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Nonassociative Tori and Applications to T-Duality

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Abstract: In this paper, we initiate the study of C^*-algebras $\mathcal{A}$ endowed with a twisted action of a locally compact abelian Lie group G, and we construct a twisted crossed product $\mathcal{A} \rtimes \mathsf{G}$, which is in general a nonassociative, noncommutative, algebra. The duality properties of this twisted crossed product algebra are studied in detail, and are applied to T-duality in Type II string theory to obtain the T-dual of a general principal torus bundle with general H-flux, which we will argue to be a bundle of noncommutative, nonassociative tori. Nonassociativity is interpreted in the context of monoidal categories of modules. We also show that this construction of the T-dual includes the other special cases already analysed in a series of papers.

1. Introduction

Recent work has revealed the strong connections between T-duality in string theory and Takai duality for C^*-algebras (as for instance discussed in the introduction to [3, 28]), but for general H-fluxes C^*-algebras are no longer adequate. In this paper we present a generalisation which permits a very precise description of the general T-dual.

Let T be a compact connected Abelian Lie group of rank ℓ with Lie algebra $\mathfrak{t}$ and let $\widehat{\mathfrak{t}}$ be the dual of $\mathfrak{t}$. Let $E \to M$ be a principal T-bundle with connection. By the Chern-Weil construction the space $\Omega(E)^{\mathsf{T}}$ of T-invariant forms on E is isomorphic to the space of forms on M with values in $\wedge\widehat{\mathfrak{t}}$, i.e.

$$\Omega^k(E)^{\mathsf{T}} \cong \bigoplus_{p=0}^{k} \Omega^p(M, \wedge^{k-p}\widehat{\mathfrak{t}}), \tag{1.1}$$

and by a classical result of Chevalley and Koszul, the de-Rham complex $(\Omega^\bullet(E), d)$ is chain homotopy equivalent to the complex $(\Omega^\bullet(M, \wedge^\bullet\widehat{\mathfrak{t}}), D)$ with a modified de-Rham differential D, and hence the associated cohomologies are isomorphic. Furthermore, we know that any class in $H^k(E)$ can be represented by a form in $\Omega^k(E)^{\mathsf{T}}$. Thus, under this isomorphism, an H-flux in $H^3(E)$ can be considered as a 4-tuple $H = (H_3, H_2, H_1, H_0)$ with $H_p \in \Omega^p(M, \wedge^{3-p}\hat{\mathfrak{t}})$ for $p = 0, 1, 2, 3$, closed under the action of D (cf. [18, 5] for more details).[1]

T-duality for principal circle bundles was treated geometrically in [2, 3], and dimensional considerations force the H_0 and H_1 components of the H-flux to vanish in this case. The T-dual turns out to be another principal circle bundle with T-dual H-flux. The arguments were extended in [4] to principal $\mathbb{T}^\ell$-bundles with H-flux satisfying the condition that the H_0 and H_1 components vanish. Then the T-dual turns out to be another principal $\mathbb{T}^\ell$-bundle with T-dual H-flux having vanishing H_0 and H_1 components. The analysis in [28, 29] shows that if one considers principal $\mathbb{T}^\ell$-bundles with H-flux satisfying the condition that just the H_0 component vanishes, then one arrives at the surprising conclusion that the T-dual bundle has to have noncommutative tori as fibres, provided the H_1 component is non-zero. The weaker condition in [28, 29] permits non-vanishing H_1, but still excludes non-zero H_0. In this paper we shall remove the last of these constraints, to allow a non-vanishing H_0 component. In this case, we arrive at the astonishing conclusion that the T-dual bundle has to have *nonassociative* tori as fibres, taking it even beyond the normal range of noncommutative geometry.

The key step is provided by a new explicit construction of a continuous trace algebra $\mathcal{B}$ having a given Dixmier-Douady invariant, whose spectrum is the total space E of the principal T-bundle, together with automorphisms β_g for $g \in \mathsf{G} = \mathfrak{t}$, which transform the spectrum in a way compatible with the G-action on E. The new features arise because in general $g \mapsto \beta_g$ is not a homomorphism but satisfies $\beta_x\beta_y = \mathrm{ad}(v(x, y))\beta_{xy}$, where $\mathrm{ad}(v)$ denotes conjugation by a unitary element of the multiplier algebra $M\mathcal{B}$. One expects the algebra associated with the T-dual to be the crossed product $\mathcal{B} \rtimes_\beta \mathsf{G}$, but the twisting forces us to take a suitably twisted crossed product $\mathcal{B} \rtimes_{\beta,v} \mathsf{G}$. Such (Leptin-)Busby-Smith twistings have long been known but, for non-trivial H_0, $v(x, y)$ is not a cocycle and that means that associativity fails in the twisted crossed product.

In Sect. 2 we give the relationship between the differential forms and the multicharacters on G which will be used in our later constructions. Section 3 reviews the generalised Busby-Smith twisted crossed products. An example of a twisting is given in Sect. 4, together with a proof that it is the only type up to stability. This is followed by an example of a nonassociative generalisation of the compact operators.

The theory of twisted induced representations is developed in Sect. 6 and then used to construct examples of algebras with given spectrum and Dixmier–Douady class in Sect. 7. In Sect. 8 it is shown that the twisted crossed product of a twisted induced algebra is isomorphic to a generalisation of the twisted compact operators. In Sect. 9, the double dual is shown to be the tensor product of the original algebra with the twisted compact operators, that is, Morita equivalent in this category to the original algebra. In Sect. 10, the mathematical results of the previous sections are used to justify the assertion that the T-dual to a general principal torus bundle with H-flux, is a bundle of nonassociative tori. The final section outlines how associativity can be restored by working in a different category, an idea which will be explored in more detail in a subsequent paper.

[1] The conclusions in this paper are valid for integral classes $H \in H^3(E, \mathbb{Z})$ as well, since this paper deals exclusively with the introduction of the additional 'degree of freedom' H_0, which does not carry torsion, on top of established results which hold in the case of torsion H. Note that H_0 can be identified with the restriction of H to a fibre. For simplicity we have chosen to formulate some of the results in terms of differential forms.

2. Differential Forms and Multicharacters

Let T be a compact connected Abelian Lie group of rank ℓ, and $E \to M$ a principal T-bundle. In essence the action of T on E provides a map from the Lie algebra $\mathfrak{t}$ to vector fields on E, which we write $X \mapsto \xi_X$, and then a p-form $f \in \Omega^p(E)^{\mathsf{T}}$ in the fibre directions defines the antisymmetric multilinear form valued function on M,

$$(\xi^* f)(X_1, X_2, \dots, X_p) = f(\xi_{X_1}, \xi_{X_2}, \dots, \xi_{X_p}). \tag{2.1}$$

For abelian Lie groups the form exponentiates to a multicharacter on $\mathsf{G} = \mathfrak{t}$,

$$\phi\left(\exp(X_1), \dots, \exp(X_p)\right) = \exp(-2\pi i(\xi^* f)(X_1, \dots, X_p)). \tag{2.2}$$

(A multicharacter is a character in each variable, and this property follows from the additivity of $\xi^* f$, and ϕ is also antisymmetric in the sense that even permutations of its variables leave it unchanged whilst odd permutations invert it.)

The multicharacter property ensures that this is always a (Moore) cocycle in $Z^p(\mathfrak{t}, \mathbb{T})$, since, for example when $p = 3$,

$$\begin{aligned}\phi(y, z, w)\phi(x, yz, w)\phi(x, y, z) &= \phi(y, z, w)\phi(x, y, w)\phi(x, z, w)\phi(x, y, z)\\ &= \phi(xy, z, w)\phi(x, y, zw).\end{aligned}$$

(It is known that every cohomology class in $H^2(\mathbb{R}^n, \mathbb{T})$ can be represented by an antisymmetric bicharacter of the form ϕ [23, 19], and for $p = 3$ it is certainly true that smooth cocycles are cohomologous to antisymmetric tricharacters.)

Example. Consider the torus bundle $\mathbb{T}^3$ over a point, with H_0 the class defined by k times the volume form $dx^1 \wedge dx^2 \wedge dx^3$. The associated antisymmetric form on $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathfrak{t} = \mathbb{R}^3$ is then given by

$$f(\mathbf{a}, \mathbf{b}, \mathbf{c}) = k[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv k\mathbf{a}.(\mathbf{b} \times \mathbf{c}), \tag{2.3}$$

whence $\phi(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \exp(-2k\pi i[\mathbf{a}, \mathbf{b}, \mathbf{c}])$.

Although we are mainly interested in the abelian groups $\mathsf{T} = \mathbb{T}^n$, $\mathsf{G} = \mathfrak{t} = \mathbb{R}^n$, and $\mathsf{N} \cong \mathbb{Z}^n$ the kernel of the exponential map $\mathfrak{t} \to \mathsf{T}$, the constructions which we present in Chapters 3 to 8 are valid for general unimodular separable locally compact groups with a tricharacter ϕ. For that reason we shall write the group composition multiplicatively, and be careful not to commute terms. However, this generalisation is less sweeping than may appear because the tricharacter ϕ on G defines a homomorphism of each variable into the abelian group $\mathbb{T}$, and so must be lifted from a tricharacter on the abelianisation $\mathsf{G}/[\mathsf{G}, \mathsf{G}]$.

Finally we note that, by definition, the Dixmier-Douady class has components which are integral 3-forms, and that means, in particular, that the tricharacter ϕ constructed from H_0 is identically 1 on $\mathsf{N} \times \mathsf{N} \times \mathsf{N}$, where N is the kernel of the exponential map. We shall assume this to be true in the general case.

3. Generalised Busby-Smith Twisted Crossed Products

As noted above we now work with a general unimodular separable locally compact group G and a closed subgroup N on which the tricharacter ϕ has trivial restriction. For any group G one can interpret $H^2(\mathsf{G}, \mathsf{A})$ as classifying central extensions of G by A, whilst, for $p > 3$, $H^p(\mathsf{G}, \mathsf{A})$ is usually interpreted in terms of crossed modules [15, 16, 6, 20, 21, 34, 26], with elements of $H^3(\mathsf{G}, \mathsf{A})$ known as MacLane–Whitehead obstructions [35, 27]. However, we shall see that these classes also arise in a C^*-algebraic context.

The H-field is usually linked to the equivariant Brauer group of a continuous trace C*-algebra $\mathcal{A}$ with spectrum E, on which a group G acts as automorphisms so that the dual action E agrees with the bundle structure. However, the equivariant Brauer group can be described entirely in terms of cohomology classes in $H^p(M, H^{3-p}(\mathsf{G}, \mathbb{T}))$ for $p = 1, 2, 3$, [11], leaving no room for H-fields with a component in $H^0(M, H^3(\mathsf{G}, \mathbb{T}))$, (for example, any non-trivial H-field on G considered as a principal G-bundle over a point). Since the representatives of H^0 are locally constant functions we shall concentrate our attention on $H^3(\mathsf{G}, \mathbb{T})$.

This suggests that we must consider a wider class of algebras or actions. In fact, inner automorphisms automatically act trivially on the spectrum, so that only homomorphisms of G to the outer automorphisms $\mathrm{Out}(\mathcal{A}) = \mathrm{Aut}(\mathcal{A})/\mathrm{Inn}(\mathcal{A})$ are interesting. However, to work with these one needs a lifting $\alpha : \mathsf{G} \to \mathrm{Aut}(\mathcal{A})$. The problem then is that $\alpha_x\alpha_y$ and α_{xy} can differ by an inner automorphism $\mathrm{ad}(u(x, y)) : a \mapsto u(x, y)au(x, y)^{-1}$, that is $\alpha_x\alpha_y = \mathrm{ad}(u(x, y))\alpha_{xy}$. We can take $u(x, y) = 1$ whenever x or y is the identity.

This is almost precisely the data needed to define a Busby–Smith (or Leptin) twisted crossed product $\mathcal{A} \rtimes_{\alpha,u} \mathsf{G}$ of $\mathcal{A}$ and G, [24, 7, 32]. Assuming that u is a measurable function on $\mathsf{G} \times \mathsf{G}$, we can define a twisted convolution product and adjoint on $C_0(\mathsf{G}, \mathcal{A})$ by

$$
\begin{aligned}
(f * g)(x) &= \int_G f(y)\alpha_y[g(y^{-1}x)]u(y, y^{-1}x)\, dy, \\
f^*(x) &= u(x, x^{-1})^{-1}\alpha_x[f(x^{-1})]^*,
\end{aligned}
$$

and complete this to get a new algebra.

The link with the algebraists' picture of $H^3(\mathsf{G}, \mathsf{A})$ arises because u is no longer a cocycle since the condition linking α and u tells us only that the adjoint actions of $u(x, y)u(xy, z)$ and $\alpha_x[u(y, z)]u(x, yz)$ coincide, so that one has a modified cocycle condition:

$$
\phi(x, y, z)u(x, y)u(xy, z) = \alpha_x[u(y, z)]u(x, yz) \tag{3.1}
$$

for some central unitary element $\phi(x, y, z) \in UZ(\mathcal{A})$. It is easy to check that ϕ is a cocycle defining an element of $H^3(\mathsf{G}, UZ(\mathcal{A}))$. (Essentially the same argument is used in [8] to explain the origin of the Gauss anomaly and Jackiw's nonassociative anomaly in quantum field theory.) When ϕ is a tricharacter one can still form a twisted crossed product.

