Commun. Math. Phys. 266, 197–210 (2006)
Digital Object Identifier (DOI) 10.1007/s00220-006-0018-0

Communications in
Mathematical Physics

On the Conserved Quantities for the Weak Solutions of the Euler Equations and the Quasi-geostrophic Equations

Dongho Chae

Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Korea. E-mail: chae@skku.edu

Received: 2 August 2005 / Accepted: 28 October 2005
Published online: 22 April 2006 –

Abstract: In this paper we obtain sufficient conditions on the regularity of the weak solutions to guarantee conservation of the energy and the helicity for the incompressible Euler equations. The regularity of the weak solutions are measured in terms of the Triebel-Lizorkin type of norms, $\dot{\mathcal{F}}^s_{p,q}$ and the Besov norms, $\dot{\mathcal{B}}^s_{p,q}$. In particular, in the Besov space case, our results refine the previous ones due to Constantin-E-Titi (energy) and the author of this paper (helicity), where the regularity is measured by a special class of the Besov space norm $\dot{\mathcal{B}}^s_{p,\infty} = \dot{\mathcal{N}}^s_p$, which is the Nikolskii space. We also obtain a sufficient regularity condition for the conservation of the L^p-norm of the temperature function in the weak solutions of the quasi-geostrophic equation.

1. Introduction and the Main Results

The Euler equations for the homogeneous incompressible fluid flows in $\mathbb{R}^n$, $n = 2, 3$, are

$$(E)\left\{\begin{array}{l} \frac{\partial v}{\partial t} + (v\cdot\nabla)v = -\nabla p, \quad (x,t)\in\mathbb{R}^n\times(0,T) \\ \mathrm{div}\ v = 0, \\ v(x,0) = v_0(x), \end{array}\right.$$

where $v = (v_1, \cdots, v_n)$, $v_j = v_j(x,t)$, $j = 1, \cdots, n$, is the velocity of the fluid flows, $p = p(x,t)$ is the scalar pressure, and v_0 is the given initial velocity satisfying div $v_0 = 0$. It is well-known that for smooth solutions of the Euler equations the energy $E(t) = \frac{1}{2}\int_{\mathbb{R}^n} |v(x,t)|^2 dx$ is preserved in time. For nonsmooth(weak) solutions it is not at all obvious that we still have energy conservation. Thus, there comes the very interesting question of how much smoothness we need to assume for the solution to have energy conservation property. Regarding this question L. Onsager conjectured that a Hölder continuous weak solution with the Hölder exponent $1/3$ preserve the energy, and this is sharp. Considering Kolmogorov's scaling argument on the energy correlation in the homogeneous turbulence the exponent $1/3$ is natural. The sufficiency part of this conjecture is proved in a positive direction by a simple but very elegant argument by Constantin-E-Titi[5], using the Besov space norm, $\dot{\mathcal{B}}^s_{3,\infty}$ with $s > 1/3$ (see below for precise definitions of the function spaces) for the velocity. Remarkably enough Shnirelman[13] later constructed an example of weak solution of 3D Euler equations, which does not preserve energy. The problem of finding the optimal regularity condition for a weak solution to have conservation property can also be considered for the helicity, which is defined by $H(t) = \int_{\mathbb{R}^n} v(x,t) \cdot \omega(x,t)dx$, where $\omega = \text{curl } v$ is the vorticity. In particular, the helicity is closely related to the topological invariants, e.g. the knottedness of vortex tubes (see [1] for the details and other significance of the helicity conservation). Thus, in [2] the author of this paper obtained a sufficient regularity condition for the helicity conservation, using the function space $\dot{\mathcal{B}}^s_{\frac{9}{5},\infty}$, $s > 1/3$, for the vorticity. One of the purposes of this paper is to refine those results, using the Triebel-Lizorkin type of spaces, $\dot{\mathcal{F}}^s_{p,q}$, and the Besov spaces $\dot{\mathcal{B}}^s_{p,q}$ with similar values for s, p, but allowing full range of values for $q \in [1,\infty]$(for a precise statement of the results see the theorems below). When we restrict $q = \infty$, our Besov space results for Euler equations reduce to the previous ones described above. On the other hand, our results for Triebel-Lizorkin type of space are completely new. We also extend our arguments to consider the L^p-norm conservation in the weak solutions of the 2D quasi-geostrophic equations.

By a weak solution of (E) in $\mathbb{R}^n \times (0,T)$ with initial data v_0 we mean a vector field $v \in C([0,T); L^2_{loc}(\mathbb{R}^n))$ satisfying the integral identity:

$$\begin{aligned}
&-\int_0^T \int_{\mathbb{R}^n} v(x,t) \cdot \frac{\partial \phi(x,t)}{\partial t} dxdt - \int_{\mathbb{R}^n} v_0(x) \cdot \phi(x,0)dx \\
&-\int_0^T \int_{\mathbb{R}^n} v(x,t) \otimes v(x,t) : \nabla \phi(x,t)dxdt \\
&-\int_0^T \int_{\mathbb{R}^n} \text{div}\, \phi(x,t) p(x,t)dxdt = 0, \qquad (1.1)
\end{aligned}$$

$$\int_0^T \int_{\mathbb{R}^n} v(x,t) \cdot \nabla \psi(x,t)dxdt = 0 \qquad (1.2)$$

for every vector test function $\phi = (\phi_1, \dots, \phi_n) \in C_0^\infty(\mathbb{R}^n \times [0,T))$, and for every scalar test function $\psi \in C_0^\infty(\mathbb{R}^n \times [0,T))$. Here we used the notation $(u \otimes v)_{ij} = u_i v_j$, and $A : B = \sum_{i,j=1}^n A_{ij} B_{ij}$ for $n \times n$ matrices A and B. In the case when we discuss the helicity conservation of the weak solution we impose further regularity for the vorticity, $\omega(\cdot,t) \in L^{\frac{3}{2}}(\mathbb{R}^3)$ for almost every $t \in [0,T]$ in order to define the helicity for such a weak solution. In order to state our main theorems we introduce function spaces. Given $0 < s < 1$, $1 \le p \le \infty, 1 \le q \le \infty$, the function space $\dot{\mathcal{F}}^s_{p,q}$ is defined by the seminorm,

$$\|f\|_{\dot{\mathcal{F}}^s_{p,q}} = \begin{cases} \left\| \left(\int_{\mathbb{R}^n} \frac{|f(x) - f(x-y)|^q}{|y|^{n+sq}} dy \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n,dx)} & \text{if } 1 \le p \le \infty, 1 \le q < \infty \\ \left\| ess \sup_{|y|>0} \frac{|f(x) - f(x-y)|}{|y|^s} \right\|_{L^p(\mathbb{R}^n,dx)} & \text{if } 1 \le p \le \infty, q = \infty \end{cases}.$$

On the other hand, the space $\dot{\mathcal{B}}^s_{p,q}$ is defined by the seminorm,

$$\|f\|_{\dot{\mathcal{B}}^s_{p,q}} = \begin{cases} \left(\int_{\mathbb{R}^n} \frac{\|f(\cdot)-f(\cdot-y)\|^q_{L^p}}{|y|^{n+sq}} dy\right)^{\frac{1}{q}} & \text{if } 1 \le p \le \infty, 1 \le q < \infty \\ ess\sup_{|y|>0} \frac{\|f(\cdot)-f(\cdot-y)\|_{L^p}}{|y|^s} & \text{if } 1 \le p \le \infty, q = \infty \end{cases}.$$

Observe that, in particular, $\dot{\mathcal{F}}^s_{\infty,\infty} = \dot{\mathcal{B}}^s_{\infty,\infty} = C^s$, which is the usual Hölder seminormed space. We also note that when $q = \infty$ we have the equivalence, $\dot{\mathcal{B}}^s_{p,\infty} = \dot{\mathcal{N}}^s_p$, which is the Nikolskii space, used in [5] and [2].

