Commun. Math. Phys. 263, 259–276 (2006)
Digital Object Identifier (DOI) 10.1007/s00220-005-1510-7

Communications in
Mathematical
Physics

Sufficiency in Quantum Statistical Inference

Anna Jenčová[1,⋆], **Dénes Petz**[2,⋆⋆]

[1] Mathematical Institute of the Slovak Academy of Sciences, Stefanikova 49, Bratislava, Slovakia. E-mail: jenca@mat,savba.sk
[2] Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, POB 127, 1364 Budapest, Hungary. E-mail: petz@renyi.hu

Received: 27 December 2004 / Accepted: 22 September 2005
Published online: 26 January 2006 –

Abstract: This paper attempts to develop a theory of sufficiency in the setting of non-commutative algebras parallel to the ideas in classical mathematical statistics. Sufficiency of a coarse-graining means that all information is extracted about the mutual relation of a given family of states. In the paper sufficient coarse-grainings are characterized in several equivalent ways and the non-commutative analogue of the factorization theorem is obtained. As an application we discuss exponential families. Our factorization theorem also implies two further important results, previously known only in finite Hilbert space dimension, but proved here in generality: the Koashi-Imoto theorem on maps leaving a family of states invariant, and the characterization of the general form of states in the equality case of strong subadditivity.

1. Introduction

A quantum mechanical system is described by a C*-algebra, the dynamical variables (or observables) correspond to the self-adjoint elements and the physical states of the system are modelled by the normalized positive functionals of the algebra, see [4, 5]. The evolution of the system $\mathcal{M}$ can be described in the **Heisenberg picture** in which an observable $A \in \mathcal{M}$ moves into $\alpha(A)$, where α is a linear transformation. α is an automorphism in the case of the time evolution of a closed system but it could be the irreversible evolution of an open system. The **Schrödinger picture** is dual, it gives the transformation of the states, the state $\varphi \in \mathcal{M}^*$ moves into $\varphi \circ \alpha$. The algebra of a quantum system is typically non-commutative but the mathematical formalism supports commutative algebras as well. A simple **measurement** is usually modelled by a family of pairwise orthogonal projections, or more generally, by a partition of unity, $(E_i)_{i=1}^n$. Since all E_i are supposed to be positive and $\sum_i E_i = I$, $\beta : \mathbb{C}^n \to \mathcal{M}$, $(z_1, z_2, \dots, z_n) \mapsto \sum_i z_i E_i$ gives a

⋆ Supported by the EU Research Training Network Quantum Probability with Applications to Physics, Information Theory and Biology and Center of Excellence SAS Physics of Information I/2/2005.
⋆⋆ Supported by the Hungarian grant OTKA T032662

positive unital mapping from the commutative C*-algebra $\mathbb{C}^n$ to the non-commutative algebra $\mathcal{M}$. Every positive unital mapping occurs in this way. The essential concept in quantum information theory is the state transformation which is affine and the dual of a positive unital mapping. All these and several other situations justify the study of positive unital mappings between C*-algebras from a quantum statistical viewpoint.

If the algebra $\mathcal{M}$ is "small" and $\mathcal{N}$ is "large", and the mapping $\alpha : \mathcal{M} \to \mathcal{N}$ sends the state φ of the system of interest to the state $\varphi \circ \alpha$ at our disposal, then loss of information takes place and the problem of statistical inference is to reconstruct the real state from partial information. In this paper we mostly consider parametric statistical models, a parametric family $\mathcal{S} := \{\varphi_\theta : \theta \in \Theta\}$ of states is given and on the basis of the partial information the correct value of the parameter should be decided. If the partial information is the outcome of a measurement, then we have statistical inference in the very strong sense. However, there are "more quantum" situations, to decide between quantum states on the basis of quantum data, see Example 4 below. The problem we discuss is not the procedure of the decision about the true state of the system but the circumstances under which this is perfectly possible.

The paper is organized as follows. In the next section we summarize the relevant basic concepts both in classical statistics and in the non-commutative framework. The first part of Sect. 3 is about sufficient subalgebras, or subsystems of a quantum system. The second part is devoted to sufficient coarse-grainings. Most of the result of this section has been known in a more restricted situation of faithful states, see Chap. 9 in [12]. The importance of the multiplicative domain of a completely positive mapping is emphasized here. This concept allows us to give a sufficient subalgebra determined by a sufficient coarse-graining.

The quantum factorization theorem of Sect. 4 is the main result of the paper. The factorization of the states corresponds to a special structure of the algebras and the sufficient coarse-grainings. We use the properties of the von Neumann entropy and of the modular group to prove this result in some infinite dimensional situations (where the essential condition is the finiteness of the von Neumann entropy). The factorization implies a generalization of the Koashi-Imoto Theorem [7].

In Sect. 5 the equality case in the strong subadditivity of the von Neumann entropy is discussed in a possibly infinite dimensional framework and the factorization result is applied.

2. Preliminaries

In this paper, C*-algebras always have a unit I. Given a C*-algebra $\mathcal{M}$, a state φ of $\mathcal{M}$ is a linear function $\mathcal{M} \to \mathbb{C}$ such that $\varphi(I) = 1 = \|\varphi\|$. (Note that the second condition is equivalent to the positivity of φ.) The books [4, 5] – among many others – explain the basic facts about C*-algebras. The class of finite dimensional full matrix algebras form a small and algebraically rather trivial subclass of C*-algebras, but from the view-point of non-commutative statistics, almost all ideas and concepts appear in this setting. A matrix algebra $M_n(\mathbb{C})$ admits a canonical trace $\mathrm{T}r$ and all states are described by their densities with respect to $\mathrm{T}r$. The correspondence is given by $\varphi(A) = \mathrm{T}r D_\varphi A \quad (A \in M_n(\mathbb{C}))$ and we can simply identify the functional φ by the density D_φ. Note that the density is a positive (semi-definite) matrix of trace 1.

Let $\mathcal{M}$ and $\mathcal{N}$ be C*-algebras. Recall that **2-positivity** of $\alpha : \mathcal{M} \to \mathcal{N}$ means that

$$\begin{bmatrix} \alpha(A) & \alpha(B) \\ \alpha(C) & \alpha(D) \end{bmatrix} \geq 0 \qquad \text{if} \qquad \begin{bmatrix} A & B \\ C & D \end{bmatrix} \geq 0$$

for 2×2 matrices with operator entries. It is well-known that a 2-positive unit-preserving mapping α satisfies the **Schwarz inequality**

$$\alpha(A^*A) \geq \alpha(A)^*\alpha(A). \tag{1}$$

A 2-positive unital mapping between C*-algebras will be called **coarse-graining**. Here are two fundamental examples.

Example 1. Let $\mathcal{X}$ be a finite set and $\mathcal{N}$ be a C*-algebra. Assume that for each $x \in \mathcal{X}$ a positive operator $E(x) \in \mathcal{N}$ is given and $\sum_x E(x) = I$. In quantum mechanics such a setting is a model for a measurement with values in $\mathcal{X}$.

The space $C(\mathcal{X})$ of function on $\mathcal{X}$ is a C*-algebra and the partition of unity E induces a coarse-graining $\alpha : C(\mathcal{X}) \to \mathcal{N}$ given by $\alpha(f) = \sum_x f(x)E(x)$. Therefore a coarse-graining defined on a commutative algebra is an equivalent way to give a measurement. (Note that the condition of 2-positivity is automatically fulfilled on a commutative algebra.) □

Example 2. Let $\mathcal{M}$ be the algebra of all bounded operators acting on a Hilbert space $\mathcal{H}$ and let $\mathcal{N}$ be the infinite tensor product $\mathcal{M} \otimes \mathcal{M} \otimes \ldots$. (To understand the essence of the example one does not need the very formal definition of the infinite tensor product.) If γ denotes the right shift on $\mathcal{N}$, then we can define a sequence α_n of coarse-grainings $\mathcal{M} \to \mathcal{N}$:

$$\alpha_n(A) := \frac{1}{n}\big(A + \gamma(A) + \cdots + \gamma^{n-1}(A)\big).$$

α_n is the quantum analogue of the **sample mean**. □

Let $(X_i, \mathcal{A}_i, \mu_i)$ be a measure space ($i = 1, 2$). Recall that a positive linear map $M : L^\infty(X_1, \mathcal{A}_1, \mu_1) \to L^\infty(X_2, \mathcal{A}_2, \mu_2)$ is called a **Markov operator** if it satisfies $M1 = 1$ and $f_n \searrow 0$ implies $Mf_n \searrow 0$. For mappings defined between von Neumann algebras, the monotone continuity is called **normality**. In the case that $\mathcal{M}$ and $\mathcal{N}$ are von Neumann algebras, a coarse-graining $\mathcal{M} \to \mathcal{N}$ will be always supposed to be normal. Our concept of coarse-graining is the analogue of the Markov operator.

We mostly mean that a coarse-graining transforms observables to observables corresponding to the **Heisenberg picture** and in this case we assume that it is unit preserving. The dual of such a mapping acts on states or on density matrices and it will be called coarse-graining as well.

We recall some well-known results from mathematical statistics, see [24] for details.

Let $(X, \mathcal{A})$ be a measurable space and let $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ be a set of probability measures on $(X, \mathcal{A})$. A sub-σ-algebra $\mathcal{A}_0 \subset \mathcal{A}$ is **sufficient** for $\mathcal{P}$ if for all $A \in \mathcal{A}$, there is an $\mathcal{A}_0$-measurable function f_A such that for all θ,

$$f_A = P_\theta(A|\mathcal{A}_0) \quad P_\theta - \text{almost everywhere},$$

that is,

$$P_\theta(A \cap A_0) = \int_{A_0} f_A \, dP_\theta \tag{2}$$

for all $A_0 \in \mathcal{A}_0$ and for all θ. It is clear from this definition that if $\mathcal{A}_0$ is sufficient then for all P_θ there is a common version of the conditional expectation $E_\theta[g|\mathcal{A}_0]$ for any measurable step function g, or, more generally, for any function $g \in \cap_{\theta\in\Theta} L_1(X, \mathcal{A}, P_\theta)$.