Proposition 3.1. *When ϕ defined as above is an antisymmetric tricharacter the twisted crossed product $\mathcal{A} \rtimes_{\alpha,u} \mathsf{G}$ satisfies the $*$-algebra identity $(f * g)^* = g^* * f^*$, and is* *associative if and only if* $\phi \equiv 1$. *In fact, we have*

$$((f * g) * h)(x) = \int_{\mathsf{G}\times\mathsf{G}} f(z)\alpha_z[g(z^{-1}y)]\alpha_y[h(y^{-1}x)]u(z, z^{-1}y)u(y, y^{-1}x)\, dydz,$$

$$(f * (g * h))(x) = \int_{\mathsf{G}\times\mathsf{G}} f(z)\alpha_z[g(z^{-1}y)]\alpha_y[h(y^{-1}x)]\phi(z, z^{-1}y, y^{-1}x) \times u(z, z^{-1}y)u(y, y^{-1}x)\, dydz.$$

Proof. Using the modified cocycle identity, one calculates that

$$(f * g)^*(x) = \int_{\mathsf{G}} \phi(x, x^{-1}y, y^{-1})^* u(y, y^{-1})^* \alpha_y[g(y^{-1})]^* \times u(x, x^{-1}y)^* \alpha_x[f(x^{-1}y)]^*\, dy, \tag{3.2}$$

whilst

$$(g^* * f^*)(x) = \int_{G} \phi(y, y^{-1}x, x^{-1}y)^* u(y, y^{-1})^* \alpha_y[g(y^{-1})]^* \times u(x, x^{-1}y)^* \alpha_x[f(x^{-1}y)]^*\, dy, \tag{3.3}$$

and for antisymmetric tricharacters both factors involving ϕ are 1.

The twisted crossed product algebra $\mathcal{A} \rtimes_{\alpha,u} \mathsf{G}$ has

$$\begin{aligned}((f * g) * h)(x) &= \int_{\mathsf{G}} (f * g)(y)\alpha_y[h(y^{-1}x)]u(y, y^{-1}x)\, dy,\\ &= \int_{\mathsf{G}\times\mathsf{G}} f(z)\alpha_z[g(z^{-1}y)]u(z, z^{-1}y)\alpha_y[h(y^{-1}x)]u(y, y^{-1}x)\, dydz\\ &= \int_{\mathsf{G}\times\mathsf{G}} f(z)\alpha_z[g(z^{-1}y)]\alpha_z\alpha_{z^{-1}y}[h(y^{-1}x)]u(z, z^{-1}y)\\ &\qquad \times u(y, y^{-1}x)\, dydz\\ &= \int_{\mathsf{G}\times\mathsf{G}} f(z)\alpha_z[g(z^{-1}y)]\alpha_{z^{-1}y}[h(y^{-1}x)]u(z, z^{-1}y)\\ &\qquad \times u(y, y^{-1}x)\, dydz,\end{aligned}$$

and, using the modified cocycle identity,

$$\begin{aligned}(f * (g * h))(x) &= \int_{\mathsf{G}} f(z)\alpha_z[(g * h)(z^{-1}x)]u(z, z^{-1}x)\, dz\\ &= \int_{\mathsf{G}\times\mathsf{G}} f(z)\alpha_z[g(z^{-1}y)]\alpha_{z^{-1}y}[h(y^{-1}x)]u(z^{-1}y, y^{-1}x)\\ &\qquad \times u(z, z^{-1}y)\, dydz,\\ &= \int_{\mathsf{G}\times\mathsf{G}} f(z)\alpha_z[g(z^{-1}y)]\alpha_{z^{-1}y}[h(y^{-1}x)]\alpha_z[u(z^{-1}y, y^{-1}x)]\\ &\qquad \times u(z, z^{-1}y)\, dydz\\ &= \int_{\mathsf{G}\times\mathsf{G}} f(z)\alpha_z[g(z^{-1}y)]\alpha_{z^{-1}y}[h(y^{-1}x)]\phi(z, z^{-1}y, y^{-1}x)\\ &\qquad \times u(z, z^{-1}y)u(y, y^{-1}x)\, dydz,\end{aligned}$$

so that the Busby-Smith twisted crossed product is nonassociative except in the case $\phi \equiv 1$. □

Henceforth we shall always take ϕ to be an antisymmetric tricharacter.

Usually Busby-Smith products are only defined when $\phi = 1$, but we shall see that much of the theory goes through without that assumption, so that this provides a means of constructing nonassociative from associative algebras. The nonassociativity becomes even more transparent when one considers a covariant representation (U, π) of $(\mathsf{G}, \mathcal{A})$ satisfying the conditions

$$U(x)\pi(a)U(x)^{-1} = \pi(\alpha_x(a)), \qquad U(x)U(y) = \pi(u(x, y))U(xy). \tag{3.4}$$

These give

$$\begin{aligned} U(x)[U(y)U(z)] &= U(x)[\pi(u(y, z))]U(yz) \\ &= \pi(u(x, yz))\pi(\alpha_x u(y, z))U(x(yz)), \end{aligned}$$

$$\begin{aligned} [U(x)U(y)]U(z) &= \pi(u(x, y))U(xy)U(z) \\ &= \pi(u(x, y))\pi(u(xy, z))U((xy)z), \end{aligned}$$

so that

$$\phi(x, y, z)U(x)[U(y)U(z)] = [U(x)U(y)]U(z). \tag{3.5}$$

In fact, $H^3(\mathsf{G})$ was already interpreted as defining a nonassociative structure in [16], and this has resurfaced in the physics literature [22, 13].

4. Generalised Packer–Raeburn Stabilisation

For a given antisymmetric tricharacter ϕ on G there is a simple example of an algebra with twisting on which G acts as automorphisms. It is derived from the imprimitivity algebra generated by multiplication and translation operators on $L^2(\mathsf{G})$. The right regular representation ρ acts on $\psi \in L^2(\mathsf{G})$ by $(\rho(x)\psi)(v) = \psi(vx)$, and we define

$$(u_\rho(y, z)\psi)(v) = \phi(v, y, z)\psi(v). \tag{4.1}$$

(We shall often be interested in the case when G is a closed subgroup of a group H, with ϕ defined on $\mathsf{H} \times \mathsf{H} \times \mathsf{H}$ and trivial on $\mathsf{G} \times \mathsf{G} \times \mathsf{G}$. Then $u_\rho(y, z)$ can be defined as a multiplication operator on $L^2(\mathsf{G})$ for general $y, z \in \mathsf{H}$, and the restriction of u_ρ to $\mathsf{G} \times \mathsf{G}$ is identically 1.)

Now

$$\begin{aligned} &\phi(x, y, z)(u_\rho(x, y)u_\rho(xy, z)u_\rho(x, yz)^{-1}\psi)(v) \\ &\quad = \phi(x, y, z)\phi(v, x, y)\phi(v, xy, z)\phi(v, x, yz)^{-1}\psi(v) \\ &\quad = \phi(vx, y, z)\psi(v) \end{aligned}$$

and

$$(\rho(x)u_\rho(y,z)\rho(x)^{-1}\psi)(v) = (u_\rho(y,z)\rho(x)^{-1}\psi)(vx)$$
$$= \phi(vx,y,z)(\rho(x)^{-1}\psi)(vx)$$
$$= \phi(vx,y,z)\psi(v), \tag{4.2}$$

so that, setting $\alpha_x = \mathrm{ad}(\rho(x))$, we get $\phi(x,y,z)u_\rho(x,y)u_\rho(xy,z)u_\rho(x,yz)^{-1} = \alpha_x[u_\rho(y,z)]$. This gives an explicit realisation of an algebra $\mathcal{C} = C_0(\mathsf{G})$ with an action of G by automorphisms and with the appropriate Busby-Smith obstruction. (When G is non-compact $u_\rho(x,y)$ is in the multiplier algebra rather than the algebra itself.) The same idea can be extended to the right regular σ-representation on $L^2(\mathsf{G})$ given by $(\rho(x)\psi)(v) = \sigma(v,x)\psi(vx)$ for any borel multiplier σ, and this makes no difference to the cocycle identity.

Naturally, one can also work with the left regular representation $(\lambda(x)\psi)(v) = \psi(x^{-1}v)$, and

$$(u_\lambda(y,z)\psi)(v) = \phi(v,y,z)^{-1}\psi(v). \tag{4.3}$$

This is useful because it links directly to the formulation used by [30] to show that twisted crossed products defined by cocycles are stably equivalent to normal crossed products. In our case u is not a cocycle, but there is nonetheless a nice generalisation of the Packer–Raeburn Theorem.

Theorem 4.1. *Let $\mathcal{A}$, G, α, u be as above. There exists a strongly continuous action β of G on $\mathcal{A}\otimes\mathcal{K}(L^2(\mathsf{G}))$ and a twisting u_λ such that (β, u_λ) are exterior equivalent to $(\alpha\otimes\mathrm{id}, u\otimes 1)$, that is, there exists $v_s = (1\otimes\lambda_s)(\mathrm{id}\otimes M(u(s,\cdot))^*)$ such that*

$$\beta_s = \mathrm{ad}(v_s)(\alpha_s\otimes\mathrm{id}), \qquad \mathrm{id}\otimes u_\lambda(s,t) = v_s\alpha_s(v_t)(u(s,t)\otimes 1)v_{st}^*. \tag{4.4}$$

Proof. The proof of Theorem 3.4 in [30] is still valid as far as the last line on p. 301. At that point the original argument uses the fact that u is a cocycle to show that it has been untwisted by the exterior equivalence. In our case u satisfies a modified cocycle identity, so that that last line (in the original notation) gives $\phi(s,t,r)^{-1}\xi(r) = (u_\lambda(s,t)\xi)(r)$. □

One similarly obtains:

Corollary 4.2. *In the situation of Theorem 4.1 one has*

$$(\mathcal{A}\rtimes_{\alpha,u}\mathsf{G})\otimes\mathcal{K}(L2(\mathsf{G})) \cong (\mathcal{A}\otimes\mathcal{K})\rtimes_{\beta,u_\lambda}\mathsf{G}\,. \tag{4.5}$$

The Packer-Raeburn stabilisation trick is used to show that up to stabilisation by tensoring with compact operators $\mathcal{A}\rtimes_{\alpha,u}\mathsf{G}$ and $\mathcal{A}\rtimes_{\beta,u_\lambda}\mathsf{G}$ are isomorphic. The original idea derived from Quigg's generalisation of Takai duality for twisted crossed products, and their β is just equivalent to Quigg's double dual $\hat{\hat{\alpha}}$. One may therefore obtain a duality theorem by the same procedure. We postpone discussion of this until Sect. 9 where we shall give a much more detailed account of duality for abelian groups.

5. Twisted Compact Operators

Before developing the theory further it is useful to give a very simple example of a nonassociative algebra, obtained by twisting the algebra of compact operators on $L^2(\mathsf{G})$ using the factor $\phi(x, y, z)$. We start with the Hilbert-Schmidt operators, realised as kernels $K(x, y)$ for $x, y \in \mathsf{G}$, with the involution $K^*(x, y) = \overline{K(y, x)}$, and norm

$$\|K\|_{HS}^2 = \int_{\mathsf{G}\times\mathsf{G}} |K(x, y)|^2\, dxdy, \tag{5.1}$$

and define the new multiplication

$$(K_1 * K_2)(x, z) = \int_{\mathsf{G}} \phi(x, y, z) K_1(x, y) K_2(y, z)\, dy. \tag{5.2}$$

This is consistent with the involution because

$$\begin{aligned}(K_1 * K_2)^*(x, z) &= \overline{(K_1 * K_2)(z, x)}\\ &= \int_{\mathsf{G}} \phi(z, y, x)^{-1} \overline{K_1(z, y)}\, \overline{K_2(y, x)}\, dy\\ &= \int_{\mathsf{G}} \phi(x, y, z) K_2^*(x, y) K_1^*(y, z)\, dy\\ &= (K_2^* * K_1^*)(x, z).\end{aligned}$$

The fact that $\phi(x, y, x) = 1$ also means that we still have

$$\|K\|_{HS}^2 = \int_{\mathsf{G}} (K^* * K)(x, x)\, dx. \tag{5.3}$$

As usual, one can define a C^*-norm using the left regular representation $\|K_1\| = \sup(\|K_1 * K_2\|_{HS}/\|K_2\|_{HS})$. By using the Cauchy-Schwarz inequality and by considering rank one projections one sees that this is equivalent to the ordinary operator norm. The twisted compact operators $\mathcal{K}_\phi(L^2(\mathsf{G}))$ are the completion of the Hilbert-Schmidt operators with respect to that norm.

Unfortunately the new multiplication is not associative unless

$$\phi(x, y, z)\phi(x, z, w) = \phi(x, y, w)\phi(y, z, w), \tag{5.4}$$

for all $x, y, z, w \in \mathsf{G}$. In that case $\phi(x, y, z) = \phi(x, y, w)\phi(y, z, w)/\phi(x, z, w)$, but conversely, whenever $\phi(x, y, z)$ has the form $\psi(x, y)\psi(y, z)/\psi(x, z)$, for some function ψ, the algebra is associative. The twisted multiplication of kernels was used in [9] to study the hyperbolic quantum Hall effect, but in that two-dimensional situation the multiplication is automatically associative. Once one gets to three dimensions the analogous algebra is nonassociative.

