Commun. Math. Phys. 264, 87–114 (2006)
Digital Object Identifier (DOI) 10.1007/s00220-005-1513-4

Communications in
**Mathematical
Physics**

Drinfeld Twists and Algebraic Bethe Ansatz of the Supersymmetric Model Associated with $U_q(gl(m|n))$

Wen-Li Yang[1,2], Yao-Zhong Zhang[2], Shao-You Zhao[2,3]

[1] Institute of Modern Physics, Northwest University, Xian 710069, P.R. China
[2] Department of Mathematics, University of Queensland, Brisbane, QLD 4072, Australia
[3] Department of Physics, Beijing Institute of Technology, Beijing 100081, China

Received: 1 March 2005 / Accepted: 22 August 2005
Published online: 31 January 2006 –

Abstract: We construct the Drinfeld twists (or factorizing F-matrices) of the supersymmetric model associated with quantum superalgebra $U_q(gl(m|n))$, and obtain the completely symmetric representations of the creation operators of the model in the F-basis provided by the F-matrix. As an application of our general results, we present the explicit expressions of the Bethe vectors in the F-basis for the $U_q(gl(2|1))$-model (the quantum t-J model).

1. Introduction

It was realized in [1] that for the XXX or XXZ spin chain systems, there exists a non-degenerate lower-triangular F-matrix (the Drinfeld twists) [2] in terms of which the R-matrix of the system is factorized:

$$R_{12}(u_1, u_2) = F_{21}^{-1}(u_2, u_1)F_{12}(u_1, u_2), \tag{1.1}$$

where the R-matrix acts on the tensor space $V \otimes V$ with V being a 2-dimensional $U_q(gl(2))$-module. In the basis provided by the N-site F-matrix, i.e. the so-called F-basis, the entries of the monodromy matrices of the models appear in completely symmetric forms. As a result the Bethe vectors of the models are dramatically simplified and can be written down explicitly. These results enabled the authors in [3, 4] to compute form factors, correlation functions [5] and spontaneous magnetizations of the systems analytically and explicitly.

The results of [1] were generalized to other systems. In [6], the Drinfeld twists associated with any finite-dimensional irreducible representations of Yangian $Y[gl(2)]$ were investigated. In [7], Albert et al constructed the F-matrix of the rational $gl(m)$ Heisenberg model, obtained a polarization free representation of its creation operators and resolved the hierarchy of its nested Bethe vectors. In [8, 9], the Drinfeld twists of the elliptic XYZ and Belavin models were constructed. Recently we have successfully constructed the Drinfeld twists for the rational $gl(m|n)$ supersymmetric model and resolved the hierarchy of its nested Bethe vectors in the F-basis [10, 11]. Quantum integrable models associated with Lie superalgebras [12–14] are physically important because they give strongly correlated fermion models of superconductivity (e.g. [15, 16]).

In this paper, we extend our results in [10, 11] to the quantum (or q-deformed) supersymmetric model associated the quantum superalgebra $U_q(gl(m|n))$ (including quantum supersymmetric t-J model as a special case). Such a generalization is *non-trivial* due to the following fact. It is well-known that the $gl(m|n)$ rational model has $gl(m|n)$ symmetry which enables one to express the creation operators $C_i(u)$ in terms of the element $T_{m+n,m+n}(u)$ of the monodromy matrix $T(u)$ and the generators of $gl(m|n)$ by (anti)commutation relations [7, 11]. However, the corresponding quantum model is not $U_q(gl(m|n))$ invariant (unless appropriate boundary conditions are imposed). One of the consequences is that the creation operators $C_i(u)$ of the quantum model *cannot* be expressed in terms of $T_{m+n,m+n}(u)$ and the generators of $U_q(gl(m|n))$ by *simple* q-(anti)commutation relations. Indeed, it is found in this paper that *extra quantum correction terms* are needed, due to the non-trivial coproduct structure of the quantum superalgebra. Having found such a new recursive relation (3.37) and constructed the factorizing F-matrices of the quantum model, we obtain the symmetric representations of the creation operators of the monodromy matrix in the F-basis. These results make possible a complete resolution of the hierarchy of the nested Bethe vectors of the $U_q(gl(m|n))$ model. As an example, we give the explicit expressions of the Bethe vectors of the quantum $t-J$ model associated with $U_q(gl(2|1))$.

The present paper is organized as follows. In Sect. 2, we introduce some basic notation on the quantum superalgebra $U_q(gl(m|n))$. In Sect. 3, we derive the recursive relation between the elements of the monodromy matrix and the generators of $U_q(gl(m|n))$. In Sect. 4, we construct the F-matrix and its inverse of the $U_q(gl(m|n))$ model. In Sect. 5, we obtain the symmetric representations of the creation operators in the F-basis. As an application of our general results, the hierarchy of the nested Bethe vectors of the $U_q(gl(2|1))$ model is resolved in Sect. 6. We conclude the paper by offering some discussions in Sect. 7. Some detailed technical derivations are given in Appendices A-B.

2. Quantum Superalgebra $U_q(gl(m|n))$

Let us fix two non-negative integers n, m such that $n + m \geq 2$ and a positive integer $N(N \geq 2)$, and a generic complex number η such that the q-deformation parameter, which is defined by $q = e^{\eta}$, is not a root of unity. Let V be a $\mathbb{Z}_2$-graded $(n+m)$-dimensional vector space with the orthonormal basis $\{|i\rangle,\ i = 1, \ldots, n+m\}$. The $\mathbb{Z}_2$-grading is chosen as: $[1] = \cdots = [m] = 1,\ [m+1] = \cdots = [m+n] = 0$.

Definition 1. The quantum superalgebra $U_q(gl(m|n))$ is a $\mathbb{Z}_2$-graded unital associative superalgebra generated by the generators $E^{i,i}$, $(i = 1, \ldots, n+m)$ and $E^{j,j+1}$, $E^{j+1,j}$ $(j = 1, \ldots, n+m-1)$ with the $\mathbb{Z}_2$-grading $[E^{i,i}] = 0$, $[E^{j+1,j}] = [E^{j,j+1}] = [j] + [j+1]$ by the relations:

$$[E^{i,i}, E^{i',i'}] = 0,\ [E^{i,i}, E^{j,j+1}] = (\delta_{i,j} - \delta_{i,j+1})E^{i,j+1},\ i' = 1, \ldots, n+m, \tag{2.1}$$

$$[E^{i,i}, E^{j+1,j}] = (\delta_{i,j+1} - \delta_{i,j})E^{j+1,j}, \tag{2.2}$$

$$[E^{j,j+1}, E^{j'+1,j'}] = (-1)^{[j]}\delta_{j,j'}\frac{q^{h^j} - q^{-h^j}}{q - q^{-1}},\ j' = 1, \ldots, n+m-1, \tag{2.3}$$

and the Serre relations:

$$\begin{aligned}
&(E^{m,m+1})^2 = (E^{m+1,m})^2 = 0,\\
&[E^{j,j+1}, E^{j',j'+1}] = [E^{j+1,j}, E^{j'+1,j'}] = 0, \ |j-j'| \geq 2,\\
&(E^{j,j+1})^2 E^{j\pm1,j\pm1+1} - (q+q^{-1}) E^{j,j+1} E^{j\pm1,j\pm1+1} E^{j,j+1}\\
&\quad + E^{j\pm1,j\pm1+1}(E^{j,j+1})^2 = 0, \ j \neq m,\\
&(E^{j+1,j})^2 E^{j\pm1+1,j\pm1} - (q+q^{-1}) E^{j+1,j} E^{j\pm1+1,j\pm1} E^{j+1,j}\\
&\quad + E^{j\pm1+1,j\pm1}(E^{j+1,j})^2 = 0, \ j \neq m,
\end{aligned} \tag{2.4}$$

where $h^j = (-1)^{[j]}E^{j,j} - (-1)^{[j+1]}E^{j+1,j+1}$. In addition to the above Serre relations, there exist also extra Serre relations [17] which we omit.

Here and throughout, we adopt the convention:

$$[x, y] = xy - (-1)^{[x][y]}yx, \quad x, y \in U_q(gl(m|n)).$$

One can easily see that the $\mathbb{Z}_2$-graded vector space V supplies the fundamental $U_q(gl(m|n))$-module and the generators of $U_q(gl(m|n))$ are represented in this space by

$$\pi(E^{i,i}) = e_{i,i}, \ \pi(E^{j,j+1}) = e_{j,j+1}, \ \pi(E^{j+1,j}) = e_{j+1,j}, \tag{2.5}$$

where $e_{i,j} \in \mathrm{End}(V)$ is the elementary matrix with elements $(e_{i,j})^l_k = \delta_{jk}\delta_{il}$.

$U_q(gl(m|n))$ is a $\mathbb{Z}_2$-graded triangular Hopf superalgebra endowed with $\mathbb{Z}_2$-graded algebra homomorphisms that are coproduct Δ: $U_q(gl(m|n)) \longrightarrow U_q(gl(m|n)) \otimes U_q(gl(m|n))$ defined by

$$\Delta(E^{i,i}) = 1 \otimes E^{i,i} + E^{i,i} \otimes 1, \ i = 1, \ldots, n+m, \tag{2.6}$$

$$\Delta(E^{j,j+1}) = 1 \otimes E^{j,j+1} + E^{j,j+1} \otimes q^{h^j}, \tag{2.7}$$

$$\Delta(E^{j+1,j}) = q^{-h^j} \otimes E^{j+1,j} + E^{j+1,j} \otimes 1, \tag{2.8}$$

and counit ϵ: $U_q(gl(m|n)) \longrightarrow \mathbb{C}$ defined by

$$\epsilon(E^{j,j+1}) = \epsilon(E^{j+1,j}) = \epsilon(E^{i,i}) = 0, \ \ \epsilon(1) = 1,$$

and a $\mathbb{Z}_2$-graded algebra antiautomorphism (antipode) S: $U_q(gl(m|n)) \longrightarrow U_q(gl(m|n))$ given by

$$S(E^{j,j+1}) = -E^{j,j+1}q^{-h^j}, \ S(E^{j+1,j}) = -q^{h^j}E^{j+1,j}, \ S(E^{i,i}) = -E^{i,i}.$$

Multiplications of tensor products are $\mathbb{Z}_2$ graded:

$$(x \otimes y)(x' \otimes y') = (-1)^{[y][x']}xx' \otimes yy',$$

for homogeneous elements $x, y, x', y' \in U_q(gl(m|n))$ and where $[x] \in \mathbb{Z}_2$ denotes the grading of x. It should be pointed out that the antipode satisfies the following equation, for homogeneous elements $x, y \in U_q(gl(m|n))$,

$$S(xy) = (-1)^{[x][y]}S(y)S(x),$$

and generalizes to inhomogeneous elements through linearity. The coproduct, counit and antipode satisfy the following relations, $\forall x \in U_q(gl(m|n))$:

$$\begin{aligned}(\Delta \otimes \mathrm{id})\Delta(x) &= (\mathrm{id} \otimes \Delta)\Delta(x),\\ (\epsilon \otimes \mathrm{id})\Delta(x) &= x = (\mathrm{id} \otimes \epsilon)\Delta(x),\\ m(S \otimes \mathrm{id})\Delta(x) &= m(\mathrm{id} \otimes S)\Delta(x) = \epsilon(x),\end{aligned} \tag{2.9}$$

where m denote the product of any two elements of $U_q(gl(m|n))$, i.e., $m(x \otimes y) = xy$ for $x, y \in U_q(gl(m|n))$.

The generators $\{E^{j,j+1}\}$ ($\{E^{j+1,j}\}$) are the simple raising (lowering) generators of $U_q(gl(m|n))$ associated with the simple roots. Thanks to the Serre relations, the other generators associated with the non-simple roots (called the non-simple generators) can be uniquely constructed through the simple ones by the following relations:

$$E^{\alpha,\gamma} = E^{\alpha,\beta}E^{\beta,\gamma} - q^{-(-1)^{[\beta]}}E^{\beta,\gamma}E^{\alpha,\beta}, \quad 1 \le \alpha < \beta < \gamma \le n+m, \tag{2.10}$$

$$E^{\gamma,\alpha} = E^{\gamma,\beta}E^{\beta,\alpha} - q^{(-1)^{[\beta]}}E^{\beta,\alpha}E^{\gamma,\beta}, \quad 1 \le \alpha < \beta < \gamma \le n+m. \tag{2.11}$$

The coproduct, counit and antipode of the non-simple generators can be obtained through those of the simple ones. Here, we give the coproduct of non-simple generators which will be used later.