In the most important case, the family $\mathcal{P}$ is **dominated**, that is there is a σ-finite measure μ such that P_θ is absolutely continuous with respect to μ for all θ, this will be denoted by $\mathcal{P} \ll \mu$. The following lemma is a useful tool in examining sufficiency.

Lemma 1. *If $\mathcal{P}$ is dominated, then there is a countable subset $\{P_1, P_2 \dots\} \subseteq \mathcal{P}$ such that $P_\theta(A) = 0$ holds for all $\theta \in \Theta$ if and only if $P_n(A) = 0$ holds for all $n \in \mathbb{N}$.*

It follows that if $\mathcal{P}$ is dominated then there is a (possibly infinite) convex combination $P_0 = \sum_n c_n P_n$, $P_n \in \mathcal{P}$, such that $\mathcal{P} \equiv P_0$, that is, $P_\theta \ll P_0$ and $P(A) = 0$ for all θ implies $P_0(A) = 0$.

For our purposes, it is more suitable to use the following characterization of sufficiency in terms of randomization.

Let $\mathcal{P}_i = \{P_{i,\theta} : \theta \in \Theta\}$ be dominated families of probability measures on $(X_i, \mathcal{A}_i)$, such that $\mathcal{P}_i \equiv \mu_i, i = 1, 2$. We say that $(X_2, \mathcal{A}_2, \mathcal{P}_2)$ is a **randomization** of $(X_1, \mathcal{A}_1, \mathcal{P}_1)$, if there exists a Markov operator $M : L^\infty(X_2, \mathcal{A}_2, \mu_2) \to L^\infty(X_1, \mathcal{A}_1, \mu_1)$, satisfying

$$\int (Mf) dP_{\theta,1} = \int f dP_{\theta,2} \qquad (\theta \in \Theta,\ f \in L^\infty(X_2, \mathcal{A}_2, \mathcal{P}_2)).$$

If also $(X_1, \mathcal{A}_1, \mathcal{P}_1)$ is a randomization of $(X_2, \mathcal{A}_2, \mathcal{P}_2)$, then $(X_1, \mathcal{A}_1, \mathcal{P}_1)$ and $(X_2, \mathcal{A}_2, \mathcal{P}_2)$ are **statistically equivalent**.

For example, let $\mathcal{P} \equiv P_0$ and let $\mathcal{A}_0 \subseteq \mathcal{A}$ be a subalgebra. Then $(X, \mathcal{A}_0, \mathcal{P}|_{\mathcal{A}_0})$ is obviously a randomization of $(X, \mathcal{A}, \mathcal{P})$, where the Markov operator is the inclusion $L^\infty(X, \mathcal{A}_0, P_0|_{\mathcal{A}_0}) \to L^\infty(X, \mathcal{A}, P_0)$. On the other hand, if $\mathcal{A}_0$ is sufficient, then the map

$$f \mapsto E[f|\mathcal{A}_0], \qquad E[f|\mathcal{A}_0] = E_\theta[f|\mathcal{A}_0], \quad P_\theta \text{ almost everywhere,}$$

is a Markov operator $L^\infty(X, \mathcal{A}, P_0) \to L^\infty(X, \mathcal{A}_0, P_0|_{\mathcal{A}_0})$ and

$$\int E[f|\mathcal{A}_0] dP_\theta|_{\mathcal{A}_0} = \int f dP_\theta \qquad (f \in L^\infty(X, \mathcal{A}, P_0),\ \theta \in \Theta).$$

We have the following characterizations of sufficient subalgebras.

Proposition 1. *Let $\mathcal{P}$ be a dominated family and let $\mathcal{A}_0 \subseteq \mathcal{A}$ be a subalgebra. The following are equivalent.*

(i) *$\mathcal{A}_0$ is sufficient for $\mathcal{P}$.*
(ii) *There exists a measure P_0 such that $\mathcal{P} \equiv P_0$ and dP_θ/dP_0 is $\mathcal{A}_0$-measurable for all θ.*
(iii) *$(X, \mathcal{A}, \mathcal{P})$ and $(X, \mathcal{A}_0, \mathcal{P}|_{\mathcal{A}_0})$ are statistically equivalent.*

Let $\mathcal{F}$ be a family of measurable mappings $(X, \mathcal{A}) \to (X_1, \mathcal{A}_1)$, then the σ-algebra generated by $\mathcal{F}$ is the sub-σ-algebra in $\mathcal{A}$, generated by $\{f^{-1}(B) : f \in \mathcal{F}, B \in \mathcal{A}_1\}$. It follows from the above proposition that if $\mathcal{P} \equiv P_0$, then the σ-algebra generated by $\{dP_\theta/dP_0 : \theta \in \Theta\}$ is sufficient for $\mathcal{P}$, moreover, it is contained in any other sufficient subalgebra in $\mathcal{A}$. Such subalgebras are called **minimal sufficient**.

A classical **sufficient statistic** for the family $\mathcal{P}$ is a measurable mapping $T : (X, \mathcal{A}) \to (X_1, \mathcal{A}_1)$ such that the sub-σ-algebra generated by T is sufficient for $\mathcal{P}$. To any statistic T, we associate a Markov operator

$$\tilde{T} : L^\infty(X_1, \mathcal{A}_1, P_0^T) \to L^\infty(X, \mathcal{A}, P_0), \quad (\tilde{T}g)(x) = g(T(x)).$$

Obviously, $(X_1, \mathcal{A}_1, \mathcal{P}^T)$ is a randomization of $(X, \mathcal{A}, \mathcal{P})$. As in the case of subalgebras, we have

Proposition 2. *The statistic $T : (X, \mathcal{A}) \to (X_1, \mathcal{A}_1)$ is sufficient for $\mathcal{P}$ if and only if $(X, \mathcal{A}, \mathcal{P})$ and $(X_1, \mathcal{A}_1, \mathcal{P}^T)$ are statistically equivalent.*

Proposition 3. *(Factorization criterion). Let $\mathcal{P} \ll \mu$. The statistic $T : (X, \mathcal{A}) \to (X_1, \mathcal{A}_1)$ is sufficient for $\mathcal{P}$ if and only if there is an $\mathcal{A}_1$-measurable function g_θ for all θ and an $\mathcal{A}$-measurable function h such that*

$$\frac{dP_\theta}{d\mu}(x) = g_\theta(T(x))h(x) \quad P_\theta - \textit{almost everywhere.}$$

We describe an important example of a dominated family.

Example 3. A set of measures $\mathcal{P} = \{P_\theta,\ \theta \in \Theta\} \ll \mu$ is an **exponential family** if there are functions $\xi_1, \dots, \xi_m : \Theta \to \mathbb{R}$ and measurable functions $T_1, \dots, T_m : X \to \mathbb{R}$ such that for all $\theta \in \Theta$,

$$\frac{dP_\theta}{d\mu}(x) = \frac{1}{Z(\theta)} \exp\left(\sum_{i=1}^{m} \xi_i(\theta)T_i(x)\right) h(x)\,,$$

where $Z(\theta) = \int \exp\left(\sum_{i=1}^{m} \xi_i(\theta)T_i(x)\right) h(x)d\mu$. The parameter space Θ is usually supposed to be an open set such that the integral $Z(\theta)$ converges for $\theta \in \Theta$.

All elements in an exponential family $\mathcal{P}$ are mutually equivalent, that is, $P_{\theta_1} \ll P_{\theta_2}$ for all $\theta_1, \theta_2 \in \Theta$. We therefore have $\mathcal{P} \equiv P_0 := P_{\theta_0}$ for arbitrary $\theta_0 \in \Theta$. If $\mathcal{P} \ll \mu$ is a family of probability measures such that the elements are mutually equivalent, then $\mathcal{P}$ is an exponential family if and only if the linear space spanned by the functions $\{\log \frac{dP}{d\mu},\ P \in \mathcal{P}\}$, is finite dimensional.

From Proposition 2 and the remarks following it, it is clear that $(T_1, \dots, T_n)$ is a sufficient statistic and the σ-algebra generated by the functions $\{T_1, \dots, T_n\}$ is minimal sufficient for $\mathcal{P}$. More about exponential families can be found, for example, in [2]. □

Next we formulate the non-commutative setting. Let $\mathcal{M}$ be a von Neumann algebra and $\mathcal{M}_0$ be its von Neumann subalgebra. Assume that a family $\mathcal{S} := \{\varphi_\theta : \theta \in \Theta\}$ of normal states are given. $(\mathcal{M}, \mathcal{S})$ is called **statistical experiment**. The subalgebra $\mathcal{M}_0 \subset \mathcal{M}$ is **sufficient** for $(\mathcal{M}, \mathcal{S})$ if for every $a \in \mathcal{M}$, there is $\alpha(a) \in \mathcal{M}_0$ such that

$$\varphi_\theta(a) = \varphi_\theta(\alpha(a)) \qquad (\theta \in \Theta) \tag{3}$$

and the correspondence $a \mapsto \alpha(a)$ is a coarse-graining. (Note that a positive mapping is automatically completely positive if it is defined on a commutative algebra.)

Example 4. Consider a bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and a family $\{\varphi_\theta : \theta \in \Theta\}$ of states on $\mathcal{H}$. Assume that the expectation value of all observables localized at A is known to us, that is, we know the restriction of φ_θ's to $B(\mathcal{H}_A)$ (or the reduced density matrices). This information is not sufficient in general to decide about θ. We impose the further condition that $\mathcal{H}_A = \mathcal{H}_L \otimes \mathcal{H}_R$ and the factorization

$$\varphi_\theta = \varphi_\theta^0 \otimes \varphi_{RB},$$

where φ_θ^0 is a state on $B(\mathcal{H}_L)$ and the state φ_{RB} of $B(\mathcal{H}_R) \otimes B(\mathcal{H}_B)$ is independent of the parameter θ. In this case the restriction of the unknown state to $B(\mathcal{H}_L)$ determines the true value of the parameter θ and φ_θ is recovered uniquely. Even in the case that we do not know all expectations of observables on A, sufficiency of $B(\mathcal{H}_L)$ means that having access to L is as good as having access to all of AB, for any estimation task.