Proposition 5.1. *The group* G *acts on the twisted algebra* $\mathcal{K}_\phi(L^2(\mathsf{G}))$ *with multiplication*

$$(K_1 * K_2)(x, z) = \int_{\mathsf{G}} \phi(x, y, z) K_1(x, y) K_2(y, z)\, dy \tag{5.5}$$

by natural $*$*-automorphisms*

$$\theta_x[K](z, w) = \phi(x, z, w) K(zx, wx), \tag{5.6}$$

and $\theta_x\theta_y = \mathrm{ad}(\sigma(x, y))\theta_{xy}$, *where* $\mathrm{ad}(\sigma(x, y))[K](z, w) = \phi(x, y, z)\phi(x, y, w)^{-1} K(z, w)$ *comes from the multiplier* $\sigma(x, y)(v) = \phi(x, y, v)$.

Proof. Using the tricharacter property of ϕ, we see that

$$\begin{aligned}(\theta_x[K_1]*\theta_x[K_2])(z,w) &= \int_{\mathsf{G}} \phi(z,v,w)\phi(x,z,v)\phi(x,v,w)K_1(zx,vx)K_2(vx,wx)dv \\ &= \int_{\mathsf{G}} \phi(zx,vx,wx)\phi(x,z,w)K_1(zx,vx)K_2(vx,wx)\,dv \\ &= \phi(x,z,w)(K_1 * K_2)(zx,wx), \\ &= \theta_x[K_1 * K_2](z,w).\end{aligned}$$

We also have

$$\begin{aligned}\theta_x[K]^*(z,w) &= \overline{\theta_x[K](w,z)} = \overline{\phi(x,z,w)K(w,z)} \\ &= \phi(x,w,z)K^*(w,z) = \theta_x[K^*](z,w).\end{aligned} \tag{5.7}$$

Moreover,

$$\begin{aligned}\theta_x\theta_y[K](z,w) &= \phi(x,z,w)(\theta_y[K])(zx,wx) \\ &= \phi(x,z,w)\phi(y,zx,wx)K(zxy,wxy) \\ &= \phi(x,y,z)\phi(x,y,w)^{-1}\theta_{xy}[K](z,w) \\ &= \mathrm{ad}(\sigma(x,y))[\theta_{xy}[K]](z,w),\end{aligned}$$

with $\sigma(x,y)(z,w) = \phi(x,y,z)\delta(z-w)$. □

Note. There is also a left-handed version of this which uses the multiplication

$$(K_1 * K_2)(x,z) = \int_{\mathsf{G}} \phi(x,y,z)^{-1}K_1(x,y)K_2(y,z)\,dy \tag{5.8}$$

and automorphisms

$$\tau_x[K](z,w) = \phi(x,z,w)K(x^{-1}z, x^{-1}w), \tag{5.9}$$

and it is this version which will appear later, in Sect. 9. Some further automorphisms of the original algebra $\mathcal{K}_\phi(L^2(\mathsf{G}))$ given by

$$\gamma_x[K](z,w) = \phi(x,z,w)^{-1}K(x^{-1}z, x^{-1}w) \tag{5.10}$$

will be useful in Sect. 8.

When G is a contractible group the twisted compact operators are just a deformation of the usual ones.

Proposition 5.2. *When* G *is a contractible group* $\mathcal{K}_\phi(L^2(\mathsf{G}))$ *is a continuous deformation of* $\mathcal{K}(L^2(\mathsf{G}))$.

Proof. Let $\{\epsilon_t : t \in [0,1]\}$ give a contraction of G onto the identity, that is $\epsilon_t : \mathsf{G} \to \mathsf{G}$ is continuous and satisfies $\epsilon_0(x) = x$ and $\epsilon_1(x)$ is the identity for all $x \in G$. We then define (for the right-handed version)

$$(K_1 *_t K_2)(x,z) = \int_{\mathsf{G}} \phi(x,\epsilon_t(y),z)K_1(x,y)K_2(y,z)\,dy, \tag{5.11}$$

so that at $t = 0$ we have the twisted and at $t = 1$ the untwisted product. □

6. Twisted Induced Algebras

The introduction of u and ϕ has far-reaching consequences, because almost all the standard procedures have to be deformed, and we shall now investigate these in more detail.

Suppose that $\mathcal{A}$ is a C*-algebra on which $C(M)$ acts as double centralisers and the subgroup N of G acts by automorphisms α_r, $(r \in \mathsf{N})$, and that for each $s, t \in \mathsf{N}$ there are unitaries $u(s,t)$ in the multiplier algebra $M(\mathcal{A})$ such that $\alpha_s\alpha_t = \mathrm{ad}(u(s,t))\alpha_{st}$, and also the modified cocycle condition (for a specified continuous tricharacter ϕ)

$$\alpha_r[u(s,t)]u(r,st) = \phi(r,s,t)u(r,s)u(rs,t). \tag{6.1}$$

Such algebras always exist, since we can take the algebra freely generated by a collection of symbols $\{u(s,t) : s, t \in \mathsf{N}\}$ and define the automorphism α_r by the formula

$$\alpha_r[u(s,t)] = \phi(r,s,t)u(r,s)u(rs,t)u(r,st)^{-1}. \tag{6.2}$$

We shall suppose that u and ϕ extend to continuous functions on G satisfying the same relations:

$$\alpha_r[u(x,y)]u(r,xy) = \phi(r,x,y)u(r,x)u(rx,y), \tag{6.3}$$

for $r \in \mathsf{N}$ and $x, y \in \mathsf{G}$. (We are mainly interested in the case when N is a maximal rank lattice in a vector group G, and then ϕ automatically extends, and in the interesting examples u does too.) Normally one would induce an algebra admitting a G-action from that containing an N-action, but that will no longer work since the induced algebra is trivial. Instead we consider the u-induced algebra $\mathcal{B} = u\text{-}\mathrm{ind}_{\mathsf{N}}^{\mathsf{G}}\mathcal{A}$ described in the next result.

Proposition 6.1. *The space* $\mathcal{B} = u\text{-}\mathrm{ind}_{\mathsf{N}}^{\mathsf{G}}\mathcal{A}$ *of functions* $f \in C_0(\mathsf{G}, \mathcal{A})$ *which satisfy* $f(rx) = \mathrm{ad}(u(r,x))^{-1}\alpha_r[f(x)]$ *for all* $x \in \mathsf{G}$ *and* $r \in \mathsf{N}$ *is not trivial, and is closed under pointwise multiplication of functions* $(f_1f_2)(x) = f_1(x)f_2(x)$ *and the involution* $f^*(x) = f(x)^*$. *The norm* $\|f\| = \sup\|f(x)\|$ *is a* C^**-norm.*

Proof. We first note that (using the cocycle condition and remembering that the adjoint action is unaffected by the central factor ϕ) functions in the space satisfy

$$\begin{aligned} f(rsx) &= \mathrm{ad}(u(r,sx))^{-1}\alpha_r[f(sx)] \\ &= \mathrm{ad}(u(r,sx))^{-1}\alpha_r([\mathrm{ad}(u(s,x))]^{-1}\alpha_s[f(x)]) \\ &= \mathrm{ad}(u(r,sx))^{-1}\alpha_r[\mathrm{ad}(u(s,x))]^{-1}\alpha_r\alpha_s[f(x)] \\ &= \mathrm{ad}(u(rs,x))^{-1}\mathrm{ad}(u(r,s))^{-1}\alpha_r\alpha_s[f(x)] \\ &= \mathrm{ad}(u(rs,x))^{-1}\alpha_{rs}[f(x)], \end{aligned}$$

showing consistency of the condition. Without u this consistency check would fail and the induced algebra would be trivial, showing why one cannot use the normal induced algebra. We shall exhibit some useful explicit functions in the induced algebra in Proposition 6.4, but there is also a general construction which is useful. For f a function $C_0(\mathsf{G})$ and a an element in $\mathcal{A}$, we define an $\mathcal{A}$-valued function $(f\Diamond a)$ on G by

$$(f\Diamond a)(x) = \int_{\mathsf{N}} f(nx)\alpha_n^{-1}[\mathrm{ad}(u(n,x))[a]]\,dn. \tag{6.4}$$

Using the cocycle identity for $\mathrm{ad}(u)$ (the obstruction ϕ is central and so disappears in the adjoint action) we then check that

$$\begin{aligned}(f\diamondsuit a)(rx) &= \int_{\mathsf{N}} f(nrx)\alpha_n^{-1}[\mathrm{ad}(u(n,rx))[a]]\,dn \\ &= \int_{\mathsf{N}} f(nrx)\alpha_n^{-1}[\mathrm{ad}(\alpha_n[u(r,x)])^{-1}\mathrm{ad}(u(n,r))\mathrm{ad}(u(nr,x))[a]]\,dn \\ &= \mathrm{ad}(u(r,x))^{-1}\int_{\mathsf{N}} f(nrx)\alpha_n^{-1}[\mathrm{ad}(u(n,r))\mathrm{ad}(u(nr,x))[a]]\,dn \\ &= \mathrm{ad}(u(r,x))^{-1}\alpha_r\int_{\mathsf{N}} f(nrx)\alpha_{nr}^{-1}[\mathrm{ad}(u(nr,x))[a]]\,dn \\ &= \mathrm{ad}(u(r,x))^{-1}\alpha_r[(f\diamondsuit a)(x)],\end{aligned}$$

showing that $(f\diamondsuit a)$ defines an element of $\mathcal{B}$.

Using the fact that α_r and $\mathrm{ad}(u(x,r))$ are automorphisms, we see that

$$\begin{aligned}(f_1 f_2)(xr) &= f_1(xr)f_2(xr) \\ &= \mathrm{ad}(u(x,r))\alpha_r[f_1(x)]\mathrm{ad}(u(x,r))\alpha_r[f_2(x)] \\ &= \mathrm{ad}(u(x,r))\alpha_r[f_1(x)f_2(x)] \\ &= \mathrm{ad}(u(x,r))\alpha_r[(f_1f_2)(x)],\end{aligned}$$

so that the space of u-induced functions is closed under the product.

Finally, exploiting the unitarity of $u(r,x)$, we have

$$\begin{aligned}f^*(rx) &= f(rx)^* \\ &= [u(r,x)^{-1}\alpha_r[f(x)]u(r,x)]^* \\ &= u(r,x)^{-1}\alpha_r[f(x)]^*u(r,x) \\ &= u(r,x)^{-1}\alpha_r[f^*(x)]u(r,x),\end{aligned}$$

so that the involution respects the constraint. Finally

$$\|f^*f\| = \sup\|(f^*f)(x)\| = \sup\|f(x)^*f(x)\| = \sup\|f(x)\|^2 \tag{6.5}$$

gives a C^*-norm. □

This shows that the induced space $\mathcal{B}$ is actually a $*$-algebra (moreover, an associative algebra, since $\mathcal{A}$ was associative). A C^*-norm can be defined much as in the usual case. Exploiting the ideas of the ordinary induced representations we can improve on this.

Theorem 6.2. *If $\mathcal{A}$ is a continuous trace algebra with spectrum $\widehat{\mathcal{A}}$ then $\mathcal{B}$ is a continuous trace algebra with spectrum $\widehat{\mathcal{B}} = \mathsf{N}\backslash(\mathsf{G}\times\widehat{\mathcal{A}})$, where the action of $r\in\mathsf{N}$ on $(x,\pi)\in\mathsf{G}\times\widehat{\mathcal{A}}$ is defined by $r(x,\pi) = (rx, \pi\circ\alpha_r^{-1}\mathrm{ad}(u(r,x)))$.*

Proof. We should start by checking that the above formula does indeed define an action, but that is essentially the same calculation just done to show that $f\diamondsuit a$ satisfies the equivariance condition for $\mathcal{B}$. The rest of the proof follows the ideas in [33, 6.16 to 6.21], but with the actions on the other sides, and using our definition of $f\diamondsuit a$, and noting that in the proof of 6.18 one needs to restrict to neighbourhoods of both s and x. This enables us to show that the representation defined by $(x,\pi) \in \mathsf{G} \times \widehat{\mathcal{A}}$, $(x,\pi) : F \mapsto \pi(F(x)$ of $\mathcal{B}$ is irreducible, because, for any $(x,a) \in \mathsf{G} \times \mathcal{A}$ we can use appropriate $f \diamondsuit a$ to find $F \in \mathcal{B}$ such that $F(x) = a$, and then use irreducibility of π. The rest of the proof in [33] is independent of the twisting. □

We next introduce automorphisms of the twisted induced algebra.