Lemma 1. *The coproduct of the non-simple generators is*

$$\begin{aligned}\Delta(E^{\gamma,\gamma-l}) = {}& q^{-\sum_{k=1}^{l} h^{\gamma-k}} \otimes E^{\gamma,\gamma-l} + E^{\gamma,\gamma-l} \otimes 1\\ &+\sum_{i=1}^{l-1}(1-q^{2(-1)^{[\gamma-l+i]}})q^{-\sum_{k=1}^{l-i} h^{\gamma-k}} E^{\gamma-l+i,\gamma-l} \otimes E^{\gamma,\gamma-l+i},\\ &\gamma-l \ge 1 \text{ and } l \ge 2,\end{aligned} \tag{2.12}$$

$$\begin{aligned}\Delta(E^{\gamma,\gamma+l}) = {}& 1 \otimes E^{\gamma,\gamma+l} + E^{\gamma,\gamma+l} \otimes q^{\sum_{k=0}^{l-1} h^{\gamma+k}}\\ &+\sum_{i=1}^{l-1}(1-q^{-2(-1)^{[\gamma+i]}})E^{\gamma+i,\gamma+l} \otimes E^{\gamma,\gamma+i} q^{\sum_{k=i}^{l-1} h^{\gamma+k}},\\ &\gamma+l \le n+m \text{ and } l \ge 2.\end{aligned} \tag{2.13}$$

Proof. This lemma can be proved by induction using the coproducts of the simple generators (2.6)-(2.8), the definitions of the non-simple generators (2.10)-(2.11) and the fact that the coproduct is an algebra homomorphism, as well as (2.1)-(2.3) and the Serre relation (2.4). □

3. Recursive Relation between Monodromy Matrix Elements and $U_q(gl(m|n))$ Generators

Let $R \in \mathrm{End}(V \otimes V)$ be the R-matrix associated with the fundamental $U_q(gl(m|n))$-module V. The R-matrix depends on the difference of two spectral parameters u_1 and u_2 associated with two copies of V, and is, in the present grading, given by [13, 14, 18]

$$
\begin{aligned}
R_{12}(u_1,u_2) &= R_{12}(u_1-u_2)\\
&= c_{12}\sum_{i=1}^{m} e_{i,i}\otimes e_{i,i} + \sum_{i=m+1}^{m+n} e_{i,i}\otimes e_{i,i} + a_{12}\sum_{i\neq j=1}^{m+n} e_{i,i}\otimes e_{j,j}\\
&\quad + b^-_{12}\sum_{i>j=1}^{m+n}(-1)^{[j]}e_{i,j}\otimes e_{j,i} + b^+_{12}\sum_{j>i=1}^{m+n}(-1)^{[j]}e_{i,j}\otimes e_{j,i}, \qquad (3.1)
\end{aligned}
$$

where

$$a_{12}=a(u_1,u_2)\equiv\frac{\sinh(u_1-u_2)}{\sinh(u_1-u_2+\eta)},\quad b^{\pm}_{12}=b^{\pm}(u_1,u_2)\equiv\frac{e^{\pm(u_1-u_2)}\sinh\eta}{\sinh(u_1-u_2+\eta)}, \qquad (3.2)$$

$$c_{12}=c(u_1,u_2)\equiv\frac{\sinh(u_1-u_2-\eta)}{\sinh(u_1-u_2+\eta)}, \qquad (3.3)$$

and η is the so-called crossing parameter. One can easily check that the R-matrix satisfies the unitary relation

$$R_{21}R_{12}=1. \qquad (3.4)$$

Let us introduce the $(N+1)$-fold tensor product space $\overbrace{V\otimes V\cdots\otimes V}^{N+1}$, whose components are labelled by $0,1\ldots,N$ from the left to the right. As usual, the 0^{th} space, denoted by V_0 (V_i for the i^{th} space), corresponds to the auxiliary space and the other N spaces constitute the quantum space $\mathcal{H}=\overbrace{V\otimes V\cdots\otimes V}^{N}$. Moreover, for each factor space $V_i, i=0,\ldots,N$, we associate a complex parameter z_i. The parameter associated with the 0^{th} space is usually called the *spectral* parameter which is set to $z_0=u$ in this paper, and the other parameters are called the *inhomogeneous* parameters. In this paper we always assume that all the complex parameters u and $\{z_i|i=1,\ldots,N\}$ are *generic* ones. Hereafter we adopt the standard notation: for any matrix $A\in\mathrm{End}(V)$, A_j (or $A_{(j)}$) is an embedding operator in the tensor product space, which acts as A on the j^{th} space and as an identity on the other factor spaces; $R_{ij}=R_{ij}(z_i,z_j)$ is an embedding operator of R-matrix in the tensor product space, which acts as an identity on the factor spaces except for the i^{th} and j^{th} ones.

The R-matrix satisfies the graded Yang-Baxter equation (GYBE)

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}. \qquad (3.5)$$

In terms of the matrix elements defined by

$$R(u)(|i'\rangle\otimes|j'\rangle)=\sum_{i,j}R(u)^{i'j'}_{ij}(|i\rangle\otimes|j\rangle),$$

the GYBE reads

$$
\begin{aligned}
&\sum_{i',j',k'}R(u_1-u_2)^{i'j'}_{ij}R(u_1-u_3)^{i''k'}_{i'k}R(u_2-u_3)^{j''k''}_{j'k'}(-1)^{[j']([i']+[i''])}\\
&=\sum_{i',j',k'}R(u_2-u_3)^{j'k'}_{jk}R(u_1-u_3)^{i'k''}_{ik'}R(u_1-u_2)^{i''j''}_{i'j'}(-1)^{[j']([i]+[i'])}.
\end{aligned}
$$

Besides the GYBE, the R-matrix satisfies the following relation:

$$R_{12}\Delta(x) = \mathcal{P}_{12}\Delta(x)\mathcal{P}_{12}^{-1}R_{12}, \tag{3.6}$$

where $\mathcal{P}_{12}$ is the superpermutation operator, i.e., $\mathcal{P}_{12}(|i\rangle \otimes |j\rangle = (-1)^{[i][j]}|j\rangle \otimes |i\rangle$.

Using the coproduct structure of $U_q(gl(m|n))$, one can define the action of $U_q(gl(m|n))$ on the $(N+1)$-fold tensor product space. For any $x \in U_q(gl(m|n))$, let us denote the action of x on the $(N+1)$-fold tensor product space by $(x)_{0\ldots N}$:

$$(x)_{0\ldots N} = \Delta^{(N)}(x) = (\mathrm{id} \otimes \Delta^{(N-1)})\Delta(x). \tag{3.7}$$

By a straightforward calculation, one has

Lemma 2.

$$(E^{i,i})_{0\ldots N} = \sum_{k=0}^{N} E^{i,i}_{(k)},\ i = 1, \ldots, n+m, \tag{3.8}$$

$$(E^{j,j+1})_{0\ldots N} = \sum_{k=0}^{N} E^{j,j+1}_{(k)} q^{\sum_{i=k+1}^{N} h^{j}_{(i)}},\ j = 1, \ldots, n+m-1, \tag{3.9}$$

$$(E^{j+1,j})_{0\ldots N} = \sum_{k=0}^{N} q^{-\sum_{i=0}^{k-1} h^{j}_{(i)}} E^{j+1,j}_{(k)},\ j = 1, \ldots, n+m-1, \tag{3.10}$$

where $E^{i,j}_{(k)}$ is the embedding of $e_{i,j}$ in the tensor product space, which acts as $e_{i,j}$ on the k^{th} space and as identity on the other factor spaces.

The actions of non-simple generators can be obtained from those of simple ones through (2.10) and (2.11).

Let $\mathcal{S}_{N+1}$ denote the permutation group of the $N+1$ space labels $(0, \ldots, N)$. The GYBE (3.5) and unitary relation (3.4) of R-matrix allow one to introduce the following mapping.

Definition 2. One can define a mapping from $\mathcal{S}_{N+1}$ to $\mathrm{End}(V_0 \otimes \mathcal{H})$ which associate in a unique way an element $R^{\sigma}_{0\ldots N} \in \mathrm{End}(V_0 \otimes \mathcal{H})$ to any element σ of the permutation group $\mathcal{S}_{N+1}$. The mapping has the following composition law:

$$R^{\sigma\sigma'}_{0\ldots N} = \mathcal{P}^{\sigma}\, R^{\sigma'}_{0\ldots N}\, (\mathcal{P}^{\sigma})^{-1}\, R^{\sigma}_{0\ldots N} = R^{\sigma'}_{\sigma(0\ldots N)}\, R^{\sigma}_{0\ldots N},\ \forall \sigma, \sigma' \in \mathcal{S}_{N+1}, \tag{3.11}$$

where $\mathcal{P}^{\sigma}$ is the $\mathbb{Z}_2$-graded permutation operator on the tensor product space, i.e., $\mathcal{P}^{\sigma}|i_0\rangle_{(0)} \ldots |i_N\rangle_{(N)} = |i_0\rangle_{(\sigma(0))} \ldots |i_N\rangle_{(\sigma(N))}$. For any elementary permutation σ_j with $\sigma_j(0, \ldots, j, j+1, \ldots, N) = (0, \ldots, j+1, j, \ldots, N)$, $j = 0, \ldots, N$, the corresponding $R^{\sigma_j}_{0\ldots N}$ is

$$R^{\sigma_j}_{0\ldots N} = R_{j\, j+1}. \tag{3.12}$$

For any element $\sigma \in \mathcal{S}_{N+1}$, the corresponding $R^{\sigma}_{0\ldots N}$ can be constructed through (3.11) and (3.12) as follows. Let σ be decomposed in a minimal way in terms of elementary permutation as $\sigma = \sigma_{\beta_1} \ldots \sigma_{\beta_p}$, where the positive integer p is the length of σ. The composition law enables one to obtain the expression of the associated $R^{\sigma}_{0\ldots N}$. The GYBE (3.5) and (3.4) guarantee the uniqueness of $R^{\sigma}_{0\ldots N}$. For the special element σ_c of $\mathcal{S}_{N+1}$,

$$\sigma_c = \sigma_0\sigma_1 \ldots \sigma_{N-1},\ \text{ namely, }\ \sigma_c(0, 1, \ldots, N) = (1, 2, \ldots, N, 0), \tag{3.13}$$

the associated $R^{\sigma_c}_{0\dots N}$ is given by

$$T(u) \equiv T_0(u) = T_{0,1\dots N}(u) = R^{\sigma_c}_{0\dots N} = R_{0N}R_{0N-1}\dots R_{01}. \tag{3.14}$$

Thus $R^{\sigma_c}_{0\dots N}$ is the quantum monodromy matrix $T(u)$ of the $U_q(gl(m|n))$ spin chain on an N-site lattice. By the GYBE, one may prove that the monodromy matrix satisfies the GYBE

$$R_{00'}(u-v)T_0(u)T_{0'}(v) = T_{0'}(v)T_0(u)R_{00'}(u-v). \tag{3.15}$$

Define the transfer matrix $t(u)$

$$t(u) = str_0 T(u), \tag{3.16}$$

where str_0 denotes the supertrace over the auxiliary space. Then the Hamiltonian of our model is given by

$$H = \frac{d\ln t(u)}{du}|_{u=0}. \tag{3.17}$$

This model is integrable thanks to the commutativity of the transfer matrix for different parameters,

$$[t(u), t(v)] = 0, \tag{3.18}$$

which can be verified by using the GYBE.

The fundamental relation (3.6) and the co-associativity (2.9) of the coproduct of $U_q(gl(m|n))$ enable one to prove the following result, using the procedure similar to that in [1] for the non-super case,

Proposition 1. *The mapping defined in Definition 2 satisfies the following relation:*

$$R^{\sigma}_{0\dots N}\ (x)_{0\dots N} = \mathcal{P}^{\sigma}\ (x)_{0\dots N}\ (\mathcal{P}^{\sigma})^{-1}\ R^{\sigma}_{0\dots N},\ \forall x\in U_q(gl(m|n)),\ \sigma\in\mathcal{S}_{N+1}. \tag{3.19}$$

One may decompose the monodromy matrix $T(u)$ in terms of the basis of $\mathrm{End}(V_0)$ as

$$T(u) = \sum_{i,j=1}^{n+m} T_{i,j}(u)E^{i,j}_{(0)} \equiv \sum_{i,j=1}^{n+m} T_{i,j}(u)e_{i,j}, \tag{3.20}$$

where the matrix elements $T_{i,j}(u)$ are operators acting on the quantum space $\mathcal{H}$ and have the $\mathbb{Z}_2$-grading: $[T_{i,j}(u)] = [e_{i,j}] = [i] + [j]$. Similarly, for the quantities defined in Lemma 2, we have the decomposition:

$$(E^{i,i})_{0\dots N} = E^{i,i}_{(0)} + \sum_{k=1}^{N} E^{i,i}_{(k)} = e_{i,i} + (E^{i,i})_{1\dots N},\ i = 1,\dots,n+m, \tag{3.21}$$

$$\begin{aligned}(E^{j,j+1})_{0\dots N} &= E^{j,j+1}_{(0)}\, q^{\sum_{k=1}^{N} h^j_{(k)}} + \sum_{k=1}^{N} E^{j,j+1}_{(k)} q^{\sum_{i=k+1}^{N} h^j_{(i)}} \\ &= e_{j,j+1}\, q^{(h^j)_{1\dots N}} + (E^{j,j+1})_{1\dots N},\ j = 1,\dots,n+m-1, \end{aligned} \tag{3.22}$$

$$\begin{aligned}(E^{j+1,j})_{0\dots N} &= E^{j+1,j}_{(0)} + q^{-h^j_{(0)}} \sum_{k=1}^{N} q^{-\sum_{i=1}^{k-1} h^j_{(i)}} E^{j+1,j}_{(k)} \\ &= e_{j+1,j} + q^{-h_j}\,(E^{j+1,j})_{1\dots N},\ j = 1,\dots,n+m-1, \end{aligned} \tag{3.23}$$

where $h_j = (-1)^{[j]}e_{j,j} - (-1)^{[j+1]}e_{j+1,j+1}$. Without confusion, hereafter we adopt the following convention:

$$e_{i,i'} = E^{i,i'}_{(0)}, \quad i, i' = 1, \ldots, n+m, \tag{3.24}$$

$$h_j = h^j_{(0)} = (-1)^{[j]}e_{j,j} - (-1)^{[j+1]}e_{j+1,j+1}, \; j = 1, \ldots, n+m-1, \tag{3.25}$$

$$E_{i,i} = (E^{i,i})_{1\ldots N} = \sum_{k=1}^{N} E^{i,i}_{(k)}, \; i = 1, \ldots, n+m, \tag{3.26}$$

$$H_j = (-1)^{[j]}E_{j,j} - (-1)^{[j+1]}E_{j+1,j+1}, \; j = 1, \ldots, n+m-1, \tag{3.27}$$

$$E_{j,j+1} = (E^{j,j+1})_{1\ldots N} = \sum_{k=1}^{N} E^{j,j+1}_{(k)} q^{\sum_{i=k+1}^{N} h^j_{(i)}}, \; j = 1, \ldots, n+m-1, \tag{3.28}$$

$$E_{j+1,j} = (E^{j+1,j})_{1\ldots N} = \sum_{k=1}^{N} q^{-\sum_{i=1}^{k-1} h^j_{(i)}} E^{j+1,j}_{(k)}, \; j = 1, \ldots, n+m-1, \tag{3.29}$$

and a similar convention for the non-simple generators. Then the operators $\{E_{i,j}\}$ are the operators which act *non-trivially* on the quantum space $\mathcal{H}$ and *trivially* (i.e. as an identity) on the auxiliary space V_0. From (3.13), we have

$$\mathcal{P}^{\sigma_c}\,(h^j)_{0\ldots N}\,(\mathcal{P}^{\sigma_c})^{-1} = h_j + H_j, \; j = 1, \ldots, n+m-1, \tag{3.30}$$

$$\mathcal{P}^{\sigma_c}\,(E^{j,j+1})_{0\ldots N}\,(\mathcal{P}^{\sigma_c})^{-1} = q^{h_j}\, E_{j,j+1} + e_{j,j+1}, \; j = 1, \ldots, n+m-1, \tag{3.31}$$

$$\mathcal{P}^{\sigma_c}\,(E^{j+1,j})_{0\ldots N}\,(\mathcal{P}^{\sigma_c})^{-1} = e_{j,j+1}q^{-H_j} + E_{j+1,j}, \; j = 1, \ldots, n+m-1. \tag{3.32}$$

