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Dyson's Constant in the Asymptotics of the Fredholm Determinant of the Sine Kernel

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Abstract: We prove that the asymptotics of the Fredholm determinant of $I - K_\alpha$, where K_α is the integral operator with the sine kernel $\frac{\sin(x-y)}{\pi(x-y)}$ on the interval $[0, \alpha]$, are given by

$$\log\det(I - K_{2\alpha}) = -\frac{\alpha^2}{2} - \frac{\log\alpha}{4} + \frac{\log 2}{12} + 3\zeta'(-1) + o(1), \qquad \alpha \to \infty.$$

This formula was conjectured by Dyson. The proof for the first and second order asymptotics was given by Widom, and higher order asymptotics have also been determined. In this paper we identify the constant (or third order) term, which has been an outstanding problem for a long time.

1. Introduction

Let K_α be the integral operator defined on $L^2[0, \alpha]$ with the kernel

$$k(x, y) = \frac{\sin(x-y)}{\pi(x-y)}. \tag{1}$$

Dyson conjectured the following asymptotic formula for the determinant $\det(I - K_{2\alpha})$,

$$\log\det(I - K_{2\alpha}) = -\frac{\alpha^2}{2} - \frac{\log\alpha}{4} + \frac{\log 2}{12} + 3\zeta'(-1) + o(1), \qquad \alpha \to \infty, \tag{2}$$

(where ζ stands for the Riemann zeta function) and provided heuristic arguments [7]. Later on Jimbo, Miwa, Môri and Sato [11] (see also [15]) showed that the function

$$\sigma(\alpha) = \alpha \frac{d}{d\alpha} \log\det(I - K_\alpha)$$

satisfies a Painlevé V equation. Widom [17, 18] determined the highest term in the asymptotics of $\sigma(\alpha)$ as $\alpha \to \infty$. Knowing these asymptotics one can derive a complete asymptotic expansion for $\sigma(\alpha)$. From this it follows by integration that the asymptotic expansion of $\det(I - K_{2\alpha})$ is given by

$$\log\det(I - K_{2\alpha}) = -\frac{\alpha^2}{2} - \frac{\log\alpha}{4} + C + \sum_{n=1}^{N} \frac{C_{2n}}{\alpha^{2n}} + O(\alpha^{2N+2}), \quad \alpha \to \infty, \quad (3)$$

with effectively computable constants C_{2n}. Clearly, the constant C cannot be obtained in this way, and its (rigorous) computation was an outstanding problem for a long time.

Only recently, Krasovsky [12] was able to make Dyson's heuristic derivation of the asymptotic formula (2) rigorous by using the so-called Riemann-Hilbert method. It is the purpose of this paper to give another proof of (2) by using a different approach.

The determinant $\det(I - K_\alpha)$ appears in random matrix theory [10]. It is related to the probabilities $E_2(n; \alpha)$ that in the bulk scaling limit of the Gaussian Unitary Ensemble an interval of length α contains precisely n eigenvalues. In particular, we have $\det(I - K_\alpha) = E_2(0; \alpha)$. Since the asymptotics of the ratios $E_2(n; \alpha)/E_2(0; \alpha)$ as $\alpha \to \infty$ can be computed at least for some n (see [4]), the asymptotics of the sine kernel determinant is of relevance also for $E_\beta(n; \alpha)$ with general n. Connections between the determinant $\det(I - K_\alpha)$ and corresponding probabilities for the Gaussian Orthogonal and Symplectic Ensembles also exist [4, 10].

A generalization of the determinant $\det(I - K_\alpha)$, where the sine kernel integral operator is considered on a finite union of finite subintervals of $\mathbb{R}$, was also studied, and results were established by Widom [18] and by Deift, Its and Zhou [6]. This generalization has a similar interpretation in random matrix theory [15].

The theory of Wiener-Hopf determinants can explain at least to some extent the reason for the difficulties one faces with the sine kernel determinant. In fact, $I - K_\alpha$ is a truncated Wiener-Hopf operator $W_\alpha(\phi)$ with the generating function equal to the characteristic function of the subset $(-\infty, -1) \cup (1, \infty)$ of $\mathbb{R}$. This generating function does not belong to the already difficult class of Fisher-Hartwig symbols [5], i.e., functions which have only a finite number of zeros or discontinuities of a certain type. Notice that even for Fisher-Hartwig symbols the proof of the (conjectured) asymptotics has not yet been achieved completely.

It is useful to take a look at the discrete analogue of the determinants, i.e., Toeplitz determinants $T_n(\chi_\alpha)$, where the generating function χ_α is equal to the characteristic function of the subarc $\{\, e^{i\theta} \,:\, \alpha < \theta < 2\pi - \alpha \,\}$ of the complex unit circle. Widom [16] proved that for fixed $\alpha \in (0, \pi)$ the asymptotics are given by

$$\det T_n(\chi_\alpha) \sim \left(\cos\frac{\alpha}{2}\right)^{n^2} \left(n \sin\frac{\alpha}{2}\right)^{-1/4} 2^{1/12} e^{3\zeta'(-1)}, \qquad n \to \infty. \quad (4)$$

Dyson's heuristic derivation relies on this formula and on the fact that discretizing the sine kernel operator yields the Toeplitz operator $T_n(\chi_{\alpha/n})$, i.e., $\lim_{n\to\infty} \det T_n(\chi_{\alpha/n}) = \det(I - K_\alpha)$. He replaces α by α/n in the right-hand side of (4) and takes the limit $n \to \infty$ to arrive at the asymptotic expression given in (2).

Krasovsky shows that this (non-rigorous) argumentation can be made rigorous. He determines the asymptotics of the derivative (in α) of $\log\det T_n(\chi_\alpha)$ together with an estimate of the error, which holds uniformly on a certain range of the parameter. Upon integration and using Widom's result to fix the constant, he arrives at the asymptotic formula (2).

The proof presented in this paper also uses the discretization idea. However, we do not use the asymptotics of a Toeplitz determinant in order to the fix the constant. Instead, we establish an exact identity between the determinant $\det(I - K_\alpha)$ and what could be considered the determinants of Wiener-Hopf-Hankel operators with Fisher-Hartwig symbols. The asymptotics of these Wiener-Hopf-Hankel determinants were determined in the paper [3]. An outline of the main ideas of our proof will be given in Sect. 3 after having introduced some necessary notation. In Sect. 4 we will establish several auxiliary results and the proof will be given in Sect. 5.

2. Basic Notation

Let us first introduce some notation. For a Lebesgue measurable subset M of the real axis $\mathbb{R}$ or of the unit circle $\mathbb{T} = \{t \in \mathbb{C} \ : \ |t| = 1\}$, let $L^p(M)$ $(1 \le p < \infty)$ stand for the space of all Lebesgue measurable p-integrable complex-valued functions. For $p = \infty$ we denote by $L^\infty(M)$ the space of all essentially bounded Lebesgue measurable functions on M.

For a function $a \in L^1(\mathbb{T})$ we introduce the $n \times n$ Toeplitz and Hankel matrices

$$T_n(a) = (a_{j-k})_{j,k=0}^{n-1}, \qquad H_n(a) = (a_{j+k+1})_{j,k=0}^{n-1}, \tag{5}$$

where

$$a_k = \frac{1}{2\pi} \int_0^{2\pi} a(e^{i\theta}) e^{-ik\theta} \, d\theta, \qquad k \in \mathbb{Z},$$

are the Fourier coefficients of a. We also introduce differently defined $n \times n$ Hankel matrices

$$H_n[b] = (b_{j+k+1})_{j,k=0}^{n-1}, \tag{6}$$

where the numbers b_k are the (scaled) moments of a function $b \in L^1[-1, 1]$, i.e.,

$$b_k = \frac{1}{\pi} \int_{-1}^{1} b(x)(2x)^{k-1} \, dx, \qquad k \ge 1.$$

Given $a \in L^\infty(\mathbb{T})$ the multiplication operator $M(a)$ acting on $L^2(\mathbb{T})$ is defined by

$$M(a) : f(t) \in L^2(\mathbb{T}) \mapsto a(t) f(t) \in L^2(\mathbb{T}).$$

We denote by P the Riesz projection

$$P : \sum_{k=-\infty}^{\infty} f_k t^k \in L^2(\mathbb{T}) \mapsto \sum_{k=0}^{\infty} f_k t^k \in L^2(\mathbb{T})$$

and by J the flip operator

$$J : f(t) \in L^2(\mathbb{T}) \mapsto t^{-1} f(t^{-1}) \in L^2(\mathbb{T}).$$

The image of the Riesz projection is equal to the Hardy space

$$H^2(\mathbb{T}) = \Big\{ f \in L^2(\mathbb{T}) \ : \ f_k = 0 \text{ for all } k < 0 \Big\}.$$

For $a \in L^\infty(\mathbb{T})$ the Toeplitz and Hankel operators are bounded linear operators defined on $H^2(\mathbb{T})$ by

$$T(a) = PM(a)P|_{H^2(\mathbb{T})}, \qquad H(a) = PM(a)JP|_{H^2(\mathbb{T})}. \tag{7}$$

The matrix representation of these operators with respect to the standard basis $\{t^n\}_{n=0}^\infty$ of $H^2(\mathbb{T})$ is given by infinite Toeplitz and Hankel matrices,

$$T(a) \cong (a_{j-k})_{j,k=0}^\infty, \qquad H(a) \cong (a_{j+k+1})_{j,k=0}^\infty. \tag{8}$$

The connection to $n \times n$ Toeplitz and Hankel matrices is given by

$$P_n T(a) P_n \cong T_n(a), \qquad P_n H(a) P_n \cong H_n(a), \tag{9}$$

where P_n is the finite rank projection operator

$$P_n : \sum_{k\geq 0} f_k t^k \in H^2(\mathbb{T}) \mapsto \sum_{k=0}^{n-1} f_k t^k \in H^2(\mathbb{T}). \tag{10}$$

An operator A acting on a Hilbert space H is called a trace class operator if it is compact and if the series constituted by the singular values $s_n(A)$ (i.e., the eigenvalues of $(A^*A)^{1/2}$ taking multiplicities into account) converges. The norm

$$\|A\|_1 = \sum_{n\geq 1} s_n(A) \tag{11}$$

makes the set of all trace class operators into a Banach space, which forms also a two-sided ideal in the algebra of all bounded linear operators on H. Moreover, the estimates $\|AB\|_1 \leq \|A\|_1\|B\|$ and $\|BA\|_1 \leq \|A\|_1\|B\|$ hold, where A is a trace class operator and B is a bounded operator with the operator norm $\|B\|$.

If A is a trace class operator, then the operator trace "trace(A)" and the operator determinant "det$(I + A)$" are well defined. For more information concerning these concepts we refer to [9].

Given $a \in L^\infty(\mathbb{R})$ we denote by $M_\mathbb{R}(a)$ the multiplication operator

$$M_\mathbb{R}(a) : f(x) \in L^2(\mathbb{R}) \mapsto a(x)f(x) \in L^2(\mathbb{R})$$

and by $W_0(a)$ the convolution operator (or, "two-sided" Wiener-Hopf operator)

$$W_0(a) = \mathcal{F} M_\mathbb{R}(a) \mathcal{F}^{-1},$$

where $\mathcal{F}$ stands for the Fourier transform on $L^2(\mathbb{R})$. The usual Wiener-Hopf operator and the "continuous" Hankel operator acting on $L^2(\mathbb{R}_+)$ are given by

$$W(a) = \Pi_+ W_0(a) \Pi_+|_{L^2(\mathbb{R}_+)}, \tag{12}$$

$$H_\mathbb{R}(a) = \Pi_+ W_0(a) \hat{J} \Pi_+|_{L^2(\mathbb{R}_+)}, \tag{13}$$

where $(\hat{J}f)(x) = f(-x)$ and $\Pi_+ = M_\mathbb{R}(\chi_{\mathbb{R}_+})$ is the projection operator on the positive real half axis. If $a \in L^1(\mathbb{R})$, then $W(a)$ and $H_\mathbb{R}(a)$ are integal operators on $L^2(\mathbb{R})$ with the kernel $\hat{a}(x-y)$ and $\hat{a}(x+y)$, respectively, where

$$\hat{a}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\xi} a(x)\, dx$$

stands for the Fourier transform of a.