Proposition 6.3. *For $y \in \mathsf{G}$ and a function $f : \mathsf{G} \to \mathcal{A}$ define $\beta_y[f](x) = \mathrm{ad}(u(x,y))$ $[f(xy)]$. Then β_y preserves the subalgebra $\mathcal{B}$ and defines a $*$-automorphism of it.*

Proof.

$$\begin{aligned}\beta_y[f](rx) &= \mathrm{ad}(u(rx,y))[f(rxy)]\\ &= \mathrm{ad}(u(rx,y))\mathrm{ad}(u(r,xy))^{-1}\alpha_r[f(xy)]\\ &= \mathrm{ad}(u(r,x))^{-1}\mathrm{ad}(\alpha_r[(u(x,y)])\alpha_r[f(xy)]\\ &= \mathrm{ad}(u(r,x))^{-1}\alpha_r([\mathrm{ad}(u(x,y))][f(xy)])\\ &= \mathrm{ad}(u(r,x))^{-1}\alpha_r[\beta_y[f](x)],\end{aligned}$$

showing that β_y satisfies the equivariance condition.

To see that these are automorphisms we need only note that

$$\begin{aligned}(\beta_y[f_1]\beta_y[f_2])(x) &= (\beta_y[f_1](x)\beta_y[f_2])(x)\\ &= \mathrm{ad}(u(x,y))[f_1(xy)]\mathrm{ad}(u(x,y))[f_2(xy)]\\ &= \mathrm{ad}(u(x,y))[f_1(xy)f_2(xy)]\\ &= \mathrm{ad}(u(x,y))[(f_1f_2)(xy)]\\ &= \beta_y[f_1f_2](x),\end{aligned}$$

as required, and compatibility with the involution is also easily checked. □

Naturally the map $y \mapsto \beta_y$ is not a homomorphism.

Proposition 6.4. *The functions $v(y,z) : x \mapsto \phi(x,y,z)u(x,y)u(xy,z)u(x,yz)^{-1}$, lie in the multiplier algebra of $\mathcal{B}$ and satisfy*

$$\beta_x[v(y,z)]v(x,yz) = \phi(x,y,z)v(x,y)v(xy,z). \tag{6.6}$$

The automorphisms defined by β satisfy the relations

$$\beta_y\beta_z = \mathrm{ad}(v(y,z))\beta_{yz}. \tag{6.7}$$

Proof. When $x \in \mathsf{N}$ we can write $v(y,z)(x) = \alpha_x[u(y,z)]$, but otherwise α_x is undefined, and we cannot reduce $v(y,z)(x) = \phi(x,y,z)^{-1}\mathrm{ad}(u(x,y))\mathrm{ad}(u(xy,z))$ $\mathrm{ad}(u(x,yz))^{-1}$. To check that $v(y,z)$ satisfies the equivariance condition for membership of $\mathcal{B}$, we calculate

$$\begin{aligned}&u(rx,y)u(rxy,z)u(rx,yz)^{-1}\\ &\quad= \phi(r,x,y)^{-1}u(r,x)^{-1}\alpha_r[u(x,y)]u(r,xy)u(rxy,z)u(rx,yz)^{-1}\\ &\quad= \phi(r,x,y)^{-1}\phi(r,xy,z)^{-1}u(r,x)^{-1}\alpha_r[u(x,y)]\alpha_r[u(xy,z)]u(r,xyz)u(rx,yz)^{-1}\\ &\quad= \phi(r,x,y)^{-1}\phi(r,xy,z)^{-1}\phi(r,x,yz)u(r,x)^{-1}\alpha_r[u(x,y)u(xy,z)]\alpha_r[u(x,yz)]^{-1}\\ &\qquad\times u(r,x)\\ &\quad= \phi(r,x,y)^{-1}\phi(r,xy,z)^{-1}\phi(r,x,yz)\mathrm{ad}(u(r,x))^{-1}\alpha_r[u(x,y)u(xy,z)u(x,yz)^{-1}]\\ &\quad= \phi(r,y,z)^{-1}\mathrm{ad}(u(r,x))^{-1}\alpha_r[u(x,y)u(xy,z)u(x,yz)^{-1}].\end{aligned}$$

From this we see that

$$\begin{aligned}[v(y,z)](rx) &= \phi(rx,y,z)\phi(r,y,z)^{-1}\mathrm{ad}(u(r,x))^{-1}\alpha_r[\phi(x,y,z)^{-1}v(y,z)(x)]\\ &= \mathrm{ad}(u(r,x))^{-1}\alpha_r[v(y,z)(x)].\end{aligned}$$

We cannot conclude that $v(y,z)$ lies in $\mathcal{B}$ since it does not satisfy the analytic condition of vanishing outside compact sets, but it is certainly in the multiplier algebra.

By making some cancellations, we obtain

$$\begin{aligned}\left(v(x,y)v(xy,z)v(x,yz)^{-1}\right)(s) &= \frac{\phi(s,x,y)\phi(s,xy,z)}{\phi(s,x,yz)}u(s,x)u(sx,y)\\ &\quad\times u(sxy,z)u(s,xyz)^{-1}u(s,x)^{-1}\\ &= \phi(s,y,z)\mathrm{ad}(u(s,x))[u(sx,y)u(sxy,z)u(sx,yz)^{-1}]\\ &= \phi(x,y,z)^{-1}\mathrm{ad}(u(s,x))[v(y,z)(sx)]\\ &= \phi(x,y,z)^{-1}\beta_x[v(y,z)](s).\end{aligned}$$

Finally we see that

$$\begin{aligned}(\beta_y\beta_z[f])(x) &= \mathrm{ad}(u(x,y))[(\beta_z[f](xy)]\\ &= \mathrm{ad}(u(x,y))\mathrm{ad}(u(xy,z)[f(xyz)]\\ &= \mathrm{ad}(u(x,y))\mathrm{ad}(u(xy,z)\mathrm{ad}(u(x,yz)^{-1})[(\beta_{yz}f(x)]\\ &= \mathrm{ad}(v(y,z))(x)[(\beta_{yz}f(x)].\end{aligned}$$

□

Corollary 6.5. *The action of* G *on* $\widehat{\mathcal{B}}$ *defined by* β *has orbit space* $\widehat{\mathcal{B}}/\mathsf{G} = \widehat{\mathcal{A}}/\mathsf{N}$*, so that* β *defines the principal* G/N*-bundle* $(\mathsf{G}\times\widehat{\mathcal{A}})/\mathsf{N} \to \widehat{\mathcal{A}}/\mathsf{N}$.

Proof. The action of $y \in \mathsf{G}$ sends $\varpi \in \widehat{\mathcal{B}}$ to $\varpi \circ \beta_y$. In the earlier notation we have

$$(x,\pi)(\beta_y[f]) = \pi(\beta_y[f](x)) = \pi(\mathrm{ad}(u(x,y))[f(xy)]), \tag{6.8}$$

and, since inner automorphisms don't affect the class of a representation, this is equivalent to (xy,π). For $r \in \mathsf{N}$ this reduces to

$$(x,\pi)(\beta_r[f]) = \pi(\mathrm{ad}(u(x,r)u(r,x)^{-1})\mathrm{ad}(u(r,x)\alpha_r[f(x)]), \tag{6.9}$$

which is equivalent to $r(x,\pi)$, showing that the subgroup N stabilises the irreducible representations of $\mathcal{B}$, so that we have a G/N bundle, and that the orbit space is $\widehat{\mathcal{A}}/\mathsf{N}$, as claimed. □

7. Algebras with Prescribed Spectrum and Dixmier-Douady Class

Before tackling the general case it is useful to consider what happens for the principal bundle G/N over a point. In that case only the component H_0 can be non-trivial, and we assume that it defines the tricharacter ϕ as before.

In fact [10] gives a universal construction for a principal projective unitary bundle over a group, but we shall give an alternative description of the algebra as a twisted induced algebra. We shall induce from N to G, and in order that the spectrum should be just G/N we induce the algebra $\mathcal{K}(L^2(\mathsf{N}))$ of compact operators. That carries the right regular representation ρ of N and twisting u_ρ described in Sect. 4. Although the restriction of u_ρ to $\mathsf{N}\times\mathsf{N}$ is 1, the extension to $\mathsf{G}\times\mathsf{G}$ gives the induced algebra a twist.

At this point it is instructive to consider why $u_\rho\text{-ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{K}(L^2(\mathsf{N})))$ has a non-vanishing Dixmier–Douady obstruction when the action of N on $\mathcal{K}(L^2(\mathsf{N}))$ is given by $\alpha_\rho = \mathrm{ad}\rho$. One might expect that the induced algebra $u_\rho\text{-ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{K}(L^2(\mathsf{N}))$ simply acts on the induced Hilbert space of square-integrable functions $\psi : \mathsf{G} \to L^2(\mathsf{N})$, which satisfy the equivariance condition

$$\psi(rx) = u_\rho(r,x)^{-1}\rho(r)\psi(x), \tag{7.1}$$

so that there is no obstruction. However, this is incorrect because the suggested equivariance condition on ψ is inconsistent, when u_ρ is not a cocycle:

$$\begin{aligned}\psi(rsx) &= u_\rho(r,sx)^{-1}\rho(r)\psi(sx)\\ &= u_\rho(r,sx)^{-1}\rho(r)u_\rho(s,x)^{-1}\rho(s)\psi(x)\\ &= u_\rho(r,sx)^{-1}\alpha_r[u_\rho(s,x)]^{-1}\rho(r)\rho(s)\psi(x)\\ &= \phi(r,s,x)^{-1}u_\rho(rs,x)^{-1}u_\rho(r,s)^{-1}\rho(rs)\psi(x).\end{aligned}$$

Now, as we noted in the last paragraph, $u_\rho(r,s) = 1$, but the presence of the argument $x \notin \mathsf{N}$ means that $\phi(r,s,x) \neq 1$, and we end up with constraints

$$u_\rho(rs,x)^{-1}\rho(rs)\psi(x) = \psi(rsx) = \phi(r,s,x)^{-1}u_\rho(rs,x)^{-1}\rho(rs)\psi(x) \tag{7.2}$$

which can be satisfied only by $\psi = 0$.

Proposition 7.1. *Let ϕ be the tricharacter of G constructed from H_0, u_ρ, ρ defined on $L^2(\mathsf{N})$ as in Sect. 4, and $\alpha_\rho = \mathrm{ad}\rho$. The algebra $u_\rho\text{-ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{K}(L^2(\mathsf{N})))$ has spectrum G/N, the G action on the spectrum is transitive with stabiliser N, and the Dixmier-Douady class is described by the 3-form H_0.*

Proof. We assume that ϕ is obtained from the class $f = H_0$ by the procedure described in Sect. 2.

Choose an open set $F \subseteq \mathsf{G}$ on which the projection π to G/N is one-one, and translates $F_i = Fx_i$ whose projections give a cover of G/N. These translates share the property that the projection to G/N is injective, and so we may choose sections $\gamma_i : \pi(F_i) \to \mathsf{G}$. The differences $\gamma_{ij}(v) = \gamma_i(v)\gamma_j(v)^{-1}$ lie in N.

The restriction of the induced algebra to algebra-valued functions on $\pi(F_i) \subseteq \mathsf{G}/\mathsf{N}$ is Morita equivalent to $C(\pi(F_i))$ via the bimodule $X_i = L^2(\pi(F_i), L^2(\mathsf{N}))$, (restriction to the subsets enables us to sidestep the earlier problem with $L^2(\mathsf{G}, L^2(\mathsf{N}))$). The actions are the obvious pointwise multiplicative actions,

$$(f\psi)(v) = f(\gamma_i(v))\psi(v) \tag{7.3}$$

and this is a imprimitivity bimodule in the sense of [33].

