0$, $-\epsilon < t < \epsilon$, with variation vector field $V = \frac{\partial}{\partial t}\big|_{t=0} = \phi N, \phi \in C^\infty(\Sigma)$. Let $\theta(t)$ denote the null expansion of Σ_t with respect to $K_t = U + N_t$, where N_t is the outer unit normal field to Σ_t in V. A computation shows [6, 3],

$$\frac{\partial \theta}{\partial t}\Big|_{t=0} = -\triangle\phi + 2\langle X, \nabla\phi \rangle + \left(Q + \text{div } X - |X|^2\right)\phi, \tag{2.3}$$

where,

$$Q = \frac{1}{2}S - \mathcal{T}(U, K) - \frac{1}{2}|\chi|^2, \tag{2.4}$$

S is the scalar curvature of Σ, X is the vector field on Σ defined by $X = \tan(\nabla_N U)$, and $\langle , \rangle$ now denotes the induced metric h on Σ.

Introducing as in [3] the operator $L = -\triangle + \langle X, \nabla() \rangle + (Q + \text{div } X - |X|^2)$, Eq. (2.3) may be expressed as,

$$\frac{\partial \theta}{\partial t}\Big|_{t=0} = L(\phi). \tag{2.5}$$

L is the stability operator associated with variations in the null expansion θ. In the time symmetric case the vector field X vanishes, and L reduces to the classical stability operator of minimal surface theory, as expected [6].

As discussed in [3], although L is not in general self adjoint, its principal eigenvalue λ_1 is real, and one can choose a principal eigenfunction ϕ which is strictly positive, $\phi > 0$. Using the eigenfunction ϕ to define our variation, we have from (2.5),

$$\frac{\partial \theta}{\partial t}\bigg|_{t=0} = \lambda_1 \phi\,. \tag{2.6}$$

The eigenvalue λ_1 cannot be negative, for otherwise (2.6) would imply that $\frac{\partial \theta}{\partial t} < 0$ on Σ. Since $\theta = 0$ on Σ, this would mean that for $t > 0$ sufficiently small, Σ_t would be outer trapped, contrary to our assumptions.

Hence, $\lambda_1 \geq 0$, and we conclude for the variation determined by the positive eigenfunction ϕ that $\frac{\partial \theta}{\partial t}\big|_{t=0} \geq 0$. By completing the square on the right hand side of Eq. (2.3), this implies that the following inequality holds:

$$-\triangle \phi + (Q + \mathrm{div}\, X)\,\phi + \phi |\nabla \ln \phi|^2 - \phi |X - \nabla \ln \phi|^2 \geq 0. \tag{2.7}$$

Setting $u = \ln \phi$, we obtain,

$$-\triangle u + Q + \mathrm{div}\, X - |X - \nabla u|^2 \geq 0\,. \tag{2.8}$$

As a side remark, note that substituting the expression for Q into (2.8) and integrating gives that the total scalar curvature of Σ is nonnegative, and in fact is positive, except under special circumstances. In four spacetime dimensions one may then apply the Gauss-Bonnet theorem to recover Hawking's theorem; in fact this is essentially Hawking's original argument. However, in higher dimensions the positivity of the total scalar curvature, in and of itself, does not provide any topological information.

To proceed with the higher dimensional case, we first absorb the Laplacian term $\triangle u = \mathrm{div}\,(\nabla u)$ in (2.8) into the divergence term to obtain,

$$Q + \mathrm{div}\,(X - \nabla u) - |X - \nabla u|^2 \geq 0\,. \tag{2.9}$$

Setting $Y = X - \nabla u$, we arrive at the inequality,

$$-Q + |Y|^2 \leq \mathrm{div}\, Y\,. \tag{2.10}$$

Given any $\psi \in C^\infty(\Sigma)$, we multiply through by ψ^2 and derive,

$$\begin{aligned}
-\psi^2 Q + \psi^2 |Y|^2 &\leq \psi^2 \mathrm{div}\, Y \\
&= \mathrm{div}\left(\psi^2 Y\right) - 2\psi \langle \nabla \psi, Y \rangle \\
&\leq \mathrm{div}\left(\psi^2 Y\right) + 2|\psi| |\nabla \psi| |Y| \\
&\leq \mathrm{div}\left(\psi^2 Y\right) + |\nabla \psi|^2 + \psi^2 |Y|^2\,.
\end{aligned} \tag{2.11}$$

Integrating the above inequality yields,

$$\int_\Sigma |\nabla \psi|^2 + Q\psi^2 \geq 0 \quad \text{for all } \psi \in C^\infty(\Sigma)\,, \tag{2.12}$$

where Q is given in (2.4).