Using (3.21)-(3.23), (3.30)-(3.32) and Lemma 1, we have

Proposition 2.

$$\begin{aligned}(E^{\gamma,\gamma-l})_{0\ldots N} &= q^{-\sum_{k=1}^{l} h_{\gamma-k}}\, E_{\gamma,\gamma-l} + e_{\gamma,\gamma-l} \\ &\quad + \sum_{i=1}^{l-1}(1 - q^{2(-1)^{[\gamma-l+i]}})q^{-\sum_{k=1}^{l-i} h_{\gamma-k}}\, e_{\gamma-l+i,\gamma-l}\, E_{\gamma,\gamma-l+i}, \; \gamma - l \geq 1 \text{ and } l \geq 2,\end{aligned} \tag{3.33}$$

$$\begin{aligned}\mathcal{P}^{\sigma_c}(E^{\gamma,\gamma-l})_{0\ldots N}(\mathcal{P}^{\sigma_c})^{-1} &= e_{\gamma,\gamma-l}q^{-\sum_{k=1}^{l} H_{\gamma-k}} + E_{\gamma,\gamma-l} \\ &\quad + \sum_{i=1}^{l-1}(1 - q^{2(-1)^{[\gamma-l+i]}})\, e_{\gamma,\gamma-l+i}\, q^{-\sum_{k=1}^{l-i} H_{\gamma-k}}\, E_{\gamma-l+i,\gamma-l}, \; \gamma - l \geq 1 \text{ and } l \geq 2,\end{aligned} \tag{3.34}$$

$$\begin{aligned}(E^{\gamma,\gamma+l})_{0\ldots N} &= E_{\gamma,\gamma+l} + e_{\gamma,\gamma+l}q^{\sum_{k=0}^{l-1} H_{\gamma+k}} \\ &\quad + \sum_{i=1}^{l-1}(1 - q^{-2(-1)^{[\gamma+i]}})e_{\gamma+i,\gamma+l}\, E_{\gamma,\gamma+i}q^{\sum_{k=i}^{l-1} H_{\gamma+k}}, \; \gamma + l \leq n+m \text{ and } l \geq 2,\end{aligned} \tag{3.35}$$

$$\mathcal{P}^{\sigma_c}\,(E^{\gamma,\gamma+l})_{0\ldots N}\,(\mathcal{P}^{\sigma_c})^{-1} = e_{\gamma,\gamma+l} + q^{\sum_{k=0}^{l-1} h_{\gamma+k}}\, E_{\gamma,\gamma+l}$$
$$+\sum_{i=1}^{l-1}(1-q^{-2(-1)^{[\gamma+i]}})e_{\gamma,\gamma+i}\, q^{\sum_{k=i}^{l-1} h_{\gamma+k}} E_{\gamma+i,\gamma+l},\ \gamma+l\leq n+m \text{ and } l\geq 2. \tag{3.36}$$

Substituting (3.35) and (3.36) into Proposition 1, we obtain our main result in this section:

Theorem 1. *The matrix elements* $T_{n+m,n+m-l}(u)\ (l = 1,\ldots,n+m-1)$ *of the monodromy matrix can be expressed in terms of* $T_{n+m,n+m}(u)$ *and the generators of* $U_q(gl(m|n))$ *by the following recursive relation:*

$$T_{n+m,n+m-l}(u) =$$
$$=\Big(q^{-(-1)^{[n+m]}}E_{n+m-l,n+m}T_{n+m,n+m}(u)-T_{n+m,n+m}(u)E_{n+m-l,n+m}\Big)q^{-\sum_{k=1}^{l}H_{n+m-k}}$$
$$-\sum_{\alpha=1}^{l-1}(1-q^{-2(-1)^{[n+m-\alpha]}})T_{n+m,n+m-\alpha}(u)E_{n+m-l,n+m-\alpha}q^{-\sum_{k=\alpha+1}^{l}H_{n+m-k}}. \tag{3.37}$$

The proof of this theorem is relegated to Appendix A.

We call the second term in the R.H.S. of (3.37) a *quantum correction term*, which vanishes in the rational limit ($q \to 1$). Moreover, such a nontrivial correction term only occurs in the higher rank models (i.e., when $n + m \geq 3$). In the rational limit: $q \to 1$, (3.37) reduces to the (anti)commutation relations used in [7, 11]. For some special values of m and n, the associated recursive relations in the present grading become:

- For the $U_q(gl(1|1))$ case:

$$T_{2,1}(u) = \Big[q^{-1}E_{1,2}T_{2,2}(u) - T_{2,2}(u)E_{1,2}\Big]q^{-H_1}. \tag{3.38}$$

- For the $U_q(gl(2|1))$ case which corresponds to the quantum $t-J$ model:

$$T_{3,2}(u) = \Big[q^{-1}E_{2,3}T_{3,3}(u) - T_{3,3}(u)E_{2,3}\Big]q^{-H_2}, \tag{3.39}$$
$$T_{3,1}(u) = \Big[q^{-1}E_{1,3}T_{3,3}(u) - T_{3,3}(u)E_{1,3}\Big]q^{-H_2-H_1}$$
$$-(1-q^2)T_{3,2}(u)E_{1,2}q^{-H_1}. \tag{3.40}$$

- For the $U_q(gl(2|2))$ case which corresponds to the quantum EKS model [15]:

$$T_{4,3}(u) = \Big[q^{-1}E_{3,4}T_{4,4}(u) - T_{4,4}(u)E_{3,4}\Big]q^{-H_3}, \tag{3.41}$$
$$T_{4,2}(u) = \Big[q^{-1}E_{2,4}T_{4,4}(u) - T_{4,4}(u)E_{2,4}\Big]q^{-H_3-H_2}$$
$$-(1-q^{-2})T_{4,3}(u)E_{2,3}q^{-H_2}, \tag{3.42}$$
$$T_{4,1}(u) = \Big[q^{-1}E_{1,4}T_{4,4}(u) - T_{4,4}(u)E_{1,4}\Big]q^{-H_3-H_2-H_1}$$
$$-(1-q^{-2})T_{4,3}(u)E_{1,3}q^{-H_2-H_1}$$
$$-(1-q^2)T_{4,2}(u)E_{1,2}q^{-H_1}. \tag{3.43}$$

4. Factorizing F-Matrices and Their Inverses

In this section, we construct the Drinfeld twists [2] (factorizing F-matrices) on the N-fold tensor product space (i.e. the quantum space $\mathcal{H}$) associated with the quantum superalgebra $U_q(gl(m|n))$.

4.1. Factorizing F-matrix. Let $\mathcal{S}_N$ be the permutation group associated with the indices $(1, \ldots, N)$ and $R^{\sigma}_{1\ldots N}$ the N-site R-matrix associated with $\sigma \in \mathcal{S}_N$. $R^{\sigma}_{1\ldots N}$ acts non-trivially on the quantum space $\mathcal{H}$ and trivially (i.e as an identity) on the auxiliary space.

Definition 3. The F-matrix $F_{1\ldots N}(z_1, \ldots, z_N)$ is an operator in $\mathrm{End}(\mathcal{H})$ and satisfies the following three properties:

I. lower-triangularity;
II. non-degeneracy;
III. factorization, namely,

$$F_{\sigma(1)\ldots\sigma(N)}(z_{\sigma(1)}, \ldots, z_{\sigma(N)})\, R^{\sigma}_{1\ldots N} = F_{1\ldots N}(z_1, \ldots, z_N), \;\forall \sigma \in \mathcal{S}_N. \quad (4.1)$$

Define the N-site F-matrix:

$$F_{1\ldots N} \equiv F_{1\ldots N}(z_1, \ldots, z_N) = \sum_{\sigma\in\mathcal{S}_N} \sum^{n+m^*}_{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(N)}=1} \prod_{j=1}^{N} P^{\alpha_{\sigma(j)}}_{\sigma(j)} S(\sigma, \alpha_\sigma) R^{\sigma}_{1\ldots N}, \quad (4.2)$$

where P^{α}_i is the embedding of the project operator P^{α} in the i^{th} space with $(P^{\alpha})_{kl} = \delta_{kl}\delta_{k\alpha}$, the sum $\sum^*$ in (4.2) is over all non-decreasing sequences of the labels $\alpha_{\sigma(i)}$:

$$\begin{aligned} \alpha_{\sigma(i+1)} &\geq \alpha_{\sigma(i)} \quad \text{if} \quad \sigma(i+1) > \sigma(i), \\ \alpha_{\sigma(i+1)} &> \alpha_{\sigma(i)} \quad \text{if} \quad \sigma(i+1) < \sigma(i), \end{aligned} \quad (4.3)$$

and $S(\sigma, \alpha_\sigma)$ is a c-number function of σ, α_σ and the element c_{ij} of the R-matrix, defined by

$$S(\sigma, \alpha_\sigma) \equiv \exp\left\{ \frac{1}{2} \sum_{l>k=1}^{N} \left(1 - (-1)^{[\alpha_{\sigma(k)}]}\right) \delta_{\alpha_{\sigma(k)},\alpha_{\sigma(l)}} \ln(1 + c_{\sigma(k)\sigma(l)}) \right\}. \quad (4.4)$$

Proposition 3. *The F-matrix $F_{1\ldots N}$ given by (4.2)-(4.4) satisfies Properties I, II, III.*

Proof. The definition of $F_{1\ldots N}$ (4.2) and the summation condition (4.3) imply that $F_{1\ldots N}$ is a lower-triangular matrix. Moreover, one can easily check that the F-matrix is non-degenerate because all diagonal elements are non-zero.

We now prove that the F-matrix (4.2) satisfies Property III. Any given permutation $\sigma \in \mathcal{S}_N$ can be decomposed into elementary ones of the group $\mathcal{S}_N$ as $\sigma = \sigma_{i_1} \ldots \sigma_{i_k}$. By (3.11), we have, if Property III holds for any elementary permutation σ_i,

$$\begin{aligned} &F_{\sigma(1\ldots N)} R^{\sigma}_{1\ldots N} = \\ &= F_{\sigma_{i_1}\ldots\sigma_{i_k}(1\ldots N)} R^{\sigma_{i_k}}_{\sigma_{i_1}\ldots\sigma_{i_{k-1}}(1\ldots N)} R^{\sigma_{i_{k-1}}}_{\sigma_{i_1}\ldots\sigma_{i_{k-2}}(1\ldots N)} \cdots R^{\sigma_{i_1}}_{1\ldots N} \\ &= F_{\sigma_{i_1}\ldots\sigma_{i_{k-1}}(1\ldots N)} R^{\sigma_{i_{k-1}}}_{\sigma_{i_1}\ldots\sigma_{i_{k-2}}(1\ldots N)} \cdots R^{\sigma_{i_1}}_{1\ldots N} \\ &= \ldots = F_{\sigma_{i_1}(1\ldots N)} R^{\sigma_{i_1}}_{1\ldots N} = F_{1\ldots N}. \end{aligned} \quad (4.5)$$

For the elementary permutation σ_i, we have

$$\begin{aligned}
&F_{\sigma_i(1\ldots N)}R^{\sigma_i}_{1\ldots N} = \\
&= \sum_{\sigma\in\mathcal{S}_N}\ \sum^{*}_{\alpha_{\sigma_i\sigma(1)}\ldots\alpha_{\sigma_i\sigma(N)}}\ \prod_{j=1}^{N} P^{\alpha_{\sigma_i\sigma(j)}}_{\sigma_i\sigma(j)} S(\sigma_i\sigma,\alpha_{\sigma_i\sigma}) R^{\sigma}_{\sigma_i(1\ldots N)} R^{\sigma_i}_{1\ldots N} \\
&= \sum_{\sigma\in\mathcal{S}_N}\ \sum^{*}_{\alpha_{\sigma_i\sigma(1)}\ldots\alpha_{\sigma_i\sigma(N)}}\ \prod_{j=1}^{N} P^{\alpha_{\sigma_i\sigma(j)}}_{\sigma_i\sigma(j)} S(\sigma_i\sigma,\alpha_{\sigma_i\sigma}) R^{\sigma_i\sigma}_{1\ldots N} \\
&= \sum_{\tilde\sigma\in\mathcal{S}_N}\ \sum^{*(i)}_{\alpha_{\tilde\sigma(1)}\ldots\alpha_{\tilde\sigma(N)}}\ \prod_{j=1}^{N} P^{\alpha_{\tilde\sigma(j)}}_{\tilde\sigma(j)} S(\tilde\sigma,\alpha_{\tilde\sigma}) R^{\tilde\sigma}_{1\ldots N},
\end{aligned} \tag{4.6}$$

where $\tilde\sigma = \sigma_i\sigma$, and the summation sequences of $\alpha_{\tilde\sigma}$ in $\sum^{*(i)}$ now have the form

$$\begin{aligned}
\alpha_{\tilde\sigma(j+1)} \geq \alpha_{\tilde\sigma(j)} \quad &\text{if} \quad \sigma_i\tilde\sigma(j+1) > \sigma_i\tilde\sigma(j), \\
\alpha_{\tilde\sigma(j+1)} > \alpha_{\tilde\sigma(j)} \quad &\text{if} \quad \sigma_i\tilde\sigma(j+1) < \sigma_i\tilde\sigma(j).
\end{aligned} \tag{4.7}$$