This example is close to typical. In the general case, however, the relation of the subalgebras $B(\mathcal{H}_L)$ and $B(\mathcal{H}_A)$ is more subtle. □

For a normal state φ on a von Neumann algebra, we denote its support projection by $\operatorname{supp}\varphi$. The following lemma is a quantum version of Lemma 1.

Lemma 2. *Assume that the von Neumann algebra $\mathcal{M}$ admits a faithful normal state ψ. Let $\mathcal{S} = \{\varphi_\theta : \theta \in \Theta\}$ be a family of normal states on $\mathcal{M}$. Then there is a sequence (φ_n) of states in $\mathcal{S}$ and a normal state*

$$\omega = \sum_{n=1}^{\infty} \lambda_n \varphi_n$$

such that $\operatorname{supp}\varphi_\theta \leq \operatorname{supp}\omega$ for all $\theta \in \Theta$.

Proof. Let $\{p_i : i \in I\}$ be a set of pairwise orthogonal projections in $\mathcal{M}$, then $\psi(p_i) > 0$ and $\psi(\sum_i p_i) \leq 1$, therefore any such set must be at most countable.

We set

$$\mathcal{P} = \{p_\theta = \operatorname{supp}\varphi_\theta : \theta \in \Theta\}$$

and show that there is a countable subset $\{p_1, p_2, \ldots\} \subset \mathcal{P}$, such that $\sup_\theta p_\theta = \sup_n p_n$.

Let $\mathcal{C}$ be the set of at most countable subsets in $\mathcal{P}$, ordered by inclusion. Consider all chains in $\mathcal{C}$, such that if $C \subset D$ in the chain, then $\sup C \neq \sup D$. It is clear that each such chain has at most countably many elements. Let $\{C_1, C_2, \ldots\}$ be a maximal such chain and let $C = \cup_n C_n = \{p_1, p_2, \ldots\}$. Then $\sup_n p_n = \sup_\theta p_\theta$. Indeed, if $\sup_n p_n \neq \sup_\theta p_\theta$, then there is an element $p \in \mathcal{P}$, such that $\sup C \neq \sup C \cup \{p\}$, which contradicts the maximality of $\{C_1, C_2, \ldots\}$.

Let now $\varphi_1, \varphi_2, \ldots$ be elements in $\mathcal{S}$ such that $\operatorname{supp}\varphi_n = p_n$. Choose a sequence $\lambda_1, \lambda_2, \ldots$ such that $\lambda_n > 0$ for all n and $\sum_n \lambda_n = 1$ and put $\omega = \sum \lambda_n \varphi_n$. Then it is clear that $\operatorname{supp}\omega = \sup_n p_n$ and $\operatorname{supp}\varphi_\theta \leq \operatorname{supp}\omega$ for all θ. □

Throughout the paper, we suppose that the hypothesis of the above lemma is satisfied, that is, the von Neumann algebras considered admit a faithful normal state. The algebra $B(\mathcal{H})$ satisfies this condition if and only if the Hilbert space $\mathcal{H}$ is separable.

When the states φ_n belong to $\mathcal{S}$ and for

$$\omega := \sum_{n=1}^{\infty} \lambda_n \varphi_n$$

the condition $\operatorname{supp}\varphi_\theta \leq \operatorname{supp}\omega$ holds for all $\theta \in \Theta$, we say that $\mathcal{S}$ **is dominated by** ω.

3. Sufficient Subalgebras and Coarse-Grainings

In the study of sufficient subalgebras monotone quasi-entropy quantities play an important role. The **relative entropy** and the **transition probability** are examples of those [15, 12].

Let φ and ω be normal states of a von Neumann algebra and let ξ_φ and ξ_ω be the representing vectors of these states from the natural positive cone (see below). Then the **transition probability** is defined as

$$P_A(\varphi, \omega) = \langle \xi_\varphi, \xi_\omega \rangle.$$

In case of density matrices this reduces to $P_A(D_1, D_2) = \mathrm{Tr}(D_1^{1/2} D_2^{1/2})$.

The next theorem is essentially Thm 9.5 from [12].

Theorem 1. *Let* $(\mathcal{M}, \{\varphi_\theta : \theta \in \Theta\})$ *be a statistical experiment and let* $\mathcal{M}_0 \subset \mathcal{M}$ *be von Neumann algebras. Assume that* $\{\varphi_\theta : \theta \in \Theta\}$ *is dominated by a faithful normal state* ω*. Then the following conditions are equivalent:*

(i) $\mathcal{M}_0$ *is sufficient for* $(\mathcal{M}, \varphi_\theta)$*.*
(ii) $P_A(\varphi_\theta, \omega) = P_A(\varphi_\theta|\mathcal{M}_0, \omega|\mathcal{M}_0)$ *for all* θ*.*
(iii) $[D\varphi_\theta, D\omega]_t = [D(\varphi_\theta|\mathcal{M}_0), D(\omega|\mathcal{M}_0)]_t$ *for every real* t *and for every* θ*.*
(iv) $[D\varphi_\theta, D\omega]_t \in \mathcal{M}_0$ *for all real* t *and every* θ*.*
(v) The generalized conditional expectation $E_\omega : \mathcal{M} \to \mathcal{M}_0$ *leaves all the states* φ_θ *invariant.*

We omit the proof but explain the conditions. Since ω is assumed to be faithful and normal, it is convenient to consider a representation of $\mathcal{M}$ on a Hilbert space $\mathcal{H}$ such that ω is induced by a cyclic and separating vector Ω. Given a normal state ψ the quadratic form $a\Omega \mapsto \psi(aa^*)$ $(a \in \mathcal{M})$ determines the relative modular operator $\Delta(\psi/\omega)$ as

$$\psi(aa^*) = \|\Delta(\psi/\omega)a\Omega\|^2 \qquad (a \in \mathcal{M}).$$

The vector $\Delta(\psi/\omega)^{1/2}\Omega$ is the representative of ψ from the so-called natural positive cone (which is actually the set of all such vectors). The Connes' cocycle

$$[D\psi, D\omega]_t = \Delta(\psi/\omega)^{\mathrm{it}}\Delta(\omega/\omega)^{-\mathrm{it}}$$

is a one-parameter family of contractions in $\mathcal{M}$, unitaries when ψ is faithful. The modular group of ω is a group of automorphisms defined as

$$\sigma_t(a) = \Delta(\omega/\omega)^{\mathrm{it}} a \Delta(\omega/\omega)^{-\mathrm{it}} \qquad (t \in \mathbb{R}).$$

The Connes' cocycle is the quantum analogue of the Radon-Nikodym derivative of measures.

The generalized conditional expectation $E_\omega : \mathcal{M} \to \mathcal{M}_0$ is defined as

$$E_\omega(a)\Omega = J_0 P J a\Omega,$$

where J is the modular conjugation on the Hilbert space $\mathcal{H}$, J_0 is that on the closure $\mathcal{H}_0$ of $\mathcal{M}_0\Omega$ and $P : \mathcal{H} \to \mathcal{H}_0$ is the orthogonal projection [1] . There are several equivalent conditions which guarantee that E_ω is a conditional expectation, for example, $\sigma_t(\mathcal{M}_0) \subset \mathcal{M}_0$ (**Takesaki's theorem**, [12]).

More generally, let $\mathcal{M}_1$ and $\mathcal{M}_2$ be von Neumann algebras and let $\sigma : \mathcal{M}_1 \to \mathcal{M}_2$ be a coarse-graining. Suppose that a normal state φ_2 is given and $\varphi_1 := \varphi_2 \circ \sigma$ is normal as well. Let Φ_i be the representing vectors in given natural positive cones and J_i be the modular conjugations $(i = 1, 2)$.

From the modular theory we know that

$$p_i := \overline{J_i \mathcal{M}_i \Phi_i}$$

is the support projection of φ_i $(i = 1, 2)$.

The dual $\sigma^*_{\varphi_2} : p_2\mathcal{M}_2 p_2 \to p_1\mathcal{M}_1 p_1$ of σ is characterized by the property

$$\langle a_1\Phi_1, J_1\sigma_{\varphi_2}(a_2)\Phi_1\rangle = \langle \sigma(a_1)\Phi_2, J_2 a_2\Phi_2\rangle \quad (a_i \in \mathcal{M}_i, i = 1, 2) \tag{4}$$

(see Prop. 8.3 in [12]).

Example 5. Let $\mathcal{M}$ be a von Neumann algebra and ω be a normal state. For $a \in \mathcal{M}^{sa}$ define the state $[\omega^a]$ as the minimizer of

$$\psi \mapsto S(\psi, \omega) - \psi(a). \tag{5}$$

We define the **quantum exponential family** as

$$\mathcal{S} = \{\varphi_\theta := [\omega^{\sum_i \theta_i a_i}], \theta \in \Theta\}, \tag{6}$$

where $a_1, a_2, \ldots, a_n$ are self-adjoint operators from $\mathcal{M}$ and $\Theta \subseteq \mathbb{R}^n$ is the parameter space. Let $\mathcal{M}$ be finite dimensional, and assume that the density of ω is written in the form e^H, $H = H^* \in \mathcal{M}$. Then the density of φ_θ is nothing else but

$$D_\theta = \frac{\exp\left(H + \sum_i \theta_i a_i\right)}{\mathrm{T}r \exp\left(H + \sum_i \theta_i a_i\right)}, \tag{7}$$

which is a direct analogue of the classical exponential family.