In order to compare this space with other more classical function spaces let us introduce the Banach space $\mathcal{F}^s_{p,q}$, $\mathcal{B}^s_{p,q}$ by defining its norm,

$$\|f\|_{\mathcal{F}^s_{p,q}} = \|f\|_{L^p} + \|f\|_{\dot{\mathcal{F}}^s_{p,q}}, \quad \|f\|_{\mathcal{B}^s_{p,q}} = \|f\|_{L^p} + \|f\|_{\dot{\mathcal{B}}^s_{p,q}},$$

respectively. We note that for $0 < s < 1$, $2 \le p < \infty$, $q = 2$, $\mathcal{F}^s_{p,2} \cong L^p_s(\mathbb{R}^n) = (1-\Delta)^{-\frac{s}{2}} L^p(\mathbb{R}^n)$, the fractional order Sobolev space (or the Bessel potential space)(see p. 163,[14]). If $\frac{n}{\min\{p,q\}} < s < 1$, $n < p < \infty$ and $n < q \le \infty$, then $\mathcal{F}^s_{p,q}$ coincides with the Triebel-Lizorkin space $F^s_{p,q}(\mathbb{R}^n)$ defined by the Littlewood-Paley decomposition(see p. 101, [15]). On the other hand, for wider range of parameters, $0 < s < 1$, $0 < p \le \infty$, $0 < q \le \infty$, $\mathcal{B}^s_{p,q}$ coincides with the Besov space $B^s_{p,q}(\mathbb{R}^n)$(see p. 110, [15]). We also note the equivalence,

$$\dot{\mathcal{F}}^s_{p,p} = \dot{\mathcal{B}}^s_{p,p}, \quad \mathcal{F}^s_{p,p} = \mathcal{B}^s_{p,p}$$

for $1 \le p \le \infty$. *Hereafter, we use the notation $\dot{X}^s_{p,q}$ (resp. $X^s_{p,q}$) to represent $\dot{\mathcal{F}}^s_{p,q}$(resp. $\mathcal{F}^s_{p,q}$) or $\dot{\mathcal{B}}^s_{p,q}$(resp. $\mathcal{B}^s_{p,q}$).*

Theorem 1.1. *Let $s > \frac{1}{3}$ and $q \in [2, \infty]$ be given. Suppose v is a weak solution of the $n-$dimensional Euler equations with $v \in C([0,T]; L^2(\mathbb{R}^n)) \cap L^3(0,T; \dot{X}^s_{3,q}(\mathbb{R}^n))$. Then, the energy is preserved in time, namely*

$$\int_{\mathbb{R}^n} |v(x,t)|^2 dx = \int_{\mathbb{R}^n} |v_0(x)|^2 dx \tag{1.3}$$

for all $t \in [0, T)$.

Theorem 1.2. *Let $s > \frac{1}{3}$, $q \in [2, \infty]$, and $r_1 \in [2, \infty]$, $r_2 \in [1, \infty]$ be given, satisfying $2/r_1 + 1/r_2 = 1$. Suppose v is a weak solution of the 3-D Euler equations with $v \in C([0,T]; L^2(\mathbb{R}^3)) \cap L^{r_1}(0,T; \dot{X}^s_{\frac{9}{2},q}(\mathbb{R}^3))$ and $\omega \in L^{r_2}(0,T; \dot{X}^s_{\frac{9}{5},q}(\mathbb{R}^3))$, where the curl operation is in the sense of distribution. Then, the helicity is preserved in time, namely*

$$\int_{\mathbb{R}^3} v(x,t) \cdot \omega(x,t) dx = \int_{\mathbb{R}^3} v_0(x) \cdot \omega_0(x) dx \tag{1.4}$$

for all $t \in [0, T)$.

Similarly to [2], as an application of the above theorem we have the following estimate from below of the vorticity by a constant depending on the initial data for the weak solutions of the 3-D Euler equations.

Corollary 1.1. *Suppose v is a weak solution of the 3-D Euler equations satisfying the conditions of Theorem 1.2. Then, we have the following estimate:*

$$\|\omega(\cdot,t)\|^2_{L^{\frac{3}{2}}} \geq CH_0, \quad \forall t \in [0,T), \tag{1.5}$$

where $H_0 = \int_{\mathbb{R}^n} v_0(x)\cdot\omega_0(x)dx$ is the initial helicity, and C is an absolute constant.

Next we are concerned with the L^p-norm conservation for the weak solutions of the 2D quasi-geostrophic equation,

$$(QG)\begin{cases} \frac{\partial\theta}{\partial t} + (v\cdot\nabla)\theta = 0, \\ v(x,t) = -\nabla^\perp \int_{\mathbb{R}^2} \frac{\theta(y,t)}{|x-y|}dy, \\ \theta(x,0) = \theta_0(x), \end{cases}$$

where $\theta(x,t)$ is a scalar function representing the temperature, $v(x,t)$ is the velocity field of the fluid, and $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$. The system (QG) is of intensive interest recently (see e.g. [4, 6, 16, 7, 8, 3], and references therein), since the equation has very similar structure to the 3-D Euler equations, and also it has direct connections to the physical phenomena in atmospheric science.

Let $p \in [2,\infty)$. By a weak solution of (QG) in $D\times(0,T)$ with initial data v_0 we mean a scalar field $\theta \in C([0,T); L^p(\mathbb{R}^2)\cap L^{\frac{p}{p-1}}(\mathbb{R}^2))$ satisfying the integral identity:

$$-\int_0^T\int_{\mathbb{R}^2}\theta(x,t)\left[\frac{\partial}{\partial t} + v\cdot\nabla\right]\phi(x,t)dxdt - \int_{\mathbb{R}^2}\theta_0(x)\phi(x,0)dx = 0 \tag{1.6}$$

$$v(x,t) = -\nabla^\perp\int_{\mathbb{R}^2}\frac{\theta(y,t)}{|x-y|}dy \tag{1.7}$$

for every test function $\phi \in C_0^\infty(\mathbb{R}^2\times[0,T))$, where $\nabla^\perp$ in (1.7) is in the sense of distribution. We note that contrary to the case of 3-D Euler equations there is a global existence result for the weak solutions of (QG) for $p = 2$([12]).

Theorem 1.3. *Let $s > \frac{1}{3}$, $p \in [2,\infty)$, $q \in [1,\infty]$, and $r_1 \in [p,\infty], r_2 \in [1,\infty]$ be given, satisfying $p/r_1 + 1/r_2 = 1$. Suppose θ is a weak solution of (QG) with $\theta \in C([0,T]; L^p(\mathbb{R}^2)\cap L^{\frac{p}{p-1}}(\mathbb{R}^2))\cap L^{r_1}(0,T; X^s_{p+1,q}(\mathbb{R}^2))$ and $v \in L^{r_2}(0,T;\dot{X}^s_{p+1,q}(\mathbb{R}^2))$. Then, the L^p norm of $\theta(\cdot,t)$ is preserved,*

$$\|\theta(t)\|_{L^p} = \|\theta_0\|_{L^p} \tag{1.8}$$

for all $t \in [0,T]$.