At this point rather standard arguments become applicable [19, 6]. Consider the eigenvalue problem,

$$-\triangle\psi + Q\psi = \mu\psi . \qquad (2.13)$$

Inequality (2.12) implies that the first eigenvalue μ_1 of (2.13) is nonnegative, $\mu_1 \geq 0$. Let $f \in C^\infty(\Sigma)$ be an associated eigenfunction; f can be chosen to be strictly positive.

Now consider Σ in the conformally related metric $\tilde{h} = f^{2/n-2}h$. The scalar curvature $\tilde{S}$ of Σ in the metric $\tilde{h}$ is given by,

$$\tilde{S} = f^{-n/(n-2)}\left(-2\Delta f + Sf + \frac{n-1}{n-2}\frac{|\nabla f|^2}{f}\right)$$

$$= f^{-2/(n-2)}\left(2\mu_1 + 2\mathcal{T}(U, K) + |\chi|^2 + \frac{n-1}{n-2}\frac{|\nabla f|^2}{f^2}\right), \qquad (2.14)$$

where, for the second equation, we have used (2.13), with $\psi = f$, and (2.4).

Since, by the dominant energy condition, $\mathcal{T}(U, K) \geq 0$, Eq. (2.14) implies that $\tilde{S} \geq 0$. If $\tilde{S} > 0$ at some point, then by well known results [15] one can conformally change $\tilde{h}$ to a metric of strictly positive scalar curvature, and the theorem follows. If $\tilde{S}$ vanishes identically then, by Eq. (2.14), $\mu_1 = 0, \mathcal{T}(U, K) \equiv 0, \chi \equiv 0$ and f is constant. Eq. (2.13), with $\psi = f$ and Eq. (2.4) then imply that $S \equiv 0$. By a result of Bourguinon (see [15]), it follows that Σ carries a metric of positive scalar curvature unless it is Ricci flat. The theorem now follows. $\square$

Concluding Remarks

1. Let Σ^{n-1} be a closed 2-sided hypersurface in the spacelike hypersurface $V^n \subset M^{n+1}$. Then there exists a neighborhood W of Σ^{n-1} in V^n such that Σ separates W into an "inside" and an "outside". Suppose Σ is marginally outer trapped, i.e., $\theta = 0$ with respect to the outer null normal to Σ. Following the terminology introduced in [3], we say that Σ is stably outermost (respectively, strictly stably outermost) provided the principal eigenvalue λ_1 of the stability operator L introduced in 2.5 satisfies $\lambda_1 \geq 0$ (resp., $\lambda_1 > 0$). It is clear from the proof that the conclusion of Theorem 2.1 remains valid for marginally outer trapped surfaces Σ that are stably outermost. Moreover the conclusion that Σ is positive Yamabe holds *without any caveat* if Σ is strictly stably outermost. To see this, note that Eq. (2.6) then implies that there exists $\epsilon > 0$ such that $\frac{\partial\theta}{\partial t}\big|_{t=0} \geq \epsilon$. Tracing through the proof using this inequality shows that (2.12) holds with Q replaced by $Q - \epsilon$. Then the parenthetical expression in Eq. (2.14) will include a $+\epsilon$ term, and so $\tilde{S}$ will be strictly positive.
2. Theorem 2.1 applies, in particular, to the marginally trapped surfaces S_R of a dynamical horizon $\mathcal{H}$ (see [2] for definitions). Indeed, by the maximum principle for marginally trapped surfaces [1], there can be no outer trapped surfaces in $\mathcal{H}$ outside of any S_R. Alternatively, it is easily checked that each S_R is stably outermost in the sense described in the previous paragraph.
3. As discussed in [6], the exceptional case in Theorem 2.1 can in effect be eliminated in the time symmetric case. In this case V^n becomes a manifold of nonnegative scalar curvature, and Σ^{n-1} is minimal. By the results in [5, 4], if Σ is locally outer area minimizing and does not carry a metric of positive scalar curvature then an outer neighborhood of Σ in V splits isometrically as a product $[0, \epsilon) \times \Sigma$. In physical terms, this means that there would be marginally outer trapped surfaces outside of Σ,

which, by a slight strengthening of our definition of 'outer apparent horizon', could not occur. (In fact, marginally outer trapped surfaces cannot occur outside the event horizon.) Under mild physical assumptions, but with dim $M \leq 8$, one can show that Σ is locally outer area minimizing; see [6, Theorem 3] for further discussion. Finally, in the asymptotically flat, but not necessarily time symmetric case, it is possible to perturb the initial data to make the dominant energy inequality strict, see [19, p. 240]. Hence, the exceptional case is unstable in this sense.

Note added in proof: We are now able to eliminate the exceptional case in the general non-time symmetric setting under conditions analogous to those referred to in remark 3 above, e.g. assuming a mild asymptotic condition and assuming there are no outer trapped or marginally outer trapped surfaces outside of Σ. Details will appear in a forthcoming paper.

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