Returning to the general case, note that the support of the states φ_θ is supp ω. For more details about perturbation of states, see Chap. 12 of [12], here we recall the analogue of (7) in the general case. We assume that the von Neumann algebra is in a standard form and the representative of ω is Ω from the positive cone. Let Δ_ω be the modular operator of ω then φ_θ of (6) is the vector state induced by the unit vector

$$\Phi_\theta := \frac{\exp \frac{1}{2}\Big(\log \Delta_\omega + \sum_i \theta_i a_i\Big)\Omega}{\Big\| \exp \frac{1}{2}\Big(\log \Delta_\omega + \sum_i \theta_i a_i\Big)\Omega\Big\|}. \tag{8}$$

(This formula holds in the strict sense if ω is faithful, since Δ_ω is invertible in this case. For non-faithful ω the formula is modified by the support projection.)

In the next theorem σ_t^ω denotes the modular automorphism group of ω, $\sigma_t^\omega(a) = \Delta_\omega^{\mathrm{it}} a \Delta_\omega^{-\mathrm{it}}$.

Theorem 2 ([14]). *Let $\mathcal{M}$ be a von Neumann algebra with a faithful normal state ω and $\mathcal{M}_0$ be a subalgebra. For $a_1, a_2, \ldots, a_n \in \mathcal{A}^{sa}$ the following conditions are equivalent:*

(i) $\mathcal{M}_0$ is sufficient for the exponential family (6).
(ii) $\sigma_t^\omega(a_i) \in \mathcal{M}_0$ for all $t \in \mathbb{R}$ and $1 \leq i \leq n$.
(iii) For the generalized conditional expectation $E_\omega : \mathcal{M} \to \mathcal{M}_0$, $E_\omega(a_i) = a_i$ holds, $1 \leq i \leq n$.

Note that Theorem 1 (iv) and Theorem 2 implies that if the subalgebra $\mathcal{M}_0$ is sufficient for the finite set of states $\{[\omega^{a_1}], \ldots, [\omega^{a_n}]\}$, then it is sufficient for the whole exponential family (6).

We will now define sufficient coarse-grainings. Let $\mathcal{N}$, $\mathcal{M}$ be C*-algebras and let $\sigma : \mathcal{N} \to \mathcal{M}$ be a coarse-graining. By Proposition 2, the classical definition of sufficiency can be generalized in the following way: we say that σ is sufficient for the statistical experiment $(\mathcal{M}, \varphi_\theta)$ if there exists a coarse-graining $\beta : \mathcal{M} \to \mathcal{N}$ such that $\varphi_\theta \circ \sigma \circ \beta = \varphi_\theta$ for every θ.

Let us recall the following well-known property of coarse-grainings, see 9.2 in [23].

Lemma 3. *Let $\mathcal{M}$ and $\mathcal{N}$ be C*-algebras and let $\sigma : \mathcal{N} \to \mathcal{M}$ be a coarse-graining. Then*

$$\mathcal{N}_\sigma := \{a \in \mathcal{N} : \sigma(a^*a) = \sigma(a)\sigma(a)^* \text{ and } \sigma(aa^*) = \sigma(a)^*\sigma(a)\} \tag{9}$$

is a subalgebra of $\mathcal{N}$ and

$$\sigma(ab) = \sigma(a)\sigma(b) \quad \text{and} \quad \sigma(ba) = \sigma(b)\sigma(a) \tag{10}$$

holds for all $a \in \mathcal{N}_\sigma$ and $b \in \mathcal{N}$.

We call the subalgebra $\mathcal{N}_\sigma$ the **multiplicative domain** of σ.

Let now $\mathcal{N}$ and $\mathcal{M}$ be von Neumann algebras and let ω be a faithful normal state on $\mathcal{M}$ such that $\omega \circ \sigma$ is also faithful. Let

$$\mathcal{N}_1 = \{a \in \mathcal{N},\ \sigma^*_\omega \circ \sigma(a) = a\}.$$

It was proved in [16] that $\mathcal{N}_1$ is a subalgebra of $\mathcal{N}_\sigma$, moreover, $a \in \mathcal{N}_1$ if and only if $\sigma(a^*a) = \sigma(a)^*\sigma(a)$ and $\sigma(\sigma_t^{\omega\circ\sigma}(a)) = \sigma_t^\omega(\sigma(a))$. The restriction of σ to $\mathcal{N}_1$ is an isomorphism onto

$$\mathcal{M}_1 = \{b \in \mathcal{M},\ \sigma \circ \sigma^*_\omega(b) = b\}.$$

The following theorem was proved in [16] in the case when φ_θ are faithful states.

Theorem 3. *Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras and let $\sigma : \mathcal{N} \to \mathcal{M}$ be a coarse-graining. Suppose that $(\mathcal{M}, \varphi_\theta)$ is a statistical experiment dominated by a state ω such that both ω and $\omega \circ \sigma$ are faithful and normal. Then the following properties are equivalent:*

(i) $\sigma(\mathcal{N}_\sigma)$ *is a sufficient subalgebra for* $(\mathcal{M}, \varphi_\theta)$.
(ii) σ *is a sufficient coarse-graining for* $(\mathcal{M}, \varphi_\theta)$.
(iii) $P_A(\varphi_\theta, \omega) = P_A(\varphi_\theta \circ \sigma, \omega \circ \sigma)$.
(iv) $\sigma([D\varphi_\theta \circ \sigma, D\omega \circ \sigma]_t) = [D\varphi_\theta, D\omega]_t$.
(v) $\mathcal{M}_1$ *is a sufficient subalgebra for* $(\mathcal{M}, \varphi_\theta)$.
(vi) $\varphi_\theta \circ \sigma \circ \sigma^*_\omega = \varphi_\theta$.

Proof. Suppose (i), then there is a coarse-graining $\gamma : \mathcal{M} \to \sigma(\mathcal{N}_\sigma)$, preserving φ_θ. It is easy to see that the restriction of σ to $\mathcal{N}_\sigma$ is invertible. Let α be the inverse of this restriction and put

$$\beta = \alpha \circ \gamma.$$

Then $\beta : \mathcal{M} \to \mathcal{N}$ is a coarse-graining such that $\varphi_\theta \circ \sigma \circ \beta = \varphi_\theta$ and (ii) is proved.

The implication (ii) $\Rightarrow$ (iii) follows from monotonicity of the transition probability, (iii) $\Rightarrow$ (iv) was proved in [12, 16] (faithfulness of φ_θ is not needed in that proof).

Suppose (iv) and denote $u_t = [D\varphi_\theta \circ \sigma, D\omega \circ \sigma]_t$, $v_t = [D\varphi_\theta, D\omega]_t$. Then we have $\sigma(u_t) = v_t$ for all t. Let $p_\theta = \operatorname{supp} \varphi_\theta$, $q_\theta = \operatorname{supp} \varphi_\theta \circ \sigma$. Putting $t = 0$ in the condition (iv), we get $\sigma(q_\theta) = p_\theta$ and

$$\sigma(u_t u_t^*) = \sigma(q_\theta) = p_\theta = v_t v_t^* = \sigma(u_t)\sigma(u_t)^*\,.$$

On the other hand, $\sigma(u_t)^*\sigma(u_t) \le \sigma(u_t^* u_t)$ by Schwartz inequality and from

$$\omega(\sigma(u_t)^*\sigma(u_t)) = \omega(v_t^* v_t) = \omega(\sigma_t^\omega(p_\theta)) = \omega(p_\theta),$$
$$\omega(\sigma(u_t^* u_t)) = \omega \circ \sigma(u_t^* u_t) = \omega \circ \sigma(\sigma_t^{\omega\circ\sigma}(q_\theta)) = \omega(p_\theta),$$

we get $\sigma(u_t^* u_t) = \sigma(u_t)^* \sigma(u_t)$. Hence $u_t \in \mathcal{N}_\sigma$ for all t. Further, by the cocycle condition and Lemma 3,

$$\sigma(\sigma_s^{\omega\circ\sigma}(u_t)) = \sigma(u_s^* u_{s+t}) = v_s^* v_{t+s} = \sigma_s^\omega(\sigma(u_t)),$$

therefore $v_t \in \mathcal{M}_1$ and by Theorem 1, $\mathcal{M}_1$ is sufficient and (v) is proved. As $\mathcal{M}_1$ is a subalgebra in $\sigma(\mathcal{N}_\sigma)$, this implies (i).

Finally, we prove that (ii) is equivalent to (vi). First, note that a coarse-graining is sufficient for $(\mathcal{M}, \varphi_\theta)$ if and only if it is sufficient for $(\mathcal{M}, \psi_\theta)$, where

$$\psi_\theta = \varepsilon\varphi_\theta + (1 - \varepsilon)\omega$$

for some $0 < \varepsilon < 1$.

As the states ψ_θ are faithful and $\omega = \sum_n \lambda_n \psi_n$, it follows from the results in [16] that σ is sufficient if and only if $\psi_\theta \circ \sigma \circ \sigma_\omega^* = \psi_\theta$ for all θ. Since, by definition, $\omega \circ \sigma \circ \sigma_\omega^* = \omega$, this is equivalent to (vi). □

The previous theorem applies to a measurement which is essentially a positive mapping $\mathcal{N} \to \mathcal{M}$ from a commutative algebra. The concept of sufficient measurement appeared also in [3]. For a non-commuting family of states, there is no sufficient measurement.

We also have the following characterization of sufficient coarse-grainings in terms of relative entropy, see [14].

Proposition 4. *Under the conditions of Theorem 3, suppose that $S(\varphi_\theta, \omega)$ is finite for all θ. Then σ is a sufficient coarse-graining if and only if*

$$S(\varphi_\theta, \omega) = S(\varphi_\theta \circ \sigma, \omega \circ \sigma).$$

The equality in inequalities for entropy quantities was studied also in [19, 20]. For density matrices, it was shown that the equality in Proposition 4 is equivalent to

$$\sigma(\log \sigma^*(D_\theta) - \log \sigma^*(D_{\omega_0})) = \log D_\theta - \log D_\omega,$$

where σ^* is the dual mapping of σ on density matrices.