2. Proof of the Main Theorems

Let $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$ be the standard mollifier with $\varphi \geq 0$, $\operatorname{supp}\varphi \subset \{x\in\mathbb{R}^n \mid |x|\leq 1\}$. Let $\varphi^\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi(\frac{x}{\varepsilon})$. Given $f \in L^1_{loc}(\mathbb{R}^n)$, we denote by $f^\varepsilon(x) = (f*\varphi^\varepsilon)(x)$.

Lemma 2.1. *Let $k \in \mathbb{N}$, $s \in (0,1)$ and $p,q \in [1,\infty]$. Then, there exist constants C depending on k,s,q,n such that the following inequalities hold:*

$$\|D^k f^\varepsilon\|_{L^p} \leq C\varepsilon^{s-k}\|f\|_{\dot{X}^s_{p,q}}, \tag{2.1}$$

$$\|f - f^\varepsilon\|_{L^p} \leq C\varepsilon^s\|f\|_{\dot{X}^s_{p,q}}. \tag{2.2}$$

Proof. By integration by part we deduce

$$
\begin{aligned}
D^k f^\varepsilon(x) &= \int_{\mathbb{R}^n} D_x^k f(x-y)\varphi^\varepsilon(y)dy = (-1)^k \int_{\mathbb{R}^n} D_y^k f(x-y)\varphi^\varepsilon(y)dy \\
&= \int_{\mathbb{R}^n} f(x-y) D_y^k \varphi^\varepsilon(y)dy = \frac{1}{\varepsilon^{n+k}} \int_{\mathbb{R}^n} f(x-y)(D^k\varphi)\left(\frac{y}{\varepsilon}\right) dy \\
&= \frac{1}{\varepsilon^{n+k}} \int_{\mathbb{R}^n} [f(x-y) - f(x)](D^k\varphi)\left(\frac{y}{\varepsilon}\right) dy,
\end{aligned}
$$

where we used the fact

$$
\int_{\mathbb{R}^n} f(x)(D^k\varphi)\left(\frac{y}{\varepsilon}\right) dy = f(x)\int_{\mathbb{R}^n} (D^k\varphi)\left(\frac{y}{\varepsilon}\right) dy = 0.
$$

Hence,

$$
\begin{aligned}
|D^k f^\varepsilon(x)| &\le \frac{1}{\varepsilon^{n+k}} \int_{\mathbb{R}^n} |f(x-y)-f(x)| \left|(D^k\varphi)\left(\frac{y}{\varepsilon}\right)\right| dy \\
&\le \frac{1}{\varepsilon^{n+k}} \left(\int_{\mathbb{R}^n} \frac{|f(x-y)-f(x)|^q}{|y|^{n+sq}} dy\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^n} \left|D^k\varphi\left(\frac{y}{\varepsilon}\right)\right|^{q'} |y|^{(\frac{n}{q}+s)q'} dy\right)^{\frac{1}{q'}} \\
&= \varepsilon^{s-k} \left(\int_{\mathbb{R}^n} \frac{|f(x-y)-f(x)|^q}{|y|^{n+sq}} dy\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^n} |D^k\varphi(y)|^{q'} |y|^{(\frac{n}{q}+s)q'} dy\right)^{\frac{1}{q'}} \\
&= C\varepsilon^{s-k} \left(\int_{\mathbb{R}^n} \frac{|f(x-y)-f(x)|^q}{|y|^{n+sq}} dy\right)^{\frac{1}{q}}, \qquad (2.3)
\end{aligned}
$$

where $1/q + 1/q' = 1$. Taking $L^p(dx)$ norm of (2.3), we obtain (2.1) with $\dot{X}^s_{p,q} = \dot{\mathcal{F}}^s_{p,q}$. In order to have the corresponding inequality for the norm of $\dot{\mathcal{B}}^s_{p,q}$, we use the Minkowski inequality and the Hölder inequality to estimate

$$
\begin{aligned}
\|D^k f^\varepsilon\|_{L^p} &\le \frac{1}{\varepsilon^{n+k}} \left\{\int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |f(x-y) - f(x)| \left|(D^k\varphi)\left(\frac{y}{\varepsilon}\right)\right| dy\right]^p dx\right\}^{\frac{1}{p}} \\
&\le \frac{1}{\varepsilon^{n+k}} \int_{\mathbb{R}^n} \|f(\cdot-y)-f(\cdot)\|_{L^p} \left|D^k\varphi\left(\frac{y}{\varepsilon}\right)\right| dy \\
&\le \frac{1}{\varepsilon^{n+k}} \left(\int_{\mathbb{R}^n} \frac{\|f(\cdot-y)-f(\cdot)\|^q_{L^p}}{|y|^{n+sq}} dy\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^n} \left|D^k\varphi\left(\frac{y}{\varepsilon}\right)\right|^{q'} |y|^{(\frac{n}{q}+s)q'} dy\right)^{\frac{1}{q'}} \\
&= C\varepsilon^{s-k}\|f\|_{\dot{\mathcal{B}}^s_{p,q}}, \qquad (2.4)
\end{aligned}
$$

where $1/q + 1/q' = 1$. Next, we prove (2.2).

$$
\begin{aligned}
|f(x) - f^\varepsilon(x)| &= \left|\int_{\mathbb{R}^n} [f(x) - f(x-y)]\varphi^\varepsilon(y)dy\right| \\
&\le \int_{\mathbb{R}^n} |f(x) - f(x-y)|\varphi^\varepsilon(y)dy
\end{aligned}
$$

$$\leq \left(\int_{\mathbb{R}^n} \frac{|f(x)-f(x-y)|^q}{|y|^{n+sq}} dy\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^n} \left|\varphi^\varepsilon(y)\right|^{q'} |y|^{(\frac{n}{q}+s)q'} dy\right)^{\frac{1}{q'}}$$
$$= C\varepsilon^s \left(\int_{\mathbb{R}^n} \frac{|f(x)-f(x-y)|^q}{|y|^{n+sq}} dy\right)^{\frac{1}{q}}, \tag{2.5}$$

where $1/q + 1/q' = 1$. Taking $L^p(dx)$ norm of (2.5), we obtain (2.2) with $\dot{X}^s_{p,q} = \dot{\mathcal{F}}^s_{p,q}$. On the other hand, using the Minkowski and the Hölder inequalities again, we have

$$\begin{aligned}\|f - f^\varepsilon\|_{L^p} &\leq \left[\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x)-f(x-y)|\varphi^\varepsilon(y) dy\right)^p dx\right]^{\frac{1}{p}} \\ &\leq \int_{\mathbb{R}^n} \|f(\cdot)-f(\cdot-y)\|_{L^p} \varphi^\varepsilon(y) dy \\ &\leq \left(\int_{\mathbb{R}^n} \frac{\|f(\cdot)-f(\cdot-y)\|^q_{L^p}}{|y|^{n+sq}} dy\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^n} \left|\varphi^\varepsilon(y)\right|^{q'} |y|^{(\frac{n}{q}+s)q'} dy\right)^{\frac{1}{q'}} \\ &= C\varepsilon^s \|f\|_{\dot{\mathcal{B}}^s_{p,q}}.\end{aligned} \tag{2.6}$$