Over the intersection $\pi(F_i)\cap\pi(F_j)$ there is an equivalence of the two bimodules X_i and X_j, given by the map

$$(g_{ij}\psi)(v) = u_\rho(\gamma_{ij}(v), \gamma_j(v))^{-1}\rho(\gamma_{ij}(v))\psi(v) \tag{7.4}$$

from X_j to X_i. To see why this works we note that

$$\begin{aligned}(g_{ij}f\psi)(v) &= u_\rho(\gamma_{ij}(v),\gamma_j(v))^{-1}\rho(\gamma_{ij}(v))f(\gamma_j(v))\psi(v)\\ &= \mathrm{ad}(u_\rho(\gamma_{ij}(v),\gamma_j(v))^{-1}\mathrm{ad}(\rho(\gamma_{ij}(v))[f(\gamma_j(v))]\\ &\quad u_\rho(\gamma_{ij}(v),\gamma_j(v))^{-1}\rho(\gamma_{ij}(v))\psi(v).\end{aligned}$$

Since $\gamma_{ij}(v)\in \mathsf{N}$ this simplifies to

$$f(\gamma_{ij}(v)\gamma_j(v))u_\rho(\gamma_{ij}(v),\gamma_j(v))^{-1}\rho(\gamma_{ij}(v))\psi(v) = (fg_{ij}\psi)(v). \tag{7.5}$$

To compute the obstruction we must compare $g_{ij}g_{jk}$ and g_{ik} on $\pi(F_i)\cap\pi(F_j)\cap\pi(F_k)$. Now we have

$$\begin{aligned}(g_{ij}g_{jk}\phi)(v) &= u_\rho(\gamma_{ij}(v),\gamma_j(v))^{-1}\rho(\gamma_{ij}(v))(g_{jk}\psi)(v)\\ &= u_\rho(\gamma_{ij}(v),\gamma_j(v))^{-1}\rho(\gamma_{ij}(v))\\ &\quad u_\rho(\gamma_{jk}(v),\gamma_k(v))^{-1}\rho(\gamma_{jk}(v))\psi(v)\\ &= u_\rho(\gamma_{ij}(v),\gamma_j(v))^{-1}\mathrm{ad}(\rho(\gamma_{ij}(v)))[u_\rho(\gamma_{jk}(v),\gamma_k(v))^{-1}]\\ &\quad \rho(\gamma_{ij}(v))\rho(\gamma_{jk}(v))\psi(v).\end{aligned}$$

Applying the modified cocycle identity we have

$$\begin{aligned}(g_{ij}g_{jk}\phi)(v) &= \phi(\gamma_{ij}(v),\gamma_{jk}(v),\gamma_k(v))^{-1}u_\rho(\gamma_{ik}(v),\gamma_k(v))^{-1}\\ &\quad u_\rho(\gamma_{ij}(v),\gamma_{jk}(v))^{-1}\rho(\gamma_{ik}(v))\psi(v),\end{aligned}$$

and finally, since $\gamma_{ij}(v),\gamma_{jk}(v)\in\mathsf{N}$ we deduce that $u_\rho(\gamma_{ij}(v),\gamma_{jk}(v)) = 1$, giving

$$(g_{ij}g_{jk}\phi)(v) = \phi(\gamma_{ij}(v),\gamma_{jk}(v),\gamma_k(v))^{-1}(g_{ik}\psi)(v). \tag{7.6}$$

This shows that the Dixmier-Douady class can be described by the Čech cocycle

$$\begin{aligned}\phi_{ijk} &= \phi(\gamma_{ij}(v),\gamma_{jk}(v),\gamma_k(v))^{-1}\\ &= \exp[2\pi i f(\gamma_{ij}(v),\gamma_{jk}(v),\gamma_k(v))]\\ &= \exp[2\pi i f(\gamma_i(v),\gamma_j(v),\gamma_k(v))],\end{aligned}$$

where the antisymmetry of f has been used in the final line. To find a form describing the de Rham cocycle we note that, since locally $\gamma_k(v)$ and v are the same and the differences γ_{ij} are in N,

$$\begin{aligned}df(\gamma_{ij}(v),\gamma_{jk}(v),\gamma_k(v)) &= f(\gamma_{ij}(v),\gamma_{jk}(v),d\gamma_k(v))\\ &= f(\gamma_{ij}(v),\gamma_{jk}(v),dv)\\ &= f(\gamma_{ij}(v),\gamma_j(v),dv) - f(\gamma_{ij}(v),\gamma_k(v),dv),\end{aligned}$$

giving an explicit expression as the difference of one-forms. Repeating this process twice we arrive at the de Rham form $f(dv,dv,dv)$ giving the class H_0. (The antisymmetry of f compensates for the antisymmetry of the exterior product to give a non-vanishing answer.) □

The same ideas can now be used to deal with the general case. However, since we want to use results for the untwisted case, we shall at this point restrict ourselves to the case of an abelian group G.

Theorem 7.2. *Let* G *be abelian and* $\mathcal{A}$ *a continuous trace algebra with an action of* $\mathsf{N} \subset \mathsf{G}$ *by locally projectively unitary automorphisms. (This includes the assumption that the restriction of* u *to* $\mathsf{N} \times \mathsf{N}$ *takes the constant value* 1*). Both* $u\text{-ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{A})$ *and* $\text{ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{A})$ *are continuous trace algebras, and the difference of their Dixmier-Douady invariants* $\delta(u\text{-ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{A})) - \delta(\text{ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{A}))$ *is the class defined by the form* $f \in \Omega^3(\mathsf{G}, \mathbb{Z})$ *associated to the tricharacter* ϕ*:*

$$\delta(u\text{-ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{A})) - \delta(\text{ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{A})) = [f]. \tag{7.7}$$

Proof. We first note that by Theorem 4.1 u is exterior equivalent to u_λ which by (4.3) has trivial restriction to $\mathsf{N} \times \mathsf{N}$. Since $\mathcal{A}$ has continuous trace, it is locally Morita equivalent to an algebra of compact operators, and all our calculations will be local. In fact we may cover $\widehat{\mathcal{A}}$ by open sets $\{U_\lambda\}$ and find Hilbert spaces $\mathcal{H}_\lambda$ such that the restriction $\mathcal{A}^{U_\lambda}$ of $\mathcal{A}$ to U_λ is Morita equivalent to $C(U_\lambda, \mathcal{K}(\mathcal{H}_\lambda))$ via some bimodule Y_λ. Combining these with the bimodules X_l used in the previous proof we take $\mathcal{X}_{(l,\lambda)} = X_l \otimes Y_\lambda$ for the restriction of the algebra to $\pi(F_l) \times U_\lambda$.

We may assume that the cover is fine enough that $\alpha_n(a)$ is equivalent to $\text{ad}(\rho_n^\lambda)(a)$ for $a \in \mathcal{A}^{U_\lambda}$, with ρ^λ a θ^λ-representation. We now define an equivalence $G_{(l,\lambda)}^{(m,\mu)}$ on the overlap of the sets $\pi(F_l) \times U_\lambda$ and $\pi(F_m) \times U_\mu$ by setting

$$(G_{(l,\lambda)}^{(m,\mu)} \psi)(v) = u(\gamma_{lm}(v), \gamma_m(v))^{-1} h_{\lambda\mu} \rho^\mu(\gamma_{lm}(v)) \psi(v), \tag{7.8}$$

where $h_{\lambda\mu}$ describes the equivalences of Y_λ and Y_μ, (which are assumed to satisfy the relationship $h_{\lambda\mu} h_{\mu\nu} = \Phi_{\lambda\mu\nu} h_{\lambda\nu}$, where Φ is a Čech cocycle describing the Dixmier-Douady class of $\mathcal{A}$.) On overlaps the projective representations ρ^λ are equivalent in the sense that $\rho^\mu(n) h_{\mu\nu} = h_{\mu\nu} \kappa_{\mu\nu}(n) \rho^\nu(n)$, for some character $\kappa_{\mu\nu} \in \widehat{\mathsf{N}}$. The adjoint actions of ρ^μ and ρ^ν are both equivalent to α.

We now calculate that

$$\begin{aligned}
(G_{(l,\lambda)}^{(m,\mu)} G_{(m,\mu)}^{(n,\nu)} \psi)(v) &= u(\gamma_{lm}(v), \gamma_m(v))^{-1} h_{\lambda\mu} \rho^\mu(\gamma_{lm}(v)) u(\gamma_{mn}(v), \\
&\qquad \gamma_n(v))^{-1} h_{\mu\nu} \rho^\nu(\gamma_{mn}(v)) \psi(v) \\
&= u(\gamma_{lm}(v), \gamma_m(v))^{-1} \alpha_{\gamma_{lm}})[u(\gamma_{mn}(v), \gamma_n(v))^{-1}] h_{\lambda\mu} h_{\mu\nu} \\
&\qquad \times \kappa_{\mu\nu}(\gamma_{lm}) \rho^\nu(\gamma_{lm}(v)) \rho^\nu(\gamma_{mn}(v)) \psi(v).
\end{aligned}$$

The first two terms combine as before to give

$$u(\gamma_{ln}(v), \gamma_n(v))^{-1} u(\gamma_{lm}(v), \gamma_{mn}(v))^{-1} = \phi(\gamma_{lm}, \gamma_{mn}, \gamma_n)^{-1} u(\gamma_{ln}(v), \gamma_n(v))^{-1}, \tag{7.9}$$

whilst the projective represention give

$$\rho^\nu(\gamma_{lm}(v)) \rho^\nu(\gamma_{mn}(v)) = \theta^\nu(\gamma_{lm}(v), \gamma_{mn}(v)) \rho^\nu(\gamma_{ln}(v)), \tag{7.10}$$

giving

$$\begin{aligned}
&(G_{(l,\lambda)}^{(m,\mu)} G_{(m,\mu)}^{(n,\nu)} \psi)(v) \\
&\quad = \phi(\gamma_{lm}, \gamma_{mn}, \gamma_n)^{-1} \kappa_{\mu\nu}(\gamma_{lm}) \theta^\nu(\gamma_{lm}(v), \gamma_{mn}(v)) \Phi_{\lambda\mu\nu} \\
&\qquad u(\gamma_{ln}(v), \gamma_n(v))^{-1} h_{\lambda\nu} \rho^\nu(\gamma_{ln}(v)) \psi(v) \\
&\quad = \phi(\gamma_{lm}, \gamma_{mn}, \gamma_n)^{-1} \kappa_{\mu\nu}(\gamma_{lm}) \theta^\nu(\gamma_{lm}(v), \gamma_{mn}(v)) \Phi_{\lambda\mu\nu} G_{(l,\lambda)}^{(n,\nu)} \psi(v),
\end{aligned}$$

from which we deduce the obstruction. All the factors except the first would be present for $\mathrm{ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{A})$, so that the difference between the Dixmier-Douady obstructions for u-$\mathrm{ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{A})$ and $\mathrm{ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{A})$ is just given by ϕ^{-1}, or as forms by f. □

Our formula shows that the obstruction has four contributions: the MacLane-Whitehead obstruction ϕ^{-1}, the Mackey obstruction θ, the Phillips-Raeburn obstruction κ, and the Dixmier-Douady obstruction Φ for $\mathcal{A}$, corresponding to H_0, H_1, H_2 and H_3.

Corollary 7.3. *Let* G *be abelian and* E *be a principal* G/N*-bundle over* M *with prescribed Dixmier-Douady invariant associated with* H*. Then there is a* u*-induced algebra with spectrum* E*, and the correct action of* G*, having the Dixmier-Douady invariant* H*. This* u*-induced algebra is not necessarily unique.*[2]

Proof. We know from [2, 4, 28] that there is an algebra $\mathrm{ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{A})$ associated with the principal G/N-bundle E and having Dixmier-Douady form $H - H_0$. This algebra is not necessarily unique, cf. [28]. With f as in the theorem we see that u-$\mathrm{ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{A})$ has Dixmier-Douady class described by the form H. An alternative approach would be to note that a continuous trace algebra with Dixmier–Douady class given by H, can be constructed as the tensor product of two continuous trace algebras with classes $H - H_0$ and with H_0, and then to apply [28] for the former and our earlier result for the class H_0. □

Although we have shown how to construct an algebra with a given Dixmier-Douady class, it is natural to wonder whether one could also find another algebra with twisting given by an ordinary cocycle. The next result shows that this is not possible.

Theorem 7.4. *Every system with the Dixmier-Douady invariant* H *is described by automorphisms whose twisting gives the same tricharacter* ϕ*.*

Proof. By the general theory we know that any other system must be exterior equivalent to the u-induced system above, that is it is described by automorphisms $\lambda_x = \mathrm{ad}(W(x))\beta_x$, and $w(x, y) = W(x)\beta_x[W(y)]v(x, y)W(xy)^{-1}$ for some $M\mathcal{B}$-valued function W on G. Since the twistings are cohomologous they must define the same class ϕ, but more explicitly we calculate that

$$\begin{aligned}
\lambda_x[w(y,z)]w(x,yz) &= \mathrm{ad}(W(x))\beta_x[W(y)\beta_y[W(z)]v(y,z)W(yz)^{-1}] \\
&\quad\; W(x)\beta_x[W(yz)]v(x,yz)W(xyz)^{-1} \\
&= W(x)\beta_x[W(y)]\beta_x\beta_y[W(z)]\beta_x[v(y,z)]v(x,yz)W(xyz)^{-1} \\
&= \phi(x,y,z)W(x)\beta_x[W(y)]\mathrm{ad}(v(x,y)) \\
&\quad\; \times\beta_{xy}[W(z)]v(x,y)v(xy,z)W(xyz)^{-1} \\
&= \phi(x,y,z)W(x)\beta_x[W(y)]v(x,y)W(xy)^{-1}W(xy) \\
&\quad\; \times\beta_{xy}[W(z)]v(xy,z)W(xyz)^{-1} \\
&= \phi(x,y,z)w(x,y)w(xy,z),
\end{aligned}$$

showing that the same cocycle ϕ arises. □

[2] The non-uniqueness reflects the fact that there may exist several liftings of the action of N on $\mathcal{A}$ to G. So, strictly speaking, u-ind does not define a functor, but u-$\mathrm{ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{A})$ just indicates the induced algebra with a prescribed action of G.

8. The Twisted Crossed Product Algebra

In this section we can return to the case of a general group G. We have argued that the dual of a bundle described by $\mathcal{B}$ with a twisted group action should be given by the twisted crossed product. For the induced algebra $u\text{-ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{A})$ its twisted crossed product with G can be calculated explicitly, and the following result shows that it is a generalisation of the twisted compact operators in Sect. 5.