Finally, let Π_α stand for the projection operator,

$$\Pi_\alpha : f(t) \in L^2(\mathbb{R}_+) \mapsto \chi_{[0,\alpha]}(x) f(x) \in L^2(\mathbb{R}_+). \tag{14}$$

The image of Π_α will be identified with the space $L^2[0, \alpha]$.

3. Main Ideas of the Proof

Before explaining the main ideas of the proof, let us introduce the functions,

$$\hat{u}_\beta(x) = \left(\frac{x-i}{x+i}\right)^\beta, \qquad \hat{v}_\beta(x) = \left(\frac{x^2}{1+x^2}\right)^\beta, \qquad x \in \mathbb{R},\ \beta \in \mathbb{C}. \tag{15}$$

These functions are continuous on $\mathbb{R} \setminus \{0\}$ with limits being equal to 1 as $x \to \pm\infty$. Moreover, the function $\hat{u}_\beta$ has a jump discontinuity at $x = 0$, while the function $\hat{v}_\beta$ has a zero, a pole, or an oscillating discontinuity at $x = 0$.

The prevailing part of the paper is devoted to prove the identity

$$\det(I - K_{2\alpha}) = \exp\left(-\frac{\alpha^2}{2}\right) \det\left[\Pi_\alpha (I + H_{\mathbb{R}}(\hat{u}_{-1/2}))^{-1} \Pi_\alpha\right] \\ \times \det\left[\Pi_\alpha (I - H_{\mathbb{R}}(\hat{u}_{1/2}))^{-1} \Pi_\alpha\right]. \tag{16}$$

Therein, as we will see, the inverse operators on the right-hand side exist and the determinants can be understood as operator determinants.

Such complicated expressions seem to promise little advantage. However, the above determinants are related to determinants of truncated Wiener-Hopf-Hankel operators [3, Prop. 3.14] . Indeed,

$$\hat{D}_\alpha^+(\beta) := \det\left[\Pi_\alpha (W(\hat{v}_\beta) + H_{\mathbb{R}}(\hat{v}_\beta)) \Pi_\alpha\right] \\ = e^{-\alpha\beta} \det\left[\Pi_\alpha (I + H_{\mathbb{R}}(\hat{u}_{-\beta}))^{-1} \Pi_\alpha\right] \tag{17}$$

if $-1/2 < \operatorname{Re}\beta < 3/2$, and

$$\hat{D}_\alpha^-(\beta) := \det\left[\Pi_\alpha (W(\hat{v}_\beta) - H_{\mathbb{R}}(\hat{v}_\beta)) \Pi_\alpha\right] \\ = e^{-\alpha\beta} \det\left[\Pi_\alpha (I - H_{\mathbb{R}}(\hat{u}_{-\beta}))^{-1} \Pi_\alpha\right] \tag{18}$$

if $-1/2 < \operatorname{Re}\beta < 1/2$. One minor complication is encountered since the definition of the Wiener-Hopf-Hankel determinants requires $\hat{v}_\beta - 1 \in L^1(\mathbb{R})$, i.e., $\operatorname{Re}\beta > -1/2$. This complication can be resolved by remarking that the right-hand side of (18) is well defined for $-3/2 < \operatorname{Re}\beta < 1/2$. Hence $\hat{D}_\alpha^-(\beta)$ can be continued by analyticity (in β) to the domain where $\operatorname{Re}\beta > -3/2$. Identity (16) can thus be rewritten as

$$\det(I - K_{2\alpha}) = \exp\left(-\frac{\alpha^2}{2}\right) \hat{D}_\alpha^+(1/2) \hat{D}_\alpha^-(-1/2). \tag{19}$$

Since the Wiener-Hopf-Hankel determinants (17) and (18) have symbols of Fisher-Hartwig type, their asymptotics might be easier to analyze. Precisely this is done in the paper [3]. Applying these results we obtain the asymptotics of $\det(I - K_{2\alpha})$.

In our proof we will not rely on (19), but only on (16). We are mentioning (19) here only because it represents a much simpler interpretation.

Let us now proceed with making some remarks on how the proof of (16) is accomplished. We are first going to discretize the sine kernel operator and obtain a Toeplitz determinant, i.e.,

$$\det(I - K_\alpha) = \lim_{n\to\infty} \det T_n(\chi_{\frac{\alpha}{n}}).$$

Using an exact identity between determinants of symmetric Toeplitz matrices and Hankel matrices, which was established in [2], we reduce the Toeplitz determinant to a Hankel determinant of the form $\det H_n[b]$ with a symbol b depending on n. This symbol is supported on a proper subinterval of $[-1, 1]$. One crucial observation is that since the entries of this type of Hankel matrix are defined as the moments of its symbol, it is possible (by pulling out an appropriate factor) to arrive at a Hankel determinant $\det H_n[b_{\alpha,n}]$ with a certain symbol $b_{\alpha,n}$ which is supported on all of $[-1, 1]$. It is interesting to observe that the factor which has been pulled out gives precisely (after taking the limit $n \to \infty$) the exponential term in (16). Hence at this point we have shown that

$$\det(I - K_\alpha) = \exp\left(-\frac{\alpha^2}{8}\right) \cdot \lim_{n\to\infty} H_n[b_{\alpha,n}].$$

The next step, which is elaborated on in Sect. 4.2, is to use two other exact determinant identities (which were proved in [2] and [3]) and to establish that

$$\det H_n[b] = \det(T_n(a) + H_n(a)) = G^n \det\left[P_n(I + H(\psi))^{-1} P_n\right]$$

for "suitably behaved" functions b, a, and ψ and a constant G (all depending on each other in an explicit way, which will be described in Sect. 4.2). At this point another complication is encountered since the function $b_{\alpha,n}$ is not "suitably behaved". We can by-pass this complication only by applying an approximation argument, which turns out to be non-trivial and is based on results of [8]. Finally, we will arrive at an identity

$$\det H_n[b_{\alpha,n}] = \det\left[P_n(I + H(\psi_{\alpha,n}))^{-1} P_n\right]$$

with certain functions $\psi_{\alpha,n}$.

The functions $\psi_{\alpha,n}$ are of such a form that they have their singularities only at the points $t = 1$ and $t = -1$. What one usually tries to do in such cases is to separate the singularities. Indeed, we will prove that the above asymptotics is equal to the asymptotics of

$$\det\left[P_n(I + H(\psi^{(1)}_{\alpha,n}))^{-1} P_n\right] \cdot \det\left[P_n(I + H(\psi^{(-1)}_{\alpha,n}))^{-1} P_n\right],$$

where the functions $\psi^{(1)}_{\alpha,n}$ and $\psi^{(-1)}_{\alpha,n}$ have singularities only at $t = 1$ and $t = -1$, respectively. Fortunately, the last two expressions are simple enough to analyze, and their limits as $n \to \infty$ equal the constants

$$\det\left[\Pi_\alpha(I + H_{\mathbb{R}}(\hat{u}_{-1/2}))^{-1}\Pi_\alpha\right] \quad \text{and} \quad \det\left[\Pi_\alpha(I - H_{\mathbb{R}}(\hat{u}_{1/2}))^{-1}\Pi_\alpha\right].$$

The different sign in front of the Hankel operators comes from the fact that $\psi^{(1)}_{\alpha,n}$ has its singularity at $t = 1$, while $\psi^{(-1)}_{\alpha,n}$ has its singularity at $t = -1$. The proof of the separation of the singularities as well as the last step requires a couple of technical results, which we will establish in Sect. 4.3.

The proof of the asymptotic formula as it was outlined here will be given in Sect. 5.

4. Auxiliary Results

4.1. Invertibility of certain operators $I + H(\psi)$. In this section we are going to prove that for several concrete (piecewise continuous) functions ψ the operator $I + H(\psi)$ is invertible on $H^2(\mathbb{T})$. Actually, apart from one case, the results that we need have already been established in [3] or can easily be derived from there.

For $\tau \in \mathbb{T}$ and $\beta \in \mathbb{C}$, let us introduce the functions

$$u_{\beta,\tau}(e^{i\theta}) = \exp(i\beta(\theta - \theta_0 - \pi)), \quad 0 < \theta - \theta_0 < 2\pi, \quad \tau = e^{i\theta_0}. \tag{20}$$

These functions are continuous on $\mathbb{T} \setminus \{\tau\}$ and have a jump discontinuity at $t = \tau$ whose size is determined by β.

Proposition 4.1. *The following operators are invertible on $H^2(\mathbb{T})$:*

$$A_1 = I + H(u_{-1/2,1}), \qquad A_2 = I - H(u_{1/2,1}),$$
$$A_3 = I - H(u_{-1/2,-1}), \qquad A_4 = I + H(u_{1/2,-1}).$$

Proof. The invertibility of A_1 and A_2 follows from Thm. 3.6 in Sect. 3.2 of [3]. The invertibility of A_3 and A_4 can be obtained by remarking that $A_3 = WA_1W$ and $A_4 = WA_2W$, where W is the operator defined by $(Wf)(t) = f(-t)$, $t \in \mathbb{T}$. □

Next we introduce the function

$$\chi(e^{i\theta}) = \begin{cases} i & \text{if } 0 < \theta < \pi \\ -i & \text{if } -\pi < \theta < 0. \end{cases} \tag{21}$$

For later use, let us observe that

$$\chi(t) = u_{-1/2,1}(t)u_{1/2,-1}(t) = -\,u_{1/2,1}(t)u_{-1/2,-1}(t). \tag{22}$$

We denote by $\mathcal{W}$ the Wiener algebra, which consists of all functions in $L^1(\mathbb{T})$ whose Fourier series are absolutely convergent. Moreover, we introduce the Banach subalgebras

$$\mathcal{W}_\pm = \{\, a \in \mathcal{W} \;:\; a_n = 0 \text{ for all } \pm n < 0 \,\}, \tag{23}$$

where a_n stand for the Fourier coefficients of a. Functions in $\mathcal{W}_\pm$ can be continued by continuity to functions which are analytic in $\{z \in \mathbb{C} : |z| < 1\}$ and $\{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$, respectively. Notice that $a \in \mathcal{W}_+$ if and only if $\tilde{a} \in \mathcal{W}_-$, where $\tilde{a}(t) := a(t^{-1})$, $t \in \mathbb{T}$.

Finally, we will denote by $G\mathcal{W}$ and $G\mathcal{W}_\pm$ the group of invertible elements in the Banach algebras $\mathcal{W}$ and $\mathcal{W}_\pm$, respectively.

Proposition 4.2. *Let $c_+ \in G\mathcal{W}_+$ and $\psi(t) = \tilde{c}_+(t)c_+^{-1}(t)\chi(t)$. Then the operator $I + H(\psi)$ is invertible on $H^2(\mathbb{T})$.*

Proof. We first use a result of Power [13] in order to determine the essential spectrum of the Hankel operator $H(\psi)$. Recall that the essential spectrum $\mathrm{sp}_{\mathrm{ess}} A$ of a bounded linear operator A acting on a Banach space X is the set of all $\lambda \in \mathbb{C}$ for which $A - \lambda I$ is not a Fredholm operator. Also recall that A is a Fredholm operator on X if its image "im A" is a closed subspace of X and if both the kernel "ker A" and the quotient space "$X/\mathrm{im}\, A$" are finite dimensional. In this case the Fredholm index of A is defined as $\dim\ker A - \dim(X/\mathrm{im}\, A)$.