□

Proof of Theorem 1.1. We note the identity,

$$\begin{aligned}(u\otimes v)^\varepsilon = u^\varepsilon \otimes v^\varepsilon &+ \int_{\mathbb{R}^n} \varphi^\varepsilon(y)\,(u(x-y)-u(x)) \otimes (v(x-y)-v(x))\,dy \\ &-(u-u^\varepsilon)\otimes(v-v^\varepsilon)\end{aligned} \tag{2.7}$$

for all $u, v \in L^2_{loc}(\mathbb{R}^n)$, which was first observed in [5]. Suppose $v(x,t)$ is a weak solution of (E). Let $\xi(t) \in C^\infty_0([0,T))$. Given $y \in \mathbb{R}^n$, choosing the test functions $\phi(x,t) = \xi(t)(\varphi^\varepsilon(x-y), 0, 0)$, $\xi(t)(0, \varphi^\varepsilon(x-y), 0)$ and $\xi(t)(0, 0, \varphi^\varepsilon(x-y))$ in (1.1), we obtain each component of

$$\frac{\partial v^\varepsilon}{\partial t} + \mathrm{div}\,(v\otimes v)^\varepsilon = -\nabla p^\varepsilon, \tag{2.8}$$

and choosing $\psi(x,t) = \xi(t)\varphi^\varepsilon(x-y)$ in (1.2), we derive div $v^\varepsilon = 0$. We take $L^2(\mathbb{R}^n)$ inner product (2.8) with v^ε. Then, integrating by part, and using the identity (2.7), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} |v^\varepsilon|^2 dx = \int_{\mathbb{R}^n} (v\otimes v)^\varepsilon : \nabla v^\varepsilon dx - \int_{\mathbb{R}^n} \nabla p^\varepsilon \cdot v^\varepsilon dx \\ &= \int_{\mathbb{R}^n} \left\{\int_{\mathbb{R}^n} \varphi^\varepsilon(y)(v(x-y)-v(x)) \otimes (v(x-y)-v(x)) dy\right\} : \nabla v^\varepsilon(x) dx \\ &\quad - \int_{\mathbb{R}^n} [(v-v^\varepsilon)\otimes(v-v^\varepsilon)] : \nabla v^\varepsilon dx \\ &:= I + II,\end{aligned} \tag{2.9}$$

where we used the facts,

$$\int_{\mathbb{R}^n} \nabla p^\varepsilon \cdot v^\varepsilon dx = -\int_{\mathbb{R}^n} p^\varepsilon \mathrm{div}\, v^\varepsilon dx = 0,$$

and

$$\int_{\mathbb{R}^n} v^\varepsilon \otimes v^\varepsilon : \nabla v^\varepsilon dx = \sum_{i,j=1}^n \int_{\mathbb{R}^n} v_i^\varepsilon v_j^\varepsilon \frac{\partial v_j^\varepsilon}{\partial x_i} dx = \frac{1}{2}\int_{\mathbb{R}^n} (v^\varepsilon \cdot \nabla)|v^\varepsilon|^2 dx$$
$$= -\frac{1}{2}\int_{\mathbb{R}^n} (\operatorname{div} v^\varepsilon)|v^\varepsilon|^2 dx = 0.$$

We estimate I and II separately:

$$\begin{aligned} I &\le \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} |\varphi^\varepsilon(y)||v(x-y)-v(x)|^2 dy \right\} |\nabla v^\varepsilon(x)| dx \\ &\le \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\varphi^\varepsilon(y)|^{q'} |y|^{(\frac{2n}{q}+2s)q'} dy \right)^{\frac{1}{q'}} \left(\int_{\mathbb{R}^n} \frac{|v(x-y)-v(x)|^q}{|y|^{n+sq}} dy \right)^{\frac{2}{q}} |\nabla v^\varepsilon(x)| dx \\ &\le C\varepsilon^{2s} \left[\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{|v(x-y)-v(x)|^q}{|y|^{n+sq}} dy \right)^{\frac{3}{q}} dx \right]^{\frac{2}{3}} \|\nabla v^\varepsilon\|_{L^3} \\ &\le C\varepsilon^{3s-1} \|v\|^3_{\dot{\mathcal{F}}^s_{3,q}}, \end{aligned} \tag{2.10}$$

where $2/q + 1/q' = 1$, $q \in [2, \infty]$, and we used (2.1) in the last step. For the estimate in $\dot{\mathcal{B}}^s_{3,q}$ norm we first use the Fubini theorem, and the use the Hölder inequality to deduce,

$$\begin{aligned} I &\le \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} |\varphi^\varepsilon(y)||v(x-y)-v(x)|^2 |\nabla v^\varepsilon(x)| dx \right\} dy \\ &\le \int_{\mathbb{R}^n} |\varphi^\varepsilon(y)| \|v(\cdot - y) - v(\cdot)\|^2_{L^3} \|\nabla v^\varepsilon\|_{L^3} dy \\ &\le \left(\int_{\mathbb{R}^n} |\varphi^\varepsilon(y)|^{q'} |y|^{(\frac{2n}{q}+2s)q'} dy \right)^{\frac{1}{q'}} \left(\int_{\mathbb{R}^n} \frac{\|v(\cdot - y)-v(\cdot)\|^q}{|y|^{n+sq}} dy \right)^{\frac{2}{q}} \|\nabla v^\varepsilon\|_{L^3} \\ &\le C\varepsilon^{2s-1} \|v\|^2_{\dot{\mathcal{B}}^s_{3,q}} \cdot \varepsilon^s \|v\|^2_{\dot{\mathcal{B}}^s_{3,q}} \\ &\le C\varepsilon^{3s-1} \|v\|^3_{\dot{\mathcal{B}}^s_{3,q}}, \end{aligned} \tag{2.11}$$

where we used (2.1) again. We note that the estimate (2.10) has obvious end point extension for $q = 2 (q' = \infty)$ and $q = \infty (q' = 1)$, although we do not write down those estimates separately. The estimate of II is simpler as follows.

$$\begin{aligned} II &\le \int_{\mathbb{R}^n} |v(x) - v^\varepsilon(x)|^2 |\nabla v^\varepsilon(x)| dx \\ &\le \|v - v^\varepsilon\|^2_{L^3} \|\nabla v^\varepsilon\|_{L^3} \\ &\le C\varepsilon^{3s-1} \|v\|^3_{\dot{X}^s_{3,q}}, \end{aligned} \tag{2.12}$$

where we used (2.1) and (2.2) directly. Taking into account the estimates (2.10)–(2.12), and integrating (2.9) over $[0, t] \subset [0, T]$, we have

$$\left| \|v^\varepsilon(t)\|^2_{L^2} - \|v_0^\varepsilon\|^2_{L^2} \right| \le C\varepsilon^{3s-1} \int_0^T \|v(\tau)\|^3_{\dot{X}^s_{3,q}} d\tau.$$