Theorem 8.1. *The twisted crossed product* $u\text{-ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{A}) \rtimes_{\beta,v} \mathsf{G}$ *is isomorphic to the* $*$*-algebra of* $\mathcal{A}$*-valued kernels on* $\mathsf{G} \times \mathsf{G}$ *satisfying*

$$K_1(rz, rw) = \phi(r, z, w)^{-1} u(r, z)^{-1} \alpha_r[K_1(z, w)] u(r, w), \tag{8.1}$$

with $K^*(z, w) = K(w, z)^*$ *and product*

$$(K_1 \star K_2)(z, w) = \int_{\mathsf{G}} K_1(z, v) K_2(v, w) \phi(z, v, w)\, dv. \tag{8.2}$$

Proof. The twisted crossed product consists of functions $F_j : \mathsf{G} \to \mathcal{B}$ with twisted convolution

$$(F_1 * F_2)(x) = \int_{\mathsf{G}} F_1(y) \beta_y[F_2(y^{-1}x)] v(y, y^{-1}x)\, dy. \tag{8.3}$$

Identifying $\mathcal{B}$ with functions from G to $\mathcal{A}$, we may give the elements of the twisted crossed product a second argument in G and, using the explicit form of the induced action and of v, get

$$\begin{aligned}(F_1 * F_2)(z, x) = \int_{\mathsf{G}} & F_1(z, y) \mathrm{ad}(u(z, y))[F_2(zy, y^{-1}x)] \\ & \times \phi(z, y, y^{-1}x) u(z, y) u(zy, y^{-1}x) u(z, x)^{-1}\, dy,\end{aligned} \tag{8.4}$$

which can be rearranged as

$$\begin{aligned}(F_1 * F_2)(z, x) u(z, x) = \int_{\mathsf{G}} & \phi(z, y, y^{-1}x) F_1(z, y) u(z, y) F_2(zy, y^{-1}x) \\ & \times u(zy, y^{-1}x)\, dy.\end{aligned} \tag{8.5}$$

We now define $K_1(z, zy) = F_1(z, y) u(z, y)$, $K_2(z, zy) = F_2(z, y) u(z, y)$ and $(K_1 \star K_2)(z, zx) = (F_1 * F_2)(z, x) u(z, x)$ to obtain

$$(K_1 \star K_2)(z, zx) = \int_{\mathsf{G}} K_1(z, zy) K_2(zy, zx) \phi(z, y, y^{-1}x)\, dy. \tag{8.6}$$

Setting $w = zx$, $v = zy$ and exploiting the antisymmetry of ϕ the result follows. We readily check that

$$\begin{aligned}F^*(z, x) u(z, x) &= v(x, x^{-1})^* \mathrm{ad}(u(z, x))[F(zx, x^{-1})]^* u(z, x) \\ &= u(zx, x^{-1})^* F(zx, x^{-1})^*,\end{aligned} \tag{8.7}$$

from which it follows that $K^*(z, w) = K(w, z)^*$.

The kernels inherit an equivariance condition from the inducing process,

$$\begin{aligned}K_1(rz, rzy) &= F_1(rz, y)u(rz, y)\\ &= \mathrm{ad}(u(r, z))^{-1}\alpha_r[F_1(z, y)]u(rz, y)\\ &= u(r, z)^{-1}\alpha_r[K_1(z, zy)u(z, y)^{-1}]u(r, z)u(rz, y)\\ &= \phi(r, z, y)^{-1}u(r, z)^{-1}\alpha_r[K_1(z, zy)]u(r, zy).\end{aligned}$$

Exploiting the antisymmetry of ϕ this gives

$$K_1(rz, rw) = \phi(r, z, w)^{-1}u(r, z)^{-1}\alpha_r[K_1(z, w)]u(r, w). \tag{8.8}$$

We note that since the original product $F_1 * F_2$ respected this equivariance condition, so does the product on kernels. The norm on the twisted crossed product can be defined from the left regular representation and so agrees with that on kernels. □

The description of the crossed product algebra in Theorem 8.1 can be made more precise for a bundle over a point described by $\mathcal{A} = \mathcal{K}(L^2(\mathsf{N}))$ as in Proposition 7.1. However, this time it is more useful to take the left handed version of the automorphisms, that is

$$\alpha_r[k](s, t) = k(r^{-1}s, r^{-1}t)\,, \tag{8.9}$$

and set $(u(s, t)\psi)(r) = \phi(r, s, t)^{-1}\psi(r)$.

Theorem 8.2. *The nonassociative torus describing the dual of a bundle over a point is isomorphic to the algebra $A_\phi = \mathcal{K}_\phi(L^2(\mathsf{G})) \rtimes_{\gamma,u} \mathsf{N}$, where γ is defined by Eq.* (5.10)*.*

Proof. We abbreviate notation for the $\mathcal{K}(L^2(\mathsf{N}))$-valued kernels by setting

$$K(z, w) : (s, t) \mapsto K(z, w; s, t), \tag{8.10}$$

so that the equivariance condition can be given explicitly as

$$K(rz, rw; s, t) = \phi(r, z, w)^{-1}\phi(r, z, s)K(z, w; r^{-1}s, r^{-1}t)]\phi(r, w, t)^{-1}, \tag{8.11}$$

or, replacing the first two arguments, as

$$K(z, w; s, t) = \phi(r, z, w)^{-1}\phi(r, z, s)K(r^{-1}z, r^{-1}w; r^{-1}s, r^{-1}t)]\phi(r, w, t)^{-1}. \tag{8.12}$$

Taking $r = s$ in this formula we get

$$\begin{aligned}K(z, w; s, t) = \phi(s, z, w)^{-1}&K(s^{-1}z, s^{-1}w; 1, s^{-1}t)\\ &\times\phi(w, s, t).\end{aligned} \tag{8.13}$$

Writing $u(x, y)$ for the multiplier associated with the automorphism of twisted kernels γ_x defined by Eq. (5.10), we have

$$K(z, w; s, t) = \gamma_s[K](z, w; 1, s^{-1}t)u(s, t) = \gamma_s[K](z, w; 1, s^{-1}t)u(s, s^{-1}t)\,. \tag{8.14}$$

This shows that the kernels can be reconstructed from their values when the third argument is 1, and in this case the product formula can be simplified. The product

$$(K_1 \star K_2)(z,w;r,t) = \int_{\mathsf{N}\times\mathsf{G}} K_1(z,v;r,s)K_2(v,w;s,t)\phi(z,v,w)\,ds dv\,, \quad (8.15)$$

reduces to

$$(K_1 \star K_2)(z,w;1,t) = \int_{\mathsf{N}\times\mathsf{G}} K_1(z,v;1,s)K_2(v,w;s,t)\phi(z,v,w)\,ds dv\,. \quad (8.16)$$

The second kernel can be rewritten using the equivariance condition to give

$$(K_1 \star K_2)(z,w;1,t) = \int_{\mathsf{G}} \{\int_{\mathsf{N}} K_1(z,v;1,s)\gamma_s[K_2](v,w;1,s^{-1}t)u(s,s^{-1}t)\,ds\} \times\phi(z,v,w)\,dv\,. \quad (8.17)$$

Identifying the kernel K_j with the $\mathcal{K}_\phi(L^2(\mathsf{G}))$-valued function $s \mapsto \{(z,w) \mapsto K_j(z,w;1,s)\}$ on N, this is just the twisted crossed product of the two functions. In other words we can identify the algebra with $\mathcal{K}_\phi(L^2(\mathsf{G})) \rtimes_{\gamma,u} \mathsf{N}$. □

In the general case of a bundle over M one needs to take $\mathcal{A} = C(M, \mathcal{K}(L^2(\mathsf{N}))$, but since the products of functions on M are all taken pointwise, there is no essential change in the calculations. The only point for caution is that, in those cases where ϕ depends on M (and consequently so also do γ and u), one is really looking at an algebra of continuous sections of a bundle over M rather than just functions. Observe that by Theorem 8.1, the nonassociative torus A_ϕ is canonically isomorphic to $u\text{-ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{K}(L^2(\mathsf{N})) \rtimes_{\beta,v} \mathsf{G}$.

Theorem 8.3. *The nonassociative torus describing the dual of a bundle over M is isomorphic to the algebra $C(M, \mathcal{K}_\phi(L^2(\mathsf{G})) \rtimes_{\gamma,u} \mathsf{N})$.*

When N is trivial this gives the twisted algebra of kernels introduced in Sect. 5, though with ϕ replaced by its inverse. More generally it provides an extension of Green's Generalised Imprimitivity Theorem [17] to the case of these twisted induced algebras.

In contrast to the twisted crossed product algebra for the dual, the correspondence algebra is associative. Provided that there is no Mackey obstruction the correspondence space is associated with the algebra $u\text{-ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{A}) \rtimes_{\beta,v} \mathsf{N}$.

Theorem 8.4. *The algebra $u\text{-ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{A}) \rtimes_{\beta,v} \mathsf{N}$ is associative.*

Proof. By Propositions 3.1 and 6.4 $u\text{-ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{A}) \rtimes_{\beta,v} \mathsf{N}$ is associative if and only if the restriction of ϕ to $\mathsf{N} \times \mathsf{N} \times \mathsf{N}$ is 1, and we have seen that this is a consequence of the integrality of the form H_0. □

For non-trivial Mackey obstruction one replaces N by the projection onto G of the centre of the central extension defined by the multiplier [19, 14], and that algebra is similarly associative.

9. The Dual Action

In this section we shall assume that G is abelian. Following the development in Sect. 5, we may define automorphisms of the kernels by

$$\tau_x[K](z,w) = \phi(x,z,w)K(x^{-1}z, x^{-1}w), \tag{9.1}$$

and these satisfy $\tau_x\tau_y = \mathrm{ad}(\widetilde{u}(x,y))^{-1}\tau_{xy}$, where

$$\mathrm{ad}(\widetilde{u}(x,y)[K])(z,w) = \phi(x,y,z)\phi(x,y,w)^{-1}K(z,w). \tag{9.2}$$

However, we are primarily interested in the case of abelian groups, and there is also a more useful action of the dual group $\widehat{\mathsf{G}}$ on the dual algebra. Its definition is motivated by the action $\widehat{\beta}_\xi$ on $F \in u\text{-ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{A}) \rtimes_{\beta,v} \mathsf{G}$ given by $\widehat{\beta}_\xi[F](z,y) = \xi(y)F(z,y)$. Rewritten in terms of kernels this leads to the following idea.

Proposition 9.1. *For $\xi \in \widehat{\mathsf{G}}$ define $\widehat{\beta}_\xi$ by*

$$\widehat{\beta}_\xi[K](z,w) = \xi(z^{-1}w)K(z,w). \tag{9.3}$$

Then $\widehat{\beta}_\xi$ is an automorphism of $u\text{-ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{A}) \rtimes_{\beta,v} \mathsf{G}$, and $\widehat{\beta}_\xi\widehat{\beta}_\eta = \widehat{\beta}_{\xi\eta}$.

Proof. We must first show that $\widehat{\beta}_\xi$ is well-defined, that is that it preserves the subspace of kernels K satisfying the equivariance condition of Theorem 8.1. In fact we see that

$$\begin{aligned}
\widehat{\beta}_\xi[K](rz, rw) &= \xi(z^{-1}w)K(rz, rw) \\
&= \xi(z^{-1}w)\phi(r,z,w)^{-1}u(r,z)^{-1}\beta_r[K(z,w)]u(r,w) \\
&= \phi(r,z,w)^{-1}u(r,z)^{-1}\beta_r[\widehat{\beta}_\xi[K](z,w)]u(r,w),
\end{aligned}$$

which gives the required equivariance condition. It is also an automorphism since

$$\begin{aligned}
(\widehat{\beta}_\xi[K_1] \star \widehat{\beta}_\xi[K_2])(z,w) &= \int_{\mathsf{G}} \xi(z^{-1}v)K_1(z,v)\xi(v^{-1}w)K_2(v,w)\phi(z,v,w)^{-1}\,dv \\
&= \int_{\mathsf{G}} \xi(z^{-1}w)K_1(z,v))K_2(v,w)\phi(z,v,w)^{-1}\,dv \\
&= \xi(z^{-1}w)(K_1 \star K_2)(v,w).
\end{aligned}$$

There is no twisting involved since it is easy to check that $\widehat{\beta}_\xi\widehat{\beta}_\eta = \widehat{\beta}_{\xi\eta}$. □

We now form the crossed product of the twisted crossed product algebra with $\widehat{\mathsf{G}}$. The elements are functions from $\widehat{\mathsf{G}}$ to the algebra of kernels described in the last section, and so may be regarded as $\mathcal{A}$-valued functions on $\mathsf{G} \times \mathsf{G} \times \widehat{\mathsf{G}}$, with multiplication

$$\begin{aligned}
(\widehat{k}_1 \star \widehat{k}_2)(z,w,\xi) &= \int_{\mathsf{G}\times\widehat{\mathsf{G}}} \widehat{k}_1(z,v,\eta)\widehat{\beta}_\eta[\widehat{k}_2](v,w,\eta^{-1}\xi)\phi(z,v,w)^{-1}\,d\eta dv \\
&= \int_{\mathsf{G}\times\widehat{\mathsf{G}}} \widehat{k}_1(z,v,\eta)\eta(v^{-1}w)\widehat{k}_2(v,w,\eta^{-1}\xi)\phi(z,v,w)^{-1}\,d\eta dv.
\end{aligned}$$

We shall denote the group-theoretic Fourier transform of $\widehat{k}$ with respect to its third argument by k (which is now a function on $\mathsf{G} \times \mathsf{G} \times \mathsf{G}$):

$$k(z,w,x) = \int_{\widehat{\mathsf{G}}} \widehat{k}(z,w,\xi)\xi(x)\,d\xi. \tag{9.4}$$