Comparing (4.7) with (4.3), we find that the only difference between them is the transposition σ_i factor in the "if" conditions. For a given $\tilde\sigma \in \mathcal{S}_N$ with $\tilde\sigma(j) = i$ and $\tilde\sigma(k) = i+1$, we now examine how the elementary transposition σ_i will affect the inequalities (4.7). If $|j-k| > 1$, then σ_i does not affect the sequence of $\alpha_{\tilde\sigma}$ at all, that is, the sign of inequality "$>$" or "$\geq$" between two neighboring root indexes is unchanged with the action of σ_i. If $|j-k| = 1$, then in the summation sequences of $\alpha_{\tilde\sigma}$, when $\tilde\sigma(j+1) = i+1$ and $\tilde\sigma(j) = i$, sign "$\geq$" changes to "$>$", while when $\tilde\sigma(j+1) = i$ and $\tilde\sigma(j) = i+1$, "$>$" changes to "$\geq$". Thus (4.3) and (4.6) differ only when equal labels $\alpha_{\tilde\sigma}$ appear. With the help of the relation $c_{21}c_{12} = 1$, one may prove that in this case the product $F_{\sigma_i(1\ldots N)}R^{\sigma_i}_{1\ldots N}$ still equals $F_{1\ldots N}$ (see [10] for a more detailed proof). Thus, we obtain

$$R^{\sigma}_{1\ldots N}(z_1,\ldots,z_N) = F^{-1}_{\sigma(1\ldots N)}(z_{\sigma(1)},\ldots,z_{\sigma(N)})F_{1\ldots N}(z_1,\ldots,z_N), \tag{4.8}$$

and the factorizing F-matrix $F_{1\ldots N}$ of $U_q(gl(m|n))$ is proved to satisfy all three properties. □

From the expression of the F-matrix, one knows that it has an even grading, i.e.,

$$[F_{1\ldots N}] = 0. \tag{4.9}$$

4.2. Inverse of the F-matrix. The non-degenerate property of the F-matrix implies that we can find the inverse matrix $F^{-1}_{1\ldots N}$. To do so, we first define

$$F^{*}_{1\ldots N} = \sum_{\sigma\in\mathcal{S}_N}\ \sum^{n+m\,**}_{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(N)}=1} S(\sigma,\alpha_\sigma) R^{\sigma^{-1}}_{\sigma(1\ldots N)} \prod_{j=1}^{N} P^{\alpha_{\sigma(j)}}_{\sigma(j)}, \tag{4.10}$$

where the sum $\sum^{**}$ is taken over all possible α_i which satisfies the following non-increasing constraints:

$$\begin{aligned}
\alpha_{\sigma(i+1)} \leq \alpha_{\sigma(i)} \quad &\text{if} \quad \sigma(i+1) < \sigma(i), \\
\alpha_{\sigma(i+1)} < \alpha_{\sigma(i)} \quad &\text{if} \quad \sigma(i+1) > \sigma(i).
\end{aligned} \tag{4.11}$$

Proposition 4. *The inverse of the F-matrix is given by*

$$F^{-1}_{1\ldots N} = F^{*}_{1\ldots N} \prod_{i<j} \Delta^{-1}_{ij}, \tag{4.12}$$

where

$$[\Delta_{ij}]^{\beta_i\beta_j}_{\alpha_i\alpha_j} = \delta_{\alpha_i\beta_i}\delta_{\alpha_j\beta_j} \begin{cases} \dfrac{\sinh(z_i - z_j)}{\sinh(z_i - z_j + \eta)} & \text{if } \alpha_i > \alpha_j \\ \dfrac{\sinh(z_j - z_i)}{\sinh(z_j - z_i + \eta)} & \text{if } \alpha_i < \alpha_j, \\ 1 & \text{if } \alpha_i = \alpha_j = m+1, \ldots, n+m, \\ \dfrac{-4\sinh^2(z_i - z_j)\cosh^2\eta}{\sinh(z_i - z_j + \eta)\sinh(z_i - z_j - \eta)} & \text{if } \alpha_i = \alpha_j = 1, \ldots, m. \end{cases} \tag{4.13}$$

Proof. We compute the product of $F_{1\ldots N}$ and $F^{*}_{1\ldots N}$. Substituting (4.2) and (4.10) into the product, we have

$$\begin{aligned} F_{1\ldots N} F^{*}_{1\ldots N} &= \sum_{\sigma\in\mathcal{S}_N} \sum_{\sigma'\in\mathcal{S}_N} \sum^{*}_{\alpha_{\sigma_1}\ldots\alpha_{\sigma_N}} \sum^{**}_{\beta_{\sigma'_1}\ldots\beta_{\sigma'_N}} S(\sigma, \alpha_\sigma) S(\sigma', \beta_{\sigma'}) \\ &\quad \times \prod_{i=1}^{N} P^{\alpha_{\sigma(i)}}_{\sigma(i)} R^{\sigma}_{1\ldots N} R^{\sigma'^{-1}}_{\sigma'(1\ldots N)} \prod_{i=1}^{N} P^{\beta_{\sigma'(i)}}_{\sigma'(i)} \\ &= \sum_{\sigma\in\mathcal{S}_N} \sum_{\sigma'\in\mathcal{S}_N} \sum^{*}_{\alpha_{\sigma_1}\ldots\alpha_{\sigma_N}} \sum^{**}_{\beta_{\sigma'_1}\ldots\beta_{\sigma'_N}} S(\sigma, \alpha_\sigma) S(\sigma', \beta_{\sigma'}) \\ &\quad \times \prod_{i=1}^{N} P^{\alpha_{\sigma(i)}}_{\sigma(i)} R^{\sigma'^{-1}\sigma}_{\sigma'(1\ldots N)} \prod_{i=1}^{N} P^{\beta_{\sigma'(i)}}_{\sigma'(i)}. \end{aligned} \tag{4.14}$$

To evaluate the R.H.S., we examine the matrix element of the R-matrix

$$\left(R^{\sigma'^{-1}\sigma}_{\sigma'(1\ldots N)}\right)^{\alpha_{\sigma(N)}\cdots\alpha_{\sigma(1)}}_{\beta_{\sigma'(N)}\cdots\beta_{\sigma'(1)}}. \tag{4.15}$$

Note that the sequence $\{\alpha_\sigma\}$ is non-decreasing and $\{\beta_{\sigma'}\}$ is non-increasing. Thus the non-vanishing condition of the matrix element (4.15) requires that α_σ and $\beta_{\sigma'}$ satisfy

$$\beta_{\sigma'(N)} = \alpha_{\sigma(1)}, \ldots, \beta_{\sigma'(1)} = \alpha_{\sigma(N)}. \tag{4.16}$$

One can verify [7] that (4.16) is fulfilled only if

$$\sigma'(N) = \sigma(1), \ldots, \sigma'(1) = \sigma(N). \tag{4.17}$$

Let $\bar{\sigma}$ be the maximal element of the $\mathcal{S}_N$ which reverses the site labels

$$\bar{\sigma}(1, \ldots, N) = (N, \ldots, 1). \tag{4.18}$$

Then from (4.17), we have

$$\sigma' = \sigma\bar{\sigma}. \tag{4.19}$$

Substituting (4.16) and (4.19) into (4.14), we have

$$F_{1\ldots N}F^{*}_{1\ldots N} = \sum_{\sigma\in\mathcal{S}_N}\sum^{*}_{\alpha_{\sigma_1}\ldots\alpha_{\sigma_N}} S(\sigma,\alpha_\sigma)S(\sigma,\alpha_\sigma)\prod_{i=1}^{N}P^{\alpha_{\sigma(i)}}_{\sigma(i)}R^{\bar{\sigma}}_{\sigma(N\ldots1)}\prod_{i=1}^{N}P^{\alpha_{\sigma(i)}}_{\sigma(i)}. \tag{4.20}$$

The decomposition of $R^{\bar{\sigma}}$ in terms of elementary R-matrices is unique modulo the GYBE. One reduces from (4.20) that FF^* is a diagonal matrix:

$$F_{1\ldots N}F^{*}_{1\ldots N} = \prod_{i<j}\Delta_{ij}. \tag{4.21}$$

Then (4.12) is a simple consequence of the above equation. □

5. Monodromy Matrix in the F-Basis

In the previous section, we have constructed the F-matrix and its inverse which act on the quantum space $\mathcal{H}$. The non-degeneracy of the F-matrix means that its column vectors also form a complete basis of $\mathcal{H}$, which is called the F-basis. In this section, we study the generators of $U_q(gl(m|n))$ and the elements of the monodromy matrix in the F-basis.

5.1. $U_q(gl(m|n))$ generators in the F-basis. The Cartan generators $\{E^{i,i}\}$ (or $\{H^j\}$) and the simple generators $\{E^{j,j+1}\}$, $\{E^{j+1,j}\}$ of $U_q(gl(m|n))$ are realized on $\mathcal{H}$ by $\{E_{i,i}\}$ (or $\{H_j\}$), $\{E_{j,j+1}\}$ and $\{E_{j+1,j}\}$, respectively, as (3.26)-(3.29). The other non-simple generators $\{E^{i,j}\}$ can be obtained from the simple ones by (2.10) and (2.11), and denote their realizations on $\mathcal{H}$ by $\{E_{i,j}\}$. Introduce the generators in the F-basis:

$$\tilde{E}_{i,j} = F_{1\ldots N}E_{i,j}F^{-1}_{1\ldots N},\; i,j = 1,\ldots,n+m. \tag{5.1}$$

Theorem 2. *In the F-basis the Cartan and the simple generators of $U_q(gl(m|n))$ are given by*

$$\tilde{E}_{i,i} = E_{i,i} = \sum_{k=1}^{N}E^{i,i}_{(k)},\; i = 1,\ldots,n+m, \tag{5.2}$$

$$\tilde{E}_{j,j+1} = \sum_{k=1}^{N}E^{j,j+1}_{(k)}\otimes_{\gamma\neq k}G^{j,j+1}_{(\gamma)}(k,\gamma),\; j = 1,\ldots,n+m-1, \tag{5.3}$$

$$\tilde{E}_{j+1,j} = \sum_{k=1}^{N}E^{j+1,j}_{(k)}\otimes_{\gamma\neq k}G^{j+1,j}_{(\gamma)}(k,\gamma),\; j = 1,\ldots,n+m-1. \tag{5.4}$$

Here the diagonal matrices $G^{\gamma,\gamma\pm1}_{(j)}(i,j)$ are:

• *For* $1 < \gamma + 1 \leq m$,

$$(G^{\gamma,\gamma+1}_{(j)}(i,j))_{kl} = \delta_{kl} \begin{cases} 2e^{-\eta}\cosh\eta, & k=\gamma, \\ (2a_{ij}\cosh\eta)^{-1}\, e^{\eta}, & k=\gamma+1, \\ 1, & \text{otherwise}, \end{cases} \tag{5.5}$$

$$(G^{\gamma+1,\gamma}_{(j)}(i,j))_{kl} = \delta_{kl} \begin{cases} 2e^{-\eta}\cosh\eta, & k=\gamma+1, \\ (2a_{ji}\cosh\eta)^{-1}\, e^{\eta}, & k=\gamma, \\ 1, & \text{otherwise}, \end{cases} \tag{5.6}$$

• *For* $\gamma = m$,

$$(G^{\gamma,\gamma+1}_{(j)}(i,j))_{kl} = \delta_{kl} \begin{cases} 2e^{-\eta}\cosh\eta, & k=\gamma, \\ e^{-\eta}, & k=\gamma+1, \\ 1, & \text{otherwise}, \end{cases} \tag{5.7}$$

$$(G^{\gamma+1,\gamma}_{(j)}(i,j))_{kl} = \delta_{kl} \begin{cases} (2a_{ji}\cosh\eta)^{-1}e^{\eta}, & k=\gamma, \\ (a_{ji})^{-1}\, e^{\eta}, & k=\gamma+1, \\ 1, & \text{otherwise}, \end{cases} \tag{5.8}$$

• *For* $1 + m \leq \gamma < n + m$,

$$(G^{\gamma,\gamma+1}_{(j)}(i,j))_{kl} = \delta_{kl} \begin{cases} (a_{ij})^{-1}\, e^{\eta}, & k=\gamma, \\ e^{-\eta}, & k=\gamma+1, \\ 1, & \text{otherwise}, \end{cases} \tag{5.9}$$

$$(G^{\gamma+1,\gamma}_{(j)}(i,j))_{kl} = \delta_{kl} \begin{cases} (a_{ji})^{-1}\, e^{-\eta}, & k=\gamma+1, \\ e^{\eta}, & k=\gamma, \\ 1, & \text{otherwise}. \end{cases} \tag{5.10}$$

Proof. Using (3.26)-(3.29), Proposition 1 and the composition law of R-matrices (3.11), one can prove the theorem. Here, without losing generality, we give the proof for the generator $\tilde{E}_{1,2}$ as an example.