The result of Power [13] says that for a piecewise continuous function b the essential spectrum of $H(b)$ is a union of intervals in $\mathbb{C}$, namely,

$$\mathrm{sp}_{\mathrm{ess}} H(b) = [0, ib_{-1}] \cup [0, -ib_1] \cup \bigcup_{\tau \in \mathbb{T}_+} \left[-i\sqrt{b_\tau b_{\bar{\tau}}}, i\sqrt{b_\tau b_{\bar{\tau}}} \right].$$

Here we use the notation $b_\tau = (b(\tau+0) - b(\tau-0))/2$ with $b(\tau \pm 0) = \lim_{\varepsilon \to \pm 0} b(\tau e^{i\varepsilon})$, and $\mathbb{T}_+ := \{\tau \in \mathbb{T} \ : \ \operatorname{Im} \tau > 0\}$. This result can also be obtained from the more general results contained in [14] and [5, Sects. 4.95-4.102].

For our function ψ we obtain $\psi_1 = i$, $\psi_{-1} = -i$, and $\psi_\tau = 0$ for $\tau \in \mathbb{T} \setminus \{1, -1\}$. Hence $\mathrm{sp}_{\mathrm{ess}} H(\psi) = [0, 1]$, which implies that $I + H(\psi)$ is a Fredholm operator. Since $H(\psi) - \lambda I$ is an invertible operator for λ sufficiently large (hence a Fredholm operator with index zero), and since the Fredholm index is invariant with respect to small perturbations, it follows that $I + H(\psi)$ has Fredholm index zero. Thus it remains to prove that the kernel of $I + H(\psi)$ is trivial.

In order to prove this we introduce for $\tau \in \mathbb{T}$ and $\beta \in \mathbb{C}$ the functions

$$\eta_{\beta,\tau}(t) = (1 - t/\tau)^\beta, \qquad \xi_{\beta,\tau}(t) = (1 - \tau/t)^\beta,$$

where these functions are analytic in an open neighborhood of $\{ z \in \mathbb{C} \ : \ |z| \le 1, \ z \ne \tau\}$ and $\{ z \in \mathbb{C} \ : \ |z| \ge 1, \ z \ne \tau\} \cup \{\infty\}$, respectively, and the branch of the power function is chosen in such a way that $\eta_{\beta,\tau}(0) = 1$ and $\xi_{\beta,\tau}(\infty) = 1$. Notice that

$$u_{\beta,\tau}(t) = \eta_{\beta,\tau}(t)\xi_{-\beta,\tau}(t), \qquad u_{\beta+n,\tau}(t) = (-t/\tau)^n u_{\beta,\tau}(t). \tag{24}$$

Finally, we introduce the Hardy space

$$\overline{H^2(\mathbb{T})} = \left\{ f \in L^2(\mathbb{T}) \ : \ f_k = 0 \text{ for all } k > 0 \right\}$$

and notice that $f \in H^2(\mathbb{T})$ if and only if $\tilde{f} \in \overline{H^2(\mathbb{T})}$.

Now suppose that $f_+ \in H^2(\mathbb{T})$ belongs to the kernel of $I + H(\psi)$. Then, by (7),

$$f_+(t) + \psi(t)t^{-1}\tilde{f}_+(t) =: f_-(t) \in t^{-1}\overline{H^2(\mathbb{T})}.$$

Using (22) and (24) we can write

$$\chi(t) = -t^{-1}u_{1/2,1}(t)u_{1/2,-1}(t) = -t^{-1}\xi_{-1/2,1}(t)\xi_{-1/2,-1}(t)\eta_{1/2,1}(t)\eta_{1/2,-1}(t),$$

and hence we obtain

$$\begin{aligned} f_0(t) &:= t\tilde{c}_+^{-1}(t)\xi_{1/2,1}(t)\xi_{1/2,-1}(t)f_+(t) - t^{-1}c_+^{-1}(t)\eta_{1/2,1}(t)\eta_{1/2,-1}(t)\tilde{f}_+(t) \\ &= t\tilde{c}_+^{-1}(t)\xi_{1/2,1}(t)\xi_{1/2,-1}(t)f_-(t). \end{aligned}$$

It is easy to see that $f_0 = -\tilde{f}_0$ and that $f_0 \in \overline{H^2(\mathbb{T})}$. Hence $f_0 = 0$, and thus

$$f_+(t) + \psi(t)t^{-1}\tilde{f}_+(t) = 0.$$

Now use again (22) and (24) to write

$$\chi(t) = tu_{-1/2,1}(t)u_{-1/2,-1}(t) = t\xi_{1/2,1}(t)\xi_{1/2,-1}(t)\eta_{-1/2,1}(t)\eta_{-1/2,-1}(t),$$

and it follows

$$c_+(t)\eta_{1/2,1}(t)\eta_{1/2,-1}(t)f_+(t) = -\tilde{c}_+(t)\xi_{1/2,1}(t)\xi_{1/2,-1}(t)\tilde{f}_+(t).$$

Therein the left-hand side belongs to $\overline{H^2(\mathbb{T})}$, whereas the right hand side belongs to $H^2(\mathbb{T})$. It follows that this expression is zero. Hence $f_+ = 0$. Thus we have proved that the kernel of $I + H(\psi)$ is trivial. □

4.2. A formula for Hankel determinants. The goal of this section is to prove the following formula:

$$\det H_n[b] = G^n \det\Big[P_n(I + H(\psi))^{-1} P_n\Big],$$

where $b \in L^1[-1, 1]$ is a (sufficiently smooth) continuous and nonvanishing function. Therein G is a constant and ψ is a function (both depending on b). This formula will allow us later to reduce a Hankel determinant to the type of determinant appearing on the right-hand side.

To start with let us recall some basic notions. A function $a \in \mathcal{W}$ is said to admit a canonical Wiener-Hopf factorization in $\mathcal{W}$ if it can be represented in the form

$$a(t) = a_-(t)a_+(t), \qquad t \in \mathbb{T}, \tag{25}$$

where $a_\pm \in G\mathcal{W}_\pm$. It is well known (see, e.g., [5]) that $a \in \mathcal{W}$ admits a canonical Wiener-Hopf factorization in $\mathcal{W}$ if and only if $a \in G\mathcal{W}$ and if the winding number of a is zero. This, in turn, is equivalent to the condition that a possesses a logarithm $\log a \in \mathcal{W}$. Clearly, one can then define the geometric mean

$$G[a] := \exp\Big(\frac{1}{2\pi}\int_0^{2\pi} \log a(e^{i\theta})\,d\theta\Big). \tag{26}$$

Notice that this definition does not depend on the particular choice of the logarithm.

Next we are going to cite two results. The first result is from [3, Prop. 3.9]. Recall that a function a defined on $\mathbb{T}$ is called even if $a = \tilde{a}$, where $\tilde{a}(t) = a(1/t)$.

Proposition 4.3. *Let $a \in G\mathcal{W}$ be an even function which possesses a canonical Wiener-Hopf factorization $a(t) = a_-(t)a_+(t)$. Define $\psi(t) = \tilde{a}_+(t)a_+^{-1}(t)$. Then $I + H(\psi)$ is invertible on $H^2(\mathbb{T})$ and*

$$\det\Big[T_n(a) + H_n(a)\Big] = G[a]^n \det\Big[P_n(I + H(\psi))^{-1} P_n\Big]. \tag{27}$$

The second result is from [2, Thm. 2.3].

Proposition 4.4. *Let $a \in L^1(\mathbb{T})$ be an even function, and let $b \in L^1[-1, 1]$ be given by*

$$b(\cos\theta) = a(e^{i\theta})\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}. \tag{28}$$

Then $\det\Big[T_n(a) + H_n(a)\Big] = \det H_n[b]$.

We remark in this connection that under the assumption (28) we have $b \in L^1[-1, 1]$ if and only if $a(e^{i\theta})(1 + \cos\theta) \in L^1(\mathbb{T})$.

Combining both results gives immediately the following theorem.

Theorem 4.5. *Let $a \in G\mathcal{W}$ be an even function which possesses a Wiener-Hopf factorization $a(t) = a_-(t)a_+(t)$. Define $\psi(t) = \tilde{a}_+(t)a_+^{-1}(t)$, and define $b \in L^1(\mathbb{T})$ by (28). Then*

$$\det H_n[b] = G[a]^n \det\Big[P_n(I + H(\psi))^{-1} P_n\Big]. \tag{29}$$

This result is, however, not yet what we need. As pointed out above we want to derive a formula in which b is a continuous and nonvanishing function on $[-1, 1]$. But in the previous theorem, the function $b(x)$ has necessarily a zero at $x = -1$ and a pole at $x = 1$ (both of order $1/2$).

The desired result reads as follows. It involves the function χ defined in (21). Notice that the operator $I + H(\psi)$ is invertible by Proposition 4.2.

Theorem 4.6. *Let $c \in GW$ be an even function which possesses a canonical Wiener-Hopf factorization $c(t) = c_-(t)c_+(t)$. Define $\psi(t) = \tilde{c}_+(t)c_+^{-1}(t)\chi(t)$ and $b(\cos\theta) = c(e^{i\theta})$. Then*

$$\det H_n[b] = G[c]^n \det\Big[P_n(I + H(\psi))^{-1}P_n\Big].$$

It is interesting to observe that this theorem follows formally from the above theorem with $a_+(t) = c_+(t)(1-t)^{-1/2}(1+t)^{1/2}$ and $a_-(t) = c_-(t)(1-t^{-1})^{-1/2}(1+t^{-1})^{1/2}$. Notice in this connection (22) and

$$u_{-1/2,1}(t) = (1-t^{-1})^{1/2}(1-t)^{-1/2}, \quad u_{1/2,-1}(t) = (1+t^{-1})^{-1/2}(1+t)^{1/2}. \qquad (30)$$

However, in order to make things precise, we have to use an approximation argument. This approximation leads us to a so-called stability problem, which is somewhat difficult to analyze. In fact, we are going to resort to results of [8] and we apply also Proposition 4.1.

In order to carry out the proof we need the following definitions and basic results. In what follows r is a number in $[0, 1)$, which is supposed to tend to 1. A (generalized) sequence of functions $a_r \in L^\infty(\mathbb{T})$ is said to converge to $a \in L^\infty(\mathbb{T})$ in measure if for each $\varepsilon > 0$ the Lebesgue measure of the set

$$\Big\{\, t \in \mathbb{T} \;:\; |a_r(t) - a(t)| \geq \varepsilon \,\Big\}$$

converges to zero as $r \to 1$. A (generalized) sequence of bounded linear operators A_r on a Banach space X is said to converge strongly on X to an operator A if $A_r x \to Ax$ as $r \to 1$ for all $x \in X$. The proof of the following lemma is straightforward (see also [3], Lemma 3.2).

Lemma 4.7. *Assume that $a_r \in L^\infty(\mathbb{T})$ are uniformly bounded and converge to $a \in L^\infty(\mathbb{T})$ in measure. Then*

$$T(a_r) \to T(a) \quad \text{and} \quad H(a_r) \to H(a)$$

strongly on $H^2(\mathbb{T})$, and the same holds for the adjoints.