The multiplication obtained by Fourier transform is (assuming appropriate normalisation of the measures)

$$\begin{aligned}(k_1 \star k_2)(z, w, x) &= \int_{\mathsf{G}\times\widehat{\mathsf{G}}\times\widehat{\mathsf{G}}} \xi(x)\widehat{k_1}(z, v, \eta)\eta(v^{-1}w)\widehat{k_2}(v, w, \eta^{-1}\xi) \\ &\quad \times\phi(z, v, w)^{-1}\, d\xi d\eta dv \\ &= \int_{\mathsf{G}\times\widehat{\mathsf{G}}\times\widehat{\mathsf{G}}} \widehat{k_1}(z, v, \eta)\eta(v^{-1}wx)\widehat{k_2}(v, w, \eta^{-1}\xi)(\eta^{-1}\xi)(x) \\ &\quad \times\phi(z, v, w)^{-1}\, d\xi d\eta dv \\ &= \int_{\mathsf{G}} k_1(z, v, v^{-1}wx)k_2(v, w, x)\phi(z, v, w)^{-1}\, dv.\end{aligned}$$

We now introduce another transformation by

$$\widetilde{k}(x; z, w) = \phi(x, z, w)^{-1}u(x, z)k(xz, xw, w^{-1})u(x, w)^{-1}. \tag{9.5}$$

Theorem 9.2. *There is an isomorphism*

$$(u\text{-ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{A}) \rtimes_{\beta,v} \mathsf{G}) \rtimes_{\widehat{\beta}} \widehat{G} \cong u\text{-ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{A}) \otimes \mathcal{K}_{\overline{\phi}}(L^2(\mathsf{G})). \tag{9.6}$$

Proof. One important property which follows from the equivariance condition for the $\mathcal{A}$-valued kernels is that

$$\begin{aligned}\widetilde{k}(rx; z, w) &= \phi(rx, z, w)^{-1}u(rx, z)k(rxz, rxw, w^{-1})u(rx, w)^{-1} \\ &= [\phi(rx, z, w)\phi(r, xz, xw)]^{-1}u(rx, z)u(r, xz)^{-1}\alpha_r[k(xz, xw, w^{-1})] \\ &\qquad \times u(r, xw)u(rx, w)^{-1} \\ &= [\phi(rx, z, w)\phi(r, xz, xw)]^{-1}\phi(r, x, z)^{-1}\phi(r, x, w) \\ &\qquad u(r, x)^{-1}\alpha_r[u(x, z)]\alpha_r[k(xz, xw, w^{-1})]\alpha_r[u(x, w)]^{-1}u(r, x) \\ &= [\phi(rx, z, w)\phi(r, xz, xw)]^{-1}\phi(r, z, x)\phi(r, x, w) \\ &\qquad \mathrm{ad}(u(r, x))^{-1}\alpha_r[u(x, z)k(xz, xw, w^{-1})u(x, w)^{-1}] \\ &= \mathrm{ad}(u(r, x))^{-1}\alpha_r[\widetilde{k}(x; z, w)],\end{aligned}$$

so that the kernels $\widetilde{k}$ satisfy the induced algebra condition with respect to x, and so can be considered elements of the algebra induced from N to G by the $\mathcal{A}$-valued kernels.

Next we consider the product on these functions

$$\begin{aligned}(\widetilde{k}_1 \star \widetilde{k}_2)(x; z, w) &= \phi(x, z, w)^{-1}u(x, z)(k_1 \star k_2)(xz, xw, w^{-1})u(x, w)^{-1} \\ &= \phi(x, z, w)^{-1} \int_{\mathsf{G}} u(x, z)k_1(xz, xv, v^{-1})k_2(xv, xw, w^{-1}) \\ &\qquad \times u(x, w)^{-1}\phi(xz, xv, xw)^{-1}\, dv \\ &= \int_{\mathsf{G}} \widetilde{k}_1(x; z, v)\widetilde{k}_2(x; v, w)\phi(z, v, w)^{-1}\, dv.\end{aligned}$$

Thus we have the pointwise product with respect to x, with the twisted multiplication of $\mathcal{K}_{\overline{\phi}}(L^2(\mathsf{G}))$ in the fibres. □

We shall show in the final section that the twisted compact operators are in a certain sense Morita equivalent to the ordinary compact operators, so that this result provides a very precise analogue of the normal duality theorem.

To complete the argument we really need to know that the double dual action $\widehat{\widehat{\beta}}$ is equivalent to the original β to within the action on the twisted compact operators. The double dual action is defined on $(u\text{-ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{A}) \rtimes_{\beta,v} \mathsf{G}) \rtimes_{\widehat{\beta}} \widehat{\mathsf{G}}$ by the same procedure used to obtain the dual action, that is multiplication by the pairing of the $\widehat{\mathsf{G}}$ variable and the group element:

$$(\widehat{\widehat{\beta}}_g[\widehat{k}](z, w, \xi) = \xi(g)\widehat{k}(z, w, \xi). \tag{9.7}$$

Theorem 9.3. *The double dual action of* G *can be written as*

$$(\widehat{\widehat{\beta}}_g[\widetilde{k}])(x; z, w) = \tau_g[(v(g, g^{-1}z)^{-1}\beta_g[\widetilde{k}]v(g, g^{-1}w)(x; z, w)]. \tag{9.8}$$

Proof. Fourier transforming the definition of the action we get

$$(\widehat{\widehat{\beta}}_g[k](z, w, x) = \int_{\widehat{\mathsf{G}}} \xi(g)\xi(x)\widehat{k}(z, w, \xi) = k(z, w, xg). \tag{9.9}$$

Using the same notation for the equivalent action on $\widetilde{k}$,

$$\begin{aligned}(\widehat{\widehat{\beta}}_g[\widetilde{k}])(x; z, w) &= \phi(x, z, w)^{-1}u(x, z)(\widehat{\widehat{\beta}}_g[k])(xz, xw, w^{-1})u(x, w)^{-1}\\ &= \phi(x, z, w)^{-1}u(x, z)k(xz, xw, w^{-1}g)u(x, w)^{-1}\\ &= \phi(xg, g^{-1}z, g^{-1}w)\phi(x, z, w)^{-1}u(x, z)u(xg, g^{-1}z)^{-1}\\ &\quad\ \widetilde{k}(xg; g^{-1}z, g^{-1}w)u(xg, g^{-1}w)u(x, w)^{-1}.\end{aligned}$$

In terms of the induced twisting we now we have

$$u(x, z)u(xg, g^{-1}z)^{-1} = \phi(x, g, g^{-1}z)^{-1}v(g, g^{-1}z)(x)^{-1}u(x, g), \tag{9.10}$$

and substituting this (and the analogous expression in w), we arrive at

$$\begin{aligned}(\widehat{\widehat{\beta}}_g[\widetilde{k}])(x; z, w) &= \phi(xg, g^{-1}z, g^{-1}w)\phi(x, z, w)^{-1}\phi(x, g, g^{-1}z)^{-1}\phi(x, g, g^{-1}w)\\ &\qquad \times v(g, g^{-1}z)(x)^{-1}\\ &\quad\ u(x, g)\widetilde{k}(xg; g^{-1}z, g^{-1}w)u(x, g)^{-1}v(g, g^{-1}w)(x)\\ &= \phi(g, z, w)v(g, g^{-1}z)(x)^{-1}\mathrm{ad}(u(x, g))[\widetilde{k}(xg; g^{-1}z, g^{-1}w)]\\ &\qquad \times v(g, g^{-1}w)(x)\\ &= \phi(g, z, w)v(g, g^{-1}z)(x)^{-1}(\beta_g[\widetilde{k}](x; g^{-1}z, g^{-1}w)]v(g, g^{-1}w)(x).\end{aligned}$$

This can be rewritten in terms of the twisted action τ_g on kernels and the adjoint action of v in the form

$$(\widehat{\widehat{\beta}}_g[\widetilde{k}])(x;z,w) = \tau_g[(v(g,g^{-1}z)^{-1}\beta_g[\widetilde{k}]v(g,g^{-1}w))(x;z,w)], \tag{9.11}$$

showing that to within an inner automorphism of the kernels one has $\beta_g \otimes \tau_g$, and up to an action on the twisted kernels one recovers β_g. □

In the application to principal $\mathbb{T}^n$-bundles one takes $\mathsf{G} = \mathbb{R}^n$. As this is contractible, Proposition 5.2 shows that $\mathcal{K}_{\overline{\phi}}(L^2(\mathsf{G}))$ is a deformation of $\mathcal{K}(L^2(\mathsf{G}))$, so that this is a close substitute for the usual duality theorem.

10. Applications to T-Duality

In this section, we apply the mathematical results of the earlier sections to determine the T-dual of principal torus bundles with general H-flux, thus generalizing earlier results in [2–4, 28, 29]. Let $\mathsf{G} = \mathbb{R}^\ell$, and $\mathsf{N} = \mathbb{Z}^\ell$ in the setup of Corollary 7.3.

Let $E \to M$ be a principal $\mathbb{T}^\ell$-bundle, and $H \in H^3(E)$ be an integral H-flux on E.[3] Then we can identify $H = (H_3, H_2, H_1, H_0)$, where $H_p \in \Omega^p(M, \wedge^{3-p}\widehat{\mathfrak{t}})$, under the isomorphism (1.1), and closed under D. Then by the results in [28, 29], there is a continuous trace C^*-algebra $\mathrm{ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{A})$ with spectrum equal to E and with Dixmier-Douady invariant equal to $(H_3, H_2, H_1, 0)$, which has an action of G that covers the given action of G on E. This action is not necessarily unique. Then by Corollary 7.3, we know that there is another continuous trace C^*-algebra $u\text{-}\mathrm{ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{A})$ with spectrum equal to E and with Dixmier-Douady invariant equal to $H = (H_3, H_2, H_1, H_0)$, which has a twisted action of G that covers the given action of G on E. Our main definition in this section is:

Definition 10.1. *The twisted crossed product $u\text{-}\mathrm{ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{A}) \rtimes_{\beta,v} \mathsf{G}$ is defined to be the T-dual to the principal $\mathbb{T}^\ell$-bundle E with H-flux H.*

We justify this definition as follows. Firstly, the T-dual of $u\text{-}\mathrm{ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{A}) \rtimes_{\beta,v} \mathsf{G}$ is the crossed product $(u\text{-}\mathrm{ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{A}) \rtimes_{\beta,v} \mathsf{G}) \rtimes_{\widehat{\beta}} \widehat{\mathsf{G}}$, which by the twisted Takai duality Theorem 9.2 is isomorphic to $u\text{-}\mathrm{ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{A}) \otimes \mathcal{K}_\phi(L^2(\mathsf{G}))$. That is, the T-dual of $u\text{-}\mathrm{ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{A}) \rtimes_{\beta,v} \mathsf{G}$ is Morita equivalent to the continuous trace algebra $u\text{-}\mathrm{ind}_{\mathsf{N}}^{\mathsf{G}}(\mathcal{A})$, so that T-duality applied twice returns us to where we started, up to Morita equivalence. In the special case when $H = H_0$, the fibre of this bundle over the point $z \in M$ is equal to the nonassociative torus A_ϕ of rank ℓ with tricharacter ϕ corresponding to H_0 (see Theorem 8.2). In the general case, but when H_0 is zero, the fibre is a stabilized noncommutative torus with invariant H_1, [28, 29], and when $H_0 = 0$ and $H_1 = 0$, then the fibre is the stabilized algebra of continuous functions on a torus, [2–4]. Thus we have the following theorem.

Theorem 10.2 (T-duality for principal torus bundles). *Let $E \to M$ be a principal $\mathbb{T}^\ell$-bundle over M, and $H \in H^3(E)$ be an integral H-flux on E. Then $H = (H_3, H_2, H_1, H_0)$, where $H_p \in \Omega^p(M, \wedge^{3-p}\widehat{\mathfrak{t}})$. Let $c_1(E) \in H^2(M, \mathfrak{t})$ denote the first Chern class of E, which determines E up to isomorphism.*

Then:

(1) If $H_0 = 0$ and $H_1 = 0$, then there is a canonical T-dual $\widehat{E}$ which is a principal $\mathbb{T}^\ell$-bundle over M first Chern class $c_1(\widehat{E}) = H_2 \in H^2(M, \widehat{\mathfrak{t}})$. $\widehat{E}$ has a T-dual H-flux $\widehat{H} = (\widehat{H}_3, \widehat{H}_2, 0, 0)$ given by $\widehat{H}_3 = H_3$ and $\widehat{H}_2 = c_1(E)$. T-duality is neatly encapsulated in the commutative diagram,

[3] The conclusions in this section are valid for integral classes $H \in H^3(E, \mathbb{Z})$ as well since the component H_0 does not carry torsion, and the remainder of the arguments is based on the results of [28, 29], which also hold for torsion H. For simplicity we state the results for differential forms only.

$$\begin{array}{ccccc} & & E \times_M \widehat{E} & & \\ & \swarrow^{p} & & \searrow^{\hat{p}} & \\ E & & & & \widehat{E} \\ & \searrow_{\pi} & & \swarrow_{\hat{\pi}} & \\ & & M & & \end{array} \qquad (10.1)$$

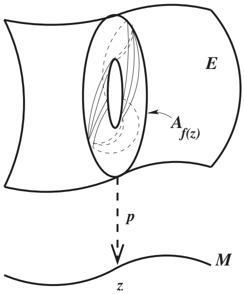

Fig. 10.1. In the diagram, the fiber over $z \in M$ is the noncommutative torus $A_{f(z)}$, which is represented by a foliated torus, with foliation angle equal to $f(z)$

(2) If $H_0 = 0$ and $H_1 \neq 0$, then the T-dual is a continuous field of (stabilized) noncommutative tori A_f over M, where the fiber over the point $z \in M$ is equal to the rank k noncommutative torus $A_{f(z)}$ (see Fig. 10.1 above). Here $f : M \to \mathbb{T}^{\binom{\ell}{2}}$ is a continuous map representing $H_1 \in [M, \mathbb{T}^{\binom{\ell}{2}}] \subset H^1(M, \wedge^2, \widehat{\mathfrak{t}})$. This map is not unique, but the nonuniqueness does not affect its K-theory.