From the expressions of $F_{1\ldots N}$ and its inverse, we have

$$\begin{aligned} \tilde{E}_{1,2} &= \sum_{\sigma,\sigma'\in\mathcal{S}_N} \sum^{*}_{\alpha_{\sigma(1)}\cdots\alpha_{\sigma(N)}} \sum^{**}_{\beta_{\sigma'(1)}\cdots\beta_{\sigma'(N)}} S(\sigma,\alpha_\sigma)S(\sigma',\beta_{\sigma'}) \\ &\quad \times \prod_{i=1}^{N} P^{\alpha_{\sigma(i)}}_{\sigma(i)} R^{\sigma}_{1\ldots N}(E^{1,2})_{1\ldots N} R^{\sigma'^{-1}}_{\sigma'(1\ldots N)} \prod_{i=1}^{N} P^{\beta_{\sigma'(i)}}_{\sigma'(i)} \prod_{i<j} \Delta^{-1}_{ij} \\ &= \sum_{\sigma,\sigma'\in\mathcal{S}_N} \sum^{*}_{\alpha_{\sigma(1)}\cdots\alpha_{\sigma(N)}} \sum^{**}_{\beta_{\sigma'(1)}\cdots\beta_{\sigma'(N)}} S(\sigma,\alpha_\sigma)S(\sigma',\beta_{\sigma'}) \\ &\quad \times \prod_{i=1}^{N} P^{\alpha_{\sigma(i)}}_{\sigma(i)} [(\mathcal{P}^{\sigma}_{1\ldots N}(E^{1,2})_{1,\ldots,N}\mathcal{P}^{\sigma^{-1}}_{1\ldots N})] R^{\sigma'^{-1}\sigma}_{\sigma'(1\ldots N)} \prod_{i=1}^{N} P^{\beta_{\sigma'(i)}}_{\sigma'(i)} \prod_{i<j} \Delta^{-1}_{ij} \end{aligned} \tag{5.11}$$

$$= \sum_{\sigma,\sigma' \in \mathcal{S}_N} \sum_{k=1}^{N} E^{1,2}_{(\sigma(l))} q^{\sum_{i=l+1}^{N} h^1_{(\sigma(i))}} \sum^{*}_{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(N)}} \sum^{**}_{\beta_{\sigma'(1)}\ldots\beta_{\sigma'(N)}} S(\sigma, \alpha_\sigma) S(\sigma', \beta_{\sigma'})$$

$$\times P^{\alpha_{\sigma(1)}=1}_{\sigma(1)} \ldots \left(P^{\alpha_{\sigma(l)}=1\to 2}_{\sigma(l)=k} \right) \ldots P^{\alpha_{\sigma(N)}}_{\sigma(N)} R^{\sigma'^{-1}\sigma}_{\sigma'(1\ldots N)} \prod_{i=1}^{N} P^{\beta_{\sigma'(i)}}_{\sigma'(i)} \prod_{i<j} \Delta^{-1}_{ij}, \qquad (5.12)$$

where in (5.11) we have used (3.19) and l stands for indices between 1 and N such that $\sigma(l) = k$. Here and below, $1 \to 2$ means that 1 is changed to 2; thus $\alpha_{\sigma(l)} = 1 \to 2$ in (5.12) means that $\alpha_{\sigma(l)} = 1$ is replaced by $\alpha_{\sigma(l)} = 2$ on the site $\sigma(l) = k$. The element of $R^{\sigma'^{-1}\sigma}_{\sigma'(1\ldots N)}$ between $P^{\alpha_{\sigma(1)}=1}_{\sigma(1)} \ldots \left(P^{\alpha_{\sigma(l)}=1\to 2}_{\sigma(l)=k}\right) \ldots P^{\alpha_{\sigma(N)}}_{\sigma(N)}$ and $P^{\beta_{\sigma'(N)}}_{\sigma'(N)} \ldots P^{\beta_{\sigma'(1)}}_{\sigma'(1)}$ is denoted as

$$\left(R^{\sigma'^{-1}\sigma}_{\sigma'(1\ldots N)} \right)^{\overset{\sigma(N)}{\alpha_{\sigma(N)}} \ldots \overset{\sigma(l)=k}{1\to 2} \ldots \overset{\sigma(1)}{1}}_{\beta_{\sigma'(N)}\ldots\beta_{\sigma'(1)}} . \qquad (5.13)$$

We call the sequence $\{\alpha_{\sigma(l)}\}$ **normal** if it is arranged according to the rules in (4.3), otherwise, we call it **abnormal**.

It is now convenient for us to discuss the non-vanishing condition of the R-matrix element (5.13). Comparing (5.13) with (4.15), we find that the difference between them lies in the k^{th} site. Because the group label in the k^{th} space has been changed, the sequence $\{\alpha_\sigma\}$ is now an abnormal sequence. However, it can be permuted to the normal sequence by some permutation $\hat{\sigma}_k$. Namely, $\alpha_{1\to 2}$ in the abnormal sequence can be moved to a suitable position by using the permutation $\hat{\sigma}_k$ according to rules in (4.3). (It is easy to verify that $\hat{\sigma}_k$ is unique by using (4.3).) Thus, by procedure similar to that in the previous section, we find that when

$$\sigma' = \hat{\sigma}_k \sigma \bar{\sigma} \quad \text{and} \quad \beta_{\sigma'(N)} = \alpha_{\sigma(1)}, \ldots, \beta_{\sigma'(1)} = \alpha_{\sigma(N)}, \qquad (5.14)$$

the R-matrix element (5.13) is non-vanishing.

Because the non-zero condition of the elementary R-matrix element $R^{i'j'}_{ij}$ is $i + j = i' + j'$, the following R-matrix elements:

$$\left(R^{\sigma'^{-1}\sigma}_{\sigma'(1\ldots N)} \right)^{\overset{\sigma(N)}{\alpha_{\sigma(N)}} \ldots \overset{\sigma(l)=k}{1} \ldots \overset{\sigma(p)}{1\to 2} \ldots \overset{\sigma(1)}{1}}_{\beta_{\sigma'(N)}\ldots\beta_{\sigma'(1)}} , \qquad (5.15)$$

with $1 \le p \le l$ are also non-vanishing.

Therefore, (5.12) becomes

$$\tilde{E}_{1,2} = \sum_{\sigma\in\mathcal{S}_N} \sum_{k=1}^{N} \sum^{*}_{\alpha_{\sigma_1}\ldots\alpha_{\sigma_N}} S(\sigma, \alpha_\sigma) S(\hat{\sigma}_k\sigma, \alpha_{\hat{\sigma}_k\sigma})$$

$$\times \Big[E^{1,2}_{(\sigma(l))} q^{\sum_{i=l+1}^{N} h^1_{(\sigma(i))}} P^{\alpha_{\sigma(1)}=1}_{\sigma(1)} \ldots P^{\alpha_{\sigma(l)}=1\to 2}_{\sigma(l)=k} \ldots P^{\alpha_{\sigma(N)}}_{\sigma(N)}$$

$$+ \ldots$$

$$+ E^{1,2}_{(\sigma(p))} q^{\sum_{i=n+1}^{N} h^1_{(\sigma(i))}} P^{\alpha_{\sigma(1)}=1}_{\sigma(1)} \ldots P^{\alpha_{\sigma(p)}=1\to 2}_{\sigma(p)} \ldots P^{\alpha_{\sigma(l)}=1}_{\sigma(l)=k} \ldots P^{\alpha_{\sigma(N)}}_{\sigma(N)}$$
$$+ \ldots$$
$$+ E^{1,2}_{(\sigma(1))} q^{\sum_{i=2}^{N} h^1_{(i)}} P^{\alpha_{\sigma(1)}=1\to 2}_{\sigma(1)} \ldots P^{\alpha_{\sigma(l)}=1}_{\sigma(l)=k} \ldots P^{\alpha_{\sigma(N)}}_{\sigma(N)} \Big]$$
$$\times R^{\bar{\sigma}\sigma^{-1}\hat{\sigma}_k^{-1}\sigma}_{\hat{\sigma}_k\sigma(N\ldots 1)} \prod_{i=1}^{N} P^{\alpha_{\hat{\sigma}_k\sigma(i)}}_{\hat{\sigma}_k\sigma(i)} \prod_{i<j} \Delta^{-1}_{ij} \tag{5.16}$$
$$= \sum_{k=1}^{N} E^{1,2}_{(k)} \otimes_{j\neq k} G^{1,2}_{(j)}(k, j), \tag{5.17}$$

where index p runs between 1 and l and $\hat{\sigma}_k$ is the element of $\mathcal{S}_N$ which permutes the first abnormal sequence in the square bracket of (5.16) to normal sequence. Using the similar procedure, one can prove the theorem for other generators. □

The non-simple generators $\tilde{E}_{\gamma,\gamma\pm l}$ (for $l \geq 2$) can be obtained through the simple ones by (2.10) and (2.11).

Some remarks are in order. Equation (5.2) implies that

$$\tilde{H}_j = H_j, \quad j = 1, \ldots, N, \tag{5.18}$$

which will be used frequently. In the rational limit: $\eta \to 0$ (or $q \to 1$), our results reduce to those in [11] and those in [7] for the special case $m = 0$.

5.2. Creation operators in the F-basis. Among the matrix elements of the mondromy matrix $T_{i,j}(u)$, the operators $T_{n+m,n+m-l}(u)$ ($l = 1, \ldots, n+m-1$) are called creation operators [5] and are usually denoted by

$$C_{n+m-l}(u) = T_{n+m,n+m-l}(u), \quad l = 1, \ldots, n+m-1. \tag{5.19}$$

In the F-basis, they become

$$\tilde{C}_{n+m-l}(u) = F_{1\ldots N} C_{n+m-l}(u) F^{-1}_{1\ldots N}, \; l = 1, \ldots, n+m-1. \tag{5.20}$$

Let us denote $T_{n+m,n+m}(u)$ by $D(u)$ and the corresponding operator in the F-basis by

$$\tilde{D}(u) = F_{1\ldots N} D(u) F^{-1}_{1\ldots N}. \tag{5.21}$$

Proposition 5. *$\tilde{D}(u)$ is a diagonal matrix given by*

$$\tilde{D}(u) = \otimes_{i=1}^{N} diag\,(a_{0i}, \ldots, a_{0i}, 1)_{(i)}\,. \tag{5.22}$$

Proof. From (3.20), we derive that

$$D(u) P_0^{n+m} = T_{n+m,n+m}(u) e_{n+m,n+m} = P_0^{n+m} T_{0,1\ldots N}(u) P_0^{n+m}. \tag{5.23}$$

Acting the F-matrix from the left on the both sides of the above equation, we have

$$\begin{aligned} F_{1\ldots N} D(u) P_0^{n+m} &= \sum_{\sigma \in \mathcal{S}_N} \sum_{\alpha_{\sigma(1)} \ldots \alpha_{\sigma(N)}}^{*} S(\sigma, \alpha_\sigma) \prod_{i=1}^{N} P_{\sigma(i)}^{\alpha_\sigma} R_{1\ldots N}^{\sigma} P_0^{m+n} T_{0,1\ldots N}(u) P_0^{m+n} \\ &= \sum_{\sigma \in \mathcal{S}_N} \sum_{\alpha_{\sigma(1)} \ldots \alpha_{\sigma(N)}}^{*} S(\sigma, \alpha_\sigma) \prod_{i=1}^{N} P_{\sigma(i)}^{\alpha_\sigma} P_0^{m+n} T_{0,\sigma(1\ldots N)}(u) P_0^{m+n} R_{1\ldots N}^{\sigma}. \end{aligned} \tag{5.24}$$

Following [7], we can split the sum $\sum^*$ according to the number of occurrences of the index $m+n$,

$$\begin{aligned} F_{1\ldots N} D(u) P_0^{n+m} = \sum_{\sigma \in \mathcal{S}_N} \sum_{k=0}^{N} \sum_{\alpha_{\sigma(1)} \ldots \alpha_{\sigma(N)}}^{*} S(\sigma, \alpha_\sigma) \prod_{j=N-k+1}^{N} \delta_{\alpha_{\sigma(j)}, m+n} P_{\sigma(j)}^{\alpha_{\sigma(j)}} \\ \times \prod_{j=1}^{N-k} P_{\sigma(j)}^{\alpha_{\sigma(j)}} P_0^{m+n} T_{0,\sigma(1\ldots N)}(u) P_0^{m+n} R_{1\ldots N}^{\sigma}. \end{aligned} \tag{5.25}$$

Consider the prefactor of $R_{1\ldots N}^{\sigma}$. We have

$$\begin{aligned} &\prod_{j=1}^{N-k} P_{\sigma(j)}^{\alpha_{\sigma(j)}} \prod_{j=N-k+1}^{N} P_{\sigma(j)}^{m+n} P_0^{m+n} T_{0,\sigma(1\ldots N)}(u) P_0^{m+n} \\ &= \prod_{j=1}^{N-k} P_{\sigma(j)}^{\alpha_{\sigma(j)}} \prod_{j=N-k+1}^{N} \left(R_{0\,\sigma(j)}\right)_{m+n\ m+n}^{m+n\ m+n} P_0^{m+n} T_{0,\sigma(1\ldots N-k)}(u) P_0^{m+n} \prod_{j=N-k+1}^{N} P_{\sigma(j)}^{m+n} \\ &= \prod_{j=1}^{N-k} P_{\sigma(j)}^{\alpha_{\sigma(j)}} P_0^{m+n} T_{0,\sigma(1\ldots N-k)}(u) P_0^{m+n} \prod_{j=N-k+1}^{N} P_{\sigma(j)}^{m+n} \\ &= \prod_{i=1}^{N-k} \left(R_{0\,\sigma(i)}\right)_{m+n\alpha_{\sigma(i)}}^{m+n\alpha_{\sigma(i)}} \prod_{j=1}^{N-k} P_{\sigma(j)}^{\alpha_{\sigma(j)}} \prod_{j=N-k+1}^{N} P_{\sigma(j)}^{m+n}\ P_0^{n+m} \\ &= \prod_{i=1}^{N-k} a_{0\,\sigma(i)} \prod_{j=1}^{N-k} P_{\sigma(j)}^{\alpha_{\sigma(j)}} \prod_{j=N-k+1}^{N} P_{\sigma(j)}^{m+n}\ P_0^{n+m}, \end{aligned} \tag{5.26}$$

where $a_{0i} = a(u, z_i)$. Substituting (5.26) into (5.25), we have

$$F_{1\ldots N} D(u) = \otimes_{i=1}^{N} \mathrm{diag}\,(a_{0i}, \ldots, a_{0i}, 1)_{(i)}\, F_{1\ldots N}. \tag{5.27}$$

This completes the proof of the proposition. □

By means of the expressions of the generators of $U_q(gl(m|n))$ in the previous subsection, combining with Theorem 1 and Proposition 5, we have

Theorem 3. *In the F-basis the creation operators $C_{n+m-l}(u)(l = 1, \ldots, n+m-1)$ are given by*

$$\tilde{C}_{n+m-l}(u) = \left(q^{-(-1)^{[n+m]}}\tilde{E}_{n+m-l,n+m}\tilde{D}(u) - \tilde{D}(u)\tilde{E}_{n+m-l,n+m}\right)q^{-\sum_{k=1}^{l} H_{n+m-k}}$$
$$-\sum_{\alpha=1}^{l-1}(1-q^{-2(-1)^{[n+m-\alpha]}})\tilde{C}_{n+m-\alpha}(u)\tilde{E}_{n+m-l,n+m-\alpha}q^{-\sum_{k=\alpha+1}^{l} H_{n+m-k}}. \tag{5.28}$$

For some special values of m and n, we have:

- For $m = 0$ and $n = 2$ (namely, the $U_q(gl(2)$ case), our general results reduce to those in [1].
- For the $U_q(gl(2|1))$ case,

$$\tilde{C}_2(u) = \left(q^{-1}\tilde{E}_{2,3}\tilde{D}(u) - \tilde{D}(u)\tilde{E}_{2,3}\right)q^{-H_2}$$
$$= \sum_{i=1}^{N}\frac{e^{-(u-z_i)}\sinh\eta}{\sinh(u-z_i+\eta)}E_{(i)}^{2,3}$$
$$\otimes_{j\neq i}\mathrm{diag}\left(\frac{\sinh(u-z_j)}{\sinh(u-z_j+\eta)}, \frac{2\sinh(u-z_j)\cosh\eta}{\sinh(u-z_j+\eta)}, 1\right)_{(j)}, \tag{5.29}$$

$$\tilde{C}_1(u) = \left(q^{-1}\tilde{E}_{1,3}\tilde{D}(u) - \tilde{D}(u)\tilde{E}_{1,3}\right)q^{-H_1-H_2} - (1-q^2)\tilde{C}_2(u)\tilde{E}_{1,2}q^{-H_1}$$
$$= \sum_{i=1}^{N}\frac{e^{-(u-z_i)}\sinh\eta}{\sinh(u-z_i+\eta)}E_{(i)}^{1,3}\otimes_{j\neq i}\mathrm{diag}\left(\frac{2\sinh(u-z_j)\cosh\eta}{\sinh(u-z_j+\eta)},\right.$$
$$\left.\frac{\sinh(u-z_j)\sinh(z_i-z_j+\eta)}{\sinh(z_i-z_j)\sinh(u-z_j+\eta)}, 1\right)_{(j)}$$
$$+\sum_{i\neq j=1}^{N}\frac{e^{z_j-u}\sinh(u-z_i)\sinh^2\eta[e^{z_j-z_i}+2\sinh(z_i-z_j)]}{\sinh(u-z_i+\eta)\sinh(u-z_j+\eta)\sinh(z_i-z_j)}E_{(i)}^{1,2}\otimes E_{(j)}^{2,3}$$
$$\otimes_{k\neq i,j}\left(\frac{2\sinh(u-z_k)\cosh\eta}{\sinh(u-z_k+\eta)}, \frac{\sinh(u-z_k)\sinh(z_i-z_k+\eta)}{\sinh(z_i-z_k)\sinh(u-z_k+\eta)}, 1\right)_{(k)}. \tag{5.30}$$

6. Bethe Vectors in the F-Basis

The explicit expressions of the creation operators of the $U_q(gl(m|n))$ monodromy matrix in the F-basis, given in the previous section, enable us to resolve the hierarchy of the nested Bethe vectors of the model associated with $U_q(gl(m|n))$. Here, we take the quantum $t-J$ model (i.e. the $U_q(gl(2|1))$-model) as an example to demonstrate the procedure. The generalization to the general $U_q(gl(m|n))$ case is straightforward.

In the framework of the standard algebraic Bethe ansatz method [19], the Bethe vector of the quantum supersymmetric t-J model is given by

$$\Omega_N = \sum_{d_1\ldots d_\alpha}(\Omega_\alpha^{(1)})^{d_1\ldots d_\alpha}C_{d_1}(v_1)\ldots C_{d_\alpha}(v_\alpha)|vac\rangle, \tag{6.1}$$

where α is a positive integer and $d_i = 1, 2$, $|vac\rangle$ is the pseudo-vacuum state

$$|vac\rangle = \otimes_{i=1}^{N} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{(i)} , \tag{6.2}$$

and $(\Omega_\alpha^{(1)})^{d_1 \ldots d_\alpha}$ are functions of the spectral parameters v_j, and are the vector components of the nested Bethe vector $\Omega^{(1)}$. The nested Bethe vector $\Omega^{(1)}$ is given by

$$\Omega_\alpha^{(1)} = C^{(1)}(v_1^{(1)})C^{(1)}(v_2^{(1)}) \cdots C^{(1)}(v_\beta^{(1)})|vac\rangle^{(1)}, \tag{6.3}$$

where β is a positive integer and $|vac\rangle^{(1)}$ is the nested pseudo-vacuum state

$$|vac\rangle^{(1)} = \otimes_{i=1}^{\alpha} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{(i)} . \tag{6.4}$$

The creation operator of the nested $U_q(gl(2))$ system $C^{(1)}(u)$ is the lower-triangular entry of the nested monodromy matrix $T^{(1)}(v^{(1)})$,

$$\begin{aligned} T^{(1)}(v^{(1)}) &= r_{0\alpha}(v^{(1)} - v_\alpha) r_{0\,\alpha-1}(v^{(1)} - v_{\alpha-1}) \ldots r_{01}(v^{(1)} - v_1) \\ &\equiv \begin{pmatrix} A^{(1)}(v^{(1)}) & B^{(1)}(v^{(1)}) \\ C^{(1)}(v^{(1)}) & D^{(1)}(v^{(1)}) \end{pmatrix}, \end{aligned} \tag{6.5}$$

and the nested R-matrix is

$$r_{12}(u_1, u_2) \equiv r_{12}(u_1 - u_2) = \begin{pmatrix} c_{12} & 0 & 0 & 0 \\ 0 & a_{12} & -b_{12}^{+} & 0 \\ 0 & -b_{12}^{-} & a_{12} & 0 \\ 0 & 0 & 0 & c_{12} \end{pmatrix}. \tag{6.6}$$

Equation (6.5) implies that the nested system is a $U_q(gl(2))$ spin chain on α-site lattice and the corresponding inhomogeneous parameters are $\{v_i | i = 1, \ldots, \alpha\}$. So, in the following, we shall adopt the same convention for the nested system as that in previous sections but the inhomogeneous parameters will be replaced by $\{v_i\}$.

Acting the associated F-matrix on the pseudo-vacuum state (6.2), one finds that the pseudo-vacuum state is invariant. It is due to the fact that only terms with roots equal to 3 will produce non-zero results. Therefore, the $U_q(gl(2|1))$ Bethe vector (6.1) in the F-basis can be written as

$$\begin{aligned} \tilde{\Omega}_N(v_1, \ldots, v_\alpha) &\equiv F_{1\ldots N} \Omega_N(v_1, \ldots, v_\alpha) \\ &= \sum_{d_1 \ldots d_\alpha} (\Omega_\alpha^{(1)})^{d_1 \ldots d_\alpha} \tilde{C}_{d_1}(v_1) \ldots \tilde{C}_{d_\alpha}(v_\alpha)|vac\rangle. \end{aligned} \tag{6.7}$$

The c-number coefficient $(\Omega_\alpha^{(1)})^{d_1 \ldots d_\alpha}$ has to be evaluated in the original basis, not in the F-basis.

6.1. Nested Bethe vectors in the F-basis. Let us first compute the nested Bethe vectors in the F-basis. For the nested R-matrix (6.6), we define the α-site F and F^*-matrices by

$$F^{(1)}_{1\ldots\alpha} = \sum_{\sigma\in\mathcal{S}_\alpha} \sum^{2\ \ *}_{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(\alpha)}=1} \prod_{j=1}^{\alpha} P^{\alpha_{\sigma(j)}}_{\sigma(j)} \bar{S}^{(1)}(c,\sigma,\alpha_\sigma) r^{\sigma}_{1\ldots\alpha}, \tag{6.8}$$

$$F^{*(1)}_{1\ldots\alpha} = \sum_{\sigma\in\mathcal{S}_\alpha} \sum^{2\ \ **}_{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(\alpha)}=1} \bar{S}^{(1)}(c,\sigma,\alpha_\sigma) r^{\sigma^{-1}}_{\sigma(1\ldots\alpha)} \prod_{j=1}^{\alpha} P^{\alpha_{\sigma(j)}}_{\sigma(j)}, \tag{6.9}$$

respectively, where the c-number function $\bar{S}^{(1)}$ is given by

$$\bar{S}^{(1)}(c,\sigma,\alpha_\sigma) \equiv \exp\{\sum_{l>k=1}^{\alpha} \delta_{\alpha_{\sigma(k)},\alpha_{\sigma(l)}} \ln(1+c_{\sigma(k)\sigma(l)})\}. \tag{6.10}$$

Therefore the inverse of the F-matrix can be represented in terms of the F^*-matrix as

$$(F^{(1)}_{1\ldots\alpha})^{-1} = F^{*(1)}_{1\ldots\alpha} \prod_{i<j} (\bar{\Delta}^{(1)}_{ij})^{-1}, \tag{6.11}$$

with

$$\begin{aligned}\bar{\Delta}^{(1)}_{ij} = \mathrm{diag}\Bigg(&\frac{4\sinh^2(v_i-v_j)\cosh^2\eta}{\sinh(v_i-v_j+\eta)\sinh(v_i-v_j-\eta)}, \frac{\sin(v_j-v_i)}{\sinh(v_j-v_i+\eta)},\\ &\frac{\sinh(v_i-v_j)}{\sinh(v_i-v_j+\eta)}, \frac{4\sinh^2(v_i-v_j)\cosh^2\eta}{\sinh(v_i-v_j+\eta)\sinh(v_i-v_j-\eta)}\Bigg).\end{aligned}$$

With the help of the F-matrix (6.8) and its inverse, one may compute the α-site $U_q(gl(2))$ generator $\tilde{E}_{1,2}$ in the F-basis,

$$\tilde{E}_{1,2} = \sum_{i=1}^{\alpha} E^{1,2}_{(i)} \otimes_{j\neq i} \left(2e^{-\eta}\cosh\eta, \frac{e^{\eta}\sinh(v_i-v_j+\eta)}{2\sinh(v_i-v_j)\cosh\eta}\right)_{(j)}. \tag{6.12}$$

Define $\tilde{D}^{(1)}(u) = F^{(1)}_{1\ldots\alpha} D^{(1)}(u) (F^{(1)}_{1\ldots\alpha})^{-1}$. From Proposition 5, we obtain

$$\tilde{D}^{(1)}(u) = \otimes_{i=1}^{\alpha} \left(\frac{\sinh(u-v_i)}{\sinh(u-v_i+\eta)}, \frac{\sinh(u-v_i-\eta)}{\sinh(u-v_i+\eta)}\right)_{(i)}. \tag{6.13}$$

The creation operator in the F-basis is then obtained with the help of the nested R-matrix (6.6) and Theorem 3 in the case of $m=2$ and $n=0$, and is given by

$$\begin{aligned}\tilde{C}^{(1)}(u) &= (q\tilde{E}_{1,2}\tilde{D}(u) - \tilde{D}(u)\tilde{E}_{1,2})q^{-H_1}\\ &= -\sum_{i=1}^{\alpha} \frac{e^{-(u-v_i)}\sinh\eta}{\sinh(u-v_i+\eta)} E^{1,2}_{(i)} \otimes_{j\neq i} \Bigg(\frac{2\sinh(u-v_j)\cosh\eta}{\sinh(u-v_j+\eta)},\\ &\qquad \frac{\sinh(v_i-v_j+\eta)\sinh(u-v_j-\eta)}{2\sinh(v_i-v_j)\sinh(u-v_j+\eta)\cosh\eta}\Bigg)_{(j)}.\end{aligned} \tag{6.14}$$

Applying $F^{(1)}_{1\ldots\alpha}$ to the nested Bethe vector $\Omega^{(1)}_\alpha$ (6.3), we obtain

$$\begin{aligned}\tilde{\Omega}^{(1)}_\alpha(v^{(1)}_1,\ldots,v^{(1)}_\beta) &\equiv F^{(1)}_{1\ldots\alpha}\Omega^{(1)}(v^{(1)}_1,\ldots,v^{(1)}_\beta)\\ &= s(c)\tilde{C}^{(1)}(v^{(1)}_1)\tilde{C}^{(1)}(v^{(1)}_2)\ldots\tilde{C}^{(1)}(v^{(1)}_\beta)|vac\rangle^{(1)}, \quad (6.15)\end{aligned}$$

where we have used $F^{(1)}_{1\ldots\alpha}|vac\rangle^{(1)} = \prod_{i<j}(1+c_{ij})|vac\rangle^{(1)} \equiv s(c)|vac\rangle^{(1)}$.

Substituting $\tilde{C}^{(1)}(v)$ into (6.15), we obtain

$$\begin{aligned}&\tilde{\Omega}^{(1)}_\alpha(v^{(1)}_1,\ldots,v^{(1)}_\beta) = s(c)\tilde{C}^{(1)}(v^{(1)}_1)\ldots\tilde{C}^{(1)}(v^{(1)}_\beta)\,|vac\rangle^{(1)}\\ &= s(c)\sum_{i_1<\ldots<i_\beta} B^{(1)}_\beta(v^{(1)}_1,\ldots,v^{(1)}_\beta|v_{i_1},\ldots,v_{i_\beta})E^{1,2}_{(i_1)}\ldots E^{1,2}_{(i_\beta)}\,|vac\rangle^{(1)}, \quad (6.16)\end{aligned}$$

where

$$\begin{aligned}&B^{(1)}_\beta(v^{(1)}_1,\ldots,v^{(1)}_\beta|v_1,\ldots,v_\beta)\\ &= \sum_{\sigma\in\mathcal{S}_\beta}\prod_{k=1}^{\beta}\left(-\frac{e^{-(v^{(1)}_k-v_{\sigma(k)})}\sinh\eta}{\sinh(v^{(1)}_k-v_{\sigma(k)}+\eta)}\right)\\ &\times\prod_{j\neq\sigma(k),\ldots,\sigma(\beta)}^{\alpha}\frac{\sinh(v_{\sigma(k)}-v_j+\eta)\sinh(v^{(1)}_k-v_j-\eta)}{2\sinh(v_{\sigma(k)}-v_j)\sinh(v^{(1)}_k-v_j+\eta)\cosh\eta}\\ &\times\prod_{l=k+1}^{\beta}\frac{2\cosh\eta\sinh(v^{(1)}_k-v_{\sigma(l)})}{\sinh(v^{(1)}_k-v_{\sigma(l)}+\eta)}. \quad (6.17)\end{aligned}$$

6.2. Bethe vectors of the quantum supersymmetric t-J model in the F-basis. Now back to the Bethe vector (6.7) of the quantum supersymmetric t-J model. As is shown in Appendix B, the Bethe vector is invariant (modulo overall factor) under the exchange of arbitrary spectral parameters:

$$\tilde{\Omega}_N(v_{\sigma(1)},\ldots,v_{\sigma(\alpha)}) = \frac{1}{c^\sigma_{1\ldots\alpha}}\tilde{\Omega}_N(v_1,\ldots,v_\alpha),\ \sigma\in\mathcal{S}_\alpha, \quad (6.18)$$

where $c^\sigma_{1\ldots\alpha}$ has the decomposition law

$$c^{\sigma'\sigma}_{1\ldots\alpha} = c^\sigma_{\sigma'(1\ldots\alpha)}c^{\sigma'}_{1\ldots\alpha}, \quad (6.19)$$

and $c^{\sigma_i}_{1\ldots\alpha} = c_{i\ i+1} \equiv c(v_i, v_{i+1})$ for an elementary permutation σ_i.