For $s > \frac{1}{3}$, passing $\varepsilon \to 0$, we have $\|v(t)\|_{L^2} = \|v_0\|_{L^2}$ if $v \in L^3(0, T; \dot{X}^s_{3,q}(\mathbb{R}^n))$. □

Proof of Theorem 1.2. Suppose $v(x,t)$ is a weak solution of (E), and $\omega = \text{curl } v$ in the sense of distribution. Let $\xi(t) \in C_0^\infty([0,T))$. Given $y \in \mathbb{R}^3$, choosing the test functions, $\phi(x,t) = \xi(t)\text{curl}_x\,(\varphi^\varepsilon(x-y),0,0)$, $\xi(t)\text{curl}_x\,(0,\varphi^\varepsilon(x-y),0)$ and $\xi(t)\text{curl}_x\,(0,0,\varphi^\varepsilon(x-y))$ in (1.1), and integrating by part, we obtain the three components of

$$\frac{\partial \omega^\varepsilon}{\partial t} + \text{div}\,(v\otimes\omega)^\varepsilon - \text{div}\,(\omega\otimes v)^\varepsilon = 0. \tag{2.13}$$

We compute

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3} v^\varepsilon\cdot\omega^\varepsilon dx &= \int_{\mathbb{R}^3}\frac{\partial v^\varepsilon}{\partial t}\cdot\omega^\varepsilon dx + \int_{\mathbb{R}^3} v^\varepsilon\cdot\frac{\partial\omega^\varepsilon}{\partial t}dx\\ &= -\int_{\mathbb{R}^3}\text{div}\,(v\otimes v)^\varepsilon\cdot\omega^\varepsilon dx - \int_{\mathbb{R}^3} v^\varepsilon\cdot\text{div}\,(v\otimes\omega)^\varepsilon dx\\ &\qquad + \int_{\mathbb{R}^3} v^\varepsilon\cdot\text{div}\,(\omega\otimes v)^\varepsilon dx\\ &= I + II + III.\end{aligned} \tag{2.14}$$

Integrating by part, and using the formula (2.7), we derive

$$\begin{aligned} I &= \int_{\mathbb{R}^3}(v\otimes v)^\varepsilon : \nabla\omega^\varepsilon dx\\ &= \int_{\mathbb{R}^3} v^\varepsilon\otimes v^\varepsilon : \nabla\omega^\varepsilon dx + \int_{\mathbb{R}^3} r_\varepsilon(v,v) : \nabla\omega^\varepsilon dx - \int_{\mathbb{R}^3}(v-v^\varepsilon)\otimes(v-v^\varepsilon) : \nabla\omega^\varepsilon dx,\end{aligned}$$

where we set

$$r_\varepsilon(u,v) = \int_{\mathbb{R}^3}\varphi^\varepsilon(y)\,(u(x-y)-u(x))\otimes(v(x-y)-v(x))\,dy.$$

Similarly,

$$\begin{aligned} II &= \int_{\mathbb{R}^3}(v\otimes\omega)^\varepsilon : \nabla v^\varepsilon dx\\ &= \int_{\mathbb{R}^3} v^\varepsilon\otimes\omega^\varepsilon : \nabla v^\varepsilon dx + \int_{\mathbb{R}^3} r_\varepsilon(v,\omega) : \nabla v^\varepsilon dx\\ &\qquad - \int_{\mathbb{R}^3}(v-v^\varepsilon)\otimes(\omega-\omega^\varepsilon) : \nabla v^\varepsilon dx,\\ III &= -\int_{\mathbb{R}^3}(\omega\otimes v)^\varepsilon : \nabla v^\varepsilon dx\\ &= -\int_{\mathbb{R}^3}\omega^\varepsilon\otimes v^\varepsilon : \nabla v^\varepsilon dx - \int_{\mathbb{R}^3} r_\varepsilon(\omega,v) : \nabla v^\varepsilon dx\\ &\qquad + \int_{\mathbb{R}^3}(\omega-\omega^\varepsilon)\otimes(v-v^\varepsilon) : \nabla v^\varepsilon dx\end{aligned}$$

respectively. Since div $v^\varepsilon = 0$, we have by integration by part,

$$\begin{aligned}\int_{\mathbb{R}^3} v^\varepsilon \otimes v^\varepsilon : \nabla\omega^\varepsilon dx &= \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} v_i^\varepsilon\, v_j^\varepsilon\, \frac{\partial \omega_j^\varepsilon}{\partial x_i} dx \\ &= -\sum_{i,j=1}^{3} \int_{\mathbb{R}^3} v_i^\varepsilon\, \frac{\partial v_j^\varepsilon}{\partial x_i}\, \omega_j^\varepsilon dx \\ &= -\int_{\mathbb{R}^3} v^\varepsilon \otimes \omega^\varepsilon : \nabla v^\varepsilon dx. \end{aligned} \tag{2.15}$$

Also, using the fact div $\omega^\varepsilon = 0$, we have by integration by part,

$$\begin{aligned}\int_{\mathbb{R}^3} \omega^\varepsilon \otimes v^\varepsilon : \nabla v^\varepsilon dx &= \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} \omega_i^\varepsilon\, v_j^\varepsilon\, \frac{\partial v_j^\varepsilon}{\partial x_i} dx \\ &= -\sum_{i,j=1}^{3} \int_{\mathbb{R}^3} \omega_i^\varepsilon\, \frac{\partial v_j^\varepsilon}{\partial x_i}\, v_j^\varepsilon dx \\ &= -\int_{\mathbb{R}^3} \omega^\varepsilon \otimes v^\varepsilon : \nabla v^\varepsilon dx = 0. \end{aligned} \tag{2.16}$$

Hence, we find that the sum of the first terms of I, II and III cancels out, and after rearrangement of the remaining terms we obtain,

$$\begin{aligned} I + II + III &= \int_{\mathbb{R}^3} r_\varepsilon(v, v) : \nabla\omega^\varepsilon dx + \int_{\mathbb{R}^3} r_\varepsilon(v, \omega) : \nabla v^\varepsilon dx \\ &\quad - \int_{\mathbb{R}^3} r_\varepsilon(\omega, v) : \nabla v^\varepsilon dx \\ &\quad - \int_{\mathbb{R}^3} (v - v^\varepsilon) \otimes (v - v^\varepsilon) : \nabla\omega^\varepsilon dx \\ &\quad - \int_{\mathbb{R}^3} (v - v^\varepsilon) \otimes (\omega - \omega^\varepsilon) : \nabla v^\varepsilon dx \\ &\quad + \int_{\mathbb{R}^3} (\omega - \omega^\varepsilon) \otimes (v - v^\varepsilon) : \nabla v^\varepsilon dx \\ &= J_1 + J_2 + J_3 + J_4 + J_5 + J_6. \end{aligned} \tag{2.17}$$