A sequence $\{A_r\}_{r\in[0,1)}$ of operators on X is called stable if there exists an $r_0 \in [0, 1)$ such that for all $r \in [r_0, 1)$ the operators A_r are invertible and $\sup_{r\in[r_0,1)} \|A_r^{-1}\|_{\mathcal{L}(X)} < \infty$.

Strong convergence of the inverses and stability are related by the following (basic) result.

Lemma 4.8. *Suppose that $A_r \to A$ strongly on X as $r \to 1$ and that A is invertible. Then $A_r^{-1} \to A^{-1}$ strongly on X as $r \to 1$ if and only if $\{A_r\}_{r\in[0,1)}$ is stable.*

Proof. One part follows from the Banach-Steinhaus Theorem, while the other part follows easily from the identity $(A_r^{-1} - A^{-1})y = A_r^{-1}(A - A_r)A^{-1}y$. □

Finally, for $r \in [0, 1)$ and $\tau \in \mathbb{T}$ we introduce the following operators $G_{r,\tau}$ acting on $L^\infty(\mathbb{T})$:

$$G_{r,\tau} : a(t) \mapsto b(t) = a\left(\tau \frac{t+r}{1+rt}\right). \tag{31}$$

Figuratively speaking, the function a is first stretched at τ and squeezed at $-\tau$, and then rotated on the unit circle such that τ is mapped into 1. It might also be illustrative to remark that for fixed $\tau, t \in \mathbb{T}$, $t \neq -1$, the sequence $(G_{r,\tau}a)(t)$ converges to $a(\tau)$ as $r \to 1$ if a is a continuous function.

Now we are ready to give the proof of Theorem 4.6.

Proof of Theorem 4.6. For $r \in [0, 1)$ introduce the even function

$$a_r(t) = c(t)\sqrt{\frac{(1-rt)(1-rt^{-1})}{(1+rt)(1+rt^{-1})}}, \qquad t \in \mathbb{T}.$$

The function b_r corresponding to a_r by means of (28) is then given by

$$b_r(x) = b(x)\sqrt{\frac{2+2x}{1+r^2+2rx}}\sqrt{\frac{1+r^2-2rx}{2-2x}}, \qquad x \in (-1, 1).$$

It is easy to verify that $b_r \to b$ in the norm of $L^1[-1, 1]$. Hence (for each fixed n)

$$H_n[b] = \lim_{r\to 1} H_n[b_r].$$

The canonical Wiener-Hopf factorization of a_r is given by $a_r(t) = a_{r,-}(t)a_{r,+}(t)$ with

$$a_{r,-}(t) = c_-(t)\frac{(1-rt^{-1})^{1/2}}{(1+rt^{-1})^{1/2}}, \qquad a_{r,+}(t) = c_+(t)\frac{(1-rt)^{1/2}}{(1+rt)^{1/2}}.$$

Upon putting

$$\psi_r(t) = \tilde{a}_{r,+}(t)a_{r,+}^{-1}(t) = \tilde{c}_+(t)c_+^{-1}(t)\left(\frac{1-rt}{1-rt^{-1}}\right)^{-1/2}\left(\frac{1+rt}{1+rt^{-1}}\right)^{1/2},$$

we conclude from Theorem 4.5 and from the fact that $G[a] = G[c]$ that

$$\det H_n[b_r] = G[c]^n \det\Big[P_n(I + H(\psi_r))^{-1}P_n\Big].$$

Hence (for n fixed)

$$\det H_n[b] = G[c]^n \lim_{r\to 1} \det\Big[P_n(I + H(\psi_r))^{-1}P_n\Big].$$

Taking account of (30) it easy to see that

$$f_r^{\pm} := \left(\frac{1 \mp rt}{1 \mp rt^{-1}}\right)^{\mp 1/2} \to u_{\mp 1/2, \pm 1} \tag{32}$$

in measure as $r \to 1$. From (22) we thus obtain that $\psi_r \to \psi$ in measure. Moreover, since $|f_r^{\pm}(t)| = 1$, Lemma 4.7 implies that $H(\psi_r) \to H(\psi)$ strongly on $H^2(\mathbb{T})$ as $r \to 1$.

We want to conclude that

$$(I + H(\psi_r))^{-1} \to (I + H(\psi))^{-1} \tag{33}$$

strongly on $H^2(\mathbb{T})$ as $r \to 1$. By Proposition 4.2 and Lemma 4.8 it is necessary and sufficient that the sequence of operators $\{I + H(\psi_r)\}_{r\in[0,1)}$ is stable.

In order to analyse this stability condition we apply the results of [8, Sects. 4.1–4.2]. These results establish the existence of certain mappings Φ_0 and Φ_τ, $\tau \in \mathbb{T}$, which are defined as

$$\Phi_0[\psi_r] := \mu\text{-}\lim_{r\to 1} \psi_r, \qquad \Phi_\tau[\psi_r] := \mu\text{-}\lim_{r\to 1} G_{r,\tau}\psi_r,$$

where μ- lim stands for the limit in measure. Because of (32) we have

$$\Phi_0[f_r^{\pm}] := \mu\text{-}\lim_{r\to 1} f_r^{\pm} = u_{\mp 1/2,\pm 1}.$$

Furthermore since $f_r^{\pm} \to u_{\mp 1/2,\pm 1}$ locally uniformly on $\mathbb{T} \setminus \{\pm 1\}$, we have

$$\Phi_\tau[f_r^{\pm}] := \mu\text{-}\lim_{r\to 1} G_{r,\tau} f_r^{\pm} = u_{\mp 1/2,\pm 1}(\tau)$$

for $\tau \neq \pm 1$. Finally,

$$\Phi_{\pm 1}[f_r^{\pm}] := \mu\text{-}\lim_{r\to 1} G_{r,\pm 1} f_r^{\pm} = \mu\text{-}\lim_{r\to 1} \left(\frac{1+rt}{1+rt^{-1}}\right)^{\pm 1/2} = u_{\pm 1/2,-1}.$$

Since $\psi_r = \tilde{c}_+ c_+^{-1} f_r^+ f_r^-$ we conclude

$$\begin{aligned}
\Phi_0[\psi_r] &= \tilde{c}_+ c_+^{-1} u_{-1/2,1} u_{1/2,-1} = \psi,\\
\Phi_1[\psi_r] &= u_{1/2,-1},\\
\Phi_{-1}[\psi_r] &= u_{-1/2,-1},\\
\Phi_\tau[\psi_r] &= \text{constant function}, \qquad \tau \in \mathbb{T} \setminus \{-1, 1\}.
\end{aligned}$$

The stability criterion in [8] (Thm. 4.2 and Thm. 4.3) says that $I + H(\psi_r)$ is stable if and only if the operators

(i) $\Psi_0[I + H(\psi_r)] = I + H(\Phi_0[\psi_r]) = I + H(\psi)$,
(ii) $\Psi_1[I + H(\psi_r)] = I + H(\Phi_1[\psi_r]) = I + H(u_{1/2,-1})$,
(iii) $\Psi_{-1}[I + H(\psi_r)] = I - H(\Phi_{-1}[\psi_r]) = I - H(u_{-1/2,-1})$,
(iv) $\Psi_\tau[I + H(\psi_r)] =$

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} M(\Phi_\tau[\psi_r]) & 0 \\ 0 & \widetilde{M(\Phi_{\bar{\tau}}[\psi_r])} \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$(\tau \in \mathbb{T}, \mathrm{Im}(\tau) > 0)$

are invertible. Clearly, by Proposition 4.1 and Proposition 4.2 this is the case. Hence the sequence $I + H(\psi_r)$ is stable and (33) follows. We conclude that (for fixed n) the $n \times n$ matrices $P_n(I + H(\psi_r))^{-1} P_n$ converge to $P_n(I + H(\psi))^{-1} P_n$ as $r \to 1$, whence it follows that the corresponding determinants converge, too. This completes the proof. □

4.3. Convergence in trace class norm. In this section we are going to prove a couple of technical results. We are mainly concerned with proving that certain sequences of operators converge in the trace norm.

Let $PC_{\pm 1}^{\mathrm{abs}}$ stand for the set of all functions on $\mathbb{T}$ which are absolutely continuous on $\mathbb{T} \setminus \{-1, 1\}$ and which possess one-sided limits at $t = 1$ and $t = -1$.

Lemma 4.9. *Let $a \in C(\mathbb{T})$ be a function such that $a' \in PC_{\pm 1}^{\mathrm{abs}}$. Then $H(a)$ is a trace class operator on $H^2(\mathbb{T})$ and*

$$\|H(a)\|_1 \leq C\Big(\|a\|_{L^\infty(\mathbb{T})} + \|a'\|_{L^\infty(\mathbb{T})} + \|a''\|_{L^1(\mathbb{T})}\Big). \tag{34}$$

Proof. From partial integration it follows that the Fourier coefficients a_k are $O(k^{-2})$ as $k \to \infty$, where the constant involved in this estimate is given in terms of the norms of a, a' and a''. We write the operator $H(a)$ as a product AB with operators A and B given by its matrix representation with respect to the standard basis by

$$A = \Big(a_{j+k+1}(1+k)^{1/2+\varepsilon}\Big)_{j,k=0}^{\infty}, \qquad B = \operatorname{diag}\Big((1+k)^{-1/2-\varepsilon}\Big)_{k=0}^{\infty}.$$

Both A and B are Hilbert-Schmidt operators if $0 < \varepsilon < 1/2$ with straightforward estimates for their norms. Hence $H(a)$ is a trace class operator, whose norm can be estimated by (34). □

Before proceeding, let us recall that Toeplitz and Hankel operators $T(a)$ and $H(a)$ satisfy the following well-known formulas:

$$T(ab) = T(a)T(b) + H(a)H(\tilde{b}), \tag{35}$$

$$H(ab) = T(a)H(b) + H(a)T(\tilde{b}), \tag{36}$$

where $a, b \in L^\infty(\mathbb{T})$ and, as before, where $\tilde{b}(t) := b(t^{-1})$.

In regard to the following proposition recall the definition (31) of the operators $G_{r,\tau}$ acting on $L^\infty(\mathbb{T})$. Their inverse are given by

$$G_{r,\tau}^{-1} : b(t) \mapsto a(t) = b\left(\frac{t\tau^{-1} - r}{1 - rt\tau^{-1}}\right). \tag{37}$$

The proof of the following proposition is very technical, which is due to the fact that we have to prove convergence in the trace norm. It might be helpful to remark the proof of convergence in the operator norm would essentially require only the first two paragraphs of the proof. (The convergence of the derivatives appearing therein would not be necessary.)

Proposition 4.10. *Let*

$$\psi_\mu^{(1)} = G_{\mu,1}^{-1}(u_{-1/2,1} - 1), \qquad \psi_\mu^{(-1)} = G_{\mu,-1}^{-1}(u_{1/2,1} - 1) \tag{38}$$

with $\mu \in [0, 1)$. Then the operators

$$H(\psi_\mu^{(1)})H(\psi_\mu^{(-1)}), \qquad H(\psi_\mu^{(-1)})H(\psi_\mu^{(1)}), \quad \text{and} \quad H(\psi_\mu^{(1)}\psi_\mu^{(-1)})$$

are trace class operators and converge to zero in the trace norm as $\mu \to 1$.

Proof. Let us first notice that (with the proper choice of the square-root),

$$\psi_\mu^{(1)}(t) = \left(-\frac{t-\mu}{1-\mu t}\right)^{-1/2} - 1, \qquad \psi_\mu^{(-1)}(t) = \left(\frac{t+\mu}{1+\mu t}\right)^{1/2} - 1. \tag{39}$$

In particular, $\psi_\mu^{(1)}$ has a jump discontinuity at $t = 1$ and vanishes at $t = -1$ while $\psi_\mu^{(-1)}$ has a jump discontinuity at $t = -1$ and vanishes at $t = 1$. Moreover, both functions are uniformly bounded and

$$\psi_\mu^{(1)} \to 0, \qquad \psi_\mu^{(-1)} \to 0, \tag{40}$$

uniformly on each compact subset of $\mathbb{T} \setminus \{1\}$ and $\mathbb{T} \setminus \{-1\}$, respectively.