Let us now show how Theorems 1 and 3 can be applied if the dominating state ω is not faithful. Suppose that $p = \operatorname{supp} \omega$, $q = \operatorname{supp} \omega \circ \sigma$. We define the map $\alpha : q\mathcal{N}q \to p\mathcal{M}p$ by $\alpha(a) = p\sigma(a)p$. Then α is a coarse-graining such that $\alpha_\omega^* = \sigma_\omega^*$ and $\varphi_\theta \circ \sigma(a) = \varphi_\theta \circ \alpha(qaq)$ for all θ. We check that α is sufficient for $(p\mathcal{M}p, \varphi_\theta|_{p\mathcal{M}p})$ if and only if σ is sufficient for $(\mathcal{M}, \varphi_\theta)$. Indeed, let $\tilde{\beta} : p\mathcal{M}p \to q\mathcal{N}q$ be a coarse-graining such that $\varphi_\theta|_{p\mathcal{M}p} \circ \alpha \circ \tilde{\beta} = \varphi_\theta|_{p\mathcal{M}p}$ and let $\beta : \mathcal{M} \to \mathcal{N}$ be defined by

$$\beta(a) = \tilde{\beta}(pap) + \omega(a)(1 - q).$$

Then β is a coarse-graining and

$$\varphi_\theta \circ \sigma \circ \beta(a) = \varphi_\theta \circ \sigma(q\beta(a)q)) = \varphi_\theta \circ \alpha \circ \tilde{\beta}(pap) = \varphi_\theta(pap) = \varphi_\theta(a).$$

The converse is proved similarly, taking $\tilde{\beta}(a) = q\beta(a)q$ for $a \in p\mathcal{M}p$.

4. Factorization

Let $\mathcal{M}$ be a von Neumann algebra and let ω be a faithful state on $\mathcal{M}$. Let $\mathcal{M}_0 \subset \mathcal{M}$ be a subalgebra and assume that it is invariant under the modular group σ_t^ω of ω. Let $\mathcal{M}_1 = \mathcal{M}_0' \cap \mathcal{M}$ be the relative commutant. We show that $\mathcal{M}_1$ is invariant under σ_t^ω as well. If $a \in \mathcal{M}_0$ and $b \in \mathcal{M}_1$, then for $t \in \mathbb{R}$, we have

$$a\sigma_t^\omega(b) = \sigma_t^\omega\big(\sigma_{-t}^\omega(a)b\big) = \sigma_t^\omega\big(b\sigma_{-t}^\omega(a)\big) = \sigma_t^\omega(b)a.$$

Hence $\mathcal{M}_1$ is invariant under σ_t^ω. Let ω_0, ω_1 be the restrictions of ω to $\mathcal{M}_0$ and $\mathcal{M}_1$. Then $\sigma_t^\omega|\mathcal{M}_0 = \sigma_t^{\omega_0}$ and $\sigma_t^\omega|\mathcal{M}_1 = \sigma_t^{\omega_1}$ are known facts in modular theory.

Recall that the **entropy** of a state φ of a C*-algebra is defined as

$$S(\varphi) := \sup\Big\{\sum_i \lambda_i S(\varphi_i, \varphi) : \sum_i \lambda_i \varphi_i = \varphi\Big\},$$

see (6.9) in [12]. For the sake of simplicity, we will suppose in the rest of this section that the state ω has finite von Neumann entropy $S(\omega)$. Then $\mathcal{M}$ must be a countable direct sum of type I factors, see Theorem 6.10. in [12]. Let τ be the canonical normal semifinite trace on $\mathcal{M}$ and let D_ω be the density of ω with respect to τ, then

$$\sigma_t^\omega(a) = D_\omega^{it} a D_\omega^{-it},\, a \in \mathcal{M}.$$

As the subalgebras $\mathcal{M}_0$ and $\mathcal{M}_1$ are invariant under σ_t^ω, we have by Proposition 6.7. in [12] that $S(\omega_0), S(\omega_1) \le S(\omega) < \infty$. It follows that both $\mathcal{M}_0$ and $\mathcal{M}_1$ must be countable direct sums of type I factors as well.

Let $D_{\omega_0} \in \mathcal{M}_0$ and $D_{\omega_1} \in \mathcal{M}_1$ be the densities of ω_0 and ω_1 with respect to the canonical traces $\tau_0 := \tau|\mathcal{M}_0$ and $\tau_1 := \tau|\mathcal{M}_1$. Then for $a \in \mathcal{M}_0$,

$$D_\omega^{it} a D_\omega^{-it} = \sigma_t^\omega(a) = \sigma_t^{\omega_0}(a) = D_{\omega_0}^{it} a D_{\omega_0}^{-it}.$$

It follows that $w_t := D_{\omega_0}^{-it} D_\omega^{it}$ is a unitary operator in $\mathcal{M}_1$ and the operators $D_{\omega_0}^{it}$ and D_ω^{is} commute for all $t, s \in \mathbb{R}$. It is easy to see that w_t is a strongly continuous one-parameter group. Moreover, we have for $a \in \mathcal{M}_1$,

$$w_t a w_t^* = D_\omega^{it} a D_\omega^{-it} = \sigma_t^{\omega_1}(a) = D_{\omega_1}^{it} a D_{\omega_1}^{-it}.$$

Therefore, the unitary $z_t = D_{\omega_1}^{-it} w_t$ is in the center of $\mathcal{M}_1$. Again, w_t and $D_{\omega_1}^{is}$ commute for all t, s and it is easy to see that $z_t = z^{it}$ for some positive element z in the center of $\mathcal{M}_1$. Putting all together, we get

$$D_\omega = D_{\omega_0} D_{\omega_1} z\,. \tag{11}$$

The following theorem is a generalization of the classical factorization theorem.

Theorem 4. *Let $(\mathcal{M}, \mathcal{S})$ be a statistical experiment dominated by a faithful normal state ω such that $S(\omega) < \infty$. Let $\mathcal{M}_0 \subset \mathcal{M}$ be a von Neumann subalgebra invariant with respect to the modular group σ_t^ω. Then $\mathcal{M}_0$ is sufficient for $\mathcal{S}$ if and only if*

$$D_\theta = D_{\theta,0} D_{\omega_1} z, \tag{12}$$

where D_θ, $D_{\theta,0}$ and D_{ω_1} are the densities of φ_θ, $\varphi_\theta|\mathcal{M}_0$ and $\omega|\mathcal{M}_0' \cap \mathcal{M}$, respectively and z is a positive operator from the center of $\mathcal{M}_0' \cap \mathcal{M}$.

Proof. By the assumptions and (11), we have $D_\omega^{it} = D_{\omega_0}^{it} D_{\omega_1}^{it} z^{it}$. If $\mathcal{M}_0$ is sufficient, then

$$u_t := D_\theta^{it} D_\omega^{-it} = [D\varphi_\theta, D\omega]_t = [D\varphi_\theta|_{\mathcal{M}_0}, D\omega_0]_t = D_{\theta,0}^{it} D_{\omega_0}^{-it},$$

hence $D_\theta^{it} = u_t D_\omega^{it} = D_{\theta,0}^{it} D_{\omega_1}^{it} z^{it}$ and (12) follows.

Conversely, let (12) be true, then $u_t = D_{\theta,0}^{it} D_{\omega_0}^{-it}$ and $\mathcal{M}_0$ is sufficient. □

The essence of the factorization (12) is that the first factor is the reduced density and the rest is independent of θ.

From Theorem 1 (iv), it follows that the subalgebra generated by the partial isometries $\{[D\varphi_\theta, D\omega]_t : t \in \mathbb{R}\}$ is **minimal sufficient**, that is, it is sufficient and contained in any sufficient subalgebra. Moreover, it is invariant under σ_t^ω. We will denote this subalgebra by $\mathcal{M}_\mathcal{S}$. By Theorem 4, we have the decompositions:

$$D_\theta = D_{\mathcal{S},\theta} D_\mathcal{R} z_\mathcal{S}, \qquad D_\omega = D_{\mathcal{S},\omega} D_\mathcal{R} z_\mathcal{S}, \tag{13}$$

where $D_{\mathcal{S},\theta}$, $D_{\mathcal{S},\omega}$ are the densities of the restrictions $\varphi_\theta|\mathcal{M}_\mathcal{S}$ and $\omega|\mathcal{M}_\mathcal{S}$ with respect to the canonical trace $\tau_\mathcal{S}$, it will be called the **$\mathcal{S}$-decomposition**. The next theorem shows that each decomposition of the form (12) is given by an invariant sufficient subalgebra and (13) is the maximal one.

Theorem 5. *Let us suppose that there is a decomposition $D_\theta = L_\theta R$, with some positive operators L_θ, R in $\mathcal{M}$, such that supp $R = I$ and R commutes with all L_θ. Let $\mathcal{M}_L$ be the von Neumann algebra generated by $\{L_\theta : \theta \in \Theta\}$. Then $\mathcal{M}_L$ is sufficient and invariant under σ_t^ω. Moreover,*

$$L_\theta = D_{\mathcal{S},\theta} R_0 \,,$$

where $D_{\mathcal{S},\theta}$ is given by (13) and $R_0 \in \mathcal{M}_L$ is a positive element commuting with all $D_{\mathcal{S},\theta}$.