(3) If $H_0 \neq 0$ and if $H = H_0$, then the T-dual is a bundle of nonassociative tori A_ϕ (cf. Theorem 8.2) over M, where ϕ is the tricharacter associated to H_0. For general H, the T-dual is a continuous field of algebras that contains both the noncommutative torus and the nonassociative torus, and moreover, the T-dual is not unique, but the nonuniqueness occurs exactly as in part (2) *above.*

Part (1) was proved in [2, 3] when $\ell = 1$ and in [4] for general ℓ.
Part (2) was proved in [28] when $\ell = 2$ and in [29] for general ℓ.
Part (3) is what has been proved in this paper.

A particular, but important case of Theorem 10.2 above is the following.

1. The T-dual of the torus $\mathbb{T}^3$ with no background flux is the dual torus $\widehat{\mathbb{T}}^3$. This remains true if the background flux is topologically trivial.

2. $(\mathbb{T}^3, k\ dx \wedge dy \wedge dz)$ considered as a trivial circle bundle over $\mathbb{T}^2$. The T-dual of $(\mathbb{T}^3, k\ dx \wedge dy \wedge dz)$ is the nilmanifold $(H_{\mathbb{R}}/H_{\mathbb{Z}}, 0)$, where $H_{\mathbb{R}}$ is the 3 dimensional

Heisenberg group and $H_{\mathbb{Z}}$ the lattice in it defined by

$$H_{\mathbb{Z}} = \left\{ \begin{pmatrix} 1 & x & \frac{1}{k}z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\}. \tag{10.2}$$

3. $(\mathbb{T}^3, k \ \ dx \wedge dy \wedge dz)$ considered as a trivial $\mathbb{T}^2$-bundle over $\mathbb{T}$. The T-dual of $(\mathbb{T}^3, k \ \ dx \wedge dy \wedge dz)$ is a continuous field of stabilized noncommutative tori, $C^*(H_{\mathbb{Z}}) \otimes \mathcal{K}$, since $\int_{\mathbb{T}^2} k\ dx \wedge dy \wedge dz \neq 0$.
4. $(\mathbb{T}^3, k\ dx \wedge dy \wedge dz)$ considered as a trivial $\mathbb{T}^3$-bundle over a point. The T-dual of $(\mathbb{T}^3, k\ dx \wedge dy \wedge dz)$ is a nonassociative torus, A_ϕ (cf. Theorem 8.2), where ϕ is the tricharacter associated to $k\ dx \wedge dy \wedge dz$, since $\int_{\mathbb{T}^3} k\ dx \wedge dy \wedge dz \neq 0$.

We end with some speculations and open problems related to the results of the paper. In Sect. 11, we propose a natural definition of K-theory for the special nonassociative algebras that are considered in this paper. These are of the form $\mathcal{A} \rtimes_{\beta,v} \mathsf{G}$, where $\mathcal{A}$ is a C^*-algebra admitting a twisted action of the Abelian group $\mathsf{G} = \mathbb{R}^\ell$. We expect an analogue of Connes–Thom isomorphism theorem in K-theory to hold, showing that the K-theories of $\mathcal{A}$ and $\mathcal{A} \rtimes_{\beta,v} \mathsf{G}$ are naturally isomorphic. This would then give further evidence that our definition of the T-dual of a principal torus bundle with H-flux is indeed correct. Finally, the K-theory of our special nonassociative algebras should be Morita invariant in our context, namely invariant under tensor product with twisted compact operators. Then the twisted Takai duality Theorem 9.2 would prove that T-duality applied twice returns us to the torus bundle with H-flux that we started out with.

It remains to also determine the topological invariants of continuous fields of noncommutative tori and bundles of nonassociative tori as in the paper. This would then enable one to give a more symmetric characterization to the T-dual, similar to part (1) of the theorem above. We have an explicit conjecture for this, the explanation for which is in [5], namely, for the continuous field of noncommutative tori A_f, there should be a "Chern class" invariant $c_1(A_f) = (H_2, H_1, 0)$ satisfying $dH_2 + c_1(E) \wedge H_1 = 0$. In this case, we can add the following to part (2) of the theorem above.

The T-dual A_f is classified by its Chern class invariant $c_1(A_f) = (H_2, H_1, 0)$ satisfying $dH_2 + c_1(E) \wedge H_1 = 0$ and $dH_1 = 0$ and has T-dual H-flux $\widehat{H} = (\widehat{H}_3, \widehat{H}_2, \widehat{H}_1, 0)$, given by $\widehat{H}_3 = H_3$, $\widehat{H}_2 = c_1(E)$ and $\widehat{H}_1 = 0$.

Similarly, for the bundle of nonassociative tori A_ϕ with tricharacter ϕ associated to H_0, there should also be a "Chern class" invariant $c_1(A_\phi) = (H_2, H_1, H_0)$ satisfying $dH_2 + c_1(E) \wedge H_1 = 0$ and $dH_1 + c_1(E) \wedge H_0 = 0$. In this case, we can add the following to part (3) of the theorem above.

The T-dual A_ϕ is classified by its Chern class invariant $c_1(A_\phi) = (H_2, H_1, H_0)$ satisfying $dH_2 + c_1(E) \wedge H_1 = 0$, $dH_1 + c_1(E) \wedge H_0 = 0$ and $dH_0 = 0$. It has T-dual H-flux $\widehat{H} = (\widehat{H}_3, \widehat{H}_2, \widehat{H}_1, \widehat{H}_0)$ given by $\widehat{H}_3 = H_3$, $\widehat{H}_2 = c_1(E)$, $\widehat{H}_1 = 0$ and $\widehat{H}_0 = 0$.

What also remains to be done is T-duality for nonabelian principal bundles, where some of the ideas of this paper and [5] apply.

11. Nonassociative Algebras and Monoidal Categories – An Outlook

Although the nonassociativity of the crossed product algebra appears to present a serious amendment to the notion of duality, that is not really the case. The fact that the same obstruction ϕ appears throughout is a signal that one should rather work in the monoidal category of $C_0(\mathsf{G})$-modules in which the isomorphism $\Phi : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ is given by the action of $\overline{\phi} \in C(\mathsf{G} \times \mathsf{G} \times \mathsf{G})$, the multiplier algebra of $C_0(\mathsf{G}) \otimes C_0(\mathsf{G}) \otimes C_0(\mathsf{G})$, [25, 12]. The cocycle identity for ϕ is equivalent to commutativity of the fundamental pentagonal diagram which ensures that all higher associators are consistent. By Fourier transforming we could identify the category as $\widehat{\mathsf{G}}$-modules rather than $C(\mathsf{G})$-modules, which fits more directly into the framework of the duality theorem. The identity object is the trivial $\widehat{\mathsf{G}}$-module 1 on $\mathbb{C}$, which certainly has the property that, for any $\widehat{\mathsf{G}}$-module U, $U \otimes 1$ and $1 \otimes U$ are naturally isomorphic to U. This is equivalent on $C(\mathsf{G})$ to evaluating a function at the identity. Because ϕ vanishes when an argument is set equal to the identity, the two obvious maps from $U \otimes (1 \otimes V) = \Phi[(U \otimes 1) \otimes V]$ to $U \otimes V$ are consistent.

An algebra $\mathcal{A}$ is a monoid in this category, and the identification Φ automatically takes care of the associativity. We can also define a left $\mathcal{A}$-module M if one has a morphism $\mathcal{A} \otimes M \to M$. A left $\mathcal{A}$-module M is said to be projective if given any surjective morphism of left $\mathcal{A}$-modules $a : E \to N$ and any morphism of left $\mathcal{A}$-modules $b : M \to N$, there is a morphism of left $\mathcal{A}$-modules $c : M \to E$ such that $a \circ c = b$. If $\mathcal{A}$ has a unit, then one can define the monoid $V(\mathcal{A})$ consisting of isomorphism classes of finitely generated projective left $\mathcal{A}$-modules under the direct sum operation. Then $K_0(\mathcal{A})$ is defined as the Grothendieck group of $V(\mathcal{A})$. If $\mathcal{A}$ does not have a unit, and $\mathcal{A}^+$ denotes $\mathcal{A}$ with a unit adjoined to it, then $K_0(\mathcal{A})$ is defined as the kernel of the canonical morphism $K_0(\mathcal{A}^+) \mapsto K_0(\mathbb{C}) \cong \mathbb{Z}$. This will be studied in detail in a subsequent paper.

An example of a monoid in the category of $\widehat{\mathsf{G}}$ modules is the algebra $\mathcal{K}_{\overline{\phi}}(L^2(\mathsf{G}))$ of twisted compact operators with the $\widehat{\mathsf{G}}$ action

$$(\xi \cdot K)(x, y) = \xi(xy^{-1})K(x, y)$$

and $L^2(\mathsf{G})$, which has the $\widehat{\mathsf{G}}$-action $(\xi \cdot \psi)(x) = \xi(x)\psi(x)$, is a module with $K \otimes \psi \mapsto K * \psi$, where

$$(K * \psi)(x) = \int_{\mathsf{G}} K(x, z)\psi(z)\, dz.$$

The $\widehat{\mathsf{G}}$ actions are compatible since

$$((\xi \cdot K) * (\xi \cdot \psi))(x) = \int_{\mathsf{G}} \xi(xz^{-1})K(x, z)\xi(z)\psi(z)\, dz = (\xi(x)(K * \psi))(x).$$

Then

$$(K_1 * (K_2 * \psi))(x) = \int_{\mathsf{G}} K_1(x, y)K_2(y, z)\psi(z)\, dydz,$$

and the alternate bracketing $(K_1 * K_2) * \psi$ must be computed as the image of $\Phi(K_1 \otimes (K_2 \otimes \psi))$, giving

$$((K_1 * K_2) * \psi)(x) = \int_{\mathsf{G}} \phi(x, y, z)^{-1} K_1(x, y)K_2(y, z)\psi(z)\, dydz,$$

consistent with the multiplication law on the twisted kernels.

One can alternately work with the $C_0(\mathsf{G})$ action rather than $\widehat{\mathsf{G}}$ but then the action on kernels requires a use of the coproduct $(\Delta f)(x, y) = f(xy)$, so that $(f \cdot K)(x, y) = f(xy^{-1})K(x, y)$.

One can similarly define right $\mathcal{A}$-modules, and also bimodules for two algebras $\mathcal{A}_1$ and $\mathcal{A}_2$. It is also possible to look at $\mathcal{A}_1$-$\mathcal{A}_2$-bimodules X which have an action of a product group $\widehat{\mathsf{G}}_1 \times \widehat{\mathsf{G}}_2$ with maps Φ_1 and Φ_2 defining the associativity properties, and are in the category of $\widehat{\mathsf{G}}_1$-modules as left $\mathcal{A}_1$-modules and in the category of $\widehat{\mathsf{G}}_2$-modules as right $\mathcal{A}_2$-modules. Such a bimodule can be used to set up a Morita equivalence between left $\mathcal{A}_2$-modules and left $\mathcal{A}_1$-modules, by mapping a left $\mathcal{A}_2$-module M to the quotient of $X \otimes V$ by the equivalence relation $(x.b) \otimes \psi \sim \Phi_2(x \otimes (b.\psi))$, for $x \in X$, $b \in \mathcal{A}_2$ and $\psi \in M$. This allows us to define Morita equivalence between algebras with different kinds of associativity. In particular, if we take $\mathcal{A}_1 = \mathcal{K}_{\overline{\phi}}(L^2(\mathsf{G}))$, $\mathcal{A}_2 = \mathcal{K}(L^2(\mathsf{G}))$, with $X = \mathcal{K}(L^2(\mathsf{G}))$, equipped with the usual right multiplication action of $\mathcal{A}_2$ and the left multiplication action of $\mathcal{A}_1$ defined above, then we have Morita equivalence between the twisted and untwisted algebras.

Clearly this is only an outline of some of the ideas arising out of this new perspective on nonassociativity, and we shall explore these in more detail in the sequel to this paper. Since posting this paper on the arXives [1] has come to our attention, which also investigates some nonassociative algebras albeit in a rather different context.

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