In order to prove the assertion for the operator $H(\psi_\mu^{(1)})H(\psi_\mu^{(-1)})$, let f and g be smooth functions on $\mathbb{T}$ with $f + g = 1$ such that $f(t)$ vanishes identically in a neighborhood of 1 (say for $|\arg t| \le \pi/3$) and $g(t)$ vanishes identically in a neighborhood of -1 (say for $|\arg t| \ge 2\pi/3$). Then (see (36))

$$\begin{aligned} H(\psi_\mu^{(1)})H(\psi_\mu^{(-1)}) &= H(\psi_\mu^{(1)})T(f)H(\psi_\mu^{(-1)}) + H(\psi_\mu^{(1)})T(g)H(\psi_\mu^{(-1)}) \\ &= H(\psi_\mu^{(1)}\tilde{f})H(\psi_\mu^{(-1)}) - T(\psi_\mu^{(1)})H(\tilde{f})H(\psi_\mu^{(-1)}) \\ &\quad + H(\psi_\mu^{(1)})H(g\psi_\mu^{(-1)}) - H(\psi_\mu^{(1)})H(g)T(\widetilde{\psi_\mu^{(-1)}}). \end{aligned}$$

Clearly, $H(\tilde{f})$ and $H(g)$ are trace class operators. Due to the afore-mentioned fact that $\psi_\mu^{(1)}$ and $\psi_\mu^{(-1)}$ are uniformly bounded and because of the convergence (40), Lemma 4.7 implies that the operators

$$H(\psi_\mu^{(1)}), \quad T(\psi_\mu^{(1)}), \quad H(\psi_\mu^{(-1)}), \quad T(\widetilde{\psi_\mu^{(1)}})$$

and their adjoints converge strongly to zero as $\mu \to 1$. We can conclude that $H(\psi_\mu^1)$ $H(\psi_\mu^{(-1)})$ is a trace class operator and converges in the trace norm to zero as soon as we have shown that

$$H(\psi_\mu^{(1)}\tilde{f}) \quad \text{and} \quad H(g\psi_\mu^{(-1)})$$

are trace class operators which converge to zero in the trace norm.

On account of Lemma 4.9 this is true if

$$\psi_\mu^{(1)}\tilde{f} \in C(\mathbb{T}), \quad (\psi_\mu^{(1)}\tilde{f})' \in PC_{\pm 1}^{\text{abs}},$$

if

$$\|\psi_\mu^{(1)}\tilde{f}\|_{L^\infty} \to 0, \quad \|(\psi_\mu^{(1)}\tilde{f})'\|_{L^\infty} \to 0, \quad \|(\psi_\mu^{(1)}\tilde{f})''\|_{L^1} \to 0,$$

and if similar statements hold for $g\psi_\mu^{(-1)}$. Due to the fact that f vanishes on a neighborhood of 1, these conditions are fulfilled if

$$\psi_\mu^{(1)}|_{\mathbb{T}_{-1}} \in C(\mathbb{T}_{-1}), \quad (\psi_\mu^{(1)})'|_{\mathbb{T}_{-1}} \in C(\mathbb{T}_{-1}), \quad (\psi_\mu^{(1)})''|_{\mathbb{T}_{-1}} \in C(\mathbb{T}_{-1}), \tag{41}$$

and if

$$\begin{gathered} \|\psi_\mu^{(1)}|_{\mathbb{T}_{-1}}\|_{L^\infty(\mathbb{T}_{-1})} \to 0, \quad \|(\psi_\mu^{(1)})'|_{\mathbb{T}_{-1}}\|_{L^\infty(\mathbb{T}_{-1})} \to 0, \\ \|(\psi_\mu^{(1)})''|_{\mathbb{T}_{-1}}\|_{L^1(\mathbb{T}_{-1})} \to 0. \end{gathered} \tag{42}$$

Here we have restricted the function $\psi_\mu^{(1)}$ to the arc $\mathbb{T}_{-1} := \{t \in \mathbb{T} \ : \ |\arg t| \geq \pi/4\}$. The corresponding (sufficient) conditions for the function $\psi_\mu^{(-1)}$ are

$$\psi_\mu^{(-1)}|_{\mathbb{T}_1} \in C(\mathbb{T}_1), \quad (\psi_\mu^{(-1)})'|_{\mathbb{T}_1} \in C(\mathbb{T}_1), \quad (\psi_\mu^{(-1)})''|_{\mathbb{T}_1} \in C(\mathbb{T}_1), \tag{43}$$

and

$$\begin{gathered} \|\psi_\mu^{(-1)}|_{\mathbb{T}_1}\|_{L^\infty(\mathbb{T}_1)} \to 0, \quad \|(\psi_\mu^{(-1)})'|_{\mathbb{T}_1}\|_{L^\infty(\mathbb{T}_1)} \to 0, \\ \|(\psi_\mu^{(-1)})''|_{\mathbb{T}_1}\|_{L^1(\mathbb{T}_1)} \to 0, \end{gathered} \tag{44}$$

where $\mathbb{T}_1 := \{t \in \mathbb{T} \ : \ |\arg t| \leq 3\pi/4\}$. It is easy to see that conditions (41) and (43) and also the first condition in (42) and in (44) are fulfilled.

We will prove the remaining conditions in a few moments, but first we will turn to the convergence of the operators $H(\psi_\mu^{(-1)})H(\psi_\mu^{(1)})$ and $H(\psi_\mu^{(1)}\psi_\mu^{(-1)})$. In regard to the operator $H(\psi_\mu^{(-1)})H(\psi_\mu^{(1)})$ we can proceed analogously and it turns out that we arrive at the same sufficient conditions (41)–(44).

As to the operator $H(\psi_\mu^{(1)}\psi_\mu^{(-1)})$ we have to show (on account of Lemma 4.9) that

$$\psi_\mu^{(1)}\psi_\mu^{(-1)} \in C(\mathbb{T}), \quad (\psi_\mu^{(1)}\psi_\mu^{(-1)})' \in PC_{\pm 1}^{\text{abs}} \tag{45}$$

and that

$$\|\psi_\mu^{(1)}\psi_\mu^{(-1)}\|_{L^\infty} \to 0, \quad \|(\psi_\mu^{(1)}\psi_\mu^{(-1)})'\|_{L^\infty} \to 0, \quad \|(\psi_\mu^{(1)}\psi_\mu^{(-1)})''\|_{L^1} \to 0. \tag{46}$$

From the facts stated at the beginning of the proof it follows that $\psi_\mu^{(1)}\psi_\mu^{(-1)}$ is continuous on $\mathbb{T}$ and that $\psi_\mu^{(1)}\psi_\mu^{(-1)}$ converges uniformly to zero on $\mathbb{T}$. Moreover, since the functions $\psi_\mu^{(\pm 1)}$ and their derivatives belong to $PC_{\pm 1}^{\text{abs}}$, it follows that the derivative of $\psi_\mu^{(1)}\psi_\mu^{(-1)}$ belongs to $PC_{\pm 1}^{\text{abs}}$, too. Thus we are left with the proof of the second and third condition in (46). We will prove these assertions by separating the singularities at $t = 1$ and $t = -1$:

$$\|(\psi_\mu^{(1)}\psi_\mu^{(-1)})'|_{\mathbb{T}_{-1}}\|_{L^\infty(\mathbb{T}_{-1})} \to 0, \quad \|(\psi_\mu^{(1)}\psi_\mu^{(-1)})''|_{\mathbb{T}_{-1}}\|_{L^1(\mathbb{T}_{-1})} \to 0, \tag{47}$$

and

$$\|(\psi_\mu^{(1)}\psi_\mu^{(-1)})'|_{\mathbb{T}_1}\|_{L^\infty(\mathbb{T}_1)} \to 0, \quad \|(\psi_\mu^{(1)}\psi_\mu^{(-1)})''|_{\mathbb{T}_1}\|_{L^1(\mathbb{T}_1)} \to 0. \tag{48}$$

Now turning back to the proof of the yet outstanding conditions in (42) and (44), we remark that the interval $\mathbb{T}_{-1}$ can be transformed into the interval $\mathbb{T}_1$ by a rotation $t \mapsto -t$. This will not precisely transform the function $\psi_\mu^{(1)}$ into the function $\psi_\mu^{(-1)}$, but into a similar function of the form (39), where only the power $1/2$ is replaced by $-1/2$. Without loss of generality we can thus confine ourselves to the proof of the conditions involving the interval $\mathbb{T}_1$, since the conditions involving the interval $\mathbb{T}_{-1}$ can be reduced to an analogous situation and can be proved in the same way.

In order to prove (48) and the last two conditions in (44) we use the linear fractional transformation

$$\sigma(x) = \frac{1 + ix}{1 - ix},$$

which maps the extended real line onto the unit circle. Clearly, $\mathbb{T}_1$ corresponds to $\sigma^{-1}(\mathbb{T}_1) = [-1-\sqrt{2}, 1+\sqrt{2}] =: I_0$. We transform the functions into

$$v_\varepsilon(x) = \psi_\mu^{(1)}(\sigma(x)), \qquad w_\varepsilon(x) = \psi_\mu^{(-1)}(\sigma(x)),$$

and we also change the parameter $\mu \in [0,1)$ into $\varepsilon = \frac{1-\mu}{1+\mu} \in (0,1]$. The conditions which we have to prove are then equivalent to

$$\|(w_\varepsilon)'|_{I_0}\|_{L^\infty(I_0)} \to 0, \qquad \|(w_\varepsilon)''|_{I_0}\|_{L^1(I_0)} \to 0 \tag{49}$$

and

$$\|(v_\varepsilon w_\varepsilon)'|_{I_0}\|_{L^\infty(I_0)} \to 0, \qquad \|(v_\varepsilon w_\varepsilon)''|_{I_0}\|_{L^1(I_0)} \to 0 \tag{50}$$

as $\varepsilon \to 0$. Introduce the functions

$$v(x) = \left(\frac{x-i}{x+i}\right)^{-1/2} - 1, \qquad w(x) = \left(\frac{1+ix}{1-ix}\right)^{1/2} - 1,$$

where v has a jump at $x = 0$ and the square-root is chosen such that $v(\pm\infty) = 0$. The function w is continuous on $\mathbb{R}$ with $w(0) = 0$ and limits at $x \to \pm\infty$. A straightforward computation implies that $v_\varepsilon(x) = v(x/\varepsilon)$ and $w_\varepsilon(x) = w(x\varepsilon)$.

The functions v and w and all of their derivatives are bounded on $\mathbb{R}$. Thus the conditions in (49) follow easily. The function w can be written as $w(x) = x\tilde{w}(x)$, where $\tilde{w}$ is a function which is locally bounded. We write

$$(v_\varepsilon w_\varepsilon)' = v'(x/\varepsilon)x\tilde{w}(\varepsilon x) + v(x/\varepsilon)\varepsilon w'(\varepsilon x),$$

and see immediately that the second term goes uniformly to zero. Moreover, $v'(x) \to 0$ as $|x| \to \infty$. Hence $xv'(x/\varepsilon)$ converges uniformly on I_0 to zero, which implies that the first term converges uniformly on I_0 to zero. Thus we have proved that $(v_\varepsilon w_\varepsilon)'$ converges uniformly on I_0 to zero as $\varepsilon \to 0$.