Proof. We have $D_\omega = \sum_n \lambda_n D_{\theta_n} = \sum_n \lambda_n L_{\theta_n} R$, hence $\sum_n \lambda_n L_{\theta_n}$ converges strongly to some positive operator $L_\omega \in \mathcal{M}_L$, such that $D_\omega = L_\omega R$. For $a \in \mathcal{M}_L$, we get

$$D_\omega^{it} a D_\omega^{-it} = L_\omega^{it} a L_\omega^{-it} \in \mathcal{M}_L,$$

and $\mathcal{M}_L$ is invariant under σ_t^ω. It follows also that there is a density operator $D_{\omega_L} \in \mathcal{M}_L$ of the restriction $\omega_L := \omega|\mathcal{M}_L$, such that $D_{\omega_L} c = L_\omega$ for some $c \in \mathcal{M}_L' \cap \mathcal{M}_L$. Moreover, it is easy to see that $\mathcal{M}_\mathcal{S} \subset \mathcal{M}_L$, so that $\mathcal{M}_L$ is sufficient and the densities of $\varphi_\theta|\mathcal{M}_L$ satisfy

$$D_{\theta,L}^{it} c^{it} = [D\varphi_\theta|\mathcal{M}_L, D\omega_L]_t D_{\omega_L}^{it} c^{it} = [D\varphi_\theta, D\omega]_t L_\omega^{it} = L_\theta^{it}.$$

By Theorem 4, there is a decomposition $D_{\theta,L} = D_{\mathcal{S},\theta} D_{\mathcal{R},L} z_L$, such that $D_{\mathcal{R},L} z_L \in \mathcal{M}_\mathcal{S}' \cap \mathcal{M}_L$. Putting all together, we get

$$L_\theta = D_{\theta,L} c = D_{\mathcal{S},\theta} R_0 \,,$$

where $R_0 = D_{\mathcal{R},L} z_L c \in \mathcal{M}_\mathcal{S}' \cap \mathcal{M}_L$. □

It is easy to see that the $\mathcal{S}$-decomposition is, up to a central element in $\mathcal{M}_{\mathcal{S}}$, the unique decomposition having the property described in the previous theorem.

Keeping the assumptions of Theorem 4, let us suppose that $\mathcal{M}$ acts on some Hilbert space $\mathcal{H}$. The relative commutant $\mathcal{M}_1 := \mathcal{M}_0' \cap \mathcal{M}$ is a countable direct sum of factors of type I, hence there is an orthogonal family of minimal central projections p_n such that $\sum_n p_n = 1$. Therefore, $z = \sum_n z_n p_n$, with some $z_n > 0$. Moreover, there is a decomposition

$$\mathcal{H} = \bigoplus_n \mathcal{H}_n^L \otimes \mathcal{H}_n^R, \qquad p_n : \mathcal{H} \to \mathcal{H}_n^L \otimes \mathcal{H}_n^R \tag{14}$$

such that, up to isomorphism,

$$\mathcal{M}_1 = \bigoplus_n \mathbb{C} I_{\mathcal{H}_n^L} \otimes B(\mathcal{H}_n^R), \qquad (\mathcal{M}_1)' = \bigoplus_n B(\mathcal{H}_n^L) \otimes \mathbb{C} I_{\mathcal{H}_n^R}$$

From $D_{\omega_1} \in \mathcal{M}_1$ and $D_{\theta,0} \in \mathcal{M}_0 \subseteq (\mathcal{M}_1)'$, we have

$$p_n D_{\omega_1} = c_n^R (I_{\mathcal{H}_n^L} \otimes D_n^R), \qquad p_n D_{\theta,0} = c_n^L(\theta)(D_n^L(\theta) \otimes I_{\mathcal{H}_n^R}),$$

where D_n^R is a density operator in $B(\mathcal{H}_n^R)$, $D_n^L(\theta)$ is a density operator in $B(\mathcal{H}_n^L)$ and $c_n^R, c_n^L(\theta) > 0$. From this, we get

$$D_\theta = D_{\theta,0} D_{\omega_1} z = \sum_n z_n p_n D_{\theta,0} p_n D_{\omega_1} = \sum_n \varphi_\theta(p_n) D_n^L(\theta) \otimes D_n^R \tag{15}$$

Let now $\mathcal{M} = B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ and let $\mathcal{M}_{\mathcal{S}} \subseteq B(\mathcal{H})$ be the minimal sufficient subalgebra. From (14) and (15), we obtain the following form of the $\mathcal{S}$-decomposition:

$$\mathcal{H} = \bigoplus_n \mathcal{H}_n^{\mathcal{S}} \otimes \mathcal{H}_n^{\mathcal{R}}, \quad p_n^{\mathcal{S}} : \mathcal{H} \to \mathcal{H}_n^{\mathcal{S}} \otimes \mathcal{H}_n^{\mathcal{R}},$$
$$D_\theta = D_{\mathcal{S},\theta} D_{\mathcal{R}} z_{\mathcal{S}} = \sum_n \varphi_\theta(p_n^{\mathcal{S}}) D_n^{\mathcal{S}}(\theta) \otimes D_n^{\mathcal{R}}.$$

Note that if the dimension of $\mathcal{H}$ is finite, then it can be shown from Theorem 5 that this gives the maximal decomposition, obtained by Koashi and Imoto in [7].

Let $\mathcal{K}$ be a Hilbert space. The next theorem shows how sufficient mappings $B(\mathcal{K}) \to B(\mathcal{H})$ can be characterized in terms of the $\mathcal{S}$-decomposition.

Theorem 6. *A coarse-graining $\alpha : B(\mathcal{K}) \to B(\mathcal{H})$ is sufficient for the experiment $(B(\mathcal{H}), \mathcal{S})$ if and only if there is a decomposition $\mathcal{K} = \bigoplus_n \mathcal{K}_n^L \otimes \mathcal{K}_n^R$, $q_n : \mathcal{K} \to \mathcal{K}_n^L \otimes \mathcal{K}_n^R$, unitaries $U_n : \mathcal{K}_n^L \to \mathcal{H}_n^{\mathcal{S}}$ and coarse-grainings $\alpha_{2,n} : B(\mathcal{K}_n^R) \to B(\mathcal{H}_n^{\mathcal{R}})$ such that*

(i) for all n, $\alpha(q_n) = p_n^{\mathcal{S}}$ and the restriction $\alpha_n := \alpha|q_n \mathcal{B}(\mathcal{K}) q_n$ has the form

$$\alpha_n = \alpha_{1,n} \otimes \alpha_{2,n}, \qquad \alpha_{1,n}(a) = U_n a U_n^*, \ a \in B(\mathcal{K}_n^L),$$

(ii) the densities are decomposed as

$$\alpha^*(D_\theta) = \alpha_\omega^*(D_{\mathcal{S},\theta}) \alpha^*(D_{\mathcal{R}} z_{\mathcal{S}}) = \sum_n \varphi_\theta(p_n^{\mathcal{S}}) U_n^* D_n^{\mathcal{S}}(\theta) U_n \otimes \alpha_{2,n}^*(D_n^{\mathcal{R}}).$$

Moreover, (ii) is the $\mathcal{S} \circ \alpha$-decomposition of $\alpha^*(D_\theta)$.

Proof. Let $\mathcal{M}_{\mathcal{S}\circ\alpha} \subset B(\mathcal{K})$ be the minimal sufficient subalgebra for the experiment $(B(\mathcal{K}), \mathcal{S} \circ \alpha)$. If α is sufficient, then by Theorem 3, $\mathcal{M}_{\mathcal{S}\circ\alpha}$ is in the multiplicative domain of α and the restriction $\alpha|\mathcal{M}_{\mathcal{S}\circ\alpha}$ is a *-isomorphism onto $\mathcal{M}_{\mathcal{S}}$. Hence, $\mathcal{M}_{\mathcal{S}\circ\alpha}$ has the same structure as $\mathcal{M}_{\mathcal{S}}$. Namely, there is a decomposition $\mathcal{K} = \bigoplus_n \mathcal{K}_n^L \otimes \mathcal{K}_n^R$ such that

$$\mathcal{M}_{\mathcal{S}\circ\alpha} = \bigoplus_n B(\mathcal{K}_n^L) \otimes \mathbb{C} I_{\mathcal{K}_n^R}$$

and the corresponding central projections satisfy $\alpha(q_n) = p_n^{\mathcal{S}}$. Moreover, there are unitaries $U_n : \mathcal{K}_n^L \to \mathcal{H}_n^{\mathcal{S}}$, such that if $a \in \mathcal{M}_{\mathcal{S}\circ\alpha}$, $a = \sum_n a_n \otimes I_{\mathcal{K}_n^R}$ for some $a_n \in B(\mathcal{K}_n^L)$, then $\alpha(a) = \sum_n U_n a_n U_n^* \otimes I_{\mathcal{H}_n^{\mathcal{R}}}$.

Let $b \in \mathcal{M}'_{\mathcal{S}\circ\alpha}$, then for $a \in \mathcal{M}_{\mathcal{S}}$,

$$\alpha(b)a = \alpha(b)\alpha(\alpha^{-1}(a)) = \alpha(b\alpha^{-1}(a)) = \alpha(\alpha^{-1}(a)b) = a\alpha(b),$$

so that $\alpha(b) \in \mathcal{M}'_{\mathcal{S}}$. Consequently, $\alpha(bq_n) = \alpha(b)p_n^{\mathcal{S}} \in \mathcal{M}'_{\mathcal{S}} p_n^{\mathcal{S}}$ and if $b_n \in B(\mathcal{K}_n^R)$, then $\alpha_n(I_{\mathcal{K}_n^L} \otimes b_n) = I_{\mathcal{H}_n^{\mathcal{S}}} \otimes b'_n$ for some $b'_n \in B(\mathcal{H}_n^{\mathcal{R}})$. It is clear that the map $\alpha_{2,n} : b_n \mapsto b'_n$ is a coarse-graining $B(\mathcal{K}_n^R) \to B(\mathcal{H}_n^{\mathcal{R}})$. We also have

$$\begin{aligned}\alpha_n(a_n \otimes b_n) = \alpha_n((a_n \otimes I_{\mathcal{K}_n^R})(I_{\mathcal{K}_n^L} \otimes b_n)) &= \alpha_n(a_n \otimes I_{\mathcal{K}_n^R})\alpha_n(I_{\mathcal{K}_n^L} \otimes b_n) \\ &= U_n a_n U_n^* \otimes \alpha_{2,n}(b_n),\end{aligned}$$

hence $\alpha_n = \alpha_{1,n} \otimes \alpha_{2,n}$ and (i) is proved.