This result is a generalization of that in [20, 21]. This invariance enables us to concentrate on a particularly simple term in the sum (6.7) of the following form with p_1 number of $d_i = 1$ and $\alpha - p_1$ number of $d_j = 2$,

$$\tilde{C}_1(v_1)\ldots\tilde{C}_1(v_{p_1})\tilde{C}_2(v_{p_1+1})\ldots\tilde{C}_2(v_\alpha). \quad (6.20)$$

In the F-basis, the commutation relation between $C_i(v)$ and $C_j(u)$, i.e. (B.4), becomes

$$\tilde{C}_i(v)\tilde{C}_j(u) = -\frac{1}{a(u,v)}\tilde{C}_j(u)\tilde{C}_i(v) + \frac{b(u,v)}{a(u,v)}\tilde{C}_j(v)\tilde{C}_i(u). \quad (6.21)$$

Then using (6.21), all $\tilde{C}_1$'s in (6.20) can be moved to the right of all $\tilde{C}_2$'s, yielding

$$\begin{aligned} &\tilde{C}_1(v_1)\ldots\tilde{C}_1(v_{p_1})\tilde{C}_2(v_{p_1+1})\ldots\tilde{C}_2(v_\alpha) = \\ &= g(v_1,\ldots,v_\alpha)\tilde{C}_2(v_{p_1+1})\ldots\tilde{C}_2(v_\alpha)\tilde{C}_1(v_1)\ldots\tilde{C}_1(v_{p_1}) + \ldots, \end{aligned} \tag{6.22}$$

where $g(v_1,\ldots,v_\alpha) = \prod_{k=1}^{p_1}\prod_{l=p_1+1}^{\alpha}(-1/a(v_l,v_k))$ is the contribution from the first term of (6.21) and "..." stands for other terms contributed by the second term of (6.21). It is easy to see that these other terms have the form

$$\tilde{C}_2(v_{\sigma(p_1+1)})\ldots\tilde{C}_2(v_{\sigma(\alpha)})\tilde{C}_1(v_{\sigma(1)})\ldots\tilde{C}_1(v_{\sigma(p_1)}), \tag{6.23}$$

with $\sigma \in \mathcal{S}_\alpha$. Substituting (6.22) into the Bethe vector (6.7), we obtain

$$\begin{aligned} \tilde{\Omega}_N^{p_1}(v_1,\ldots,v_\alpha) = (\Omega_\alpha^{(1)})^{11\ldots12\ldots2}\prod_{k=1}^{p_1}\prod_{l=p_1+1}^{\alpha}\left(-\frac{1}{a(v_l,v_k)}\right) \\ \times\,\tilde{C}_2(v_{p_1+1})\ldots\tilde{C}_2(v_\alpha)\tilde{C}_1(v_1)\ldots\tilde{C}_1(v_{p_1})|vac\rangle + \ldots, \end{aligned} \tag{6.24}$$

where and below, we use the up-index p_1 to denote the Bethe vector corresponding to the quantum number p_1. All other terms in (6.24) (denoted as " ... ") are to be obtained from the first term by the permutation (exchange) symmetry. Then we have,

$$\begin{aligned} &\tilde{\Omega}_N^{p_1}(v_1,\ldots,v_\alpha) = \\ &= \frac{1}{p_1!(\alpha-p_1)!}\sum_{\sigma\in\mathcal{S}_\alpha} c^\sigma_{1\ldots\alpha}(\Omega_\alpha^{(1),\sigma})^{11\ldots12\ldots2}\prod_{k=1}^{p_1}\prod_{l=p_1+1}^{\alpha}\left(-\frac{1}{a(v_{\sigma(l)},v_{\sigma(k)})}\right) \\ &\times\,\tilde{C}_2(v_{\sigma(p_1+1)})\ldots\tilde{C}_2(v_{\sigma(\alpha)})\tilde{C}_1(v_{\sigma(1)})\ldots\tilde{C}_1(v_{\sigma(p_1)})\,|vac\rangle\,, \end{aligned} \tag{6.25}$$

where $(\Omega_\alpha^{(1),\sigma})^{11\ldots12\ldots2} \equiv (\hat{f}_\sigma\Omega_\alpha^{(1)})^{11\ldots12\ldots2}$ with $\hat{f}_\sigma$ defined by (B.1) in Appendix B.

We now show that $(\Omega_\alpha^{(1)})^{1\ldots12\ldots2}$ in (6.25), which has to be evaluated in the original basis, is invariant (modulo an overall factor) under the action of the $U_q(gl(2))$ F-matrix, i.e.

$$(\Omega_\alpha^{(1)})^{11\ldots12\ldots2} = (t(c))^{-1}(\tilde{\Omega}_\alpha^{(1)})^{11\ldots12\ldots2}, \tag{6.26}$$

where the scalar factor $t(c)$ is

$$t(c) = \prod_{j>i=1}^{p_1}(1+\bar{c}_{ij})\prod_{j>i=p_1+1}^{\alpha}(1+\bar{c}_{ij}),\ \ \bar{c}_{ij} = c(v_i,v_j), \tag{6.27}$$

so that it can be expressed in the form of (6.16).

Write the nested pseudo-vacuum vector in (6.4) as

$$|vac\rangle^{(1)} \equiv |2\cdots 2\rangle^{(1)}, \tag{6.28}$$

where the number of 2 is α. Then the nested Bethe vector (6.15) can be rewritten as

$$\Omega_\alpha^{(1)}(v_1^{(1)}\dots v_{p_1}^{(1)}) \equiv |\Omega_\alpha^{(1)}\rangle = \sum_{d_1\dots d_\alpha} (\Omega_\alpha^{(1)})^{d_1\dots d_\alpha}|d_1\dots d_\alpha\rangle^{(1)}. \tag{6.29}$$

Acting the $U_q(gl(2))$ F-matrix $F^{(1)}_{1\dots\alpha}$ from the left on the above equation, we have

$$\tilde{\Omega}_\alpha^{(1)}(v_1^{(1)}\dots v_{p_1}^{(1)}) \equiv |\tilde{\Omega}_\alpha^{(1)}\rangle = F^{(1)}_{1\dots\alpha}|\Omega_\alpha^{(1)}\rangle = \sum_{d_1\dots d_\alpha} (\tilde{\Omega}_\alpha^{(1)})^{d_1\dots d_\alpha}|d_1\dots d_\alpha\rangle^{(1)}. \tag{6.30}$$

It follows that

$$\begin{aligned}
(\tilde{\Omega}_\alpha^{(1)})^{1\dots 12\dots 2} &= \langle 1\dots 12\dots 2|\tilde{\Omega}_\alpha^{(1)}\rangle = \langle 1\dots 12\dots 2|F^{(1)}_{1\dots\alpha}|\Omega_\alpha^{(1)}\rangle \\
&= \langle 1\dots 12\dots 2|\sum_{\sigma\in\mathcal{S}_\alpha}\sum^{*}_{\alpha_{\sigma(1)}\dots\alpha_{\sigma(\alpha)}}\prod_{j=1}^{\alpha} P^{\alpha_{\sigma(j)}}_{\sigma(j)}\bar{S}^{(1)}(c,\sigma,\alpha_\sigma)R^{\sigma}_{1\dots\alpha}|\Omega_\alpha^{(1)}\rangle \qquad (6.31)\\
&= \langle 1\dots 12\dots 2|\left\{\sum^{*}_{\alpha_{\sigma(1)}\dots\alpha_{\sigma(\alpha)}}\prod_{j=1}^{\alpha} P^{\alpha_{\sigma(j)}}_{\sigma(j)}\right\}\Bigg|_{\sigma=id}\bar{S}^{(1)}(c,\sigma,\alpha_\sigma)|\Omega_\alpha^{(1)}\rangle \qquad (6.32)\\
&= t(c)\langle 1\dots 12\dots 2|\Omega_\alpha^{(1)}\rangle = t(c)(\Omega_\alpha^{(1)})^{1\dots 12\dots 2}, \qquad (6.33)
\end{aligned}$$

where the scalar factor $t(c)$ is given by (6.27).

Summarizing, we propose the following form of the $U_q(gl(2|1))$ Bethe vector

$$\begin{aligned}
&\tilde{\Omega}_N^{p_1}(v_1,\dots,v_\alpha) = \\
&= \frac{1}{p_1!(\alpha-p_1)!}\sum_{\sigma\in\mathcal{S}_\alpha} c^{\sigma}_{1\dots\alpha}\prod_{i=1}^{p_1}\prod_{j=p_1+1}^{\alpha}\frac{2\sinh(v_{\sigma(i)}-v_{\sigma(j)})\cosh\eta}{\sinh(v_{\sigma(i)}-v_{\sigma(j)}+\eta)} \\
&\times B^{(1)}_{p_1}(v_1^{(1)},\dots,v_{p_1}^{(1)}|v_{\sigma(1)},\dots,v_{\sigma(p_1)}) \\
&\times\prod_{k=1}^{p_1}\prod_{l=p_1+1}^{\alpha}\left(-\frac{1}{a(v_{\sigma(l)},v_{\sigma(k)})}\right)\tilde{C}_2(v_{\sigma(p_1+1)})\dots\tilde{C}_2(v_{\sigma(\alpha)}) \\
&\times\tilde{C}_1(v_{\sigma(1)})\dots\tilde{C}_1(v_{\sigma(p_1)})\,|vac\rangle\,.
\end{aligned} \tag{6.34}$$

Substituting (5.29) and (5.30) into the above relation, we finally have

Proposition 6. *The nested Bethe vector of the quantum $t-J$ model is given by*

$$\tilde{\Omega}_N^{p_1}(v_1,\ldots,v_\alpha) = \frac{1}{p_1!(\alpha-p_1)!}\sum_{i_1<\ldots<i_{p_1}}\sum_{i_{p_1+1}<\ldots<i_\alpha} B_{\alpha,p_1}(v_1,\ldots,v_\alpha;v_1^{(1)},\ldots,v_{p_1}^{(1)}|z_{i_1},\ldots,z_{i_\alpha}) \times\prod_{j=p_1+1}^{\alpha}E_{(i_j)}^{2,3}\prod_{j=1}^{p_1}E_{(i_j)}^{1,3}|vac\rangle, \quad (6.35)$$

where $\{i_1,i_2,\ldots,i_{p_1}\}\cap\{i_{p_1+1},i_{p_1+2},\ldots,i_\alpha\}=\varnothing$ *and*

$$B_{\alpha,p_1}(v_1,\ldots,v_\alpha;v_1^{(1)},\ldots,v_{p_1}^{(1)}|z_{i_1},\ldots,z_{i_\alpha}) = \sum_{\sigma\in S_\alpha}c_{1\ldots\alpha}^{\sigma}\prod_{i=1}^{p_1}\prod_{j=p_1+1}^{\alpha}\frac{2\sinh(v_{\sigma(i)}-v_{\sigma(j)})\cosh\eta}{\sinh(v_{\sigma(i)}-v_{\sigma(j)}+\eta)} \times\prod_{k=1}^{p_1}\prod_{l=p_1+1}^{\alpha}\left(-\frac{\sinh(v_{\sigma(l)}-z_{i_k})\sinh(v_{\sigma(l)}-v_{\sigma(k)}+\eta)}{\sinh(v_{\sigma(l)}-v_{\sigma(k)})\sinh(v_{\sigma(l)}-z_{i_k}+\eta)}\right) \times B^*_{\alpha-p_1}(v_{\sigma(p_1+1)},\ldots,v_{\sigma(\alpha)}|z_{i_{p_1+1}},\ldots,z_{i_\alpha}) \times B^{(1)}_{p_1}(v_1^{(1)},\ldots,v_{p_1}^{(1)}|v_{\sigma(1)},\ldots,v_{\sigma(p_1)})B^*_{p_1}(v_{\sigma(1)},\ldots,v_{\sigma(p_1)}|z_{i_1},\ldots,z_{i_{p_1}}). \quad (6.36)$$

Here the function $B_p^*(v_1,\ldots,v_p|z_1,\ldots,z_p)$ *is given*

$$B_p^*(v_1,\ldots,v_p|z_1,\ldots,z_p) = \sum_{\sigma\in\mathcal{S}_p} sign(\sigma)\prod_{k=1}^{p}\frac{e^{-(v_k-z_{\sigma(k)})}\sinh\eta}{\sinh(v_k-z_{\sigma(k)}+\eta)}\prod_{l=k+1}^{p}\frac{2\sinh(v_k-z_{\sigma(l)})\cosh\eta}{\sinh(v_k-z_{\sigma(l)}+\eta)}. \quad (6.37)$$

7. Discussions

We have constructed the factorizing F-matrices for the supersymmetric model associated with quantum superalgebra $U_q(gl(m|n))$ with generic m and n, which includes the quantum supersymmetric t-J model as a special case. We have obtained the completely symmetric representations for the creation operators of the model in the F-basis. Our results make possible a complete resolution of the hierarchy of its nested Bethe vectors. As an example, we have given the explicit expressions of the Bethe vectors of the quantum $t-J$ model in F-basis. Our results are new even for the special $m=0$ case, which give the results of the general model associated with $U_q(gl(n))$. (For $m=0$ and $n=2$, our results reduce to those in [1].)

Authors in [22] solved the quantum inverse problem of the supersymmetric t-J model in the original basis. Namely, they reconstructed the local operators ($E_{(k)}^{i,j}$) in terms of operators figuring in the $gl(2|1)$ monodromy matrix. Their results should be generalizable to the $U_q(gl(2|1))$ case. Then together with the results of the present paper in the F-basis one should be able to get the exact representations of form factors and correlation functions of the quantum supersymmetric t-J model. These are under investigation and results will be reported elsewhere.

Acknowledgements. This work was financially supported by the Australia Research Council. S. -Y. Zhao has also been supported by the UQ Postdoctoral Research Fellowship.