Finally, we write the second derivative as

$$(v_\varepsilon w_\varepsilon)'' = \varepsilon^{-1}v''(x/\varepsilon)x\tilde{w}(\varepsilon x) + 2v'(x/\varepsilon)w'(\varepsilon x) + \varepsilon^2 v(x/\varepsilon)w''(\varepsilon x). \tag{51}$$

The $L^1(I_0)$-norm of the first term can be estimated by a constant times

$$\int_{I_0} |v''(x/\varepsilon)x/\varepsilon|dx \le \varepsilon \int_{\mathbb{R}} |xv''(x)|\, dx,$$

which converges to zero. The $L^1(I_0)$-norm of the second term can be estimated by a constant times

$$\int_{I_0} |v'(x/\varepsilon)|dx \le \varepsilon \int_{\mathbb{R}} |v'(x)|\, dx$$

and also converges to zero. The last term converges to zero even uniformly. Hence we have proved the conditions (50) and the proof is complete. □

In addition to the operators $G_{\mu,\tau}$ we introduce operators

$$R_{\mu,\tau} : f \in H^2(\mathbb{T}) \mapsto g(t) = \frac{\sqrt{1-\mu^2}}{1+\mu t} G_{\mu,\tau}(f) \in H^2(\mathbb{T}), \tag{52}$$

where $\mu \in [0,1)$ and $\tau \in \{-1,1\}$.

Lemma 4.11. *For each* $\tau \in \{-1, 1\}$, *the operator* $R_{\mu,\tau}$ *is unitary on* $H^2(\mathbb{T})$. *Moreover,* $R_{\mu,\tau} H(a) R^*_{\mu,\tau} = \tau H(G_{\mu,\tau} a)$ *for all* $a \in L^\infty(\mathbb{T})$.

Proof. We can define the operators $R_{\mu,\tau}$ also on $L^2(\mathbb{T})$. In [8, Sect. 5.1] it is proved that $R_{\mu,\tau}$ are unitary on $L^2(\mathbb{T})$ and that

$$R_{\mu,\tau} P R^*_{\mu,\tau} = P, \quad R_{\mu,\tau} M(a) R^*_{\mu,\tau} = M(G^{-1}_{\mu,\tau} a), \quad R_{\mu,\tau} J R^*_{\mu,\tau} = \tau J.$$

These statements imply the desired assertions. □

In connection with the following proposition recall that the operators $I + H(u_{-1/2,1})$ and $I - H(u_{1/2,1})$ are invertible on $H^2(\mathbb{T})$ (see Proposition 4.1).

Moreover, for $\alpha > 0$ and $n \in \mathbb{N}$ define the functions

$$h_\alpha(t) = \exp\left(-\frac{\alpha(1-t)}{2(1+t)}\right), \tag{53}$$

$$h_{\alpha,n}(t) = \left(\frac{t + \mu_{\alpha,n}}{1 + \mu_{\alpha,n} t}\right)^n, \tag{54}$$

where $\mu_{\alpha,n} \in [0, 1)$ is any sequence. (This sequence will be specified later on.) Finally, introduce the functions

$$\psi^{(1)}_{\alpha,n} = G^{-1}_{\mu_{\alpha,n},1}(u_{-1/2,1} - 1), \qquad \psi^{(-1)}_{\alpha,n} = G^{-1}_{\mu_{\alpha,n},-1}(u_{1/2,1} - 1). \tag{55}$$

Proposition 4.12. *Let* $\alpha > 0$ *be fixed, and consider (53), (54), and (55). Assume that*

$$\mu_{\alpha,n} = 1 - \frac{\alpha}{2n} + O(n^{-2}), \quad \text{as } n \to \infty. \tag{56}$$

Then the following is true:

(i) The operators $H(\psi^{(1)}_{\alpha,n})$ *and* $H(\psi^{(-1)}_{\alpha,n})$ *are unitarily equivalent to the operators* $H(u_{-1/2,1})$ *and* $-H(u_{1/2,1})$, *respectively.*

(ii) The operators

$$P_n(I + H(\psi^{(1)}_{\alpha,n}))^{-1} P_n - P_n$$

are unitarily equivalent to the operators

$$A_n = H(h_{\alpha,n})(I + H(u_{-1/2,1}))^{-1} H(h_{\alpha,n}) - H(h_{\alpha,n})^2,$$

which are trace class operators and converge as $n \to \infty$ *in the trace norm to*

$$A = H(h_\alpha)(I + H(u_{-1/2,1}))^{-1} H(h_\alpha) - H(h_\alpha)^2.$$

(iii) The operators

$$P_n(I + H(\psi^{(-1)}_{\alpha,n}))^{-1} P_n - P_n$$

are unitarily equivalent to the operators

$$B_n = H(h_{\alpha,n})(I - H(u_{1/2,1}))^{-1} H(h_{\alpha,n}) - H(h_{\alpha,n})^2,$$

which are trace class operators and converge as $n \to \infty$ *in the trace norm to*

$$B = H(h_\alpha)(I - H(u_{1/2,1}))^{-1} H(h_\alpha) - H(h_\alpha)^2.$$

Proof. (i) We employ Lemma 4.11 in order to conclude that

$$H(\psi^{(1)}_{\alpha,n}) = R^*_{\mu_{\alpha,n},1} H(u_{-1/2,1}) R_{\mu_{\alpha,n},1}, \qquad H(\psi^{(-1)}_{\alpha,n}) = -R^*_{\mu_{\alpha,n},-1} H(u_{1/2,1}) R_{\mu_{\alpha,n},-1}.$$

(ii) We first introduce the operator $W_n = H(t^n)$ and remark that $W_n^2 = P_n$ and $W_n P_n = P_n W_n = W_n$. It is easily seen that the operator $P_n(I + H(\psi^{(1)}_{\alpha,n}))^{-1} P_n - P_n$ is unitarily equivalent to the operator $W_n(I + H(\psi^{(1)}_{\alpha,n}))^{-1} W_n - W_n^2$ by means of the unitary and selfadjoint operator $W_n + (I - P_n)$.

Now we use the unitary equivalence established in (i) in connection with the fact that $R_{\mu_{\alpha,n},1} W_n R^*_{\mu_{\alpha,n},1} = R_{\mu_{\alpha,n},1} H(t^n) R^*_{\mu_{\alpha,n},1} = H(h_{\alpha,n})$ (see again Lemma 4.11). Notice that $h_{\alpha,n} = G_{\mu_{\alpha,n},1}(t^n)$. This implies the unitary equivalence to A_n.

In order to prove the convergence $A_n \to A$ in the trace norm we write

$$A_n = H(h_{\alpha,n})(I + H(u_{-1/2,1}))^{-1} H(u_{-1/2,1}) H(h_{\alpha,n}).$$

The function $h_{\alpha,n}$ is uniformly bounded and converges (along with all its derivatives) uniformly on each compact subset of $\mathbb{T} \setminus \{-1\}$ to the function h_α. Hence (by Lemma 4.7)

$$H(h_{\alpha,n}) \to H(h_\alpha), \qquad T(\widetilde{h_{\alpha,n}}) \to T(\widetilde{h_\alpha})$$

strongly on $H^2(\mathbb{T})$. The same holds for their adjoints.

Next we claim that all operators $H(u_{-1/2,1})H(h_{\alpha,n})$ are trace class operators and converge in the trace norm to $H(u_{-1/2,1})H(h_\alpha)$. To see this we choose two smooth functions f and g on $\mathbb{T}$ which vanish identically in a neighborhood of -1 and 1, respectively, such that $f + g = 1$. Then we decompose

$$\begin{aligned} H(u_{-1/2,1})H(h_{\alpha,n}) &= H(u_{-1/2,1})T(f)H(h_{\alpha,n}) + H(u_{-1/2,1})T(g)H(h_{\alpha,n}) \\ &= H(u_{-1/2,1})H(fh_{\alpha,n}) - H(u_{-1/2,1})H(f)T(\widetilde{h_{\alpha,n}}) \\ &\quad + H(u_{-1/2,1}\tilde{g})H(h_{\alpha,n}) - T(u_{-1/2,1})H(\tilde{g})H(h_{\alpha,n}). \end{aligned}$$

The Hankel operators $H(f)$ and $H(\tilde{g})$ are both trace class and so are the operators $H(fh_{\alpha,n})$ and $H(u_{-1/2,1}\tilde{g})$ since the generating functions are smooth.

Moreover, $fh_{\alpha,n} \to fh_\alpha$ uniformly and the same holds for the derivatives. Hence $H(fh_{\alpha,n}) \to H(fh_\alpha)$ in the trace norm by Lemma 4.9. Along with the strong convergence noted above, it follows that $H(u_{-1/2,1})H(h_{\alpha,n})$ converges in the trace norm to

$$\begin{aligned} &H(u_{-1/2,1})H(fh_\alpha) - H(u_{-1/2,1})H(f)T(\widetilde{h_\alpha}) + H(u_{-1/2,1}\tilde{g})H(h_\alpha) \\ &\quad - T(u_{-1/2,1})H(\tilde{g})H(h_\alpha), \end{aligned}$$

which is trace class and equal to $H(u_{-1/2,1})H(h_\alpha)$.

(iii) The proof of these assertions is analogous. The only (slight) difference is that $R_{\mu_{\alpha,n},-1} W_n R^*_{\mu_{\alpha,n},-1} = R_{\mu_{\alpha,n},-1} H(t^n) R^*_{\mu_{\alpha,n},-1} = (-1)^{n+1} H(h_{\alpha,n})$ as $G_{\mu_{\alpha,n},-1}(t^n) = (-1)^n h_{\alpha,n}$. The possibly different sign at this place does not effect the argument. □

5. Proof of the Asymptotic Formula

In this section we are going to prove the asymptotic formula (2).

Our first goal is to discretize the Wiener-Hopf operator $I - K_\alpha$, which will lead us to a Toeplitz operator. Here and in what follows χ_α stands for the characteristic function of the subarc $\{e^{i\theta} : \alpha < \theta < 2\pi - \alpha\}$ of $\mathbb{T}$.

Proposition 5.1. *For each $\alpha > 0$ we have*

$$\det(I - K_\alpha) = \lim_{n\to\infty} \det T_n(\chi_{\frac{\alpha}{n}}). \tag{57}$$

Proof. Recall that the operator K_α is the integral operator on $L^2[0,\alpha]$ with the kernel $K(x-y)$, where

$$K(x) = \frac{\sin x}{\pi x}.$$

Introduce the $n \times n$ matrices

$$A_n = \left[\frac{\alpha}{n} K\left(\frac{\alpha(j-k)}{n}\right)\right]_{j,k=0}^{n-1}, \quad B_n = \left[\frac{\alpha}{n}\int_0^1\int_0^1 K\left(\frac{\alpha(j-k+\xi-\eta)}{n}\right) d\xi\, d\eta\right]_{j,k=0}^{n-1}.$$

By the mean value theorem the entries of $A_n - B_n$ can be estimated uniformly by $O(n^{-2})$, whence it follows that the Hilbert-Schmidt norm of $A_n - B_n$ is $O(n^{-1})$. Since the Hilbert-Schmidt norm of the $n\times n$ identity matrix is $O(\sqrt{n})$, we obtain that the trace norm of $A_n - B_n$ is $O(1/\sqrt{n})$.