Let $D_{\theta,1}$ be the density of the restriction of $\varphi_\theta \circ \alpha$ to $\mathcal{M}_{\mathcal{S}\circ\alpha}$ with respect to the canonical trace $\tau_1 = \tau_{\mathcal{S}} \circ \alpha$. Then it is clear that $D_{\theta,1} = \alpha^{-1}(D_{\mathcal{S},\theta}) = \alpha_\omega^*(D_{\mathcal{S},\theta})$. Let $a \in B(\mathcal{K})$, then

$$\mathrm{Tr}\alpha^*(D_\theta)a = \mathrm{Tr} D_\theta \alpha(a) = \mathrm{Tr}\alpha(D_{\theta,1}) D_{\mathcal{R}} z_{\mathcal{S}} \alpha(a) = \mathrm{Tr} D_{\theta,1} \alpha^*(D_{\mathcal{R}} z_{\mathcal{S}})a$$

and (ii) follows.

Conversely, if (i) and (ii) are satisfied, then it is quite clear that the Connes' cocycles satisfy $\alpha([D\varphi_\theta \circ \alpha, D\omega \circ \alpha]_t) = [D\varphi_\theta, D\omega]_t$ and α is sufficient. □

Corollary 1. *Let $\mathcal{H}$ and $\mathcal{K}$ be finite dimensional and suppose that $\alpha : B(\mathcal{K}) \to B(\mathcal{H})$ is a completely positive map, with the Kraus representation $\alpha(a) = \sum_i V_i a V_i^*$. Then α is sufficient for $(B(\mathcal{H}), \mathcal{S})$ if and only if there is a decomposition $\mathcal{K} = \bigoplus_n \mathcal{K}_n^L \otimes \mathcal{K}_n^R$ and*

$$V_i = \sum_n U_n \otimes L_{i,n},$$

where $U_n : \mathcal{K}_n^L \to \mathcal{H}_n^{\mathcal{S}}$ are unitary and $L_{i,n} : \mathcal{K}_n^R \to \mathcal{H}_n^{\mathcal{R}}$ are linear maps such that $\sum_i L_{i,n} L_{i,n}^* = I_{H_n^{\mathcal{R}}}$.

Proof. By Theorem 6, if α is sufficient, then there is a decomposition $\mathcal{K} = \bigoplus_n \mathcal{K}_n^L \otimes \mathcal{K}_n^R$ and the corresponding projections q_n satisfy $\alpha(q_n a q_m) = p_n^{\mathcal{S}} \alpha(a) p_m^{\mathcal{S}}$. Consequently

$$\alpha(a) = \sum_{n,m} p_n^{\mathcal{S}} \alpha\left(\sum_{k,l} q_k a q_l\right) p_m^{\mathcal{S}} = \sum_{n,m} p_n^{\mathcal{S}} \alpha(q_n a q_m) p_m^{\mathcal{S}}$$

$$= \sum_i \left(\sum_n p_n^{\mathcal{S}} V_i q_n\right) a \left(\sum_m q_m V_i^* p_m^{\mathcal{S}}\right)$$

Let $V_{i,n} := p_n^{\mathcal{S}} V_i q_n$, then $V_{i,n} : B(\mathcal{K}_n^L \otimes \mathcal{K}_n^R) \to B(\mathcal{H}_n^{\mathcal{S}} \otimes \mathcal{H}_n^{\mathcal{R}})$ and

$$\sum_i V_{i,n} a V_{i,n}^* = \alpha_n(a), \qquad a \in B(\mathcal{K}_n^L \otimes \mathcal{K}_n^R).$$

Moreover, there are unitaries $U_n : \mathcal{K}_n^L \to \mathcal{H}_n^{\mathcal{S}}$ and coarse-grainings $\alpha_{2,n} : B(\mathcal{K}_n^R) \to B(\mathcal{H}_n^{\mathcal{R}})$ such that $\alpha_n = \alpha_{1,n} \otimes \alpha_{2,n}$, in fact, it is easy to see that $\alpha_{2,n}$ have to be completely positive. This implies that there are linear maps $K_{i,n} : \mathcal{K}_n^R \to \mathcal{H}_n^{\mathcal{R}}$, $\sum_i K_{i,n} K_{i,n}^* = I_{\mathcal{H}_n^{\mathcal{R}}}$, such that

$$\alpha_n(a) = \sum_i \left(U_n \otimes K_{i,n}\right) a (U_n \otimes K_{i,n})^*$$

is another Kraus representation of α_n. Hence there are $\{\mu_{i,j}^n\}$, $\sum_i \mu_{i,j}^n \bar{\mu}_{i,k}^n = \delta_{j,k}$, such that $V_{i,n} = U_n \otimes \sum_j \mu_{i,j}^n K_{j,n}$. Similarly, there are $\nu_{i,j}$, $\sum_i \nu_{i,j} \bar{\nu}_{i,k} = \delta_{j,k}$, such that

$$V_i = \sum_j \nu_{i,j} (\sum_n V_{j,n}) = \sum_n U_n \otimes L_{i,n},$$

where $L_{i,n} = \sum_{j,k} \nu_{i,j} \mu_{j,k}^n K_{k,n}$. The converse is obvious. □

As another corollary, we obtain a result previously proved in [7].

Corollary 2. *Let $\mathcal{H}$ be a finite dimensional Hilbert space and let $\alpha : B(\mathcal{H}) \to B(\mathcal{H})$ be completely positive, with Kraus representation $\alpha(a) = \sum_i V_i a V_i^*$. Then α preserves all states $\varphi_\theta \in \mathcal{S}$ if and only if*

$$V_i = \sum_n I_{\mathcal{H}_n^{\mathcal{S}}} \otimes L_{i,n},$$

where $\sum_i L_{i,n} L_{i,n}^* = I_{\mathcal{H}_n^{\mathcal{R}}}$ and $L_{i,n}$ commutes with $D_n^{\mathcal{R}}$ for all i, n.

Proof. Let α satisfy $\varphi_\theta \circ \alpha = \varphi_\theta$ for all θ, then α is obviously sufficient and by Theorem 6 and Corollary 1, $V_i = \sum_n U_n \otimes L_{i,n}$ and

$$D_\theta = \alpha^*(D_\theta) = \sum_n \varphi_\theta(p_n^{\mathcal{S}}) U_n^* D_n^{\mathcal{S}}(\theta) U_n \otimes \alpha_{2,n}^*(D_n^{\mathcal{R}}).$$

It follows that $U_n D_n^{\mathcal{S}}(\theta) U_n^* = D_n^{\mathcal{S}}(\theta)$ and $\alpha_{2,n}^*(D_n^{\mathcal{R}}) = \sum_i L_{i,n}^* D_n^{\mathcal{R}} L_{i,n} = D_n^{\mathcal{R}}$ for all θ and n. By construction of the $\mathcal{S}$-decomposition, the operators $D_n^{\mathcal{S}}(\theta)$ generate $B(\mathcal{H}_n^{\mathcal{S}})$, hence $U_n = I_{\mathcal{H}_n^{\mathcal{S}}}$. Moreover, the operator $D_n^{\mathcal{R}}$ is in the fixed point space of $\alpha_{2,n}^*$ if and only if it commutes with the Kraus operators $L_{i,n}$ for all i, [6].

The converse statement is obvious. □

5. Strong Subadditivity of Entropy

Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and let ω_{ABC} be a normal state on $B(\mathcal{H})$ with restrictions ω_B, ω_{AB} and ω_{BC}. The von Neumann entropies satisfies the **strong subadditivity**

$$S(\omega_{ABC}) + S(\omega_B) \leq S(\omega_{AB}) + S(\omega_{BC}), \tag{16}$$

which was obtained by Lieb and Ruskai [8]. A concise proof using the Jensen operator inequality is contained in [15] and [11] is a didactical presentation of the same ideas. As we want to investigate the case of equality mostly, we suppose below that all the involved entropies are finite. The case of equality was studied in several papers recently but always restricted to finite dimensional Hilbert spaces [6, 9]. Our aim now is to allow infinite dimensional spaces.

The strong subadditivity is equivalent to

$$S(\omega_{AB}, \varphi \otimes \omega_B) \leq S(\omega_{ABC}, \varphi \otimes \omega_{BC}), \tag{17}$$

where φ is any state on $B(\mathcal{H}_A)$ of finite entropy. This inequality is a consequence of monotonicity of the relative entropy. Clearly, the equality in (16) is equivalent to equality in (17).

Theorem 7. *Let ω_{ABC} be a normal state on $B(\mathcal{H})$ such that the von Neumann entropies $S(\omega_{ABC})$, $S(\omega_{BC})$ and $S(\omega_B)$ are finite. Suppose that*

$$S(\omega_{ABC}) + S(\omega_B) = S(\omega_{AB}) + S(\omega_{BC}).$$

Then there is a decomposition $p_B\mathcal{H}_B = \bigoplus_n \mathcal{H}^L_{nB} \otimes \mathcal{H}^R_{nB}$, $p_B = \operatorname{supp} \omega_B$, such that the density operator of ω_{ABC} satisfies

$$D_{ABC} = \sum_n \omega_B(p_n) D^L_n \otimes D^R_n, \tag{18}$$

where $D^L_n \in B(\mathcal{H}_A) \otimes B(\mathcal{H}^L_{nB})$ and $D^R_n \in B(\mathcal{H}^R_{nB}) \otimes B(\mathcal{H}_C)$ are density operators and $p_n \in B(\mathcal{H}_B)$ are the orthogonal projections $\mathcal{H}_B \to \mathcal{H}^L_{nB} \otimes \mathcal{H}^R_{nB}$.

Proof. Let φ be a faithful state on $B(\mathcal{H}_A)$ with finite entropy. Then equality in the strong subadditivity is equivalent to

$$S(\omega_{AB}, \varphi \otimes \omega_B) = S(\omega_{ABC}, \varphi \otimes \omega_{BC}).$$

Let p and q be the support projection of $\varphi \otimes \omega_{BC}$ and $\varphi \otimes \omega_B$, respectively. Then $p = I_A \otimes p_{BC}$ and $q = I_A \otimes p_B$, where $p_{BC} = \operatorname{supp} \omega_{BC}$ and $p_B = \operatorname{supp} \omega_B$. Consider the restricted experiment $(pB(\mathcal{H})p, \mathcal{S})$, where $\mathcal{S} := \{\omega_{ABC}|pB(\mathcal{H})p, \varphi \otimes \omega_{BC}|pB(\mathcal{H})p\}$. Then $\mathcal{S}$ is dominated by the faithful state $\varphi \otimes \omega_{BC}|pB(\mathcal{H})p$ with finite entropy, and the results of Sect. 4 apply.