Appendix A: Proof of Theorem 1

Proposition 2 allows one to derive the following equations:

$$\begin{aligned}
(E^{n+m-l,n+m})_{0\ldots N} &= E_{n+m-l,n+m} + e_{n+m-l,n+m} q^{\sum_{k=1}^{l} H_{n+m-k}} \\
&\quad + \sum_{\alpha=1}^{l-1} (1 - q^{-2(-1)^{[n+m-\alpha]}}) e_{n+m-\alpha,n+m}\, E_{n+m-l,n+m-\alpha} q^{\sum_{k=1}^{\alpha} H_{n+m-k}}, \\
&\qquad l = 1, \ldots, n+m-1,
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
\mathcal{P}^{\sigma_c} (E^{n+m-l,n+m})_{0\ldots N} (\mathcal{P}^{\sigma_c})^{-1} &= e_{n+m-l,n+m} + q^{\sum_{k=1}^{l} h_{n+m-k}}\, E_{n+m-l,n+m} \\
&\quad + \sum_{i=1}^{l-1} (1 - q^{-2(-1)^{[n+m-\alpha]}}) e_{n+m-l,n+m-\alpha}\, q^{\sum_{k=1}^{\alpha} h_{n+m-k}} E_{n+m-\alpha,n+m}, \\
&\qquad l = 1, \ldots, n+m-1.
\end{aligned} \tag{A.2}$$

Taking $x = E^{n+m-l,n+m}$ and $\sigma = \sigma_c$ and using (3.14), then (3.19) becomes

$$T(u)\,(E^{n+m-l,n+m})_{0\ldots N} = \mathcal{P}^{\sigma_c}\,(E^{n+m-l,n+m})_{0\ldots N}\,(\mathcal{P}^{\sigma_c})^{-1}\, T(u). \tag{A.3}$$

Substituting (3.20), (A.1) and (A.2) into the above equation, we have, for the L.H.S. of the resulting relation,

$$\begin{aligned}
\text{L.H.S.} &= \sum_{i=1}^{n+m} (-1)^{([i]+[n+m-l])([n+m-l]+[n+m]+1)} e_{i,n+m} T_{i,n+m-l}(u) q^{\sum_{k=1}^{l} H_{n+m-k}} \\
&\quad + \sum_{i,j=1}^{n+m} e_{i,j} (-1)^{[i]+[j]} T_{i,j}(u) E_{n+m-l,n+m} \\
&\quad + \sum_{i=1}^{n+m} \sum_{\alpha=1}^{l-1} (1 - q^{-2(-1)^{[n+m-\alpha]}}) (-1)^{([i]+[n+m-\alpha])([n+m-\alpha]+[n+m]+1)} \\
&\quad \times e_{i,n+m} T_{i,n+m-\alpha}(u) E_{n+m-l,n+m-\alpha} q^{\sum_{k=1}^{\alpha} H_{n+m-k}}.
\end{aligned} \tag{A.4}$$

Similarly for the R.H.S. of the resulting relation, we obtain

$$\begin{aligned}\text{R.H.S.} = &\sum_{i,j=1}^{n+m}(-1)^{([n+m-l]+[n+m]+1)([i]+[j])}q^{\sum_{k=1}^{l}h_{n+m-k}}e_{i,j}E_{n+m-l,n+m}T_{i,j}(u)\\ &+\sum_{i=1}^{n+m}(-1)^{[n+m]+[i]}e_{n+m-l,i}T_{n+m,i}(u)\\ &+\sum_{i=1}^{n+m}\sum_{\alpha=1}^{l-1}(1-q^{-2(-1)^{[n+m-\alpha]}})(-1)^{([n+m-\alpha]+[i])([n+m-\alpha]+[n+m]+1)}\\ &\times e_{n+m-l,n+m-\alpha}q^{\sum_{k=1}^{\alpha}h_{n+m-k}}e_{n+m-\alpha,i}E_{n+m-\alpha,n+m}T_{n+m-\alpha,i}(u).\end{aligned}\tag{A.5}$$

Comparing the coefficients of the $e_{n+m,n+m}$ term on both sides, we obtain

$$\begin{aligned}&q^{-(-1)^{[n+m]}}E_{n+m-l,n+m}T_{n+m,n+m}(u)=\\ &\quad= T_{n+m,n+m}(u)E_{n+m-l,n+m}+T_{n+m,n+m-l}(u)q^{\sum_{k=1}^{l}H_{n+m-k}}\\ &\qquad+\sum_{\alpha=1}^{l-1}(1-q^{-2(-1)^{[n+m-\alpha]}})T_{n+m,n+m-\alpha}(u)E_{n+m-l,n+m-\alpha}q^{\sum_{k=1}^{\alpha}H_{n+m-k}},\end{aligned}\tag{A.6}$$

which leads to the recursive relation (3.37).

Appendix B: The Exchange Symmetry of the Bethe Vector

For the Bethe vector $\Omega_N(v_1,\ldots,v_\alpha)$ of the quantum supersymmetric t-J model, we define the exchange operator $\hat{f}_\sigma=\hat{f}_{\sigma_{i_l}}\ldots\hat{f}_{\sigma_{i_k}}$ by

$$\hat{f}_\sigma\Omega_N(v_1,v_2,\ldots,v_\alpha)=\Omega_N(v_{\sigma(1)},v_{\sigma(2)},\ldots,v_{\sigma(\alpha)}),\tag{B.1}$$

where $\sigma\in\mathcal{S}_\alpha$, and $\{\sigma_i\}$ are the elementary permutations of $\mathcal{S}_\alpha$.

We first study the exchange symmetry for the elementary exchange operator $\hat{f}_{\sigma_i}$ which exchanges the parameter v_i and v_{i+1}. Acting $\hat{f}_{\sigma_i}$ on the Bethe vector of $U_q(gl(2|1))$ (6.7), we have

$$\begin{aligned}&\hat{f}_{\sigma_i}\Omega_N(v_1,v_2,\ldots,v_\alpha)=\Omega_N(v_1,\ldots,v_{i+1},v_i,\ldots,v_\alpha)\\ &=\sum_{d_1,\ldots,d_\alpha}(\Omega_\alpha^{(1),\sigma_i})^{d_1\ldots d_\alpha}C_{d_1}(v_1)\ldots C_{d_i}(v_{i+1})C_{d_{i+1}}(v_i)\ldots C_{d_\alpha}(v_\alpha)|vac\rangle,\end{aligned}\tag{B.2}$$

where $\{(\Omega_\alpha^{(1),\sigma_i})^{d_1\ldots d_\alpha}\}$ are the vector components of the nested Bethe vector $\Omega_\alpha^{(1),\sigma_i}$ constructed by the nested monodromy matrix

$$T^{(1),\sigma_i}(u)=L_\alpha^{(1)}(u,v_\alpha)\ldots L_{i+1}^{(1)}(u,v_i)L_i^{(1)}(u,v_{i+1})\ldots L_1^{(1)}(u,v_1),\tag{B.3}$$

where the local L-operator is defined by $L_i^{(1)}(u,v)=r_{0i}(u,v)$.

From the GYBE (3.15), one can derive the commutation relation between $C_i(u)$ and $C_j(v)$, which is given by

$$C_i(u)C_j(v) = \sum_{k,l} \check{r}(u,v)_{ij}^{kl} C_k(v)C_l(u). \tag{B.4}$$

Here the braided r-matrix $\check{r}(u,v) \equiv \mathcal{P}r(u,v)$, $\mathcal{P}$ permutes the tensor product spaces of the 2-dimensional $U_q(gl(2))$-module. Then, by (B.4), (B.2) becomes

$$\begin{aligned}\hat{f}_{\sigma_i}\Omega_N(v_1,v_2,\ldots,v_\alpha) = \sum_{d_1,\ldots,d_\alpha} &(\Omega_\alpha^{(1),\sigma_i})^{d_1\ldots d_\alpha} C_{d_1}(v_1)\ldots \\ &\times(\check{r}(v_{i+1},v_i))_{d_i d_{i+1}}^{k\ l} C_k(v_i)C_l(v_{i+1})\ldots C_{d_\alpha}(v_\alpha)|vac\rangle.\end{aligned} \tag{B.5}$$

We now compute the action of $(\check{r}(v_{i+1},v_i))_{d_i d_{i+1}}^{k\ l}$ on $(\Omega^{(1),\sigma_i})^{d_1\ldots d_\alpha}$. One checks that the $\check{r}$-matrix satisfies the YBE,

$$\begin{aligned}&\check{r}_{i\,i+1}(v_{i+1},v_i)L_{i+1}^{(1)}(u,v_i)L_i^{(1)}(u,v_{i+1}) \\ &\quad = L_{i+1}^{(1)}(u,v_{i+1})L_i^{(1)}(u,v_i)\check{r}_{i\,i+1}(v_{i+1},v_i)\,.\end{aligned} \tag{B.6}$$

Therefore, acting $\check{r}$ on $T^{(1),\sigma_i}(u)$, we have

$$\check{r}_{i\,i+1}(v_{i+1},v_i)T^{(1),\sigma_i}(u) = T^{(1)}(u)\check{r}_{i\,i+1}(v_{i+1},v_i). \tag{B.7}$$

Thus, because

$$\check{r}_{i\,i+1}(v_{i+1},v_i)\,v_2\otimes v_2 = c_{i\,i+1}(v_{i+1},v_i)v_2\otimes v_2 = \frac{1}{c_{i\,i+1}}v_2\otimes v_2,$$

we obtain

$$\sum_{d_i d_{i+1}} (\check{r}(v_{i+1},v_i))_{d_i d_{i+1}}^{k\ l}(\Omega_\alpha^{(1),\sigma_i})^{d_1\ldots d_i d_{i+1}\ldots d_\alpha} = \frac{1}{c_{i\,i+1}}(\Omega_\alpha^{(1)})^{d_1\ldots kl\ldots d_\alpha}. \tag{B.8}$$

Changing the indices k, l to d_i, d_{i+1}, respectively, and substituting the above relation into (B.5), we obtain the exchange symmetric relation of the Bethe vector of $U_q(gl(2|1))$,

$$\hat{f}_{\sigma_i}\Omega_N(v_1,v_2,\ldots,v_\alpha) = \frac{1}{c_{i\,i+1}}\Omega_N(v_1,v_2,\ldots,v_\alpha), \tag{B.9}$$

for the elementary permutation operator σ_i.

It follows that under the action of the exchange operator f_σ,

$$\hat{f}_\sigma\Omega_N(v_1,v_2,\ldots,v_\alpha) = \frac{1}{c_{1\ldots\alpha}^\sigma}\Omega_N(v_1,v_2,\ldots,v_\alpha), \tag{B.10}$$

where $c_{1\ldots\alpha}^\sigma$ is defined in (6.19).

References

1. Maillet, J.M., Sanchez de Santos, I.: Drinfel'd twists and algebraic Bethe ansatz. http://arxiv.org/q-alg/9612012, 1996
2. Drinfeld, V.G.: Constant quasi-classical solutions of the Yang-Baxter quantum equation. Sov. Math. Dokl. **28**, 667 (1983)
3. Kitanine, N., Maillet, J.M., Terras, V.: Form factors of the XXZ Heisenberg spin–1/2 finite chain. Nucl. Phys. **B 554**, 647 (1999)
4. Izergin, A.G., Kitanine, N., Maillet, J.M., Terras, V.: Spontaneous magnetization of the XXZ Heisenberg spin-1/2 chain. Nucl. Phys. **B 554**, 679 (1999)
5. Korepin, V.E., Bogoliubov, N.M., Izergin, A.G.: *Quantum Inverse Scattering Method and correlation Function*. Cambridge: Cambridge Univ. Press, 1993
6. Terras, V.: Drinfel'd twists and functional Bethe ansatz. Lett. Math. Phys. **48**, 263 (1999)
7. Albert, T.-D., Boos, H., Flume, R., Ruhlig, K.: Resolution of the nested hierarchy for the rational $sl(n)$ models. J. Phys. **A 33**, 4963 (2000)
8. Albert, T.-D., Boos, H., Flume, R., Poghossian, R.H., Ruhlig, K.: An F-twisted XYZ model. Lett. Math. Phys. **53**, 201 (2000)
9. Albert, T.-D., Ruhlig, K.: Polarization free generators for the Belavin model. J. Phys. **A 34**, 1569 (2001)
10. Yang, W.-L., Zhang, Y.-Z., Zhao, S.-Y.: Drinfeld twists and algebraic Bethe Ansatz of the supersymmetric t-J model. JHEP **12**, 038 (2004)
11. Zhao, S.-Y., Yang, W.-L., Zhang, Y.-Z.: Drinfeld twists and symmetric Bethe vectors of supersymmetric Fermion model. JSTAT, P04005 (2005)
12. Perk, J.H.H., Schultz, C.L.: New families of commuting transfer matrices in q-state vertex models, *Phys. Lett.* **A84**, 407 (1981)
13. Kulish, P.P., Sklyanin, E.K.: On solutions of the Yang-Baxter equation. J. Soviet Math. **19**, 1596 (1982)
14. Kulish, P.P.: Integrable graded magnets. J. Soviet Math. **35**, 2648 (1986)
15. Essler, F.H.L., Korepin, V.E., Schoutens, K.: New exactly solvable model of the strongly correlated electrons motivated by high-T_c superconductivity. Phys. Rev. Lett. **68**, 2960 (1992)
16. Bracken, A.G., Gould, M.D., Links, J.R., Zhang, Y.-Z.: A new supersymmetric and exactly solvable model of correlated electrons. Phys. Rev. Lett. **74**, 2768 (1995)
17. Yamane, H.: On defining relations of the affine Lie superalgebras and their quantized universal enveloping superalgebras. http://arxiv.org/abs/q-alg/9603015, 1996
18. Bazhanov, V.V., Shadrikov, A.G.: Trigonometric sotions of the triangle equations and simple Lie superalgebras. Theor. Math. Phys. **73**, 1302 (1988)
19. Faddeev, L.D., Sklyanin, E.K., Takhtajan, L.A.: The quantum inverse scattering method. Theor. Math. Phys. **40**, 688 (1979)
20. Takhtajan, L.A.: Quantum inverse scattering method and algebraized matrix Bethe-Ansata. J. Sov. Math. **23**, 2470 (1983)
21. de Vega, H.J.: Yang-Baxter algebras, integrable theories and quantum groups. Int. J. Mod. Phys. **A 44**, 2371 (1989)
22. Göhmann, F., Korepin, V.E.: Solution of the quantum inverse problem. J. Phys. **A 33**, 1199 (2000)

Communicated by L.Takhtajan