We estimate (2.17) term by term starting from J_1:

$$\begin{aligned} |J_1| &= \left| \int_{\mathbb{R}^3} \left\{ \int_{\mathbb{R}^3} \varphi^\varepsilon(y)(v(x-y) - v(x)) \otimes (v(x-y) - v(x)) dy \right\} : \nabla\omega^\varepsilon(x) dx \right| \\ &\le \int_{\mathbb{R}^3} \left\{ \int_{\mathbb{R}^3} \varphi^\varepsilon(y) |v(x-y) - v(x)|^2 dy \right\} |\nabla\omega^\varepsilon(x)| dx \\ &\le \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} |\varphi^\varepsilon(y)|^{q'} |y|^{(\frac{6}{q}+2s)q'} dy \right)^{\frac{1}{q'}} \left(\int_{\mathbb{R}^3} \frac{|v(x-y) - v(x)|^q}{|y|^{3+sq}} dy \right)^{\frac{2}{q}} |\nabla\omega^\varepsilon(x)| dx \end{aligned}$$

$$\le C\varepsilon^{2s}\left[\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}\frac{|v(x-y)-v(x)|^q}{|y|^{3+sq}}dy\right)^{\frac{9}{2q}}dx\right]^{\frac{4}{9}}\|\nabla\omega^\varepsilon\|_{L^{\frac{9}{5}}}$$
$$\le C\varepsilon^{3s-1}\|v\|^2_{\dot{\mathcal{F}}^s_{\frac{9}{2},q}}\|\omega\|_{\dot{\mathcal{F}}^s_{\frac{9}{5},q}}, \tag{2.18}$$

where $1/q+1/q'=1$, and we used (2.1) in the last step. For the Besov space norm estimate we use the Fubini theorem and the Hölder inequality as previously:

$$|J_1|\le\int_{\mathbb{R}^3}\left\{\int_{\mathbb{R}^3}\varphi^\varepsilon(y)|v(x-y)-v(x)|^2|\nabla\omega^\varepsilon(x)|dx\right\}dy$$
$$\le\int_{\mathbb{R}^3}\varphi^\varepsilon(y)\|v(\cdot-y)-v(\cdot)\|^2_{L^{\frac{9}{2}}}\|\nabla\omega^\varepsilon\|_{L^{\frac{9}{5}}}dy$$
$$\le\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}|\varphi^\varepsilon(y)|^{q'}|y|^{(\frac{6}{q}+2s)q'}dy\right)^{\frac{1}{q'}}\left(\int_{\mathbb{R}^3}\frac{\|v(\cdot-y)-v(\cdot)\|^q_{L^{\frac{9}{2}}}}{|y|^{3+sq}}dy\right)^{\frac{2}{q}}\|\nabla\omega^\varepsilon\|_{L^{\frac{9}{5}}}$$
$$\le C\varepsilon^{3s-1}\|v\|^2_{\dot{\mathcal{B}}^s_{\frac{9}{2},q}}\|\omega\|_{\dot{\mathcal{B}}^s_{\frac{9}{5},q}}. \tag{2.19}$$

We estimate J_2 as follows:

$$|J_2|=\left|\int_{\mathbb{R}^3}\left\{\int_{\mathbb{R}^3}\varphi^\varepsilon(y)(v(x-y)-v(x))\otimes(\omega(x-y)-\omega(x))dy\right\}:\nabla v^\varepsilon(x)dx\right|$$
$$\le\int_{\mathbb{R}^3}\left\{\int_{\mathbb{R}^3}\varphi^\varepsilon(y)|v(x-y)-v(x)||\omega(x-y)-\omega(x)|dy\right\}|\nabla v^\varepsilon(x)|dx$$
$$\le\left(\int_{\mathbb{R}^3}|\varphi^\varepsilon(y)|^{q'}|y|^{(\frac{6}{q}+2s)q'}dy\right)^{\frac{1}{q'}}$$
$$\times\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}\frac{|v(x-y)-v(x)|^q}{|y|^{3+sq}}dy\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}^3}\frac{|\omega(x-y)-\omega(x)|^q}{|y|^{3+sq}}dy\right)^{\frac{1}{q}}|\nabla v^\varepsilon(x)|dx$$
$$\le C\varepsilon^{2s}\left[\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}\frac{|v(x-y)-v(x)|^q}{|y|^{3+sq}}dy\right)^{\frac{9}{2q}}dx\right]^{\frac{2}{9}}$$
$$\times\left[\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}\frac{|\omega(x-y)-\omega(x)|^q}{|y|^{3+sq}}dy\right)^{\frac{9}{5q}}dx\right]^{\frac{5}{9}}\|\nabla v^\varepsilon\|_{L^{\frac{9}{2}}}$$
$$\le C\varepsilon^{3s-1}\|v\|^2_{\dot{\mathcal{F}}^s_{\frac{9}{2},q}}\|\omega\|_{\dot{\mathcal{F}}^s_{\frac{9}{5},q}}, \tag{2.20}$$

where $2/q+1/q'=1$. Next, we estimate in the Besov norm:

$$|J_2|\le\int_{\mathbb{R}^3}\left\{\int_{\mathbb{R}^3}\varphi^\varepsilon(y)|v(x-y)-v(x)||\omega(x-y)-\omega(x)||\nabla v^\varepsilon(x)|dx\right\}dy$$
$$\le\int_{\mathbb{R}^3}\varphi^\varepsilon(y)\|v(\cdot-y)-v(\cdot)\|_{L^{\frac{9}{2}}}\|\omega(\cdot-y)-\omega(\cdot)\|_{L^{\frac{9}{5}}}\|\nabla v^\varepsilon\|_{L^{\frac{9}{2}}}dy$$

$$\leq \left(\int_{\mathbb{R}^3} |\varphi^\varepsilon(y)|^{q'} |y|^{(\frac{6}{q}+2s)q'} dy\right)^{\frac{1}{q'}}$$

$$\times \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} \frac{\|v(\cdot - y) - v(\cdot)\|^q_{L^{\frac{9}{2}}}}{|y|^{3+sq}} dy\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^3} \frac{\|\omega(\cdot - y) - \omega(\cdot)\|^q_{L^{\frac{9}{5}}}}{|y|^{3+sq}} dy\right)^{\frac{1}{q}} \|\nabla v^\varepsilon\|_{L^{\frac{9}{2}}}$$

$$\leq C\varepsilon^{3s-1} \|v\|^2_{\dot{\mathcal{B}}^s_{\frac{9}{2},q}} \|\omega\|_{\dot{\mathcal{B}}^s_{\frac{9}{5},q}}. \tag{2.21}$$

The estimate of J_3 is similar to that of J_2, and we have

$$|J_3| \leq C\varepsilon^{3s-1} \|v\|^2_{\dot{X}^s_{\frac{9}{2},q}} \|\omega\|_{\dot{X}^s_{\frac{9}{5},q}}. \tag{2.22}$$

We estimate J_4 as follows:

$$\begin{aligned} |J_4| &= \left|\int_{\mathbb{R}^3} [(v - v^\varepsilon) \otimes (v - v^\varepsilon)] : \nabla\omega^\varepsilon dx\right| \\ &\leq \int_{\mathbb{R}^3} |v - v^\varepsilon|^2 |\nabla\omega^\varepsilon| dx \leq \|v - v^\varepsilon\|^2_{L^{\frac{9}{2}}} \|\nabla\omega^\varepsilon\|_{L^{\frac{9}{5}}} \\ &\leq C\varepsilon^{3s-1} \|v\|^2_{\dot{X}^s_{\frac{9}{2},q}} \|\omega\|_{\dot{X}^s_{\frac{9}{5},q}}, \end{aligned} \tag{2.23}$$

where we used (2.1) and (2.2). Similarly, we estimate J_5:

$$\begin{aligned} |J_5| &= \left|\int_{\mathbb{R}^3} [(v - v^\varepsilon) \otimes (\omega - \omega^\varepsilon)] : \nabla v^\varepsilon dx\right| \\ &\leq \int_{\mathbb{R}^3} |v - v^\varepsilon||\omega - \omega^\varepsilon||\nabla v^\varepsilon| dx \leq \|v - v^\varepsilon\|_{L^{\frac{9}{2}}} \|\omega - \omega^\varepsilon\|_{L^{\frac{9}{5}}} \|\nabla v^\varepsilon\|_{L^{\frac{9}{2}}} \\ &\leq C\varepsilon^{3s-1} \|v\|^2_{\dot{X}^s_{\frac{9}{2},q}} \|\omega\|_{\dot{X}^s_{\frac{9}{5},q}}. \end{aligned} \tag{2.24}$$