Let $\alpha : qB(\mathcal{H}_{AB})q \to pB(\mathcal{H})p$ be the map

$$\alpha : a \mapsto p(a \otimes I_C)p,$$

then α is a coarse-graining and

$$\begin{aligned} &S(\omega_{ABC}|pB(\mathcal{H}))p, \varphi \otimes \omega_{BC}|pB(\mathcal{H})p) \\ &= S(\omega_{ABC}|pB(\mathcal{H}))p \circ \alpha, \varphi \otimes \omega_{BC}|pB(\mathcal{H})p \circ \alpha). \end{aligned}$$

By Proposition 4, α is sufficient for the experiment $(pB(\mathcal{H})p, \mathcal{S})$, equivalently, the subalgebra $\mathcal{M}_1$ is sufficient for this experiment, by Theorem 3, (v). Since $\mathcal{M}_1$ is invariant under $\sigma_t^{\varphi\otimes\omega_{BC}}$, the factorization results now imply that

$$D_{ABC} = D^L D^R z,$$

where $D^L \in \mathcal{M}_1$, $D^R \in \mathcal{M}_1'$ are density operators and $z \geq 0$ is in the center of $\mathcal{M}_1$. We will now investigate the subalgebra $\mathcal{M}_1$.

Let $\mathcal{N}_1 = \{a \in qB(\mathcal{H}_{AB})q,\ \alpha^*_{\varphi\otimes\omega_{BC}} \circ \alpha(a) = a\}$, then $\mathcal{M}_1 = \alpha(\mathcal{N}_1)$. Each $a \in \mathcal{N}_1$ is in the multiplicative domain of α, that is

$$p(a^*a \otimes I_C)p = p(a^* \otimes I_C)p(a \otimes I_C)p. \tag{19}$$

This implies $(I - p)(a \otimes I_C)p = p(a \otimes I_C)(I - p) = 0$, hence (19) is satisfied if and only if p commutes with $a \otimes I_C$. There is some $D_0^L \in \mathcal{N}_1$, such that $D^L = p(D_0^L \otimes I_C)p = (D_0^L \otimes I_C)p$. Further, we clearly have $p\mathcal{M}_1'p \subseteq (\mathcal{N}_1 \otimes I_C)'$. From this and from $\operatorname{supp}\omega_{ABC} \leq p$ we get

$$D_{ABC} = pD_{ABC}p = (D_0^L \otimes I_C)pD^R pz = (D_0^L \otimes I_C)D_0^R z_0,$$

where D_0^R is an element in $\mathcal{N}_1' \otimes B(\mathcal{H}_C)$ and z_0 is in the center of $\mathcal{N}_1 \otimes I_C$.

On the other hand, $\mathcal{N}_1$ is the algebra of elements $a \in qB(\mathcal{H}_{AB})q$, satisfying (19) and

$$p(\sigma_t^{\varphi\otimes\omega_B}(a) \otimes I_C)p = \sigma_t^{\varphi\otimes\omega_{BC}}(a \otimes I_C).$$

Since $p = I_A \otimes p_{BC}$, $q = I_A \otimes p_B$ and $\mathcal{N}_1$ is invariant under $\sigma_t^{\varphi\otimes\omega_B}$, it is easy to see that $\mathcal{N}_1 = B(\mathcal{H}_A) \otimes \mathcal{N}_B$, where $\mathcal{N}_B$ is a subalgebra in $p_B B(\mathcal{H}_B)p_B$, invariant under $\sigma_t^{\omega_B}$. Therefore, $S(\omega_B|\mathcal{N}_B) \leq S(\omega_B) < \infty$ and we obtain a decomposition $p_B\mathcal{H}_B = \bigoplus_n \mathcal{H}_{nB}^L \otimes \mathcal{H}_{nB}^R$ such that

$$\mathcal{N}_B = \bigoplus_n B(\mathcal{H}_{nB}^L) \otimes \mathbb{C}I_{\mathcal{H}_{nB}^R} \qquad \mathcal{N}_B' = \bigoplus \mathbb{C}I_{\mathcal{H}_{nB}^L} \otimes B(\mathcal{H}_{nB}^R),$$

which implies (18). □

The structure (18) of the density matrix ω_{ABC} is similar to the finite dimensional situation discussed in [6, 9], however the direct sum decomposition may be infinite.

When the state ω_{ABC} is pure, Eq. (18) simplifies. In this case, the strong subadditivity reduces to

$$S(\omega_{AC}) \leq S(\omega_A) + S(\omega_C),$$

which is simply the subadditivity. The equality holds here if $\omega_{AC} = \omega_A \otimes \omega_C$. Since the purification of a product state is a product vector, we have the product structure (18), without the summation over n. Note that this kind of states were discussed in [21].

The decomposition (18) has a continuous version formulated in terms of direct integrals (see [13] for references about the direct integral of fields of Hilbert spaces and operators or [22]). Let (X, μ) be a measure space. Assume that for $x \in X$ density matrices $D^L(x) \in B(\mathcal{H}_A) \otimes B(\mathcal{H}^L(x))$ and $D^R(x) \in B(\mathcal{H}^R(x)) \otimes B(\mathcal{H}_C)$ such that $\mathcal{H}^L(x)$ and $\mathcal{H}^R(x)$ are measurable fields of Hilbert spaces and the operator fields $D^L(x)$ and $D^R(x)$ are measurable as well, $x \in X$. Given a probability density $p(x)$ on X,

$$\omega_{ABC} := \int_{\oplus} p(x)D^L(x) \otimes D^R(x)\, d\mu(x) \tag{20}$$

is a density on the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, where

$$\mathcal{H}_B := \int_{\oplus} \mathcal{H}^L(x) \otimes \mathcal{H}^R(x)\, d\mu(x)\,.$$

Then $B(\mathcal{H}_A) \otimes B(\mathcal{H}_B)$ is a sufficient subalgebra for the states ω_{ABC} and $\omega_A \otimes \omega_{BC}$. If the measure μ is not atomic, then $S(\omega_{ABC}) = \infty$.

References

1. Accardi, L., Cecchini, C.: Conditional expectations in von Neumann algebras and a theorem of Takesaki. J. Funct. Anal. **45**, 245–273 (1982)
2. Barndorff-Nielsen, O.: *Information and exponential families in statistical theory*. Wiley Series in Probability and Mathematical Statistics. Chichester: John Wiley & Sons, Ltd., 1978
3. Barndorff-Nielsen, O.E., Gill, R., Jupp, P.E.: On quantum statistical inference. J. R. Stat. Soc. Ser. B Stat. Methodol. **65**, 775–816 (2003)
4. Blank, J., Exner, P., Havliček, M.: *Hilbert space operators in quantum physics*. New York: American Institute of Physics, 1994
5. Bratteli, O., Robinson, D.W.: *Operator algebras and quantum statistical mechanics. 1. C*- and W*–algebras, symmetry groups, decomposition of states*. 2nd ed., Texts and Monographs in Physics, New York: Springer Verlag, 1987
6. Hayden, P., Jozsa, R., Petz, D., Winter, A.: Structure of states which satisfy strong subadditivity of quantum entropy with equality. Commun. Math. Phys. **246**, 359–374 (2004)
7. Koashi, M., Imoto, N.: Operations that do not disturb partially known quantum states. Phys. Rev. A, **66**, 022318 (2002)
8. Lieb, E.H., Ruskai, M.B.: Proof of the strong subadditivity of quantum mechanical entropy. J. Math. Phys. **14**, 1938–1941 (1973)
9. Mosonyi, M., Petz, D.: Structure of sufficient quantum coarse-grainings. Lett. Math. Phys. **68**, 19–30 (2004)
10. Nielsen, M.A., Chuang, I.L.: *Quantum Computation and Quantum Information*. Cambridge: Cambridge University Press, 2000
11. Nielsen, M.A., Petz, D.: A simple proof of the strong subadditivity inequality. Quantum Information & Computation **5**, 507–513 (2005)
12. Ohya, M., Petz, D.: *Quantum Entropy and Its Use*. Heidelberg:Springer-Verlag, 1993, 2nd edition, 2004
13. Petz, D.: Direct integral of multifunctions into von Neumann algebras. Studia Sci. Math. Hungar. **18**, 239–245 (1978)
14. Petz, D.: Sufficient subalgebras and the relative entropy of states of a von Neumann algebra. Commun. Math. Phys. **105**, 123–131 (1986)
15. Petz, D.: Quasi-entropies for finite quantum systems. Rep. Math. Phys. **21**, 57–65 (1986)
16. Petz, D.: Sufficiency of channels over von Neumann algebras. Quart. J. Math. Oxford, **39**, 907–1008 (1988)
17. Petz, D.: Geometry of canonical correlation on the state space of a quantum system. J. Math. Phys. **35**, 780–795 (1994)
18. Petz, D.: Discrimination between states of a quantum system by observations. J Funct. Anal. **120**, 82–97 (1994)
19. Petz, D.: Monotonicity of quantum relative entropy revisited. Rev. Math. Phys. **15**, 79–91 (2003)
20. Ruskai, M.B.: Inequalities for quantum entropy: A review with conditions with equality. J. Math. Phys. **43**, 4358–4375 (2002)
21. Schumacher, B.W., Nielsen, M.A.: Quantum data processing and error correction. Phys. Rev. A **54**, 2629–2635 (1996)
22. Schwartz, J.T.: *W*-algebras*. New York-London-Paris: Gordon and Breach Science Publishers, 1967
23. Strătilă, Ş: *Modular theory of operator algebras*. Tunbridge Wells: Abacus Press, 1981
24. Strasser, H.: *Mathematical theory of statistics. Statistical experiments and asymptotic decision theory*. Berlin: Walter de Gruyter, 1985

Communicated by M.B. Ruskai