The Fourier coefficients of $1 - \chi_{\frac{\alpha}{n}}$ are

$$[1-\chi_{\frac{\alpha}{n}}]_k = \begin{cases} \dfrac{\alpha}{\pi n} & \text{if } k = 0 \\ \dfrac{\sin(\frac{k\alpha}{n})}{\pi k} & \text{if } k \neq 0. \end{cases}$$

Hence it follows that $T_n(\chi_{\frac{\alpha}{n}}) = I_n - A_n$. Introduce the isometry

$$U_{\alpha,n} : \{x_k\}_{k=0}^{n-1} \in \mathbb{C}^n \mapsto \sqrt{\frac{n}{\alpha}} \sum_{k=0}^{n-1} x_k \chi_{[\frac{\alpha k}{n}, \frac{\alpha(k+1)}{n}]} \in L^2[0,\alpha],$$

and remark that

$$U_{\alpha,n}^* : f \in L^2[0,\alpha] \mapsto \left\{\sqrt{\frac{n}{\alpha}} \int_0^\alpha f(x)\chi_{[\frac{\alpha k}{n}, \frac{\alpha(k+1)}{n}]}\, dx\right\}_{k=0}^{n-1} \in \mathbb{C}^n.$$

It can be verified straightforwardly that $U_{\alpha,n}^* K_\alpha U_{\alpha,n} = B_n$. Hence

$$\begin{aligned} \det(I - K_\alpha) &= \det(I_n - U_{n,\alpha}^* K_\alpha U_{n,\alpha}) = \det(I_n - B_n) \\ &\sim \det(I_n - A_n) = \det T_n(\chi_{\frac{\alpha}{n}}) \end{aligned}$$

as $n \to \infty$. This completes the proof. □

The following result has been established in [2, Cor. 2.5].

Proposition 5.2. *Let $b \in L^1[-1,1]$ and suppose that $b_0(x) = b_0(-x)$, where*

$$b_0(x) = b(x)\sqrt{\frac{1-x}{1+x}}.$$

Then $\det H_n[b] = \det T_n(d)$ *with* $d(e^{i\theta}) = b_0(\cos\frac{\theta}{2})$.

We use this result in order to reduce our Toeplitz determinant $\det T_n(\chi_{\frac{\alpha}{n}})$ to a Hankel determinant.

Proposition 5.3. *We have*

$$\det T_n(\chi_{\frac{\alpha}{n}}) = (\varrho_{\alpha,n})^{n^2} \det H_n[b_{\alpha,n}], \tag{58}$$

where

$$b_{\alpha,n}(x) = \sqrt{\frac{1+\varrho_{\alpha,n}x}{1-\varrho_{\alpha,n}x}}, \qquad \varrho_{\alpha,n} = \cos\left(\frac{\alpha}{2n}\right). \tag{59}$$

Proof. We apply Proposition 5.2 with $d(e^{i\theta}) = \chi_{\frac{\alpha}{n}}(e^{i\theta})$, $b_0(x) = \chi_{[-\varrho_{\alpha,n},\varrho_{\alpha,n}]}(x)$, and

$$b(x) = \sqrt{\frac{1+x}{1-x}}\chi_{[-\varrho_{\alpha,n},\varrho_{\alpha,n}]}(x).$$

It follows that $\det T_n(\chi_{\frac{\alpha}{n}}) = \det H_n[b]$. The entries of $H_n[b]$ are the moments $[b]_{1+j+k}$, $0 \le j,k \le n-1$. A simple computation gives

$$[b]_k = \frac{1}{\pi}\int_{-1}^{1} b(x)(2x)^{k-1}\,dx = \frac{(\varrho_{\alpha,n})^k}{\pi}\int_{-1}^{1}\sqrt{\frac{1+\varrho_{\alpha,n}y}{1-\varrho_{\alpha,n}y}}(2y)^{k-1}dy = (\varrho_{\alpha,n})^k[b_{\alpha,n}]_k.$$

Now we can pull out certain diagonal matrices from the left and the right of $H_n[b]$ to obtain the matrix $H_n[b_{\alpha,n}]$. The determinants of the diagonal matrices give the factor $(\varrho_{\alpha,n})^{n^2}$. □

In the following result we use the function

$$\psi_{\alpha,n}(t) = \left(\frac{1-\mu_{\alpha,n}t}{1-\mu_{\alpha,n}t^{-1}}\right)^{1/2}\left(\frac{1+\mu_{\alpha,n}t^{-1}}{1+\mu_{\alpha,n}t}\right)^{1/2}\chi(t), \tag{60}$$

where $\chi(t)$ is given by (22) and where

$$\mu_{\alpha,n} = \frac{1-\sqrt{1-\varrho_{\alpha,n}^2}}{\varrho_{\alpha,n}} \tag{61}$$

with $\varrho_{\alpha,n}$ given by (59). For later use remark that $\mu_{\alpha,n} \in [0,1)$ satisfies condition (56).

Proposition 5.4. *We have*

$$\lim_{n\to\infty}\det T_n(\chi_{\frac{\alpha}{n}}) = e^{-\frac{\alpha^2}{8}}\lim_{n\to\infty}\det\Big[P_n(I+H(\psi_{\alpha,n}))^{-1}P_n\Big]. \tag{62}$$

Proof. We use Proposition 5.3. Since $\varrho_{\alpha,n} = 1 - \frac{\alpha^2}{8n^2} + O(n^{-4})$ it is readily verified that $(\varrho_{\alpha,n})^{n^2} \to e^{-\frac{\alpha^2}{8}}$. We obtain

$$\lim_{n\to\infty} \det T_n(\chi_{\frac{\alpha}{n}}) = e^{-\frac{\alpha^2}{8}} \lim_{n\to\infty} \det H_n[b_{\alpha,n}].$$

Now we employ Theorem 4.6 with

$$c(e^{i\theta}) = \sqrt{\frac{1 + \varrho_{\alpha,n}\cos\theta}{1 - \varrho_{\alpha,n}\cos\theta}}.$$

Obviously, (since $\varrho_{\alpha,n} = 2\mu_{\alpha,n}/(1+\mu_{\alpha,n}^2)$)

$$c(t) = \sqrt{\frac{(1+\mu_{\alpha,n}t)(1+\mu_{\alpha,n}t^{-1})}{(1-\mu_{\alpha,n}t)(1-\mu_{\alpha,n}t^{-1})}},$$

whence we conclude that $G[c] = 1$ and that the canonical Wiener-Hopf factorization of c is given by $c(t) = c_-(t)c_+(t)$ with

$$c_-(t) = \left(\frac{1+\mu_{\alpha,n}t^{-1}}{1-\mu_{\alpha,n}t^{-1}}\right)^{1/2}, \qquad c_+(t) = \left(\frac{1+\mu_{\alpha,n}t}{1-\mu_{\alpha,n}t}\right)^{1/2}.$$

Furthermore,

$$\tilde{c}_+(t)c_+^{-1}(t) = \left(\frac{1-\mu_{\alpha,n}t}{1-\mu_{\alpha,n}t^{-1}}\right)^{1/2}\left(\frac{1+\mu_{\alpha,n}t^{-1}}{1+\mu_{\alpha,n}t}\right)^{1/2}.$$

It follows that

$$\det H_n[b_{\alpha,n}] = \det\Big[P_n(I + H(\psi_{\alpha,n}))^{-1}P_n\Big].$$

This implies the desired assertion. □

In the following proposition we identify the limit of the determinant

$$\det\Big[P_n(I + H(\psi_{\alpha,n}))^{-1}P_n\Big]$$

as $n \to \infty$. Recall the definitions (20), (53), and Proposition 4.1.

Proposition 5.5. *We have*

$$\begin{aligned}\lim_{n\to\infty} &\det\Big[P_n(I + H(\psi_{\alpha,n}))^{-1}P_n\Big] \\ &= \det\Big[H(h_\alpha)(I + H(u_{-1/2,1}))^{-1}H(h_\alpha)\Big]\det\Big[H(h_\alpha)(I - H(u_{1/2,1}))^{-1}H(h_\alpha)\Big],\end{aligned} \tag{63}$$

where all expressions on the right-hand side are well defined.

Proof. First of all we remark that the right-hand side is well defined. The inverses exist due to Proposition 4.1. Notice that $H(h_\alpha)^2$ is a projection operator. This can be seen as follows. Formulas (35) together with the fact that $\tilde{h}_\alpha = h_\alpha^{-1}$ imply that $I = T(h_\alpha)T(h_\alpha^{-1}) + H(h_\alpha)H(h_\alpha)$ and $0 = H(h_\alpha^{-1})T(h_\alpha^{-1}) + T(h_\alpha^{-1})H(h_\alpha) = T(h_\alpha^{-1})H(h_\alpha)$, whence

$$H(h_\alpha)^3 = (I - T(h_\alpha)T(h_\alpha^{-1}))H(h_\alpha) = H(h_\alpha).$$

We consider the operators $H(h_\alpha)(I \pm H(u_{\mp 1/2,1}))^{-1}H(h_\alpha)$ as being restricted onto the image of $H(h_\alpha)^2$. We can complement these operators with the projection $I - H(h_\alpha)^2$ without changing the value of the corresponding determinant,

$$\begin{aligned}\det&\Big[H(h_\alpha)(I \pm H(u_{\mp 1/2,1}))^{-1}H(h_\alpha)\Big]\\ &= \det\Big[I + H(h_\alpha)(I \pm H(u_{\mp 1/2,1}))^{-1}H(h_\alpha) - H(h_\alpha)^2\Big].\end{aligned}$$

By Proposition 4.12(ii)-(iii) we see that this last operator determinant is well-defined.

With $\mu = \mu_{\alpha,n}$ given by (61) we obtain from (22), (60) and (37) that

$$\psi_{\alpha,n}(t) = \left(\frac{t-\mu}{1-\mu t}\right)^{-1/2}\left(\frac{t+\mu}{1+\mu t}\right)^{1/2} = G_{\mu,1}^{-1}(u_{-1/2,1})G_{\mu,-1}^{-1}(u_{1/2,1}).$$

Introduce the functions $\psi_{\alpha,n}^{(\pm 1)}$ by (55). Then

$$\psi_{\alpha,n} = (\psi_{\alpha,n}^{(1)} + 1)(\psi_{\alpha,n}^{(-1)} + 1).$$

Proposition 4.10 implies that

$$H(\psi_{\alpha,n}) = H(\psi_{\alpha,n}^{(1)}) + H(\psi_{\alpha,n}^{(-1)}) + H(\psi_{\alpha,n}^{(-1)})H(\psi_{\alpha,n}^{(1)}) + o_1(1),$$

where $o_1(1)$ stands for a sequence of operators converging in the trace norm to zero as $n \to \infty$. By Proposition 4.12(i) and Proposition 4.1, the operators $I + H(\psi_{\alpha,n}^{(1)})$ and $I + H(\psi_{\alpha,n}^{(-1)})$ are invertible and their inverses are uniformly bounded. Hence

$$(I + H(\psi_{\alpha,n}))^{-1} = (I + H(\psi_{\alpha,n}^{(1)}))^{-1}(I + H(\psi_{\alpha,n}^{(-1)}))^{-1} + o_1(1).$$

Using the formula $(I + A)^{-1} = I - (I + A)^{-1}A = I - A(I + A)^{-1}$, we can write this as

$$\begin{aligned}(I + H(\psi_{\alpha,n}))^{-1} =& -I + (I + H(\psi_{\alpha,n}^{(1)}))^{-1} + (I + H(\psi_{\alpha,n}^{(-1)}))^{-1}\\ &+ (I+H(\psi_{\alpha,n}^{(1)}))^{-1}H(\psi_{\alpha,n}^{(1)})H(\psi_{\alpha,n}^{(-1)})(I+H(\psi_{\alpha,n}^{(-1)}))^{-1} + o_1(1).\end{aligned}$$