The estimates of J_6 is similar to that of J_5, and we have

$$|J_6| \leq C\varepsilon^{3s-1} \|v\|^2_{\dot{X}^s_{\frac{9}{2},q}} \|\omega\|_{\dot{X}^s_{\frac{9}{5},q}}. \tag{2.25}$$

Taking into account the estimates (2.18)–(2.25), and integrating (2.17) over $[0, t] \subset [0, T]$, we have

$$\begin{aligned} &\left|\int_{\mathbb{R}^3} v^\varepsilon(x,t) \cdot \omega^\varepsilon(x,t) dx - \int_{\mathbb{R}^3} v_0^\varepsilon(x) \cdot \omega_0^\varepsilon(x) dx\right| \\ &\leq C\varepsilon^{3s-1} \int_0^T \|v(t)\|^2_{\dot{X}^s_{\frac{9}{2},q}} \|\omega(t)\|_{\dot{X}^s_{\frac{9}{5},q}} dt \\ &\leq C\varepsilon^{3s-1} \left(\int_0^T \|v(t)\|^{r_1}_{\dot{X}^s_{\frac{9}{2},q}} dt\right)^{\frac{2}{r_1}} \left(\int_0^T \|\omega(t)\|^{r_1}_{\dot{X}^s_{\frac{9}{5},q}} dt\right)^{\frac{1}{r_2}}, \end{aligned}$$

where $2/r_1 + 1/r_2 = 1$. Passing $\varepsilon \to 0$, we find that

$$\int_{\mathbb{R}^3} \omega(x,t) \cdot v(x,t) dx = \int_{\mathbb{R}^3} \omega_0(x) \cdot v_0(x) dx$$

for $s > \frac{1}{3}$. □

Proof of Corollary 1.1. We estimate the helicity,

$$\int_{\mathbb{R}^3} v(x,t)\cdot\omega(x,t)dx \le \|v(\cdot,t)\|_{L^3}\|\omega(\cdot,t)\|_{L^{\frac{3}{2}}}$$
$$\le C\|\nabla v(\cdot,t)\|_{L^{\frac{3}{2}}}\|\omega(\cdot,t)\|_{L^{\frac{3}{2}}} \le C\|\omega(\cdot,t)\|^2_{L^{\frac{3}{2}}}, \quad (2.26)$$

where we used the Sobolev inequality and the Calderon-Zygmund inequality. Combining (2.26) with Theorem 1.2, we obtain the desired conclusion. □

Proof of Theorem 1.3. Suppose $\theta(x,t)$ is a weak solution of (QG) in the sense of (1.6)–(1.7). Let $\xi(t)\in C_0^\infty([0,T))$. Given $y\in\mathbb{R}^2$, choosing the test function $\phi(x,t)=\xi(t)\varphi^\varepsilon(x-y)$ in (1.6) we obtain

$$\frac{\partial\theta^\varepsilon}{\partial t}+\text{div}\,(v\theta)^\varepsilon=0. \quad (2.27)$$

We take the $L^2(\mathbb{R}^2)$ inner product (2.27) with $\theta^\varepsilon|\theta^\varepsilon|^{p-2}$. Then, integrating by part, and using the identity (2.7), we obtain

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2}|\theta^\varepsilon|^p dx &= (p-1)\int_{\mathbb{R}^2}(v\theta)^\varepsilon\cdot\nabla\theta^\varepsilon|\theta^\varepsilon|^{p-2}dx\\ &= (p-1)\int_{\mathbb{R}^2}\left\{\int_{\mathbb{R}^2}\varphi^\varepsilon(y)(v(x-y)-v(x))(\theta(x-y)-\theta(x))dy\right\}\\ &\qquad\cdot\nabla\theta^\varepsilon(x)|\theta^\varepsilon(x)|^{p-2}dx\\ &\quad-(p-1)\int_{\mathbb{R}^2}[(v-v^\varepsilon)(\theta-\theta^\varepsilon)]\cdot\nabla\theta^\varepsilon(x)|\theta^\varepsilon(x)|^{p-2}dx\\ &:=(p-1)[I+II],\end{aligned} \quad (2.28)$$

where we used the fact,

$$\int_{\mathbb{R}^2} v^\varepsilon\theta^\varepsilon\cdot\nabla\theta^\varepsilon|\theta^\varepsilon|^{p-2}dx=\frac{1}{p}\int_{\mathbb{R}^2}(v^\varepsilon\cdot\nabla)|\theta^\varepsilon|^p dx=-\frac{1}{p}\int_{\mathbb{R}^2}(\text{div}\,v^\varepsilon)|\theta^\varepsilon|^p dx=0.$$

We estimate I and II separately:

$$\begin{aligned}I &\le \int_{\mathbb{R}^2}\left\{\int_{\mathbb{R}^2}|\varphi^\varepsilon(y)||v(x-y)-v(x)||\theta(x-y)-\theta(x)|dy\right\}|\nabla\theta^\varepsilon(x)||\theta^\varepsilon(x)|^{p-2}dx\\ &\le \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}|\varphi^\varepsilon(y)|^{q'}|y|^{(\frac{4}{q}+2s)q'}dy\right)^{\frac{1}{q'}}\left(\int_{\mathbb{R}^2}\frac{|v(x-y)-v(x)|^q}{|y|^{2+sq}}dy\right)^{\frac{1}{q}}\\ &\quad\times\left(\int_{\mathbb{R}^2}\frac{|\theta(x-y)-\theta(x))|^q}{|y|^{2+sq}}dy\right)^{\frac{1}{q}}|\nabla\theta^\varepsilon(x)||\theta^\varepsilon(x)|^{p-2}dx\\ &\le C\varepsilon^{2s}\left[\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{|v(x-y)-v(x)|^q}{|y|^{2+sq}}dy\right)^{\frac{p+1}{q}}dx\right]^{\frac{1}{p+1}}\\ &\quad\times\left[\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{|\theta(x-y)-\theta(x)|^q}{|y|^{2+sq}}dy\right)^{\frac{p+1}{q}}dx\right]^{\frac{1}{p+1}}\|\nabla\theta^\varepsilon\|_{L^{p+1}}\|\theta^\varepsilon\|^{p-2}_{L^{p+1}}\\ &\le C\varepsilon^{3s-1}\|v\|_{\dot{\mathcal{F}}^s_{p+1,q}}\|\theta\|^2_{\dot{\mathcal{F}}^s_{p+1,q}}\|\theta\|^{p-2}_{L^{p+1}},\end{aligned} \quad (2.29)$$

where $1/q + 1/q' = 1$, $q \in [1, \infty]$, and we used (2.1) in the last step. The estimate in the Besov space norm is the following.

$$\begin{aligned}
I &\le \int_{\mathbb{R}^2} \left\{ \int_{\mathbb{R}^2} |\varphi^\varepsilon(y)||v(x-y) - v(x)||\theta(x-y) - \theta(x)||\nabla\theta^\varepsilon(x)||\theta^\varepsilon(x)|^{p-2} dx \right\} dy \\
&\le \int_{\mathbb{R}^2} \varphi^\varepsilon(y) \|v(\cdot - y) - v(\cdot)\|_{L^{p+1}} \|\theta(\cdot - y) - \theta(\cdot)\|_{L^{p+1}} \|\nabla\theta^\varepsilon\|_{L^{p+1}} \|\theta^\varepsilon\|_{L^{p+1}}^{p-2} dy \\
&\le \left(\int_{\mathbb{R}^2} |\varphi^\varepsilon(y)|^{q'} |y|^{(\frac{4}{q}+2s)q'} dy \right)^{\frac{1}{q'}} \left(\int_{\mathbb{R}^2} \frac{\|v(\cdot - y) - v(\cdot)\|_{L^{p+1}}^q}{|y|^{2+sq}} dy \right)^{\frac{1}{q}} \\
&\quad \times \left(\int_{\mathbb{R}^2} \frac{\|\theta(\cdot - y) - \theta(\cdot)\|_{L^{p+1}}^q}{|y|^{2+sq}} dy \right)^{\frac{1}{q}} \|\nabla\theta^\varepsilon\|_{L^{p+1}} \|\theta^\varepsilon\|_{L^{p+1}}^{p-2} \\
&\le C\varepsilon^{3s-1} \|v\|_{\dot{\mathcal{B}}^s_{p+1,q}} \|\theta\|^2_{\dot{\mathcal{B}}^s_{p+1,q}} \|\theta\|^{p-2}_{L^{p+1}}.
\end{aligned} \tag{2.30}$$

Using (2.1) and (2.2) directly, we estimate

$$\begin{aligned}
II &\le \int_{\mathbb{R}^3} |v(x) - v^\varepsilon(x)||\theta(x) - \theta^\varepsilon(x)||\nabla\theta^\varepsilon(x)||\theta^\varepsilon(x)|^{p-2} dx \\
&\le \|v - v^\varepsilon\|_{L^{p+1}} \|\theta - \theta^\varepsilon\|_{L^{p+1}} \|\nabla\theta^\varepsilon\|_{L^{p+1}} \|\theta^\varepsilon\|^{p-2}_{L^{p+1}} \\
&\le C\varepsilon^{3s-1} \|v\|_{\dot{X}^s_{p+1,q}} \|\theta\|^2_{\dot{X}^s_{p+1,q}} \|\theta\|^{p-2}_{L^{p+1}}.
\end{aligned} \tag{2.31}$$

Taking into account the estimates (2.29)–(2.31), and integrating (2.28) over $[0, t] \subset [0, T]$, we have

$$\begin{aligned}
\left| \|\theta^\varepsilon(t)\|^p_{L^p} - \|\theta^\varepsilon_0\|^p_{L^p} \right| &\le C\varepsilon^{3s-1} \int_0^T \|v(\tau)\|_{\dot{X}^s_{p+1,q}} \|\theta(\tau)\|^2_{\dot{X}^s_{p+1,q}} \|\theta(\tau)\|^{p-2}_{L^{p+1}} d\tau \\
&\le C\varepsilon^{3s-1} \int_0^T \|v(\tau)\|_{\dot{X}^s_{p+1,q}} \|\theta(\tau)\|^p_{X^s_{p+1,q}} d\tau \\
&\le C\varepsilon^{3s-1} \left(\int_0^T \|v(\tau)\|^{r_2}_{\dot{X}^s_{p+1,q}} d\tau \right)^{\frac{1}{r_2}} \left(\int_0^T \|\theta(\tau)\|^{r_1}_{X^s_{p+1,q}} d\tau \right)^{\frac{p}{r_1}},
\end{aligned}$$

where $p/r_1 + 1/r_2 = 1$. For $s > \frac{1}{3}$, passing $\varepsilon \to 0$, we have $\|\theta(t)\|_{L^p} = \|\theta_0\|_{L^p}$, if $v \in L^{r_2}(0, T; \dot{X}^s_{p+1,q}(\mathbb{R}^2))$ and $\theta \in L^{r_1}(0, T; X^s_{p+1,q}(\mathbb{R}^2))$. $\square$

Acknowledgements. This research was supported by KOSEF Grant no. R01-2005-000-10077-0.

References

1. Arnold, V.I., Khesin, B.A.: Topological Methods in Hydrodynamics. Berlin Heidelberg New York: Springer-Verlag Inc., 1998
2. Chae, D.: Remarks on the Helicity of the 3-D Incompressible Euler Equations. Commun. Math. Phys. **240**(3), 501–507 (2003)
3. Chae, D.: The quasi-geostrophic equation in the Triebel-Lizorkin spaces. Nonlinearity **16**, 479–495 (2003)

4. Constantin, P.: Geometric Statistics in Turbulence. SIAM Rev. **36**, 73–98 (1994)
5. Constantin, P., E, W., Titi, E.: Onsager's Conjecture on the Energy Conservation for Solutions of Euler's Equations. Commun. Math. Phys. **165**(1), 207–209 (1994)
6. Constantin, P., Majda, A., Tabak, E.: Formation of strong fronts in the 2-D quasi-geostrophic thermal active scalar. Nonlinearity **7**, 1459–1533 (1994)
7. Córdoba, D.: Nonexistence of simple hyperbolic blow-up for the quasi-geostrophic equation. Ann. of Math. **148**, 1135–1152 (1998)
8. Córdoba, D., Fefferman, C.: Growth of solutions for QG and 2D Euler equations. J. Amer. Math. Soc. **15**(3), 665–670 (2002)
9. Duchon, J., Robert, R.: Inertial Energy Dissipation for Weak Solutions of Incompressible Euler and Navier-Stokes Equations. Nonlinearity **13**, 249–255 (2000)
10. Majda, A., Bertozzi, A.: Vorticity and Incompressible Flow. Cambridge: Cambridge Univ. Press, 2002
11. Moffat, H.K., Tsinober, A.: Helicity in Laminar and Turbulent Flow. Ann. Rev. Fluid Mech. **24**, 281–312 (1992)
12. Resnick, S.: Dynamical Problems in Non-linear Advective Partial Differential Equations. Ph.D Thesis, University of Chicago, 1995
13. Shnirelman, A.: Weak Solutions with Decreasing Energy of Incompressible Euler Equations. Commun. Math. Phys., **210**, 541–603
14. Stein, E.M.: Singular Integrals and Differentiability Properties of Functions. Princeton, NJ: Princeton Univ. Press, 1970
15. Triebel, H.: Theory of Function Spaces. Boston: Birkäuser Verlag, 1983
16. Wu, J.: Inviscid limits and regularity estimates for the solutions of the 2-D dissipative Quasi-geostrophic equations. Indiana Univ. Math. J. **46**(4), 1113–1124 (1997)

Communicated by P. Constantin