It follows that

$$\begin{aligned}P_n(I + H(\psi_{\alpha,n}))^{-1}P_n =& -P_n + P_n(I + H(\psi_{\alpha,n}^{(1)}))^{-1}P_n + P_n(I + H(\psi_{\alpha,n}^{(-1)}))^{-1}P_n\\ &+ P_n(I+H(\psi_{\alpha,n}^{(1)}))^{-1}H(\psi_{\alpha,n}^{(1)})H(\psi_{\alpha,n}^{(-1)})(I+H(\psi_{\alpha,n}^{(-1)}))^{-1}P_n\\ &+ o_1(1).\end{aligned}$$

Since $I - P_n = I - H(t^n)^2 = T(t^n)T(t^{-n})$ (see (35)), we have

$$\begin{aligned} H(\psi_{\alpha,n}^{(1)})(I - P_n)H(\psi_{\alpha,n}^{(-1)}) &= H(\psi_{\alpha,n}^{(1)})T(t^n)T(t^{-n})H(\psi_{\alpha,n}^{(-1)}) \\ &= T(t^{-n})H(\psi_{\alpha,n}^{(1)})H(\psi_{\alpha,n}^{(-1)})T(t^n) = o_1(1), \end{aligned}$$

where we used also Proposition 4.10 and (36). Hence we obtain

$$\begin{aligned} &P_n(I + H(\psi_{\alpha,n}))^{-1}P_n \\ &\quad = -P_n + P_n(I + H(\psi_{\alpha,n}^{(1)}))^{-1}P_n + P_n(I + H(\psi_{\alpha,n}^{(-1)}))^{-1}P_n \\ &\qquad + P_n(I + H(\psi_{\alpha,n}^{(1)}))^{-1}H(\psi_{\alpha,n}^{(1)})P_nH(\psi_{\alpha,n}^{(-1)})(I + H(\psi_{\alpha,n}^{(-1)}))^{-1}P_n + o_1(1) \\ &\quad = P_n(I + H(\psi_{\alpha,n}^{(1)}))^{-1}P_n(I + H(\psi_{\alpha,n}^{(-1)}))^{-1}P_n + o_1(1). \end{aligned}$$

Therein we employ again $(I + A)^{-1} = I - (I + A)^{-1}A = I - A(I + A)^{-1}$. Proposition 4.12(ii)-(iii) implies the uniform boundedness of $P_n(I + H(\psi_{\alpha,n}^{(1)}))^{-1}P_n$ and $P_n(I + H(\psi_{\alpha,n}^{(-1)}))^{-1}P_n$. In connection with the well-known formula

$$|\det(I + A) - \det(I + B)| \;\leq\; \|A - B\|_1 \exp(\max\{\|A\|_1, \|B\|_1\}),$$

this proves that

$$\begin{aligned} &\lim_{n\to\infty} \det\left[P_n(I + H(\psi_{\alpha,n}))^{-1}P_n\right] \\ &\quad = \lim_{n\to\infty} \det\left[P_n(I + H(\psi_{\alpha,n}^{(1)}))^{-1}P_n\right] \det\left[P_n(I + H(\psi_{\alpha,n}^{(-1)}))^{-1}P_n\right]. \end{aligned}$$

These determinants can be written as

$$\det\left[P_n(I + H(\psi_{\alpha,n}^{(\pm 1)}))^{-1}P_n\right] = \det\left[I + P_n(I + H(\psi_{\alpha,n}^{(\pm 1)}))^{-1}P_n - P_n\right],$$

and now the convergence in the trace norm stated in Proposition 4.12(ii)-(iii) implies the desired assertion. We remark in this connection that the sequence $\mu_{\alpha,n}$ defined in (61) satisfies condition (56). □

In regard to the next result, recall the definition (15) of the functions $\hat{u}_\beta$.

Theorem 5.6. *We have*

$$\begin{aligned} \det(I - K_\alpha) = \exp\left(-\frac{\alpha^2}{8}\right) &\det\left[\Pi_{\frac{\alpha}{2}}(I + H_{\mathbb{R}}(\hat{u}_{-1/2}))^{-1}\Pi_{\frac{\alpha}{2}})\right] \\ &\times \det\left[\Pi_{\frac{\alpha}{2}}(I - H_{\mathbb{R}}(\hat{u}_{1/2}))^{-1}\Pi_{\frac{\alpha}{2}}\right], \end{aligned} \tag{64}$$

where all expressions on the right-hand side are well defined.

Proof. We combine Proposition 5.1 with Proposition 5.4 and Proposition 5.5 to conclude that

$$\begin{aligned} \det(I - K_\alpha) = \exp\left(-\frac{\alpha^2}{8}\right) &\det\left[H(h_\alpha)(I + H(u_{-1/2,1}))^{-1}H(h_\alpha)\right] \\ &\times \det\left[H(h_\alpha)(I - H(u_{1/2,1}))^{-1}H(h_\alpha)\right]. \end{aligned}$$

Let us now introduce a unitary transform $S = \mathcal{F}U$, i.e.,

$$S : H^2(\mathbb{T}) \xrightarrow{U} H^2(\mathbb{R}) \xrightarrow{\mathcal{F}} L^2(\mathbb{R}_+),$$

where $(Uf) = \frac{1}{\sqrt{\pi}(1-ix)} f\left(\frac{1+ix}{1-ix}\right)$, $\mathcal{F}$ is the Fourier transform, and $H^2(\mathbb{R})$ is the Hardy space on $\mathbb{R}$, i.e., the set of all functions $f \in L^2(\mathbb{R})$ for which $(\mathcal{F}f)(x) = 0$ for $x < 0$. Using the definitions (7) and (13) it is straightforward to prove that $H_{\mathbb{R}}(a) = SH(b)S^*$, where

$$a(x) = b\left(\frac{1+ix}{1-ix}\right). \tag{65}$$

In particular,

$$H_{\mathbb{R}}(\hat{u}_\beta) = SH(u_{\beta,1})S^*, \qquad H_{\mathbb{R}}(e^{ix\alpha/2}) = SH(h_\alpha)S^*.$$

It remains to remark that $H(e^{ix\alpha/2})^2 = \Pi_{\alpha/2}$. The invertibility of $I \pm H_{\mathbb{R}}(\hat{u}_{\mp 1/2})$ follows from the invertibility of $I \pm H(u_{\mp 1/2})$. □

Now we state the following asymptotic formulas for the two operator determinants appearing on the right-hand side of (64). For convenience we make a change in variables $\alpha \mapsto 2\alpha$.

These formulas are proved in [3] (see Sect. 3.6). In these formulas, $G(z)$ stands for the Barnes G-function [1]. In regard to the second fomula, the simple computation $G(3/2) = G(1/2)\Gamma(1/2) = G(1/2)\pi^{1/2}$ has to be done.

Theorem 5.7. *The following asymptotic formulas hold,*

$$\det\left[\Pi_\alpha(I + H_{\mathbb{R}}(\hat{u}_{-1/2}))^{-1}\Pi_\alpha\right] \sim \alpha^{-1/8}\pi^{1/4}2^{1/4}G(1/2), \qquad \alpha \to \infty, \tag{66}$$

$$\det\left[\Pi_\alpha(I - H_{\mathbb{R}}(\hat{u}_{1/2}))^{-1}\Pi_\alpha\right] \sim \alpha^{-1/8}\pi^{1/4}2^{-1/4}G(1/2), \qquad \alpha \to \infty. \tag{67}$$

Combining the previous results we get the desired asymptotic formula.

Theorem 5.8. *The asymptotic formula*

$$\log\det(I - K_{2\alpha}) = -\frac{\alpha^2}{2} - \frac{\log\alpha}{4} + C + o(1), \qquad \alpha \to \infty, \tag{68}$$

holds with the constant

$$C = \frac{\log 2}{12} + 3\zeta'(-1). \tag{69}$$

Proof. The previous two theorems give the asymptotic formula

$$\det(I - K_{2\alpha}) \sim \exp\left(-\frac{\alpha^2}{2}\right)\alpha^{-1/4}\pi^{1/2}(G(1/2))^2, \qquad \alpha \to \infty. \tag{70}$$

We can express $G(1/2)$ in terms of $\zeta'(-1)$, where ζ is Riemann's zeta function. According to [1, p. 290] we have

$$\log G(1/2) = -\frac{\log\pi}{4} + \frac{1}{8} - \frac{3\log A}{2} + \frac{\log 2}{24}$$

with $A = \exp(-\zeta'(-1) + 1/12)$ being Glaisher's constant. Hence

$$2\log G(1/2) = -\frac{\log\pi}{2} + 3\zeta'(-1) + \frac{\log 2}{12},$$

which implies the desired asymptotic formula (68) with the constant (69). □

References

1. Barnes, E.B.: The theory of the G-function. Quart. J. Pure Appl. Math. **XXXI**, 264–313 (1900)
2. Basor, E.L., Ehrhardt, T.: Some identities for determinants of structured matrices. Linear Algebra Appl. **343/344**, 5–19 (2002)
3. Basor, E.L., Ehrhardt, T.: On the asymptotics of certain Wiener-Hopf-plus-Hankel determinants. New York J. Math. **11**, 171–203 (2005)
4. Basor, E.L., Tracy, C.A., Widom, H.: Asymptotics of level-spacing distributions for random matrices. Phys. Rev. Lett. **69**(1), 5–8 (1992)
5. Böttcher, A., Silbermann, B.: *Analysis of Toeplitz operators*. Berlin: Springer, 1990
6. Deift, P.A., Its, A.R., Zhou, X.: A Riemann-Hilbert approach to asymptotic problems arising in the theory of random matrix models, and also in the theory of integrable statistical mechanics. Ann. Math. **146**, 149–235 (1997)
7. Dyson, F.J.: Fredholm determinants and inverse scattering problems. Commun. Math. Phys. **47**, 171–183 (1976)
8. Ehrhardt, T., Silbermann, B.: Approximate identities and stability of discrete convolution operators with flip. Operator Theory: Adv. Appl., Vol. **110**, 103–132 (1999)
9. Gohberg, I., Krein, M.G.: Introduction to the theory of linear nonselfadjoint operators in Hilbert space. Trans. Math. Monographs **18**, Providence, R.I.: Amer. Math. Soc., 1969
10. Metha, M.L.: *Random Matrices*. 6$^{\text{th}}$ ed., San Diego: Academic Press, 1994
11. Jimbo, M., Miwa, T., Môri, Y., Sato, M.: Density matrix of an impenetrable Bose gas and the fifth Painlevé transcendent. Phys. D **1**(1), 80–158 (1980)
12. Krasovsky, I. V.: Gap probability in the spectrum of random matrices and asymptotics of polynomials orthogonal on an arc of the unit circle. Int. Math. Res. Not. (25), 1249–1272 (2004)
13. Power, S.: The essential spectrum of a Hankel operator with piecewise continuous symbol. Michigan Math. J. **25**(1), 117–121 (1978)
14. Power, S.: C*-algebras generated by Hankel operators and Toeplitz operators. J. Funct. Anal. **31**(31), 52–68 (1979)
15. Tracy, C.A., Widom, H.: Introduction to random matrices. In: *Lecture Notes in Physics*, Vol. **424**, Berlin: Springer, 1993, pp. 103–130
16. Widom, H.: The strong Szegö limit theorem for circular arcs. Indiana Univ. Math. J. **21**, 277–283 (1971/1972)
17. Widom, H.: The asymptotics of a continuous analogue of orthogonal polynomials. J. Appr. Theory **77**, 51–64 (1994)
18. Widom, H.: Asymptotics for the Fredholm determinant of the sine kernel on a union of intervals. Commun. Math. Phys. **171**, 159–180 (1995)

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